CHAIN MAPS BETWEEN GRUENBERG RESOLUTIONS

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By
Müge Fidan
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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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Let G be a finite group. For a given presentation of $G = \langle F|R \rangle$, Gruenberg gives a construction of a projective resolution for $\mathbb{Z}$ as a $\mathbb{Z}G$-module. This resolution, which is called Gruenberg resolution, only depends on the ideals $I_F := \ker\{\mathbb{Z}F \to \mathbb{Z}\}$ and $J := \ker\{\mathbb{Z}F \to \mathbb{Z}G\}$ (see [1]). We write standard resolution as a Gruenberg resolution by following the construction of Gruenberg [2]. We get an explicit chain map formula between Gruenberg resolution for standard presentation and the Gruenberg resolution for the usual presentation of a cyclic group. Then we write an explicit chain map formula between any two Gruenberg resolutions. We also give some calculations with Gruenberg resolution.

Keywords: projective resolution, cohomology, Gruenberg resolution, chain map.

ÖZET

GRUENBERG ÇÖZÜCÜLERİ ARASINDAKİ ZİNCİR DÖNÜŞÜMLERİ

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G sonlu bir grup olsun. Verilen $G = \langle F|R \rangle$'nin takdimi için, Gruenberg bir $\mathbb{Z}G$-modül olarak $\mathbb{Z}$'nin bir projektif çözücüsünün nasıl inşa edilebileceğini gösteriyor. Gruenberg çözücü olarak adlandırılan bu çözücü sadece $I_F := \ker\{\mathbb{Z}F \to \mathbb{Z}\}$ ve $J := \ker\{\mathbb{Z}F \to \mathbb{Z}G\}$ ideallarına bağlıdır [1]. Gruenberg'in metodunu kullanarak Gruenberg çözücü olarak standart takdimi kullanarak çözücüyü elde ediyoruz [2]. Standart takdim için Gruenberg çözücü ile devirli grubun takdimi için Gruenberg çözücü arasındaki zincir dönüşümünün açık formülünü elde ediyoruz. Daha sonra herhangi iki Gruenberg çözücü arasındaki zincir dönüşümünün nasıl bulunacağını gösteriyoruz. Ayrıca Gruenberg çözücü ile bazı hesaplamalar yapıyoruz.

Anahtar sözcükler: projektif çözücü, kohomoloji, Gruenberg çözücü, zincir dönüşümü .

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I would like to dedicate this thesis to my father, Seyit İbrahim Fidan, who I lost in the last year, and my mother, Sevim Fidan.

Contents

Chapter 1

Introduction

Let G be a group. A presentation of G is a short exact sequences of groups

$$1 \to R \to F \xrightarrow{\pi} G \to 1$$

where F is a free group. By using a given presentation, Gruenberg gave a construction of a projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module in [1] and [2]. The Gruenberg resolution depends only on augmentation ideal $I_F := \ker\{\mathbb{Z}F \xrightarrow{\epsilon} \mathbb{Z}\}$ and the two-sided ideal $J := \ker\{\mathbb{Z}F \xrightarrow{\pi^*} \mathbb{Z}G\}$. The Gruenberg resolution is a $\mathbb{Z}G$-free resolution of $\mathbb{Z}$ of the form

$$\cdots \longrightarrow J^2/J^3 \longrightarrow JI_F/J^2I_F \longrightarrow J/J^2 \longrightarrow I_F/JI_F \xrightarrow{\pi^*} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0.$$

Let us denote $F_{2k} := J^k/J^{k+1}$ and $F_{2k-1} := J^{k-1}I_F/J^kI_F$ for $k \geq 1$. Using standard presentation, Gruenberg presented a $\mathbb{Z}G$-free resolution of $\mathbb{Z}$ expressing with bar notation in [2]. Then he showed the differentials induced by inclusions $d_k : F_{2k} \to F_{2k-1}$ is, in fact, the same as the differentials in the standard resolution.

Since any two projective resolutions of $\mathbb{Z}$ are chain homotopy equivalent, Gruenberg resolutions are unique up to chain homotopy equivalence. We find an explicit formula to write chain map between Gruenberg resolution for standard presentation and the Gruenberg resolution for the usual presentation of a cyclic group. After this result, we realize Hanke and et al. [3] described a chain map from the standard resolution to the resolution for the cyclic group. Our chain map formula coincides with their formula up to choice of generators. However, we replaced this with a more natural formula. To write this formula, we used the Fox derivative. It is defined in the Chapter 3.3. Consider $1 \to R \to F \to G \to 1$ be a standard presentation of cyclic group with order n where F is free on $X = \{x_1, x_g, x_{g^2}, \dots, x_{g^{n-1}}\}$ and R is free on $Y = \{y_{(a,b)} = x_{ab}^{-1} x_a x_b\}$ and $\pi(x_g) = g$. The first theorem of this thesis is the following:

Theorem 1.0.1. *There is a chain map between the Gruenberg resolution for standard presentation and the Gruenberg resolution for the presentation of cyclic group, that is, for all $n \geq 0$ the diagrams*

$$\begin{tikzcd}
\cdots \arrow[r] & J^k I_F/J^{k+1} I_F \arrow[r, "d_{2k+1}"] \arrow[d, "f_{2k+1}"] & J^k/J^{k+1} \arrow[r] \arrow[d, "f_{2k}"] & \cdots \arrow[r] & J/J^2 \arrow[r, "d_2"] \arrow[d, "f_2"] & I_F/JI_F \arrow[r, "d_1"] \arrow[d, "f_1"] & \mathbb{Z}G \arrow[r, "\epsilon"] \arrow[d, "id"] & \mathbb{Z} \arrow[r] \arrow[d, "id"] & 0 \\
\cdots \arrow[r] & \mathbb{Z}G \arrow[r, "g-1"] & \mathbb{Z}G \arrow[r, "N"] & \cdots \arrow[r] & \mathbb{Z}G \arrow[r, "N"] & \mathbb{Z}G \arrow[r, "g-1"] & \mathbb{Z}G \arrow[r, "\epsilon"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

commute with

$$f_1([g^i]) = \frac{\partial g^i}{\partial g} \text{ and } f_2([g^{i_1}|g^{i_2}]) = \begin{cases} g^{i_1+i_2-n} & , \text{ if } i_1 + i_2 \geq n \\ 0 & , \text{ otherwise} \end{cases}$$

and there is a formula for chain maps such that

$$\begin{aligned} f_{2k}(\alpha) &= f_2([g^{i_1}|g^{i_2}]) \dots f_2([g^{i_{2k-1}}|g^{i_{2k}}]) && \text{for } \alpha \in J^k/J^{k+1} \\ f_{2k+1}(\alpha) &= f_{2k}([g^{i_1}| \dots |g^{i_{2k}}]) f_1([g^{i_{2k+1}}]) && \text{for } \alpha \in J^k I_F/J^{k+1} I_F. \end{aligned}$$

More generally, we explain how to write chain maps for any two Gruenberg resolutions. There is a commutative diagram between any two presentations of group G. The mapping between generators is denoted by f_F and the mapping between relations is denoted by f_R in the following diagram

$$\begin{tikzcd}
1 \arrow[r] & R_1 \arrow[r, "q_1"] \arrow[d, "f_R"] & F_1 \arrow[r, "\pi_1"] \arrow[d, "f_F"] & G \arrow[r] \arrow[d, "id"] & 1 \\
1 \arrow[r] & R_2 \arrow[r, "q_2"] & F_2 \arrow[r, "\pi_2"] & G \arrow[r] & 1.
\end{tikzcd}$$

Let F_1,F_2 be free on $X_1 = \langle x_i \rangle$ and $X_2 = \langle a_i \rangle$, respectively. Also R_1, R_2 be free on $Y_1 = \langle y_i \rangle$ and $Y_2 = \langle b_i \rangle$, respectively. To write chain map between their corresponding Gruenberg resolutions, we define a convolution $*$ in the Equation 4.1. The second theorem of this thesis is the following:

Theorem 1.0.2. *Given a map of presentations (f_F, f_R). There is a chain map f_*, associated to a map of presentations, between their corresponding resolutions, that is, for all $n \geq 0$ the diagrams*

$$\begin{tikzcd}
\cdots \arrow[r] & J_1^k/J_1^{k+1} \arrow[r, "d_{2k}"] \arrow[d, "f_{2k}"] & J_1^{k-1}I_{F_1}/J_1^kI_{F_1} \arrow[r] \arrow[d, "f_{2k-1}"] & \cdots \arrow[r] & J_1/J_1^2 \arrow[r, "d_2"] \arrow[d, "f_2"] & I_{F_1}/J_1I_{F_1} \arrow[r, "d_1"] \arrow[d, "f_1"] & \mathbb{Z}G \arrow[r, "\epsilon"] \arrow[d, "id"] & \mathbb{Z} \arrow[r] \arrow[d, "id"] & 0 \\
\cdots \arrow[r] & J_2^k/J_2^{k+1} \arrow[r, "\delta_{2k}"] & J_2^{k-1}I_{F_2}/J_2^kI_{F_2} \arrow[r] & \cdots \arrow[r] & J_2/J_2^2 \arrow[r, "\delta_2"] & I_{F_2}/J_2I_{F_2} \arrow[r, "\delta_1"] & \mathbb{Z}G \arrow[r, "\epsilon"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

commute where

$$f_1(x_i - 1) = \sum_j \frac{\partial(f_F(x_i))}{\partial a_j}(a_j - 1)$$
$$f_2(y_i - 1) = \sum_j \frac{\partial f_R(y_i)}{\partial b_j}(b_j - 1)$$

and there is a formula for chain map such that

$$f_{2k}(\alpha) = f_2(y_{i_1} - 1) * \cdots * f_2(y_{i_k} - 1) \qquad \text{for } \alpha \in J^k/J^{k+1}$$
$$f_{2k-1}(\alpha) = f_2(y_{i_1} - 1) * \cdots * f_2(y_{i_{k-1}} - 1) * f_1(x_{i_k} - 1) \quad \text{for } \alpha \in J^{k-1}I_F/J^kI_F.$$

The thesis is organized as follows:

In Chapter 2, we give the necessary definitions and propositions from homological algebra including the definitions of the projective resolution and group cohomology.

Chapter 3 is divided into three sections: Gruenberg resolution, Standard resolution as a Gruenberg resolution and Fox calculus. In the first section, we begin with several lemmas which are necessary to show that Gruenberg resolution is a $\mathbb{Z}G$-free resolution of $\mathbb{Z}$. In the second section, we introduce Gruenberg resolution for standard presentation. In the last section, we introduce the Fox derivative. We write our chain map formula with using the Fox derivative, hence it is crucial for this thesis.

Chapter 4 is the main chapter of this thesis. This chapter includes two sections. The proofs of the Theorem 1.0.1 and 1.0.2, which are mentioned above, are in the first and second section, respectively.

The last chapter includes our calculations with Gruenberg resolution. The first group we considered is $C_2 \times C_2$. We show that our chain map formula gives a chain map between Gruenberg resolutions for $C_2 \times C_2$. In the following sections, we calculate the cohomology group of S_3 using Gruenberg resolution.

Chapter 2

Projective Resolutions and Group Cohomology

This chapter contains background information on group cohomology. In the first section, we give some definitions and propositions which are used in the following chapters. In the second section, we give the algebraic definition for group cohomology. The main references for this chapter are [4], [5], [6], and [7].

2.1 Projective resolutions

Definition 2.1.1. *Let R be an arbitrary ring. A sequence*

$$\cdots \to F_{i+1} \xrightarrow{f_{i+1}} F_i \xrightarrow{f_i} F_{i-1} \xrightarrow{f_{i-1}} \cdots$$

of R-module homomorphisms is called exact at F_i if $\mathrm{Im}(f_{i+1}) = \ker(f_i)$ where $\mathrm{Im}(f_{i+1})$ denotes the image of f_{i+1} and $\ker(f_i)$ is the kernel of f_i. A sequence of homomorphisms is called an ***exact sequence*** *if it is an exact at F_i for all i. Let A, B, and C be R-modules. An exact sequence of the form*

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

is called a ***short exact sequence****. For a short exact sequence, we have*

1. α *is injective because of exactness at* A,

2. β *is surjective because of exactness at* C, *and*

3. $\mathrm{Im}(\alpha) = \ker(\beta)$ *because of exactness at* B.

Example 2.1.2. *Consider* $A = \mathbb{Z}$ *and* $C = \mathbb{Z}/n\mathbb{Z}$. *An extension of* $\mathbb{Z}/n\mathbb{Z}$ *by* $\mathbb{Z}$ *is given by the short exact sequence*

$$0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \to 0,$$

where $\times n$ *denotes the multiplication by* n *and* π *denotes natural projection.*

Definition 2.1.3. *A short exact sequence*

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

is called split if there is a map $\gamma : C \to B$ *with* $\beta \circ \gamma = id_C$.

Proposition 2.1.4. *A short exact sequence* $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *is split, then there is an* R*-module complement to* $\alpha(A) \in B$, *that is,* $B = A \oplus C$.

Definition 2.1.5. *An* R*-module* P *is projective if given any homomorphism* $f : P \to N$ *and any epimorphism* $\varphi : M \to N$, *there is a homomorphism* $\hat{f} : P \to M$ *such that* $\varphi \circ \hat{f} = f$, *i.e. the following diagram commutes*

$$\begin{tikzcd}
& P \arrow[dl, dashed, "\hat{f}"'] \arrow[d, "f"] & \\
M \arrow[r, two heads, "\varphi"'] & N \arrow[r] & 0.
\end{tikzcd}$$

Definition 2.1.6. *Let* C_* *be a sequence of* R*-modules*

$$\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \cdots \to C_1 \xrightarrow{d_1} C_0 \to 0.$$

If the composition of any two consecutive maps is zero, that is, $d_{n-1} \circ d_n = 0$ *for all* $n \geq 0$, *then the complex is called a* ***chain complex***. *The* n*-th homology group of* C_* *are defined as the* R*-module*

$$H_n(C_*) = \frac{\ker(d_n)}{\mathrm{Im}(d_{n+1})}.$$

We define a cochain complex as follows: Let C^* be a sequence of R-module:

$$0 \to C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-2}} C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \xrightarrow{d^{n+1}} \cdots .$$

If the composition of any two consecutive maps is zero, that is, $d^{n+1} \circ d^n = 0$ for all $n \geq 0$, then the complex is called a ***cochain complex****. The n-th cohomology group of C^* are defined as the R-module*

$$H^n(C^*) = \frac{\ker(d^n)}{\operatorname{Im}(d^{n-1})}.$$

Definition 2.1.7. *Let $A_* = \{A_n\}$ and $B_* = \{B_n\}$ be chain complexes. A chain map of complexes $f : A_* \to B_*$ is a sequence of R-module homomorphism $f_n : A_n \to B_n$ such that for every n the following diagram commutes:*

$$\begin{tikzcd}
\cdots \arrow[r, "d_{n+2}"] & A_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] & A_n \arrow[r, "d_n"] \arrow[d, "f_n"] & A_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r, "\delta_{n+2}"] & B_{n+1} \arrow[r, "\delta_{n+1}"] & B_n \arrow[r, "\delta_n"] & B_{n-1} \arrow[r, "\delta_{n-1}"] & \cdots
\end{tikzcd}$$

i.e. for all n, $f_{n-1} \circ d_n = \delta_n \circ f_n$.

Let $g : A_* \to B_*$ be a chain map. If there is a map $s : A_* \to B_*$ satisfying $f_n - g_n = \delta_{n+1} \circ s_n + s_{n-1} \circ d_n$ for all n, then f and g are called homotopic and denoted by $f \simeq g$. Hence the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & A_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & A_n \arrow[r, "d_n"] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] \arrow[dl, dashed, "s_n"'] & A_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] \arrow[dl, dashed, "s_{n-1}"'] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\delta_{n+1}"] & B_n \arrow[r, "\delta_n"] & B_{n-1} \arrow[r, "\delta_{n-1}"] & \cdots
\end{tikzcd}$$

Remark 2.1.8.

1. *A chain map $f : A_* \to B_*$ is said to be* ***null-homotopic*** *if f is homotopic to the zero map.*

2. *A chain map f is called* ***homotopy equivalence*** *if there is another chain map $g : B_* \to A_*$ such that $f \circ g \simeq id_{A_*}$ and $g \circ f \simeq id_{B_*}$. Then A_* and B_* are called* ***homotopy equivalent****.*

3. *If* $f_* : A_* \to B_*$ *and* $g_* : B_* \to C_*$ *are two chain maps, then* $(g_* \circ f_*) : A_* \to C_*$ *is a chain map.*

Definition 2.1.9. *Let N be an R-module. A* ***projective resolution*** *of N is an exact sequence*

$$\cdots \to P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} N \to 0$$

such that each P_i is a projective R-module.

Every R-module has a projective resolution. To see this, take P_0 as the free (hence projective) R-module on a set of generators of N. By the universal property of free modules, we define an R-module homomorphism $\epsilon : P_0 \to N$. Since ϵ is surjective, the resolution begins with $P_0 \xrightarrow{\epsilon} N$ which is exact. Let $K_0 = \ker \epsilon$ and P_1 be any free module mapping onto the submodule K_0, that is, $P_1 \twoheadrightarrow K_0 \to 0$ is exact. Define $d_1 : P_1 \to P_0$ to be the composition $P_1 \twoheadrightarrow K_0 \to P_0$. Then

$$\begin{tikzcd} P_1 \arrow[r, "d_1"] \arrow[dr] & P_0 \arrow[r, two heads, "\epsilon"] & N \arrow[r] & 0 \\ & K_0 \arrow[u] & & \end{tikzcd}$$

is exact since $\mathrm{Im}(d_1) = K_0 = \ker(\epsilon)$. Inductively, we can take n-th stage a free R-module P_{n+1} which is mapping onto the submodule K_n. Then we obtain a projective resolution of N.

Consider the projective resolution $P_* \to N$ of N where

$$P_* : \cdots \to P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \to 0.$$

Note that $H_n(P_*) = 0$ if $n > 0$ and $H_0(P_*) \cong N$. Given an R-module M, we have cochain complex

$$0 \longrightarrow \mathrm{Hom}_R(P_0, M) \xrightarrow{d^1} \mathrm{Hom}_R(P_1, M) \xrightarrow{d^2} \cdots$$

$$\cdots \longrightarrow \mathrm{Hom}_R(P_{n-1}, M) \xrightarrow{d^n} \mathrm{Hom}_R(P_n, M) \xrightarrow{d^{n+1}} \cdots$$

obtained by applying $\mathrm{Hom}_R(-, M)$. Now we define

$$\mathrm{Ext}^n_R(N, M) = \frac{\ker(d^{n+1})}{\mathrm{Im}(d^n)}$$

for $n \geq 1$ and $\mathrm{Ext}^0_R(A,M) = \ker(d^1)$. This group is called the n-th **Ext group**.

Again consider P_* of N and take tensor product with M to obtain

$$\cdots \to M \otimes P_{n+1} \xrightarrow{1\otimes d_{n+1}} M \otimes P_n \xrightarrow{1\otimes d_n} M \otimes P_{n-1} \xrightarrow{1\otimes d_{n-1}} \cdots \xrightarrow{1\otimes d_1} M \otimes P_0 \to 0.$$

Define

$$\mathrm{Tor}^R_n(M,A) = \frac{\ker(1\otimes d_n)}{\mathrm{Im}(1\otimes d_{n+1})}$$

for $n \geq 1$ and $\mathrm{Tor}^R_0(M,A) = \dfrac{M \otimes P_0}{\mathrm{Im}(1 \otimes d_1)}$. This group is called the n-th **Tor group**.

Proposition 2.1.10. *[5, Proposition 4, page 781] Let $f : A \to B$ be a map of R-modules and $P_* \to A$ be a projective resolution of A and $Q_* \to B$ be a resolution of B. Then there is a chain map $\hat{f} : P_* \to Q_*$ such that the following diagram commutes:*

$$\begin{tikzcd}
\cdots \arrow[r, "d_{n+1}"] & P_n \arrow[r, "d_n"] \arrow[d, "\hat{f}_n"] & P_{n-1} \arrow[r] \arrow[d, "\hat{f}_{n-1}"] & \cdots \arrow[r] & P_1 \arrow[r, "d_1"] \arrow[d, "\hat{f}_1"] & P_0 \arrow[r, "\varphi"] \arrow[d, "\hat{f}_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r, "\delta_{n+1}"] & Q_n \arrow[r, "\delta_n"] & Q_{n-1} \arrow[r] & \cdots \arrow[r] & Q_1 \arrow[r, "\delta_1"] & Q_0 \arrow[r, "\psi"] & B \arrow[r] & 0.
\end{tikzcd}$$

Moreover, $\hat{f}$ is unique up to chain homotopy.

Proof. Let P_n be projective for all n and let φ be surjective. The bottom sequence is exact. First we need to show existence of $\hat{f}$. Since P_0 is projective and ψ is surjective, there exist a map $\hat{f}_0$ such that the first square commutes, that is, $\psi \circ \hat{f}_0 = f \circ \varphi$. By induction, suppose we have constructed $\hat{f}_{n-1}$ such that $\delta_{n-1} \circ \hat{f}_{n-1} = \hat{f}_{n-2} \circ d_{n-1}$. Then

$$\delta_{n-1} \circ \hat{f}_{n-1} \circ d_n = \hat{f}_{n-2} \circ d_{n-1} \circ d_n = 0$$

so that $\hat{f}_{n-1} \circ d_n$ maps P_n to $\ker \delta_{n-1} = \mathrm{Im}\, \delta_n$. Then we obtain $\hat{f}_n$ such that $\delta_n \circ \hat{f}_n = \hat{f}_{n-1} \circ d_n$ since P_n is projective. Now we need to show uniqueness up to chain homotopy. Suppose there is another map $\hat{g} : P_* \to Q_*$. Then $\psi(\hat{f}_0 - \hat{g}_0) = 0$. Define maps $s_n : P_n \to Q_{n+1}$. We need to show that there exist a map $s_n : P_n \to Q_{n+1}$ such that $d_{n+1} \circ s_n = \hat{f}_n - \hat{g}_n - s_{n-1} \circ d_n$.

$$\begin{tikzcd}
\cdots \arrow[r, "d_{n+2}"] & P_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "\hat{f}_{n+1}"'] & P_n \arrow[r, "d_n"] \arrow[d, "\hat{f}_n"'] \arrow[dl, dashed, "s_n"'] & P_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "\hat{f}_{n-1}"'] \arrow[dl, dashed, "s_{n-1}"'] & \cdots \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] \arrow[d, "\hat{f}_1"'] & P_0 \arrow[r, "\varphi"] \arrow[d, "\hat{f}_0"'] \arrow[dl, dashed, "s_0"'] & A \arrow[r] \arrow[d, "f"'] \arrow[dl, dashed, "s_{-1}"'] & 0 \\
\cdots \arrow[r, "\delta_{n+2}"] & Q_{n+1} \arrow[r, "d_{n+1}"] & Q_n \arrow[r, "\delta_n"] & Q_{n-1} \arrow[r, "\delta_{n-1}"] & \cdots \arrow[r, "\delta_2"] & Q_1 \arrow[r, "\delta_1"] & Q_0 \arrow[r, "\psi"] & B \arrow[r] & 0
\end{tikzcd}$$

Since P_0 is projective and ψ is surjective, we obtain $s_0 : P_0 \to Q_1$ such that $\delta_1 \circ s_0 = \hat{f}_0 - \hat{g}_0$. Inductively, suppose s_{n-1} such that $\delta_n \circ s_{n-1} + s_{n-2} \circ d_{n-1} = \hat{f}_{n-1} - \hat{g}_{n-1}$. Then

$$\begin{aligned}
\delta_n \circ (\hat{f}_n - \hat{g}_n - s_{n-1} \circ d_n) &= (\hat{f}_{n-1} - \hat{g}_{n-1}) \circ d_n - \delta_n \circ s_{n-1} \circ d_n \\
&= (\delta_n \circ s_{n-1} + s_{n-2} \circ d_{n-1}) \circ d_n - \delta_n \circ s_{n-1} \circ d_n \\
&= s_{n-2} \circ d_{n-1} \circ d_n = 0
\end{aligned}$$

Since $\operatorname{Im} \delta_{n+1} = \ker \delta_n$ and P_n is projective, there exist a map $s_n : P_n \to Q_{n+1}$ such that $d_{n+1} \circ s_n = \hat{f}_n - \hat{g}_n - s_{n-1} \circ d_n$. $\square$

Corollary 2.1.11. *Any two projective resolutions of A are chain homotopy equivalent.*

Proof. Suppose $P_* \to A$ and $Q_* \to A$ are projective resolutions of A.

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d, "f_2"] & P_1 \arrow[r, "d_1"] \arrow[d, "f_1"] & P_0 \arrow[r, "d_0"] \arrow[d, "f_0"] & A \arrow[r] \arrow[d, "id"] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r, "\delta_2"] \arrow[d, "g_2"] & Q_1 \arrow[r, "\delta_1"] \arrow[d, "g_1"] & Q_0 \arrow[r, "\delta_0"] \arrow[d, "g_0"] & A \arrow[r] \arrow[d, "id"] & 0 \\
\cdots \arrow[r] & P_2 \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "d_0"] & A \arrow[r] & 0.
\end{tikzcd}$$

By previous lemma, the identity map $id : A \to A$ can be lifted. Then there exist chain maps $f : P_* \to Q_*$ and $g : Q_* \to P_*$. The composition of these chain maps $g \circ f : P_* \to P_*$ induces the identity map on A. Also $id : P_* \to P_*$ induces identity map on A. Using previous lemma, $g \circ f \simeq id_{P_*}$. Similarly, we get $f \circ g \simeq id_{Q_*}$. Hence we conclude P_* and Q_* are chain homotopy equivalent. $\square$

2.2 Group cohomology

Let R be the commutative ring and G be a group. The group ring of G over R, RG, is the set of all formal sums, $\sum_{i=1}^{n} a_i g_i$ where $a_i \in R$ and where only finitely many of the a_i's are nonzero. Addition is defined by

$$\sum_{i=1}^{n} a_i g_i + \sum_{i=1}^{n} b_i g_i = \sum_{i=1}^{n} (a_i + b_i) g_i$$

and multiplication is defined by

$$(\sum_{g\in G} a_g g)(\sum_{h\in G} b_h h) = \sum_{g,h\in G} a_g b_h (gh) = \sum_{k\in G} (\sum_{g\in G} a_g b_{g^{-1}k}) k.$$

Let $R = \mathbb{Z}$. A $\mathbb{Z}$-module A is an abelian group A. G-action on an abelian group A is a map

$$G \times A \longrightarrow A$$
$$(g, a) \longrightarrow g \cdot a$$

such that, for all $a, b \in A$ and $g, h \in G$,

1. $1 \cdot a = a$
2. $g \cdot (a + b) = g \cdot a + g \cdot b$
3. $g \cdot (h \cdot a) = (gh) \cdot a$.

In other words, an abelian group A on which G acts as group automorphisms is the same as a module over $\mathbb{Z}G$. Denote the set of elements of A which are fixed by all the elements of G, that is,

$$A^G = \{a \in A : ga = a \text{ for all } g \in G\}.$$

In general a short exact sequence

$$0 \to A \to B \to C \to 0$$

of G-modules induces an exact sequence

$$0 \to A^G \to B^G \to C^G. \tag{2.1}$$

To observe that this sequence is exact, we have following lemma:

Lemma 2.2.1. *Let A be a $\mathbb{Z}G$-module. Suppose that $\operatorname{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A)$ is the group of all $\mathbb{Z}G$-module homomorphism from $\mathbb{Z}$ to A. Then $A^G \cong \operatorname{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A)$.*

Proof.

$$\begin{array}{rcl} \operatorname{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A) & \to & A^G \\ \\ f : \mathbb{Z} \to A & \to & f(1) \end{array}$$

Any $\mathbb{Z}G$-module homomorphism $f : \mathbb{Z} \to A$ is uniquely determined by its value on 1. Say $f_x(1) = x$. Then $x = f_x(1) = f_x(g \cdot 1) = g \cdot f_x(1) = g \cdot x$ for all $g \in G$. Hence $x \in A^G$ and $f_x \to x$ is an isomorphism. Then $A^G \cong \operatorname{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A)$. □

By this lemma, any projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module with trivial action gives a long exact sequence extending (2.1). One of this type resolution is the projective resolution of $\mathbb{Z}$

$$\cdots \to F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \to \cdots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

Denote $F_n = \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}G \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Z}G$ n-th fold tensor product for $n \geq 0$ which is a free $\mathbb{Z}$-module generated by the $(n+1)$-tuples $(g_0, g_1, \cdots, g_n)$ where $g_i \in G$ for all i. This F_n is equipped with $\mathbb{Z}G$-module structure via the diagonal action by $g \cdot (g_0, \cdots, g_n) = (gg_0, \cdots, gg_n)$. The map $\epsilon : F_0 \to \mathbb{Z}$ is the augmentation map $\epsilon(\sum_{g \in G} a_g g) = \sum_{g \in G} a_g$ and the maps d_n for $n \geq 1$ is defined by

$$d_n(g_0, g_1, \ldots, g_n) = \sum_{i=0}^{n} (-1)^i (g_0, \ldots, \hat{g}_i, \ldots, g_n). \tag{2.2}$$

This projective resolution with this basis and this boundary maps is called **the standard resolution of $\mathbb{Z}$ over $\mathbb{Z}G$**.

$$(d_n \circ d_{n+1})(g_0, \ldots, g_{n+1}) = \sum_{j=0}^{n+1} \sum_{k=0, k \neq j}^{n+1} (-1)^{j+k-\delta(j,k)} (g_0, \ldots, \hat{g}_j, \ldots, \hat{g}_k, \ldots, g_{n+1})$$

where

$$\delta(j,k) = \begin{cases} 0, & \text{if } k < j \\ 1, & \text{if } k > j. \end{cases}$$

Since each possible $(n-1)$-tuple appears twice in the summation with opposite sign, we have $d_n \circ d_{n+1} = 0$. Hence if we show $\ker d_{n-1} = \operatorname{Im} d_n$ for all n, we will show the standard resolution is exact. Define $s_n : F_{n-1} \to F_n$ by $s_n(g_1, \ldots, g_n) = (1, g_1, \ldots, g_n)$. Then

$$\begin{aligned}(d_n \circ s_n)(g_1, \ldots, g_n) &= d_n(1, g_1, \ldots, g_n) \\ &= (g_1, \ldots, g_n) - \sum_{j=0}^{n} (-1)^j (1, g_1, \ldots, \hat{g}_j, \ldots, g_n) \\ &= (g_1, \ldots, g_n) - (s_{n-1} \circ d_{n-1})(g_1, \ldots, g_n).\end{aligned}$$

Hence $d_n \circ s_n + s_{n-1} \circ d_{n-1} = id_{F_{n-1}}$. If $\alpha \in \ker d_{n-1}$, then $d_n(s_n(\alpha)) = \alpha$, which means $\alpha \in \operatorname{Im} d_n$. Since it is clear that $\operatorname{Im} \alpha \subset \ker d_{n-1}$, we have $\ker d_{n-1} = \operatorname{Im} d_n$ for all n.

Let F_n' be a free $\mathbb{Z}G$-module with G-action $g \cdot (g_0, g_1, \ldots, g_n) = (gg_0, g_1, \ldots, g_n)$. Its basis is in the form $(1, g_1, g_1 g_2, \ldots, g_1 g_2 \ldots g_n)$. Define the bar notation such that

$$\begin{aligned}(g_0, g_1, \ldots, g_n) &= g_0[g_0^{-1} g_1 | g_1^{-1} g_2 | \ldots | g_{n-1}^{-1} g_n] \\ [g_1 | g_2 | \ldots | g_n] &= (1, g_1, g_1 g_2, \ldots, g_1 g_2 \ldots g_n).\end{aligned}$$

Then there is an isomorphism of $\mathbb{Z}G$-module between F_n and F_n' such that

$$\begin{tikzcd} (g_0, g_1, \ldots, g_n) \arrow[r, leftrightarrow] \arrow[d, "g"] & (g_0, g_0^{-1} g_1, \ldots, g_{n-1}^{-1} g_n) \arrow[d, "g"] \\ (gg_0, gg_1, \ldots, gg_n) \arrow[r, leftrightarrow] & (gg_0, g_0^{-1} g_1, \ldots, g_{n-1}^{-1} g_n). \end{tikzcd}$$

Using the differentials formula in 2.2 gives the following formula for the maps $d_n : F_n' \to F_{n-1}'$

$$\begin{aligned}d_n[g_1 | \ldots | g_n] &= g_1[g_2 | \ldots | g_n] \\ &\quad + \sum_{i=1}^{n-1} (-1)^i [g_1 | \ldots | g_{i-1} | g_i g_{i+1} | g_{i+2} | \ldots | g_n] \\ &\quad + (-1)^n [g_1 | \ldots | g_{n-1}].\end{aligned}$$

Hence this projective sequence with this basis and this boundary maps is called **bar resolution**.

Consider the resolution with deleting first term

$$\cdots \to F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \to \cdots \to F_1 \xrightarrow{d_1} F_0 \to 0.$$

Applying $\mathrm{Hom}_{\mathbb{Z}G}(-, M)$, we get

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}G}(F_0, M) \xrightarrow{d^1} \mathrm{Hom}_{\mathbb{Z}G}(F_1, M) \xrightarrow{d^2} \mathrm{Hom}_{\mathbb{Z}G}(F_2, M) \xrightarrow{d^3} \cdots .$$

We can identify $\mathrm{Hom}_{\mathbb{Z}G}(F_n, M)$ with the set of functions from $G^n = G \times G \times \cdots \times G$ to M. Let $C^n(G, M)$ be the collection of all maps from G^n to M for $n \geq 1$ and $C^0(G, M) = M$. The elements of $C^n(G, M)$ is called n-cochains. We can define the coboundary map from $C^n(G, M)$ to $C^{n+1}(G, M)$ by

$$\begin{aligned} d^n(f)(g_1, \ldots, g_n) = {} & g_1 f(g_2, \ldots, g_n) \\ & + \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_n) \\ & + (-1)^{n+1} f(g_1, \ldots, g_{n-1}). \end{aligned}$$

The elements of $\ker d^n$ which is denoted by $Z^n(G, M)$ are called the n-cocyles and the elements of $\mathrm{Im}\, d^{n-1}$ which is denoted by $B^n(G, M)$ are called the n-coboundaries. Then the n-th cohomology group of G with coefficient M is equal to quotient of n-cocyles over n-coboundaries, that is,

$$H^n(G, M) = \frac{Z^n(G, M)}{B^n(G, M)}.$$

Example 2.2.2. *Let $G = \langle x \rangle$ be a cyclic group of order n and $N = 1 + x + \cdots + x^{n-1}$. We have*

$$(1 + x + \cdots + x^{n-1})(x - 1) = (x - 1)(1 + x + \cdots + x^{n-1}) = x^n - 1 = 0.$$

Then there is a free resolution with the maps $N(x - 1) = (x - 1)N = 0$ in the followings:

$$\cdots \to \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{x-1} \mathbb{Z}G \to \cdots \to \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{x-1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

After deleting $\mathbb{Z}$, apply $\mathrm{Hom}_{\mathbb{Z}G}(-, M)$. *Since* $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, M) = M$, *we get a chain complex*

$$0 \to M \xrightarrow{x-1} M \xrightarrow{N} M \xrightarrow{x-1} M \to \cdots .$$

Hence we can calculate the cohomology group of G with coefficient M. Since

$$\ker(x-1) = \{m \in M : (x-1)m = 0\} \text{ which is denoted by } M^G \text{ and}$$

$$\ker(N) = \{m \in M : Nm = 0\} \text{ which is denoted by } {}_N M,$$

we have $H^0(G, M) = M^G$ *for* $n = 0$ *and*

$$H^n(G, M) = \begin{cases} \dfrac{\ker(N)}{\mathrm{Im}(x-1)} = \dfrac{{}_N M}{(x-1)M}, & \text{if } n \text{ is odd} \\ \\ \dfrac{\ker(T)}{\mathrm{Im}(N)} = \dfrac{M^G}{NM}, & \text{if } n \text{ is even}, n > 0. \end{cases}$$

If we take $M = \mathbb{Z}$, *then* $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, \mathbb{Z}) = \mathbb{Z}$. *Then we get a chain complex*

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \ldots .$$

Then

$$H^n(G, \mathbb{Z}) = \begin{cases} \mathbb{Z}/n\mathbb{Z}, & \text{if } n \text{ is even}, n > 0 \\ 0, & \text{if } n \text{ is odd} \\ \mathbb{Z}, & \text{if } n=0 . \end{cases}$$

Chapter 3

Gruenberg Resolution

Gruenberg [2] gave an explicit method for obtaining a projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module by using a given presentation of a group G. In this chapter, we explain how to construct a Gruenberg resolution. We follow [1], [2], [7], [8]. Then we introduce Fox derivative [9] which we use for writing differentials of a Gruenberg resolution.

3.1 Gruenberg resolution

Let G be a group. Let X be any set and Y be a set of words on X. A group G has generators X and relations Y if $G \cong F/R$ where F is free group with basis X and R is the normal subgroup of F generated by Y. The ordered pair $\langle F|R\rangle$ is called a **presentation** of G [10]. Equivalently, a presentation of G is a short exact sequences of groups

$$1 \to R \to F \xrightarrow{\pi} G \to 1$$

where F is a free group. Recall that the augmentation map $\epsilon : \mathbb{Z}G \to \mathbb{Z}$ is defined as a map which takes every element of G to $1 \in \mathbb{Z}$. The kernel of ϵ is an ideal which is called augmentation ideal of $\mathbb{Z}G$ and is denoted by I_G. Similarly, the augmentation ideal of $\mathbb{Z}F$ will be denoted by I_F. The map $\pi : F \to G$ in the presentation induces the surjective ring map

$$\begin{array}{rcl} \pi^* : & \mathbb{Z}F \longrightarrow \mathbb{Z}G \\ & \sum_{f\in F} a_f f \mapsto \sum_{f\in F} a_f \pi(f). \end{array}$$

The kernel of this map is two-sided ideal in $\mathbb{Z}F$ and is denoted by J.

Lemma 3.1.1. *Let F be free on a set X. Consider the word $\alpha = x_1^{\epsilon_1} \cdots x_n^{\epsilon_n} \in F$ where $\epsilon_i = \pm 1$ for $1 \leq i \leq n$. Then $\alpha - 1$ is an F-linear combination of $X - 1 = \{x_i - 1 : x_i \in X \text{ for } 1 \leq i \leq n\}$.*

Proof. It is clear for $n = 1$. Suppose the statement is true for $n = k$. We need to show it is true for $n = k+1$. Using the following identities $x^{-1} - 1 = -x^{-1}(x-1)$ and $x_1x_2 - 1 = x_1(x_2 - 1) + (x_1 - 1)$, if $\alpha = x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_{k+1}^{\epsilon_{k+1}}$, then we have

$$\begin{aligned} \alpha - 1 &= x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_k^{\epsilon_k}(x_{k+1}^{\epsilon_{k+1}} - 1) + (x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_k^{\epsilon_k} - 1) \\ &= \begin{cases} x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_k^{\epsilon_k}(x_{k+1} - 1) + (x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_k^{\epsilon_k} - 1) & if\ \epsilon_{k+1} = 1 \\ -x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_k^{\epsilon_k} x_{k+1}^{-1}(x_{k+1} - 1) + (x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_k^{\epsilon_k} - 1) & if\ \epsilon_{k+1} = -1. \end{cases} \end{aligned}$$

Since $(x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_k^{\epsilon_k} - 1)$ can be written F-linear combination of $X - 1$ by the induction assumption and $(x_{k+1} - 1) \in X - 1$, $(x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_{k+1}^{\epsilon_{k+1}} - 1)$ can be written a linear combination of the set $X - 1$. Hence $\alpha - 1$ is an F-linear combination of $X - 1$. □

Definition 3.1.2. *Let F be a group and A be a $\mathbb{Z}F$-module. A mapping $d : F \to A$ is called a derivation if $d(xy) = d(x) + xd(y)$ for all $x, y \in F$.*

Lemma 3.1.3. *Let A be an $\mathbb{Z}F$-module, $d : F \to A$ be any map and $h : I_F \to A$ be a map with $h(x - 1) = d(x)$. Then d is a derivation if and only if h is a $\mathbb{Z}F$-module homomorphism. Hence* $\mathrm{Hom}_{\mathbb{Z}F}(I_F, A) \cong \mathrm{Der}(F, A)$*, where* $\mathrm{Der}(F, A)$ *is a set of all derivations.*

Proof. Suppose d is derivation. Then $d(xy) = d(x) + xd(y)$ for all $x, y \in F$. Since $d(x) = h(x - 1)$, $h(xy - 1) = d(xy) = h(x - 1) + xh(y - 1)$ for all $x, y \in F$. Then

$h(xy-1)-h(x-1)=xh(y-1)$ for all $x,y\in F$. Since $h(xy-1)-h(x-1)=h((xy-1)-(x-1))=h(xy-x)=h(x(y-1))$, $h(x(y-1))=xh(y-1)$ for all $x,y\in F$, which means h is a module homomorphism.

Suppose h is a module homomorphism. Then $h(x(y-1))=xh(y-1)$ for all $x,y\in F$. This implies $h(xy-1)-h(x-1)=xh(y-1)$ for all $x,y\in F$. By using the definition of the map h, we get $d(xy)-d(x)=xd(y)$ for all $x,y\in F$. Then we obtain a function satisfying the derivation formula $d(xy)=xd(y)+d(x)$ for all $x,y\in F$. Hence d is a derivation. □

Proposition 3.1.4. *If F is free on a set X, then I_F is free as a left $\mathbb{Z}F$-module on $X-1$.*

Proof. Take an element $\alpha=\sum_{f\in F}a_f f\in\mathbb{Z}F$. If $\alpha\in I_F$, then $\sum_{f\in F}a_f=0$.

$$\alpha=\alpha-0=\sum_{f\in F}a_f f-\sum_{f\in F}a_f=\sum_{f\in F}a_f(f-1).$$

Then I_F is generated by all $f-1$ as an abelian group. By Lemma 3.1.1, any element $f-1$ can be written as a linear combination of $X-1$. Hence $X-1$ generates I_F as a $\mathbb{Z}F$-module. To prove freeness, we need to show I_F satisfies the universal property of a free $\mathbb{Z}F$-module with $X-1$, that is, given a map $h:X-1\to A$, where A is a $\mathbb{Z}F$-module, there exists $\mathbb{Z}F$-module homomorphism $\hat{h}:I_F\to A$ such that the following diagram commutes

$$\begin{tikzcd} X-1 \arrow[r, hook] \arrow[dr, "h"'] & I_F \arrow[d, "\hat{h}"] \\ & A. \end{tikzcd}$$

Let $A\rtimes F$ denote the semidirect product defined by

$$(a_1,f_1)(a_2,f_2)=(a_1+f_1\cdot a_2,f_1f_2)$$

where $a_1,a_2\in A$ and $f_1,f_2\in F$. Let $\mu:X\to A\rtimes F$ denote the map defined by $\mu(x)=(h(x-1),x)$. Since F is free on X, μ extends to a group homomorphism $\hat{\mu}:F\to A\rtimes F$ with $\pi\hat{\mu}=id$ where $\pi:A\rtimes F\to F$ is defined by $\pi(a,f)=f$. Let $d:F\to A$ be the map defined by $\hat{\mu}(f)=(d(f),f)$. By the definiton of semidirect product, $\hat{\mu}(x)\hat{\mu}(y) = (d(x) + xd(y), xy)$. Since $\hat{\mu}$ must satisfy $\hat{\mu}(xy) = \hat{\mu}(x)\hat{\mu}(y)$, we get $d(xy) = d(x) + xd(y)$, which means that d is derivation. By Lemma 3.1.3, we have $\mathrm{Hom}(I_F, A) \cong \mathrm{Der}(F, A)$. This means there exist a map $\hat{h} : I_F \to A$ with $\hat{h}(f - 1) = d(f)$. Hence

$$(h(x-1), x) = \mu(x) = \hat{\mu}(x) = (d(x), x) \text{ for all } x \in X$$

Then $\hat{h}(x - 1) = h(x - 1)$, which shows that h extends to a $\mathbb{Z}F$-module homomorphism $\hat{h}$. $\square$

Proposition 3.1.5. *Let F be a free group with basis X, and R be a normal subgroup of F with basis Y. Then $J = \ker(\mathbb{Z}F \xrightarrow{\pi^*} \mathbb{Z}G)$ is a free $\mathbb{Z}F$-module with basis $Y - 1 = \{y - 1 | y \in Y\}$.*

Proof. It is clear $Y - 1 \subseteq J$. To show that J is ideal generated by $Y - 1$, choose left transversal T of R in F. Then $F = \cup_{t\in T} tR$. Take any element $\alpha \in \mathbb{Z}F$, then $\alpha = \sum_{i,j} m_{i,j} t_i r_j$ where $t_i r_j \in F$, $m_{i,j} \in \mathbb{Z}$. If $\alpha \in J$,

$$0 = \pi^*(\alpha) = \sum_{i,j} m_{i,j}\pi(t_i)$$

where $\pi(t_i)$ are distinct.

$$\alpha = \alpha - 0 = \sum_i (\sum_j m_{i,j} t_i (r_j - 1))$$

is F-linear combination of elements of form $r - 1$ with $r \in R$. Since $rr' - 1 = r(r' - 1) + (r - 1)$ and $r^{-1} - 1 = -r^{-1}(r - 1)$, J is generated as $\mathbb{Z}F$-module by $Y - 1$. Similarly, using $F = \bigcup Rt$, J is generated by $Y - 1$ as right $\mathbb{Z}F$-module. To show freeness, assume $\sum \alpha_i (y_i - 1) = 0$, where $\alpha_i \in \mathbb{Z}F$. Then $\alpha_i = \sum_i t_i \beta_{i,j}$ where $\beta_{i,j} \in \mathbb{Z}R$ and $\beta_{i,j} = \sum m_{j,k} r_k$ where $m_{j,k} \in \mathbb{Z}, r_k \in R$.

$$0 = \sum_{j,k} t_j m_{j,k} r_k \Longrightarrow m_{j,k} = 0.$$

Then the coset representation of t_j are independent over $\mathbb{Z}R$. Hence we have $\sum_i \beta_{i,j}(y_i - 1) = 0$ for each j. By Proposition 3.1.4, J is a free left $\mathbb{Z}F$-module with basis $Y - 1$. $\square$

Lemma 3.1.6. *If A is a left ideal of $\mathbb{Z}F$, then A/JA is a left G-module.*

Proof. If $a \in A$ and $\pi(f) = g \in G$, then $g(a + Ja) = fa + JA$. If $\pi(f_1) = \pi(f)$, then $f_1 - f \in \ker(\pi^*) = J$. Hence this is well-defined since $(f_1 - f)a \in JA$. $\square$

Lemma 3.1.7. *If A is free as a left ideal of $\mathbb{Z}F$ on S, then A/JA is G-free on $S + JA$.*

Proof. Since we have $A = \bigoplus_{s \in S} \mathbb{Z}F(s)$ and $JA = \bigoplus_{s \in S} Js$, we have

$$A/JA \cong \bigoplus_{s \in S} \mathbb{Z}F(s)/Js \cong \bigoplus_{s \in S} (\mathbb{Z}F/J)(s).$$

$\square$

Lemma 3.1.8. *If A is free as a two-sided ideal of $\mathbb{Z}F$ on S, B is free as a left ideal of $\mathbb{Z}F$ on T, then AB is free as a left ideal on ST.*

Proof. Suppose A and B are free as left ideal of $\mathbb{Z}F$ on S and T respectively. Then AB is generated by all elements $s_i \lambda t_j, \lambda \in \mathbb{Z}F$. Since A is two-sided ideal, $s_i \lambda \in A$ can be written as a sum of νs_k, with $\nu \in \mathbb{Z}F$. Then $s_i t_j$ generates AB as left ideal. Suppose $\sum_{i,j} \lambda_{i,j} s_i t_j = 0$, then we have $\sum_i \lambda_{i,j} s_i = 0$ for each j by the independence of t_j and then $\lambda_{i,j} = 0$ for each i and j by the independence of s_i. Hence $\{s_i t_j\}$ generates AB freely. $\square$

Theorem 3.1.9. *Let $1 \to R \to F \xrightarrow{\pi} G \to 1$ be a given presentation of group G where F is a free group. Then the following is a free resolution of $\mathbb{Z}$:*

$$\cdots \to J^2/J^3 \to JI_F/J^2I_F \to J/J^2 \to I_F/JI_F \to \mathbb{Z}G \to \mathbb{Z} \to 0$$

where $\mathbb{Z}G \to \mathbb{Z}$ is the unit augmentaiton map and $I_F/JI_F \to \mathbb{Z}G$ is induced by π and other maps are all induced by inclusions.

Proof. Denote $F_{2n} = J^n/J^{n+1}$ and $F_{2n-1} = J^{n-1}I_F/J^nI_F$ for all $n \geq 1$. Since $J^{n+1} \subseteq J^nI_F$ and $d_{2n} : J^n/J^{n+1} \to J^{n-1}I_F/J^nI_F$, d_{2n} is a composition of $J^n/J^{n+1} \to J^n/J^nI_F \to J^{n-1}I_F/J^nI_F$. Since $J^nI_F \trianglelefteq J^n$ and $J^{n+1} \trianglelefteq J^n$, we have

$$\begin{aligned}
\ker d_{2n} &= \{\alpha + J^{n+1} \in J^n/J^{n+1} | d_{2n}(\alpha + J^{n+1}) = J^n I_F\} \\
&= \{\alpha + J^{n+1} \in J^n/J^{n+1} | \alpha J^n I_F = J^n I_F\} \\
&= \{\alpha + J^{n+1} \in J^n/J^{n+1} | \alpha \in J^n I_F\} \\
&= J^n I_F/J^{n+1}, \\
\mathrm{Im}(d_{2n}) &= J^n/J^n I_F.
\end{aligned}$$

Since $J^{n+1}I_F \subseteq J^{n+1}$ and $d_{2n+1} : J^n I_F/J^{n+1}I_F \to J^n/J^{n+1}$, d_{2n+1} is a composition of $J^n I_F/J^{n+1}I_F \to J^n I_F/J^{n+1} \to J^n/J^{n+1}$. Similarly, we have

$$\begin{aligned}
\ker d_{2n+1} &= \{\alpha + J^{n+1}I_F \in J^n/J^{n+1}I_F | d_{2n+1}(\alpha + J^{n+1}I_F) = J^{n+1}\} \\
&= \{\alpha + J^{n+1}I_F \in J^n/J^{n+1}I_F | \alpha + J^{n+1} = J^{n+1}\} \\
&= \{\alpha + J^{n+1}I_F \in J^n/J^{n+1}I_F | \alpha \in J^{n+1}\} \\
&= J^{n+1}/J^{n+1}I_F, \\
\mathrm{Im}\, d_{2n+1} &= J^n I_F/J^{n+1}.
\end{aligned}$$

Then $\ker d_{2n} = \mathrm{Im}\, d_{2n+1}$ and $\ker d_{2n+1} = \mathrm{Im}\, d_{n+2}$. This gives us exactness at all F_n. We need to check exactness at first three terms. Take the sequence

$$I_F/JI_F \xrightarrow{\pi^*} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0.$$

Take a basis element $(x-1) \in I_F/JI_F$. Since $\mathrm{Im}(\pi^*(x-1)) = \pi^*(x)-1 = \ker(\epsilon)$, the image of π^* is in the kernel of ϵ. Then this sequence is exact. □

Theorem 3.1.10. *If F is free on X and R is free on Y, then J^n/J^{n+1} is G-free on the following set of cosets of elements*

$$\{(y_1-1)(y_2-1)\dots(y_n-1) + J^{n+1}, \text{ where } y_1, ..., y_n \in Y\}$$

and $J^{n-1}I_F/J^nI_F$ is G-free on the cosets of all elements

$$\{(y_1-1)(y_2-1)\dots(y_{n-1}-1)(x-1) + J^nI_F, \text{ where } x \in X \text{ and } y_1, \dots, y_{n-1} \in Y\}.$$

Proof. Since F is free on X and R is free on Y, by Propositions 3.1.4 and 3.1.5, we have I_F is free on $X-1$ and J is free on $Y-1$. By applying the Lemma 3.1.8, J^n is free on $(y_1-1)(y_2-1)\dots(y_n-1)$ and $J^{n-1}I_F$ is free on

$(y_1-1)(y_2-1)\dots(y_{n-1}-1)(x-1)$ where $x \in X$ and $y_1,\dots,y_n \in Y$. By Lemma 3.1.7, J^n/J^{n+1} is $G-$free on

$$(y_1-1)(y_2-1)\dots(y_n-1)+J^{n+1}$$

and $J^{n-1}I_F/J^nI_F$ is $G-$free on

$$(y_1-1)(y_2-1)\dots(y_{n-1}-1)(x-1)+J^nI_F.$$

□

Example 3.1.11. *Let $G=\langle g\rangle$ be cyclic group with order n. Let*

$$1 \to R \to F \xrightarrow{\pi} G \to 1$$

be a presentation of G with $F=\langle x\rangle$ and $R=\langle x^n\rangle$ and $\pi(x)=g$. By using Gruenberg resolution, we get

$$\cdots \longrightarrow F_{2k+1} \xrightarrow{d_{2k+1}} F_{2k} \xrightarrow{d_{2k}} F_{2k-1} \xrightarrow{d_{2k-1}} \cdots \longrightarrow F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{\pi^*} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

where $F_{2k} = J^k/J^{k+1}$ and $F_{2k-1} = J^{k-1}I_F/J^kI_F$. Then F_{2k} is free on generator $x_{2k} = (x^n-1)^k + J^{k+1}$ and F_{2k-1} is free on generator $x_{2k-1} = (x^n-1)^{k-1}(x-1)+J^kI_F$. Since

$$(x^n-1)^k = (1+x+\cdots+x^{n-1})(x^n-1)^{k-1}(x-1),$$

we have

$$\begin{aligned} d_{2k}(x_{2k}+J^{k+1}) &= (1+x+\cdots+x^{n-1})((x^n-1)^{k-1}(x-1)+J^kI_F) \\ &= (1+x+\cdots+x^{n-1})(x_{2k-1}). \end{aligned}$$

Hence d_{2k} is the multiplication by $(1+\cdots+x^{n-1}) := N$. We write $d_{2k}=N$. Since F_{2k+1} is free on generator $x_{2k+1} = (x^n-1)^k(x-1)+J^{k+1}I_F$ and $d_{2k+1}(x_{2k+1}) = (x-1)x_{2k}$, then d_{2k+1} is the multiplication by $x-1$. We write $d_{2k+1}=x-1$. Then this resolution becomes

$$\cdots \xrightarrow{N} \mathbb{Z}G \xrightarrow{g-1} \mathbb{Z}G \xrightarrow{N} \cdots \xrightarrow{g-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{g-1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0.$$

3.2 Standard resolution as a Gruenberg resolution

Let G be any group and F be free group on $\{x_g : g \in G^*\}$ where G^* denotes the set of all non-identity elements of G. The map $\pi : x_g \to g$ gives a presentation

$$1 \longrightarrow R \longrightarrow F \xrightarrow{\pi} G \longrightarrow 1$$

which is called the **standard presentation**. Take $x_1 = 1_F$. Define $y_{(g,h)} = x_{gh}^{-1} x_g x_h$.

Lemma 3.2.1. *If $w = x_{g_1} x_{g_2} \dots x_{g_n} \in R$, then w can be written as a product of elements in $Y = \{y_{(g,h)} | g, h \in G^*\}$. Moreover, R is free on the set $Y = \{y_{(g,h)} | g, h \in G^*\}$.*

Proof. First take $w = x_{g_1} x_{g_2}$. Then

$$x_{g_1} x_{g_2} = x_{g_1 g_2} x_{g_1 g_2}^{-1} x_{g_1} x_{g_2} = x_{g_1 g_2} y_{(g_1, g_2)}$$

If we take $w = x_{g_1} x_{g_2} x_{g_3}$, then

$$\begin{aligned} x_{g_1} x_{g_2} x_{g_3} &= x_{g_1} x_{g_2 g_3} y_{(g_2, g_3)} \\ &= x_{g_1 g_2 g_3} x_{g_1 g_2 g_3}^{-1} x_{g_1} x_{g_2 g_3} y_{(g_2, g_3)} \\ &= x_{g_1 g_2 g_3} y_{(g_1, g_2 g_3)} y_{(g_2, g_3)}. \end{aligned}$$

Claim that $w = x_{g_1} x_{g_2} \dots x_{g_k} = x_{g_1 g_2 \dots g_k} y_{(g_1, g_2 \dots g_k)} \dots y_{(g_{k-1}, g_k)}$. Suppose the statement is true for $k = n - 1$. Then we need to show it holds for $k = n$. Take $w = x_{g_1} x_{g_2} \dots x_{g_n}$.

$$\begin{aligned} x_{g_1} x_{g_2} \dots x_{g_n} &= x_{g_1} x_{g_2 g_3 \dots g_n} y_{(g_2, g_3 \dots g_n)} \dots y_{(g_{n-1}, g_n)} \\ &= x_{g_1 g_2 \dots g_n} x_{g_1 g_2 \dots g_n}^{-1} x_{g_1} x_{g_2 \dots g_n} y_{(g_2, g_3 \dots g_n)} \dots y_{(g_{n-1}, g_n)} \\ &= x_{g_1 g_2 \dots g_n} y_{(g_1, g_2 \dots g_n)} y_{(g_2, g_3 \dots g_n)} \dots y_{(g_{n-1}, g_n)}. \end{aligned}$$

Since $w = x_{g_1} x_{g_2} \dots x_{g_n} \in R$ and $\pi(w) = 1$, $x_{g_1 g_2 \dots g_n} = x_1 = 1$. From this we obtain that $w \in \langle y_{(g,h)} \rangle$.

Take $T = \{x_h : h \in G^*\}$ is a Schreier transversal for R in F and $\phi(x_g x_h) = x_{gh}$. Then $y_{(g,h)} = \phi(x_g x_h)^{-1} x_g x_h$ where $x_h \in T$ and $x_g \in X$. If any $x_{h_1}, x_{h_2} \in T$ and $x_g \in X$, $x_{h_1}^{-1} x_g x_{h_2} = w \in R$ implies $x_g x_{h_2} = x_{h_1} w$. Then $\phi(x_g x_{h_2}) = \phi(x_{h_1} w) = x_{h_1}$. R is free on the set $Y = \{y_{(g,h)} : g, h \in G^*\}$ by Nielsen-Schreier Theorem [10]. □

Since R is free on the set Y, $Y - 1 = \{y_{(g,h)} - 1 : g, h \in G^*\}$ is a set of free generators for J. Then we denote

$$\begin{aligned}(g,h) &= x_g x_h - x_{gh} \in J \\ &= x_{gh}(y_{(g,h)} - 1).\end{aligned}$$

Since $(g,h) = 0$ if and only if $g = 1$ or $h = 1$, $(g,h) \neq 0$ generate J freely. Define

$$[g_1|g_2|\dots|g_{2k}] = (g_1, g_2)\dots(g_{2k-1}, g_{2k}) + J^{k+1}$$
$$[g_1|g_2|\dots|g_{2k-1}] = (g_1, g_2)\dots(g_{2k-3}, g_{2k-2})(x_{g_{2k-1}} - 1) + J^k I_F.$$

Then

$$[g_1|g_2|\dots|g_{2k}] \in F_{2k} = J^k/J^{k+1}$$
$$[g_1|g_2|\dots|g_{2k-1}] \in F_{2k-1} = J^{k-1} I_F / J^k I_F.$$

Hence for $1 \neq g_j \in G$, $[g_1|g_2|\dots|g_{2k}]$ generate F_{2k} freely and $[g_1|g_2|\dots|g_{2k-1}]$ generate F_{2k-1} freely. It is clear $[g_1|g_2|\dots|g_j] = 0$ if $g_m = 1$ for some $1 < m \leq j$.

Proposition 3.2.2. *If d_k is the differential $F_k \to F_{k-1}$ in the Gruenberg resolution for the standard presentation of G, then for all $k > 0$*

$$\begin{aligned}d_k([g_1|g_2|\dots|g_k]) =& g_1[g_2|\dots|g_k] \\ &+ \sum_{i=1}^{k-1} (-1)^i [g_1|g_2|\dots|g_i g_{i+1}|\dots|g_k] \\ &+ (-1)^k [g_1|g_2|\dots|g_{k-1}].\end{aligned}$$

Proof. Consider the Gruenberg resolution for the standard presentation

$$\cdots \longrightarrow F_{2k+1} \xrightarrow{d_{2k+1}} F_{2k} \xrightarrow{d_{2k}} F_{2k-1} \xrightarrow{d_{2k-1}} \cdots \longrightarrow F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0.$$

If $k = 1$, then $d_1 : F_1 \to F_0$ is induced by $\pi : F \to G$. Hence we have $d_1[g_1] = d_1((x_{g_1} - 1) + JI_F) = g_1[\,] - [\,] = g_1 - 1$. Then the statement is true for $k = 1$. For the case $k = 2$, we have

$$\begin{aligned} d_2([g_1|g_2]) &= d_2((g_1, g_2) + J^2) \\ &= d_2((x_{g_1}x_{g_2} - x_{g_1g_2}) + J^2) \\ &= x_{g_1}(x_{g_2} - 1) + (x_{g_1} - 1) - (x_{g_1g_2} - 1) + JI_F \\ &= g_1[g_2] - [g_1g_2] + [g_1] \end{aligned}$$

Then $d_2([g_1|g_2]) = g_1[g_2] - [g_1g_2] + [g_1]$. Consider the case $k = 3$,

$$\begin{aligned} d_3([g_1|g_2|g_3]) &= d_3((g_1, g_2)(x_{g_3} - 1) + J^2I_F) \\ &= d_3((x_{g_1}x_{g_2} - x_{g_1g_2})x_{g_3} - (g_1, g_2) + J^2I_F) \\ &= d_3(x_{g_1}x_{g_2}x_{g_3} - x_{g_1g_2}x_{g_3} - (g_1, g_2) + J^2I_F) \\ &= x_{g_1}(x_{g_2}x_{g_3} - x_{g_2g_3}) + (x_{g_1}x_{g_2g_3} - x_{g_1g_2g_3}) \\ &\quad - (x_{g_1g_2}x_{g_3} - x_{g_1g_2g_3}) - (g_1, g_2) + J^2 \\ &= g_1[g_2, g_3] + [g_1|g_2g_3] - [g_1g_2|g_3] - [g_1|g_2]. \end{aligned}$$

Thus $d_3([g_1|g_2|g_3]) = g_1[g_2|g_3] - [g_1g_2|g_3] + [g_1|g_2g_3] - [g_1|g_2]$.

Using induction on k, we can conclude the proof. Suppose the statement is true for all $m < k$. Consider the case $m = k$. First suppose k is even. Then take $k = 2n$ the map $d_{2n} : F_{2n} \to F_{2n-1}$. Abusing the notation,

$$\begin{aligned} [g_1|g_2|\dots|g_{2n}] &= (g_1, g_2)\dots(g_{2n-3}, g_{2n-2})(g_{2n-1}, g_{2n}) + J^{n+1} \\ &= (g_1, g_2)\dots\big(g_{2n-1}[g_{2n}] - [g_{2n-1}g_{2n}] + [g_{2n-1}]\big) + J^nI_F, \text{ by } k = 2 \\ &= (g_1, g_2)\dots\big([g_{2n-1}][g_{2n}] + [g_{2n}] - [g_{2n-1}g_{2n}] + [g_{2n-1}]\big) + J^nI_F \\ &= [g_1|\dots|g_{2n-1}][g_{2n}] + [g_1|\dots|g_{2n-2}|g_{2n}] \\ &\quad - [g_1|\dots|g_{2n-1}g_{2n}] + [g_1|\dots|g_{2n-1}]. \end{aligned}$$

By induction on n, the definition of d_{2n-1} satisfies,

$$\begin{aligned} [g_1|g_2|\dots|g_{2n-1}][g_{2n}] =& g_1[g_2|\dots|g_{2n-1}|g_{2n}] \\ &+ \sum_{i=1}^{2n-2}(-1)^i[g_1|g_2|\dots|g_ig_{i+1}|\dots|g_{2n}] \\ &- [g_1|g_2|\dots|g_{2n-2}|g_{2n}]. \end{aligned}$$

Using this equality

$$\begin{aligned}[g_1|\dots|g_n] &= g_1[g_2|\dots|g_{2n}] + \sum_{i=1}^{2n-2}(-1)^i[g_1|\dots|g_ig_{i+1}|\dots|g_{2n}] - [g_1|\dots|g_{2n-2}|g_{2n}] \\ &\quad + [g_1|\dots|g_{2n-2}|g_{2n}] - [g_1|\dots|g_{2n-1}g_{2n}] + [g_1|\dots|g_{2n-1}] \\ &= g_1[g_2|\dots|g_{2n}] + \sum_{i=1}^{2n-1}(-1)^i[g_1|\dots|g_ig_{i+1}|\dots|g_{2n}] + [g_1|\dots|g_{2n-1}],\end{aligned}$$

that is, $d_{2n}([g_1|g_2|\dots|g_{2n}]) = g_1[g_2|\dots|g_{2n}] + \sum_{i=1}^{2n-1}(-1)^i[g_1|\dots|g_ig_{i+1}|\dots|g_{2n}] + [g_1|g_2|\dots|g_{2n-1}]$.

Now suppose k is odd. Then take $k = 2n+1$ the map $d_{2n+1} : F_{2n+1} \to F_{2n}$

$$\begin{aligned}[g_1|g_2|\dots|g_{2n+1}] &= (g_1, g_2)\dots(g_{2n-1}, g_{2n})(x_{g_{2n+1}} - 1) + J^{k+1}I_F \\ &= (g_1, g_2)\dots(g_{2n-3}, g_{2n-2})(g_{2n-1}[g_{2n}|g_{2n+1}] - [g_{2n-1}g_{2n}|g_{2n+1}] \\ &\quad + [g_{2n-1}|g_{2n}g_{2n+1}] - [g_{2n-1}|g_{2n}]) + J^{k+1}, \text{ by } k = 3 \\ &= (g_1, g_2)\dots(g_{2n-3}, g_{2n-2})([g_{2n-1}][g_{2n}|g_{2n+1}] + [g_{2n}|g_{2n+1}] \\ &\quad - [g_{2n-1}g_{2n}|g_{2n+1}] + [g_{2n-1}|g_{2n}g_{2n+1}] - [g_{2n-1}|g_{2n}]) + J^{k+1} \\ &= [g_1|\dots|g_{2n-2}|g_{2n-1}][g_{2n}|g_{2n+1}] \\ &\quad + (g_1, g_2)\dots(g_{2n-3}, g_{2n-2})([g_{2n}|g_{2n+1}] - [g_{2n-1}g_{2n}|g_{2n+1}] \\ &\quad + [g_{2n-1}|g_{2n}g_{2n+1}] - [g_{2n-1}|g_{2n}]) + J^{k+1}.\end{aligned}$$

By the case $k = 2n$,

$$\begin{aligned}[g_1|\dots|g_{2n+1}] &= [g_1|\dots|g_{2n-2}|g_{2n-1}][g_{2n}|g_{2n+1}] + [g_1|\dots|g_{2n}|g_{2n+1}] \\ &\quad - [g_1|\dots|g_{2n-1}g_{2n}|g_{2n+1}] + [g_1|\dots|g_{2n-1}|g_{2n}g_{2n+1}] - [g_1|\dots|g_{2n-1}|g_{2n}]\end{aligned}$$

Using the definition of d_{2n-1}

$$\begin{aligned}[g_1|\dots|g_{2n-2}|g_{2n-1}][g_{2n}|g_{2n+1}] =& g_1[g_2|\dots|g_{2n+1}] \\ &+ \sum_{i=1}^{2n-2}(-1)^i[g_1|g_2|\dots|g_ig_{i+1}|\dots|g_{2n+1}] \\ &+ (-1)^k[g_1|g_2|\dots|g_{2n-2}|g_{2n}|g_{2n+1}].\end{aligned}$$

By combining above equalities

$$\begin{aligned}
[g_1|\dots|g_{2n+1}] &= g_1[g_2|\dots|g_{2n+1}] + \sum_{i=1}^{2n-2}(-1)^i[g_1|\dots|g_i|g_{i+1}|\dots|g_{2n+1}] \\
&\quad - [g_1|\dots|g_{2n-2}|g_{2n}|g_{2n+1}] + [g_1|\dots|g_{2n-2}|g_{2n}|g_{2n+1}] \\
&\quad - [g_1|\dots|g_{2n-1}g_{2n}|g_{2n+1}] + [g_1|\dots|g_{2n}g_{2n+1}] - [g_1|\dots|g_{2n}] \\
&= g_1[g_2|\dots|g_{2n}|g_{2n+1}] + \sum_{i=1}^{2n}(-1)^i[g_1|\dots|g_i|g_{i+1}|\dots|g_{2n+1}] \\
&\quad - [g_1|\dots|g_{2n}].
\end{aligned}$$

Then

$$\begin{aligned}
d_{2n+1}([g_1|\cdots|g_{2n+1}]) = g_1[g_2|\dots|g_{2n}|g_{2n+1}] + \sum_{i=1}^{2n}(-1)^i[g_1|\dots|g_i|g_{i+1}|\dots|g_{2n+1}] \\
- [g_1|\dots|g_{2n-1}|g_{2n}].
\end{aligned}$$

Hence we have the statement is true for all k. □

3.3 Fox calculus

Let F be free group on X and $\mathbb{Z}F$ be the group ring of F. The **derivation** D is an additive map from $\mathbb{Z}F$ into itself such that for all $u, v \in \mathbb{Z}F$,

1. $D(u+v) = D(v) + D(v)$
2. $D(u \cdot v) = D(u) \cdot \epsilon(v) + u \cdot D(v)$ where $\epsilon : \mathbb{Z}F \to \mathbb{Z}$ is an augmentation map.

Alternatively, we can say the derivation D is an additive map from $\mathbb{Z}F$ into itself such that for all $g, h \in F$,

$$D(gh) = D(g) + gD(h).$$

By definition we have the following consequences

1. $D(a) = 0$ for $a \in \mathbb{Z}$

2. $D(\sum a_g g) = \sum a_g D(g)$

3. $D(g^{-1}) = -g^{-1}D(g)$ where $g \in F$.

For each generator $x_j \in X$, the **Fox Derivative** of F with respect to x_j, that is $\dfrac{\partial}{\partial x_j} : F \to \mathbb{Z}F$, is defined by the rules

1. $\dfrac{\partial x_k}{\partial x_j} = \begin{cases} 1, & \text{if } x_k = x_j \\ 0, & \text{otherwise} \end{cases}$

2. For any $w_1, w_2 \in F$,

$$\frac{\partial(w_1 w_2)}{\partial x_j} = \frac{\partial w_1}{\partial x_j} + w_1 \frac{\partial w_2}{\partial x_j}.$$

Fox [9] proved that there is unique derivation from $u \in \mathbb{Z}F$ to Du mapping $x_1, x_2, \dots$ into prescribed elements $D(x_1), D(x_2), \dots \in \mathbb{Z}F$ with the following formula

$$Du = \sum_j \frac{\partial u}{\partial x_j} D(x_j).$$

Consider the augmentation map given by $\epsilon : x_j \to 1$. Then $u \to u - \epsilon(u)$ is a derivation mapping from $x_1, x_2, \dots$ into $x_1 - 1, x_2 - 1, \dots$. By using the above formula, we get the fundamental formula

$$u - \epsilon(u) = \sum_j \frac{\partial u}{\partial x_j}(x_j - 1).$$

Hence any element $u \in \mathbb{Z}F$ can be obtained from $\epsilon(u)$ and the Fox derivatives with respect to each j.

By using the definition of Fox Derivative, we can obtain the result for $n \geq 1$

$$\frac{\partial x^n}{\partial x} = \frac{\partial x}{\partial x} + x\frac{\partial x}{\partial x} + x^2\frac{\partial x}{\partial x} + \cdots + x^{n-1}\frac{\partial x}{\partial x} = 1 + x + \cdots + x^{n-1}.$$

Since $x^n x^{-n} = 1$ and $\dfrac{\partial 1}{\partial x} = 0$,

$$0 = \frac{\partial(x^n x^{-n})}{\partial x} = \frac{\partial x^n}{\partial x} + x^n \frac{\partial x^{-n}}{\partial x}.$$

From this equality, we have $\dfrac{\partial x^{-n}}{\partial x} = \dfrac{-1-x-\cdots-x^{n-1}}{x^n} = -x^{-n}-\cdots-x^{-1}$. Then we get the formula

$$\frac{\partial x^n}{\partial x} = \begin{cases} 1+x+\cdots+x^{n-1} & , if\ n \geq 1 \\ 0 & ,\ n=0 \\ -x^n - x^{n-1} - \cdots - x^{-1} & , if\ n \leq -1. \end{cases}$$

More generally, take a word $u \in F$ such that $u = u_0 x_i^{p_1} u_1 x_i^{p_2} \cdots u_{q-1} x_i^{p_q} u_q$ where all $p_k \neq 0$ and all u_k do not include the generator x_i for $1 \leq k \leq q$. From the formula of the Fox derivative, we have

$$\begin{aligned} \frac{\partial u}{\partial x_i} &= u_0 \frac{\partial x_i^{p_1}}{\partial x_i} + u_0 x_i^{p_1} u_1 \frac{\partial x_i^{p_2}}{\partial x_i} + \cdots + u_0 x_i^{p_1} \cdots u_{q-1} \frac{\partial x_i^{p_q}}{\partial x_i} \\ &= u_0(1 + x_i + \cdots + x_i^{p_1-1}) + u_0 x_i^{p_1} u_1 (1 + x_i + \cdots + x_i^{p_2-1}) + \ldots \\ &\quad + u_0 x_i^{p_1} \cdots u_{q-1}(1 + x_i + \cdots + x_i^{p_q-1}). \end{aligned}$$

For example, consider $u = x_2 x_1^2 x_2^{-2}$.

$$\frac{\partial u}{\partial x_1} = x_2(1+x_1)$$

$$\frac{\partial u}{\partial x_2} = 1 - x_2 x_1^2 x_2^{-2}(1+x_2).$$

Then

$$\begin{aligned} Du &= \sum_j \frac{\partial u}{\partial x_j} D(x_j) \\ &= \frac{\partial u}{\partial x_1}(x_1 - 1) + \frac{\partial u}{\partial x_2}(x_2 - 1) \\ &= x_2(1+x_1)(x_1-1) + (1 - x_2 x_1^2 x_2^{-2}(1+x_2)(x_2-1)) \\ &= x_2 x_1^2 x_2^{-2} - 1, \end{aligned}$$

which satisfy the fundamental formula.

Also consider the map $d_2 : J/J^2 \to I_F/JI_F$ where J is free on $Y - 1 = \{y_i - 1 : y_i \in Y\}$ and I_F is free on $X - 1 = \{x_i - 1 : x_i \in X\}$. Take an element $(y_j - 1) + J^2$. Using the fundamental formula, we get

$$y_j - 1 = \sum_i \frac{\partial(y_j - 1)}{\partial x_i}(x_i - 1).$$

We write d_2 using Fox derivative

$$d_2(y_j - 1) = \sum_i \frac{\partial(y_j - 1)}{\partial x_i}(x_i - 1).$$

For example, the map $d_2 : J/J^2 \to I_F/JI_F$ from the Gruenberg resolution of cyclic group with order n. Take an element $x_2 = (x^n - 1) + J^2$. Using the fundamental formula, we get

$$(x^n - 1) = \frac{\partial x^n}{\partial x}(x - 1).$$

Then d_2 is multiplication by $\dfrac{\partial x^n}{\partial x}$, which is the same what we found in Example 3.1.11

We will use the Fox derivative in order to write down differentials for the Gruenberg resolutions in the following chapters.

Chapter 4

Chain Maps between two Gruenberg Resolutions

As we mention earlier in Corallay 2.1.11, projective resolutions are unique up to chain homotopy equivalence. Hence there exist a chain map between any two Gruenberg resolutions. In the first section, we give an explicitly chain map between Gruenberg resolution for standard presentation and the Gruenberg resolution for the presentation of cyclic group. In the second section, we give a chain map for any two Gruenberg resolutions. In the third section, we mention an application of chain map between standard resolution and the periodic resolution of cyclic group [3].

4.1 The case of cyclic group

In this section, we write down an explicit chain map between the Gruenberg resolution for standard presentation and the Gruenberg resolution for the presentation of cyclic group.

Consider G is cyclic group with order k. Then for all $n \geq 0$ the diagrams

$$\begin{tikzcd}
\cdots \arrow[r] & J^k/J^{k+1} \arrow[r, "d_{2k}"] \arrow[d, "f_{2k}"] & J^{k-1}I_F/J^kI_F \arrow[r] \arrow[d, "f_{2k-1}"] & \cdots \arrow[r] & J/J^2 \arrow[r, "d_2"] \arrow[d, "f_2"] & I_F/JI_F \arrow[r, "d_1"] \arrow[d, "f_1"] & \mathbb{Z}G \arrow[r, "\epsilon"] \arrow[d, "id"] & \mathbb{Z} \arrow[r] \arrow[d, "id"] & 0 \\
\cdots \arrow[r] & \mathbb{Z}G \arrow[r, "N"] & \mathbb{Z}G \arrow[r, "g-1"] & \cdots \arrow[r] & \mathbb{Z}G \arrow[r, "N"] & \mathbb{Z}G \arrow[r, "g-1"] & \mathbb{Z}G \arrow[r, "\epsilon"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

commutes. Since G is cyclic, we have

$$\begin{aligned}
G &= \langle g : g^n = 1 \rangle \\
F &= \mathbb{Z} \cdot g \\
R &= \mathbb{Z} \cdot g^n.
\end{aligned}$$

Consider F is free on $X = \{x_1, x_g, x_{g^2}, \ldots, x_{g^{n-1}}\}$ and R is free on $Y = \{y_{(a,b)} = x_{ab}^{-1} x_a x_b\}$. Using Proposition 3.1.4 and 3.1.5, we have I_F is free as a left $\mathbb{Z}F$-module on $X - 1$ and J is free $\mathbb{Z}F$-module on $Y - 1$. By using Theorem 3.1.10, we have $J^{k-1}I_F/J^kI_F$ is G-free on the cosets of all elements $(y_1 - 1)(y_2 - 1)\ldots(y_{k-1} - 1)(x - 1)$, where $x \in X, y_i \in Y$ for $1 \leq i \leq k$ and J^k/J^{k+1} is G-free on the cosets of all elements $(y_1 - 1)(y_2 - 1)\ldots(y_k - 1)$, where $y_i \in Y$ for $1 \leq i \leq k$. Take an element $\alpha \in I_F/JI_F$. Let $\alpha = [g^i]$ where $1 \leq i < n$.

$$\begin{tikzcd}
(x_{g^i} - 1) = [g^i] \arrow[r, "d_1"] \arrow[d, "f_1"] & g^i[\,] - [\,] \arrow[d, "id"] \\
f_1([g^i]) \arrow[r, "g-1"] & g^i - 1.
\end{tikzcd}$$

Since this diagram commutes, $f_1([g^i]) = \dfrac{g^i - 1}{g - 1}$. By using Fox derivative, we conclude $f_1([g^i]) = \dfrac{\partial g^i}{\partial g}$ where $1 \leq i < n$.

Take an element $\alpha \in J/J^2$. Let $\alpha = [g^{i_1}|g^{i_2}]$ where $1 \leq i_1, i_2 < n$.

$$\begin{tikzcd}
x_{g^{i_1}g^{i_2}}(y_{(g^{i_1},g^{i_2})} - 1) = [g^{i_1}|g^{i_2}] \arrow[r, "d_2"] \arrow[d, "f_2"] & g^{i_1}[g^{i_2}] - [g^{i_1}g^{i_2}] + [g^{i_1}] \arrow[d, "f_1"] \\
f_2([g^{i_1}|g^{i_2}]) \arrow[r, "N"] & g^{i_1}\dfrac{\partial g^{i_2}}{\partial g} - \dfrac{\partial (g^{i_1}g^{i_2})}{\partial g} + \dfrac{\partial g^{i_1}}{\partial g}.
\end{tikzcd}$$

For $i_1 + i_2 < n$ we have $g^{i_1}g^{i_2} = g^{i_1+i_2}$, then using the Fox derivative

$$\frac{\partial(g^{i_1+i_2})}{\partial g} = g^{i_1}\frac{\partial g^{i_2}}{\partial g} + \frac{\partial g^{i_1}}{\partial g}.$$

Then

$$g^{i_1}\frac{\partial g^{i_2}}{\partial g} - \frac{\partial(g^{i_1}g^{i_2})}{\partial g} + \frac{\partial g^{i_1}}{\partial g} = 0.$$

For $i_1 + i_2 \geq n$ we have $g^{i_1}g^{i_2} = g^{i_1+i_2-n}$, then we need to find $g^{i_1}\frac{\partial g^{i_2}}{\partial g} - \frac{\partial(g^{i_1+i_2-n})}{\partial g} + \frac{\partial g^{i_1}}{\partial g}$. Using the Fox derivative we have

$$\frac{\partial(g^{i_1+i_2})}{\partial g} = \frac{\partial(g^{i_2+i_2-n}g^n)}{\partial g} = \frac{\partial(g^{i_1+i_2-n})}{\partial g} + g^{i_1+i_2-n}\frac{\partial g^n}{\partial g}.$$

Since $\frac{\partial(g^{i_1+i_2})}{\partial g} = \frac{\partial g^{i_1}}{\partial g} + g^{i_1}\frac{\partial g^{i_2}}{\partial g}$, we have

$$g^{i_1+i_2-n}\frac{\partial g^n}{\partial g} = \frac{\partial g^{i_1}}{\partial g} - \frac{\partial(g^{i_1+i_2-n})}{\partial g} + g^{i_1}\frac{\partial g^{i_2}}{\partial g}.$$

This means

$$g^{i_1}\frac{\partial g^{i_2}}{\partial g} - \frac{\partial(g^{i_1}g^{i_2})}{\partial g} + \frac{\partial g^{i_1}}{\partial g} = \begin{cases} g^{i_1+i_2-n}\frac{\partial g^n}{\partial g} & , if\ i_1 + i_2 \geq n \\ 0 & , otherwise. \end{cases}$$

Since $f_2([g^{i_1}|g^{i_2}]) = \dfrac{g^{i_1}\frac{\partial g^{i_2}}{\partial g} - \frac{\partial(g^{i_1}g^{i_2})}{\partial g} + \frac{\partial g^{i_1}}{\partial g}}{N}$, we have

$$f_2([g^{i_1}|g^{i_2}]) = \begin{cases} g^{i_1+i_2-n} & , \ if\ i_1 + i_2 \geq n \\ 0 & , \ otherwise. \end{cases}$$

Since $f_2([g^{i_1}|g^{i_2}]) = g^{i_1+i_2-n}f_2(y_{(g^{i_1},g^{i_2})} - 1)$, we have

$$f_2(y_{(g^{i_1},g^{i_2})} - 1) = \begin{cases} 1 & , \ if\ i_1 + i_2 \geq n \\ 0 & , \ otherwise. \end{cases}$$

Take $\alpha \in JI_F/J^2I_F$. Let $\alpha = [g^{i_1}|g^{i_2}|g^{i_3}]$.

$$\begin{tikzcd} {[g^{i_1}|g^{i_2}|g^{i_3}]} \arrow[r, "d_3"] \arrow[d, "f_3"] & g^{i_1}[g^{i_2}|g^{i_3}] - [g^{i_1}g^{i_2}|g^{i_3}] + [g^{i_1}|g^{i_2}g^{i_3}] - [g^{i_1}|g^{i_2}] \arrow[d, "f_2"] \\ f_3([g^{i_1}|g^{i_2}|g^{i_3}]) \arrow[r, "g-1"] & g^{i_1}f_2([g^{i_2}|g^{i_3}]) - f_2([g^{i_1}g^{i_2}|g^{i_3}]) + f_2([g^{i_1}|g^{i_2}g^{i_3}]) - f_2([g^{i_1}|g^{i_2}]). \end{tikzcd}$$

Denote $g^{i_1}f_2([g^{i_2}|g^{i_3}]) - f_2([g^{i_1}g^{i_2}|g^{i_3}]) + f_2([g^{i_1}|g^{i_2}g^{i_3}]) - f_2([g^{i_1}|g^{i_2}]) = A$. Then $A = g^{i_1}g^{i_2+i_3-n}f_2(y_{(g^{i_2},g^{i_3})} - 1) - g^{i_1+i_2+i_3-n}f_2(y_{(g^{i_1}g^{i_2},g^{i_3})} - 1) + g^{i_1+i_2+i_3-n}f_2(y_{(g^{i_1},g^{i_2}g^{i_3})} - 1) - g^{i_1+i_2-n}f_2(y_{(g^{i_1},g^{i_2})} - 1)$. By the following table

	A
$i_1+i_2<n$ and $i_2+i_3<n$ and $i_1+i_2+i_3<n$	0
$i_1+i_2<n$ and $i_2+i_3<n$ and $i_1+i_2+i_3\geq n$	0
$i_1+i_2\geq n$ and $i_2+i_3<n$	$g^{i_1+i_2-n}(g^{i_3}-1)$
$i_1+i_2<n$ and $i_2+i_3\geq n$	0
$i_1+i_2\geq n$ and $i_2+i_3\geq n$	$g^{i_1+i_2-n}(g^{i_3}-1)$

we get

$$A = \begin{cases} g^{i_1+i_2-n}(g^{i_3}-1) & , \; if \;\; i_1+i_2 \geq n \\ 0 & , \; otherwise. \end{cases}$$

Hence

$$\begin{aligned} f_3([g^{i_1}|g^{i_2}|g^{i_3}]) = \frac{A}{g-1} &= \frac{g^{i_1+i_2-n}f_2(y_{(g^{i_1},g^{i_2})}-1)(g^{i_3}-1)}{g-1} \\ &= f_2([g^{i_1}|g^{i_2}])f_1([g^{i_3}]) \end{aligned}$$

Here is a question: “Can we generalize the remaining maps in terms of f_1 and f_2? ”

Remark 4.1.1. *Let G be cyclic group with order n. For the chain map between the Gruenberg resolution for standard presentation and the Gruenberg resolution for the presentation of cyclic group, take any element $[g^i|g^j|g^k] \in JI_F/J^2I_F$. Then $f_2(d_3([g^i|g^j|g^k])) = f_2[g^i|g^j](g^k-1)$. This follows from commutativity of the diagram at $F_3 = JI_F/J^2I_F$. Similarly if the diagram commutes at F_{2k-1}, we have*

$$f_{2k-2}(d_{2k-1}[g^{i_1}|\dots|g^{i_{2k-1}}]) = f_{2k-2}([g^{i_1}|\dots|g^{i_{2k-2}}])(g^{i_{2k-1}}-1).$$

Theorem 4.1.2. *There is a chain map between the Gruenberg resolution for standard presentation and the Gruenberg resolution for the presentation of cyclic* *group, that is, for all $n \geq 0$ the diagrams*

$$\begin{tikzcd}
\cdots \arrow[r] & J^kI_F/J^{k+1}I_F \arrow[r, "d_{2k+1}"] \arrow[d, "f_{2k+1}"] & J^k/J^{k+1} \arrow[r] \arrow[d, "f_{2k}"] & \cdots \arrow[r] & J/J^2 \arrow[r, "d_2"] \arrow[d, "f_2"] & I_F/JI_F \arrow[r, "d_1"] \arrow[d, "f_1"] & \mathbb{Z}G \arrow[r, "\epsilon"] \arrow[d, "id"] & \mathbb{Z} \arrow[r] \arrow[d, "id"] & 0 \\
\cdots \arrow[r] & \mathbb{Z}G \arrow[r, "g-1"] & \mathbb{Z}G \arrow[r, "N"] & \cdots \arrow[r] & \mathbb{Z}G \arrow[r, "N"] & \mathbb{Z}G \arrow[r, "g-1"] & \mathbb{Z}G \arrow[r, "\epsilon"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

commute with

$$f_1([g^i]) = \frac{\partial g^i}{\partial g} \text{ and } f_2([g^{i_1}|g^{i_2}]) = \begin{cases} g^{i_1+i_2-n} & , \text{ if } i_1+i_2 \geq n \\ 0 & , \text{ otherwise} \end{cases}$$

and there is a formula for chain maps such that

$$\begin{aligned} f_{2k}(\alpha) &= f_2([g^{i_1}|g^{i_2}]) \dots f_2([g^{i_{2k-1}}|g^{i_{2k}}]) && \text{for } \alpha \in J^k/J^{k+1} \\ f_{2k+1}(\alpha) &= f_{2k}([g^{i_1}|\dots|g^{i_{2k}}])f_1([g^{i_{2k+1}}]) && \text{for } \alpha \in J^kI_F/J^{k+1}I_F. \end{aligned}$$

Proof. We have f_3 as a base step. Assume the following equality

$$f_{2k-1}[g^{i_1}|\dots|g^{i_{2k-1}}] = f_{2k-2}([g^{i_1}|\dots|g^{i_{2k-2}}])f_1([g^{i_{2k-1}}]).$$

We need to show f_{2k} holds the statement. Take $\alpha = [g^{i_1}|\dots|g^{i_{2k}}] \in F_{2k}$.

$$\begin{tikzcd}
{[g^{i_1}|\dots|g^{i_{2k}}]} \arrow[r, "d_{2k}"] \arrow[d, "f_{2k}"] & d_{2k}([g^{i_1}|\dots|g^{i_{2k}}]) \arrow[d, "f_{2k-1}"] \\
f_{2k}([g^{i_1}|\dots|g^{i_{2k}}]) \arrow[r, "N"] & f_{2k-1}(d_{2k}([g^{i_1}|\dots|g^{i_{2k}}])).
\end{tikzcd}$$

Denote $A = f_{2k-1}(d_{2k}([g^{i_1}|\dots|g^{i_{2k}}]))$. By the definition of d_{2k}, which is given in the Proposition 3.2.2, we have

$$\begin{aligned} A &= g^{i_1}f_{2k-1}([g^{i_2}|\dots|g^{i_{2k}}]) + \textstyle\sum_{j=1}^{2k-1}(-1)^j f_{2k-1}([g^{i_1}|\dots|g^{i_j}g^{i_{j+1}}|\dots|g^{i_{2k}}]) \\ &\quad + f_{2k-1}([g^{i_1}|\dots|g^{i_{2k-1}}]) \\ &= g^{i_1}f_{2k-1}([g^{i_2}|\dots|g^{i_{2k}}]) + \textstyle\sum_{j=1}^{2k-2}(-1)^j f_{2k-1}([g^{i_1}|\dots|g^{i_j}g^{i_{j+1}}|\dots|g^{i_{2k}}]) \\ &\quad - f_{2k-1}([g^{i_1}|\dots|g^{i_{2k-1}}g^{i_{2k}}]) + f_{2k-1}([g^{i_1}|\dots|g^{i_{2k-1}}]) \end{aligned}$$

By using the induction assumption,

$$\begin{aligned} A &= g^{i_1}f_{2k-2}([g^{i_2}|\dots|g^{i_{2k-1}}])f_1([g^{i_{2k}}]) \\ &\quad + \textstyle\sum_{j=1}^{2k-2}(-1)^j f_{2k-2}([g^{i_1}|\dots|g^{i_j}g^{i_{j+1}}|\dots|g^{i_{2k-1}}])f_1([g^{i_{2k}}]) \\ &\quad - f_{2k-2}([g^{i_1}|\dots|g^{i_{2k-2}}])f_1([g^{i_{2k-1}}g^{i_{2k}}]) + f_{2k-2}([g^{i_1}|\dots|g^{i_{2k-2}}])f_1([g^{i_{2k-1}}]) \\ &= f_{2k-2}(d_{2k-1}[g^{i_1}|\dots|g^{i_{2k-1}}])f_1([g^{i_{2k}}]) + f_{2k-2}([g^{i_1}|\dots|g^{i_{2k-2}}])f_1([g^{i_{2k}}]) \\ &\quad - f_{2k-2}([g^{i_1}|\dots|g^{i_{2k-2}}])f_1([g^{i_{2k-1}}g^{i_{2k}}]) + f_{2k-2}([g^{i_1}|\dots|g^{i_{2k-2}}])f_1([g^{i_{2k-1}}]) \end{aligned}$$

Using the Remark 4.1.1 and the fact that diagram commutes at F_{2k-1}, we have

$$\begin{aligned}
A \ &= f_{2k-2}([g^{i_1}|\dots|g^{i_{2k-2}}])(g^{i_{2k-1}}-1)f_1([g^{i_{2k}}]) + f_{2k-2}([g^{i_1}|\dots|g^{i_{2k-2}}])f_1([g^{i_{2k}}]) \\
&\quad - f_{2k-2}([g^{i_1}|\dots|g^{i_{2k-2}}])f_1([g^{i_{2k-1}}]) + f_{2k-2}([g^{i_1}|\dots|g^{i_{2k-2}}])f_1([g^{i_{2k-1}}]) \\
&= f_{2k-2}([g^{i_1}|\dots|g^{i_{2k-2}}])\big(g^{i_{2k-1}}f_1([g^{i_{2k}}]) - f_1([g^{i_{2k-1}}g^{i_{2k}}]) + f_1([g^{i_{2k-1}}])\big), \\
&= f_{2k-2}([g^{i_1}|\dots|g^{i_{2k-2}}])g^{i_{2k-1}+i_{2k}-n}\frac{\partial(g^n)}{\partial g}.
\end{aligned}$$

Then

$$\begin{aligned}
f_{2k}([g^{i_1}|\dots|g^{i_{2k}}]) \ &= \frac{f_{2k-2}([g^{i_1}|\dots|g^{i_{2k-2}}])g^{i_{2k-1}+i_{2k}-n}\dfrac{\partial(g^n)}{\partial g}}{N} \\
&= f_{2k-2}([g^{i_1}|\dots|g^{i_{2k-2}}])f_2([g^{i_{2k-1}}|g^{i_{2k}}]),
\end{aligned}$$

which is what we want.

Now take $\alpha \in F_{2k+1} = J^k I_F / J^{k+1} I_F$. Then $\alpha = [g^{i_1}|\dots|g^{i_{2k+1}}]$.

$$\begin{tikzcd}
{[g^{i_1}|\dots|g^{i_{2k+1}}]} \arrow[r, "d_{2k+1}"] \arrow[d, "f_{2k+1}"] & d_{2k+1}([g^{i_1}|\dots|g^{i_{2k+1}}]) \arrow[d, "f_{2k}"] \\
f_{2k+1}([g^{i_1}|\dots|g^{i_{2k+1}}]) \arrow[r, "g-1"] & f_{2k}(d_{2k+1}([g^{i_1}|\dots|g^{i_{2k+1}}])).
\end{tikzcd}$$

By using the Remark 4.1.1 and the fact that the diagram commutes at F_{2k}, we have

$$f_{2k}(d_{2k+1}([g^{i_1}|\dots|g^{i_{2k+1}}])) = f_{2k}([g^{i_1}|\dots|g^{i_{2k}}])(g^{i_{2k+1}}-1)$$

Using commutativity of above diagram,

$$\begin{aligned}
f_{2k+1}([g^{i_1}|\dots|g^{i_{2k+1}}]) &= f_{2k}([g^{i_1}|\dots|g^{i_{2k}}])\frac{(g^{i_{2k+1}}-1)}{g-1} \\
&= f_{2k}([g^{i_1}|\dots|g^{i_{2k}}])f_1([g^{i_{2k+1}}]),
\end{aligned}$$

which is what we want. We showed chain maps between these two Gruenberg resolutions depend on f_1 and f_2. $\square$

4.2 The general case

A map of presentations $\langle F_1, R_1\rangle \longrightarrow \langle F_2, R_2\rangle$ is a pair of maps f_F, f_R such that the following diagram

$$\begin{tikzcd}
1 \arrow[r] & R_1 \arrow[r, "q_1"] \arrow[d, "f_R"] & F_1 \arrow[r, "\pi_1"] \arrow[d, "f_F"] & G \arrow[r] \arrow[d, "id"] & 1 \\
1 \arrow[r] & R_2 \arrow[r, "q_2"] & F_2 \arrow[r, "\pi_2"] & G \arrow[r] & 1
\end{tikzcd}$$

commutes. Let F_1,F_2 be free group with basis $X_1 = \langle x_i\rangle$ and $X_2 = \langle a_i\rangle$, respectively. Assume R_1, R_2 are generated by $Y_1 = \langle y_i\rangle$ and $Y_2 = \langle b_i\rangle$, respectively. Since above diagram commutes, we have

$$\pi_1(x_i - 1) = \pi_2(\sum_j \frac{\partial(f_F(x_i))}{\partial a_j}(a_j - 1))$$

$$\sum_j \frac{\partial(f_F(q_1(y_i - 1)))}{\partial a_j}(a_j - 1) = q_2(\sum_j \frac{\partial f_R(y_i - 1)}{\partial b_j}(b_j - 1)).$$

There exist a chain map between Gruenberg resolutions of these presentations such that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & J_1^k/J_1^{k+1} \arrow[r, "d_{2k}"] \arrow[d, "f_{2k}"] & J_1^{k-1}I_{F_1}/J_1^kI_{F_1} \arrow[r] \arrow[d, "f_{2k-1}"] & \cdots \arrow[r] & J_1/J_1^2 \arrow[r, "d_2"] \arrow[d, "f_2"] & I_{F_1}/J_1I_{F_1} \arrow[r, "d_1"] \arrow[d, "f_1"] & \mathbb{Z}G \arrow[r, "\epsilon"] \arrow[d, "id"] & \mathbb{Z} \arrow[r] \arrow[d, "id"] & 0 \\
\cdots \arrow[r] & J_2^k/J_2^{k+1} \arrow[r, "\delta_{2k}"] & J_2^{k-1}I_{F_2}/J_2^kI_{F_2} \arrow[r] & \cdots \arrow[r] & J_2/J_2^2 \arrow[r, "\delta_2"] & I_{F_2}/J_2I_{F_2} \arrow[r, "\delta_1"] & \mathbb{Z}G \arrow[r, "\epsilon"] & \mathbb{Z} \arrow[r] & 0.
\end{tikzcd}$$

Take an element $\alpha \in I_{F_1}/J_1I_{F_1}$.

$$\begin{tikzcd}
\alpha = (x_i - 1) \arrow[r, "\pi_1"] \arrow[d, "f_1"] & \pi_1(x_i - 1) \arrow[d, "id"] \\
\sum_j \lambda_j(a_j - 1) \arrow[r, "\pi_2"] & \pi_2(\sum_j \frac{\partial f_F(x_i)}{\partial a_j}(a_j - 1)).
\end{tikzcd}$$

By the commutativity of this diagram, we have $\lambda_j = \dfrac{\partial f_F(x_i)}{\partial a_j}$. Hence

$$f_1(x_i - 1) = \sum_j \frac{\partial f_F(x_i)}{\partial a_j}(a_j - 1).$$

Take an element $\alpha \in J_1/J_1^2$.

$$\begin{tikzcd}
\alpha = (y_i - 1) \arrow[r, "q_1"] \arrow[d, "f_2"] & q_1(y_i - 1) \arrow[d, "f_1"] \\
\sum_k \lambda_k (b_k - 1) \arrow[r, "q_2"] & \sum_j \dfrac{\partial f_F(q_1(y_i - 1))}{\partial a_j}(a_j - 1).
\end{tikzcd}$$

Since $\sum_j \dfrac{\partial f_F(q_1(y_i-1))}{\partial a_j}(a_j - 1) = q_2(\sum_k \dfrac{\partial f_R(y_i-1)}{\partial b_k}(b_k - 1))$, we get $\lambda_k = \dfrac{\partial f_R(y_i-1)}{\partial b_k}$ by using commutativity. Hence

$$f_2(y_i - 1) = \sum_k \frac{\partial f_R(y_i)}{\partial b_k}(b_k - 1).$$

Note that the maps f_1 and f_2 can be written directly using Fox derivative. The commutativity f_1 and f_2 comes from the maps $J_1 \to J_2$ and $I_{F_1} \to I_{F_2}$.

Take an element $\alpha \in J_1 I_{F_1}/J_1^2 I_{F_1}$.

$$\begin{tikzcd}
\alpha = (y_i - 1)(x_j - 1) \arrow[r, "d_3"] \arrow[d, "f_3"] & \sum_k u_k^{i,j}(y_k - 1) \arrow[d, "f_2"] \\
\sum_{s,t} \lambda_{s,t}^{i,j}(b_s - 1)(a_t - 1) \arrow[r, "\delta_3"] & \sum_k u_k^{i,j} \sum_l \dfrac{\partial f_R(y_k)}{\partial b_l}(b_l - 1).
\end{tikzcd}$$

Since we do not know whether group is commutative or not, we cannot say $f_2(y_i - 1) f_1(x_j - 1) = f_3(\alpha)$ such that

$$\sum_s \frac{\partial f_R(y_i)}{\partial b_s}(b_s - 1) \sum_t \frac{\partial f_F(x_j)}{\partial a_t}(a_t - 1) = \sum_s \frac{\partial f_R(y_i)}{\partial b_s} \sum_t \frac{\partial f_F(x_j)}{\partial a_t}(b_s - 1)(a_t - 1).$$

To get rid of this, we define an operation $*$, convolution, such that

$$(b_s - 1) * \sum_t \frac{\partial f_F(x_j)}{\partial a_t} = \sum_n \frac{\eth(\partial f_F(x_j))}{\partial b_n}(b_n - 1) \tag{4.1}$$

where $\eth$ denotes the noncommutative Fox derivative. Using convolution, we have

$$\begin{aligned}
f_2(y_i - 1) * f_1(x_j - 1) &= \sum_s \frac{\partial f_R(y_i)}{\partial b_s}(b_s - 1) * \sum_t \frac{\partial f_F(x_j)}{\partial a_t}(a_t - 1) && (4.2) \\
&= \sum_{n,t} \frac{\partial f_R(y_i)}{\partial b_s} \frac{\eth(\partial f_F(x_j))}{\partial b_n}(b_n - 1)(a_t - 1). && (4.3)
\end{aligned}$$

Then we can take $\lambda_{n,t}^{i,j} = \dfrac{\partial f_R(y_i)}{\partial b_s}\dfrac{\eth(\partial f_F(x_j))}{\partial b_n}$. Now we need to show this diagram commutes. We have the following equations

$$(y_i - 1)(x_j - 1) = \sum_k u_k^{i,j}(y_k - 1) \tag{4.4}$$

Taking Fox derivative of 4.4,

$$\begin{aligned}\sum_s \frac{\partial f_R(y_i)}{\partial b_s}(b_s - 1) * \sum_t \frac{\partial f_F(x_j)}{\partial a_t}(a_t - 1) &= \sum_k u_k^{i,j} \sum_l \frac{\partial f_R(y_k)}{\partial b_l}(b_l - 1) \\ &= \sum_l (\sum_k u_k^{i,j} \frac{\partial f_R(y_k)}{\partial b_l})(b_l - 1).\end{aligned}$$

By the Equations 4.2 and 4.3, we have

$$\sum_{n,t} \frac{\partial f_R(y_i)}{\partial b_s}\frac{\eth(\partial f_F(x_j))}{\partial b_n}(b_n - 1)(a_t - 1) = \sum_l (\sum_k u_k^{i,j} \frac{\partial f_R(y_k)}{\partial b_l})(b_l - 1),$$

which means this diagram commutes. Hence we can write f_3 in terms of f_1 and f_2 such that

$$f_3(\alpha) = f_2(y_i - 1) * f_1(x_j - 1) \text{ where } \alpha = (y_i - 1)(x_j - 1).$$

Theorem 4.2.1. *Given a map of presentation (f_F, f_R). There is a chain map f_*, associated to map of presentation, between their corresponding resolutions, that is, for all $n \geq 0$ the diagrams*

$$\begin{tikzcd} \cdots \arrow[r] & J_1^k/J_1^{k+1} \arrow[r, "d_{2k}"] \arrow[d, "f_{2k}"] & J_1^{k-1}I_{F_1}/J_1^k I_{F_1} \arrow[r] \arrow[d, "f_{2k-1}"] & \cdots \arrow[r] & J_1/J_1^2 \arrow[r, "d_2"] \arrow[d, "f_2"] & I_{F_1}/J_1 I_{F_1} \arrow[r, "d_1"] \arrow[d, "f_1"] & \mathbb{Z}G \arrow[r, "\epsilon"] \arrow[d, "id"] & \mathbb{Z} \arrow[r] \arrow[d, "id"] & 0 \\ \cdots \arrow[r] & J_2^k/J_2^{k+1} \arrow[r, "\delta_{2k}"] & J_2^{k-1}I_{F_2}/J_2^k I_{F_2} \arrow[r] & \cdots \arrow[r] & J_2/J_2^2 \arrow[r, "\delta_2"] & I_{F_2}/J_2 I_{F_2} \arrow[r, "\delta_1"] & \mathbb{Z}G \arrow[r, "\epsilon"] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

commute where

$$f_1(x_i - 1) = \sum_j \frac{\partial f_F(x_i)}{\partial a_j}(a_j - 1)$$

$$f_2(y_i - 1) = \sum_j \frac{\partial f_R(y_i)}{\partial b_j}(b_j - 1).$$

Furthermore, there is a formula for chain maps such that

$$\begin{aligned} f_{2k}(\alpha) &= f_2(y_{i_1} - 1) * \cdots * f_2(y_{i_k} - 1) && \textit{for } \alpha \in J^k/J^{k+1} \\ f_{2k-1}(\alpha) &= f_2(y_{i_1} - 1) * \cdots * f_2(y_{i_{k-1}} - 1) * f_1(x_{i_k} - 1) && \textit{for } \alpha \in J^{k-1}I_F/J^k I_F. \end{aligned}$$

Proof. We have f_3 as a base step. Assume the statement is true for f_{2k-2}. We need to show it is also true for f_{2k-1}. Take an element in $J_1^{k-1}I_{F_1}/J_1^kI_{F_1}$, say $\alpha = (y_{i_1}-1)\dots(y_{i_{k-1}}-1)(x_{i_k}-1)$.

$$\begin{tikzcd}
\alpha \arrow[rr, "d_{2k-1}"] \arrow[d, "f_{2k-1}"] & & \sum_j u_j^i e_j \arrow[d, "f_{2k-2}"] \\
f_{2k-1}(\alpha) \arrow[r, "\delta_{2k-1}"] & \sum_{j,l} u_j^i f_2(y_{j_1}-1) * \dots * f_2(y_{j_{k-1}}-1), &
\end{tikzcd}$$

where $e_j = (y_{j_1}-1)\dots(y_{j_{k-1}}-1)$. Take $f_{2k-1}(\alpha) = \sum \lambda_m^i \hat{e}_m$ where $\hat{e}_m = (b_{m_1}-1)\dots(b_{m_{k-1}}-1)(a_{m_k}-1)$.

$$\sum_{m,i} \frac{\partial f_R(y_{i_1})}{\partial b_{m_1}}(b_{m_1}-1)\cdots \sum_{m,i}\frac{\partial f_F(x_{i_k})}{\partial a_{m_k}}(a_{m_k}-1) = \sum_n \frac{\partial f_R(y_{i_1})}{\partial b_{m_1}} \dots \frac{\eth(\partial f_F)}{\partial b_{n_k}}\hat{e}_n, \tag{4.5}$$

where $\hat{e}_n = (b_{n_1}-1)\dots(b_{n_{k-1}}-1)(a_{m_k}-1)$. Hence

$$\lambda_n^i = \frac{\partial f_R(y_{i_1})}{\partial b_{m_1}} \dots \frac{\eth(\partial f_R)}{\partial b_{n_{k-1}}}\frac{\eth(\partial f_R)}{\partial b_{n_k}}.$$

We have the following equation

$$(y_{i_1}-1)\dots(y_{i_{k-1}}-1)(x_{i_k}-1) = \sum_j u_j^i (y_{j_1}-1)\dots(y_{j_{k-1}}-1). \tag{4.6}$$

Taking Fox derivative of 4.6,

$$\sum_{m,i} \frac{\partial f_R(y_{i_1})}{\partial b_{m_1}}(b_{m_1}-1)\cdots \sum_{m,i}\frac{\partial f_F(x_{i_k})}{\partial a_{m_k}}(a_{m_k}-1) = \sum_{l,j} u_j^i f_2(y_{j_1}-1) * \dots * f_2(y_{j_{k-1}}-1).$$

By the Equation 4.5, we have

$$\sum_n \frac{\partial f_R(y_{i_1})}{\partial b_{m_1}} \dots \frac{\eth(\partial f_F)}{\partial b_{n_k}}\hat{e}_n = \sum_{l,j} u_j^i f_2(y_{j_1}-1) * \dots * f_2(y_{j_{k-1}}-1).$$

Then we can write f_{2n-1} in terms of f_1 and f_2 such that

$$f_{2n-1}(\alpha) = f_2(y_{i_1}-1) * \dots * f_2(y_{i_{k-1}}-1) * f_1(x_{i_k}-1)$$

where $\alpha = (y_{i_1}-1)\dots(y_{i_{k-1}}-1)(x_{i_k}-1)$.

Now we need to show it is also true for f_{2k}. Take an element in J_1^k/J_1^{k+1}, say $\alpha = (y_{i_1}-1)\dots(y_{i_k}-1)$.

$$\begin{tikzcd}
\alpha \arrow[rr, "d_{2k}"] \arrow[d, "f_{2k}"] & & \sum_j u_j^i e_j \arrow[d, "f_{2k-1}"] \\
f_{2k}(\alpha) \arrow[r, "\delta_{2k}"] & \sum_{j,l} u_j^i f_2(y_{j_1}-1) * \dots * & f_2(y_{j_{k-1}}-1) * f_1(x_{j_k}-1),
\end{tikzcd}$$

where $e_j = (y_{j_1}-1)\dots(y_{j_{k-1}}-1)(x_{j_k}-1)$. Take $f_{2k}(\alpha) = \sum \lambda_m^i \hat{e}_m$ where $\hat{e}_m = (b_{m_1}-1)\dots(b_{m_{k-1}}-1)(b_{m_k}-1)$.

$$\sum_{m,i} \frac{\partial f_R(y_{i_1})}{\partial b_{m_1}}(b_{m_1}-1)\cdots\sum_{m,i}\frac{\partial f_R(y_{i_k})}{\partial b_{m_k}}(b_{m_k}-1) = \sum \frac{\partial f_R(y_{i_1})}{\partial b_{m_1}}\dots\frac{\eth(\partial f_R(y_{i_k}))}{\partial b_{n_k}}\hat{e}_n \quad (4.7)$$

where $\hat{e}_n = (b_{n_1}-1)\dots(b_{n_{k-1}}-1)(b_{m_k}-1)$. Hence

$$\lambda_n^i = \frac{\partial f_R(y_{i_1})}{\partial b_{m_1}}\dots\frac{\eth(\partial f_R)}{\partial b_{n_{k-1}}}\frac{\eth(\partial f_R)}{\partial b_{n_k}}.$$

We have the following equations

$$(y_{i_1}-1)\dots(y_{i_k}-1) = \sum_j u_j^i (y_{j_1}-1)\dots(y_{j_{k-1}}-1)(x_{j_k}-1). \qquad (4.8)$$

Taking Fox derivative of 4.8, we have

$$\sum_{m,i}\frac{\partial f_R(y_{i_1})}{\partial b_{m_1}}(b_{m_1}-1)\cdots\sum_{m,i}\frac{\partial f_R(y_{i_k})}{\partial b_{m_k}}(b_{m_k}-1) = \sum_{l,j} u_j f_2(y_{j_1}-1) * \dots * f_1(x_{j_k}-1).$$

By the Equation 4.7, we have

$$\sum_n \frac{\partial f_R(y_{i_1})}{\partial b_{m_1}}\dots\frac{\eth(\partial f_R)}{\partial b_{n_k}}\hat{e}_n = \sum_{l,j} u_j^i f_2(y_{j_1}-1) * \dots * f_1(x_{j_k}-1).$$

Then we can write f_{2n} in terms of f_1 and f_2 such that

$$f_{2n}(\alpha) = f_2(y_{i_1}-1) * \dots * f_2(y_{i_k}-1) \text{ where } \alpha = (y_{i_1}-1)\dots(y_{i_k}-1).$$

Hence given formula for chain maps is true for all f_*. $\square$

4.3 Applications

Hanke et al. [3] described a chain map from standard resolution to the resolution for the cyclic group. In their paper, R is a commutative ring with identity and $\Lambda = R[\mathbb{Z}_k]$ is group ring of $\mathbb{Z}_k$ over R. The standard resolution of R is denoted by $S_\bullet$ with modules $S_r := \Lambda \otimes_R \cdots \otimes_R \Lambda$ and particular basis which is given by

$$[h_1|h_2|\dots|h_r] := e \otimes h_1 \otimes h_1h_2 \otimes \cdots \otimes h_1h_2\dots h_r$$

where $h_1, h_2, \dots, h_r \in \mathbb{Z}_k$. The boundary is given by

$$\begin{aligned}\partial_r([h_1|h_2|\dots|h_r]) =& h_1[h_2|\dots|h_r] \\ &+ \sum_{i=1}^{r-1}(-1)^i[h_1|h_2|\dots|h_ih_{i+1}|\dots|h_r] \\ &+ (-1)^k[h_1|h_2|\dots|h_{r-1}]\end{aligned}$$

The resolution of cyclic group is denoted by $M_\bullet$ with $M_i := \Lambda$ for all $i \geq 0$ and the boundary maps are defined by

$$\partial_r = \begin{cases} m_\sigma, & \text{if } r \text{ is even} \\ m_\tau, & \text{if } r \text{ is odd,} \end{cases}$$

where m_σ denotes multiplication by $\sigma = e + g + \cdots + g^{k-1}$ and m_τ denotes multiplication by $\tau = g - e$. They defined some notion to describe a chain map $S_\bullet \to M_\bullet$. Take an element $h_0 \otimes \cdots h_{2s} \in S_{2s}$ and it can be written by bar notation such that $h_0 \otimes \cdots h_{2s} = g^{a_0}[g^{a_1}|\cdots|g^{a_{2s}}]$ where $0 \leq a_i < k$ for all $0 \leq i \leq 2s$. They call the element $h_0 \otimes \cdots h_{2s} \in S_{2s}$ strongly alternating if $g^{a_0}[g^{a_1}|\cdots|g^{a_{2s}}]$ satisfies $a_{2i+1} + a_{2i+2} \geq k$ for all $0 \leq i \leq s-1$. Take an element $h_0 \otimes \cdots h_{2s+1} \in S_{2s+1}$. This element is called strongly alternating if there is an $a \in \mathbb{Z}_k$ such that $a \otimes h_0 \otimes \cdots \otimes h_{2s+1}$ is strongly alternating. The Λ-linear maps $f_r : S_r \to \Lambda$ are given by

$$f_{2n}([h_1|\dots|h_{2n}]) = \begin{cases} 1, & \text{if } [h_1|\dots|h_{2n}] \text{ is strongly alternating} \\ 0, & otherwise \end{cases}$$

$$f_{2n+1}([h_1|\dots|h_{2n+1}]) = \sigma_i f_{2n}([h_2|\dots|h_{2n+1}]) \text{ for } h_1 = g^i,\ 0 \leq i < n. \tag{4.9}$$

Then they give a proposition:

Proposition 4.3.1. *The collection of the maps $f_r : S_r \to M_r$ is a chain map from standard resolution to the minimal resolution, i.e for all $s \geq 0$ the diagrams*

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & S_{2s+2} \arrow[r, "\partial"] \arrow[d, "f_{2s+2}"] & S_{2s+1} \arrow[r, "\partial"] \arrow[d, "f_{2s+1}"] & S_{2s} \arrow[r, "\partial"] \arrow[d, "f_{2s}"] & \cdots \\
\cdots \arrow[r, "\partial"] & M_{2s+2} \arrow[r, "\partial"] & M_{2s+1} \arrow[r, "\partial"] & M_{2s} \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

commute:

$$f_{2s}(\partial c) = \tau f_{2s+1}(c) \qquad \text{for } c \in S_{2s+1}$$

$$f_{2s+1}(\partial c) = \sigma f_{2s+2}(c) \qquad \text{for } c \in S_{2s+2}.$$

Using this explicit chain map formula, they proved the "Combinatorial Stokes Formula" which is Theorem 4.1, page 11 in [3].

Our chain map formula does not coincides with their formula in 4.9 since we work with normalized standard resolution. However $\tau f_1(c) = g^r - 1$ for $c = [g^r]$ in their paper, that is, $f_1(c) = \dfrac{\partial g^r}{\partial g}$ which is the same with our formula for f_1. The notion of strongly alternating is also the same as our condition for unnormalized f_2. Remember we find

$$f_2(y_{(g^{i_1},g^{i_2})} - 1) = \begin{cases} 1 & , \ if \ i_1 + i_2 \geq n \\ 0 & , \ otherwise. \end{cases}$$

and

$$f_2([g^{i_1}|g^{i_2}]) = \begin{cases} g^{i_1+i_2-n} & , \ if \ i_1 + i_2 \geq n \\ 0 & , \ otherwise. \end{cases}$$

Hence if we use $[g^{i_1}|g^{i_2}] = (y_{(g^{i_1},g^{i_2})} - 1)$ instead of $x_{g^{i_1}g^{i_2}}(y_{(g^{i_1},g^{i_2})} - 1)$, then our formulas coincides.

Chapter 5

Computations with Gruenberg Resolution

In this chapter, we give some computations using Gruenberg resolution. In the first section, we write differentials of the Gruenberg resolution for a given presentation of $C_2 \times C_2$. This presentation is written according to the Example 3.15 in [8], page 62. After calculating differentials, we write a chain map between Gruenberg resolutions of two given presentations of $C_2 \times C_2$. In the following section, we write differentials for S_3 this time. Then we calculate the cohomology group of S_3 in two different ways.

5.1 Gruenberg resolution for $C_2 \times C_2$

Let $G = C_2 \times C_2$. Consider F is free on $X = \{x_1, x_2\}$. We have a presentation such that

$$1 \to R \to F \to G \to 1 \tag{5.1}$$

where R is free on $Y = \{x_1^2, x_2^2, x_1^{-1}x_2^2x_1, x_2^{-1}x_1x_2x_1, x_1^{-1}x_2^{-1}x_1x_2\}$. Denote $r_1 = x_1^2$, $r_2 = x_2^2$, $r_3 = x_1^{-1}x_2^2x_1$, $r_4 = x_2^{-1}x_1x_2x_1$, $r_5 = x_1^{-1}x_2^{-1}x_1x_2$. Then using Proposition 3.1.4 and 3.1.5 we see that $x_1 - 1$ and $x_2 - 1$ are generators for I_F and

$r_1-1, r_2-1, r_3-1, r_4-1, r_5-1$ are generators for J. Consider the Gruenberg resolution

$$\cdots \to J^2/J^3 \xrightarrow{\partial_4} JI_F/J^2I_F \xrightarrow{\partial_3} J/J^2 \xrightarrow{\partial_2} I_F/JI_F \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

Using the Fox derivative, we can write the map ∂_2 from $\{r_i-1\}+J^2$ where $1 \le i \le 5$ to $\{x_1-1, x_2-1\}+JI_F$ as follows

$$\begin{aligned}
\partial_2(r_1-1) &= \partial_2(x_1^2-1) = (1+x_1)(x_1-1)\\
\partial_2(r_2-1) &= \partial_2(x_2^2-1) = (1+x_2)(x_2-1)\\
\partial_2(r_3-1) &= \partial_2(x_1^{-1}x_2^2x_1-1) = (-x_1^{-1}+x_1^{-1}x_2^2)(x_1-1)+(x_1^{-1}+x_1^{-1}x_2)(x_2-1)\\
&= x_1^{-1}\big((x_2^2-1)(x_1-1)+(1+x_2)(x_2-1)\big)\\
\partial_2(r_4-1) &= \partial_2(x_2^{-1}x_1x_2x_1-1) = (x_2^{-1}+x_2^{-1}x_1x_2)(x_1-1)+(-x_2^{-1}+x_2^{-1}x_1)(x_2-1)\\
&= x_2^{-1}\big((1+x_1x_2)(x_1-1)+(x_1-1)(x_2-1)\big)\\
\partial_2(r_5-1) &= \partial_2(x_1^{-1}x_2^{-1}x_1x_2-1) = x_1^{-1}x_2^{-1}\big((1-x_2)(x_1-1)+(x_1-1)(x_2-1)\big)
\end{aligned}$$

Using this map and relations, we can get the map ∂_3 from $(r_i-1)(x_j-1)+J^2I_F$ to $(r_i-1)+J^2$ where $1 \le i \le 5$ and $1 \le j \le 2$ in the following way

$$\begin{aligned}
\partial_3((r_1-1)(x_1-1)) &= (x_1-1)(r_1-1)\\
\partial_3((r_1-1)(x_2-1)) &= x_2(x_2^{-1}x_1x_2x_1(x_1^{-1}x_2^{-1}x_1x_2-1))+x_2(x_2^{-1}x_1x_2x_1-1)-(r_1-1)\\
&= x_2(r_4(r_5-1)+(r_4-1))-(r_1-1)\\
\partial_3((r_2-1)(x_1-1)) &= x_1(r_3-1)-(r_2-1)\\
\partial_3((r_2-1)(x_2-1)) &= (x_2-1)(r_2-1)\\
\partial_3((r_3-1)(x_1-1)) &= x_1^{-1}\big((x_2^2-1)(r_1-1)+(r_2-1)\big)-(r_3-1)\\
\partial_3((r_3-1)(x_2-1)) &= x_2(x_2^{-1}x_1^{-1}x_2^3x_1(x_1^{-1}x_2^{-1}x_1x_2-1)+x_2^{-1}x_1^{-1}x_2x_1(x_1^{-1}x_2^2x_1-1)\\
&\quad -x_2^{-1}x_1^{-1}x_2x_1(x_1^{-1}x_2^{-1}x_1x_2-1))-(r_3-1)\\
&= x_2(r_5^{-1}r_3(r_5-1)+r_5^{-1}(r_3-1)-r_5^{-1}(r_5-1))-(r_3-1)\\
\partial_3((r_4-1)(x_1-1)) &= (x_2^{-1}+x_2^{-1}x_1x_2+x_2^{-1}x_1x_2x_1-1)(x_1-1)\\
&\quad +(-x_2^{-1}+x_2^{-1}x_1)(x_2-1)-(r_4-1)\\
&= x_1(x_1^{-1}x_2^{-1}x_1x_2(r_1-1)+(r_5-1))-(r_4-1)
\end{aligned}$$

$$
\begin{aligned}
\partial_3((r_4-1)(x_2-1)) &= x_2(x_2^{-2}x_1^2x_1^{-1}x_2^2x_1(x_1^{-1}x_2^{-1}x_1x_2-1)+x_2^{-2}x_1^2(x_1^{-1}x_2^2x_1-1)) \\
&\quad + x_2(x_2^{-2}(x_1^2-1)-x_2^{-2}(x_2^2-1))-(r_4-1) \\
&= x_2(r_2^{-1}r_1r_3(r_5-1)+r_2^{-1}r_1(r_3-1)+r_2^{-1}(r_1-1) \\
&\quad - r_2^{-1}(r_2-1))-(r_4-1) \\
\partial_3((r_5-1)(x_1-1)) &= x_1(x_1^{-2}(r_4-1)-x_1^{-2}(r_1-1))-(r_5-1) \\
\partial_3((r_5-1)(x_2-1)) &= x_2(x_2^{-1}x_1^{-1}x_2x_1x_1^{-1}x_2^{-2}x_1(x_2^2-1)) \\
&\quad - x_2(x_2^{-1}x_1^{-1}x_2x_1x_1^{-1}x_2^{-2}x_1x_1^{-1}x_2^{-2}x_1(x_1^{-1}x_2^2x_1-1)) \\
&\quad - x_2(x_2^{-1}x_1^{-1}x_2x_1(x_1^{-1}x_2^{-1}x_1x_2-1))-(r_5-1) \\
&= x_2(r_5^{-1}r_3^{-1}(r_2-1)-r_5^{-1}r_3^{-1}(r_3-1)-r_5^{-1}(r_5-1))-(r_5-1)
\end{aligned}
$$

With using this differentials, we can write down the higher differentials for Gruenberg resolution for $C_2 \times C_2$.

5.2 A chain map between Gruenberg resolutions of $C_2 \times C_2$

In this time, we write a chain map between the Gruenberg resolution of standard presentation and the Gruenberg resolution of $C_2 \times C_2$.
Let $1 \to R_1 \to F_1 \to C_2 \times C_2 \to 1$ be standard presentation of $C_2 \times C_2$. Take another presentation of $C_2 \times C_2$ which is stated 5.1. Then we have a commutative diagram:

$$
\begin{tikzcd}
1 \arrow[r] & R_1 \arrow[r, "q_1"] \arrow[d, "f_R"] & F_1 \arrow[r, "\pi_1"] \arrow[d, "f_F"] & C_2 \times C_2 \arrow[r] \arrow[d, "id"] & 1 \\
1 \arrow[r] & R \arrow[r, "q_2"] & F \arrow[r, "\pi_2"] & C_2 \times C_2 \arrow[r] & 1.
\end{tikzcd}
$$

Then we have

$$
\begin{aligned}
f_F(x_a) &= x_1\\
f_F(x_b) &= x_2\\
f_F(x_{ab}) &= f_F(x_{ba}) = x_2x_1\\
f_R(y_{(a,a)}) &= r_1\\
f_R(y_{(a,b)}) &= r_5\\
f_R(y_{(a,ab)}) &= f_R(y_{(a,ba)}) = r_4\\
f_R(y_{(b,b)}) &= r_2\\
f_R(y_{(b,a)}) &= 1\\
f_R(y_{(b,ab)}) &= f_R(y_{(b,ba)}) = r_3.
\end{aligned}
$$

By using Gruenberg resolutions of these presentations, there exist a chain map such that the following diagram commutes:

$$
\begin{tikzcd}
\cdots \arrow[r] & J_1^2/J_1^3 \arrow[r, "d_4"] \arrow[d, "f_4"] & J_1I_{F_1}/J_1^2I_{F_1} \arrow[r, "d_3"] \arrow[d, "f_3"] & J_1/J_1^2 \arrow[r, "d_2"] \arrow[d, "f_2"] & I_{F_1}/J_1I_{F_1} \arrow[r, "d_1"] \arrow[d, "f_1"] & \mathbb{Z}G \arrow[r, "\epsilon"] \arrow[d, "id"] & \mathbb{Z} \arrow[r] \arrow[d, "id"] & 0\\
\cdots \arrow[r] & J_2^2/J_2^3 \arrow[r, "\partial_4"] & J_2I_{F_2}/J_2^2I_{F_2} \arrow[r, "\partial_3"] & J_2/J_2^2 \arrow[r, "\partial_2"] & I_{F_2}/J_2I_{F_2} \arrow[r, "\partial_1"] & \mathbb{Z}G \arrow[r, "\epsilon"] & \mathbb{Z} \arrow[r] & 0.
\end{tikzcd}
$$

Then we have

$$
\begin{aligned}
f_1(x_a - 1) &= x_1 - 1\\
f_1(x_b - 1) &= x_2 - 1\\
f_1(x_{ab} - 1) &= x_2(x_1 - 1) + (x_2 - 1).
\end{aligned}
$$

Hence it is equal to what we defined in Chapter 4.2 such that

$$f_1(x_g - 1) = \sum_i \frac{\partial f_F(x_g - 1)}{\partial x_i}(x_i - 1).$$

We also get

$$
\begin{aligned}
f_2(y_{(a,a)} - 1) &= (r_1 - 1)\\
f_2(y_{(a,b)} - 1) &= (r_5 - 1)\\
f_2(y_{(a,ab)} - 1) &= f_2(y_{(a,ba)} - 1) = (r_4 - 1)\\
f_2(y_{(b,b)} - 1) &= (r_2 - 1)\\
f_2(y_{(b,a)} - 1) &= 0\\
f_2(y_{(b,ab)} - 1) &= f_2(y_{(b,ba)} - 1) = (r_3 - 1).
\end{aligned}
$$

Then again it is equal to what we defined in Chapter 4.2 such that

$$f_2(y_{(g,h)} - 1) = \sum_i \frac{\partial f_R(y_{(g,h)})}{\partial y_i}(y_i - 1).$$

Take an element $\alpha \in J_1 I_{F_1}/J_1^2 I_{F_1}$. We define earlier f_3 such that $f_3(\alpha) = f_2(y_{(g,h)} - 1) * f_1(x_g - 1)$ where $\alpha = (y_{(g,h)} - 1)(x_g - 1)$. This convolution is equal to our differentials which is calculated above, that is,

$$f_3(\alpha) = \partial_3(\sum_{i,j} \frac{\partial f_R(y_{(g,h)} - 1)}{\partial r_i} \frac{\partial f_F(x_g - 1)}{\partial x_j}).$$

1. Take an element $\alpha = (y_{a,a} - 1)(x_a - 1)$.

$$\begin{tikzcd}
\alpha = (y_{a,a} - 1)(x_a - 1) \arrow[r] \arrow[d, "f_3"] & x_a(y_{(a,a)} - 1) - (y_{(a,a)} - 1) \arrow[d, "f_2"] \\
\sum_{i,j} \frac{\partial f_R(y_{(a,a)})}{\partial r_i}(r_i - 1)\frac{\partial f_F(x_a)}{\partial x_j}(x_j - 1) \arrow[r, "\partial_3"] & x_1(r_1 - 1) - (r_1 - 1).
\end{tikzcd}$$

Since $\frac{\partial f_R(y_{(a,a)})}{\partial r_i} = (r_1 - 1)$ and $\frac{\partial f_F(x_a)}{\partial x_j} = (x_1 - 1)$, $\partial_3((r_1 - 1)(x_1 - 1)) = (x_1 - 1)(r_1 - 1)$. Hence this diagram commutes.
2. Take an element $\alpha = (y_{a,a} - 1)(x_b - 1)$.

$$\begin{tikzcd}
\alpha = (y_{a,a} - 1)(x_b - 1) \arrow[r] \arrow[d, "f_3"] & x_a x_{ab}(y_{(a,b)} - 1) + x_b(y_{a,ab} - 1) - (y_{(a,a)} - 1) \arrow[d, "f_2"] \\
f_2(y_{(a,a)} - 1) * f_1(x_b - 1) \arrow[r, "\partial_3"] & x_1 x_2 x_1(r_5 - 1) + x_2(r_4 - 1) - (r_1 - 1).
\end{tikzcd}$$

Since $f_2(y_{(a,a)} - 1) * f_1(x_b - 1) = (r_1 - 1) * (x_2 - 1)$,

$$\begin{aligned}
\partial_3((r_1 - 1)(x_2 - 1)) &= x_2(r_4(r_5 - 1) + (r_4 - 1)) - (r_1 - 1) \\
&= x_1 x_2 x_1(r_5 - 1) + x_2(r_4 - 1) - (r_1 - 1).
\end{aligned}$$

Hence this diagram commutes.
3. Take an element $\alpha = (y_{a,a} - 1)(x_{ab} - 1)$.

$$\begin{tikzcd}
\alpha = (y_{a,a} - 1)(x_{ab} - 1) \arrow[r] \arrow[d, "f_3"] & x_a x_b(y_{(a,ab)} - 1) + x_{ab}(y_{(a,b)} - 1) - (y_{(a,a)} - 1) \arrow[d, "f_2"] \\
f_2(y_{(a,a)} - 1) f_1(x_{ab} - 1) \arrow[r, "\partial_3"] & x_1 x_2(r_4 - 1) + x_2 x_1(r_5 - 1) - (r_1 - 1).
\end{tikzcd}$$

Since $f_2(y_{(a,a)}-1)*f_1(x_{ab}-1)=(r_1-1)*(x_2(x_1-1)+(x_1-1))$,

$$\begin{aligned}
\partial_3((r_1-1)(x_2(x_1-1)+(x_2-1))) &= x_2(r_4(r_5-1)+(r_4-1))(x_1-1) \\
&\quad + x_2(r_4(r_5-1)+(r_4-1))-(r_1-1) \\
&= x_2r_4(x_1^{-1}((r_4-1)-(r_1-1))-(r_5-1) \\
&\quad + x_2(x_1(r_5-1)+x_2^{-1}x_1x_2(r_1-1)-(r_4-1)) \\
&\quad + x_2r_4(r_5-1)+x_2(r_4-1)-(r_1-1) \\
&= x_1x_2(r_4-1)-x_1x_2(r_1-1)-x_1x_2x_1(r_5-1) \\
&\quad + x_2x_1(r_5-1)+x_1x_2(r_1-1) \\
&\quad - x_2(r_4-1)+x_1x_2x_1(r_5-1) \\
&\quad + x_2(r_4-1)-(r_1-1) \\
&= x_1x_2(r_4-1)+x_2x_1(r_5-1)-(r_1-1).
\end{aligned}$$

Hence this diagram commutes.

4. Take an element $\alpha=(y_{b,b}-1)(x_a-1)$.

$$\begin{tikzcd}
\alpha=(y_{(b,b)}-1)(x_a-1) \arrow[r] \arrow[d, "f_3"] & x_ax_{ba}(y_{(b,a)}-1)+x_a(y_{(b,ba)}-1)-(y_{(b,b)}-1) \arrow[d, "f_2"] \\
f_2(y_{(b,b)}-1)*f_1(x_a-1) \arrow[r, "\partial_3"] & x_2x_1x_2(0)+x_1(r_3-1)-(r_2-1).
\end{tikzcd}$$

Since $f_2(y_{(b,b)}-1)*f_1(x_a-1) = (r_2-1)*(x_1-1)$, $\partial_3((r_2-1)(x_1-1)) = x_1(r_3-1)-(r_2-1)$. Hence this diagram commutes.

5. Take an element $\alpha=(y_{b,b}-1)(x_b-1)$.

$$\begin{tikzcd}
\alpha=(y_{(b,b)}-1)(x_b-1) \arrow[r] \arrow[d, "f_3"] & x_b(y_{(b,b)}-1)-(y_{(b,b)}-1) \arrow[d, "f_2"] \\
f_2(y_{(b,b)}-1)*f_1(x_b-1) \arrow[r, "\partial_3"] & x_2(r_2-1)-(r_2-1).
\end{tikzcd}$$

Since $f_2(y_{(b,b)}-1)*f_1(x_b-1) = (r_2-1)*(x_2-1)$, $\partial_3((r_2-1)(x_2-1)) = (x_2-1)(r_2-1)$. Hence this diagram commutes.

6. Take an element $\alpha=(y_{b,b}-1)(x_{ab}-1)$.

$$\begin{tikzcd}
\alpha=(y_{(b,b)}-1)(x_{ab}-1) \arrow[r] \arrow[d, "f_3"] & x_bx_a(y_{(b,ab)}-1)+x_{ba}(y_{(b,a)}-1)-(y_{(b,b)}-1) \arrow[d, "f_2"] \\
f_2(y_{(b,b)}-1)*f_1(x_{ab}-1) \arrow[r, "\partial_3"] & x_2x_1(r_3-1)-x_2x_1(0)-(r_2-1).
\end{tikzcd}$$

Since $f_2(y_{(b,b)} - 1) * f_1(x_{ab} - 1) = (r_2 - 1) * (x_2(x_1 - 1) + (x_2 - 1))$,

$$\begin{aligned}\partial_3((r_2-1)(x_2(x_1-1)+(x_1-1)) &= x_2(r_2-1)(x_1-1)+(r_2-1)(x_2-1)\\ &= x_2(x_1(r_3-1)-(r_2-1))+(x_2-1)(r_2-1)\\ &= x_2x_1(r_3-1)-(r_2-1).\end{aligned}$$

Hence this diagram commutes.

7. Take an element $\alpha = (y_{a,b} - 1)(x_a - 1)$.

$$\begin{tikzcd} \alpha = (y_{(a,b)}-1)(x_a-1) \arrow[r] \arrow[d, "f_3"] & d_3(\alpha) \arrow[d, "f_2"] \\ f_2(y_{(a,b)}-1) * f_1(x_a-1) \arrow[r, "\partial_3"] & f_2(d_3(\alpha)). \end{tikzcd}$$

$(y_{(a,b)} - 1)(x_a - 1) = x_{ab}^{-1}x_ax_{ba}(y_{(b,a)} - 1) - x_{ab}^{-1}x_b(y_{(ab,a)} - 1) + x_{ab}^{-1}x_b(y_{(a,ba)} - 1) - (y_{(a,b)} - 1)$ By the definition of f_2, we have

$$\begin{aligned}d_3(\alpha) &= x_1^{-1}x_2^{-1}x_1x_2x_1(0) - x_1^{-1}x_2^{-1}x_2(r_1-1) + x_1^{-1}x_2^{-1}x_2(r_4-1) - (r_5-1)\\ &= -x_1^{-1}(r_1-1) + x_1^{-1}(r_4-1) - (r_5-1)\end{aligned}$$

Since $f_2(y_{(a,b)} - 1) * f_1(x_a - 1) = (r_5 - 1) * (x_1 - 1)$, $\partial_3((r_5 - 1)(x_1 - 1)) = x_1^{-1}(r_4 - 1) - x_1^{-1}(r_1 - 1) - (r_5 - 1)$. Hence this diagram commutes.

Similarly, we can write f_3 in terms of f_2 and f_1 for any other elements in $J_1I_{F_1}/J_1^2I_{F_1}$. For cyclic group, we write chain maps in terms of f_2 and f_1 without any convolution because in this case generator and relation commute. Although $C_2 \times C_2$ is commutative group, the chain map of $C_2 \times C_2$ is not commutative because generators and relations does not commute. Hence we need the convolution in order to write explicitly.

5.3 Gruenberg resolution for S_3

Let $G = S_3$. Consider F is free on $X = \{x, y\}$. We have a presentation such that

$$1 \to R \to F \to G \to 1$$

where R is free on $Y = \{x^2, y^3, xyxy, x^{-1}y^{-1}xy^2, yxyx, y^{-1}xy^{-1}x, x^{-1}y^3x, y^{-1}x^{-1}y^{-1}x\}$. Denote $r_1 = x^2$, $r_2 = y^3$, $r_3 = xyxy$, $r_4 = x^{-1}y^{-1}xy^2$, $r_5 = yxyx$, $r_6 = y^{-1}xy^{-1}x$, $r_7 = x^{-1}y^3x$, $r_8 = y^{-1}x^{-1}y^{-1}x$. Then using Proposition 3.1.4 and 3.1.5 we have $x-1$ and $y-1$ are generators for I_F and $r_1-1, r_2-1, r_3-1, r_4-1, r_5-1, r_6-1, r_7-1, r_8-1$ are generators for J. Consider

$$\cdots \to J^2/J^3 \xrightarrow{d_4} JI_F/J^2I_F \xrightarrow{d_3} J/J^2 \xrightarrow{d_2} I_F/JI_F \xrightarrow{d_1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

Using Fox derivative, we can write the map d_2 from $\{r_i - 1\} + J^2$ where $1 \leq i \leq 8$ to $\{x-1, y-1\} + JI_F$ with abusing notation in the following way

$$\begin{aligned}
d_2(r_1 - 1) &= d_2(x^2 - 1) = (1+x)(x-1)\\
d_2(r_2 - 1) &= d_2(y^3 - 1) = (1+y+y^2)(y-1)\\
d_2(r_3 - 1) &= d_2(xyxy - 1) = (1+xy)(x-1) + (x+xyx)(y-1)\\
d_2(r_4 - 1) &= d_2(x^{-1}y^{-1}xy^2 - 1)\\
&= x^{-1}(y^{-1}-1)(x-1) + (-x^{-1}y^{-1} + x^{-1}y^{-1}x + x^{-1}y^{-1}xy)(y-1)\\
d_2(r_5 - 1) &= d_2(yxyx - 1) = y(1+xy)(x-1) + (1+yx)(y-1)\\
d_2(r_6 - 1) &= d_2(y^{-1}xy^{-1}x - 1) = y^{-1}(1+xy^{-1})(x-1) - y^{-1}(1+xy^{-1})(y-1)\\
d_2(r_7 - 1) &= d_2(x^{-1}y^3x - 1) = x^{-1}(-1+y^3)(x-1) + x^{-1}(1+y+y^2)(y-1)\\
d_2(r_8 - 1) &= d_2(y^{-1}x^{-1}y^{-1}x - 1) = y^{-1}x^{-1}(y^{-1}-1)(x-1) - y^{-1}(1+x^{-1}y^{-1})(y-1).
\end{aligned}$$

Using relations, we can get the map d_3 from $(r_i - 1)(x-1) + J^2I_F$ and $(r_i-1)(y-1) + J^2I_F$ to $(r_i - 1) + J^2$ where $1 \leq i \leq 8$ with abusing notation

$$\begin{aligned}
d_3((r_1-1)(x-1)) &= (x-1)(r_1-1)\\
d_3((r_1-1)(y-1)) &= y[r_6r_1^{-1}(r_3-1) - r_6r_1^{-1}(r_1-1) + (r_6-1)] - (r_1-1)\\
d_3((r_2-1)(x-1)) &= x(r_7-1) - (r_2-1)\\
d_3((r_2-1)(y-1)) &= (y-1)(r_2-1)
\end{aligned}$$

$$
\begin{aligned}
d_3((r_3-1)(x-1)) &= x(r_5-1)-(r_3-1)\\
d_3((r_3-1)(y-1)) &= y[r_6r_7(r_4-1)+r_6(r_7-1)+(r_6-1)]-(r_3-1)\\
d_3((r_4-1)(x-1)) &= x[r_1^{-1}r_6(r_7-1)+r_1^{-1}(r_6-1)-r_1^{-1}(r_1-1)]-(r_4-1)\\
d_3((r_4-1)(y-1)) &= y[r_8(r_2-1)+(r_8-1)]-(r_4-1)\\
d_3((r_5-1)(x-1)) &= x[r_1^{-1}r_3(r_1-1)+r_1^{-1}(r_3-1)-r_1^{-1}(r_1-1)]-(r_5-1)\\
d_3((r_5-1)(y-1)) &= y(r_3-1)-(r_5-1)\\
d_3((r_6-1)(x-1)) &= x[r_4r_2^{-1}(r_1-1)-r_4r_2^{-1}(r_2-1)+(r_4-1)]-(r_6-1)\\
d_3((r_6-1)(y-1)) &= y[r_2^{-1}r_5r_7^{-1}r_1^{-1}(r_3-1)-r_2^{-1}r_5r_7^{-1}r_1^{-1}(r_1-1)]\\
&\quad +y[-r_2^{-1}r_5r_7^{-1}(r_7-1)+r_2^{-1}(r_5-1)-r_2^{-1}(r_2-1)]-(r_6-1)\\
d_3((r_7-1)(x-1)) &= x[r_1^{-1}r_2(r_1-1)+r_1^{-1}(r_2-1)-r_1^{-1}(r_1-1)]-(r_7-1)\\
d_3((r_7-1)(y-1)) &= y[r_8r_7r_1^{-1}(r_3-1)-r_8r_7r_1^{-1}(r_1-1)+r_8(r_7-1)+(r_8-1)]\\
&\quad -(r_7-1)\\
d_3((r_8-1)(x-1)) &= x[r_5^{-1}(r_1-1)-r_5^{-1}(r_5-1)]-(r_8-1)\\
d_3((r_8-1)(y-1)) &= y[r_4^{-1}r_7^{-1}r_1^{-1}(r_3-1)-r_4^{-1}r_7^{-1}r_1^{-1}(r_1-1)-r_4^{-1}r_7^{-1}(r_7-1)]\\
&\quad +y[-r_4^{-1}(r_4-1)]-(r_8-1).
\end{aligned}
$$

With using this differentials, we can write down the higher differentials for Gruenberg resolution for S_3.

5.4 Cohomology groups of S_3

In [11], Swan constructed a resolution for S_3. In this section we examine this construction. Then we calculate cohomology groups of S_3. After this, we will again calculate the cohomology groups of S_3 by using Gruenberg resolution. First we write the presentation of S_3 in the following way:

$$S_3 = \langle x, y | x^2 = 1, xyx = y^2 \rangle.$$

The relation $y^3 = 1$ can be omitted since

$$y = x^2yx^2 = xxyxx = xy^2x = xyxxyx = y^2y^2 = y^4.$$

Then we have resolution such that

$$0 \to \mathbb{Z} \xrightarrow{d_4} C_3 \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where $C_0 = \mathbb{Z}G$ has a single generator with $\epsilon(g) = 1$, C_1 is generated by $x - 1$ and $y - 1$, C_2 is generated by $x^2 - 1$ and $xyx - y^2$ and C_3 is generated by e.

$$\begin{tikzcd}
& & \mathbb{Z}G \arrow[r, "x+1"] & \mathbb{Z}G \arrow[dr, "x-1"] & & & \\
\mathbb{Z}G \arrow[r] & \mathbb{Z}G \arrow[ur, "x-1"] \arrow[dr, "y(x+1)(y-1)"'] & & & \mathbb{Z}G \arrow[r, "\epsilon"] & \mathbb{Z} \arrow[r] & 0. \\
& & \mathbb{Z}G \arrow[r, "x-1-y"] \arrow[uur, "1+xy"] & \mathbb{Z}G \arrow[ur, "y-1"] & & &
\end{tikzcd}$$

Apply $\mathrm{Hom}_{\mathbb{Z}G}(-, \mathbb{Z})$, then we have

$$\begin{tikzcd}
& & \mathbb{Z} \arrow[r, "2"] \arrow[ddr, "2"] & \mathbb{Z} \arrow[dr, "0"] & & & \\
0 \arrow[r] & \mathbb{Z} \arrow[ur, "0"] \arrow[dr, "0"] & & & \mathbb{Z} \arrow[r, "|G|"] & \mathbb{Z} \arrow[ur] \arrow[dr] & \cdots \\
& & \mathbb{Z} \arrow[r, "1"] & \mathbb{Z} \arrow[ur, "0"] & & &
\end{tikzcd}$$

$$\begin{aligned}
H^0(G, \mathbb{Z}) &= \ker d_1 = \mathbb{Z} \\
H^1(G, \mathbb{Z}) &= \frac{\ker d_2}{\mathrm{Im}\, d_1} = 0 \\
H^2(G, \mathbb{Z}) &= \frac{\ker d_3}{\mathrm{Im}\, d_2} = \mathbb{Z}/2 \\
H^3(G, \mathbb{Z}) &= \frac{\ker d_4}{\mathrm{Im}\, d_3} = 0 \\
H^4(G, \mathbb{Z}) &= \frac{\ker d_5}{\mathrm{Im}\, d_4} = \mathbb{Z}/|G| = \mathbb{Z}/6.
\end{aligned}$$

Since it is 4-periodic, we have

$$H^n(G, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & n = 0 \\ \mathbb{Z}/2, & n = 2 \mod (4) \\ \mathbb{Z}/6, & n = 0 \mod (4) \\ 0, & n \text{ is odd.} \end{cases}$$

In the Chapter 5.3 we calculate differentials of the Gruenberg resolution of S_3

$$\cdots \to J^2/J^3 \xrightarrow{d_4} JI_F/J^2I_F \xrightarrow{d_3} J/J^2 \xrightarrow{d_2} I_F/JI_F \xrightarrow{d_1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

Then

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d^1} \mathbb{Z}^2 \xrightarrow{d^2} \mathbb{Z}^8 \xrightarrow{d^3} \mathbb{Z}^{16} \xrightarrow{d^4} \mathbb{Z}^{64} \xrightarrow{d^5} \cdots$$

obtained by applying $\mathrm{Hom}_{\mathbb{Z}G}(-, \mathbb{Z})$. By our calculations in the Chapter 5.3, we have following

$$d^1 = 0,$$

$$d^2 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 2 & 2 \\ 0 & 1 \\ 2 & 2 \\ 2 & -2 \\ 0 & 3 \\ 0 & -2 \end{bmatrix}$$

$$d^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & -1 & 1 & 0 & 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ -1 & 0 & 1 & -1 & 0 & 0 & -1 & -1 \end{bmatrix}$$

Then denote $\hat{d^3}$ by its reduced row echelon form.

$$\hat{d^3} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 3/2 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3/2 \end{bmatrix}$$

Then

$$\begin{aligned} \ker d_1 &= \mathbb{Z} \\ \operatorname{Im} d_1 &= 0 \\ \ker d_2 &= 0 \\ \operatorname{Im} d_2 &= \mathbb{Z} \oplus \mathbb{Z} \\ \ker d_3 &= \mathbb{Z}. \end{aligned}$$

Hence

$$\begin{aligned} H^0(G, \mathbb{Z}) &= \ker d_1 = \mathbb{Z} \\ H^1(G, \mathbb{Z}) &= \frac{\ker d_2}{\operatorname{Im} d_1} = 0 \\ H^2(G, \mathbb{Z}) &= \frac{\ker d_3}{\operatorname{Im} d_2} = \mathbb{Z}/2. \end{aligned}$$

For higher cohomology groups can be calculated with using higher differentials.

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