GRAPH PROBLEMS IN CALL MODELS AND SWITCHING NETWORKS

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Abdullah Atmaca
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We certify that we have read this dissertation and that in our opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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ABSTRACT

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In the first part of this dissertation, we focus on graph problems that arise in call models. Such models are used to study the combinatorial properties of certain types of calls that include unicast, multicast, and bicast interconnections. Here we focus on bicast calls, and provide closed-form expressions for the number of unlabeled bicast calls when either the number of callers or number of receivers is fixed to 2 or 3. We then obtain lower and upper bounds on the number of such calls by solving an open problem in graph theory, namely counting the number of unlabeled bipartite graphs. Next, these results are extended to left (right) set labeled and set labeled bipartite graphs. In the second part of the dissertation, we focus on wiring and routing problems for one-sided, binary tree switching networks. Specifically, we reduce the $O(n)$ time complexity of the routing algorithm for the one-sided, binary tree switching networks to $O(\lg n)$. We also present a new wiring algorithm for one-sided, binary tree switching networks. Finally, an algorithm is presented to locate the cluster in which the terminals of the corresponding one-sided binary tree switching network are paired. The time complexity of this algorithm is shown to be $O(\lg n)$.

Keywords: Bipartite graphs, Polya's counting theorem, cycle index polynomial, switching networks, call models.

ÖZET

ÇAĞRI MODELLERİ VE ANAHTARLAMA AĞLARINDA ÇİZGE PROBLEMLERİ

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Bu tezin ilk bölümünde, çağrı modellerinde ortaya çıkan çizge problemlerine odaklanılmaktadır. Bu tür modeller, tekli çağrı, çoklu çağrı ve karşılıklı çoklu çağrı bağlantılarını içeren bazı çağrı tiplerinin kombinatoryel özelliklerini incelemek için kullanılır. Burada, karşılıklı çoklu çağrılara odaklanıyoruz ve arayanların sayısı veya alıcıların sayısı 2 veya 3'e sabitlendiğinde etiketsiz karşılıklı çoklu çağrıların sayısı için kapalı form ifadeleri sağlıyoruz. Bu durumda, çizge teorisinde açık bir problemi çözerek, yani etiketsiz iki parçalı çizgeleri sayarak bu tür çağrıların sayısıyla ilgili alt ve üst sınırlar elde ediyoruz. Daha sonra, bu sonuçlar, sol(sağ) tarafı küme olarak etiketli ve iki tarafı da küme olarak etiketli iki parçalı çizgelere genişletilmektedir. Tezin ikinci bölümünde, tek taraflı, ikili ağaç anahtarlama ağları için bağlama ve yönlendirme problemlerine odaklanıyoruz. Özellikle, tek taraflı, ikili ağaç anahtarlama ağları için yönlendirme algoritmasının $O(n)$ hesaplama zamanını $O(\lg n)$'e düşürüyoruz. Tek taraflı, ikili ağaç anahtarlama ağları için yeni bir bağlama algoritması da sunuyoruz. Son olarak, bağlama tasarımı verilen tek taraflı, ikili ağaç anahtarlama ağının terminallerinin eşleştirildiği kümenin yerini belirlemek için bir algoritma sunulmuştur. Bu algoritmanın zaman karmaşıklığının $O(\lg n)$ olduğu gösterilmiştir.

Anahtar sözcükler: İki parçalı çizgeler, Polya sayma teoremi, döngü endeks polinomu, anahtarlama ağları, çağrı modelleri.

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Contents

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List of Symbols

$B_u(n,r)$	The set of unlabeled bipartite graphs with n left and r right vertices.
S_n	The symmetric group of permutations of degree n.
Z_{S_n}	The cycle index polynomial of S_n.
$B_x(n,r)$	The set of all left-set-labeled bipartite graphs with n left and r right vertices.
$B_x(n,r,i)$	The set of (n,r)-left-set-labeled bipartite graphs in each of which the degrees of exactly i left vertices are greater than 0.
$\overline{B}_x(n,r)$	The set of all (n,r)-unlabeled bipartite graphs in each of which the degrees of all left vertices are greater than 0.
$B_{xy}(n,r)$	The set of all (n,r)-set-labeled bipartite graphs.
$\overline{B}_{xy}(i,j)$	The set of all (i,j)-unlabeled bipartite graphs such that there is no vertex in the graph that has degree of 0.

Chapter 1

Introduction

This dissertation is concerned with the solutions of certain graph problems in combinatorial call models and switching networks. Switching networks have beeen extensively investigated in connection with communication systems, and multiprocessing and parallel computing [1–18]. Our work is motivated by the recent renewed interest in such call models and networks due to the introduction of on-chip systems, especially network-on-chip architectures. In particular, one-sided switching networks were reported in [19] as a possible network architecture for an on-chip network. Two problems were introduced in this connection to reduce the area/volume complexity of the targeted on-chip network. One deals with wiring replicates of terminals that represent cores, while the other is concerned with routing connection requests. Even though some solutions were provided for both of these problems in [19–21], these solutions place a restriction on the number of terminals in the case of wiring, and an exact excessive time complexity in the case of routing. Both these issues have been addressed in this dissertation and resolved. On the other hand, call models have been introduced in [19] to classify switching networks. Effectively, they capture the multiplicities of calls and ordering of callers and/or receivers under various call scenarios. Each of these call models can be represented by a bipartite graph with certain conditions. Broadly speaking, three call models distinguish between the multiplicity properties of calls, and three more conditions are added to each of these call models to characterize the ordering between the callers and receivers. A comprehensive description of these different call models can be found in [19]. The enumeration of calls leads to the crosspoint complexity of switching networks that can realize the set of calls defined by such call models using a logarithmic transformation. One problem that has been highlighted in [19] is the enumeration of the corresponding bipartite graphs that represent the call models of interest. A number of formulas have been provided in [19], but the enumerations of bipartite graphs in some of the call models have not been concluded with asymptotic closed-form formulas. One of the main contributions of this dissertation is to settle this problem. More specifically, we provide both exact and asymptotic formulas that count the number of bipartite graphs in the aforementioned call models; in particular for unlabeled bicast, left(right) set labeled and set labeled bicast calls.

The contributions of this dissertation are stated below:

1. An exact closed-form expression for the number of unlabeled bipartite graphs one of whose parts consists of two vertices.
2. An exact closed-form expression for the number of unlabeled bipartite graphs one of whose parts consists of three vertices.
3. The solution of the long-standing open problem that was stated in 1973 [22]. Specifically, a lower bound on the number of unlabeled bipartite graphs, and an upper bound within a factor of two of the lower bound have been established using Polya's Counting Theorem.
4. The results in (1), (2), and (3) have been extended to left (right) set labeled and set labeled bipartite graphs.
5. The $O(n)$ time complexity of the routing algorithm given in [20] for the one-sided, binary tree switching network has been reduced to $O(\lg n)$.
6. A new wiring algorithm has been given for one-sided, binary tree switching networks.

7. An algorithm has been presented to locate the cluster in which the terminals of the corresponding one-sided, binary tree switching network are paired. The time complexity of this algorithm is $O(\lg n)$.

The rest of this dissertation is organized as follows. In Chapters 2 and 3, enumerations of unlabeled bipartite graphs are considered. We give exact results for the size of two families of unlabeled bipartite graphs in Chapter 2 and derive lower and upper bounds on the number of unlabeled bipartite graphs in Chapter 3. We extend these calculations to left set labeled and set labeled bipartite graphs in Chapter 4. In Chapter 5, we reduce the time complexity of a routing method in one-sided switching network. We also introduce a new wiring method to cluster replicates of cores into clusters and provide a new routing algorithm for this wiring architect. Finally, we conclude the dissertation in Chapter 6. The appendix lists computer programs that implement the proposed algorithms and provides some sample results.

Chapter 2

Counting Two Families of Unlabeled Bipartite Graphs[1]

In this chapter, we present two results that provide the exact number of distinct unlabeled bipartite graphs when the cardinality of one of the sets of vertices is fixed to 2 or 3.

2.1 Preliminary Facts

This problem has been investigated in connection with the enumeration of unlabeled bipartite graphs and binary matrices [22]. Our work has been motivated in part by a counting problem that arises in the representation of calls in interconnection networks [19]. Let (I, O, E) denote a graph with two disjoint sets of vertices, I, called *left vertices* and a set of vertices, O, called *right vertices*, where each edge in E connects a left vertex with a right vertex. We let $n = |I|$, $r = |O|$, and refer to such a graph as an (n, r)-bipartite graph. Let $G_1 = (I, O, E_1)$ and $G_2 = (I, O, E_2)$ be two (n, r)-bipartite graphs, and $\alpha : I \to I$ and $\beta : O \to O$ be both bijections. The pair (α, β) is an isomorphism between G_1 and G_2 provided

[1] **This is joint work with Prof. A. Yavuz Oruç.**

that $((\alpha(v_1), \beta(v_2)) \in E_2$ if and only if $(v_1, v_2) \in E_1$, $\forall v_1 \in I, \forall v_2 \in O$. It is easy to establish that this mapping induces an equivalence relation, and partitions the set of 2^{nr} (n, r)-bipartite graphs into equivalence classes. This equivalence relation captures the fact that the vertices in I and O are unlabeled, and so each class of (n, r)-bipartite graphs can be represented by any one of the graphs in that class without identifying the vertices in I and O. Let $B_u(n, r)$ denote any set of (n, r)-bipartite graphs that contains exactly one such graph from each of the equivalence classes of (n, r)-bipartite graphs induced by the isomorphism we defined. It is easy to see that determining $|B_u(n, r)|$ amounts to an enumeration of non-isomorphic (n, r)-bipartite graphs that will henceforth be referred to as unlabeled (n, r)-bipartite graphs. Figure 2.1 depicts the unlabeled bipartite graphs for $n = 3$ and $r = 2$.

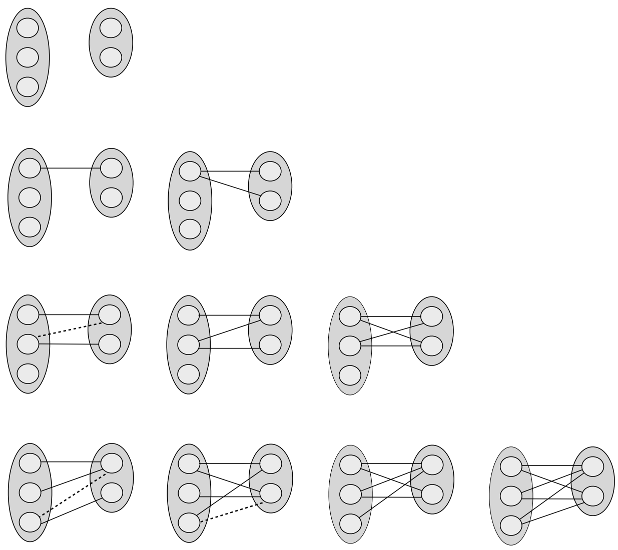

Figure 2.1: Unlabeled bipartite graphs for $n = 3$ and $r = 2$

In [22], Harrison used Pólya's counting theorem to obtain an expression to compute the number of non-equivalent $n \times r$ binary matrices. This expression contains a nested sum, in which one sum is carried over all partitions of n while the other is carried over all partitions of r, where the argument of the nested sum involves factorial, exponentiation and greatest common divisor (gcd) computations. He further established that this formula also enumerates the number of unlabeled (n, r)-bipartite graphs. A number of results indirectly related to Harrison's work appeared in the literature [23–26]. In particular, the set $B_u(n, r)$ in our work coincides with the set of bicolored graphs described in Section 2 in [23]. Whereas Harary [23] provides a counting polynomial for the number of bicolored graphs, we focus on the asymptotic behavior of $|B_u(n, r)|$. Counting polynomials for other families of bipartite graphs were also reported in [24]. Likewise, Hanlon [25], and Gainer-Dewar and Gessel [26] provide generating functions for related bipartite graph counting problems without an asymptotic analysis as provided in our contributions. The species and category theory approach in [26] leads to a summation formula for the number of unlabeled bipartite graphs with v vertices. This formula is similar to the expression in (8) in [22] except that the latter formula counts the number of unlabeled bipartite graphs whose vertices are divided into two disjoint sets as in the model that we used in our research. As such, for fixed n and r, the set $B_u(n, r)$ forms a subset of the set of unlabeled bipartite graphs with v vertices that are counted in [25,26], where $v = n+r$. It should also be mentioned that some results on asymptotic enumeration of certain families of bipartite graphs (binary matrices) have been reported (see for example, [27–30]).

That $|B_u(1, r)| = r+1$ trivially holds. Exact closed form expressions for $|B_u(n, r)|$ for $n = 2$, $n = 3$, and any integer $r > n$ will be given in the remaining of this chapter.

Let S_n denote the symmetric group of permutations of degree n acting on set $N = \{1, 2, \cdots, n\}$. Suppose that the $n!$ permutations in S_n are indexed by $1, 2, \cdots, n!$ in some arbitrary, but fixed manner. The cycle index polynomial of S_n is defined as follows([31], see p. 35, Eqn. 2.2.1):

$$Z_{S_n}(x_1, x_2, \cdots, x_n) = \frac{1}{n!} \sum_{m=1}^{n!} \prod_{k=1}^{n} x_k^{p_{m,k}} \tag{2.1}$$

where $p_{m,k}$ denotes the number of cycles of length k in the disjoint cycle representation of the m^{th} permutation in S_n, and $\sum_{k=1}^{n} k p_{m,k} = n, \forall m = 1, 2, \cdots, n!$.

Let $S_n \times S_r$ denote the direct product of symmetric groups S_n and S_r acting on $N = \{1, 2, \cdots, n\}$ and $R = \{1, 2, \cdots, r\}$, respectively, where n and r are positive integers such that $n < r$. It can be inferred from Harrison ([32], Lemma 4.1 and Theorem 4.2) that the cycle index polynomial of $S_n \times S_r$ is given by

$$Z_{S_n \times S_r}(x_1, x_2, \cdots, x_{nr}) = Z_{S_n}(x_1, x_2, \cdots, x_n) \boxtimes Z_{S_r}(x_1, x_2, \cdots, x_r), \qquad (2.2)$$

where $\boxtimes$ is a particular polynomial multiplication that distributes over ordinary addition, and in which the multiplication $X_m \bigodot X_t$ of two product terms, $X_m = x_1^{p_{m,1}} x_2^{p_{m,2}} \cdots x_n^{p_{m,n}}$ and $X_t = x_1^{q_{t,1}} x_2^{q_{t,2}} \cdots x_r^{q_{t,r}}$ in Z_{S_n} and Z_{S_r}, respectively, is defined as

$$X_m \bigodot X_t = \prod_{k=1}^{n} \prod_{j=1}^{r} x_{\mathrm{lcm}(k,j)}^{p_{m,k} q_{t,j} \gcd(k,j)}. \qquad (2.3)$$

Note that we will not display the zero powers of $x_1, x_2, \cdots$ in a cycle index polynomial. We will use the same convention for all other cycle index polynomials throughout the thesis. The lcm(a,b) and gcd(a,b) denote least common multiple and greatest common divisor of a and b.

Harrison further proved that [22]:

$$|B_u(n, r)| = Z_{S_n \times S_r}(\underbrace{2, 2, .., 2}_{nr}) \qquad (2.4)$$

when $n \neq r$. As noted in [22], $n = r$ case involves a different cycle index polynomial and will be omitted here as well.

We need one more fact that can be found in Harary ([31], p. 36) in order to compute $|B_u(2, r)|$ and $|B_u(3, r)|$:

$$Z_{S_r}(x_1, x_2, \ldots\ldots, x_r) = \frac{1}{r} \sum_{i=1}^{r} x_i Z_{S_{r-i}}(x_1, x_2, \ldots\ldots, x_{r-i}) \qquad (2.5)$$

where $Z_{S_0}() = 1$.

2.2 A Closed-Form Expression for $|B_u(2,r)|$

We use Polya's counting theorem (See [33]), in particular Harrison's cycle index formulation in [22] to compute $|B_u(2,r)|$.

We calculate $|B_u(2,r)|$ as follows:

$$
\begin{aligned}
|B_u(2,r)| &= Z_{S_2\times S_r}(2,2,\ldots,2), && (2.6)\\
&= [Z_{S_2}(x_1,x_2)\boxtimes Z_{S_r}(x_1,x_2,\ldots,x_r)](2,\ldots,2), && (2.7)\\
&= \left[\left(\frac{1}{2}\left(x_1^2+x_2\right)\right)\boxtimes Z_{S_r}(x_1,x_2,\ldots,x_r)\right](2,\ldots,2), && (2.8)\\
&= \frac{1}{2}\left[x_1^2\boxtimes Z_{S_r}(x_1,x_2,\ldots,x_r)+x_2\boxtimes Z_{S_r}(x_1,x_2,\ldots,x_r)\right](2,\ldots,2), && (2.9)\\
&= \frac{1}{2}\left\{\left[x_1^2\boxtimes\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r}x_j^{q_{t,j}}\right](2,\ldots,2)+\left[x_2\boxtimes\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r}x_j^{q_{t,j}}\right](2,\ldots),\right\}, && (2.10)\\
&= \frac{1}{2}\left\{\left[\frac{1}{r!}\sum_{t=1}^{r!}x_1^2\bigodot\prod_{j=1}^{r}x_j^{q_{t,j}}\right](2,\ldots,2)+\left[\frac{1}{r!}\sum_{t=1}^{r!}x_2\bigodot\prod_{j=1}^{r}x_j^{q_{t,j}}\right](\ldots)\right\}, && (2.11)\\
&= \frac{1}{2}\left\{\left[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r}x_{\mathrm{lcm}(1,j)}^{2q_{t,j}\gcd(1,j)}\right](2,\ldots,2)+\left[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r}x_{\mathrm{lcm}(2,j)}^{q_{t,j}\gcd(2,j)}\right](\ldots).\right\}, && (2.12)\\
&= \frac{1}{2}\left\{\left[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r}x_j^{2q_{t,j}}\right](2,\ldots,2)+\left[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r}x_{\mathrm{lcm}(2,j)}^{q_{t,j}\gcd(2,j)}\right](2,\ldots,2)\right\}, && (2.13)\\
&= \frac{1}{2}\left\{\left[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r}2^{2q_{t,j}}\right]+\left[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r}2^{q_{t,j}\gcd(2,j)}\right]\right\}, && (2.14)\\
&= \frac{1}{2}\left\{\left[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r}(2^2)^{q_{t,j}}\right]+\left[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{\text{odd } j}2^{q_{t,j}}\prod_{\text{even } j}(2^2)^{q_{t,j}}\right]\right\}, && (2.15)\\
&= \frac{1}{2}\left\{\left[Z_{S_r}(2^2,2^2,\ldots,2^2)\right]+\left[Z_{S_r}(2,2^2,2,2^2,\ldots)\right]\right\}. && (2.16)
\end{aligned}
$$

Thus, we have reduced the computation of $|B_u(2,r)|$ to computing the two terms in Eqn. 2.16. These computations are carried out in the next two lemmas.

Lemma 1.

$$Z_{S_r}(2^2,2^2,\ldots,2^2)=\binom{r+3}{r}. \qquad (2.17)$$

Proof. Using Eqn. 2.5, we have

$$rZ_{S_r}(2^2, 2^2, \ldots, 2^2) = \sum_{i=1}^{r} 2^2 Z_{S_{r-i}}(2^2, 2^2, \ldots, 2^2), \tag{2.18}$$

$$(r-1)Z_{S_{r-1}}(2^2, 2^2, \ldots, 2^2) = \sum_{i=1}^{r-1} 2^2 Z_{S_{r-1-i}}(2^2, 2^2, \ldots, 2^2). \tag{2.19}$$

Subtracting the second equation from the first one and simplifying it gives

$$rZ_{S_r}(2^2, 2^2, \ldots, 2^2) - (r-1)Z_{S_{r-1}}(2^2, 2^2, \ldots, 2^2) = 4Z_{S_{r-1}}(2^2, 2^2, \ldots, 2^2), \tag{2.20}$$

$$Z_{S_r}(2^2, 2^2, \ldots, 2^2) = (\frac{r+3}{r})Z_{S_{r-1}}(2^2, 2^2, \ldots, 2^2). \tag{2.21}$$

Expanding the last equation recursively, we obtain

$$Z_{S_r}(2^2, 2^2, \ldots, 2^2) = (\frac{r+3}{r})(\frac{r+2}{r-1})Z_{S_{r-2}}(2^2, 2^2, \ldots, 2^2), \tag{2.22}$$

$$= (\frac{r+3}{r})(\frac{r+2}{r-1})(\frac{r+1}{r-2})\ldots(\frac{4}{1})Z_{S_0}(). \tag{2.23}$$

Noting that $Z_{S_0}() = 1$ proves the statement, i.e.,

$$Z_{S_r}(2^2, 2^2, \ldots, 2^2) = \binom{r+3}{r}. \quad \square \tag{2.24}$$

Lemma 2.

$$Z_{S_r}(2, 2^2, 2, 2^2, \ldots) = \frac{2r^2 + 8r + 7 + (-1)^r}{8}. \tag{2.25}$$

Proof. By Eqn. 2.5,

$$rZ_{S_r}(2, 2^2, \ldots) = \sum_{\text{odd } i}^{r-\beta_1} 2Z_{S_{r-i}}(2, 2^2, \ldots) + \sum_{\text{even } i}^{r-\beta_2} 2^2 Z_{S_{r-i}}(2, 2^2, \ldots), \tag{2.26}$$

where $\beta_1 = 1, \beta_2 = 0$ if r is even and $\beta_1 = 0, \beta_2 = 1$ if r is odd. Similarly, for $r-2$,

$$(r-2)Z_{S_{r-2}}(2, 2^2, \ldots) = \sum_{\text{odd } i}^{r-2-\beta_1} 2Z_{S_{r-2-i}}(2, 2^2, \ldots) + \sum_{\text{even } i}^{r-2-\beta_2} 2^2 Z_{S_{r-2-i}}(2, 2^2, \ldots). \tag{2.27}$$

Subtracting the second equation from the first one and rearranging the terms gives

$$rZ_{S_r}(2, 2^2, \ldots) = 2Z_{S_{r-1}}(2, 2^2, \ldots) + (r+2)Z_{S_{r-2}}(2, 2^2, \ldots). \tag{2.28}$$

We now use induction and this recurrence to prove that Eqn. 2.25 holds.

Basis $r = 0$. Substituting $r = 0$ in Eqn. 2.25 gives 1 as it should since $Z_{S_0}() = 1$. $r = 1$. Substituting $r = 1$ in Eqn. 2.25 gives

$$Z_{S_1}(2) = \frac{2(1)^2 + 8(1) + 7 + (-1)^1}{8} = 2, \tag{2.29}$$

and this agrees with Eqn. 2.5, i.e., $Z_{S_1}(2) = \frac{1}{1}(2Z_{S_0}()) = 2$.

Induction Step:

Suppose that Eqn. 2.25 holds for $r - 2$ and $r - 1$. Then by Eqn. 2.28, we have

$$\begin{aligned} rZ_{S_r}(2, 2^2, \ldots) &= 2Z_{S_{r-1}}(2, 2^2, \ldots) + (r+2)Z_{S_{r-2}}(2, 2^2, \ldots), && (2.30)\\ &= 2\frac{2(r-1)^2 + 8(r-1) + 7 + (-1)^{r-1}}{8} \\ &+ (r+2)\frac{2(r-2)^2 + 8(r-2) + 7 + (-1)^{r-2}}{8}, && (2.31)\\ &= r\frac{2r^2 + 8r + 7 + (-1)^r}{8}, && (2.32)\end{aligned}$$

that agrees with Eqn. 2.25. □

Finally, by combining Lemmas 1 and 2, we have

Theorem 1.

$$|B_u(2, r)| = \frac{2r^3 + 15r^2 + 34r + 22.5 + 1.5\,(-1)^r}{24}. \quad \square \tag{2.33}$$

2.3 A Closed-Form Expression for $|B_u(3,r)|$

We proceed as in the computation of $|B_u(2,r)|$.

$$
\begin{aligned}
|B_u\,(3,r)\,| &= Z_{S_3\times S_r}(2,2,\ldots\ldots,2), & (2.34)\\
&= \left[Z_{S_3}(x_1,x_2,x_3)\boxtimes Z_{S_r}(x_1,x_2,\ldots\ldots,x_r)\right](2,2,...,2), & (2.35)\\
&= \left[\left(\frac{1}{6}\left(x_1^3+3x_1x_2+2x_3\right)\right)\boxtimes Z_{S_r}(x_1,x_2,\ldots\ldots,x_r)\right](2,2,...,2), & (2.36)\\
&= \frac{1}{6}\left[x_1^3\boxtimes Z_{S_r}(x_1,x_2,\ldots\ldots,x_r)\right](2,2,...,2)+\\
&\quad \frac{1}{6}\left[3x_1x_2\boxtimes Z_{S_r}(x_1,x_2,\ldots\ldots,x_r)\right](2,2,...,2)+\\
&\quad \frac{1}{6}\left[2x_3\boxtimes Z_{S_r}(x_1,x_2,\ldots\ldots,x_r)\right](2,\ldots,2), & (2.37)\\
&= \frac{1}{6}\Bigg\{\left[x_1^3\boxtimes\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r}x_j^{q_{t,j}}\right](2,\ldots,2)+\left[3x_1x_2\boxtimes\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r}x_j^{q_{t,j}}\right](2,\ldots,2)+\\
&\quad \left[2x_3\boxtimes\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r}x_j^{q_{t,j}}\right](2,\ldots,2)\Bigg\}, & (2.38)\\
&= \frac{1}{6}\Bigg\{\left[\frac{1}{r!}\sum_{t=1}^{r!}x_1^3\bigodot\prod_{j=1}^{r}x_j^{q_{t,j}}\right](2,\ldots,2)+\left[\frac{3}{r!}\sum_{t=1}^{r!}x_1x_2\bigodot\prod_{j=1}^{r}x_j^{q_{t,j}}\right](2,\ldots,2)+\\
&\quad \left[\frac{2}{r!}\sum_{t=1}^{r!}x_3\bigodot\prod_{j=1}^{r}x_j^{q_{t,j}}\right](2,\ldots,2)\Bigg\}, & (2.39)\\
&= \frac{1}{6}\Bigg\{\left[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r}x_{\mathrm{lcm}(1,j)}^{3q_{t,j}\gcd(1,j)}\right](2,\ldots,2)+\left[\frac{3}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r}x_{\mathrm{lcm}(1,j)}^{q_{t,j}\gcd(1,j)}x_{\mathrm{lcm}(2,j)}^{q_{t,j}\gcd(2,j)}\right](\ldots)+\\
&\quad \left[\frac{2}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r}x_{\mathrm{lcm}(3,j)}^{q_{t,j}\gcd(3,j)}\right](2,\ldots,2)\Bigg\}, & (2.40)\\
&= \frac{1}{6}\Bigg\{\left[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r}x_j^{3q_{t,j}}\right](2,\ldots,2)+\left[\frac{3}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r}x_j^{q_{t,j}}x_{\mathrm{lcm}(2,j)}^{q_{t,j}\gcd(2,j)}\right](2,2,...,2)+\\
&\quad \left[\frac{2}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r}x_{\mathrm{lcm}(3,j)}^{q_{t,j}\gcd(3,j)}\right](2,\ldots,2)\Bigg\}, & (2.41)\\
&= \frac{1}{6}\Bigg\{\left[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r}2^{3q_{t,j}}\right]+\left[\frac{3}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r}2^{q_{t,j}}2^{q_{t,j}\gcd(2,j)}\right]+\left[\frac{2}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r}2^{q_{t,j}\gcd(3,j)}\right]\Bigg\}, & (2.42)
\end{aligned}
$$

$$= \frac{1}{6}\Bigg\{\Big[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r}(2^3)^{q_{t,j}}\Big]+3\Big[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{\text{odd } j}(2^2)^{q_{t,j}}\prod_{\text{even } j}(2^3)^{q_{t,j}}\Big]+2\Big[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j \bmod 3=0}(2^3)^{q_{t,j}}\prod_{j \bmod 3\neq 0}2^{q_{t,j}}\Big]\Bigg\}, \quad (2.43)$$

$$= \frac{1}{6}\Bigg\{\Big[Z_{S_r}(2^3,2^3,\ldots,2^3)\Big]+3\Big[Z_{S_r}(2^2,2^3,2^2,2^3,\ldots)\Big]+2\Big[Z_{S_r}(2,2,2^3,2,2,2^3,\ldots)\Big]\Bigg\}. \quad (2.44)$$

Thus, we have reduced the computation of $|B_u(3,r)|$ to computing the three terms in Eqn. 2.44. These computations are carried out in the next three lemmas.

Lemma 3.

$$Z_{S_r}(2^3,2^3,\ldots,2^3) = \binom{r+7}{r}. \quad (2.45)$$

Proof. Using Eqn. 2.5, we have

$$rZ_{S_r}(2^3,2^3,\ldots,2^3) = \sum_{i=1}^{r} 2^3 Z_{S_{r-i}}(2^3,2^3,\ldots,2^3), \quad (2.46)$$

$$(r-1)Z_{S_{r-1}}(2^3,2^3,\ldots,2^3) = \sum_{i=1}^{r-1} 2^3 Z_{S_{r-1-i}}(2^3,2^3,\ldots,2^3). \quad (2.47)$$

Subtracting the second equation from the first one and simplifying it give

$$rZ_{S_r}(2^3,2^3,\ldots,2^3) - (r-1)Z_{S_{r-1}}(2^3,2^3,\ldots,2^3) = 8Z_{S_{r-1}}(2^3,2^3,\ldots,2^3), \quad (2.48)$$

$$Z_{S_r}(2^3,2^3,\ldots,2^3) = (\frac{r+7}{r})Z_{S_{r-1}}(2^3,2^3,\ldots,2^3). \quad (2.49)$$

Expanding the last equation recursively, we obtain

$$Z_{S_r}(2^3,2^3,\ldots,2^3) = (\frac{r+7}{r})(\frac{r+6}{r-1})Z_{S_{r-2}}(2^3,2^3,\ldots,2^3), \quad (2.50)$$

$$= (\frac{r+7}{r})(\frac{r+6}{r-1})(\frac{r+5}{r-2})\ldots(\frac{8}{1})Z_{S_0}(). \quad (2.51)$$

Noting that $Z_{S_0}() = 1$ proves the statement, i.e.,

$$Z_{S_r}(2^3,2^3,\ldots,2^3) = \binom{r+7}{r}. \quad \square \quad (2.52)$$

Lemma 4.

$$Z_{S_r}(2^2,2^3,2^2,2^3,\ldots) = \frac{(r+4)\left(2r^4+32r^3+172r^2+352r+15\left(-1\right)^r+225\right)}{960}. \tag{2.53}$$

Proof. We consider two cases:

Case 1: $r \bmod 2 = 0$.

By Eqn. 2.5,

$$rZ_{S_r}(2^2,2^3,2^2,2^3,\ldots) = \sum_{\text{odd } i}^{r-1} 2^2 Z_{S_{r-i}}(2^2,2^3,2^2,2^3,\ldots) + \sum_{\text{even } i}^{r} 2^3 Z_{S_{r-i}}(2^2,2^3,2^2,2^3,\ldots), \tag{2.54}$$

and

$$(r-2)Z_{S_{r-2}}(2^2,2^3,2^2,2^3,\ldots) = \sum_{\text{odd } i}^{r-3} 2^2 Z_{S_{r-2-i}}(2^2,2^3,2^2,2^3,\ldots) + \sum_{\text{even } i}^{r-2} 2^3 Z_{S_{r-2-i}}(2^2,2^3,2^2,2^3,\ldots). \tag{2.55}$$

Subtracting the second equation from the first one and rearranging the terms give

$$rZ_{S_r}(2^2,2^3,2^2,2^3,\ldots) = 4Z_{S_{r-1}}(2^2,2^3,2^2,2^3,\ldots) + (r+6)Z_{S_{r-2}}(2^2,2^3,2^2,2^3,\ldots). \tag{2.56}$$

Case 2: $r \bmod 2 = 1$.

Again by Eqn. 2.5,

$$rZ_{S_r}(2^2,2^3,2^2,2^3,\ldots) = \sum_{\text{odd } i}^{r} 2^2 Z_{S_{r-i}}(2^2,2^3,2^2,2^3,\ldots) + \sum_{\text{even } i}^{r-1} 2^3 Z_{S_{r-i}}(2^2,2^3,2^2,2^3,\ldots), \tag{2.57}$$

$$(r-2)Z_{S_{r-2}}(2^2,2^3,\ldots) = \sum_{\text{odd } i}^{r-2} 2^2 Z_{S_{r-2-i}}(2^2,2^3,2^2,2^3,\ldots) + \sum_{\text{even } i}^{r-3} 2^3 Z_{S_{r-2-i}}(2^2,2^3,2^2,2^3,\ldots). \tag{2.58}$$

Subtracting the second equation from the first one, and rearranging the terms give

$$rZ_{S_r}(2^2, 2^3, 2^2, 2^3, \ldots) = 4Z_{S_{r-1}}(2^2, 2^3, 2^2, 2^3, \ldots) + (r+6)Z_{S_{r-2}}(2^2, 2^3, \ldots). \tag{2.59}$$

Hence, we obtain the same recurrence for both even and odd r. We now use induction and this recurrence to prove that Eqn. 2.53 holds.

Basis $r = 0$. Substituting $r = 0$ in (2.53) gives 1 as it should since $Z_{S_0}() = 1$.

$r = 1$. Substituting $r = 1$ in (2.53) gives

$$Z_{S_1}(2^2) = \frac{(1+4)\left(2(1)^4 + 32(1)^3 + 172(1)^2 + 352(1) + 15\,(-1)^1 + 225\right)}{960} = 4, \tag{2.60}$$

and this agrees with Eqn. 2.5, i.e., $Z_{S_1}(2^2) = \frac{1}{1}\left(2^2 Z_{S_0}()\right) = 2^2 = 4$.

Induction Step:

Suppose that Eqn. 2.53 holds for $r-2$ and $r-1$.Then by Eqn. 2.59, we have

$$rZ_{S_r}(2^2, 2^3, 2^2, 2^3, \ldots)$$

$$= 4Z_{S_{r-1}}(2^2, 2^3, 2^2, 2^3, \ldots) + (r+6)Z_{S_{r-2}}(2^2, 2^3, 2^2, 2^3, \ldots), \tag{2.61}$$

$$= \frac{4(r+3)\left(2(r-1)^4 + 32(r-1)^3 + 172(r-1)^2 + 352(r-1) + 15(-1)^{(r-1)} + 225\right)}{960} + \frac{(r+6)(r+2)\left(2(r-2)^4 + 32(r-2)^3 + 172(r-2)^2 + 352(r-2) + 15(-1)^{(r-2)} + 225\right)}{960}, \tag{2.62}$$

$$= \frac{8r^5 + 120r^4 + 640r^3 + 1440r^2 + [1212 - 60(-1)^r]r - 180(-1)^r + 180}{960} + \frac{2r^6 + 32r^5 + 180r^4 + 400r^3 + [193 + 15(-1)^r]r^2 + [120(-1)^r - 312]r + 180(-1)^r - 180}{960}, \tag{2.63}$$

$$= \frac{2r^6 + 40r^5 + 300r^4 + 1040r^3 + [1633 + 15(-1)^r]r^2 + [900 + 60(-1)^r]r}{960}, \tag{2.64}$$

$$= r\frac{(r+4)\left(2r^4 + 32r^3 + 172r^2 + 352r + 15\,(-1)^r + 225\right)}{960}, \tag{2.65}$$

that agrees with Eqn. 2.53. □

Lemma 5.

$$Z_{S_r}(2,2,2^3,2,2,2^3,\ldots) = \begin{cases} \frac{(r^3+12r^2+45r+54)}{54} & \text{if } r \bmod 3 = 0, \\ \frac{(r^3+12r^2+45r+50)}{54} & \text{if } r \bmod 3 = 1, \\ \frac{(r^3+12r^2+39r+28)}{54} & \text{if } r \bmod 3 = 2. \end{cases} \tag{2.66}$$

Proof. We consider three cases:

Case 1: $r \bmod 3 = 0$. Using Eqn. 2.5, we have

$$\begin{aligned} rZ_{S_r}(2,2,2^3,\ldots) = & \sum_{i \bmod 3=0}^{r} 2^3 Z_{S_{r-i}}(2,2,2^3,\ldots) + \sum_{i \bmod 3=1}^{r-2} 2Z_{S_{r-i}}(2,2,2^3,\ldots) \\ & + \sum_{i \bmod 3=2}^{r-1} 2Z_{S_{r-i}}(2,2,2^3,\ldots), \end{aligned} \tag{2.67}$$

$$\begin{aligned} (r-3)Z_{S_{r-3}}(2,2,2^3,\ldots) = & \sum_{i \bmod 3=0}^{r-3} 2^3 Z_{S_{r-3-i}}(2,2,2^3,\ldots) + \sum_{i \bmod 3=1}^{r-5} 2Z_{S_{r-3-i}}(2,2,2^3,\ldots) \\ & + \sum_{i \bmod 3=2}^{r-4} 2Z_{S_{r-3-i}}(2,2,2^3,\ldots). \end{aligned} \tag{2.68}$$

Subtracting Eqn. 2.68 from Eqn. 2.67 gives

$$\begin{aligned} rZ_{S_r}(2,2,2^3,\ldots) - (r-3)Z_{S_{r-3}}(2,2,2^3,\ldots) &= 2Z_{S_{r-1}}(2,2,2^3,\ldots) \\ &+ 2Z_{S_{r-2}}(2,2,2^3,\ldots) + 8Z_{S_{r-3}}(2,2,2^3,\ldots), \end{aligned} \tag{2.69}$$

$$rZ_{S_r}(2,2,2^3,\ldots) = 2Z_{S_{r-1}}(2,2,2^3,\ldots) + 2Z_{S_{r-2}}(2,2,2^3,\ldots) + (r+5)Z_{S_{r-3}}(2,2,2^3,\ldots). \tag{2.70}$$

Case 2, 3: $r \bmod 3 = 1, r \bmod 3 = 2$. We omit the derivations for these two cases as it is not difficult to show that these two cases also lead to the recurrence in Eqn. 2.70.

Now we use the recurrences given in Eqns. 2.5 and 2.70 to prove Eqn. 2.66 by induction on r.

Basis ($r=0$). Substituting $r=0$ in Eqn. 2.66 gives 1 as it should since $Z_{S_0}()=1$.

($r=1$). Substituting $r=1$ in Eqn. 2.66 gives 2 as it should since $Z_{S_1}(2)= \frac{1}{1}\left(2Z_{S_0}()\right)=2$ by Eqn. 2.5.

($r=2$). Substituting $r=2$ in Eqn. 2.66 gives 3 as it should since

$Z_{S_2}(2,2)=\frac{1}{2}\left(2Z_{S_1}(2)+2Z_{S_0}()\right)=\frac{4+2}{2}=3$ by Eqn. 2.5.

(r=3). Substituting $r=3$ in Eqn. 2.66 gives 6 as it should since

$Z_{S_3}(2,2,2^3)=\frac{1}{3}\left(2Z_{S_2}(2,2)+2Z_{S_1}(2)+2^3Z_{S_0}()\right)=\frac{6+4+8}{3}=6$ by Eqn. 2.5.

Induction Step: Suppose that Eqn. 2.66 holds for $r-1, r-2$, and $r-3$ and $r \bmod 3=0$. Then by Eqn. 2.70,

$$
\begin{aligned}
& rZ_{S_r}(2,2,2^3,\ldots) \\
&= 2Z_{S_{r-1}}(2,2,2^3,\ldots)+2Z_{S_{r-2}}(2,2,2^3,\ldots)+(r+5)Z_{S_{r-3}}(2,2,2^3,\ldots), && (2.71)\\
&= \frac{2[(r-1)^3+12(r-1)^2+39(r-1)+28]}{54}+\frac{2[(r-2)^3+12(r-2)^2+45(r-2)+50]}{54}\\
&+\frac{(r+5)[(r-3)^3+12(r-3)^2+45(r-3)+54]}{54}, && (2.72)\\
&= \frac{2r^3+18r^2+36r+2r^3+12r^2+18r}{54}+\frac{r^4+8r^3+15r^2}{54}, && (2.73)\\
&= \frac{r^4+12r^3+45r^2+54r}{54}, && (2.74)\\
&= \frac{r(r^3+12r^2+45r+54)}{54}, && (2.75)
\end{aligned}
$$

as stated in Eqn. 2.66. The other two cases are shown to hold similarly and omitted. □

By combining Lemmas 3, 4, and 5 we have

Theorem 2.

$$
|B_u(3,r)| = \begin{cases}
\frac{1}{6}\left[A(r)+\frac{2(r^3+12r^2+45r+54)}{54}\right] & \textit{if } r \bmod 3=0,\\[2ex]
\frac{1}{6}\left[A(r)+\frac{2(r^3+12r^2+45r+50)}{54}\right] & \textit{if } r \bmod 3=1,\\[2ex]
\frac{1}{6}\left[A(r)+\frac{2(r^3+12r^2+39r+28)}{54}\right] & \textit{if } r \bmod 3=2,
\end{cases} \qquad (2.76)
$$

where $A(r) = \binom{r+7}{r} + \frac{3(r+4)\left(2r^4+32r^3+172r^2+352r+15(-1)^r+225\right)}{960}$.

This method of computation can be extended to $|B_u(n,r)|$ for $n \geq 4$, but the solutions of resulting recurrences become significantly more complex to obtain closed form formulas.

2.4 An Elementary Counting for $|B_u(2,r)|$

In this section, we will provide an elementary proof for Theorem 1.

Theorem 3. For even r,

$$|B_u(2,r)| = 1 + r + \frac{1}{16}(r^3 + 8r^2 + 4r) + \frac{1}{48}(r^3 + 6r^2 + 8r). \tag{2.77}$$

Proof. The first two terms in the formula count the number of unlabeled $(2,r)$-bipartite graphs in which one of the left vertices has zero degree and the degree of the other vertex varies between 0 and r. To count the remaining unlabeled $(2,r)$-bipartite graphs, we note that each left vertex can be connected up to r right vertices. Let a_i denote the first left vertex[2], where i indicates that a_i is connected to i right vertices, $1 \leq i \leq r$, and b_j denote the second left vertex, where j indicates that b_j is connected to j right vertices, $1 \leq j \leq r$. Let (a_i, b_j)-bipartite graph refer to any unlabeled bipartite graph in which the two left vertices are connected to i and j right vertices, respectively. Since reordering the left vertices in any $(2,r)$-bipartite graph results in an equivalent $(2,r)$-bipartite graph, any two (a_i, b_j)-bipartite graph and (a_j, b_i)-graph, $1 \leq i, j \leq r$ are equivalent, and they should therefore be counted only once. This counting constraint will be enforced by requiring that $1 \leq j \leq i \leq r$. Now, there exist exactly two distinct unlabeled (a_i, b_1)-bipartite graphs for each $i, 1 \leq i \leq r-1$, and there exists one (a_r, b_1)-bipartite graph (See Figure 2.2). Thus, there exist exactly $2(r-1)+1$ unlabeled (a_i, b_j)-bipartite graphs, $1 \leq i \leq r$ and $j = 1$. Extending the construction to the next case, it is not difficult to see that there exist $3(r -$

[2] This labeling of vertices is used to keep track of their degrees during the counting process and it does not mean that the resulting bipartite graphs are labeled.

3) + 2 + 1 (a_i, b_j)-bipartite graphs, $2 \leq i \leq r$ and $j = 2$, and in general, there exist $(j+1)(r-2j+1)+\sum_{k=1}^{j} k = (j+1)(r-3j/2+1)$ (a_i, b_j)-bipartite graphs, $1 \leq j \leq r/2$. Summing the last expression for $1 \leq j \leq r/2$ gives the third term in the formula in Eqn. 2.77. What remains unaccounted for are (a_i, b_j)-bipartite graphs, $r/2+1 \leq j \leq i \leq r$. These are counted using a similar argument, i.e., by creating new (a_i, b_j)-bipartite graphs by first fixing j to $r/2+1$, then to $r/2+2$, and so on, while varying i between j and r in each case. It can be shown that the first case leads to $r/2$ unlabeled $(2, r)$-bipartite graphs, i.e., $(a_i, b_{r/2+1})$-bipartite graphs, $r/2+1 \leq i \leq r$, the second case leads to $r/2-1$ new unlabeled $(2, r)$-bipartite graphs, i.e., $(a_i, b_{r/2+2})$-bipartite graphs, $r/2+2 \leq i \leq r$, and the last case leads to just one new unlabeled $(2, r)$-bipartite graph, i.e., (a_r, b_r)-bipartite graph. Summing these up gives $\sum_{j=r/2+1}^{r} \sum_{i=j}^{r} r-i+1 = \frac{1}{48}(r^3+6r^2+8r)$, i.e., the last term in Eqn. 2.77 and the statement follows. □

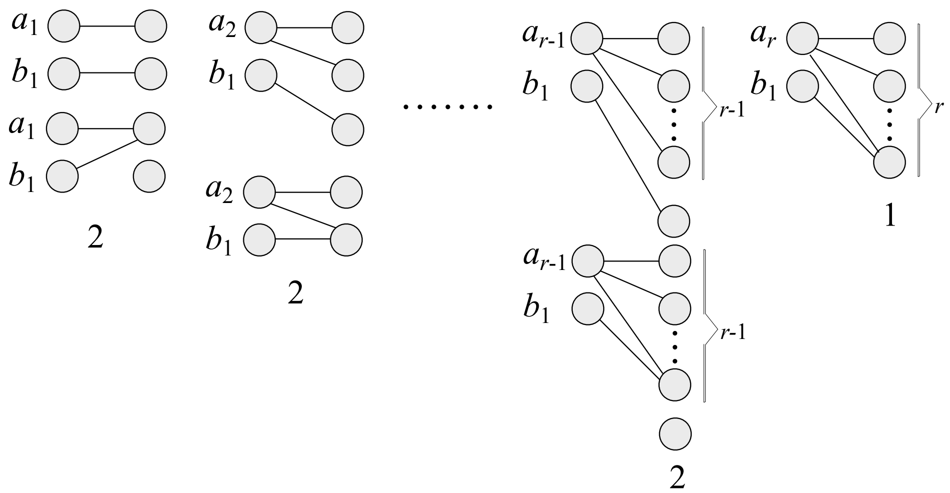

Figure 2.2: Construction of unlabeled $(2, r)$-bipartite graphs with exactly two left vertices, $1 \leq i \leq r$.

The odd r case is a direct corollary of the theorem.

Corollary 1. For odd r,

$$|B_u(2, r)| = 1 + r + \frac{1}{16}(r^3 + 7r^2 - r - 7) + \frac{1}{48}(r^3 + 9r^2 + 23r + 15). \quad (2.78)$$

Proof. To obtain Eqn. 2.78, it is sufficient to replace each occurrence of $r/2$ in the proof of theorem by $(r-1)/2$.

Remark 1. It is noted that the $|B_u(2,r)|$ formulas in Eqns. 2.77 and 2.78 can be combined to give Eqn. 2.33. Moreover, $|B_u(2, 2i-2)|$ coincides with the i^{th} hexagonal pyramidal number (see the integer sequence, A002412 in [34]), when $i = 1, 2, 3, \ldots$. □

Extension of the elementary proof of the formula $|B_u(n,r)|$ when $n = 3$ case remains open. Moreover an exact computation $|B_u(n,r)|$ for $n \geq 4$ also remains open. In the next chapter, we will establish a two sided inequality for $|B_u(n,r)|$ for any $n \neq r$.

Chapter 3

Bounds for Unlabeled Bipartite Graphs[1]

In this chapter, we provide a two sided inequality for $|B_u(n,r)|$ that then establishes an asympthotic formula for the same. More precisely, we prove

$$\frac{\binom{r+2^n-1}{r}}{n!} \leq |B_u(n,r)| \leq \frac{2\binom{r+2^n-1}{r}}{n!}, \; n < r. \tag{3.1}$$

We begin with our lower bound.

3.1 A Lower Bound for $|B_u(n,r)|$

From Eqns. 2.2 and 2.4 we know that

$$\begin{aligned} |B_u(n,r)| &= Z_{S_n \times S_r}(2,2,\ldots,2), & (3.2)\\ &= [Z_{S_n}(x_1,x_2,\cdots,x_n) \boxtimes Z_{S_r}(x_1,x_2,\cdots,x_r)](2,2,\ldots,2). & (3.3) \end{aligned}$$

[1] **This is joint work with Prof. A. Yavuz Oruç.**

One of the terms in $Z_{S_n}(x_1, x_2, \cdots, x_n)$ is $\frac{1}{n!}(x_1^n)$ and it is associated with the identity permutation in S_n. Using this fact, we find

$$
\begin{aligned}
|B_u(n,r)| &= Z_{S_n \times S_r}(2,2,\ldots,2), && (3.4)\\
&= [Z_{S_n}(x_1,x_2,\cdots,x_n) \boxtimes Z_{S_r}(x_1,x_2,\cdots,x_r)](2,2,\ldots,2), && (3.5)\\
&= \left[\left(\frac{1}{n!}(x_1^n+\ldots)\right) \boxtimes Z_{S_r}(x_1,x_2,\cdots,x_r)\right](2,2,\ldots,2), && (3.6)\\
&= \left[\left(\frac{1}{n!}x_1^n\right) \boxtimes Z_{S_r}(x_1,x_2,\ldots,x_r)\right](2,2,\ldots,2)+\ldots, && (3.7)\\
&= \frac{1}{n!}\left\{\left[x_1^n \boxtimes \frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r} x_j^{q_{t,j}}\right](2,2,...,2)\right\}+\ldots, && (3.8)\\
&= \frac{1}{n!}\left\{\left[\frac{1}{r!}\sum_{t=1}^{r!} x_1^n \bigodot \prod_{j=1}^{r} x_j^{q_{t,j}}\right](2,2,...,2)\right\}+\ldots, && (3.9)\\
&= \frac{1}{n!}\left\{\left[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r} x_{\mathrm{lcm}(1,j)}^{nq_{t,j}\gcd(1,j)}\right](2,2,...,2)\right\}+\ldots, && (3.10)\\
&= \frac{1}{n!}\left\{\left[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r} x_j^{nq_{t,j}}\right](2,2,...,2)\right\}+\ldots, && (3.11)\\
&= \frac{1}{n!}\left\{\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r} 2^{nq_{t,j}}\right\}+\ldots, && (3.12)\\
&= \frac{1}{n!}\left\{\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r} (2^n)^{q_{t,j}}\right\}+\ldots, && (3.13)\\
&= \frac{1}{n!}\left\{Z_{S_r}(2^n,2^n,\ldots,2^n)\right\}+\ldots. && (3.14)
\end{aligned}
$$

This proves

$$|B_u(n,r)| \geq \frac{1}{n!} Z_{S_r}(2^n,2^n,\ldots,2^n). \quad \square \tag{3.15}$$

Proposition 1.

$$Z_{S_r}(2^n,2^n,\ldots,2^n) = \binom{r+2^n-1}{r} \tag{3.16}$$

Proof. Using Eqn. 2.5, we have

$$rZ_{S_r}(2^n, 2^n, \ldots, 2^n) = \sum_{i=1}^{r} 2^n Z_{S_{r-i}}(2^n, 2^n, \ldots, 2^n), \tag{3.17}$$

and

$$(r-1)Z_{S_{r-1}}(2^n, 2^n, \ldots, 2^n) = \sum_{i=1}^{r-1} 2^n Z_{S_{r-1-i}}(2^n, 2^n, \ldots, 2^n). \tag{3.18}$$

Subtracting the second equation from the first one gives

$$rZ_{S_r}(2^n, 2^n, \ldots, 2^n) - (r-1)Z_{S_{r-1}}(2^n, 2^n, \ldots, 2^n) = 2^n Z_{S_{r-1}}(2^n, 2^n, \ldots, 2^n), \tag{3.19}$$

$$rZ_{S_r}(2^n, 2^n, \ldots, 2^n) = (r + 2^n - 1)Z_{S_{r-1}}(2^n, 2^n, \ldots, 2^n), \tag{3.20}$$

$$Z_{S_r}(2^n, 2^n, \ldots, 2^n) = (\frac{r + 2^n - 1}{r})Z_{S_{r-1}}(2^n, 2^n, \ldots, 2^n). \tag{3.21}$$

Expanding the last equation inductively, we obtain

$$Z_{S_r}(2^n, 2^n, \ldots, 2^n) = (\frac{r + 2^n - 1}{r})(\frac{r + 2^n - 2}{r-1})Z_{S_{r-2}}(2^n, 2^n, \ldots, 2^n), \tag{3.22}$$

$$Z_{S_r}(2^n, 2^n, \ldots, 2^n) = (\frac{r + 2^n - 1}{r})(\frac{r + 2^n - 2}{r-1})(\frac{r + 2^n - 3}{r-2})Z_{S_{r-3}}(2^n, 2^n, \ldots, 2^n), \tag{3.23}$$

$$Z_{S_r}(2^n, 2^n, \ldots, 2^n) = (\frac{r + 2^n - 1}{r})(\frac{r + 2^n - 2}{r-1})(\frac{r + 2^n - 3}{r-2})\ldots(\frac{2^n}{1})Z_{S_0}(). \tag{3.24}$$

Noting that $Z_{S_0}() = 1$, and combining the product terms together, we obtain

$$Z_{S_r}(2^n, 2^n, \ldots, 2^n) = \binom{r + 2^n - 1}{r}. \quad \square \tag{3.25}$$

Combining Proposition 1 with Eqn. 3.15 proves the lower bound.

Theorem 4.

$$|B_u(n,r)| \geq \frac{\binom{r+2^n-1}{r}}{n!}. \quad \square \tag{3.26}$$

3.2 An Upper Bound for $|B_u(n,r)|$

We first note that $|B_u(1,r)| = r+1 = \binom{r+2^1-1}{r}/1! \leq 2\binom{r+2^1-1}{r}/1!$. Hence the upper bound that is claimed in the beginning of this chapter holds for $n = 1$. Proving that it also holds for $n \geq 2$ requires a more careful analysis of the terms in

$$Z_{S_n}(x_1, x_2, \cdots, x_n) \boxtimes Z_{S_r}(x_1, x_2, \cdots, x_r). \tag{3.27}$$

We first express $Z_{S_n}(x_1, x_2, \cdots, x_n)$ as

$$Z_{S_n}(x_1, x_2, \ldots, x_n) = Z_{S_n}[1] + Z_{S_n}[2] + \ldots + Z_{S_n}[n!], \tag{3.28}$$

where

$$Z_{S_n}[1] = \frac{1}{n!}x_1^n, \tag{3.29}$$

$$Z_{S_n}[2] = \frac{1}{n!}x_1^{n-2}x_2. \tag{3.30}$$

The first term is associated with the identity permutation and the second term is associated with any one of the permutations in which all but two of the elements in $N = 1, 2, \cdots, n$ are fixed to themselves. The remaining $Z_{S_n}[i] = \frac{1}{n!}\prod_{k=1}^{n} x_k^{p_{i,k}}, 3 \leq i \leq n!$ terms represent all the other product terms in the cycle index polynomial of S_n with no particular association with the permutations in S_n. Similarly, we set $Z_{S_r}(x_1, x_2, \ldots, x_r) = \frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r} x_j^{q_{t,j}}$ without identifying the actual product terms with any particular permutation in S_r.

The following equations obviously hold as the sum of the lengths of all the cycles in any cycle disjoint representation of a permutation in S_n and S_r must be n and r, respectively.

$$\sum_{k=1}^{n} kp_{i,k} = n, 1 \le i \le n!, \tag{3.31}$$

$$\sum_{j=1}^{r} jq_{t,j} = r, 1 \le t \le r!. \tag{3.32}$$

Now we can proceed with the computation of the upper bound for $|B_u(n,r)|$. First, we note that

$$\begin{aligned}
|B_u(n,r)| =& Z_{S_n \times S_r}(2,2,2,\ldots,2), & (3.33)\\
=& \left[Z_{S_n}(x_1,x_2,\ldots,x_n) \boxtimes Z_{S_r}(x_1,x_2,\cdots,x_r)\right](2,2,\ldots,2), & (3.34)\\
=& \left[(Z_{S_n}[1] + Z_{S_n}[2] + \ldots + Z_{S_n}[n!]) \boxtimes Z_{S_r}(x_1,x_2,\ldots,x_r)\right](2,2,\ldots,2), & (3.35)\\
=& \left[Z_{S_n}[1] \boxtimes Z_{S_r}(x_1,x_2,\ldots,x_r)\right](2,2,\ldots,2)\\
&+ \left[Z_{S_n}[2] \boxtimes Z_{S_r}(x_1,x_2,\ldots,x_r)\right](2,2,\ldots,2)\\
&+ \ldots + \left[Z_{S_n}[n!] \boxtimes Z_{S_r}(x_1,x_2,\ldots,x_r)\right](2,2,\ldots,2). & (3.36)
\end{aligned}$$

The first term in Eqn. 3.36 is directly computed from Proposition 1. Thus, it suffices to upper bound each of the remaining terms in Eqn. 3.36 to upper bound $|B_u(n,r)|$. This will be established by proving $\left[Z_{S_n}[2] \boxtimes Z_{S_r}(x_1,x_2,\ldots,x_r)\right](2,2,\ldots,2) \ge \left[Z_{S_n}[i] \boxtimes Z_{S_r}(x_1,x_2,\ldots,x_r)\right](2,2,\ldots,2)$, $\forall i, 3 \le i \le n!$. We first need some preliminary facts.

Lemma 6. For all $i, 1 \le i \le n!$,

$$[Z_{S_n}[i] \boxtimes Z_{S_r}(x_1,x_2,\ldots,x_r)](2,\ldots,2) = \frac{1}{n!} Z_{S_r}(2^{\sum_{k=1}^{n} p_{i,k}\gcd(k,1)},\ldots,2^{\sum_{k=1}^{n} p_{i,k}\gcd(k,r)}). \tag{3.37}$$

Proof.

$$[Z_{S_n}[i] \boxtimes Z_{S_r}(x_1,\ldots,x_r)](2,\ldots,2) = \left[\frac{1}{n!}\prod_{k=1}^{n} x_k^{p_{i,k}} \boxtimes \left(\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r} x_j^{q_{t,j}}\right)\right](2,\ldots,2), \tag{3.38}$$

$$= \left[\frac{1}{n!r!}\sum_{t=1}^{r!}\prod_{k=1}^{n} x_k^{p_{i,k}} \bigodot \prod_{j=1}^{r} x_j^{q_{t,j}}\right](2,\ldots,2), \tag{3.39}$$

$$= \left[\frac{1}{n!r!}\sum_{t=1}^{r!}\prod_{j=1}^{r}\prod_{k=1}^{n} x_{\mathrm{lcm}(k,j)}^{p_{i,k}q_{t,j}\mathrm{gcd}(k,j)}\right](2,\ldots,2), \tag{3.40}$$

$$= \frac{1}{n!r!}\sum_{t=1}^{r!}\prod_{j=1}^{r}\prod_{k=1}^{n} 2^{p_{i,k}q_{t,j}\mathrm{gcd}(k,j)}, \tag{3.41}$$

$$= \frac{1}{n!}\left[\frac{1}{r!}\sum_{t=1}^{r!}\prod_{j=1}^{r}(2^{\sum_{k=1}^{n} p_{i,k}\mathrm{gcd}(k,j)})^{q_{t,j}}\right], \tag{3.42}$$

$$= \frac{1}{n!} Z_{S_r}(2^{\sum_{k=1}^{n} p_{i,k}\mathrm{gcd}(k,1)},\ldots,2^{\sum_{k=1}^{n} p_{i,k}\mathrm{gcd}(k,r)}). \quad \square \tag{3.43}$$

Corollary 2.

$$[Z_{S_n}[2] \boxtimes Z_{S_r}(x_1,x_2,\ldots,x_r)](2,\ldots,2) = \tfrac{1}{n!} Z_{S_r}(2^{n-1},2^n,2^{n-1},2^n,\ldots). \tag{3.44}$$

Proof. By definition, $p_{2,1} = n-2, p_{2,2} = 1, p_{2,k} = 0, 3 \le k \le n$. Substituting these into the last equation in Lemma 6 proves the statement. $\square$

Lemma 7.

$$\textstyle\sum_{k=1}^{n} p_{i,k} \le n-1, \forall i, 2 \le i \le n!. \tag{3.45}$$

Proof. Recall from Eqn. 3.31 that $\sum_{k=1}^{n} kp_{i,k} = n,\ \forall i, 1 \le i \le n!$. Hence $\sum_{k=1}^{n} p_{i,k} = n - \sum_{k=1}^{n}(k-1)p_{i,k}$, and so the maximum value of $\sum_{k=1}^{n} p_{i,k}$ occurs when $\sum_{k=1}^{n}(k-1)p_{i,k}$ is minimized. Furthermore, at least one of $p_{i,k}, \forall i, 2 \le i \le n!$ must be ≥ 1 for some $k \ge 2$ since none of the permutations we consider is the identity. Thus, $\sum_{k=1}^{n}(k-1)p_{i,k} \ge 1$ and the statement follows. $\square$

Lemma 8. If $\sum_{k=1}^{n} p_{i,k}\text{gcd}(k, \alpha+1) = n$, then $\sum_{k=1}^{n} p_{i,k}\text{gcd}(k, \alpha) \leq n-1$, $\forall i, 2 \leq i \leq n!$ and for any integer $\alpha \geq 2$.

Proof. If $\sum_{k=1}^{n} p_{i,k}\text{gcd}(k, \alpha+1) = n$ as stated in the lemma, then we must have $\text{gcd}(k, \alpha+1) = k$ where $p_{i,k} \geq 1$, $\forall i, 2 \leq i \leq n!$. Therefore $k \leq \alpha + 1$. Now if $k = \alpha + 1$, then trivially $\text{gcd}(k, \alpha) < k$. On the other hand if $k < \alpha + 1$, then $\alpha + 1$ must be a multiple of k. Therefore, α can not be a multiple of k for any $k \geq 2$. At this point we find that $\text{gcd}(k, \alpha) < k$, $\forall k, 2 \leq k \leq n$. Since as in the previous lemma, none of the permutations we consider is the identity, at least one of $p_{i,k}, \forall i, 2 \leq i \leq n!$ must be ≥ 1 for some $k \geq 2$ and so we conclude that $\sum_{k=1}^{n} p_{i,k}\text{gcd}(k, \alpha) \leq n-1$. □

Lemma 9.

$$Z_{S_r}(2^{n-1}, 2^n, \ldots) \geq Z_{S_{r-1}}(2^{n-1}, 2^n, \ldots), \tag{3.46}$$

for $2 \leq n$.

Proof. Using Eqn. 2.5, we get

$$rZ_{S_r}(2^{n-1}, 2^n, \ldots) = \sum_{\textbf{odd } i}^{r-\beta_1} 2^{n-1} Z_{S_{r-i}}(2^{n-1}, 2^n, \ldots) + \sum_{\textbf{even } i}^{r-\beta_2} 2^n Z_{S_{r-i}}(2^{n-1}, 2^n, \ldots), \tag{3.47}$$

where $\beta_1 = 1, \beta_2 = 0$ if r is even and $\beta_1 = 0, \beta_2 = 1$ if r is odd.

Similarly, for $r-1$,

$$(r-1)Z_{S_{r-1}}(2^{n-1}, 2^n, \ldots) = \sum_{\textbf{odd } i}^{r-1-\beta_2} 2^{n-1} Z_{S_{r-1-i}}(2^{n-1}, 2^n, \ldots) + \sum_{\textbf{even } i}^{r-1-\beta_1} 2^n Z_{S_{r-1-i}}(2^{n-1}, 2^n, \ldots). \tag{3.48}$$

Subtracting Eqn. 3.48 from Eqn. 3.47 gives

$$
\begin{aligned}
&rZ_{S_r}(2^{n-1},2^n,\ldots) - (r-1)Z_{S_{r-1}}(2^{n-1},2^n,\ldots) \\
&\quad = \sum_{\textbf{even } i}^{r-\beta_2} 2^n Z_{S_{r-i}}(2^{n-1},2^n,\ldots) - \sum_{\textbf{odd } i}^{r-1-\beta_2} 2^{n-1} Z_{S_{r-1-i}}(2^{n-1},2^n\ldots) \\
&\qquad + \sum_{\textbf{odd } i}^{r-\beta_1} 2^{n-1} Z_{S_{r-i}}(2^{n-1},2^n,\ldots) - \sum_{\textbf{even } i}^{r-1-\beta_1} 2^n Z_{S_{r-1-i}}(2^{n-1},2^n,\ldots), \qquad (3.49)
\end{aligned}
$$

$$
\begin{aligned}
&rZ_{S_r}(2^{n-1},2^n,\ldots) - (r-1)Z_{S_{r-1}}(2^{n-1},2^n,\ldots) \\
&\quad = \sum_{\textbf{even } i}^{r-\beta_2} 2^{n-1} Z_{S_{r-i}}(2^{n-1},2^n,\ldots) + 2^{n-1} Z_{S_{r-1}}(2^{n-1},2^n,\ldots) - \sum_{\textbf{even } i}^{r-1-\beta_1} 2^{n-1} Z_{S_{r-1-i}}(2^{n-1},2^n,\ldots), \\
&\qquad\qquad (3.50)
\end{aligned}
$$

$$
\begin{aligned}
&rZ_{S_r}(2^{n-1},2^n,\ldots) = (r-1+2^{n-1})Z_{S_{r-1}}(2^{n-1},2^n\ldots) \\
&\quad + 2^{n-1}\left(\sum_{\textbf{even } i}^{r-\beta_2} Z_{S_{r-i}}(2^{n-1},2^n,\ldots) - \sum_{\textbf{even } i}^{r-1-\beta_1} Z_{S_{r-1-i}}(2^{n-1},2^n,\ldots)\right). \qquad (3.51)
\end{aligned}
$$

We now prove the lemma by induction on r.

Basis $r = 1$. By Eqn. 2.5, $Z_{S_1}(2^{n-1}) = 2^{n-1}Z_{S_0}() = 2^{n-1}$. So we have $Z_{S_1}(2^{n-1}) = 2^{n-1} \geq Z_{S_0}() = 1$ for $2 \leq n$.

Induction Step. Suppose that the lemma holds from 1 to $r-1$. That is, $Z_{S_{r-i}} - Z_{S_{r-i-1}} \geq 0, 1 \leq i \leq r-1$. Now if r is even then the difference of the two sums in Eqn. 3.51 becomes $(Z_{S_{r-2}} - Z_{S_{r-3}}) + (Z_{S_{r-4}} - Z_{S_{r-5}}) \ldots + (Z_{S_2} - Z_{S_1}) + Z_{S_0}$, which is clearly ≥ 0 by the induction hypothesis. Therefore,

$$rZ_{S_r}(2^{n-1},2^n,\ldots) \geq (r-1+2^{n-1})Z_{S_{r-1}}(2^{n-1},2^n,\ldots), \qquad (3.52)$$

$$Z_{S_r}(2^{n-1},2^n,\ldots) \geq Z_{S_{r-1}}(2^{n-1},2^n,\ldots), n \geq 2. \qquad (3.53)$$

On the other hand, if r is odd then the difference of the two sums in the same equation becomes $(Z_{S_{r-2}} - Z_{S_{r-3}}) + (Z_{S_{r-4}} - Z_{S_{r-5}}) \ldots + (Z_{S_2} - Z_{S_1}) + (Z_{S_1} - Z_{S_0})$, which is again ≥ 0, and the statement follows in this case as well. □

We now are ready to prove that

$$[Z_{S_n}[2] \boxtimes Z_{S_r}(x_1, x_2, \ldots, x_r)](2, \ldots, 2) \geq [Z_{S_n}[i] \boxtimes Z_{S_r}(x_1, x_2, \ldots, x_r)](2, \ldots, 2), \tag{3.54}$$

$\forall i, 2 \leq i \leq n!$ and $\forall n, n < r$.

Theorem 5.

$$[Z_{S_n}[2] \boxtimes Z_{S_r}(x_1, x_2, \ldots, x_r)](2, 2, \ldots, 2) \geq [Z_{S_n}[i] \boxtimes Z_{S_r}(x_1, x_2, \ldots, x_r)](2, 2, \ldots, 2) \tag{3.55}$$

$\forall i, 2 \leq i \leq n!$ and $\forall n, n < r$.

Proof. Using Lemma 6 and Corollary 2 it suffices to show that

$$Z_{S_r}(2^{n-1}, 2^n, \ldots) \geq Z_{S_r}(2^{\sum_{k=1}^{n} p_{i,k} \gcd(k,1)}, \ldots, 2^{\sum_{k=1}^{n} p_{i,k} \gcd(k,r)}). \tag{3.56}$$

We prove the statement by induction on r.

Basis: $(r = 1)$. By Eqn. 2.5, $Z_{S_1}(2^{n-1}) = 2^{n-1} Z_{S_0}() = 2^{n-1}$. Similarly, by Eqn. 2.5, $Z_{S_1}(2^{\sum_{k=1}^{n} p_{i,k} \gcd(k,1)}) = 2^{\sum_{k=1}^{n} p_{i,k} \gcd(k,1)} Z_{S_0}() = 2^{\sum_{k=1}^{n} p_{i,k}}$. Given that $\sum_{k=1}^{n} p_{i,k} \leq n-1$ by Lemma 7, we have $2^{\sum_{k=1}^{n} p_{i,k}} \leq 2^{n-1}$, and hence the statement holds in this case.

Induction Step: First, by Eqn. 2.5,

$$Z_{S_r}(2^{n-1}, 2^n, \ldots) = \frac{1}{r} \begin{bmatrix} 2^{n-1} Z_{S_{r-1}}(2^{n-1}, 2^n, \ldots) \\ +2^n Z_{S_{r-2}}(2^{n-1}, 2^n, \ldots) \\ +2^{n-1} Z_{S_{r-3}}(2^{n-1}, 2^n, \ldots) \\ \vdots \\ +2^{\beta} Z_{S_0}() \end{bmatrix}, \tag{3.57}$$

where $\beta = n$ if r is even and $\beta = n - 1$ if r is odd. Similarly,

$$Z_{S_r}(2^{\sum_{k=1}^{n} p_{i,k} \gcd(k,1)}, \ldots, 2^{\sum_{k=1}^{n} p_{i,k} \gcd(k,r)}) = \frac{1}{r} \begin{bmatrix} 2^{\sum_{k=1}^{n} p_{i,k} \gcd(k,1)} Z_{S_{r-1}}(2^{\sum_{k=1}^{n} p_{i,k} \gcd(k,1)}, \ldots) \\ +2^{\sum_{k=1}^{n} p_{i,k} \gcd(k,2)} Z_{S_{r-2}}(2^{\sum_{k=1}^{n} p_{i,k} \gcd(k,1)}, \ldots) \\ +2^{\sum_{k=1}^{n} p_{i,k} \gcd(k,3)} Z_{S_{r-3}}(2^{\sum_{k=1}^{n} p_{i,k} \gcd(k,1)}, \ldots) \\ \vdots \\ +2^{\sum_{k=1}^{n} p_{i,k} \gcd(k,r)} Z_{S_0}() \end{bmatrix}. \tag{3.58}$$

Subtracting Eqn. 3.58 from Eqn. 3.57, we have

$$
\begin{aligned}
&Z_{S_r}(2^{n-1}, 2^n, \ldots) - Z_{S_r}(2^{\sum_{k=1}^n p_{i,k}\gcd(k,1)}, \ldots, 2^{\sum_{k=1}^n p_{i,k}\gcd(k,r)}) \\
=&\frac{1}{r}\begin{bmatrix} 2^{n-1}Z_{S_{r-1}}(2^{n-1}, 2^n, \ldots) \\ +2^n Z_{S_{r-2}}(2^{n-1}, 2^n, \ldots) \\ +2^{n-1}Z_{S_{r-3}}(2^{n-1}, 2^n, \ldots) \\ \vdots \\ +2^{\beta}Z_{S_0}() \end{bmatrix} - \frac{1}{r}\begin{bmatrix} 2^{\sum_{k=1}^n p_{i,k}\gcd(k,1)}Z_{S_{r-1}}(2^{\sum_{k=1}^n p_{i,k}\gcd(k,1)}, 2^{\sum_{k=1}^n p_{i,k}\gcd(k,2)}, \ldots) \\ +2^{\sum_{k=1}^n p_{i,k}\gcd(k,2)}Z_{S_{r-2}}(2^{\sum_{k=1}^n p_{i,k}\gcd(k,1)}, 2^{\sum_{k=1}^n p_{i,k}\gcd(k,2)}, \ldots) \\ +2^{\sum_{k=1}^n p_{i,k}\gcd(k,3)}Z_{S_{r-3}}(2^{\sum_{k=1}^n p_{i,k}\gcd(k,1)}, 2^{\sum_{k=1}^n p_{i,k}\gcd(k,2)}, \ldots) \\ \vdots \\ +2^{\sum_{k=1}^n p_{i,k}\gcd(k,r)}Z_{S_0}() \end{bmatrix}.
\end{aligned}
\tag{3.59}
$$

Thus, it suffices to show that the right hand side of the above equation is ≥ 0, or

$$
\begin{gathered}
2^{n-1}Z_{S_{r-1}}(2^{n-1}, 2^n, \ldots) - 2^{\sum_{k=1}^n p_{i,k}\gcd(k,1)}Z_{S_{r-1}}(2^{\sum_{k=1}^n p_{i,k}\gcd(k,1)}, 2^{\sum_{k=1}^n p_{i,k}\gcd(k,2)}, \ldots) \\
+2^n Z_{S_{r-2}}(2^{n-1}, 2^n, \ldots) - 2^{\sum_{k=1}^n p_{i,k}\gcd(k,2)}Z_{S_{r-2}}(2^{\sum_{k=1}^n p_{i,k}\gcd(k,1)}, 2^{\sum_{k=1}^n p_{i,k}\gcd(k,2)}, \ldots \\
+2^{n-1}Z_{S_{r-3}}(2^{n-1}, 2^n, \ldots) - 2^{\sum_{k=1}^n p_{i,k}\gcd(k,3)}Z_{S_{r-3}}(2^{\sum_{k=1}^n p_{i,k}\gcd(k,1)}, 2^{\sum_{k=1}^n p_{i,k}\gcd(k,2)}, \ldots) \\
\vdots \\
+2^{\beta}Z_{S_0}() - 2^{\sum_{k=1}^n p_{i,k}\gcd(k,r)}Z_{S_0}() \geq 0.
\end{gathered}
\tag{3.60}
$$

Now by induction hypothesis, Eqn. 3.56 holds for $1, 2, \cdots, r-1$. Thus, Eqn. 3.60 can be replaced by

$$
\begin{gathered}
2^{n-1}Z_{S_{r-1}}(2^{n-1}, 2^n, \ldots) - 2^{\sum_{k=1}^n p_{i,k}\gcd(k,1)}Z_{S_{r-1}}(2^{n-1}, 2^n, \ldots) \\
+2^n Z_{S_{r-2}}(2^{n-1}, 2^n, \ldots) - 2^{\sum_{k=1}^n p_{i,k}\gcd(k,2)}Z_{S_{r-2}}(2^{n-1}, 2^n, \ldots) \\
+2^{n-1}Z_{S_{r-3}}(2^{n-1}, 2^n, \ldots) - 2^{\sum_{k=1}^n p_{i,k}\gcd(k,3)}Z_{S_{r-3}}(2^{n-1}, 2^n, \ldots) \\
\vdots \\
+2^{\beta}Z_{S_0}() - 2^{\sum_{k=1}^n p_{i,k}\gcd(k,r)}Z_{S_0}() \geq 0.
\end{gathered}
\tag{3.61}
$$

Moreover, invoking Lemma 7 gives

$$
\begin{aligned}
&2^{n-1}Z_{S_{r-1}}(2^{n-1}, 2^n, \ldots) - 2^{\sum_{k=1}^n p_{i,k}\gcd(k,1)}Z_{S_{r-1}}(2^{n-1}, 2^n \ldots) \\
&\qquad \geq 2^{n-1}Z_{S_{r-1}}(2^{n-1}, 2^n, \ldots) - 2^{n-1}Z_{S_{r-1}}(2^{n-1}, 2^n, \ldots) = 0.
\end{aligned}
\tag{3.62}
$$

Hence the difference in the first line in Eqn. 3.61 ≥ 0, and therefore it is sufficient to show that

$$\begin{gathered} 2^n Z_{S_{r-2}}(2^{n-1}, 2^n, \ldots) - 2^{\sum_{k=1}^n p_{i,k}\text{gcd}(k,2)} Z_{S_{r-2}}(2^{n-1}, 2^n, \ldots) \\ +2^{n-1} Z_{S_{r-3}}(2^{n-1}, 2^n, \ldots) - 2^{\sum_{k=1}^n p_{i,k}\text{gcd}(k,3)} Z_{S_{r-3}}(2^{n-1}, 2^n, \ldots) \\ \vdots \\ +2^{\beta} Z_{S_0}() - 2^{\sum_{k=1}^n p_{i,k}\text{gcd}(k,r)} Z_{S_0}() \geq 0. \end{gathered} \tag{3.63}$$

To prove this inequality, we will combine four terms in pairs of consecutive lines for the remaining $r-1$ lines by considering two cases. If r is odd then $\beta = n-1$ and no extra line remains in this pairing. Thus, for all even $\alpha, 2 \leq \alpha \leq r-1$, it suffices to prove

$$\begin{gathered} 2^n Z_{S_{r-\alpha}}(2^{n-1}, 2^n, \ldots) - 2^{\sum_{k=1}^n p_{i,k}\text{gcd}(k,\alpha)} Z_{S_{r-\alpha}}(2^{n-1}, 2^n \ldots), \\ +2^{n-1} Z_{S_{r-\alpha-1}}(2^{n-1}, 2^n, \ldots) - 2^{\sum_{k=1}^n p_{i,k}\text{gcd}(k,\alpha+1)} Z_{S_{r-\alpha-1}}(2^{n-1}, 2^n \ldots) \geq 0. \end{gathered} \tag{3.64}$$

or,

$$\begin{gathered} 2^n Z_{S_{r-\alpha}}(2^{n-1}, 2^n, \ldots) - 2^{\sum_{k=1}^n p_{i,k}k} Z_{S_{r-\alpha}}(2^{n-1}, 2^n \ldots) \\ +2^{n-1} Z_{S_{r-\alpha-1}}(2^{n-1}, 2^n, \ldots) - 2^{\sum_{k=1}^n p_{i,k}\text{gcd}(k,\alpha+1)} Z_{S_{r-\alpha-1}}(2^{n-1}, 2^n \ldots) \geq 0. \end{gathered} \tag{3.65}$$

Now if $\sum_{k=1}^n p_{i,k}\text{gcd}(k, \alpha+1) \leq n-1$, then

$$\begin{aligned} &2^n Z_{S_{r-\alpha}}(2^{n-1}, 2^n, \ldots) - 2^{\sum_{k=1}^n p_{i,k}k} Z_{S_{r-\alpha}}(2^{n-1}, 2^n, \ldots) \\ &+ 2^{n-1} Z_{S_{r-\alpha-1}}(2^{n-1}, 2^n, \ldots) - 2^{n-1} Z_{S_{r-\alpha-1}}(2^{n-1}, 2^n, \ldots) \\ &\geq 2^n Z_{S_{r-\alpha}}(2^{n-1}, 2^n, \ldots) - 2^n Z_{S_{r-\alpha}}(2^{n-1}, 2^n, \ldots) + 2^{n-1} Z_{S_{r-\alpha-1}}(2^{n-1}, 2^n, \ldots) \\ &- 2^{n-1} Z_{S_{r-\alpha-1}}(2^{n-1}, 2^n, \ldots) = 0. \end{aligned} \tag{3.66}$$

On the other hand, if $\sum_{k=1}^n p_{i,k}\text{gcd}(k, \alpha+1) = n$, then we prove Eqn. 3.64 by noting that $\sum_{k=1}^n p_{i,k}\text{gcd}(k, \alpha) \leq n-1$ by Lemma 8. Thus,

$$\begin{aligned} &2^n Z_{S_{r-\alpha}}(2^{n-1}, 2^n, \ldots) - 2^{n-1} Z_{S_{r-\alpha}}(2^{n-1}, 2^n, \ldots) \\ &+ 2^{n-1} Z_{S_{r-\alpha-1}}(2^{n-1}, 2^n, \ldots) - 2^n Z_{S_{r-\alpha-1}}(2^{n-1}, 2^n, \ldots) \\ &= 2^{n-1} Z_{S_{r-\alpha}}(2^{n-1}, 2^n, \ldots) - 2^{n-1} Z_{S_{r-\alpha-1}}(2^{n-1}, 2^n, \ldots) \\ &2^{n-1} \left[Z_{S_{r-\alpha}}(2^{n-1}, 2^n, \ldots) - Z_{S_{r-\alpha-1}}(2^{n-1}, 2^n, \ldots) \right]. \end{aligned} \tag{3.67}$$

Now by Lemma 9, $Z_{S_{r-\alpha}}(2^{n-1}, 2^n, \ldots) \geq Z_{S_{r-\alpha-1}}(2^{n-1}, 2^n, \ldots)$ and the statement is proved for odd $r, n < r$. For even r, the last line in Eqn. 3.63 is left out in the pairing of consecutive lines and $\beta = n$. In this case we have $2^n Z_{S_0}() - 2^{\sum_{k=1}^{n} p_{i,k} \gcd(k,r)} Z_{S_0}() \geq 2^n Z_{S_0}() - 2^{\sum_{k=1}^{n} p_{i,k} k} Z_{S_0}() = 2^n Z_{S_0}() - 2^n Z_{S_0}() = 0$ and the statement follows. □

Theorem 6.

$$[Z_{S_n}[2] \boxtimes Z_{S_r}(x_1, x_2, \ldots, x_r)]\,(2, 2, \ldots, 2) \leq \frac{\binom{r+2^n-1}{r}}{n!(n!-1)} \tag{3.68}$$

where $2 \leq n < r$.

Proof. By Corollary 2

$$[Z_{S_n}[2] \boxtimes Z_{S_r}(x_1, x_2, \ldots, x_r)]\,(2, 2, \ldots, 2) = \frac{1}{n!} Z_{S_r}(2^{n-1}, 2^n, \ldots). \tag{3.69}$$

Thus, to prove the theorem, it is sufficient to show

$$\frac{1}{n!} Z_{S_r}(2^{n-1}, 2^n, 2^{n-1}, 2^n, \ldots) \leq \frac{\binom{r+2^n-1}{r}}{n!(n!-1)} \tag{3.70}$$

where $2 \leq n < r$.

Now, using Eqn. 2.5, we get

$$rZ_{S_r}(2^{n-1}, 2^n, \ldots) = \sum_{\textbf{odd } i}^{r-\beta_1} 2^{n-1} Z_{S_{r-i}}(2^{n-1}, 2^n, \ldots) + \sum_{\textbf{even } i}^{r-\beta_2} 2^n Z_{S_{r-i}}(2^{n-1}, 2^n, \ldots) \tag{3.71}$$

where $\beta_1 = 1, \beta_2 = 0$ if r is even and $\beta_1 = 0, \beta_2 = 1$ if r is odd. Similarly, for $r - 2$,

$$(r-2)Z_{S_{r-2}}(2^{n-1}, 2^n, \ldots) = \sum_{\textbf{odd } i}^{r-2-\beta_1} 2^{n-1} Z_{S_{r-2-i}}(2^{n-1}, 2^n, \ldots) + \sum_{\textbf{even } i}^{r-2-\beta_2} 2^n Z_{S_{r-2-i}}(2^{n-1}, 2^n, \ldots). \tag{3.72}$$

Subtracting Eqn. 3.72 from Eqn. 3.71 gives

$$rZ_{S_r}(2^{n-1},2^n,\ldots)-(r-2)Z_{S_{r-2}}(2^{n-1},2^n,\ldots)$$
$$=2^{n-1}Z_{S_{r-1}}(2^{n-1},2^n,\ldots)+2^nZ_{S_{r-2}}(2^{n-1},2^n,\ldots), \tag{3.73}$$
$$rZ_{S_r}(2^{n-1},2^n,\ldots)=2^{n-1}Z_{S_{r-1}}(2^{n-1},2^n,\ldots)+(r-2+2^n)Z_{S_{r-2}}(2^{n-1},2^n,\ldots), \tag{3.74}$$
$$Z_{S_r}(2^{n-1},2^n,\ldots)=\frac{1}{r}\left[2^{n-1}Z_{S_{r-1}}(2^{n-1},2^n,\ldots)+(r-2+2^n)Z_{S_{r-2}}(2^{n-1},2^n,\ldots)\right]. \tag{3.75}$$

We will use induction on r and the recurrence given in Eqn. 3.75 to prove this inequality.

Basis

Case 1: $r=3$. Recall that

$$Z_{S_n}[2]=\frac{1}{n!}x_1^{n-2}x_2, \tag{3.76}$$
$$Z_{S_3}(x_1,x_2,x_3)=\frac{1}{3!}(x_1^3+3x_1x_2+2x_3). \tag{3.77}$$

Thus,

$$[Z_{S_n}[2]\boxtimes Z_{S_3}(x_1,x_2,x_3)](2,2,\ldots,2)$$
$$=\left[\frac{1}{n!}(x_1^{n-2}x_2)\boxtimes\frac{1}{3!}(x_1^3+3x_1x_2+2x_3)\right](2,2,\ldots,2), \tag{3.78}$$
$$=\frac{1}{3!n!}\left[(x_1^{n-2}x_2)\bigodot x_1^3+(x_1^{n-2}x_2)\bigodot(3x_1x_2)+(x_1^{n-2}x_2)\bigodot 2x_3\right](2,2,\ldots,2), \tag{3.79}$$
$$=\frac{1}{3!n!}\left[x_1^{3(n-2)}x_2^3+3x_1^{n-2}x_2x_2^{n-2}x_2^2+2x_3^{n-2}x_6\right](2,2,\ldots,2), \tag{3.80}$$
$$=\frac{1}{3!n!}\left[2^{3n-3}+3\times2^{2n-1}+2^n\right]\le\frac{\binom{r+2^n-1}{r}}{n!(n!-1)}. \tag{3.81}$$

for $n=2$ and $r=3$.

Case 2: $r = 4$. In this case we have

$$[Z_{S_n}[2] \boxtimes Z_{S_4}(x_1, x_2, x_3, x_4)]\,(2, \ldots, 2)$$

$$= \left[\frac{1}{n!}(x_1^{n-2}x_2) \boxtimes \frac{1}{4!}(x_1^4 + 6x_1^2x_2 + 3x_2^2 + 8x_1x_3 + 6x_4)\right](2, \ldots, 2), \tag{3.82}$$

$$= \frac{1}{4!n!}\left[(x_1^{n-2}x_2) \bigodot x_1^4 + (x_1^{n-2}x_2) \bigodot (6x_1^2x_2) + (x_1^{n-2}x_2) \bigodot 3x_2^2 \right.$$

$$\left. + (x_1^{n-2}x_2) \bigodot (8x_1x_3) + (x_1^{n-2}x_2) \bigodot 6x_4\right](2, \ldots, 2), \tag{3.83}$$

$$= \frac{1}{4!n!}\left[x_1^{4(n-2)}x_2^4 + 6x_1^{2(n-2)}x_2^{n-2}x_2^2x_2^2 + 3x_1^{2(n-2)}x_2^4 + 8x_1^{n-2}x_3^{n-2}x_2x_6 + 6x_4^{n-2}x_4^2\right](2, \ldots, 2), \tag{3.84}$$

$$= \frac{1}{4!n!}\left[2^{4n-4} + 6 \times 2^{3n-2} + 3 \times 2^{2n} + 8 \times 2^{2n-2} + 6 \times 2^n\right], \tag{3.85}$$

$$= \frac{1}{4!n!}\left[2^{4n-4} + 6 \times 2^{3n-2} + 5 \times 2^{2n} + 6 \times 2^n\right]. \tag{3.86}$$

Now, given that $r = 4$, the only possible values of n are 2 and 3. If $n = 2$ then:

$$[Z_{S_n}[2] \boxtimes Z_{S_4}(x_1, x_2, x_3, x_4)]\,(2, 2, \ldots, 2) = \frac{1}{4!n!}\left[2^{4n-4} + 6 \times 2^{3n-2} + 5 \times 2^{2n} + 6 \times 2^n\right], \tag{3.87}$$

$$= \frac{1}{4!2!}\left[2^4 + 6 \times 2^4 + 5 \times 2^4 + 6 \times 2^2\right], \tag{3.88}$$

$$= \frac{16 + 96 + 80 + 24}{4!2!} = 4.5, \tag{3.89}$$

$$\leq \frac{\binom{r+2^n-1}{r}}{n!(n!-1)} = \frac{\binom{7}{4}}{2!(2!-1)} = \frac{35}{2} = 17.5. \tag{3.90}$$

On the other hand, if $n = 3$ then:

$$[Z_{S_n}[2] \boxtimes Z_{S_4}(x_1, x_2, x_3, x_4)]\,(2, \ldots, 2) = \frac{1}{4!n!}\left[2^{4n-4} + 6 \times 2^{3n-2} + 5 \times 2^{2n} + 6 \times 2^n\right], \tag{3.91}$$

$$= \frac{1}{4!3!}\left[2^8 + 6 \times 2^7 + 5 \times 2^6 + 6 \times 2^3\right], \tag{3.92}$$

$$[Z_{S_n}[2] \boxtimes Z_{S_4}(x_1, x_2, x_3, x_4)]\,(2, \ldots, 2) = \frac{256 + 768 + 320 + 48}{4!3!} = \frac{29}{3}, \tag{3.93}$$

$$\leq \frac{\binom{r+2^n-1}{r}}{n!(n!-1)} = \frac{\binom{11}{4}}{3!(3!-1)} = \frac{330}{30} = 11. \tag{3.94}$$

Induction Step: Suppose that Eqn. 3.70 holds for all values from 3 to $r-1$. Using the recurrence given in Eqn. 3.75 and the induction hypothesis for $r-1$ and $r-2$ we get

$$\frac{1}{n!} Z_{S_r}(2^{n-1}, 2^n, \ldots) = \frac{1}{n!r}\left[2^{n-1} Z_{S_{r-1}}(2^{n-1}, 2^n, \ldots) + (r-2+2^n) Z_{S_{r-2}}(2^{n-1}, 2^n, \ldots)\right], \tag{3.95}$$

$$= \frac{2^{n-1}}{n!r} Z_{S_{r-1}}(2^{n-1}, 2^n, \ldots) + \frac{r-2+2^n}{n!r} Z_{S_{r-2}}(2^{n-1}, 2^n, \ldots), \tag{3.96}$$

$$\leq \frac{2^{n-1}}{r} \frac{\binom{r+2^n-2}{r-1}}{n!(n!-1)} + \frac{r-2+2^n}{r} \frac{\binom{r+2^n-3}{r-2}}{n!(n!-1)}, \tag{3.97}$$

$$\leq \frac{2^{n-1}}{n!(n!-1)r} \frac{(r+2^n-2)!}{(r-1)!(2^n-1)!} + \frac{r-2+2^n}{n!(n!-1)r} \frac{(r+2^n-3)!}{(r-2)!(2^n-1)!}, \tag{3.98}$$

$$\leq \frac{2^{n-1}}{n!(n!-1)r} \frac{(r+2^n-2)!}{(r-1)!(2^n-1)!} + \frac{(r-1)(r+2^n-2)!}{n!(n!-1)r!(2^n-1)!}, \tag{3.99}$$

$$\leq \frac{(r+2^n-2)!(r+2^{n-1}-1)}{n!(n!-1)r!(2^n-1)!} \leq \frac{(r+2^n-2)!(r+2^n-1)}{n!(n!-1)r!(2^n-1)!}, \tag{3.100}$$

$$\leq \frac{(r+2^n-1)!}{n!(n!-1)r!(2^n-1)!} = \frac{1}{n!(n!-1)} \binom{r+2^n-1}{r}, \tag{3.101}$$

$$\leq \frac{1}{n!(n!-1)} \binom{r+2^n-1}{r}, \tag{3.102}$$

and this completes the proof. □

Combining Theorems 5 and 6 concludes the upper bound calculation.

Theorem 7.

$$|B_u(n,r)| \leq \frac{2\binom{r+2^n-1}{r}}{n!}. \quad (3.103)$$

Proof.

$$|B_u(n,r)| = Z_{S_n \times S_r}(2,2,\dots,2), \quad (3.104)$$

$$= [Z_{S_n}(x_1,x_2,\dots,x_n) \boxtimes Z_{S_r}(x_1,x_2,\dots,x_r)]\,(2,2,\dots,2), \quad (3.105)$$

$$= [(Z_{S_n}[1] + Z_{S_n}[2] + \dots + Z_{S_n}[n!]) \boxtimes Z_{S_r}(x_1,x_2,\dots,x_r)]\,(2,2,\dots,2), \quad (3.106)$$

$$= [(Z_{S_n}[1]) \boxtimes Z_{S_r}(x_1,\dots,x_r)]\,(2,\dots,2) + [(Z_{S_n}[2]) \boxtimes Z_{S_r}(x_1,\dots,x_r)]\,(2,\dots,2)$$
$$+ \dots + [(Z_{S_n}[n!]) \boxtimes Z_{S_r}(x_1,\dots,x_r)]\,(2,\dots,2), \quad (3.107)$$

$$\leq [(Z_{S_n}[1]) \boxtimes Z_{S_r}(x_1,\dots,x_r)]\,(2,\dots,2) + [(Z_{S_n}[2]) \boxtimes Z_{S_r}(x_1,\dots,x_r)]\,(2,\dots,2)$$
$$+ \dots + [(Z_{S_n}[2]) \boxtimes Z_{S_r}(x_1,x_2,\dots,x_r)]\,(2,2,\dots,2), \quad (3.108)$$

$$\leq \frac{\binom{r+2^n-1}{r}}{n!} + (n!-1)\frac{\binom{r+2^n-1}{r}}{n!(n!-1)} = \frac{2\binom{r+2^n-1}{r}}{n!}. \quad \square \quad (3.109)$$

Table 3.1 lists $\ln|B_u(n,r)|$ along with the natural logarithms of lower and upper bounds for $1 \leq n < r \leq 15$.

Remark 2. It should be mentioned that, if $r < n$, using the relation $|B_u(n,r)| = |B_u(r,n)|$ gives

$$|B_u(n,r)| \leq 2\frac{\binom{n+2^r-1}{n}}{r!}. \quad (3.110)$$

Likewise, if $r < n$, Theorem 4 and $|B_u(n,r)| = |B_u(r,n)|$ together imply

$$|B_u(n,r)| \geq \frac{\binom{n+2^r-1}{n}}{r!}. \quad (3.111)$$

Furthermore, if $r = n$, using the cycle index representation of bi-colored graphs provided in Section 3 in [23] and Theorem 4 gives

$$|B_u(n,n)| \geq \frac{\binom{n+2^n-1}{n}}{2n!}. \quad (3.112)$$

The Z' term in the cycle index representation of bi-colored graphs in [23] prevents us from deriving an upper bound for $|B_u(n,n)|$ that is a constant multiple of the lower bound in this case. On the other hand, an obvious upper bound for $|B_u(n,n)|$ can be derived by setting $r = n+1$ in the inequality in Theorem 7.

n \ r	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1.09861 1.09861 1.79176	1.38629 1.38629 2.07944	1.60944 1.60944 2.30259	1.79176 1.79176 2.48491	1.94591 1.94591 2.63906	2.07944 2.07944 2.77259	2.19722 2.19722 2.89037	2.30259 2.30259 2.99573	2.3979 2.3979 3.09104	2.48491 2.48491 3.17805	2.56495 2.56495 3.2581	2.63906 2.63906 3.3322	2.70805 2.70805 3.4012	2.77259 2.77259 3.46574
2		2.30259 2.56495 2.99573	2.83321 3.09104 3.55535	3.3322 3.52636 4.02535	3.73767 3.91202 4.43082	4.09434 4.2485 4.78749	4.40672 4.55388 5.10595	4.70048 4.82831 5.39363	4.96284 5.0814 5.65599	5.20401 5.31321 5.89715	5.42495 5.52943 6.1203	5.63479 5.7301 6.32794	5.82895 5.91889 6.52209	6.01127 6.09582 6.70441
3			4.00733 4.46591 4.70048	4.8828 5.24702 5.57595	5.65599 5.95584 6.34914	6.34914 6.59851 7.04229	6.97728 7.18841 7.67089	7.55276 7.73368 8.24617	8.08364 8.24012 8.77678	8.57622 8.71276 9.26936	9.03575 9.1562 9.7289	9.46653 9.57345 10.1597	9.872 9.96754 10.5651	10.255 10.3409 10.9481
4				6.4708 6.9594 7.16395	7.72356 8.08641 8.41671	8.86869 9.14238 9.56184	9.92471 10.1349 10.6179	10.9056 11.0692 11.5987	11.8219 11.9512 12.515	12.6821 12.7855 13.3752	13.493 13.5767 14.1861	14.2603 14.3287 14.9534	14.9885 15.045 15.6816	15.6816 15.7287 16.3748
5					9.87164 10.2603 10.5648	11.5633 11.826 12.2565	13.1474 13.3276 13.8406	14.6391 14.7645 15.3322	16.0501 16.1388 16.7432	17.3899 17.4535 18.083	18.6662 18.7124 19.3593	19.8854 19.9195 20.5785	21.053 21.0784 21.7461	22.1736 22.1927 22.8667
6						14.3253 14.5771 15.0185	16.5086 16.6637 17.2017	18.588 18.6849 19.2811	20.5759 20.6372 21.269	22.482 22.5215 23.1752	24.3146 24.3403 25.0078	26.0804 26.0974 26.7736	27.7852 27.7965 28.4783	29.4338 29.4415 30.127
7							19.9011 20.0463 20.5942	22.6165 22.6996 23.3097	25.2339 25.282 25.927	27.7633 27.7915 28.4564	30.2128 30.2295 30.906	32.5895 32.5995 33.2827	34.8992 34.9053 35.5924	37.147 37.1507 37.8401
8								26.6393 26.7201 27.3324	29.9164 29.9604 30.6096	33.102 33.1261 33.7952	36.2043 36.2177 36.8975	39.2304 39.2378 39.9235	42.186 42.1902 42.8792	45.0764 45.0788 45.7696
9									34.5644 34.6096 35.2575	38.4241 38.4479 39.1173	42.1988 42.2114 42.892	45.8953 45.902 46.5885	49.5197 49.5233 50.2128	53.0769 53.0789 53.7701
10										43.693 43.7187 44.3861	48.1502 48.1635 48.8434	52.5284 52.5353 53.2216	56.8335 56.837 57.5266	61.0705 61.0723 61.7636
11											54.0381 54.0528 54.7312	59.1036 59.1111 59.7967	64.0955 64.0993 64.7886	69.0189 69.0208 69.712
12												65.6106 65.6191 66.3038	71.2925 71.2968 71.9856	76.9056 76.9078 77.5988
13													78.4205 78.4254 79.1137	84.7251 84.7275 85.4182
14														92.4768 92.4797 93.17

Table 3.1: Exact values of $\ln|B_u(n,r)|, 1 \leq n < r \leq 15$, and natural logarithms of lower and upper bounds.

We conclude this chapter by stating that the constant factor of 2 in the upper bound can potentially be reduced further. However this will require a further reduction in Inequality 3.68, but reducing $1/(n! - 1)$ in this inequality further seems difficult. Another alternative for reducing the constant in the upper bound would be to bound the remaining terms by a different technique, and this remains to be settled with further research.

Chapter 4

Labeled Bipartite Graphs [1]

In this chapter, we will extend the bounds given in previous chapter to labeled bipartite graphs. In particular, we introduce left(right)-set-labeled and set-labeled bipartite graphs and provide asymptotic bounds for their sizes.

4.1 Preliminary Facts

We begin with some definitions. An (n, r)-bipartite graph is called left(right)-set-labeled if its left(right) vertices are distinguishable up to subsets of all left(right) vertices. An (n, r)-bipartite graph is called set-labeled if left vertices are distinguishable up to subsets of all left vertices and right vertices are distinguishable up to subsets of all right vertices.

[1] **This is joint work with Prof. A. Yavuz Oruç.**

4.2 Left-Set-Labeled Bipartite Graphs

In this section, we start with counting four families of left-set-labeled bipartite graphs and then provide a lower and an upper bound on the number of left-set-labeled bipartite graphs. Right-set-labeled bipartite graphs are counted similarly and their counting is omitted.

Let $B_x(n,r)$ be the set of all (n,r)-left-set-labeled bipartite graphs. Let $B_x(n,r,i)$ be the set of (n,r)-left-set-labeled bipartite graphs in each of which the degrees of exactly i left vertices are greater than 0. Let $\overline{B}_x(i,r)$ be the set of all (i,r)-unlabeled bipartite graphs in each of which the degrees of all left vertices are greater than 0. It follows that

$$|B_x(n,r,i)| = \binom{n}{i}|\overline{B}_x(i,r)|. \tag{4.1}$$

$$|B_x(n,r)| = 1 + \sum_{i=1}^{n} |B_x(n,r,i)|. \tag{4.2}$$

$$|B_x(n,r)| = 1 + \sum_{i=1}^{n} \binom{n}{i}|\overline{B}_x(i,r)|. \tag{4.3}$$

4.2.1 Counting Left-Set-Labeled Bipartite Graphs

Recall that $B_u(n,r)$ denote the set of all unlabeled (n,r)-bipartite graphs. In chapter 2, it was stated that $|B_u(1,r)| = r+1$. Dropping the empty graph yields $|\overline{B}_x(1,r)| = r$, and hence $|B_x(1,r)| = r+1$. To compute $|B_x(n,r)|$, $n \geq 2$ we note the following identities.

$$|B_u(n,r)| = 1 + \sum_{i=1}^{n} |\overline{B}_x(i,r)|, \tag{4.4}$$

$$|B_u(n-1,r)| = 1 + \sum_{i=1}^{n-1} |\overline{B}_x(i,r)|. \tag{4.5}$$

Hence

$$|\overline{B}_x(n,r)| = |B_u(n,r)| - |B_u(n-1,r)|. \tag{4.6}$$

Now we can calculate $|\overline{B}_x(2,r)|$ using Eqn. 4.6 and Theorem 1,

$$\begin{aligned} |\overline{B}_x(2,r)| &= |B_u(2,r)| - |B_u(1,r)|, && (4.7)\\ &= \frac{2r^3+15r^2+34r+22.5+1.5\,(-1)^r}{24} - (r+1), && (4.8) \end{aligned}$$

and similarly $|\overline{B}_x(3,r)|$ using Eqn. 4.6 and Theorem 2,

$$|\overline{B}_x(3,r)| = |B_u(3,r)| - |B_u(2,r)|, \tag{4.9}$$

$$= \begin{cases} \frac{1}{6}\left[\binom{r+7}{7} + \frac{3(r+4)\left(2r^4+32r^3+172r^2+352r+15(-1)^r+225\right)}{960} + \frac{2(r^3+12r^2+45r+54)}{54}\right] \\ \qquad -\frac{2r^3+15r^2+34r+22.5+1.5(-1)^r}{24} \text{ if } r \bmod 3 = 0, \\ \frac{1}{6}\left[\binom{r+7}{7} + \frac{3(r+4)\left(2r^4+32r^3+172r^2+352r+15(-1)^r+225\right)}{960} + \frac{2(r^3+12r^2+45r+50)}{54}\right] \\ \qquad -\frac{2r^3+15r^2+34r+22.5+1.5(-1)^r}{24} \text{ if } r \bmod 3 = 1, \\ \frac{1}{6}\left[\binom{r+7}{7} + \frac{3(r+4)\left(2r^4+32r^3+172r^2+352r+15(-1)^r+225\right)}{960} + \frac{2(r^3+12r^2+39r+28)}{54}\right] \\ \qquad -\frac{2r^3+15r^2+34r+22.5+1.5(-1)^r}{24} \text{ if } r \bmod 3 = 2. \end{cases} \tag{4.10}$$

Using Eqns. 4.3 and 4.8, we can easily calculate $|B_x(2,r)|$.

$$\begin{aligned} |B_x(2,r)| &= 1 + \sum_{i=1}^{2}\binom{2}{i}|\overline{B}_x(i,r)| = 1 + 2|\overline{B}_x(1,r)| + |\overline{B}_x(2,r)|, && (4.11)\\ &= 1 + 2r + \frac{2r^3+15r^2+34r+22.5+1.5\,(-1)^r}{24} - (r+1), && (4.12)\\ &= \frac{2r^3+15r^2+58r+22.5+1.5\,(-1)^r}{24}. && (4.13) \end{aligned}$$

Similarly, using Eqns. 4.3 and 4.10 we can calculate $|B_x(3,r)|$ as follows.

$$\begin{aligned} |B_x(3,r)| &= 1 + \sum_{i=1}^{3}\binom{3}{i}|\overline{B}_x(i,r)|, && (4.14)\\ &= 1 + 3|\overline{B}_x(1,r)| + 3|\overline{B}_x(2,r)| + |\overline{B}_x(3,r)|, && (4.15)\\ &= 1 + 3r + 3\Big(\frac{2r^3+15r^2+34r+22.5+1.5\,(-1)^r}{24} - (r+1)\Big) + |\overline{B}_x(3,r)|, && (4.16) \end{aligned}$$

and replacing $|\overline{B}_x(3,r)|$ by the formula in Eqn. 4.10 gives us

$$|B_x(3,r)| = \begin{cases} \frac{1}{6}\left[\binom{r+7}{7} + \frac{3(r+4)\left(2r^4+32r^3+172r^2+352r+15(-1)^r+225\right)}{960} + \frac{2(r^3+12r^2+45r+54)}{54}\right] \\ \qquad + \frac{4r^3+30r^2+68r-3+3(-1)^r}{24} \text{ if } r \bmod 3 = 0, \\ \frac{1}{6}\left[\binom{r+7}{7} + \frac{3(r+4)\left(2r^4+32r^3+172r^2+352r+15(-1)^r+225\right)}{960} + \frac{2(r^3+12r^2+45r+50)}{54}\right] \\ \qquad + \frac{4r^3+30r^2+68r-3+3(-1)^r}{24} \text{ if } r \bmod 3 = 1, \\ \frac{1}{6}\left[\binom{r+7}{7} + \frac{3(r+4)\left(2r^4+32r^3+172r^2+352r+15(-1)^r+225\right)}{960} + \frac{2(r^3+12r^2+39r+28)}{54}\right] \\ \qquad + \frac{4r^3+30r^2+68r-3+3(-1)^r}{24} \text{ if } r \bmod 3 = 2. \end{cases} \tag{4.17}$$

These results can be extended so that the number of left vertices becomes n while the number of right vertices is reduced to 2 or 3 as follows.

$$|\overline{B}_x(i,1)| = |B_u(i,1)| - |B_u(i-1,1)|, \tag{4.18}$$

$$= (i+1) - i = 1. \tag{4.19}$$

This leads to

$$|B_x(n,1)| = 1 + \sum_{i=1}^{n} \binom{n}{i} |\overline{B}_x(i,1)| = 2^n. \tag{4.20}$$

Similarly, $|\overline{B}_x(i,2)|$ can be calculated using (4.6).

$$|\overline{B}_x(i,2)| = |B_u(i,2)| - |B_u(i-1,2)|, \tag{4.21}$$

$$= |B_u(2,i)| - |B_u(2,i-1)|, \tag{4.22}$$

$$= \frac{2i^3+15i^2+34i+22.5+1.5(-1)^i}{24} - \frac{2(i-1)^3+15(i-1)^2+34(i-1)+22.5+1.5(-1)^{i-1}}{24}, \tag{4.23}$$

$$= \frac{6i^2+24i+21+3(-1)^i}{24}. \tag{4.24}$$

This leads to

$$
\begin{aligned}
|B_x(n,2)| &= 1 + \sum_{i=1}^{n} \binom{n}{i} |\overline{B}_x(i,2)|, && (4.25)\\
&= 1 + \sum_{i=1}^{n} \binom{n}{i} \frac{6i^2 + 24i + 21 + 3(-1)^i}{24}, && (4.26)\\
&= n(n+1)2^{n-4} + n2^{n-1} + 7(2^{n-3}). && (4.27)
\end{aligned}
$$

$|\overline{B}_x(i,3)|$ can similarly be computed as follows.

$$
\begin{aligned}
|\overline{B}_x(i,3)| &= |B_u(i,3)| - |B_u(i-1,3)|, && (4.28)\\
&= |B_u(3,i)| - |B_u(3,i-1)|, && (4.29)\\
&= \begin{cases}
\frac{1}{6}\Big[\binom{i+7}{7} - \binom{i+6}{7} + \frac{3(i+4)(2i^4+32i^3+172i^2+352i+15(-1)^i+225)}{960} \\
\quad - \frac{3(i+3)(2(i-1)^4+32(i-1)^3+172(i-1)^2+352(i-1)-15(-1)^i+225)}{960} \\
\quad + \frac{2(i^3+12i^2+45i+54)}{54} - \frac{2((i-1)^3+12(i-1)^2+39(i-1)+28)}{54}\Big] & \text{if } i \bmod 3 = 0,\\
\frac{1}{6}\Big[\binom{i+7}{7} - \binom{i+6}{7} + \frac{3(i+4)\left(2i^4+32i^3+172i^2+352i+15(-1)^i+225\right)}{960} \\
\quad - \frac{3(i+3)(2(i-1)^4+32(i-1)^3+172(i-1)^2+352(i-1)-15(-1)^i+225)}{960} \\
\quad + \frac{2(i^3+12i^2+45i+50)}{54} - \frac{2((i-1)^3+12(i-1)^2+45(i-1)+54)}{54}\Big] & \text{if } i \bmod 3 = 1,\\
\frac{1}{6}\Big[\binom{i+7}{7} - \binom{i+6}{7} + \frac{3(i+4)(2i^4+32i^3+172i^2+352i+15(-1)^i+225)}{960} \\
\quad - \frac{3(i+3)(2(i-1)^4+32(i-1)^3+172(i-1)^2+352(i-1)-15(-1)^i+225)}{960} \\
\quad + \frac{2(i^3+12i^2+39i+28)}{54} - \frac{2((i-1)^3+12(i-1)^2+45(i-1)+50)}{54}\Big] & \text{if } i \bmod 3 = 2,
\end{cases} && (4.30)
\end{aligned}
$$

$$
= \begin{cases} \frac{1}{6}\left[\binom{i+6}{6} + \frac{10i^4+140i^3+680i^2+1330i+30i(-1)^i+105(-1)^i+855}{320} \right. \\ \qquad \left. + \frac{6i^2+54i+108}{54}\right] \text{ if } i \bmod 3 = 0, \\ \frac{1}{6}\left[\binom{i+6}{6} + \frac{10i^4+140i^3+680i^2+1330i+30i(-1)^i+105(-1)^i+855}{320} \right. \\ \qquad \left. + \frac{6i^2+42i+60}{54}\right] \text{ if } i \bmod 3 = 1, \\ \frac{1}{6}\left[\binom{i+6}{6} + \frac{10i^4+140i^3+680i^2+1330i+30i(-1)^i+105(-1)^i+855}{320} \right. \\ \qquad \left. + \frac{6i^2+30i+24}{54}\right] \text{ if } i \bmod 3 = 2. \end{cases} \tag{4.31}
$$

This leads to

$$
|B_x(n,3)| = 1 + \sum_{i=1}^{n} \binom{n}{i} |\overline{B}_x(i,3)|, \tag{4.32}
$$

$$
= \begin{cases} \frac{3\ 2^n n^6+171\ 2^n n^5+3765\ 2^n n^4+41265\ 2^n n^3+14787\ 2^{n+4} n^2}{829440} \\ \quad + \frac{12\left(2560(-1)^{n/3}+55077\ 2^n\right)n+880\left(128(-1)^{n/3}+763\ 2^n\right)}{829440} \text{ if } n \bmod 3 = 0, \\ \frac{3\ 2^n n^6+171\ 2^n n^5+3765\ 2^n n^4+41265\ 2^n n^3+14787\ 2^{n+4} n^2}{829440} \\ \quad + \frac{165231\ 2^{n+2} n-80\left(1280(-1)^{\frac{n+2}{3}}-8393\ 2^n\right)}{829440} \text{ if } n \bmod 3 = 1, \\ \frac{3\ 2^n n^6+171\ 2^n n^5+3765\ 2^n n^4+41265\ 2^n n^3+14787\ 2^{n+4} n^2}{829440} \\ \quad + \frac{12\left(2560(-1)^{\frac{n+1}{3}}+55077\ 2^n\right)n+80\left(128(-1)^{\frac{n+1}{3}}+8393\ 2^n\right)}{829440} \text{ if } n \bmod 3 = 2, \end{cases} \tag{4.33}
$$

where $n \geq 2$.

4.2.2 A Lower Bound for $|B_x(n,r)|$

Proposition 2.

$$
|B_x(n,r)| \geq 1 + \sum_{i=1}^{n} \binom{n}{i} \binom{i+2^r-2}{i} \Big/ r!. \tag{4.34}
$$

Proof. The first term 1 given in lower bound formula counts for the left-set-labeled bipartite graph where there are no edges between vertices. Consider $2^r - 1$ subsets of r right vertices excluding empty set. Let X_i be an arbitrary but fixed subset of left n vertices of size $i, 1 \leq i \leq n$. Each one-to-one pairing of X_i with any i of the $2^r - 1$ subsets of right vertices constitute a left-set-labeled bipartite graphs. The number of such pairings is given by $\binom{i+2^r-2}{i}$. This formula counts the number of i-selections from a set of $2^r - 1$ distinct elements. Since right vertices are indistinguishable up to $r!$ of these may be equivalent under a permutation of r vertices. Therefore, there exist at least $\binom{i+2^r-2}{i}\Big/r!$ distinct left-set-label bipartite graphs associated with X_i. Since there exist $\binom{n}{i}$ i-subset of left vertices, we must have at least $\binom{n}{i}\binom{i+2^r-2}{i}\Big/r!$ distinct left-set-labeled bipartite graphs between subsets of size i left vertices and r right vertices. Summing this for i from 1 to n establishes the lower bound in the statement. $\square$

It is difficult to obtain a closed form formula for the sum in Proposition 2, but it can be approximated by computing the maximum value of the argument of the sum with respect to i as shown in the following proposition.

Proposition 3.

$$|B_x(n,r)| \geq \binom{n}{i_{max}}\binom{i_{max}+2^r-2}{i_{max}}\Big/r!, \tag{4.35}$$

where $i_{max} = \frac{1}{4}\Big[(a^2+6an+n^2)^{1/2} - a - n\Big]$ and $a = 2^r - 2$.

Proof. The inequality

$$|B_x(n,r)| \geq \binom{n}{i_{max}}\binom{i_{max}+2^r-2}{i_{max}}\Big/r! \tag{4.36}$$

holds, where i_{max} is the value of i, $1 \leq i_{max} \leq n$ that maximizes the expression $\binom{n}{i}\binom{i+2^r-2}{i}\Big/r!$.

Differentiating the argument of the sum and setting equal to 0 gives $i_{max} = \frac{1}{4}\Big[(a^2+6an+n^2)^{1/2} - a - n\Big] + n/2$ where $a = 2^r - 2$. $\square$

4.2.3 Upper Bounds for $|B_x(n,r)|$

Proposition 4.

$$|B_x(n,r)| \leq 1 + \sum_{i=1}^{n} \binom{n}{i} \left[2\binom{r+2^i-1}{r} \Big/ i! - \binom{r+2^{i-1}-1}{r} \Big/ (i-1)! \right], \quad (4.37)$$

where $n < r$.

Proof. The first term 1 given in the upper bound formula counts for left-set-labeled bipartite graph in which there are no edges between vertices. For the remaining sum, we use Theorem 7, Eqns. 4.3 and 4.6 to obtain

$$|B_x(n,r)| = 1 + \sum_{i=1}^{n} \binom{n}{i} |\overline{B}_x(i,r)|, \quad (4.38)$$

$$= 1 + \sum_{i=1}^{n} \binom{n}{i} \Big[|B_u(i,r)| - |B_u(i-1,r)| \Big], \quad (4.39)$$

$$\leq 1 + \sum_{i=1}^{n} \binom{n}{i} \left[2\binom{r+2^i-1}{r} \Big/ i! - \binom{r+2^{i-1}-1}{r} \Big/ (i-1)! \right], \quad (4.40)$$

and this establishes the upper bound in the statement. □

Proposition 5. For all $n, r \geq 2$,

$$|B_x(n,r)| \leq 1 + 2\sum_{i=1}^{r-1} \binom{n}{i} \binom{r+2^i-1}{r} \Big/ i! + 2\binom{n}{r}\binom{r+2^r}{r+1} \Big/ r!$$
$$+2\sum_{i=r+1}^{n} \binom{n}{i} \binom{i+2^r-1}{i} \Big/ r!. \quad (4.41)$$

Proof. Again the first term 1 given in the upper bound formula counts for the left-set-labeled bipartite graph where there are no edges between vertices. For the remaining sum, we note $|\overline{B}_x(i,r)|$ can not exceed $|B_u(i,r)|$, since every bipartite graph in $\overline{B}_x(i,r)$ is also in $B_u(i,r)$, $1 \le i \le n$.

$$
\begin{aligned}
|B_x(n,r)| &= 1 + \sum_{i=1}^{n} \binom{n}{i} |\overline{B}_x(i,r)|, & (4.42)\\
&\le 1 + \sum_{i=1}^{n} \binom{n}{i} |B_u(i,r)|, & (4.43)\\
&\le 1 + 2\sum_{i=1}^{r-1} \binom{n}{i} \binom{r+2^i-1}{r} \Big/ i! + \sum_{i=r}^{r} \binom{n}{i} |B_u(i,r)| \\
&\quad +2 \sum_{i=r+1}^{n} \binom{n}{i} \binom{i+2^r-1}{i} \Big/ r!. & (4.44)
\end{aligned}
$$

Since we do not have any upper bound formula for $|B_u(i,r)|$ when $i = r$, we will use $|B_u(r,r)| \le |B_u(r+1,r)|$.

$$
\begin{aligned}
|B_x(n,r)| &\le 1 + 2\sum_{i=1}^{r-1} \binom{n}{i} \binom{r+2^i-1}{r} \Big/ i! + 2\binom{n}{r}\binom{r+2^r}{r+1} \Big/ r! \\
&\quad +2 \sum_{i=r+1}^{n} \binom{n}{i} \binom{i+2^r-1}{i} \Big/ r!, & (4.45)
\end{aligned}
$$

and this establishes the upper bound in the statement. □

4.3 Set-Labeled Bipartite Graphs

In this section, we start with counting two families of set-labeled bipartite graphs and then provide a lower and an upper bound on the number of set-labeled bipartite graphs.

First of all we start with some definitions. Let $B_{xy}(n,r)$ be the set of all (n,r)-set-labeled bipartite graphs. Also $\overline{B}_{xy}(i,j)$ be the set of all (i,j)-unlabeled bipartite graphs such that there is no vertex in the graph that has degree of 0. We have

$$
|B_{xy}(n,r)| = 1 + \sum_{i=1}^{n} \sum_{j=1}^{r} \binom{n}{i} \binom{r}{j} |\overline{B}_{xy}(i,j)|. \qquad (4.46)
$$

We could use a notation that is similar to the notation $B_x(n, r, i)$ of the prior section for completeness. However, the relation between $\overline{B}_{xy}(i, j)$ and $B_{xy}(n, r, i, j)$ is obvious and omitted here.

4.3.1 Counting Set-Labeled Bipartite Graphs

Clearly, $|\overline{B}_{xy}(1, 1)| = |\overline{B}_{xy}(1, j)| = |\overline{B}_{xy}(i, 1)| = 1$. $|\overline{B}_x(i, j)$ and $|\overline{B}_x(i, j-1)|$ can be witten as follows.

$$|\overline{B}_x(i, j)| = \sum_{k=1}^{j} |\overline{B}_{xy}(i, k)|, \tag{4.47}$$

$$|\overline{B}_x(i, j-1)| = \sum_{k=1}^{j-1} |\overline{B}_{xy}(i, k)|. \tag{4.48}$$

Hence

$$|\overline{B}_{xy}(i, j)| = |\overline{B}_x(i, j)| - |\overline{B}_x(i, j-1)|. \tag{4.49}$$

Now we can calculate $|\overline{B}_{xy}(i, 2)|$ using Eqn. 4.49.

$$\begin{aligned}
|\overline{B}_{xy}(i, 2)| &= |\overline{B}_x(i, 2)| - |\overline{B}_x(i, 1)|, && (4.50)\\
&= \frac{6i^2 + 24i + 21 + 3(-1)^i}{24} - 1, && (4.51)\\
&= \frac{6i^2 + 24i - 3 + 3(-1)^i}{24}. && (4.52)
\end{aligned}$$

Using Eqns. 4.46 and 4.52, we can calculate $|B_{xy}(n, 2)|$ as follows.

$$\begin{aligned}
|B_{xy}(n, 2)| &= 1 + \sum_{i=1}^{n} \binom{n}{i} \sum_{j=1}^{2} \binom{2}{j} |\overline{B}_{xy}(i, j)|, && (4.53)\\
&= 1 + \sum_{i=1}^{n} \binom{n}{i} \left[\binom{2}{1} |\overline{B}_{xy}(i, 1)| + \binom{2}{2} |\overline{B}_{xy}(i, 2)| \right], && (4.54)\\
&= 1 + \sum_{i=1}^{n} \binom{n}{i} \frac{6i^2 + 24i + 45 + 3(-1)^i}{24}, && (4.55)\\
&= \frac{15}{8}(2^n) + 2^{n-1}n + 2^{n-4}n(n+1) - 1. && (4.56)
\end{aligned}$$

Similarly, we can calculate $|\overline{B}_{xy}(i,3)|$ using Eqn. 4.49 as follows.

$$|\overline{B}_{xy}(i,3)| = |\overline{B}_x(i,3)| - |\overline{B}_x(i,2)|. \tag{4.57}$$

Now we will use Eqns. 4.24 and 4.57 to compute $|B_{xy}(n,3)|$.

$$\begin{aligned}
|B_{xy}(n,3)| &= 1 + \sum_{i=1}^{n} \binom{n}{i} \sum_{j=1}^{3} \binom{3}{j} |\overline{B}_{xy}(i,j)|, && (4.58)\\
&= 1 + \sum_{i=1}^{n} \binom{n}{i} \left[\binom{3}{1} |\overline{B}_{xy}(i,1)| + \binom{3}{2} |\overline{B}_{xy}(i,2)| + \binom{3}{3} |\overline{B}_{xy}(i,3)| \right], && (4.59)\\
&= 1 + \sum_{i=1}^{n} \binom{n}{i} \left[3 + 3|\overline{B}_{xy}(i,2)| + |\overline{B}_x(i,3)| - |\overline{B}_x(i,2)| \right], && (4.60)\\
&= 1 + \sum_{i=1}^{n} \binom{n}{i} \left[3 + \frac{3(6i^2+24i-3+3(-1)^i)}{24} - \frac{6i^2+24i+21+3(-1)^i}{24} \right] \\
&\quad + \sum_{i=1}^{n} \binom{n}{i} |\overline{B}_x(i,3)|, && (4.61)\\
&= 1 + \sum_{i=1}^{n} \binom{n}{i} \left[\frac{i^2}{2} + 2i + \frac{(-1)^i}{4} + \frac{7}{4} \right] + \sum_{i=1}^{n} \binom{n}{i} |\overline{B}_x(i,3)|, && (4.62)\\
&= 1 + 2^{n-3}n^2 + 9 \times 2^{n-3}n + 7 \times 2^{n-2} - 2 + \sum_{i=1}^{n} \binom{n}{i} |\overline{B}_x(i,3)|. && (4.63)
\end{aligned}$$

Rearranging terms using Eqn. 4.33 gives us

$$\begin{aligned}
|B_{xy}(n,3)| &= 2^{n-3}n^2 + 9 \times 2^{n-3}n + 7 \times 2^{n-2} - 2 + |B_x(n,3)| && (4.64)\\
&= \begin{cases}
\frac{3\times 2^n n^6 + 171\times 2^n n^5 + 3765\times 2^n n^4 + 41265\times 2^n n^3 + 21267\times 2^{n+4} n^2}{829440} \\
\quad + \frac{12\left(2560(-1)^{n/3} + 132837\times 2^n\right)n + 80\left(1408(-1)^{n/3} + 26537\times 2^n - 20736\right)}{829440} & \text{if } n \bmod 3 = 0,\\[2ex]
\frac{3\times 2^n n^6 + 171\times 2^n n^5 + 3765\times 2^n n^4 + 41265\times 2^n n^3 + 21267\times 2^{n+4} n^2 + 398511\times 2^{n+2} n}{829440} \\
\quad - \frac{80\left(1280(-1)^{\frac{n+2}{3}} - 26537\times 2^n + 20736\right)}{829440} & \text{if } n \bmod 3 = 1,\\[2ex]
\frac{3\times 2^n n^6 + 171\times 2^n n^5 + 3765\times 2^n n^4 + 41265\times 2^n n^3 + 21267\times 2^{n+4} n^2}{829440} \\
\quad + \frac{12\left(2560(-1)^{\frac{n+1}{3}} + 132837\times 2^n\right)n + 80\left(128(-1)^{\frac{n+1}{3}} + 26537\times 2^n - 20736\right)}{829440} & \text{if } n \bmod 3 = 2.
\end{cases} && (4.65)
\end{aligned}$$

4.3.2 Bounds for Set-Labeled Bipartite Graphs

In this section, we provide both a lower and upper bound on the number of set-labeled bipartite graphs. But before we need some preliminary facts to give our bounds.

Using Theorems 4 and 7, and Eqn. 3.112 we can write following inequalities for $|B_u(i,j)|$ and $|B_u(i-1,j)|$ as follows.

$$\frac{\binom{j+2^i-1}{j}}{i!} \leq |B_u(i,j)| \leq \frac{2\binom{j+2^i-1}{j}}{i!}, \tag{4.66}$$

$$\frac{\binom{j+2^{i-1}-1}{j}}{(i-1)!} \leq |B_u(i-1,j)| \leq \frac{2\binom{j+2^{i-1}-1}{j}}{(i-1)!}, \tag{4.67}$$

where $i < j$,

$$\frac{\binom{i+2^j-1}{i}}{j!} \leq |B_u(i,j)| \leq \frac{2\binom{i+2^j-1}{i}}{j!}, \tag{4.68}$$

$$\frac{\binom{i+2^j-2}{i-1}}{j!} \leq |B_u(i-1,j)| \leq \frac{2\binom{i+2^j-2}{i-1}}{j!}, \tag{4.69}$$

where $j < i-1$,

$$\frac{\binom{i+2^{i-1}-1}{i}}{(i-1)!} \leq |B_u(i,j)| \leq \frac{2\binom{i+2^{i-1}-1}{i}}{(i-1)!}, \tag{4.70}$$

$$\frac{\binom{i+2^{i-1}-2}{i-1}}{2(i-1)!} \leq |B_u(i-1,j)| \leq \frac{2\binom{i+2^{i-1}-1}{i}}{(i-1)!}, \tag{4.71}$$

where $j = i-1$ and

$$\frac{\binom{i+2^i-1}{i}}{2(i!)} \leq |B_u(i,j)| \leq \frac{2\binom{i+2^i}{i+1}}{i!}, \tag{4.72}$$

$$\frac{\binom{i+2^{i-1}-1}{i}}{(i-1)!} \leq |B_u(i-1,j)| \leq \frac{2\binom{i+2^{i-1}-1}{i}}{(i-1)!}, \tag{4.73}$$

where $j = i$.

Using these inequalities with Eqn. 4.6, we can bound $|\overline{B}_x(i,j)|$ as follows.

$$\frac{\binom{j+2^i-1}{j}}{i!} - \frac{2\binom{j+2^{i-1}-1}{j}}{(i-1)!} \leq |\overline{B}_x(i,j)| \leq \frac{2\binom{j+2^i-1}{j}}{i!} - \frac{\binom{j+2^{i-1}-1}{j}}{(i-1)!}, i < j, \tag{4.74}$$

$$\frac{\binom{i+2^j-1}{i}}{j!} - \frac{2\binom{i+2^j-2}{i-1}}{j!} \leq |\overline{B}_x(i,j)| \leq \frac{2\binom{i+2^j-1}{i}}{j!} - \frac{\binom{i+2^j-2}{i-1}}{j!}, j < i-1, \tag{4.75}$$

$$\frac{\binom{i+2^{i-2}-1}{i}}{(i-2)!} - \frac{2\binom{i+2^{i-2}-2}{i-1}}{(i-2)!} \leq |\overline{B}_x(i,i-2)| \leq |\overline{B}_x(i,j)| \leq \frac{2\binom{i+2^{i-1}-1}{i}}{(i-1)!} - \frac{\binom{i+2^{i-1}-2}{i-1}}{2(i-1)!}, j = i-1, \tag{4.76}$$

$$\frac{\binom{i+2^i-2}{i-1}}{i!} - \frac{2\binom{i+2^{i-1}-2}{i-1}}{(i-1)!} \leq |\overline{B}_x(i,i-1)| \leq |\overline{B}_x(i,j)| \leq \frac{2\binom{i+2^i}{i+1}}{i!} - \frac{\binom{i+2^{i-1}-1}{i}}{(i-1)!}, j = i. \tag{4.77}$$

Proposition 6. Let $B_{xy}(n,r)$ be the set of all (n,r)-set-labeled bipartite graphs.

$$|B_{xy}(n,r)| \geq 1 + \sum_{i=1}^{n} \binom{n}{i} \left(\sum_{j=1}^{i-2} \binom{r}{j} \left[\frac{\binom{i+2^j-1}{i}}{j!} - \frac{2\binom{i+2^j-2}{i-1}}{j!} - \frac{2\binom{i+2^{j-1}-1}{i}}{(j-1)!} + \frac{\binom{i+2^{j-1}-2}{i-1}}{(j-1)!} \right] \right.$$
$$\left. + \sum_{j=i+2}^{r} \binom{r}{j} \left[\frac{\binom{j+2^i-1}{j}}{i!} - \frac{2\binom{j+2^{i-1}-1}{j}}{(i-1)!} - \frac{2\binom{j+2^i-2}{j-1}}{i!} + \frac{\binom{j+2^{i-1}-2}{j-1}}{(i-1)!} \right] \right). \tag{4.78}$$

Proof. Substituting $|\overline{B}_{xy}(i,j)|$ in Eqn. 4.46 using Eqn. 4.49 gives

$$|B_{xy}(n,r)| = 1 + \sum_{i=1}^{n} \sum_{j=1}^{r} \binom{n}{i} \binom{r}{j} |\overline{B}_{xy}(i,j)|, \tag{4.79}$$

$$= 1 + \sum_{i=1}^{n} \sum_{j=1}^{r} \binom{n}{i} \binom{r}{j} \left[|\overline{B}_x(i,j)| - |\overline{B}_x(i,j-1)| \right]. \tag{4.80}$$

Now replacing $|\overline{B}_x(i,j)|$ with its lower bound, $|\overline{B}_x(i,j-1)|$ with its upper bound and ignoring the cases where $i-1 \leq j \leq i+1$ lead to

$$|B_{xy}(n,r)| \geq 1 + \sum_{i=1}^{n} \binom{n}{i} \left(\sum_{j=1}^{i-2} \binom{r}{j} \left[\frac{\binom{i+2^j-1}{i}}{j!} - \frac{2\binom{i+2^j-2}{i-1}}{j!} - \frac{2\binom{i+2^{j-1}-1}{i}}{(j-1)!} + \frac{\binom{i+2^{j-1}-2}{i-1}}{(j-1)!} \right] \right.$$
$$\left. + \sum_{j=i+2}^{r} \binom{r}{j} \left[\frac{\binom{j+2^i-1}{j}}{i!} - \frac{2\binom{j+2^{i-1}-1}{j}}{(i-1)!} - \frac{2\binom{j+2^i-2}{j-1}}{i!} + \frac{\binom{j+2^{i-1}-2}{j-1}}{(i-1)!} \right] \right). \tag{4.81}$$

□

Remark 3. It is noted that using the cases where $i-1 \leq j \leq i+1$ will likely give negative terms. Therefore, they are not included in the computation of the lower bound.

Proposition 7. Let $B_{xy}(n, r)$ be the set of all (n, r)-set-labeled bipartite graphs.

$$|B_{xy}(n,r)| \leq 1 + 2\sum_{i=1}^{n}\binom{n}{i}\left[\sum_{j=1}^{i-1}\binom{r}{j}\binom{i+2^j-1}{i}\Big/j! + \binom{r}{i}\binom{i+2^i}{i+1}\Big/i! + \sum_{j=i+1}^{r}\binom{r}{j}\binom{j+2^i-1}{j}\Big/i!\right]. \tag{4.82}$$

Proof. For the upper bound, we note $|\overline{B}_{xy}(i, j)|$ can not exceed $|B_u(i, j)|$, since every bipartite graph in $\overline{B}_{xy}(i, j)$ is also in $B_u(i, j)$, $1 \leq i \leq n$, $1 \leq j \leq r$.

$$|B_{xy}(n,r)| = 1 + \sum_{i=1}^{n}\sum_{j=1}^{r}\binom{n}{i}\binom{r}{j}|\overline{B}_{xy}(i,j)|, \tag{4.83}$$

$$\leq 1 + \sum_{i=1}^{n}\sum_{j=1}^{r}\binom{n}{i}\binom{r}{j}|B_u(i,j)|, \tag{4.84}$$

$$\leq 1 + 2\sum_{i=1}^{n}\binom{n}{i}\left[\sum_{j=1}^{i-1}\binom{r}{j}\binom{i+2^j-1}{i}\Big/j! + \binom{r}{i}\binom{i+2^i}{i+1}\Big/i! + \sum_{j=i+1}^{r}\binom{r}{j}\binom{j+2^i-1}{j}\Big/i!\right], \tag{4.85}$$

gives the expression in the statement. Note that the second term in the last inequality has been obtained by replacing $j = i$ by $j = i + 1$. □

Chapter 5

Two Problems in Network Wiring and Switching

5.1 Motivation

The results we presented up until now have been concerned with enumeration problems in combinatorial call models [19]. In this chapter, our focus will shift to problems which arise in one-sided switching network wiring and routing [19]. The next section describes a routing algorithm that improves the time complexity of the routing algorithm [20]. Our new wiring algorithm is presented in Section 5.3. The reader is referred to [21] for the description of one-sided switching networks.

5.2 Routing in One-Sided, Binary Tree Switches

Oruç [20] described a self-routing algorithm for one-sided, binary tree switches whose terminals are paired using the cyclic permutation group wiring method. The following three assumptions were made by this algorithm:

1. The number of terminals n is a power of 2.
2. The clusters are labeled from top to bottom by $\lg n$-bit numbers.
3. Terminal i requests terminal j using a $\lg n$-bit cluster address, $A_i = a_{i,\lg n-1} \dots a_{i,1} a_{i,0}$, where $0 \leq i \leq n-1$.

The steps of the algorithm in [20] are as follows:

- Compute the index h_i of i in $p^{(j-i) \mod n}$.
- If h_i is even then $A_i = (j-i) \mod n$; else $A_i = (i-j) \mod n$.
- Route i to cluster A_i by decoding A_i over the binary tree that connects i to the $n-1$ clusters in the one-sided, binary tree switch.

This algorithm has $O(n)$ time complexity and this time complexity is contributed by the first step. In what follows we reduce this time complexity to order $O(\lg n)$.

Let

- n be a power of 2,
- $T = \{0, 1, 2, \dots, n-1\}$ be set of positive integers,
- $p = (0\,1\,2 \dots n-1)$ denote the permutation that maps i to $i+1 \mod n$, $i \in T$,
- G be the cyclic group of permutations generated by p.

It is assumed that the first cycle of each permutation starts with 0. Also index of first element in each permutation starts with 0.

Lemma 10.

Let x be an odd integer where $1 \leq x \leq n-1$. The index of element $i \in T$ in p^x is even if element i is even and index of element $i \in T$ in p^x is odd if element i is odd.

Proof. As stated by Oruç [20], p^x maps i to $i + x \mod n$ where $1 \leq x \leq n - 1$. Since x is odd if i is even then $i + x$ is odd. Similarly, if element i is odd then $i + x$ is even. Thus odd numbers follow even numbers and even numbers follow odd numbers in p^x. Moreover, p^x is a cycle of length n because n is a power of 2, x is an odd number implying that $\text{lcm}(n, x) = nx$. Since each permutation begins with 0, i.e., an even number and its index value is even we conclude that the index of i in p^x is even if i is even and index of i in p^x is odd if i is odd. □

Lemma 11.

Let $n = 2^a$ and x be an even integer, $1 \leq x \leq n - 1$ such that $x = 2^b \times B$, where B is an odd positive integer and b is a nonnegative integer. If 2^{b+1} divides $i - (i \mod 2^b)$ then index of i in p^x is even otherwise index of i in p^x is odd.

Proof.

We first prove the statement for the first cycle of p^x. Let h_i denote the index of i and suppose i is in the first cycle of p^x.

$$x \times h_i \mod n = i, \tag{5.1}$$

$$x \times h_i = i + n \times A, \tag{5.2}$$

where A is a natural number. Now replacing x by $2^b \times B$ and n by 2^a gives

$$2^b \times B \times h_i = i + 2^a \times A. \tag{5.3}$$

Dividing both sides of the equation by 2^b gives

$$B \times h_i = \frac{i}{2^b} + 2^{a-b} \times A. \tag{5.4}$$

Note that i is divisible by 2^b by Eqn. 5.4. Therefore $i - (i \mod 2^b) = i$. Since $x < n$, then $b < a$, $a - b > 0$ and $2^{a-b} \times A$ is even. Thus, if 2^{b+1} divides i then h_i is even otherwise h_i is odd.

Now suppose i appears in a cycle other than the first cycle of p^x.

We first note that p^x moves 0 to itself by shifting it αx places such that $\alpha x \bmod n = 0$. Hence $\alpha x = \text{lcm}(x, n)$ so that the length of the first cycle, $\alpha = \text{lcm}(x, n)/x$. Furthermore $\text{lcm}(x, n)/x = n/2^b$ since

$$\begin{aligned} \text{lcm}(x, n)/x &= \text{lcm}(2^b \times B, 2^a)/x, && (5.5) \\ &= (2^a \times B)/x, && (5.6) \\ &= n/2^b. && (5.7) \end{aligned}$$

Since x is divisible by 2^b, none of the elements in the first cycle can be 1 through $2^b - 1$. Using the wiring mechanism of Oruç [20], we fixed the first element of the second cycle to 1. This second cycle has the same structure as the first cycle such that all values in the first cycle are incremented by one. For the remaining cycles we use the same construction. At the end, we get 2^b cycles of length $n/2^b$ in p^x and first value of each cycle starts with numbers from 0 to $2^b - 1$, respectively.

Without loss of generality, pick an element i in the cycle that begins with j, $0 \leq j \leq 2^b - 1$. Using Eqn. 5.3 we have

$$2^b \times B \times h_{i-j} = i - j + 2^a \times A \tag{5.8}$$

where $i - j$ is an element of the first cycle and the parity of h_i is equal to parity of h_{i-j} by the construction described above and by the fact that all cycles are even length ([20], Appendix B). Since $j < 2^b$ and by Eqn. 5.8, $i - j$ is divisible by 2^b. Thus, we have

$$j = i \mod 2^b. \tag{5.9}$$

By combining Eqn. 5.8 with Eqn. 5.9 we get

$$2^b \times B \times h_{i-j} = i - (i \mod 2^b) + 2^a \times A. \tag{5.10}$$

Applying the same rules given for the first cycle to Eqn. 5.10 concludes the proof. □

Remark 4. The time complexity of the calculation of b in x is $O(\lg n)$.

Proof. Recall that $x = 2^b \times B < n$ and B is an odd number. We divide x by 2 and continue until the remainder is 1. This establishes the proof. □

Combining Lemmas 10 and 11 with Remark 4 shows that our proposed method reduces the time complexity of computing the parity of h_i from $O(n)$ to $O(\lg n)$.

5.3 A New Wiring Method

The routing method described in preceding section applies to the wiring scheme in [20]. In this section, we present a new wiring method that relaxes the assumption that n be a power of 2. Our construction amounts to designing a $(n-1) \times \lfloor n/2 \rfloor$-matrix in which each row consists of pairs of numbers 0 through $n-1$. If n is a power of 2, the number of rows is given by $n-1$, otherwise is fixed to at most to the smallest power of 2 that is greater than n. The construction proceeds by initializing a list of numbers from 0 through $n-1$ in order. We then proceed to partition the numbers into two groups and pair the i^{th} element of the first group with the i^{th} element of the second group. If the number of elements in the original list is not even, we place $\lceil n/2 \rceil$ of the elements in the first group and the remaining $\lfloor n/2 \rfloor$ of the elements in the second group. In this case the last element of the first group is not paired with any element in the second group. After this operation, the generated pairs are placed in the first row of our matrix. Next, we fixed the places of elements of the second group as before and rotate the elements of the first group by one place to the right in a cyclic manner. Then we pair the two groups as before and place the generated pairs into the second row of our matrix. We continue this process until the rotation step brings the first group back into its original order. At the end of the process the first $\lceil n/2 \rceil$ rows of our matrix will be determined. This entire process is repeated recursively on each half of the list of elements, generating remaining $n/4, n/8, \ldots$ rows of our matrix, and the process is terminated when the pairs of the last row is generated.

That this algorithm generates all $n(n-1)/2$ pairs of the replicates of n terminals can be argued as follows. The pairs are generated in such a way that each element in the first half is combined with all elements of the second half by the rotation steps. Since the algorithm works recursively, this observation applies throughout the entire process and thus all the pairs are generated.

Remark 5. Time complexity of this algorithm is given by the following recurrence.

$$T(n) = 2T(n/2) + O(n^2). \tag{5.11}$$

The solution of this recurrence is

$$T(n) = O(n^2). \tag{5.12}$$

This algorithm is implemented in C++ and listed in the Appendix with some examples.

5.4 The Routing Algorithm

The wiring method described in preceding section results in a one-sided switching network where $n(n-1)/2$ pairs of replicates of n terminals are partitioned into clusters of pairs. These clusters correspond to the rows of our matrix constructed during the wiring process. This is illustrated in Figure 5.1 for $n = 8$. We now provide an algorithm to determine the cluster in which the acquired pair is located.

Let (i, j), where $0 \le i < j \le n-1$, be such a pair and the index of first cluster be 0. If $i < n/2 \le j$, then pair (i, j) belongs to cluster

$$(i-j) \mod \lceil \frac{n}{2} \rceil. \tag{5.13}$$

We note that the formula works simply by the fact that $(i-j) \mod \lceil \frac{n}{2} \rceil$ increases by one as we traverse from cluster 0 towards $\lceil n/2 \rceil - 1$.

Now suppose $i < j < n/2$ or $n/2 \le i < j$. In the first case, if $i < n/4 \le j$, then pair (i, j) belongs to cluster

$$\lceil n/2 \rceil + \left((i-j) \mod \left\lceil \frac{n}{4} \right\rceil \right). \tag{5.14}$$

In the latter case, if $i < 3n/4 \le j$, then pair (i, j) belongs to cluster

$$\lceil n/2 \rceil + \left((i-j) \mod \left\lceil \frac{\lfloor n/2 \rfloor}{2} \right\rceil \right). \tag{5.15}$$

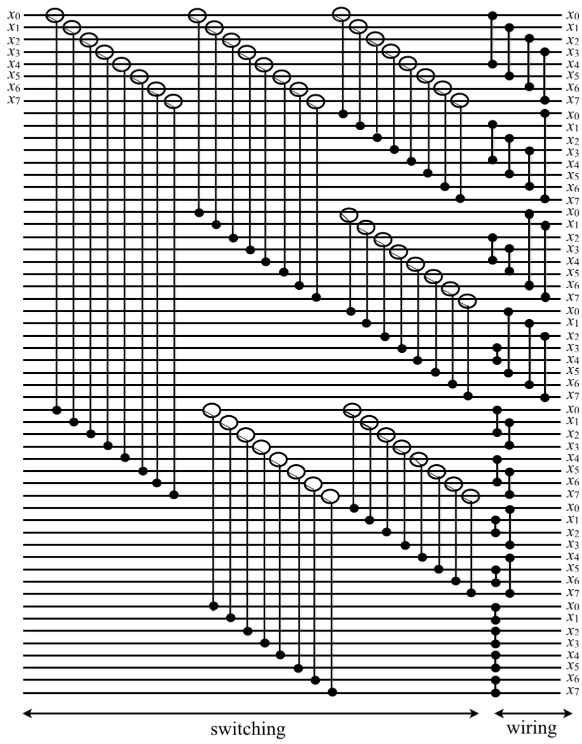

Figure 5.1: An 8 terminal, one-sided switch wiring.

In both cases, if the auxiliary condition fails, then the same process is recursively applied until the cluster is found.

An implementation of this algorithm in C++ along with some examples is given in the Appendix.

Remark 6. Time complexity of this routing algorithm is given by the following recurrence.

$$T(n) = T(n/2) + O(1). \tag{5.16}$$

The solution of this recurrence is

$$T(n) = O(\lg n). \tag{5.17}$$

Chapter 6

Concluding Remarks

In the first part of this dissertation, we used Polya's Counting Theorem to settle an open question that was posed by Harrison in [22] in 1973. We proved an asymptotic formula for the number of distinct unlabeled bipartite graphs with n left and r right vertices, where $n \neq r$. We also obtained exact formulas for the same when one of n or r is fixed to 2 or 3. These results have also been extended to bipartite graphs in which left and/or right vertices are partially labeled.

In the second part of the dissertation, we dealt with wiring and routing problems that arise in the design of one-sided switching networks. We reduced the time complexity of the routing algorithm for a one-sided switching network with a particular clustering scheme. We also introduced a new wiring method and described a routing algorithm for it as well.

6.1 Future Work

The results presented in this dissertation suggest several problems of interest. In particular, the following counting problems, regarding unlabeled bipartite graphs remain open:

1. give a closed-form formula for $|B_u(n, r)|$ for $n \geq 4$,
2. give a closed-form formula for $|B_u(n, r)|$ for $n = r$, and
3. provide tighter lower and upper bounds for $|B_u(n, r)|$.

As for wiring and routing in one-sided switching networks, one possible future direction of research is to extend the cluster model so as to allow multiple connections between the replicates. This will likely lead to smaller number of columns of wiring. At the same time, it will also likely increase the time complexity of the routing algorithm. It will be worthwhile to investigate this trade-off between the two key parameters to optimize the design of one-sided, binary tree switching networks.

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Appendix A

Implementation of Proposed Wiring Method

The following C++ code implements the wiring method proposed in Section 5.3.

```
// Wiring stage: fillMatrix function recursively fills up the matrix
   which will be used for routing stage.
void fillMatrix(int start, int end,int rowIndex, vector<vector<Pair> >
   &matrix) {

  int size = end - start + 1;
  int half = size / 2;
  if (size % 2 == 1)
    half++;
  // Initialize subgroup of numbers
  int startIndexOfSecondGroup = start + half;
  int sizeOfSecondGroup = size - half;
  int columnIndexPadding = start / 2;

  int shift = 0;
  for (int row = 0; row < half; row++) {
    for (int i = 0; i < half; i++) {
```

```
            if (i == sizeOfSecondGroup) {
               break;
            }
            int pairIndex1 = start + ((i + shift) % half);
            int pairIndex2 = startIndexOfSecondGroup + i;
            matrix[rowIndex + row][columnIndexPadding + i] =
                Pair(pairIndex1, pairIndex2);
         }
         shift++;
      }

      if (size <= 2)
         return;

      // Recursion for first half.
      fillMatrix(start, startIndexOfSecondGroup-1, rowIndex+half, matrix);

      // Recursion for second half.
      if (size - half > 1)
         fillMatrix(startIndexOfSecondGroup, end, rowIndex + half, matrix);
}
```

Appendix B

Implementation of Proposed Routing Method

The following C++ code implements the routing method proposed in Section 5.4.

```
// Routing stage: getClusterId function queries the row id of the
    given pair of terminals recursively.
int getClusterId(Pair pair, int start, int end, int rowIndex) {
   int size = end - start + 1;
   int half = size / 2;
   if (size % 2 == 1)
      half++;
   int startIndexOfSecondGroup = start + half;
   int firstElementOfSecondGroup = startIndexOfSecondGroup;
   if(pair.x<firstElementOfSecondGroup&&pair.y>=firstElementOfSecondGroup){
      return modFunction(pair.x - pair.y, half) + rowIndex;
   } else if (pair.y < firstElementOfSecondGroup) {
      return getClusterId(pair, start, startIndexOfSecondGroup - 1,
   rowIndex + half);
   } else {
      return getClusterId(pair, startIndexOfSecondGroup, end, rowIndex
          + half);
   }
}
```

Appendix C

Complete Implementation of Proposed Wiring and Routing Method

---

```
// Sample test for wiring and routing stages.
#include <iostream>
#include <string>
#include <sstream>
#include <vector>
using namespace std;

// Pair class holds the pair of terminals.
class Pair {
   public:
      int x, y;
   public:
      Pair(int x0 = -1, int y0 = -1) {
         if (x0 < y0) {
            x = x0;
            y = y0;
```

```
        } else {
            x = y0;
            y = x0;
        }
    }
    string toString() {
        return "(" + std::to_string(x) + "," + std::to_string(y) + ")";
    }
    bool isNull() {
        if (x == -1 || y == -1) {
            return true;
        } else {
            return false;
        }
    }
};

// Function declarations
void fillMatrix(int start, int end, int rowIndex, vector<vector<Pair>
    > &matrix);
int getClusterId(Pair pair, int start, int end, int rowIndex);
int modFunction(int val, int mod);
void printMatrix(vector<vector<Pair> > &matrix);
int smallestPowerOfTwoAtLeastGivenNumber(int n);

int main() {
    cout << "Please enter the number of terminals:" << endl;
    int n;
    try {
        cin >> n;
    } catch (std::exception& e) {
        cout << "Invalid input." << endl;
        return -1;
    }
```

```
if (n < 2) {
   cout << "Invalid input." << endl;
   return -1;
}

int rows = smallestPowerOfTwoAtLeastGivenNumber(n) - 1;
vector<vector<Pair> > matrix(rows, vector<Pair>(n / 2));
int start = 0;
int end = n - 1;
int rowIndex = 0;
fillMatrix(start, end, rowIndex, matrix);
printMatrix(matrix);
cout << "Matrix has been filled up. Enter pair of terminals as x,y
    to retrieve cluster id. Enter -1 to exit." << endl;

while (true) {
   std::string pairStr;
   cin >> pairStr;
   if (pairStr.compare("-1") == 0)
      break;

   // Read inputs from console.
   std::replace(pairStr.begin(), pairStr.end(), ',', ' ');
   std::vector<int> pairArray;
   std::stringstream ss(pairStr);
   int temp;
   while (ss >> temp)
      pairArray.push_back(temp);

   int e1 = pairArray[0];
   int e2 = pairArray[1];
   Pair pair(e1, e2);
   if (pair.y >= n || pair.x < 0 || e1 == e2) {
      cout << "Invalid input." << endl;
      continue;
```

```
    }
    cout << "Cluster Id: " << getClusterId(pair, start, end,
        rowIndex) << endl;
  }
  return 0;
}

// modFunction supports negative mod calculations.
int modFunction(int val, int mod) {
  while (val < 0) {
    val += mod;
  }
  return val % mod;
}

void printMatrix(vector<vector<Pair> > &matrix) {
  for (int i = 0; i < matrix.size(); ++i) {
    bool textWritten = false;
    for (int j = 0; j < matrix[i].size(); ++j) {
      if (matrix[i][j].isNull() == false) {
        if (textWritten == false) {
          cout << "Cluster " << i << ": ";
          textWritten = true;
        }
        cout << matrix[i][j].toString();
      }
    }
    if (textWritten)
      cout << endl;
  }
  cout << endl;
}

// smallestPowerOfTwoAtLeastGivenNumber returns the smallest number
   which is a power of 2 and greater than or equal to the given number.
```

```
int smallestPowerOfTwoAtLeastGivenNumber(int n) {
   int x = 2;
   while (x < n) {
      x = x * 2;
   }
   return x;
}
```

Appendix D

Sample Runs and Examples

```
Please enter the number of terminals: 6

Wiring Output:
Cluster 0: (0,3)(1,4)(2,5)
Cluster 1: (1,3)(2,4)(0,5)
Cluster 2: (2,3)(0,4)(1,5)
Cluster 3: (0,2)(3,5)
Cluster 4: (1,2)(4,5)
Cluster 5: (0,1)(3,4)

Routing Queries:
5,3
Cluster Id: 3
1,2
Cluster Id: 4
2,4
Cluster Id: 1
0,3
Cluster Id: 0
```

```
Please enter the number of terminals: 8

Wiring Output:
Cluster 0: (0,4)(1,5)(2,6)(3,7)
Cluster 1: (1,4)(2,5)(3,6)(0,7)
Cluster 2: (2,4)(3,5)(0,6)(1,7)
Cluster 3: (3,4)(0,5)(1,6)(2,7)
Cluster 4: (0,2)(1,3)(4,6)(5,7)
Cluster 5: (1,2)(0,3)(5,6)(4,7)
Cluster 6: (0,1)(2,3)(4,5)(6,7)

Routing Queries:
2,5
Cluster Id: 1
4,6
Cluster Id: 4
4,7
Cluster Id: 5

Please enter the number of terminals: 13
Wiring Output:
Cluster 0: (0,7)(1,8)(2,9)(3,10)(4,11)(5,12)
Cluster 1: (1,7)(2,8)(3,9)(4,10)(5,11)(6,12)
Cluster 2: (2,7)(3,8)(4,9)(5,10)(6,11)(0,12)
Cluster 3: (3,7)(4,8)(5,9)(6,10)(0,11)(1,12)
Cluster 4: (4,7)(5,8)(6,9)(0,10)(1,11)(2,12)
Cluster 5: (5,7)(6,8)(0,9)(1,10)(2,11)(3,12)
Cluster 6: (6,7)(0,8)(1,9)(2,10)(3,11)(4,12)
Cluster 7: (0,4)(1,5)(2,6)(7,10)(8,11)(9,12)
Cluster 8: (1,4)(2,5)(3,6)(8,10)(9,11)(7,12)
Cluster 9: (2,4)(3,5)(0,6)(9,10)(7,11)(8,12)
Cluster 10: (3,4)(0,5)(1,6)(7,9)(10,12)
Cluster 11: (0,2)(1,3)(4,6)(8,9)(11,12)
Cluster 12: (1,2)(0,3)(5,6)(7,8)(10,11)
Cluster 13: (0,1)(2,3)(4,5)
```

```
Routing Queries:
7,12
Cluster Id: 8
1,9
Cluster Id: 6
10,11
Cluster Id: 12
```