GRAVITATIONAL WAVES FROM BINARY BLACK HOLE SYSTEMS IN INSPIRALLING PHASE

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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Binary black hole systems have three phases: inspiralling, merger and ringdown phases. This thesis is a review of methods developed to study the inspiralling phase analytically. Those methods include post Minkowskian (PM) and post Newtonian (PN) expansions of the metric, which are perturbative expansions in G and $1/c$ respectively. By applying these expansions the general solution with reasonable boundary conditions is derived, and its near and far zone limits are studied. This solution is observed to be a multipolar expansion, consisting of multipole moments that cannot be calculated directly. By using PN and PM expansions together, those multipole moments are calculated. At last, the metric is expanded to a 1 PN order and equations of motion for a binary black hole system in inspiralling phase is calculated to 1 PN order.

Keywords: General Relativity, Post Newtonian Expansion, Post Minkowskian Expansion, Inspiralling Binary Black Holes.

ÖZET

SARMALLANAN KARADELİK ÇİFTLERİNDEN GELEN KÜTLEÇEKİM DALGALARI

Efekan Kökcü
Fizik, Yüksek Lisans
Tez Danışmanı: Metin Gürses
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Karadelik çiftlerinin üç evresi vardır: sarmallanma, birleşme ve dengelenme evreleri. Bu tez, sarmallanma evresinin incelenmesi için yapılan analitik yaklaşımların bir derlemesidir. Bu yaklaşımlar, metriğin Minkowski ve Newton ötesi açılımları gibi pertürbatif açılımları içerir. Bu açılımlar sırasıyla G ve $1/c$ üzerindendir. Bu açılımların var olduğu varsayımıyla Einstein alan denklemlerinin bazı makul sınır şartları altında genel çözümü türetilmiş ve bu çözümün yakın ve uzak bölgelerdeki davranışı çalışılmıştır. Gözlenmiştir ki bu genel çözüm, bir dizi doğrudan hesaplanamayan çok kutup momenti tarafından temsil edilebilmektedir. Bu çokkutup momentleri hem Minkowski ötesi hem de Newton ötesi yaklaşımların aynı anda kullanılması sayesinde hesaplanabilmiştir. En son bölümde metrik 1 Newton ötesi seviyesine kadar açılmış ve sarmallanan karadelik çiftlerinin hareket denklemleri yine aynı 1 Newton ötesi seviyesine kadar hesaplanmıştır.

Anahtar sözcükler: Genel Görelilik, Newton Ötesi Açılım, Minkowski Ötesi Açılım, Sarmallanan Karadelik Çiftleri.

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Chapter 1

Introduction

Our knowledge about the farther places in our universe consists only of astronomical observations. For a long time, most of the astronomical observations have ever been made are made via one agent, the electromagnetic waves. By using them we have observed the expansion of the universe, galaxies, many star systems, and many other objects that emit electromagnetic radiation or interact with the electromagnetic field. However, there may be many other types of events occurring in our universe that do not have any electromagnetic trace, therefore we were blind to those events. This situation is changed after the first direct observation of gravitational waves by *Laser Interferometer Gravitational-Wave Observatory* (LIGO) collaboration. In fact, the first observation is a binary black hole collision, which does not have any charged matter involved, and therefore did not generate electromagnetic radiation.

Gravitational waves were predicted by Einstein almost a century before they were observed by LIGO, in [1, 2]. Their existence can be traced back to the principle that states nothing can move faster than the speed of light. If you consider a field generated by its source, and this source is changing, then its effect cannot be instantaneous, it should move in space. The metric field, as will be defined in chapter 2, is the main concern of General Relativity (GR), and assumed to be determined by energy-momentum densities. As the name stands, GR is a relativistic theory following the principle above. Therefore, it is logical to expect that gravitational waves would be produced for certain changes in the source. For electromagnetism, for an arbitrary change in the charge and current distribution, the generated electromagnetic wave can easily be calculated. However, it is not the case in GR due to its extremely nonlinear structure of field equations called Einstein Field Equations (EFE).

Nonlinear structure of GR makes it almost impossible to find an exact solution to EFE for a given source distribution unless the distribution is assumed to have symmetries such as spherical or cylindrical symmetry. Binary black hole merger event includes two point particles moving around, and does not enjoy any such symmetry. Therefore, Einstein made a perturbative approach in both of his papers [1, 2], and assumed an almost flat space-time metric

$$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}, \tag{1.1}$$

where $\eta_{\mu\nu}$ is the metric of the Minkowskian (flat) spacetime and $\gamma_{\mu\nu}$ is the small perturbation. Plugging this in EFE, he found a very nice wave equation. To linear order in $\gamma_{\mu\nu}$, the calculations have shown that at large distances, the leading order term of $\gamma_{\mu\nu}$ is like

$$\gamma_{ij} \sim \frac{2G}{c^4 r}\ddot{Q}_{ij}\left(t - \frac{r}{c}\right), \tag{1.2}$$

where G is the Newton gravitational constant, c is the speed of light and Q_{ij} is the Newtonian quadrupole moment. Observe that two powers of the $1/c$ are generated by the two time derivatives on the quadrupole moment. This is one of the crucial results that show the difference between gravitational and electromagnetic radiation. Electromagnetic radiation starts with the first time derivative of dipole moment. Therefore there is an additional $1/c$ factor that makes the effect of the gravitational wave much smaller than the electromagnetic wave. Besides, the gravitational interaction constant G is also much smaller than its analog in electromagnetic theory, which means it looks like a dream to observe gravitational waves directly, as Einstein also was of the opinion once. However, weakness of gravitational waves has a good aspect too: since the interaction constant G is small, this means that if a gravitational wave is produced, it will not be altered by the matter content between us and the source, therefore we can get more clean information than electromagnetic waves.

How as impossible as it was thought maybe, an indirect effect of gravitational waves from the quadrupole formula (1.2) has been measured in the observation of PSR1913 + 16, made by Hulse and Taylor [3, 4]. When the energy loss formula obtained from (1.2) is used to calculate the period change of the pulsar, it can be seen that even in the linear order, GR is within 0.2 percent range of the experimental data [5]. This result, even if it is indirect, gives a striking proof of the existence of gravitational waves, and generated a motivation to calculate the period change as accurate as possible in the late 70s [6]. Yet, there is still a desire for a direct observation the gravitational waves.

As it was stated, gravitational waves are too weak. Therefore, the experimental data is buried in white noise caused by any vibration we can think of, including the seismic vibrations of the earth. Then, to observe gravitational waves directly, the experimental data should be analyzed very carefully. To do this, we better know what the theory predicts in the first place so that we can match one of those theoretical predictions to the data by checking if the difference between the experimental data and the theoretical prediction is a white noise, as the LIGO and VIRGO collaboration does for data analysis. They have a large library of theoretical predictions or templates.

In order to find what the theory predicts, it should be determined that which approximation scheme should be chosen. For this, physical aspects of the motion of the binaries should be studied first. In binary systems of two massive objects, there are three phases that are introduced. They are called inspiralling, merger, and ringdown phases [7]. Inspiralling phase is while those two objects are spinning around each other and well separated. Merger phase is the time interval that the source is still generating gravitational waves with large amplitude, but is not well separated. The ringdown phase is the phase that the source is just slightly out of equilibrium and slowly decays into a stationary solution by radiating gravitational waves. In the case of binary black hole mergers, the inspiralling phase would be before the horizons touch, merger phase is the short interval that the horizon evolves wildly, and the ringdown phase would be the last interval which the resulting spacetime is just a perturbated Kerr black hole solution. Therefore, as it can be guessed, the inspiral and the ringdown phase can be studied via perturbation theory, and the predictions of GR can be approximated order by order. However, there can be wild non-perturbative effects occurring in the merger phase, meaning that only numerical calculations can be used to investigate the merger. The exception is the miraculous effective one body approach developed by T. Damour and A. Buonanno [8]. That approach is based on the perturbative approach that is developed for the inspiralling phase but turns the perturbation series by a perfect re-summation so that the waveform generated by it matches to the numerical results even in the merger phase. This astonishing result made it easier to calculate gravitational waveforms and enlarged the library of templates of LIGO in a considerable amount.

Let us go back to the methods that are motivated by the pulsar PSR1913+16. Since the pulsar is still in the inspiralling phase, in the calculations motivated by it, the source is considered to be a slowly moving source and perturbation on $1/c$, which is called the Post Newtonian (PN) expansion, is applied the most. However, even when calculating the metric, there are divergent terms that appear in PN expansion [9]. Therefore, another perturbative approach is developed, which is expanding not in $1/c$ but in G. This is called the Post Minkowskian (PM) expansion due to the fact that G measures the coupling to the gravity, and alters the Minkowskian spacetime into a curved spacetime. In both of these approaches, a field variable is defined, as Einstein did as in equation (1.1), but in a slightly different way:

$$h^{\mu\nu} = \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}, \tag{1.3}$$

where g is the metric determinant as it will be introduced in next chapter. Then PN and PM expansions on this field are applied as

$$h^{\mu\nu} = Gh_1^{\mu\nu} + G^2 h_2^{\mu\nu} + ..., \tag{1.4}$$

and

$$h^{\mu\nu} = \frac{1}{c^2}h_{(2)}^{\mu\nu} + \frac{1}{c^4}h_{(4)}^{\mu\nu} + ... \,. \tag{1.5}$$

Both of these expansions have their limits of validity. PM expansion is an expansion over the nonlinearity of the GR. Therefore, if these nonlinear effects are as large as linear effects, this expansion would not be an appropriate approach. Therefore, this expansion is valid only if the distance from the source is much larger than the Schwarzschild radius GM/c^2 where M is the characteristic mass of the source. PN expansion, on the other hand, is just an expansion in v/c where v is the characteristic speed of the source. This expansion fails if the source moves ultra-relativistically. But this is not the only way that PN expansion can fail. If we look at (1.2), we can see that its PN expansion would be a Taylor series around $r = 0$:

$$\gamma_{ij} \sim \frac{2G}{c^4 r}\Big(\ddot{Q}_{ij}(t) - \frac{r}{c}\dddot{Q}_{ij}(t) + \frac{r^2}{2c^2}Q^{(4)}_{ij}(t) + ...\Big), \tag{1.6}$$

therefore only valid around $r = 0$. That means, this expansion cannot be used for large distances. If we define, as it was defined in any radiative theory, the near zone as $r \ll \lambda$ and the far zone as $r \gg \lambda$ where λ is the characteristic wavelength of the radiation that the source is emitting, then we see that PN expansion can only be used in the near zone. Considering PS1913+16 and general inspiralling binaries, the matter content is packed: after a distance, there is no source. Let us define this distance as a, meaning for $r > a$, there is no energy nor momentum. In inspiralling phase, when the objects are well separated, obviously we have $a > GM/c^2$. For slowly moving sources, it can be shown that $a \ll \lambda$. Therefore we find that the PM expansion can be used for $r > a$. Distances and their scales are roughly sketched in figure 1.

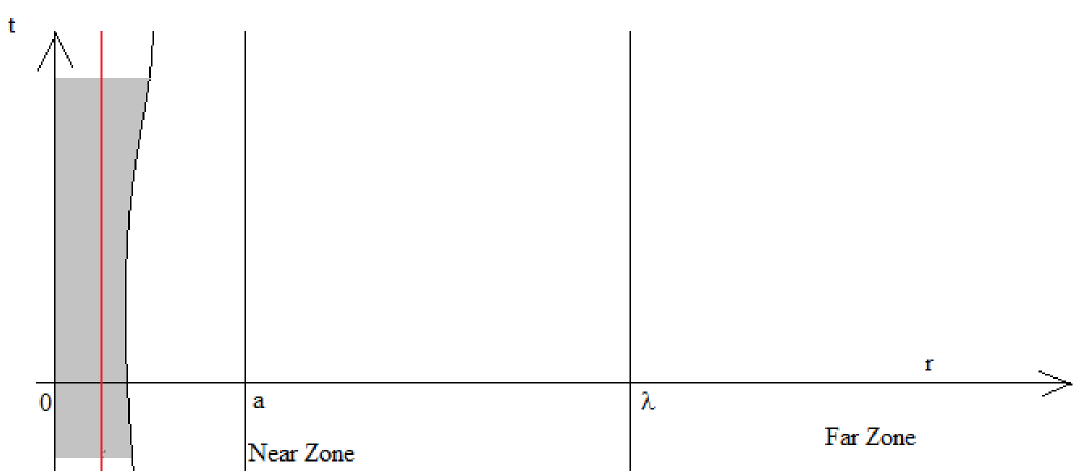

Figure 1.1: The grey area is where the matter is, the red line inside the grey area is the Schwarzschild radius GM/c^2.

This thesis is a review of the PN and PM methods. In the second chapter, a brief introduction of General Relativity will be given. Only necessary mathematical and physical aspects of the theory will be introduced, such as metric, curvature, geodesic equations and EFE. For more detail, [10, 11] can be applied. In the third chapter, mostly the calculations in [6] will be discussed. We will construct the metric in the region $r > a$ by using PM approach and study its behavior in near and far zone limits. In the fourth chapter, we will be reconstructing the metric in the far zone in such a way that PM expansion will be more accurate by adding the correction coming from the interaction of the gravitational wave and the characteristic mass of the source, as done in [12]. In the fifth chapter, we will consider the PN approach, and solve for the PN expansion of metric for a general distribution of energy and momentum. In the sixth and final chapter, we will derive the equation of motion for an inspiralling binary system and study their behavior to 1PN ($O(1/c^2)$) accuracy.

Chapter 2

A Brief Introduction to General Relativity

In this chapter, a quick summary of the theory of General Relativity will be given. For more details, see [10, 11].

2.1 Metric, Connection and Geodesics

In Einstein's theory of General Relativity (GR), the universe is assumed to be a pseudo Riemannian manifold, i.e. it admits a line element in the form

$$ds^2 = g_{\mu\nu}(x^\alpha)dx^\mu dx^\nu, \tag{2.1}$$

where $g_{\mu\nu}$ are the components of the metric tensor, dx^μ is the difference of μth coordinate between the chosen infinitely close points and ds is the interval or distance between those points. The components of the metric tensor are assumed to be sufficient many times differentiable functions of coordinates and symmetric in two indices i.e. $g_{\mu\nu} = g_{\nu\mu}$. Equation (2.1) is an extended version of the Pythagorean Theorem we have for a two-dimensional plane which can be applied to curved spaces. In this expression, as in all the expressions in this thesis, Einstein summation convention is applied. For example, $A_\mu B^\mu = A_0B^0 + A_1B^1 + A_2B^2 + A_3B^3$. We will use greek letters for spacetime indices that can take values $0, 1, 2, 3$ and Latin letters for space indices with values $1, 2, 3$.

In Riemannian manifolds, ds^2 is always positive definite, because the metric is considered as a matrix which has all of its eigenvalues to be positive. Signs of the eigenvalues of this matrix is called the signature of the metric, and Riemannian spaces have signature $(+ + ...+)$. However, pseudo Riemannian manifolds has metric with signature $(- + +...+)$ or $(+ - -...-)$. The choice between them is just a matter of convention which does not affect the essence of the calculations. Here, the former $(- + +...+)$ signature will be used.

Under a coordinate change, ds^2 is desired to remain the same since what name you give to the points should not affect the distance between them. Therefore, under a coordinate change $\overline{x}^\mu = \overline{x}^\mu(x^\alpha)$, we have $d\overline{s}^2 = ds^2$. Then since

$$d\overline{x}^\mu = \frac{\partial \overline{x}^\mu}{\partial x^\lambda} dx^\lambda, \tag{2.2}$$

we will have

$$\begin{aligned} d\overline{s}^2 &= \overline{g}_{\mu\nu}(\overline{x}^\alpha) d\overline{x}^\mu d\overline{x}^\nu \\ &= \Big\{ \overline{g}_{\mu\nu}(\overline{x}^\alpha) \frac{\partial \overline{x}^\mu}{\partial x^\lambda} \frac{\partial \overline{x}^\nu}{\partial x^\sigma} \Big\} dx^\lambda dx^\sigma \\ &= g_{\lambda\sigma}(x^\alpha) dx^\lambda dx^\sigma. \end{aligned} \tag{2.3}$$

Since dx^μ can be chosen arbitrarily, this gives us, under the given coordinate transformation

$$g_{\lambda\sigma}(x^\alpha) = \frac{\partial \overline{x}^\mu}{\partial x^\lambda} \frac{\partial \overline{x}^\nu}{\partial x^\sigma} \overline{g}_{\mu\nu}(\overline{x}^\alpha). \tag{2.4}$$

Anything that transforms by getting multiplied with the coordinate transformation matrix $\frac{\partial \overline{x}^\mu}{\partial x^\lambda}$ under the coordinate transformations, such as (2.2) or (2.4), is called a tensor. This $g_{\mu\nu}$ is called the metric tensor.

We are talking any coordinate transformations, not only the linear ones. Thus, the matrix $\frac{\partial \overline{x}^\nu}{\partial x^\sigma}$ generally depends on the position. Considering this dependence, generally, a partial derivative of a tensor will NOT be a tensor, because

$$\frac{\partial \overline{A}^\nu}{\partial \overline{x}^\mu} = \frac{\partial x^\alpha}{\partial \overline{x}^\mu} \frac{\partial}{\partial x^\alpha} \Big(\frac{\partial \overline{x}^\nu}{\partial x^\beta} A^\beta \Big) = \frac{\partial x^\alpha}{\partial \overline{x}^\mu} \frac{\partial \overline{x}^\nu}{\partial x^\beta} \frac{\partial A^\beta}{\partial x^\alpha} + \frac{\partial x^\alpha}{\partial \overline{x}^\mu} \frac{\partial}{\partial x^\alpha} \Big(\frac{\partial \overline{x}^\nu}{\partial x^\beta} \Big) A^\beta. \tag{2.5}$$

If it were a tensor, the second term that is proportional to the derivative of the matrix would not be there. If the tensor A had n more indices, there would be n more of those terms. That is why covariant derivative is defined. Covariant derivative is defined as

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\lambda} A^\lambda, \tag{2.6}$$

where $\Gamma^\nu_{\mu\lambda}$ is called connection. The connection transforms in such a way that it generates negative of those additional terms, so the covariant derivative transforms as a tensor.

In pseudo Riemann manifolds, additionally the following is assumed:

$$\nabla_\sigma g_{\mu\nu} = 0. \tag{2.7}$$

From this assumption, connection terms can be found uniquely as

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}), \tag{2.8}$$

where $g^{\mu\nu}$ is the inverse metric, simply calculated by matrix inversion, satisfying $g^{\mu\sigma} g_{\sigma\nu} = \delta^\mu_\nu$ where δ^μ_ν is the Kronecker delta symbol.

Last thing in this section to be mentioned is the geodesics. Let us have two points, A and B in our pseudo Riemanian manifold with metric tensor $g_{\mu\nu}$. Let us have a curve passing from both of these points. A geodesic between those points is the curve or curves that give the extremum for the total interval or length of itself. Which means, geodesic is a curve that solves

$$\delta I = \delta \int_A^B \mathrm{d}s = \delta \int_A^B \sqrt{g_{\mu\nu} dx^\mu dx^\nu} = 0. \tag{2.9}$$

If we apply the variation carefully, we get

$$\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{ds} \frac{dx^\sigma}{ds} = 0, \tag{2.10}$$

as the equation that a geodesic curve must satisfy. By defining the four velocity as

$$u^\mu = c \frac{dx^\mu}{ds}. \tag{2.11}$$

we can rewrite (2.10) in a more compact form

$$u^\lambda \nabla_\lambda u^\mu = 0. \tag{2.12}$$

For the future sections, let us put the geodesic equation in a simpler form. Let us define

$$u_\mu = g_{\mu\nu} u^\nu, \tag{2.13}$$

If we use the definiton of the connection (2.8), the geodesic equation takes the following form

$$\frac{du_\alpha}{ds} = \frac{u^\mu u^\nu}{2c} \partial_\alpha g_{\mu\nu} \tag{2.14}$$

which can be written with only one derivative term of the metric instead of three. terms as in (2.10).

The metric and the connection, as we have seen, carry the information about the geometry of pseudo-Riemannian manifolds. Geometry is not only distances and geodesics, there is more information that can be extracted from the metric. In the next session, we will look at it.

2.2 Curvature Tensors

If we closely look at the covariant derivative, we can easily see that they do not commute when we are applying it to a generic tensor. Let us specifically consider it for a vector. It can be shown that

$$[\nabla_\mu, \nabla_\nu] A^\lambda = R^\lambda_{\sigma\nu\mu} A^\sigma, \tag{2.15}$$

for

$$R^\lambda_{\mu\nu\sigma} = \partial_\nu \Gamma^\lambda_{\mu\sigma} - \partial_\sigma \Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\nu\alpha} \Gamma^\alpha_{\mu\sigma} - \Gamma^\lambda_{\alpha\sigma} \Gamma^\alpha_{\nu\mu}. \tag{2.16}$$

Those $R^\lambda_{\mu\nu\sigma}$ obviously transform as a tensor and are components of the Riemann curvature tensor. The Riemann tensor gives us the specific information about how much the spacetime is curved. Defining

$$R_{\tau\mu\nu\sigma} = g_{\tau\lambda} R^\lambda_{\mu\sigma\nu}. \tag{2.17}$$

The Riemann tensor satisfies the symmetry relations

$$R_{\lambda\mu\nu\sigma} = -R_{\lambda\mu\sigma\nu} = -R_{\mu\lambda\nu\sigma} = R_{\nu\sigma\lambda\mu}, \tag{2.18}$$

and the Bianchi identities

$$\begin{gathered} R^{\lambda}_{\mu\nu\sigma} + R^{\lambda}_{\nu\sigma\mu} + R^{\lambda}_{\sigma\mu\nu} = 0, \\ \nabla_{\alpha} R^{\lambda}_{\sigma\mu\nu} + \nabla_{\mu} R^{\lambda}_{\sigma\nu\alpha} + \nabla_{\nu} R^{\lambda}_{\sigma\alpha\mu} = 0. \end{gathered} \tag{2.19}$$

A beautiful fact about the Riemann curvature tensor is, it is equal to zero if and only if spacetime we consider is flat. This can easily be observed. A spacetime is called flat only if there exists a coordinate system so that the metric tensor does not depend oncoordinates. In that case, it is obvious that the connection is zero, and therefore Riemann tensor is zero. The inverse is also true.

There are two more curvature tensors constructed from the Riemann curvature tensor: Ricci tensor and Ricci scalar. Ricci tensor is defined as

$$R_{\mu\sigma} = R^{\nu}_{\mu\nu\sigma} = \partial_{\nu} \Gamma^{\nu}_{\mu\sigma} - \partial_{\sigma} \Gamma^{\nu}_{\mu\nu} + \Gamma^{\nu}_{\nu\alpha} \Gamma^{\alpha}_{\mu\sigma} - \Gamma^{\nu}_{\alpha\sigma} \Gamma^{\alpha}_{\nu\mu}. \tag{2.20}$$

It is a symmetric rank 2 tensor just like the metric tensor

$$R_{\mu\nu} = R_{\nu\mu}. \tag{2.21}$$

Ricci scalar is defined as the trace of the Ricci curvature tensor

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{2.22}$$

It can be shown from Bianchi identities that Ricci tensor satisfies

$$\nabla_{\mu} \left(R^{\mu\nu} - \frac{R}{2} g^{\mu\nu} \right) = 0. \tag{2.23}$$

Due to being divergenceless, the tensor inside of the covariant derivative in the above equation is called Einstein tensor. Here, we will show it as

$$G^{\mu\nu} = R^{\mu\nu} - \frac{R}{2} g^{\mu\nu}. \tag{2.24}$$

So far we have given geometrical structures of the spacetime, now we will use them for gravitational physics.

2.3 Einstein Field Equations

In GR, spacetime is generally curved, free particles follow geodesic of the spacetime, thus their trajectories satisfy the geodesic equation (2.10). If we carefully look at this equation, we would see that the metric tensor is the quantity that is replaced by the gravitational potential in Newtonian gravity because, in the Newtonian limit, this equation reduces to the force equation $\vec{a} = -\vec{\nabla}\Phi$.Thus, we need an equation or set of equations to replace the Poisson equation in the Newtonian gravity. This set of equations is the Einstein field equations

$$G^{\mu\nu} = R^{\mu\nu} - \frac{R}{2}g^{\mu\nu} = \frac{8\pi G}{c^4}T^{\mu\nu}, \tag{2.25}$$

where G is the Newton gravitational constant, c is the speed of light and $T^{\mu\nu}$ is the energy-momentum density tensor or energy-momentum tensor to be short.

Chapter 3

Multipolar Post Minkowskian Solution

In this chapter, mostly the calculations in [6] will be discussed. As in [6], we will expand the metric in G and try to show that there exists a general form of solution to the Einstein field equations by using an iterative approach. After that, we will examine the behavior of the metric we found in the near and far zone limits.

3.1 Post Minkowskian Expansion

We want to solve the Einstein field equations for a given $T^{\mu\nu}$ in various regions such as far zone and near zone. Since the equations are extremely nonlinear, only thing to do to solve them is to use perturbation theory. As done in [7, 6, 9, 12], let us define the following variable to use for expanding our theory as a perturbation on the following field variable

$$h^{\mu\nu} = \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}, \tag{3.1}$$

where $\eta^{\mu\nu} = diag(-1, +1, +1, +1)$ and g is the determinant of the matrix $g_{\mu\nu}$. This variable somehow measures the difference between our spacetime and the flat Minkowskian spacetime with metric $\eta_{\mu\nu}$. With this choice of variable, if we apply the deDonder gauge

$$\partial_\mu h^{\mu\nu} = 0, \tag{3.2}$$

then we get the Einstein tensor to be [7, 6, 9, 12]

$$\begin{aligned} 2|g|G^{\mu\nu} = \Box h^{\mu\nu} &+ h^{\lambda\sigma}\partial_\lambda\partial_\sigma h^{\mu\nu} + g_{\sigma\beta}\partial_\lambda h^{\mu\beta}\partial^\nu h^{\sigma\lambda} + g_{\lambda\beta}\partial^\mu h^{\sigma\beta}\partial_\sigma h^{\nu\lambda} \\ &- g^{\alpha\sigma}g_{\lambda\beta}\partial_\alpha h^{\mu\beta}\partial_\sigma h^{\nu\lambda} - \partial_\sigma h^{\alpha\mu}\partial_\alpha h^{\sigma\nu} - \frac{1}{2}g_{\sigma\beta}g^{\mu\nu}\partial_\lambda h^{\alpha\beta}\partial_\alpha h^{\sigma\lambda} \\ &- \frac{1}{2}\partial_\tau h^{\alpha\beta}\partial_\kappa h^{\sigma\lambda}\Big(g^{\tau\mu}g^{\kappa\nu} - \frac{g^{\tau\kappa}g^{\mu\nu}}{2}\Big)\Big(g_{\lambda\beta}g_{\sigma\alpha} - \frac{g_{\alpha\beta}g_{\lambda\sigma}}{2}\Big), \end{aligned} \tag{3.3}$$

where the d'Alembertian is defined as $\Box h^{\mu\nu} = \eta^{\alpha\beta}\partial_\alpha\partial_\beta h^{\mu\nu}$. If we define

$$\begin{aligned} \Lambda^{\mu\nu} = -h^{\lambda\sigma}\partial_\lambda\partial_\sigma h^{\mu\nu} &- g_{\sigma\beta}\partial_\lambda h^{\mu\beta}\partial^\nu h^{\sigma\lambda} - g_{\lambda\beta}\partial^\mu h^{\sigma\beta}\partial_\sigma h^{\nu\lambda} \\ &+ g^{\alpha\sigma}g_{\lambda\beta}\partial_\alpha h^{\mu\beta}\partial_\sigma h^{\nu\lambda} + \partial_\sigma h^{\alpha\mu}\partial_\alpha h^{\sigma\nu} + \frac{1}{2}g_{\sigma\beta}g^{\mu\nu}\partial_\lambda h^{\alpha\beta}\partial_\alpha h^{\sigma\lambda} \\ &+ \frac{1}{2}\partial_\tau h^{\alpha\beta}\partial_\kappa h^{\sigma\lambda}\Big(g^{\tau\mu}g^{\kappa\nu} - \frac{g^{\tau\kappa}g^{\mu\nu}}{2}\Big)\Big(g_{\lambda\beta}g_{\sigma\alpha} - \frac{g_{\alpha\beta}g_{\lambda\sigma}}{2}\Big), \end{aligned} \tag{3.4}$$

for later convenience, then the Einstein equations become

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}|g|T^{\mu\nu} + \Lambda^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu}. \tag{3.5}$$

Since we have the Einstein equations written as $\Box h^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu}$, the first thing that comes in mind is to transform that equation into an integral equation under reasonable conditions, to solve it iteratively. To prevent the incoming waves, we impose the following:

- $T^{\mu\nu}$ is of spatially compact support: for $r = |\vec{x}| > a$, $T^{\mu\nu} = 0$.
- $T^{\mu\nu}$ is a $C^\infty(\mathbb{R}^4)$ function.
- Source is post Newtonian, it can be expanded in $\frac{1}{c}$.
- For some T, we have, for $t < -T$ space to be asymptotically flat, and

$$\frac{\partial h^{\mu\nu}}{\partial t} = 0.$$

With these conditions, integral version of that wave equation becomes

$$h^{\mu\nu}(\vec{x},t) = -\frac{4G}{c^4}\int \mathrm{d}^3x' \frac{\tau^{\mu\nu}(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c})}{|\vec{x}-\vec{x}'|}. \tag{3.6}$$

As it was already stated, this is not a solution but an integral equation, since $\tau^{\mu\nu}$ is a functional of $h^{\mu\nu}$. But we can solve it by iteration to any order in G. There is only one problem, what if that integral diverges at some order? We somehow need to regularize that integral, but that is not the way we will proceed. Let us assume we can expand $h^{\mu\nu}$ in G, though. Then we would find

$$h^{\mu\nu} = Gh_1^{\mu\nu} + G^2h_2^{\mu\nu} + ... + G^n h_n^{\mu\nu} + ... \;. \tag{3.7}$$

This is called the post Minkowskian (PM) expansion. If we put this PM expansion of $h^{\mu\nu}$ in the definiton of $\Lambda^{\mu\nu}$, we also find

$$\Lambda^{\mu\nu} = G^2\Lambda_2^{\mu\nu} + G^3\Lambda_3^{\mu\nu} + ... + G^n\Lambda_n^{\mu\nu} + ... \;. \tag{3.8}$$

Expansion starts with G^2 since $\Lambda^{\mu\nu}$ is at least quadratic in $h^{\mu\nu}$. In the region outside of the matter source, from (3.5), the Einstein field equations takes the form

$$\Box h^{\mu\nu} = \Lambda^{\mu\nu}. \tag{3.9}$$

By using the PM expansions of h and Λ,

$$\begin{aligned} \Box h_1^{\mu\nu} =& 0, \\ \Box h_n^{\mu\nu} =& \Lambda_n^{\mu\nu}, \quad n \geq 2. \end{aligned} \tag{3.10}$$

Now, since the structure of $\Lambda^{\mu\nu}$ is like $\partial h \partial h + h \partial\partial h$, nth order term of $\Lambda^{\mu\nu}$ cannot include nth or higher order terms of $h^{\mu\nu}$. Then we have $\Lambda_n^{\mu\nu} = \Lambda_n^{\mu\nu}[h_1, h_2, ..., h_{n-1}]$. That means, if we know $h_1, h_2, ..., h_{n-1}$, then we know the source of h_n, thus we have at least a hope to find a solution for h_n.

3.2 Linearized Solution

In this section, we will solve the Einstein field equations only for the outside of the matter source, to linear order in G. Thus, we will consider only

$$\Box h_1^{\mu\nu} = 0, \tag{3.11}$$

with the boundary conditions above. As it is done in [6], we will use

$$f(\vec{n}) = \sum_l \hat{f}_L \hat{n}^L, \tag{3.12}$$

expansion where $n^i = x^i/r$ and $r = \sqrt{x^i x^i}$. Here, A_L represents a tensor with l spatial index, which actually is $A_{i_1 i_2 \ldots i_l}$ with $i_1, \ldots, i_l = 1, 2, 3$. Special cases such as n_L and ∂_L means $n_L = n_{i_1} \ldots n_{i_l}$ and $\partial_L = \partial_{i_1} \ldots \partial_{i_l}$ respectively. If the tensor has a hat on it $\hat{A}_L$, or is written as $A_{<L>}$, this represents symmetric traceless part of the tensor A_L. This $A_{<L>}$ type of tensors are called symmetric and tracefree (STF) tensor. In a similar fashion, $A_{(L)}$ represents the symmetric part of A_L. To be more clear, we have $A_{(ij)} = (A_{ij} + A_{ji})/2$ and for $A = \delta_{ij} A_{ij}$, we have $A_{<ij>} = A_{(ij)} - \delta_{ij} A/3$. Finally, $A_L B_L$ means both tensor multiplied and all indices of those tensors are contracted. For example, for $l = 3$ we have $A_L B_L = A_{ijk} B_{ijk}$

This expansion is exactly the same with the expansion done by using spherical harmonic $Y_l^m(\theta, \phi)$ functions, and just like it, it is unique [6]

$$\hat{f}_L = \frac{(2l+1)!!}{4\pi l!} \int \mathrm{d}\Omega \hat{n}^L f(\vec{n}). \tag{3.13}$$

Therefore, we can expand $h_1^{\mu\nu}$ on its angle dependence as

$$h_1^{\mu\nu}(\vec{r}, t) = \sum_l h_{1L}^{\mu\nu}(r, t) \hat{n}^L. \tag{3.14}$$

It is easy to show that (3.11) yields

$$\left\{ \frac{\partial^2}{\partial r^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{l(l+1)}{r^2} \right\} h_{1L}^{\mu\nu}(r, t) = 0. \tag{3.15}$$

Introducing

$$\begin{aligned} u &= t - \frac{r}{c}, \\ v &= t + \frac{r}{c}, \end{aligned} \tag{3.16}$$

and defining

$$f(u, v) = \frac{h_{1L}(r, t)}{(v-u)^l}, \tag{3.17}$$

without spacetime indices for convenience, we find that $f(u, v)$ must satisfy

$$\left\{ (v-u) \frac{\partial}{\partial u} \frac{\partial}{\partial v} + (l+1) \frac{\partial}{\partial u} - (l+1) \frac{\partial}{\partial v} \right\} f(u, v) = 0. \tag{3.18}$$

This is an Euler Darboux type of differential equation, and its general solution is

$$f(u,v) = \frac{\partial^{2l}}{\partial u^l \partial v^l}\Big\{\frac{U(u)+V(v)}{v-u}\Big\}, \tag{3.19}$$

where U and V are arbitrary, sufficiently many times differentiable functions of u and v respectively [6]. We are looking for a solution that is constant in the past. Therefore, it is obvious that we should have $V(v) = constant$, and we can choose this constant to be zero without loss of generality. Putting the definitions of u and v, one gets

$$\frac{\partial}{\partial u}\frac{\partial}{\partial v} = \frac{1}{4}\Big(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial r^2}\Big). \tag{3.20}$$

Since we have

$$\begin{aligned} 0 = \Box\Big(\frac{U(t-\frac{r}{c})}{r}\Big) &= \Big(\frac{\partial^2}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{2}{r}\frac{\partial}{\partial r}\Big)\Big(\frac{U(t-\frac{r}{c})}{r}\Big) \\ &= \Big(\frac{4}{c^2}\frac{\partial}{\partial u}\frac{\partial}{\partial v} + \frac{2}{r}\frac{\partial}{\partial r}\Big)\Big(\frac{U(t-\frac{r}{c})}{r}\Big), \end{aligned} \tag{3.21}$$

then we can write

$$f(u,v) = \frac{(-1)^l c^{2l}}{2^l}\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^l\Big\{\frac{U(u)}{v-u}\Big\}. \tag{3.22}$$

Therefore from (3.17), after rescaling the function U and putting the indices back we find

$$h^{\mu\nu}_{1L}(r,t) = r^l\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^l\Big(\frac{U^{\mu\nu}_{1L}(t-\frac{r}{c})}{r}\Big). \tag{3.23}$$

This can be written as [6]

$$\hat{n}^L h^{\mu\nu}_{1L}(r,t) = \hat{\partial}_L\Big(\frac{U^{\mu\nu}_{1L}(t-\frac{r}{c})}{r}\Big). \tag{3.24}$$

Thus, general solution is

$$h^{\mu\nu}_1(\vec{x},t) = \sum_l \hat{\partial}_L\left(\frac{U^{\mu\nu}_L(t-\frac{r}{c})}{r}\right). \tag{3.25}$$

This solution can be considered as an infinite series. However, not to deal with the convergence issues, it will be considered as a finite sum, which will terminate at some l_0 value. Here, $U^{\mu\nu}_L$ are $C^\infty(\mathbb{R})$ and for $u < -T$, we have $U^{\mu\nu}_L(u)$ constant, which means they are past constant functions. When we decompose those $U^{\mu\nu}_L$

into the irreducible representations of the rotation group $SO(3)$, we then have, for $u = t - \frac{r}{c}$

$$
\begin{aligned}
h_1^{00}(\vec{x},t) &= \sum_{l\geq 0} \partial_L \frac{A_L(u)}{r}, \\
h_1^{i0}(\vec{x},t) &= \sum_{l\geq 0} \partial_{iL} \frac{B_L(u)}{r} + \sum_{l\geq 1} \left(\partial_{L-1} \frac{C_{L-1i}(u)}{r} + \epsilon_{iab}\partial_{aL-1} \frac{D_{bL-1}(u)}{r} \right), \\
h_1^{ij}(\vec{x},t) &= \sum_{l\geq 0} \left(\partial_{ijL} \frac{E_L(u)}{r} + \delta_{ij}\partial_L \frac{F_L(u)}{r} \right) \\
&\quad + \sum_{l\geq 1} \left(\partial_{L-1(i} \frac{G_{j)L-1}(u)}{r} + \epsilon_{ab(i}\partial_{j)aL-1} \frac{H_{bL-1}(u)}{r} \right) \\
&\quad + \sum_{l\geq 2} \left(\partial_{L-2} \frac{I_{ijL-2}(u)}{r} + \partial_{aL-2} \left(\frac{1}{r} \epsilon_{ab(i} J_{j)bL-2}(u) \right) \right),
\end{aligned}
\tag{3.26}
$$

where ϵ_{ijk} is the totally antisymmetric Levi Civita symbol with $\epsilon_{123} = 1$, $A_L, B_L, ..., I_L$ are the symmetric and trace free tensors that depend linearly on $U_L^{00}, U_L^{0i}, U_L^{ij}$,. The above general solution satisfies only (3.10) only. To satisfy the gauge condition (3.2), those functions $A_L, B_L, ..., I_L$ should satisfy

$$
\begin{aligned}
&\dot{A} + \ddot{B} = 0, && \dot{A}_L + \ddot{B}_L + C_L = 0, (l \geq 1), \\
&\dot{B} + \ddot{E} + F = 0, && \dot{B}_L + \ddot{E}_L + F_L + \frac{1}{2}G_L = 0, (l \geq 1), \\
&\dot{C}_i + \frac{1}{2}\ddot{G}_i = 0, && \dot{C}_L + \frac{1}{2}\ddot{G}_L + I_L = 0, (l \geq 2), \\
&\dot{D}_i + \frac{1}{2}\ddot{H}_i = 0, && \dot{D}_L + \frac{1}{2}\ddot{H}_L + \frac{1}{2}\ddot{J}_L = 0, (l \geq 2),
\end{aligned}
\tag{3.27}
$$

where $\dot{A} = \frac{1}{c}\frac{\partial A}{\partial t}$. How can we make them simpler? Recall that we still have not specified the gauge entirely. Consider a coordinate transformation: $x^\mu = \widetilde{x}^\mu + G w^\mu$. Then, if we put the PM expansion in (2.4), we find

$$
h_1^{\mu\nu} = \widetilde{h}_1^{\mu\nu} + \partial^\mu w^\nu + \partial^\nu w^\mu - \eta^{\mu\nu}\partial_\tau w^\tau. \tag{3.28}
$$

If both satisfy the deDonder gauge $\partial_\mu h^{\mu\nu} = \partial_\mu \widetilde{h}^{\mu\nu} = 0$, then we have $\Box w^\mu = 0$. We then can write w in terms of STF multipole moments, as we have done for h_1:

$$
\begin{aligned}
w^0(\vec{x},t) &= \sum_{l\geq 0} \partial_L \frac{W_L(u)}{r}, \\
w^i(\vec{x},t) &= \sum_{l\geq 0} \partial_{iL} \frac{X_L(u)}{r} + \sum_{l\geq 1} \left(\partial_{L-1} \frac{Y_{L-1i}(u)}{r} + \epsilon_{iab}\partial_{aL-1} \frac{Z_{bL-1}(u)}{r} \right).
\end{aligned}
\tag{3.29}
$$

This gauge transformation can then be written as a relation between the multipole moments $\widetilde{A}_L, \widetilde{B}_L, ..., \widetilde{J}_L, A_L, B_L, ..., J_L$ and W_L, X_L, Y_L, Z_L as:

$$\begin{aligned}
&A_L = \widetilde{A}_L - \dot{W}_L + \ddot{X}_L + Y_L, (l \geq 1), && A = \widetilde{A} - \dot{W} + \ddot{X}, \\
&B_L = \widetilde{B}_L + W_L - \dot{X}_L, (l \geq 0), && C_L = \widetilde{C}_L - \dot{Y}_L, (l \geq 1), \\
&D_L = \widetilde{D}_L - \dot{Z}_L, (l \geq 1), && E_L = \widetilde{E}_L + 2X_L, (l \geq 0), \\
&F_L = \widetilde{F}_L - \ddot{X}_L - \dot{W}_L - Y_L, (l \geq 1), && F = \widetilde{F} - \ddot{X} - \dot{W} - Y, \\
&G_L = \widetilde{G}_L + 2Y_L, (l \geq 1), && H_L = \widetilde{H}_L + 2Z_L, (l \geq 1).
\end{aligned} \tag{3.30}$$

Here, by choosing

$$\begin{aligned}
X_L &= -\frac{1}{2}\widetilde{E}_L, & Y_L &= -\frac{1}{2}\widetilde{G}_L, \\
Z_L &= -\frac{1}{2}\widetilde{H}_L, & W_L &= -\widetilde{B}_L - \frac{1}{2}\dot{\widetilde{E}}_L,
\end{aligned} \tag{3.31}$$

and writing

$$A_L = -\frac{4}{c^2}\frac{(-1)^l}{l!}M_L, \quad D_L = \frac{4}{c^3}\frac{(-1)^l l}{(l+1)!}S_L, \tag{3.32}$$

we find

$$\begin{aligned}
h_1^{00}(\vec{x}, t) =& -\frac{4}{c^2}\sum_{l\geq 0}\frac{(-1)^l}{l!}\partial_L\frac{M_L(u)}{r}, \\
h_1^{i0}(\vec{x}, t) =& \frac{4}{c^2}\sum_{l\geq 1}\frac{(-1)^l}{l!}\left(\partial_{L-1}\frac{\dot{M}_{L-1i}(u)}{r} + \frac{1}{c}\frac{l}{l+1}\epsilon_{iab}\partial_{aL-1}\frac{S_{bL-1}(u)}{r}\right), \\
h_1^{ij}(\vec{x}, t) =& -\frac{4}{c^2}\sum_{l\geq 2}\frac{(-1)^l}{l!}\left(\partial_{L-2}\frac{\ddot{M}_{ijL-2}(u)}{r} + \frac{2}{c}\frac{l}{l+1}\partial_{aL-2}\left(\frac{1}{r}\epsilon_{ab(i}\dot{S}_{j)bL-2}(u)\right)\right),
\end{aligned} \tag{3.33}$$

with

$$\dot{M} = \dot{M}_i = \dot{S}_i = 0, \tag{3.34}$$

to satisfy the gauge condition. With that choice of coordinates, we see that M_L and S_L multipole moments are actually enough to represent h_1 since other W_L,..., Z_L moments carry information about the coordinate choice, which is pure gauge. The solution in (3.33) written only by M_L and S_L is called the canonical solution. Coefficients in front of the A_L and D_L moments are chosen so that if the source is moving slowly and has sufficiently weak self-gravity, M_L and S_L are mass and angular momentum multipole moments related to the source as

$$M_L(u) = \int \mathrm{d}^3x' \rho(\vec{x}', u)\widehat{x}'^L, \tag{3.35}$$

and

$$S_L(u) = \int \mathrm{d}^3x' \rho(\vec{x}', u) x'_b v_a(\vec{x}', u) \epsilon_{ba<i_1} x'_{i_2 i_3 \dots i_l>}, \tag{3.36}$$

where $\rho(\vec{x}', u)$ is the rest mass density of the matter and $v_a(\vec{x}', u)$ is the coordinate velocity field of the matter. We remark here that not all powers of c are explicitly shown, there are also powers of c in the dot derivatives, since we defined it as $\dot{A} = \frac{1}{c}\frac{\partial A}{\partial t}$.

3.3 Particular Multipolar Post-Minkowskian Solution

We have solved h_1, now we will try to show that there is a solution for h_n by using the method given in [6], i.e. we will solve (3.10):

$$\begin{aligned} \Box h_n^{\mu\nu} &= \Lambda_n^{\mu\nu}[h_m, m < n], \\ \partial_\mu h_n^{\mu\nu} &= 0, \end{aligned} \tag{3.37}$$

for $n > 2$. The first thing that appears in mind is to transform this equation into an integral equation with the boundary conditions and write

$$h_n^{\mu\nu}(\vec{x}, t) = \Box_R^{-1}[\Lambda_n^{\mu\nu}](\vec{x}, t) = -\frac{1}{4\pi} \int \mathrm{d}^3x' \frac{\Lambda_n^{\mu\nu}(\vec{x}', t_{ret})}{|\vec{x} - \vec{x}'|}, \tag{3.38}$$

where $t_{ret} = t - \frac{|\vec{x}-\vec{x}'|}{c}$. However, that integral may diverge for two reasons: the first one is the region of integration is not finite, and the second one is the source $\Lambda^{\mu\nu}$ is divergent at $r = 0$. In order to overcome the former reason, we will divide $h^{\mu\nu}$ into two pieces:

$$h^{\mu\nu} = h_S^{\mu\nu} + h_D^{\mu\nu}, \tag{3.39}$$

where S stands for static, D stands for dynamic part, and the static part is defined as

$$h_S^{\mu\nu}(\vec{r}) = h^{\mu\nu}(\vec{r}, -T). \tag{3.40}$$

Here the name static part really fits because boundary conditions imply that for all $t < -T$, we have $h^{\mu\nu}(\vec{r}, t) = h^{\mu\nu}(\vec{r}, -T)$. We also can divide Λ into static and dynamic pieces. After putting PM expansions of both sides in (3.37), the equations that static and dynamic parts should satisfy become

$$\begin{aligned} \Delta h^{\mu\nu}_{nS} &= \Lambda^{\mu\nu}_{nS}, \\ \Box h^{\mu\nu}_{nD} &= \Lambda^{\mu\nu}_{nD}. \end{aligned} \tag{3.41}$$

The solution for the dynamic part can be written as an integral equation, where Λ_D is zero for past retarded times, thus it is zero for points that are far enough from the source. This gives the integral a compact support, thus effectively the integral is to be calculated in a finite region. That leaves only $r' = 0$ as a region that possibly make the integral for the dynamic part divergent.

Equation of the static part can be solved without problem and without any need for actually doing the integral, by using one of the definitions of inverse Laplacian:

$$\Delta^{-1}\frac{\hat{n}^L}{r^a} = \frac{\hat{n}^L r^{2-a}}{(l-a+3)(2-l-a)}. \tag{3.42}$$

That definition has no meaning for $l + a = 2$ and $l = a - 3$. In order to avoid that, we can multiply whatever function we want to take the inverse Laplacian of, with r^B where B is a complex number. Then the denominator in (3.42) will not be zero for some values of B, and we can hope to apply analytical continuation. As an example, let us try to solve for $h^{\mu\nu}_{2S}$. We know that $h^{\mu\nu}_{1S}$ is

$$h^{\mu\nu}_{1S} = \sum_l \widehat{\partial}_L\left(\frac{U^{\mu\nu}_L(-T-\frac{r}{c})}{r}\right) = \sum_l U^{\mu\nu}_L(-T)\widehat{\partial}_L\left(\frac{1}{r}\right). \tag{3.43}$$

Then by taking derivatives of $1/r$, we can write it as

$$h^{\mu\nu}_{1S} = \sum_{a\geq 1,l} F^{\mu\nu}_{aL}\frac{\hat{n}^L}{r^a}. \tag{3.44}$$

If we take this sum as a finite sum, we can write Λ_{2s} as a finite sum too:

$$\Lambda^{\mu\nu}_{2S} = \sum_{a\geq 2,l} K^{\mu\nu}_{aL}\frac{\hat{n}^L}{r^{a+2}}. \tag{3.45}$$

There $a \geq 2$ since we have two h_{1S} multiplied, and that $+2$ is there since we took two derivatives. We will remind those constraints when it is time, from now on, let us not write them for convenience. Then we have

$$\Delta\widetilde{h}^{\mu\nu}_{2S}(B) = r^B\Lambda^{\mu\nu}_{2S}, \tag{3.46}$$

$$\widetilde{h}^{\mu\nu}_{2S}(B) = \Delta^{-1}\sum_{l,a} K^{\mu\nu}_{aL}\frac{\hat{n}^L}{r^{a+2-B}} = \sum_{l,a} K^{\mu\nu}_{aL}\Delta^{-1}\frac{\hat{n}^L}{r^{a+2-B}}. \tag{3.47}$$

Using (3.42)

$$\widetilde{h}^{\mu\nu}_{2S}(B) = \sum_{l,a} K^{\mu\nu}_{aL}\frac{\hat{n}^L r^{-a+B}}{(l-a+B+1)(B-l-a)}. \tag{3.48}$$

Obviously this is an analytical function of B. As we can see, if $l = a-1$ terms exits, this function have a first degree pole at $B = 0$. Thus, its Laurent expansion is like

$$\widetilde{h}^{\mu\nu}_{2S}(B;\vec{r},t) = \sum_{k\geq -1} C^{\mu\nu}_k(\vec{r},t)B^k. \tag{3.49}$$

Then we have

$$\Delta\widetilde{h}^{\mu\nu}_{2S}(B) = \sum_{k\geq -1} B^k \Delta C^{\mu\nu}_k = r^B\Lambda^{\mu\nu}_{2S}, \tag{3.50}$$

and, by expanding r^B in B, we have this set of equations:

$$\Delta C^{\mu\nu}_{-1} = 0, \quad \Delta C^{\mu\nu}_k = \frac{\log^k r}{n!}\Lambda^{\mu\nu}_{2S}, \tag{3.51}$$

We will define the zeroth coefficient of a Laurent expansion as finite part, which will be shown as

$$C^{\mu\nu}_0 = \underset{B=0}{\mathrm{FP}}\sum_{k\geq -1} C^{\mu\nu}_k(\vec{r},t)B^k = \underset{B=0}{\mathrm{FP}}\,\Delta^{-1}(r^B\Lambda^{\mu\nu}_{2S}). \tag{3.52}$$

Now, it is obvious that at each step in this expansion, we may get another $\log r$ generated, thus it looks like $h^{\mu\nu}_{nS}$ can be chosen as a $n-1$ degree polynomial in $\log r$. But there still remains another question: $C^{\mu\nu}_0$ satisfies our equation, but does it satisfy the gauge condition? We have the following theorem:

Theorem 1 *We can expand $h^{\mu\nu}_{nS}$ around $r = 0$ as*

$$h^{\mu\nu}_{nS}(\vec{r}) = \sum_{l,a\geq n,p<n} F^{\mu\nu}_{napL}\hat{n}^L r^{-a}\log^p r. \tag{3.53}$$

We will prove it by induction. It is obvious that the statement holds for $n = 1$. Now, assume that it is true for all $m < n$. Then due to its structure, we can write $\Lambda^{\mu\nu}_{nS}$ as a $n-2$ degree polynomial of $\log r$:

$$\Lambda^{\mu\nu}_{nS}(\vec{r}) = \sum_{l,a\geq n,p<n-1} K^{\mu\nu}_{napL}\hat{n}^L r^{-a-2}\log^p r. \tag{3.54}$$

Here, due to the structure of $\Lambda \sim h\partial\partial h + \partial h\partial h$, we have r^{-a-2} with $a \leq n$. That -2 is there for taking two derivatives, and $a \leq n$ since we assume that the leading power of $1/r$ for h_{mS} is at least m for all $m < n$. We can write

$$r^B \Lambda^{\mu\nu}_{nS}(\vec{r}) = \sum_{l,a\geq n,p<n-1} K^{\mu\nu}_{napL} \frac{\partial^p}{\partial B^p}(\hat{n}^L r^{-a-2+B}). \tag{3.55}$$

Then we have

$$\begin{aligned} \widetilde{h}^{\mu\nu}_{nS}(B) &= \sum_{l,a\geq n,p<n-1} K^{\mu\nu}_{napL} \frac{\partial^p}{\partial B^p} \Delta^{-1}(\hat{n}^L r^{-a-2+B}) \\ &= \sum_{l,a\geq n,p<n-1} \frac{\partial^p}{\partial B^p} \frac{K^{\mu\nu}_{napL} \hat{n}^L r^{-a+B}}{(l-a+B+1)(B-l-a)}. \end{aligned} \tag{3.56}$$

In the same way, after applying the Laurent expansion, we see that the zeroth order coefficient, which is defined as the finite part, is a solution. In order to take the zeroth order coefficient out of a term inside the summation, we need to take coefficient of B^{p+1} inside for pole terms, B^p for other terms. They will introduce $\log^{p+1} r$ and $\log^p r$ respectively. Since we have $p < n-1$, then we just showed that the finite part can contain powers of $\log r$ up to n only. Then we have

$$q^{\mu\nu} = \underset{B=0}{\mathrm{FP}}\, \widetilde{h}^{\mu\nu}_{nS}(B) = \sum_{l,a\geq n,p<n} F^{\mu\nu}_{napL} \hat{n}^L r^{-a} \log^p r. \tag{3.57}$$

Now, this is a solution to the equation, but does it satisfy our choice of gauge? Let us calculate its divergence. Knowing that our definition of inverse Laplacian satisfies

$$\Delta^{-1}(r^{B-a}\hat{n}^L) = -\frac{1}{4\pi} \int \mathrm{d}^3 r' \frac{r'^{B-a}\hat{n}^L}{|\vec{r} - \vec{r'}|}, \tag{3.58}$$

for $a - l - 1 < Re(B) < a - l$, then we easily have

$$\partial_\mu \Delta^{-1}(r^{B-a}\hat{n}^L) = \Delta^{-1}\partial_\mu(r^{B-a}\hat{n}^L), \tag{3.59}$$

within that region for $l \geq 2$. Since both sides are analytic functions of B, we can extend the region that the equality holds to almost all values of B, except poles of that function, which are integers. For $l = 0, 1$, it can be explicitly proven without difficulty. Then, using this result, we have

$$\begin{aligned} \partial_\mu q^{\mu\nu} &= \underset{B=0}{\mathrm{FP}}\, \partial_\mu \widetilde{h}^{\mu\nu}_{nS}(B) = \underset{B=0}{\mathrm{FP}}\, \partial_\mu \Delta^{-1}(r^B \Lambda^{\mu\nu}_{nS}) \\ &= \underset{B=0}{\mathrm{FP}}\, \Delta^{-1}(B r^{B-1} n^i \Lambda^{i\nu}_{nS} + r^B \partial_\mu \Lambda^{\mu\nu}_{nS}). \end{aligned} \tag{3.60}$$

Since we have $\partial_\mu \Lambda^{\mu\nu}_{nS} = 0$,

$$\begin{aligned}\partial_\mu q^{\mu\nu} &= \underset{B=0}{\mathrm{FP}}\, \Delta^{-1}(B r^{B-1} n^i \Lambda^{i\nu}_{nS}) \\ &= \underset{B=0}{\mathrm{FP}} \sum_{l,a\geq n,p<n-1} B K^{i\nu}_{napL} \frac{\partial^p}{\partial B^p} \Delta^{-1}(n^i \hat{n}^L r^{-a-3+B}).\end{aligned} \tag{3.61}$$

Now, finite part is getting the coefficient of B^0 of the whole, which is the coefficient of B^{-1} of the summation. Thus

$$\partial_\mu q^{\mu\nu} = \underset{B=0}{\mathrm{Res}} \sum_{l,a\geq n,p<n-1} K^{i\nu}_{napL} \frac{\partial^p}{\partial B^p} \Delta^{-1}(n^i \hat{n}^L r^{-a-3+B}). \tag{3.62}$$

Now, $n^i \hat{n}^L$ can be written as sum of $\hat{n}^S$ for $s = l-1, l+1$. In the summation, we have $l = 0, 1, 2, \ldots.$ For $l = 0$, we only have $s = 1$ term, thus we have $s = 0, 1, 2, \ldots$ too. Then, we can rewrite the sum as

$$\begin{aligned}\partial_\mu q^{\mu\nu} &= \underset{B=0}{\mathrm{Res}} \sum_{s,a\geq n,p<n-1} H^{\nu}_{napS} \frac{\partial^p}{\partial B^p} \Delta^{-1}(\hat{n}^S r^{-a-3+B}) \\ &= \underset{B=0}{\mathrm{Res}} \sum_{s,a\geq n,p<n-1} H^{\nu}_{napS} \frac{\partial^p}{\partial B^p} \frac{\hat{n}^S r^{-a-1+B}}{(B-a+s)(B-a-1-s)}.\end{aligned} \tag{3.63}$$

Since $a \geq n$ and $s \geq 0$, poles can only come from $a = s$ and $p = 0$ terms. Remember that we can actually write those H coefficients in the summation as a multiplication of Kronecker deltas, Levi Civita tensors and n multipole moments:

$$H^{\nu}_{napS} \approx (\delta\epsilon) M_{L_1} S_{L_2} \ldots Z_{L_n}. \tag{3.64}$$

Now, if this expression has w W_L, x X_L and z Z_L multipoles, then dimensional analysis gives us

$$a = n + \sum_{i=1}^{n} l_i + w + 2x + z. \tag{3.65}$$

Here, H^{ν}_{napS} has one space-time index ν, and s space indices i_1, ..., i_s that it is symmetric and traceless. Therefore for $\nu = 0$, it already belongs to a spin s representation of SO(3). For $\nu = 1$, its spin is $s-1$, s or $s+1$. From the structure of the H and addition rule of angular momenta

$$spin \leq \sum_{i=1}^{n} l_i. \tag{3.66}$$

We know that spin can take values $s-1$, s or $s+1$. However, since we only know one of those spins exists (we assume that coefficient gives a contribution to the sum), the strongest argument we can have is

$$s-1 \leq \sum_{i=1}^{n} l_i. \tag{3.67}$$

Combining this with (3.65)

$$n \leq a-s-w-2x-z+1 \leq a-s+1. \tag{3.68}$$

Thus, we can have $a=s$ only for $n \leq 1$. Therefore, for $n>1$, the sum cannot have any pole at $B=0$, thus its residue is zero, then $p^{\mu\nu}$ is divergenceless, therefore it can be chosen as a solution for $h^{\mu\nu}_{nS}$. The statement in the theorem is already satisfied by our solution for $n=1$, thus the theorem is proved $\Box$.

If we consider the dynamic part, we have these equations:

$$\begin{aligned} \Box h^{\mu\nu}_{nD} &= \Lambda^{\mu\nu}_{nD}, \\ \partial_\mu h^{\mu\nu}_{nD} &= 0. \end{aligned} \tag{3.69}$$

First guess for the solution is

$$h^{\mu\nu}_{nD} = \Box^{-1}_R \Lambda^{\mu\nu}_{nD}, \tag{3.70}$$

but obviously, $\Lambda^{\mu\nu}_{nD}$ is divergent at $r=0$, thus integral will diverge. This is the only possible source for the integral to diverge, since the integral has compact support. We will try to solve it by applying the same regularization procedure we have done to the static part, as done in [6].

$$\widetilde{h}^{\mu\nu}_{nD}(B) = \Box^{-1}_R [r^B \Lambda^{\mu\nu}_{nD}]. \tag{3.71}$$

It is obvious that if we choose $Re(B)$ large enough, the integral we expect to see as derivative of $\widetilde{h}^{\mu\nu}_{nD}(B)$ would converge. Therefore, that integral is the derivative, and $\widetilde{h}^{\mu\nu}_{nD}(B)$ is analytic on B with derivative

$$\frac{\partial \widetilde{h}^{\mu\nu}_{nD}(B)}{\partial B} = \Box^{-1}_R [r^{B-1} \log r \ \Lambda^{\mu\nu}_{nD}], \tag{3.72}$$

where $Re(B)$ is large enough. Now, observe the following

$$\Box_R^{-1} F = \Delta^{-1} F + \frac{1}{c^2} \Box_R^{-1} \Delta^{-1} \ddot{F}, \tag{3.73}$$

where F is a past-zero function. d'Alembertian of both sides are the same, and both sides are past zero. By using the uniqueness of the solution to the wave equation, we conclude that those expressions must be equal. By using this equality, we can expand the retarded integral of Λ_D as inverse Laplacians to some order. Let us do the integral for $n = 2$ case, to get some feeling again. We know that we can write Λ_{2D} as sum of terms $F(t)\widehat{n}^L r^a$. Then, it is enough to investigate their integrals for now:

$$\begin{aligned} &\Box_R^{-1}(F(t)\widehat{n}^L r^{a+B}) = \Delta^{-1}(F(t)\widehat{n}^L r^{a+B}) + \frac{1}{c^2}\Box_R^{-1}\Delta^{-1}(\ddot{F}(t)\widehat{n}^L r^{a+B}) \\ =&\frac{F(t)\widehat{n}^L r^{a+B+2}}{(a+B+2-l)(a+B+3+l)} + \frac{1}{c^2}\Box_R^{-1}\frac{\ddot{F}(t)\widehat{n}^L r^{a+B+2}}{(a+B+2-l)(a+B+3+l)}. \end{aligned} \tag{3.74}$$

If we apply the same equality again and again to the retarded integral that will appear on the right side, the power of r' in the integral will increase. In that way, regardless of the value of $Re(B)$, the integral can be made convergent by doing this iteration sufficiently many times. Then its derivative with respect to B is also convergent, therefore exists, hence the integral can be made an analytical function of B. All other inverse Laplacian terms are obviously analytical functions of B. Therefore we find that $\widetilde{h}^{\mu\nu}_{2D}(B)$ is an analytical function of B on $\mathbb{C} - \mathbb{Z}$. By applying Laurent expansion to $\widetilde{h}^{\mu\nu}_{2D}(B)$, it again will be found that the zeroth coefficient of the Laurent expansion solves the equation. It can be shown that poles of Λ_{2D} are first degree poles. Therefore if $B = 0$ is a pole, while calculating FP, a $\log r$ term will appear. It appears that a $\log r$ will be generated in every iteration, and FP $\widetilde{h}^{\mu\nu}_{nD}(B)$ will be an $n-1$th degree polynomial of $\log r$, just as in the static part case. Let us prove the following theorem:

Theorem 2 *We can expand $h^{\mu\nu}_{nD}$ around $r = 0$ for any positive integer N as*

$$h^{\mu\nu}_{nD}(\vec{r}, t) = \sum_{l,a,p<n} F^{\mu\nu}_{napL}(t)\hat{n}^L r^{-a} \log^p r + S_N(\vec{r}, t), \tag{3.75}$$

where $F(t)$ functions are infinitely many times differentiable past zero functions, *and $S_N(\vec{r},t)$ is past zero, infinitely many times differentiable in time and N times differentiable in position, and $O(r^N)$ function for $r \to 0$.*

We will prove by induction again. The statement is true for $n = 1$. Let us assume that it is true for all $m < n$. Since $h^{\mu\nu}_{mS}$ is also a $m-1$ degree polynomial of $\log r$, we will have $\Lambda^{\mu\nu}_{nD}$ as a $n-2$ degree polynomial of $\log r$, plus a remainder term coming from S_N functions:

$$\Lambda^{\mu\nu}_{nD}(\vec{r},t) = \sum_{l,a,p<n-1} K^{\mu\nu}_{napL}(t)\hat{n}^L r^{-a-2}\log^p r + R_N(\vec{r},t). \tag{3.76}$$

Here, $K(t)$ functions are past zero infinitely many times differentiable functions, and R_N is past zero, infinitely many times differentiable in time, $N-2$ times differentiable in position, and $O(r^{N-2})$ for $r \to 0$. Then we find

$$r^B\Lambda^{\mu\nu}_{nD}(\vec{r},t) = \sum_{l,a,p<n-1} K^{\mu\nu}_{napL}(t)\frac{\partial^p}{\partial B^p}(\hat{n}^L r^{-a-2+B}) + r^B R_N. \tag{3.77}$$

d'Alembert inverse of the R_N reminder term can be shown to be a past zero, infinitely many times differentiable in time, N times differentiable in position, and $O(r^N)$ function for $r \to 0$ limit. Therefore, that directly will make a contribution to the S_N term of h_n. We know how to take the d'Alembert inverse of the terms in the finite sum from (3.74). After applying it, it would be clear that $\widetilde{h}^{\mu\nu}_{nD}(B)$ is an analytical function of B except the remainder part. d'Alembert inverse of the remainder terms will be analytical functions of B and convergent at $B = 0$, due to the reasons that are explained for the discussion for h_{2D}. That means $\widetilde{h}^{\mu\nu}_{nD}(B)$ is really an analytical function of B except the integer values, hence we can do a Laurent expansion. It can be shown that the zeroth coefficient of the Laurent expansion around $B = 0$ is again a solution to the d'Alembert equation (3.69).

If we look at the expansion of the d'Alembert inverse in terms of inverse Laplacian, poles of $\widetilde{h}^{\mu\nu}_{nD}(B)$ are integer values of B, and all of them are first degree poles. Therefore, if $B = 0$ is a pole, it is first degree. Therefore we need the coefficient of B^{p+1} of the term inside the derivatives with respect to B if the term has a pole at $B = 0$, and B^p if it has no pole. Those generate $\log^{p+1} r$ ve $\log^p r$ respectively. Since we have $p < n-1$, this means that the maximum degree of the $\log r$ terms can be n. Then we get

$$p^{\mu\nu} = \underset{B=0}{\mathrm{FP}}\, \widetilde{h}^{\mu\nu}_{nD}(B) = \sum_{l,a,p<n} F^{\mu\nu}_{napL}(t)\hat{n}^L r^{-a}\log^p r + S_N(\vec{r},t). \tag{3.78}$$

This is a solution for the d'Alembert wave equation we have. But does it satisfy the gauge condition? Divergence of $p^{\mu\nu}$ can be calculated to be

$$\partial_\nu p^{\mu\nu} = \underset{B=0}{\mathrm{FP}}\ \Box_R^{-1}[Br^{B-1}\Lambda^{\mu i}_n\ n_i]. \tag{3.79}$$

Now, $\Box h^{\mu\nu}_n = \Lambda^{\mu\nu}_n$. Thus $\Box(h^{\mu\nu}_n - p^{\mu\nu}) = 0$. Let us define $q^{\mu\nu} = h^{\mu\nu}_n - p^{\mu\nu}$, then

$$\begin{aligned} &\Box q^{\mu\nu} = 0, \\ &\partial_\nu q^{\mu\nu} = -\partial_\nu p^{\mu\nu}. \end{aligned} \tag{3.80}$$

Since we have both $\Box q^{\mu\nu} = 0$ and $\Box\partial_\nu p^{\mu\nu}_n = 0$, we can write $\partial_\nu p^{\mu\nu}_n$ and $q^{\mu\nu}$ by as a multipol eexpansion, since they satisfy d'Alembert equation with our specific boundary conditions:

$$\begin{aligned} \partial_\nu p^{0\nu}(\vec{x},t) &= \sum_{l\geq 0}\partial_L\frac{A_L(u)}{r}, \\ \partial_\nu p^{i\nu}(\vec{x},t) &= \sum_{l\geq 0}\partial_{iL}\frac{B_L(u)}{r} + \sum_{l\geq 1}\left(\partial_{L-1}\frac{C_{L-1i}(u)}{r} + \epsilon_{iab}\partial_{aL-1}\frac{D_{bL-1}(u)}{r}\right), \end{aligned} \tag{3.81}$$

and

$$\begin{aligned} q^{00}(\vec{x},t) =& \sum_{l\geq 0}\partial_L\frac{\widetilde{A}_L(u)}{r}, \\ q^{i0}(\vec{x},t) =& \sum_{l\geq 0}\partial_{iL}\frac{\widetilde{B}_L(u)}{r} + \sum_{l\geq 1}\left(\partial_{L-1}\frac{\widetilde{C}_{L-1i}(u)}{r} + \epsilon_{iab}\partial_{aL-1}\frac{\widetilde{D}_{bL-1}(u)}{r}\right), \\ q^{ij}(\vec{x},t) =& \sum_{l\geq 0}\left(\partial_{ijL}\frac{\widetilde{E}_L(u)}{r} + \delta_{ij}\partial_L\frac{\widetilde{F}_L(u)}{r}\right) \\ &+ \sum_{l\geq 1}\left(\partial_{L-1(i}\frac{\widetilde{G}_{j)L-1}(u)}{r} + \epsilon_{ab(i}\partial_{j)aL-1}\frac{\widetilde{H}_{bL-1}(u)}{r}\right) \\ &+ \sum_{l\geq 2}\left(\partial_{L-2}\frac{\widetilde{I}_{ijL-2}(u)}{r} + \partial_{aL-2}\left(\frac{1}{r}\epsilon_{ab(i}\widetilde{J}_{j)bL-2}(u)\right)\right). \end{aligned} \tag{3.82}$$

If we put those multipole expansions into the equation $\partial_\nu q^{\mu\nu} = -\partial_\nu p^{\mu\nu}$ we find these relations among multipole moments:

$$
\begin{aligned}
& -A = \dot{\widetilde{A}} + \ddot{\widetilde{B}}, && -A_L = \dot{\widetilde{A}}_L + \ddot{\widetilde{B}}_L + \widetilde{C}_L \quad (l \geq 1), \\
& -B = \dot{\widetilde{B}} + \ddot{\widetilde{E}} + \widetilde{F}, && -B_L = \dot{\widetilde{B}}_L + \ddot{\widetilde{E}}_L + \widetilde{F}_L + \frac{1}{2}\widetilde{G}_L \quad (l \geq 1), \\
& -C_i = \dot{\widetilde{C}}_i + \frac{1}{2}\ddot{\widetilde{G}}_i, && -C_L = \dot{\widetilde{C}}_L + \frac{1}{2}\ddot{\widetilde{G}}_L + \widetilde{I}_L \quad (l \geq 2), \\
& -D_i = \dot{\widetilde{D}}_i + \frac{1}{2}\ddot{\widetilde{H}}_i, && -D_L = \dot{\widetilde{D}}_L + \frac{1}{2}\ddot{\widetilde{H}}_L + \widetilde{J}_L \quad (l \geq 2).
\end{aligned}
\tag{3.83}
$$

Those relations does not yield to a unique solution. However, if we seek for the solution with minimum number of summations, we get

$$
\begin{aligned}
q^{00} &= -\frac{A^{(-1)}}{r} - \partial_i \frac{A_i^{(-1)}}{r} + \partial_i \frac{C_i^{(-2)}}{r}, \\
q^{0i} &= -\frac{C_i^{(-1)}}{r} - \epsilon_{iab}\partial_a \frac{D_b^{(-1)}}{r} - \sum_{l\geq 2} \partial_{L-1} \frac{A_{iL-1}}{r}, \\
q^{ij} &= -\delta_{ij}\Big(\frac{B}{r} + \partial_i \frac{B_i}{r}\Big) + \sum_{l\geq 2}\Big(\partial_{L-2}\frac{\dot{A}_{ijL-2}}{r} + 2\delta_{ij}\partial_L \frac{B_L}{r} - 6\partial_{L-1(i}\frac{B_{j)L-1}}{r} \\
&\qquad + 3\partial_{L-2}\frac{\ddot{B}_{ijL-2}}{r} - \partial_{L-2}\frac{C_{ijL-2}}{r} - 2\partial_{aL-2}\frac{\epsilon_{ab(i}D_{j)bL-2}}{r}\Big),
\end{aligned}
\tag{3.84}
$$

as our solution [6]. Here $\dot{A} = \frac{1}{c}\frac{\partial A}{\partial t}$ and $A^{(-1)}(u) = \int_{-\infty}^{u} \mathrm{d}u' A(u')$. Then

$$
h_{nD}^{\mu\nu} = p^{\mu\nu} + q^{\mu\nu}, \tag{3.85}
$$

satisfies both $\Box h_{nD}^{\mu\nu} = \Lambda_{nD}^{\mu\nu}$ and $\partial_\nu h_{nD}^{\mu\nu} = 0$ gauge condition. Moreover, since $q^{\mu\nu}$ has no $\log r$ term in it, just like p, h_{nD} is also an $n-1$th degree polynomial of $\log r$, which finalizes the proof $\Box$.

The solution we found with this method will be called the particular multipolar post Minkowskian (MPM) solution, and it will be shown as a functional of $M = \{M_L, S_L\}$ and $W = \{W_L, X_L, Y_L, Z_L\}$ multipole moments as

$$
\sqrt{-g}\,g_{part}^{\mu\nu} = \mathfrak{g}_{part}^{\mu\nu} = \mathfrak{g}_{part}^{\mu\nu}[M, W]. \tag{3.86}
$$

3.4 Construction of a General MPM Solution

In this section, we will show that the most general solution can be written as the particular solution we have found in the last section, by using the methods in [6]. For convenience, let M stand for all multipole moments $\{M, W\}$.

The solution given in the last section was merely a particular solution, so it actually is not a general solution to field equations. Let us construct the general solution order by order. We already now the general solution for h_1. Then, general solution to the first order is given by

$$\mathfrak{g}^{\mu\nu}_{gen} = \eta^{\mu\nu} + G\ h^{\mu\nu}_1[M_1] + O(G^2) = \mathfrak{g}^{\mu\nu}_{part}[M_1] + O(G^2), \tag{3.87}$$

so the particular solution is general solution to the first order. Let's increase accuracy, and solve for second order contribution $\Box h^{\mu\nu}_{2\ gen} = \Lambda^{\mu\nu}_2$. If we subtract our particular solution from $h^{\mu\nu}_{2\ gen}$, then

$$\begin{aligned} &\Box(h^{\mu\nu}_{2\ gen} - h^{\mu\nu}_{2\ part}) = 0, \\ &\partial_\nu(h^{\mu\nu}_{2\ gen} - h^{\mu\nu}_{2\ part}) = 0. \end{aligned} \tag{3.88}$$

These are the equations for the first order, and general solution to them is just $h^{\mu\nu}_1$ with other arbitrary multipole moments, call $\{M_2, W_2\}$. Then we have

$$h^{\mu\nu}_{2\ gen} = h^{\mu\nu}_{2\ part}[M_1] + h^{\mu\nu}_1[M_2]. \tag{3.89}$$

Then

$$\begin{aligned} \mathfrak{g}^{\mu\nu}_{gen} &= \eta^{\mu\nu} + G\ h^{\mu\nu}_1[M_1] + G^2(h^{\mu\nu}_{2\ part}[M_1] + h^{\mu\nu}_1[M_2]) + O(G^3) \\ &= \eta^{\mu\nu} + Gh^{\mu\nu}_1[M_1 + G\ M_2] + G^2\ h^{\mu\nu}_{2\ part}[M_1] + O(G^3) \\ &= \mathfrak{g}^{\mu\nu}_{part}[M_1 + G\ M_2] + O(G^3). \end{aligned} \tag{3.90}$$

Let us assume that the particular solution is general up to order $k-1$ and define $M^{(k)} = \sum_{i=1}^{k} G^{i-1} M_i$. Then we assume

$$\mathfrak{g}^{\mu\nu}_{gen} = \mathfrak{g}^{\mu\nu}_{part}[M^{(k-1)}] + O(G^k). \tag{3.91}$$

Next equations we have to solve are $\Box h_k^{\mu\nu} = \Lambda_k^{\mu\nu}$ and $\partial_\nu h_k^{\mu\nu} = 0$. If we subtract the particular solution, then again

$$\begin{aligned}
&\Box(h_{k\ gen}^{\mu\nu} - h_{k\ part}^{\mu\nu}) = 0, \\
&\partial_\nu(h_{k\ gen}^{\mu\nu} - h_{k\ part}^{\mu\nu}) = 0.
\end{aligned} \tag{3.92}$$

General solution to them is again $h_1^{\mu\nu}$, thus we have

$$h_{k\ gen}^{\mu\nu} = h_{k\ part}^{\mu\nu}[M^{(k-1)}] + h_1^{\mu\nu}[M_k], \tag{3.93}$$

then we have

$$\begin{aligned}
\mathfrak{g}_{gen}^{\mu\nu} &= \eta^{\mu\nu} + G\ h_1^{\mu\nu}[M^{(k-1)}] + ... + G^{k-1} h_{k-1\ part}^{\mu\nu}[M^{(k-1)}] \\
&+ G^k(h_{k\ part}^{\mu\nu}[M^{(k-1)}] + h_1^{\mu\nu}[M_k]) + O(G^{k+1}) \\
&= \eta^{\mu\nu} + Gh_1^{\mu\nu}[M^{(k)}] + ... + G^k h_{k\ part}^{\mu\nu}[M^{(k-1)}] + O(G^{k+1}) \\
&= \mathfrak{g}_{part}^{\mu\nu}[M^{(k)}] + O(G^{k+1}).
\end{aligned} \tag{3.94}$$

Therefore, we conclude that the most general MPM solution to Einstein's field equations can be written as

$$\mathfrak{g}_{gen}^{\mu\nu} = \mathfrak{g}_{part}^{\mu\nu}[M^{(n-1)}, W^{(n-1)}] + O(G^n), \tag{3.95}$$

to any order n we choose [6]. We can symbolically write it as

$$\mathfrak{g}_{gen}^{\mu\nu} = \mathfrak{g}_{part}^{\mu\nu}\Big[\sum_{i=1}^{\infty} G^{i-1} M_i, \sum_{i=1}^{\infty} G^{i-1} W_i\Big]. \tag{3.96}$$

This means, any MPM metric can be represented via 6 multipole moment sets $\{M, S, W, X, Y, Z\}$. This number will go down to two in next sections.

3.5 Generality of the Canonical Solution

I the last section, it has been shown that we can represent the most general MPM expansion solution with 6 multipole moments by using our particular solution. In this section, by using the methods in [6], it will be shown that only mass and angular momentum multipole moments, M_L and S_L are sufficient to represent a general MPM geometry, i.e. we will show that general solution is related to a canonical solution via a coordinate transformation.

Canonical metric is defined as $\mathfrak{g}^{\mu\nu}_{can} = \mathfrak{g}^{\mu\nu}_{part}[M, W = 0]$, which means it does not include gauge multiple moments. To show that a general metric can be written in this form, instead of constructing a coordinate transformation, we will reconstruct a general solution out of canonical solution. The only difference from the last section will be that while constructing the general solution at a given order, we will apply a coordinate transformation that cancels the gauge multipole moments at the corresponding order. Here, it will be done only to the second order since the proof can be fulfilled by following the steps from the last section.

To the first order, the most general MPM solution is given as

$$\mathfrak{g}^{\mu\nu}_{gen} = \eta^{\mu\nu} + Gh^{\mu\nu}_1[M_1, W_1] + O(G^2). \tag{3.97}$$

We already know that the coordinate transformation $x^\mu \to x^\mu - Gw^\mu[W_1]$ will cancel those gauge moments. Then we have

$$\mathfrak{g}^{\mu\nu}_{gen} = \eta^{\mu\nu} + Gh^{\mu\nu}_1[M_1, 0] + O(G^2). \tag{3.98}$$

Let's continue to second order. We will solve the same equation as we have solved in the last section, the conclusion is

$$\begin{aligned}\mathfrak{g}^{\mu\nu}_{gen} &= \eta^{\mu\nu} + G\ h^{\mu\nu}_1[M_1, 0] + G^2(h^{\mu\nu}_{2\ part}[M_1, 0] + h^{\mu\nu}_1[M_2, W_2]) + O(G^3)\\ &= \eta^{\mu\nu} + Gh^{\mu\nu}_1[M_1 + G\ M_2, G\ W_2] + G^2\ h^{\mu\nu}_{2\ part}[M_1, 0] + O(G^3).\end{aligned} \tag{3.99}$$

Now, we will apply a coordinate transformation $x^\mu \to x^\mu - Gw^\mu[GW_2]$. That will generate an $O(G)$ change in any of the terms, so changes in the second order term can be neglected. Thus we have

$$\begin{aligned}\mathfrak{g}^{\mu\nu}_{gen} &= \eta^{\mu\nu} + Gh^{\mu\nu}_1[M_1 + G\ M_2, 0] + G^2\ h^{\mu\nu}_{2\ part}[M_1, 0] + O(G^3)\\ &= \mathfrak{g}^{\mu\nu}_{part}[M_1 + G\ M_2, 0] + O(G^3).\end{aligned} \tag{3.100}$$

Assume this goes all the way to the nth term. For the $n+1$th term, homogenous h_1 solution will generate a multipole moment $G^n W_{n+1}$, to be added to the first term. Then we will apply a coordinate transformation to cancel it. Effect on all terms will be $O(G^n)$, so any change in the gauge moments can be neglected except the change on the first term. We know how the coordinate transformation effects the first term, it will cancel $G^n W_{n+1}$ moment on it if we choose it to do so. We can erase it from all other terms, since that change can be neglected. Therefore, we can write

$$\mathfrak{g}^{\mu\nu}_{gen} = \mathfrak{g}^{\mu\nu}_{part}[M^{(n-1)}, 0] + O(G^n) = \mathfrak{g}^{\mu\nu}_{can}[M^{(n-1)}] + O(G^n), \tag{3.101}$$

to any order n [6]. This can be summarized symbolically as

$$\mathfrak{g}^{\mu\nu}_{gen} = \mathfrak{g}^{\mu\nu}_{can}\Big[\sum_{i=1}^{\infty} G^{i-1} M_i\Big]. \tag{3.102}$$

That means only mass and angular momentum multipole moments are sufficient to describe a generic MPM geometry.

3.6 Far Zone Structure of MPM Solution

In any radiation field emitted by a source, we have three characteristic regions: $r \ll \lambda$ (near zone), $r \sim \lambda$ (transition zone) and $r \gg \lambda$ (far zone), where λ is the characteristic wavelength of the radiation. Since GR is a nonlinear theory, we have another characteristic distance that measures the strength of nonlinear terms: $\frac{GM}{c^2}$ where M is the characteristic mass of the source. In this section, we will study the far-zone behavior of the MPM solution we have found.

We will be taking $r \to \infty$ limit. However, if we keep t constant, then taking r to be very large will make $u = t - r/c$ to be smaller than $-T$, so it will lead to solutions from the past with no radiation. That happens since we are taking the limit such that we look at the field at spatial infinity, which is beyond the reach of the radiation coming from the source. To observe the radiation, since radiation moves away with speed of light, we should look at future null infinity instead. Therefore we should take the $r \to \infty$ limit while keeping $u = t - r/c$ constant

The expansion proven in Theorem 2 is not useful for taking the far zone limit since it involves the Taylor expansions of multipole moments like $M_L(t - r/c)$ around $r \sim 0$. Thus, as done in [6], MPM solution will be reconstructed in a more favorable form to take the far zone limit. From this point in this section, we will take $c = 1$. Thus we will take $r \to \infty$ limit with $u = t - r$ kept constant.

In order to solve (3.37), another iterative expansion of d'Alembertian similar to (3.74) will be used

$$\begin{aligned}\Box_R^{-1}(\hat{n}^Q r^{B-k} F(t-r)) &= -\frac{\hat{n}^Q}{2(B-k+2)} r^{B-k+1} F^{(-1)}(t-r) \\ &+ \frac{(B-k+1-q)(B-k+2+q)}{2(B-k+2)} \Box_R^{-1}(\hat{n}^Q r^{B-k-1} F^{(-1)}(t-r)),\end{aligned} \tag{3.103}$$

where $F(u)$ is an infinitely many times differentiable past zero function of u. This equality again can be proved by using the uniqueness of the wave equation solutions. After applying this identity let us say s many times, we will have a remainder term proportional to $\Box_R^{-1}(\hat{n}^Q r^{B-k-s} F^{(-s)}(t-r))$. By using the angular expansion of the Green function of d'Alembertian, it has been shown that if $f(r, u)$ is a past zero function that behaves like $O(1/r^{N+1})$ in the far zone limit, then

$$\underset{B=0}{\mathrm{FP}}\, \Box_R^{-1}(\hat{n}^Q r^B f(r, t-r)) = \hat{\partial}_Q \Big(\frac{G(t-r)}{r} \Big) + \hat{n}^Q g(t, t-r), \tag{3.104}$$

where $G(u)$ is a past zero and $C^\infty(\mathbb{R})$ function and $g(r, u)$ is a past zero function that behaves like $O(1/r^N)$ in the far zone limit [6]. By using these facts and the pattern of the proof of theorem 2, the following theorem can be proved

Theorem 3 *We can expand $h_n^{\mu\nu}$ in the far zone limit for any positive integer N as*

$$h_n^{\mu\nu}(\vec{x}, t) = \sum_q \widehat{n}^Q \left(\sum_{\substack{1 \le a \le N \\ 0 \le p < n}} F^{\mu\nu}_{apnNQ}(t-r)\, \frac{\log^p r}{r^a} + I_N^Q(r, t-r) \right), \tag{3.105}$$

where $F(t-r)$ functions are infinitely many times differentiable past constant functions, and $I_N(r, t)$ is past zero, infinitely many times differentiable in $t - r = u$, and $O\big(\frac{1}{r^N}\big)$ function for $r \to \infty$ and $u = constant$.

Therefore again, h_n is an $n-1$th degree polynomial of $\log r$ in far zone limit too, as we had in $r \to 0$ limit in Theorem 2. As indicated in [6], these $\log r$ terms are merely a coordinate effect, and there should be a coordinate transformation that transforms (3.105) to an expansion only in $1/r$. This is because, by another approach, a metric with $1/r$ expansion has already been found in the far zone [13, 14, 15, 16]. Besides, even if it feels parallel to the last sections, the result of the Theorem 3 is undesirable. It is because in the given expansion, there will be terms that are proportional to $(\log r)^{n-1}/r$ in h_n [12]. That means, for large r, we have $h_{n+1} \sim \log r \; h_n \gg h_n$, which is undesirable for a perturbative expansion. Therefore, to make this MPM solution useful in the far zone, this coordinate transformation that transforms the expansion in Theorem 3 into an expansion in $1/r$ will be studied in chapter 4.

3.7 Near Zone Structure of the MPM Solution

In this section, the near zone behavior of the general MPM solution will be discussed, by using the methods done in [6]. The MPM solution obviously works only for outside of the world tube that contains matter: $r > a$. In order to examine near zone behavior of the metric, we need $a < r \ll \lambda$. To have the multipolar expansion be valid in the near zone, we need $\frac{a}{c} \ll \frac{\lambda}{c} = P$, where P represents the characteristic period of the motion of the source, which can be thought as the time between two maxima of the radiation pattern. Thus, we must have a source that satisfies $\frac{a}{P} \ll c$, which means that the source should move a lot slower than the speed of light in our coordinates. Otherwise, we cannot use our MPM solution in near zone. Besides, we also need $r \gg \frac{GM}{c^2}$ since we want the expansion in G work with a couple of terms. And lastly, we need to work with a large number of multipoles to say $r > a$ only, instead of $r \gg a$. In short, we have $a < r \ll \lambda$ and $r \gg \frac{GM}{c^2}$, and for that, the source should be a slow moving source.

From Theorem 2, we know that $h_n^{\mu\nu}$ can be expanded as

$$h_n^{\mu\nu}(\vec{x},t) = \sum_{\substack{a<N \\ 0\leq p<n}} F_{apnNQ}^{\mu\nu}(t)\ \widehat{n}^Q r^a \log^p r + S_N(\vec{x},t), \tag{3.106}$$

where S_N is described in the statement of the theorem. For convenience, the general canonical solution will be considered, which means $F(t)$ coefficients consist of M_L and S_L moments, their time derivatives, Kronecker delta and Levi Civita symbols. This expansion is valid for $r \sim 0$. In order to use this expansion to find the behaviour of the metric for $a < r \ll \lambda$, we can scale the r variable and replace it with r/λ. Then if we define $\tilde{r} = \frac{r}{\lambda}$, if the expansion can be constructed in the same way for $\tilde{r}$, then the near zone limit corresponds to the region that expansion is valid, which is $\tilde{r} \sim 0$. But, how to replace r with $\frac{r}{\lambda}$? If we look at h_1, we would observe that this replacement corresponds to scaling of multipole moments:

$$\begin{aligned} M_L &\rightarrow \frac{M_L}{\lambda^{l+3}}, \\ S_L &\rightarrow \frac{S_L}{\lambda^{l+4}}. \end{aligned} \tag{3.107}$$

After that we can re-construct h_n for $n > 1$ as it has been done before. If we define new spatial coordinates as $\tilde{x}^i = x^i/\lambda$, we then find $\tilde{\partial}_i = \lambda \partial_i$. For convenience, let us define $\tilde{\partial}_0 = \frac{\lambda}{c}\partial/\partial t$. If we write $h_n \rightarrow \tilde{h}_n$, then the new equations become

$$\begin{aligned} \frac{1}{\lambda^2}\Big(- \frac{\lambda^2}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial \tilde{x}^i}\frac{\partial}{\partial \tilde{x}^i}\Big)\tilde{h}_n = \tilde{\Lambda}_n &\sim \partial\tilde{h}\partial\tilde{h} + \tilde{h}\partial\partial\tilde{h} \\ &= \frac{1}{\lambda^2}(\tilde{\partial}\tilde{h}\tilde{\partial}\tilde{h} + \tilde{h}\tilde{\partial}\tilde{\partial}\tilde{h}). \end{aligned} \tag{3.108}$$

Thus we have

$$\tilde{\Box}\tilde{h}_n \sim \tilde{\partial}\tilde{h}\tilde{\partial}\tilde{h} + \tilde{h}\tilde{\partial}\tilde{\partial}\tilde{h}. \tag{3.109}$$

Structure of the equation is the same with (3.37) apart from the exchange of c and c/λ in the definition of d'Alembertian. Therefore Theorem 2 will still be valid for $\tilde{h}_n$, which means we can expand $\tilde{h}_n$ around $\tilde{r} = \frac{r}{\lambda} \sim 0$ as

$$\tilde{h}_n^{\mu\nu}(\tilde{x}^i,t) = \sum_{\substack{a<N \\ 0\leq p<n}} \tilde{F}_{apnNQ}^{\mu\nu}(t)\ \widehat{n}^Q \tilde{r}^a \log^p \tilde{r} + S_N(\tilde{x}^i,t), \tag{3.110}$$

where $\tilde{F}(t)$ functions are multiplications of multipole moments, their time derivatives, Kronecker deltas, Levi Civita tensors and powers of $1/\lambda$. If we define

$$E_n = M_{L_1}^{(a_1)} M_{L_2}^{(a_2)} ... M_{L_{n-s}}^{(a_{n-s})} S_{L_{n-s+1}}^{(a_{n-s+1})} ... S_{L_n}^{(a_n)}, \tag{3.111}$$

and

$$
\begin{aligned}
s(E_n) &= s, \\
b(E_n) &= s + \sum_{a=1}^{n} l_a,
\end{aligned}
\tag{3.112}
$$

and show the part of h_n that is proportional to a particular combination E_n as h_{E_n}:

$$
\tilde{h}_{E_n}(\tilde{x}^i, t) = \sum_{\substack{a<N \\ 0\leq p<n}} E_n(t)\ \widehat{n}^Q \tilde{r}^a \log^p \tilde{r}, \tag{3.113}
$$

then we would have

$$
\tilde{h}_n(\tilde{x}^i, t) = \sum_{E_n} h_{E_n}(\tilde{x}^i, t) + S_N(\tilde{x}^i, t), \tag{3.114}
$$

and

$$
h_n(\vec{x}, t) = \sum_{E_n} \frac{1}{\lambda^{3n+b(E_n)}} h_{E_n}\Big(\frac{\vec{x}}{\lambda}, t\Big) + S_N\big(\frac{\vec{x}}{\lambda}, t\big). \tag{3.115}
$$

Therefore we would end up with an expansion like

$$
h_n(\vec{x}, t) = \sum_{E_n} \frac{1}{\lambda^{3n+b(E_n)}} \left(\sum_{\substack{a<N \\ 0<p\leq n-1}} E_n(t) \widehat{n}^Q \Big(\frac{r}{\lambda}\Big)^a \log^p \Big(\frac{r}{\lambda}\Big) \right) + S_N\Big(\frac{\vec{x}}{\lambda}, t\Big), \tag{3.116}
$$

where all space and spacetime indices in E_n are kept hidden for convenience. This is an expansion around $r/\lambda \sim 0$, which corresponds to the near zone limit as described at the beginning of this section.

From (3.116), we can get the expansion of the metric in $1/c$, which is called the Post Newtonian (PN) expansion. We will not calculate the Post Newtonian terms, we will just derive the PN expansion structure. This subject will be discussed in a more detailed way in chapter 5, where one can find calculation of the terms to 1 PN order (to $O(1/c^2)$). As it was stated in the first paragraph of this section, (3.116) is valid only for slowly moving sources, which means the characteristic velocity of the source is much smaller than the speed of light $v \ll c$. If we choose a unit system that gives us numerically $a = O(1)$ and $P = O(1)$, then we would have $v \sim \frac{a}{P} = O(1)$ and $\lambda = cP = O(c)$. Therefore, numerically, we can put $\lambda = c$ in (3.116). Using the fact that $S_N(\vec{r}, t) = O(r^N)$ for $r \to o$, we find that in

$r = O(1)$ region the PN behaviour of the metric has the following structure

$$h_n(c) \sim \sum_{\substack{a<N \\ 0<p\leq n-1}} \left(\frac{1}{c}\right)^a \log^p\left(\frac{1}{c}\right) + O\left(\frac{1}{c^N}\right), \tag{3.117}$$

for any N. That shows only c^{-n} terms are not enough in post-Newtonian expansion of the metric. There are terms that include powers of log$c$ too. This is a result of nonlinearity of GR, and physically exist in near zone.

Those terms were actually predicted in [9]. Futamase tried to expand in $1/c$, or to be more precise, in $\epsilon \sim \frac{v}{c}$ where v is the characteristic speed of the source. It has been shown that some of those coefficients give logarithmically divergent integrals. Those retarded integrals can be calculated a bit more carefully so that they are actually proportional to powers of $\log \epsilon$. Also, it has been shown that those $\log \epsilon$ terms exist only if the multipole moment appearing in front of the term gives a contribution to radiation. This means that if quadrupole moment radiates, $\log \epsilon$ terms start at the quadrupole level, which is ϵ^5 term, otherwise they start at the octopole and so on.

Let us try to understand this result that relates logarithmic terms to radiation. In order to trace the source of a $\log r$ term back, we need to look for some specific terms in Λ. Let us specify the term we are looking for in h_2 as $F(t) r^s \log r$, and let us only focus on mass multipole moments M_L for convenience, since this is just to understand not to prove the relation between radiation and the logarithmic terms. Thus, we want spin zero terms $\partial h_1 \partial h_1 + h_1 \partial\partial h_1$, which are just r^a terms. Only for $a = -2, -3$ we have Laplace inverses that are proportional to $\log r$:

$$\begin{aligned} \Delta^{-1} r^{-2} &= \log r, \\ \Delta^{-1} r^{-3} &= -\frac{\log r}{r}. \end{aligned} \tag{3.118}$$

If we check the expansion of d'Alembertian inverse in terms of Laplace inverse, we see that only for $a \leq -2$ integers, we have $\log r$ terms. If $a = -2, -3$ there are no additional time derivatives, but if $a = -4, -5$ two additional time derivatives we have, and for $a = -6, -7$ we have four derivatives, and so on. As it will be explained later, we want as few time derivatives hitting on the multipole moments as possible. Therefore, we should investigate $a = -2, -3$ situation. If $a = -2$, we need only $1/r$ terms in h_1 and all derivatives in the structure on the lambda should hit the multipole moments. If $a = -3$, only one of those derivatives can hit the r^{-1} term in the h_1. In either way, we need the examine the $1/r$ term in h_1:

$$\begin{aligned} h_1^{00}(\vec{x}, t) &= -\frac{4}{c^2} \sum_{l \geq 0} \frac{(-1)^l}{l!} \partial_L \frac{M_L(u)}{r} \\ &= -\frac{4}{c^2} \frac{1}{r} \sum_{l \geq 0} \frac{(-1)^l}{l!} \partial_L M_L(u) + O\left(\frac{1}{r^2}\right). \end{aligned} \tag{3.119}$$

Let us choose the case $a = -2$. Then the terms in Λ_2 we are interested in are $\sum(\partial_L \dot{M}_L(u))^2/r^2$ and $\sum(\partial_L M_L(u)\partial_L \ddot{M}_L(u))/r^2$. Therefore, $\log r$ terms coming from $a = -2$ choice in h_2 are like

$$\begin{aligned} h_2 &\sim \sum \big((\partial_L \dot{M}_L(u))^2 + (\partial_L M_L(u)\partial_L \ddot{M}_L(u))\big) \log r \\ &\sim \sum \big((\widehat{n}^L {M_L}^{(l+1)}(u))^2 + (\widehat{n}^L M_L^{(l)}(u))(\widehat{n}^L({M_L}^{(l+2)}(u))\big) \log r. \end{aligned} \tag{3.120}$$

Obviously, to have a contribution like this, we need to have for at least one value of l that $l+1$th time derivative of M_L is not equal to zero. It is easy to see that this is the case for $a = -3$ case too. For smaller a values, we get stronger conditions like $M_L^{(s)}$ is nonzero for $s > l + 1$. Therefore, we find that $\log r$ contribution from mass multipole moment M_L exists only if $M_L^{(l+1)} \neq 0$.

What indicates that if there is radiation or not? Radiation means we have a nonzero energy flux to infinity. Here, this means $\Lambda \sim r^{-2}$ for large r. If we want to see this radiation in G^2 order, this means that $\Lambda_2 \sim r^{-2}$, therefore we should have $h_1 \sim r^{-1}$ for large r and $\dot{h}_1$ is nonzero. Immediately from (3.119), this means that at least for one value of l, $M_L^{(l+1)} \neq 0$. Therefore, if there is radiation, then there should be a $\log r$ term in the corresponding order; and visa versa.

Chapter 4

Gravitational Radiation

In this chapter, as in [12], we will reconstruct the metric in such a way that it will not contain $\log r$ terms in the far zone expansion. After that, we will discuss the behavior of this metric and the multipole moments in the far zone when there is gravitational radiation. In the last section, we will derive the formula of the energy carried by the gravitational waves.

4.1 Radiative Metric

Theorem 3 implies that the metric will have logarithmic terms in the far zone limit. However, these terms are merely a coordinate effect, since the metric in the far zone can be constructed as an expansion in inverse of the radial coordinate, $1/R$ only, as in [13, 14, 15] and [16]. In other and more precise words, there are coordinates $(T, \vec{X})$ such that we have

$$H_n^{\mu\nu}(\vec{X}, T) = \sum_{a<N} F^{\mu\nu}_{apnNQ}(T-R)\, \frac{\widehat{N}^Q}{R^a} + K_N(\vec{X}, T-R), \tag{4.1}$$

where H_n is the perturbation variable we have after the coordinate transformation, $R = \sqrt{X^i X^i}$, $N^i = \frac{X^i}{R}$ and K_N are infinitely many times differentiable in time, past zero functions that behaves as $O\big(\frac{1}{R^N}\big)$ in the far zone limit $R \to \infty$

with $T - R$ kept constant.

This will be shown by separating the $h^{\mu\nu}$ into static and dynamic parts, as done in [6] and [12]. Firstly, we will prove a stronger version of theorem 1 which can be found in [6]:

Theorem 4 *We can expand $h^{\mu\nu}_{nS}$ around $r = 0$ as*

$$h^{\mu\nu}_{nS}(\vec{r}) = \sum_{l,a \geq n} F^{\mu\nu}_{naL} \hat{n}^L r^{-a}. \tag{4.2}$$

We already know that the statement of the theorem is satisfied for $n = 1$, since we have

$$h^{\mu\nu}_{1S}(\vec{r}) = \sum_{l} \partial_L \Big(\frac{U^{\mu\nu L}_S}{r} \Big) = \sum_l U^{\mu\nu L}_S \partial_L \Big(\frac{1}{r} \Big) = \sum_l \frac{\tilde{U}^{\mu\nu L}_S \hat{n}^L}{r^{l+1}}. \tag{4.3}$$

Now, let us assume that the statement is satisfied for all $m < n$. Then we would find that

$$\Lambda^{\mu\nu}_{nS}(\vec{r}) = \sum_{l,a \geq n} K^{\mu\nu}_{naL} \hat{n}^L r^{-a-2}. \tag{4.4}$$

Proceeding like we have done in theorem 1, we define

$$\tilde{h}^{\mu\nu}_{nS}(B) = \Delta^{-1}(r^B \Lambda^{\mu\nu}_{nS}) = \sum_{l,a \geq n} K^{\mu\nu}_{naL} \Delta^{-1}(\hat{n}^L r^{B-a-2}). \tag{4.5}$$

Using (3.42)

$$\tilde{h}^{\mu\nu}_{nS}(B) = \sum_{l,a \geq n} K^{\mu\nu}_{naL} \frac{\hat{n}^L r^{B-a}}{(B + l - a + 1)(B - l - a)}. \tag{4.6}$$

In Theorem 1, it was proven that the h_{nS}, the nth coefficient of the expansion of h_S can be obtained by taking the finite part of $\tilde{h}_{nS}(B)$. Since we have $a > n$ and $l \geq 0$, we cannot have $a + l = 0$. Then the only possible way to generate logarithmic terms is the existence of the terms satisfying $l = a - 1$.

We know we can write the K coefficients as

$$K^{\mu\nu}_{naL} \sim (\epsilon\delta) M_{L_1} S_{L_2} ... Z_{L_n}. \tag{4.7}$$

Dimensional analysis again gives us

$$a = n + \sum_{i=1}^{n} l_i + w + 2x + z, \tag{4.8}$$

where there are w many W_L, x many X_L and z many Z_L multipole moments in the K coefficient. Without loss of generality, we can assume that all gauge multipole moments are zero and consider only the canonical solution since general solution can be written by using the canonical solution plus a small coordinate transformation as indicated in (3.102). Then we can just write

$$a = n + \sum_{i=1}^{n} l_i. \tag{4.9}$$

Now, we see that $K^{\mu\nu}_{naL}$ has two spacetime and l space indices, and it is already symmetric and traceless on those space indices. Therefore, for $\mu\nu = 00$, it already belongs to the spin l representation f the $SO(3)$ group. For $\mu\nu = 0i$ it belongs to spin $l-1, l$ or $l+1$; for $\mu\nu = ij$ to spin $l-2, ..., l+2$ representations. Therefore, the minimum spin it can have is $l-2$. However, it is a product of n STF tensors. Due to angular momentum addition rules, we have

$$l - 2 \leq spin \leq \sum_{i=1}^{n} l_i. \tag{4.10}$$

If we put this to the result of the dimensional analysis, we have

$$n \leq a - l + 2. \tag{4.11}$$

Then, if a pole exists in (4.6), which means if we have $l = a - 1$ for at least a term, then we have

$$n \leq 3. \tag{4.12}$$

Thus, to prove the Theorem 4 we need to consider only the cases $n = 1$, $n = 2$ and $n = 3$. We know it is true for $n = 1$. Let us check $n = 2$ and 3.

Using (4.3), we find that

$$\Lambda_{2S} \sim \sum_{pq} \partial_P\Big(\frac{1}{r}\Big)\partial_Q\Big(\frac{1}{r}\Big) \sim \sum_{pq} \frac{\hat{n}^P \hat{n}^Q}{r^{p+q+2}}. \tag{4.13}$$

Then, as proceeding like in the Theorem 1, we have

$$\tilde{h}_{2S}(B) = \Delta^{-1}(r^B \Lambda_{2S}) \sim \Delta^{-1} \sum_{pq} \frac{\hat{n}^P \hat{n}^Q}{r^{p+q+2-B}}. \tag{4.14}$$

Now, we need to rewrite $\hat{n}^P \hat{n}^Q$ in terms of $\hat{n}^K$ in order to be able to use the definition of the inverse Laplacian in (3.42). Since there are no antisymmetric indices in $\hat{n}^P$, this expansion cannot include Levi Civita tensor, it can only has Kronecker delta. With this information, it is easy to show that

$$\hat{n}^P \hat{n}^Q \sim \sum_{\substack{|p-q| \le k \le p+q \\ p+q-k=even}} \hat{n}^K. \tag{4.15}$$

Then we have

$$\tilde{h}_{2S}(B) \sim \Delta^{-1} \sum_{a-k=even} \frac{\hat{n}^K}{r^{a-B}} \sim \sum_{a-k=even} \frac{\hat{n}^K r^{2+B-a}}{(B+k-a+3)(B+2-k-a)}. \tag{4.16}$$

Now, we know that $a = p + q + 2$ and $p, q \ge 1$. Therefore we cannot have $2 - k - a = 0$. We also have $k - a$ to be an even integer, therefore we cannot have $k - a + 3 = 0$. Thus, no term in the expansion of $\tilde{h}_{2S}(B)$ has a pole on $B = 0$. Then the finite part means just putting $B = 0$, and no $\log r$ term is generated. Therefore, the statement is also satisfied for the case $n = 2$, and we can write

$$h^{\mu\nu}_{2S}(\vec{r}) = \sum_{l,a \ge 2} F^{\mu\nu}_{aL} \hat{n}^L r^{-a}. \tag{4.17}$$

For $n = 3$, we should consider only spin 2 terms in Λ^{ij}, since we need spin of the coefficient to be exactly $l-2$ to have $n = 3$. Therefore the critical terms that can generate $\log r$ terms for $n = 3$ case are

$$\begin{aligned}\Lambda^{ij}_{3critical} = \sum_{l_1,l_2,l_3} \frac{\hat{n}^{L_1L_2L_3ij}}{r^{l_1+l_2+l_3+5}} \{ & A M_{L_1} M_{L_2} M_{L_3} + B M_{L_1} M_{L_2} S_{L_3} \\ & + C M_{L_1} S_{L_2} S_{L_3} + D S_{L_1} S_{L_2} S_{L_3} \},\end{aligned} \tag{4.18}$$

where we have $l = l_1 + l_2 + l_3 + 2$. We can immediately say that $B = D = 0$ by using parity transformations. We need the distance to be the same under $x^i \to -x^i$, therefore we have $h^{00} \to h^{00}$, $h^{0i} \to -h^{0i}$ and $h^{ij} \to h^{ij}$ under parity. Therefore $h^{ij}_3 \to h^{ij}_3$ and $\Lambda^{ij}_3 \to \Lambda^{ij}_3$ as well. From (3.33), we find

$$M_L \to (-1)^l M_L, \quad S_L \to (-1)^{l+1} S_L. \tag{4.19}$$

Therefore, terms with odd number of S_L in (4.18) will generate -1 under parity. Since we have $\Lambda_3^{ij} \to \Lambda_3^{ij}$, those terms should not exist, and $B = D = 0$. If we put h_1 and h_2 in the definition of Λ_3, after a lengthy calculation, $A = C = 0$ will also be found. Thus,

$$\Lambda^{ij}_{3critical} = 0, \tag{4.20}$$

and there are no $\log r$ terms in h_3^{ij}, which finalizes the proof $\Box$.

Thus, we actually do not need the radiative coordinates to get rid of these logarithm terms for the static part. However, there exist logarithmic terms in the far zone expansion of the metric which comes from the dynamic part. At the end of [6] and at the beginning of [12], the origin of these $\log r$ terms is discussed. The idea that these discussions yielded is that the $\log r$ terms exist because the far zone limit is taken in a wrong way. $u = t - r = constant$ surfaces are not the surfaces that the radiation will follow. The reason is those surfaces are not null surfaces! Radiation moves with speed of light do not mean that its coordinate velocity is c, it means that it will follow null surfaces. When there is a source at the center, to the first order in G, our metric will only have a contribution of h_1, which will alter the null surfaces. With that contribution, we have $\sqrt{-g}g^{\mu\nu} = h^{\mu\nu} + \eta^{\mu\nu}$. Metric determinant is $|g| = 1 + \eta_{\mu\nu}h^{\mu\nu} + O(h^2)$. Defining $h = \eta_{\mu\nu}h^{\mu\nu}$ then we have

$$g^{\mu\nu} = \eta^{\mu\nu}\left(1 - \frac{h}{2}\right) + h^{\mu\nu} + O(h^2), \tag{4.21}$$

and

$$g_{\mu\nu} = \eta_{\mu\nu}\left(1 + \frac{h}{2}\right) - h^{\mu\nu} + O(h^2). \tag{4.22}$$

If we just consider leading order post Newtonian terms, since $h^{00} \sim O(c^{-2})$, $h^{0i} \sim O(c^{-3})$ and $h^{ij} \sim O(c^{-4})$, we have

$$g_{\mu\nu} = \eta_{\mu\nu}\left(1 - \frac{h^{00}}{2}\right) - h_{\mu\nu} + O(h^2). \tag{4.23}$$

The line element will be

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &= -\left(1 + \frac{h^{00}}{2}\right)c^2dt^2 + \left(1 - \frac{h^{00}}{2}\right)dx^idx^i + O(c^{-3}). \end{aligned} \tag{4.24}$$

Null paths are the solutions of $ds^2 = 0$. Let us consider a path that does not rotate around the center, which means it goes away direcly from the center. Then for $r = \sqrt{x^i x^i}$, we have

$$0 = ds^2 = -\left(1 + \frac{h^{00}}{2}\right)c^2 dt^2 + \left(1 - \frac{h^{00}}{2}\right)dr^2 + O(c^{-3}). \tag{4.25}$$

Thus

$$cdt = \left(1 - \frac{h^{00}}{2}\right)dr + O(c^{-3}). \tag{4.26}$$

Considering only leading order terms in $\frac{1}{r}$

$$\begin{aligned} h_1^{00}(\vec{x}, t) &= -\frac{4}{c^2}\sum_{l\geq 0}\frac{(-1)^l}{l!}\partial_L\frac{M_L(u)}{r} \\ &\sim -\frac{4}{c^2}\left(\frac{M}{r} - \partial_i\left(\frac{M_i}{r}\right) + \partial_i\partial_j\left(\frac{M_{ij}}{2r}\right)\right) \\ &= -\frac{4}{c^2}\left(\frac{M}{r} - M_i\partial_i\frac{1}{r} + M_{ij}\partial_i\partial_j\frac{1}{2r}\right) + O(c^{-3}). \end{aligned} \tag{4.27}$$

Then

$$\begin{aligned} cdt &= \left(1 + \frac{2G}{c^2}\left(\frac{M}{r} + \frac{M_i n^i}{r^2} + \frac{3M_{ij}\widehat{n}^{ij}}{2r^4}\right)\right)dr + O(c^{-3}) \\ &= dr + \frac{2G}{c^2}\left(M d\log r - M_i n^i d\left(\frac{1}{r}\right) - \frac{M_{ij}\widehat{n}^{ij}}{2}d\left(\frac{1}{r^3}\right)\right) + O(c^{-3}). \end{aligned} \tag{4.28}$$

Thus the equation for null geodesics is

$$ct - r - \frac{2G}{c^2}\left(M\log r - \frac{M_i n^i}{r} - \frac{M_{ij}\widehat{n}^{ij}}{2r^3}\right) = constant + O(c^{-3}). \tag{4.29}$$

Therefore the far zone limit is not $r \to \infty$ with $t - r = constant$, but instead $t - r - 2GM\log r = constant$. Thus, we see that there is a logarithmic contribution that cannot be neglected. But since this term is proportional to G, and we were expanding metric in terms of G, we get $\log r$ terms. This is simply the Taylor expansion of multipole moments $M_L(t - r - 2GM\log r)$ on G, plus the coordinate transformations [6].

The result we found gives a clue about the coordinate transformation we need to apply to make logarithm terms disappear. If we write the transformation as

$X^\mu \to x^\mu + \xi^\mu$ and expand it as

$$\xi^\mu = G\xi_1^\mu + G^2\xi_2^\mu + ... + G^n\xi_n^\mu + ..., \tag{4.30}$$

we found that

$$\xi_1^\mu = 2M\eta^{0\mu}\log\left(\frac{r}{r_0}\right), \tag{4.31}$$

should be chosen, where r_0 is an arbitrary lengthscale that we will equate to 1 for convenience, as we have done to the logarithmic terms we encountered in the last chapter. After this transformation

$$H_1^{\mu\nu} = h_1^{\mu\nu} + \partial^\mu\xi_1^\nu + \partial^\nu\xi_1^\mu - \eta^{\mu\nu}\partial_\lambda\xi_1^\lambda. \tag{4.32}$$

Obviously, since the transformation only includes derivatives of ξ^μ and we can write h_1 as a sum of terms like $n^L r^{-a}F(t-r)$ where F is past zero function, so we can write H_1 as a sum of terms like that.

Now, we want to calculate H_n, but first observe that $(T, \vec{X})$ coordinate system is not harmonic:

$$\partial_\nu H_1^{\mu\nu} = \Box\xi^\mu = \frac{2M}{r^2}\eta^{0\mu} \neq 0. \tag{4.33}$$

Therefore to find H_n, the method we have done so far is not enough. We need to modify it a bit. Firstly, the equation we need to solve becomes a little different than (3.5). It will be

$$\Box H^{\mu\nu} = \partial^\mu\Phi^\nu + \partial^\nu\Phi^\mu - \eta^{\mu\nu}\partial_\lambda\Phi^\lambda + \Omega^{\mu\nu}[H], \tag{4.34}$$

where $\Phi^\mu = \partial_\nu H^{\mu\nu}$, and $\Omega^{\mu\nu}$ is a divergenceless functional of $H^{\mu\nu}$ which is given as [10]

$$\begin{aligned}
\Omega^{\mu\nu} &= \partial_\lambda\partial_\sigma(H^{\mu\lambda}H^{\nu\sigma} - H^{\lambda\sigma}H^{\mu\nu}) + \partial_\lambda H^{\mu\nu}\partial_\sigma H^{\sigma\lambda} - \partial_\lambda H^{\mu\lambda}\partial_\sigma H^{\nu\sigma} \\
&\quad - g_{\sigma\beta}g^{\nu\tau}\partial_\lambda H^{\mu\beta}\partial_\tau H^{\sigma\lambda} - g_{\lambda\beta}g^{\mu\tau}\partial_\tau H^{\sigma\beta}\partial_\sigma H^{\nu\lambda} \\
&\quad + g^{\alpha\sigma}g_{\lambda\beta}\partial_\alpha H^{\mu\beta}\partial_\sigma H^{\nu\lambda} + \frac{1}{2}g_{\sigma\beta}g^{\mu\nu}\partial_\lambda H^{\alpha\beta}\partial_\alpha H^{\sigma\lambda} \\
&\quad + \frac{1}{2}\partial_\tau H^{\alpha\beta}\partial_\kappa H^{\sigma\lambda}\Big(g^{\tau\mu}g^{\kappa\nu} - \frac{g^{\tau\kappa}g^{\mu\nu}}{2}\Big)\Big(g_{\lambda\beta}g_{\sigma\alpha} - \frac{g_{\alpha\beta}g_{\lambda\sigma}}{2}\Big).
\end{aligned} \tag{4.35}$$

If we apply PM expansion on both sides of (4.34), we will have this set of equations

$$\begin{aligned}
\Phi_n^\mu &= \partial_\nu H_n^{\mu\nu}, \\
\Box H_n^{\mu\nu} &= \partial^\mu\Phi_n^\nu + \partial^\nu\Phi_n^\mu - \eta^{\mu\nu}\partial_\lambda\Phi_n^\lambda + \Omega_n^{\mu\nu}[H_m; 1 \leq m < n],
\end{aligned} \tag{4.36}$$

for $n = 1, 2, 3, ...$ Observe that the former can be considered as a result of the latter if Φ^μ_n are past zero functions, so we have a freedom to choose Φ^μ_n from past zero functions. Now, let us show that we can choose Φ^μ_n in such a way that $H^{\mu\nu}_n$ can be written as an expansion on $1/R$:

Theorem 5 *There exists a choice of Φ^μ_n such that we can expand $H^{\mu\nu}_n$ as*

$$H^{\mu\nu}_n(\vec{X}, T) = \sum_q \widehat{N}^Q \left(\sum_{1 \le a \le N} \frac{F^{\mu\nu}_{anNQ}(T - R)}{R^a} + I^Q_N(R, T - R) \right), \tag{4.37}$$

where $F(T - R)$ functions are infinitely many times differentiable past constant functions, and $I_N(R, T)$ is past zero, infinitely many times differentiable in $T - R = U$, and $O\left(\frac{1}{R^N}\right)$ function for $R \to \infty$ and $U = constant$ limit.

In the proof, for convenience, instead of T and R, we will use t and r, since we will not be considering the Harmonic coordinates in the proof. Now, let us consider the static part first. It is obvious from the form of the coordinate transformation (4.32) that if we apply this coordinate transformation from the beginning by using (2.4), the derivatives of ξ^μ and the derivatives of the metric on the ξ_μ direction will be included. Since we know that h_{cannS} has no logarithms in it, and their derivative on the ξ_μ direction is zero because they are not time dependent, obviously we find that H_{nS} has no logarithmic terms in it.

We can continue with the dynamic part. We know that we can write H_1 in the given form in the theorem from (4.32). Let us proceed with $n = 2$. We want to solve

$$\Box H^{\mu\nu}_{2D} = \partial^\mu \Phi^\nu_2 + \partial^\nu \Phi^\mu_2 - \eta^{\mu\nu} \partial_\lambda \Phi^\lambda_2 + \Omega^{\mu\nu}_{2D}[H_1]. \tag{4.38}$$

We know that the solution to this equation is

$$H^{\mu\nu}_2 = \mathrm{FP}\, \Box^{-1}_R \left(\partial^\mu \Phi^\nu_2 + \partial^\nu \Phi^\mu_2 - \eta^{\mu\nu} \partial_\lambda \Phi^\lambda_2 + \Omega^{\mu\nu}_2[H_1] \right) + q^{\mu\nu}_2, \tag{4.39}$$

where q_2 satisfies $\Box q_2 = 0$ and is there to adjust the divergence of H_2 as we used it in Theorem 2 before in equation (3.80). Here, we should always remember that the vector field Φ_2 can arbitrarily be chosen. Let us track the logarithmic terms that can be generated from Ω_{2D} in the far zone limit. If we expand H_1 in Ω_{2D}, we get

$$\Omega_{2D}(\vec{x},t) = \sum_l \hat{n}^L \Big(\sum_{2\le a<N+1} \frac{G_{La}(t-r)}{r^a} + J^L_{N+1}(r,t-r) \Big), \tag{4.40}$$

where G_{La} are past zero functions and J^L_N are like the I^Q_N functions in the theorem. Therefore, we have

$$\mathrm{FP}\,\Box_R^{-1}\Omega_{2D} = \sum_{\substack{l\ge 0\\ 2\le a<N}} \underset{B=0}{\mathrm{FP}}\,\Box_R^{-1}\big(r^{B-a}\hat{n}^L G_{La}(t-r)\big) + \sum_l \hat{n}^L \tilde{J}^L_N(r,t-r), \tag{4.41}$$

where $\tilde{J}_N$ are like I^Q_N in the theorem. Now, to have logarithmic terms from the FP of the retarded integral, that integral should have a pole on $B=0$. Therefore, from the far zone expansion of inverse d'Alembertian (3.103), we see that for $a\ge 3$ in (4.41) there will be no $\log r$ term generated. Then the only logarithmic contribution comes from $a=2$. Therefore we should choose Φ_2 in such a way that it cancels the r^{-2} term in Ω_{2D} in (4.38).

Due to the structure of $\Omega_2 \sim \partial H_1 \partial H_1 + H_1 \partial\partial H_1$, we see that r^{-2} term of it includes only time derivatives of $1/r$ term in H_1. Let us define the $1/r$ term of the harmonic coordinate metric as

$$h_1(\vec{x},t) = \frac{z(t-r,\vec{n})}{r} + O\Big(\frac{1}{r^2}\Big), \tag{4.42}$$

which means

$$\begin{aligned}
z^{00}(u,\vec{n}) &= -4\sum_{l\ge 2}\frac{n_L}{l!}M_L^{(l)}(u),\\
z^{0i}(u,\vec{n}) &= -4\sum_{l\ge 2}\frac{n_{L-1}}{l!}M_{iL-1}^{(l)}(u)\\
&\quad + 4\sum_{l\ge 2}\frac{l}{(l+1)!}\epsilon_{iab}n_{aL-1}S_{bL-1}^{(l)}(u),\\
z^{ij}(u,\vec{n}) &= -4\sum_{l\ge 2}\frac{n_{L-2}}{l!}M_{ijL-1}^{(l)}(u)\\
&\quad + 8\sum_{l\ge 2}\frac{l}{(l+1)!}n_{aL-2}\epsilon_{ab(i}S_{j)bL-2}^{(l)}(u).
\end{aligned} \tag{4.43}$$

Note that since all terms include at least second time derivative of the multipole moments, $z^{\mu\nu}$ are past zero functions. Then if we define the $1/r$ component of the radiative metric H_1 as

$$H_1(\vec{x},t) = \frac{K(t-r,\vec{n})}{r} + O\Big(\frac{1}{r^2}\Big), \tag{4.44}$$

from (4.32) and definition of Φ_1, for $k^0 = 1$ and $k^i = n^i = x^i/r$ we find

$$K^{\mu\nu}(t-r,\vec{n}) = 2M(\eta^{0\mu}k^\nu + \eta^{0\nu}k^\mu) + z^{\mu\nu}(t-r,\vec{n}). \tag{4.45}$$

Then, if we define the r^{-2} component of Ω_{2D} as

$$\Omega_{2D}(\vec{x},t) = \frac{A_2(t-r,\vec{n})}{r^2} + O\Big(\frac{1}{r^3}\Big), \tag{4.46}$$

we find

$$A_2^{\mu\nu} = k^\mu k^\nu\Big(\frac{1}{2}\dot{z}^{\alpha\beta}\dot{z}_{\alpha\beta} - \frac{1}{4}\dot{z}^\alpha_\alpha\dot{z}^\beta_\beta\Big), \tag{4.47}$$

where $\dot{z}$ is time derivative of z, and $z_{\mu\nu} = \eta_{\mu\alpha}z^\alpha_\nu = \eta_{\mu\alpha}\eta_{\nu\beta}z^{\alpha\beta}$. If we introduce for convenience, as in [12]

$$\mathcal{E}(t-r,\vec{n}) = \int_{-\infty}^{t-r} ds\Big(\frac{1}{4}\dot{z}^{\alpha\beta}\dot{z}_{\alpha\beta} - \frac{1}{8}\dot{z}^\alpha_\alpha\dot{z}^\beta_\beta\Big)(s,\vec{n}), \tag{4.48}$$

then we have

$$A_2^{\mu\nu}(t-r,\vec{n}) = 4k^\mu k^\nu\dot{\mathcal{E}}(t-r,\vec{n}). \tag{4.49}$$

If we put this in (4.38), we get

$$\Box H_{2D}^{\mu\nu} = \partial^\mu\Phi_2^\nu + \partial^\nu\Phi_2^\mu - \eta^{\mu\nu}\partial_\lambda\Phi_2^\lambda + \frac{4}{r^2}k^\mu k^\nu\dot{\mathcal{E}}(t-r,\vec{n}) + O\Big(\frac{1}{r^3}\Big). \tag{4.50}$$

If we stare at this equation long enough, it becomes obvious that

$$\Phi_2^\mu = \frac{2}{r^2}k^\mu\mathcal{E}(t-r,\vec{n}), \tag{4.51}$$

would be a solution that leaves no r^{-2} terms on the right hand side.

$$\begin{aligned}\partial_\nu\Phi_2^\mu &= \frac{2}{r^2}k^\mu\partial_\nu\mathcal{E}(t-r,\vec{n}) + O\Big(\frac{1}{r^3}\Big)\\ &= \frac{2}{r^2}k^\mu\partial_\nu(t-r)\dot{\mathcal{E}}(t-r,\vec{n}) + O\Big(\frac{1}{r^3}\Big).\end{aligned} \tag{4.52}$$

$\partial_\nu(t-r)$ is 1 for $\nu = 0$ and n^i for $\nu = i$. Therefore we have

$$\partial_\nu(t-r) = -k_\nu. \tag{4.53}$$

Putting it in the equation above,

$$\partial_\nu \Phi_2^\mu = -\frac{2}{r^2} k^\mu k_\nu \dot{\mathcal{E}}(t-r,\vec{n}) + O\Big(\frac{1}{r^3}\Big). \tag{4.54}$$

Thus

$$\begin{aligned}\partial^\mu \Phi_2^\nu + \partial^\nu \Phi_2^\mu - \eta^{\mu\nu}\partial_\lambda \Phi_2^\lambda &= -\frac{2}{r^2}(2k^\mu k^\nu - \eta^{\mu\nu}k^\lambda k_\lambda)\dot{\mathcal{E}}(t-r,\vec{n}) + O\Big(\frac{1}{r^3}\Big) \\ &= -\frac{4}{r^2}k^\mu k^\nu \dot{\mathcal{E}}(t-r,\vec{n}) + O\Big(\frac{1}{r^3}\Big),\end{aligned} \tag{4.55}$$

which means we have

$$\Box H_{2D}^{\mu\nu} = O\Big(\frac{1}{r^3}\Big). \tag{4.56}$$

Therefore, Theorem 5 is true for $n = 2$ too. Now, as outlined in [7], let us reproduce the proof in [17] and try to cancel the r^{-2} terms in Ω_{nD} for $n \geq 3$. The crucial reason of getting this result is the fact that we can write the r^{-2} term of $\Omega_{2D}^{\mu\nu}$ in the following form

$$\Omega_{2D}^{\mu\nu}(\vec{x},t) = \frac{4k^\mu k^\nu \dot{\mathcal{E}}(t-r,\vec{n})}{r^2} + O\Big(\frac{1}{r^3}\Big). \tag{4.57}$$

That means, if we can write r^{-2} terms in $\Omega_{nD}^{\mu\nu}$ in the same form, it is obvious that we can find a Φ_n to cancel the logarithmic terms in H_{nD}!.

We have $\partial_\mu H_1^{\mu\nu} = O(1/r^2)$ and $\partial_\mu H_2^{\mu\nu} = O(1/r^2)$. Let us assume that for all $m < n$. To calculate the r^{-2} term in Ω_n, we need to take only two H_m multiplied, and only the r^{-1} parts of them should be considered. Since Ω is at least quadratic in H, the terms that involve the divergence $\partial_\mu H_m^{\mu\nu}$ can be neglected, because $H^{\alpha\beta}\partial_\mu H_m^{\mu\nu} = O(r^{-3})$ in the far zone limit. We also know that

$$\partial_\sigma H_m^{\mu\nu} = -k_\sigma \dot{H}_m^{\mu\nu} + O\Big(\frac{1}{r^2}\Big). \tag{4.58}$$

Then we can write Ω_n as

$$\Omega_n^{\mu\nu} = \frac{k^\mu k^\nu}{2}\Big(\eta_{\lambda\beta}\eta_{\sigma\alpha} - \frac{\eta_{\alpha\beta}\eta_{\lambda\sigma}}{2}\Big)\sum_{m\leq n-1} \dot{H}_m^{\alpha\beta}\dot{H}_{n-m}^{\sigma\lambda} + O\Big(\frac{1}{r^3}\Big). \tag{4.59}$$

Thus, we can write the dynamic part of it as

$$\Omega_{nD}^{\mu\nu} = \frac{4k^\mu k^\nu \mathcal{E}_n(t-r,\vec{n})}{r^2} + O\Big(\frac{1}{r^3}\Big). \tag{4.60}$$

Then we can choose

$$\Phi_n^\mu = \partial_\nu H_{nD}^{\mu\nu} = \frac{2}{r^2} k^\mu \mathcal{E}_n(t-r,\vec{n}), \tag{4.61}$$

to cancel the logarithm terms in H_n as we have done for $n = 2$. Besides, we have found $\partial_\nu H_{nD}^{\mu\nu} = O(r^{-2})$. Since we already have $H_{nS}^{\mu\nu} = O(r^{-1})$, then we have $\partial_\nu H_{nS}^{\mu\nu} = O(r^{-2})$. Therefore we have found $\partial_\nu H_n^{\mu\nu} = O(r^{-2})$ as an additional result $\Box$.

As given in [12], there is a better proof that also shows that we can make $H_n = O(r^{-2})$ for $n \geq 2$. The significance of this additional constraint will be discussed at the end of this section. We can write this additional constraint as a separate theorem:

Theorem 6 *There exists a choice of coordinates such that the statement of Theorem 5 is true, and we have*

$$H_n^{\mu\nu}(\vec{X},T) = O\Big(\frac{1}{R^2}\Big), \quad n \geq 2, \tag{4.62}$$

in the far zone limit.

Let us continue to the proof above from (4.56). To avoid confusion, we will represent H_2 defined in Theorem 5 by $\tilde{H}_2$. We want to cancel the r^{-1} term in $\tilde{H}_2$. Let us define that term as

$$\tilde{H}_2^{\mu\nu} = -\frac{Z^{\mu\nu}(t-r,\vec{n})}{r} + O\Big(\frac{1}{r^2}\Big). \tag{4.63}$$

Now, we know that

$$\partial_\mu \tilde{H}_2^{\mu\nu} = O\Big(\frac{1}{r^2}\Big). \tag{4.64}$$

If we put the definition above in it, we find

$$k_\mu \dot{Z}^{\mu\nu}(t-r,\vec{n}) = \frac{\partial}{\partial t}(k_\mu Z^{\mu\nu}(t-r,\vec{n})) = 0. \tag{4.65}$$

From Theorem 1, Theorem 4 and our choices of Φ_1^μ and Φ_2^μ, we can say that the static part of H_2 is $O(r^{-2})$. Therefore we have this $Z^{\mu\nu}(t-r,\vec{n})$ to be a past zero function. Then we have

$$k_\mu Z^{\mu\nu}(t-r,\vec{n}) = 0. \tag{4.66}$$

If we write the angular or STF expansion of $Z^{\mu\nu}$ as

$$Z^{\mu\nu}(t-r,\vec{n}) = \sum_{l\geq 0} n^L \tilde{Z}_L^{\mu\nu}(t-r), \tag{4.67}$$

then we can re-write the r^{-1} term of H_2 as

$$\begin{aligned}
\tilde{H}_2^{\mu\nu} &= -\frac{Z^{\mu\nu}(t-r,\vec{n})}{r} + O\Big(\frac{1}{r^2}\Big) \\
&= -\frac{1}{r}\sum_{l\geq 0} n^L \tilde{Z}_L^{\mu\nu}(t-r) + O\Big(\frac{1}{r^2}\Big) \\
&= -\sum_{l\geq 0} \partial_L\Big(\frac{(-1)^l \tilde{Z}_L^{\mu\nu(-l)}(t-r)}{r}\Big) + O\Big(\frac{1}{r^2}\Big).
\end{aligned} \tag{4.68}$$

Then if we define

$$p^{\mu\nu} = \sum_{l\geq 0} \partial_L\Big(\frac{(-1)^l \tilde{Z}_L^{\mu\nu(-l)}(t-r)}{r}\Big), \tag{4.69}$$

we will have

$$\tilde{H}_2^{\mu\nu} = -p^{\mu\nu} + O\Big(\frac{1}{r^2}\Big). \tag{4.70}$$

Now, from (3.25) , it is obvious that we have

$$\Box p^{\mu\nu} = 0, \tag{4.71}$$

for $r > 0$. If we write $p^{\mu\nu}$ in terms of irreducable STF multipole moments as

$$\begin{aligned}
p_1^{00}(\vec{x},t) &= \sum_{l\geq 0} \partial_L \frac{A_L(u)}{r}, \\
p_1^{i0}(\vec{x},t) &= \sum_{l\geq 0} \partial_{iL}\frac{B_L(u)}{r} + \sum_{l\geq 1}\left(\partial_{L-1}\frac{C_{L-1i}(u)}{r} + \epsilon_{iab}\partial_{aL-1}\frac{D_{bL-1}(u)}{r}\right), \\
p_1^{ij}(\vec{x},t) &= \sum_{l\geq 0}\left(\partial_{ijL}\frac{E_L(u)}{r} + \delta_{ij}\partial_L\frac{F_L(u)}{r}\right) \\
&\quad + \sum_{l\geq 1}\left(\partial_{L-1(i}\frac{G_{j)L-1}(u)}{r} + \epsilon_{ab(i}\partial_{j)aL-1}\frac{H_{bL-1}(u)}{r}\right) \\
&\quad + \sum_{l\geq 2}\left(\partial_{L-2}\frac{I_{ijL-2}(u)}{r} + \partial_{aL-2}\Big(\frac{1}{r}\epsilon_{ab(i}J_{j)bL-2}(u)\Big)\right),
\end{aligned} \tag{4.72}$$

then by calculating the $1/r$ term of it, we find $Z^{\mu\nu}$ to be

$$
\begin{aligned}
Z_1^{00}(u,\vec{n}) &= \sum_{l\geq 0}(-1)^l n_L A_L^{(l)}(u),\\
Z_1^{i0}(u,\vec{n}) &= -\sum_{l\geq 0}(-1)^l n_{iL} B_L^{(l+1)}(u),\\
&\quad -\sum_{l\geq 1}(-1)^l \left(n_{L-1} C_{L-1i}^{(l-1)}(u) - \epsilon_{iab} n_{aL-1} D_{bL-1}^{(l)}(u)\right)\\
Z_1^{ij}(u,\vec{n}) &= \sum_{l\geq 0}(-1)^l \left(n_{ijL} E_L^{(l+2)}(u) + \delta_{ij} n_L F_L^{(l)}(u)\right)\\
&\quad +\sum_{l\geq 1}(-1)^l \left(n_{L-1(i} G_{j)L-1}^{(l)}(u) - \epsilon_{ab(i} n_{j)aL-1} H_{bL-1}^{(l+1)}(u)\right)\\
&\quad +\sum_{l\geq 2}(-1)^l \left(n_{L-2} I_{ijL-2}^{(l-2)}(u) - n_{aL-2}\epsilon_{ab(i} J_{j)bL-2}^{(l-1)}(u)\right).
\end{aligned}
\tag{4.73}
$$

Then from (4.66), we find

$$
\begin{aligned}
&A + \dot{B} = 0, && A_L^{(l)} + B_L^{(l+1)} + C_L^{(l-1)} = 0, (l\geq 1),\\
&\dot{B} + \ddot{E} + F = 0, && B_L^{(l+1)} + E_L^{(l+2)} + F_L^{(l)} + \frac{1}{2}G_L^{(l)} = 0, (l\geq 1),\\
&C_i + \frac{1}{2}\dot{G}_i = 0, && C_L^{(l-1)} + \frac{1}{2}G_L^{(l)} + I_L^{(l-2)} = 0, (l\geq 2),\\
&\dot{D}_i + \frac{1}{2}\ddot{H}_i = 0, && D_L^{(l)} + \frac{1}{2}H_L^{(l+1)} + \frac{1}{2}J_L^{(l-1)} = 0, (l\geq 2).
\end{aligned}
\tag{4.74}
$$

Considering the fact that these $A_L, B_L, ..., J_L$ functions are past zero functions, we can take anti-derivatives of each equality in (4.74). After taking anti-derivatives of those equations, they turn into the equations in (3.27) which were the conditions for h_1 to satisfy de Donder gauge condition. Thus, from (4.74), we simply get

$$\partial_\mu p^{\mu\nu} = 0. \tag{4.75}$$

Therefore we can define

$$H_2^{\mu\nu} = \tilde{H}_2^{\mu\nu} + p^{\mu\nu}, \tag{4.76}$$

as a solution to (4.38), which satisfies the theorem for $n = 2$.

For $n \geq 3$, let us assume that for all $2 \leq m \leq n$ that the theorem is true. Then we would find that $\Omega_n = O(r^{-3})$, which gives $\Omega_{nD} = O(r^{-3})$. Therefore there will not be any logarithmic term in H_n. The r^{-1} term in the $\tilde{H}_n$ again can be thrown away just like it has been done for $n = 2$ case. Therefore the theorem is true for $n \geq 3$ too, which finalizes the proof $\square$.

In this section, we have proven that if specific coordinate transformations are done, the metric can be expanded in the far zone as a power series of $1/R$ only. We will show this radiative metric defined in the Theorem 6 as

$$H^{\mu\nu} = H^{\mu\nu}[M_L, S_L], \tag{4.77}$$

where the multipole moments are used in the calculation of H_1

$$H_1^{\mu\nu} = h^{\mu\nu}_{can\ 1}[M_L, S_L] + \partial^\mu \xi^\nu + \partial^\nu \xi^\mu - \eta^{\mu\nu} \partial_\lambda \xi^\lambda, \tag{4.78}$$

where $\xi^\mu = 2M\eta^{0\mu} \log r$ as already stated many times. This solution will be called the radiative metric, and the multipole moments it depends on are called the radiative multipole moments, which will be shown to be different than the source multipole moments in the next section.

Thanks to the additional constraint that is proven in the Theorem 6, these multipole moments are guaranteed to be the moments that would directly contribute to the gravitational wave energy, as will be discussed in section 4.3. This is because the $1/r$ term in the metric does not get any contribution from H_n for $n > 1$, and the energy that the radiation carries is directly the result of the $1/r$ term of the metric, as it has been discussed at the end of the section 3.7 and will be discussed in section 4.3. Since these moments are directly related to the energy of the gravitational waves, these multipole moments should be calculated as accurately as possible. Thus, in the next section, the main idea and the outline of the derivation of the relation between the radiative and the source multipole moments will be given.

4.2 Radiative Multipole Moments

In this section, we will relate the radiative multipole moments to source multipole moments by using the methods in [12] and [18]. For this purpose, let us reconsider what kind of coordinate transformations we have applied to the canonical solution in the proof of the Theorem 5 and 6. Starting with the canonical metric $h_{can1}[M_L, S_L]$, or in short $h_{can1}[M]$, we first applied

$$X^\alpha = x^\alpha + G\xi^\alpha = x^\alpha + G\frac{2M}{c^2}\eta^{0\alpha}\log r, \tag{4.79}$$

so that the linearized part of the metric solution will truly correspond to the spherical wave solutions, i.e. constant $T - R$ surfaces are the null surfaces that the waves follow. Then, in order to cancel the logarithmic terms that will appear in the quadratic part $\tilde{H}_2$, we have chosen its divergence to be

$$\partial_\mu \tilde{H}_2^{\mu\nu} = \Phi_2^\mu = \frac{2}{r^2}k^\mu \mathcal{E}(t-r,\vec{n}), \tag{4.80}$$

where $\mathcal{E}$ is defined in (4.48) as

$$\mathcal{E}(t-r,\vec{n}) = \int_{-\infty}^{t-r} ds\Big(\frac{1}{4}\dot{z}^{\alpha\beta}\dot{z}_{\alpha\beta} - \frac{1}{8}\dot{z}_\alpha^\alpha\dot{z}_\beta^\beta\Big)(s,\vec{n}). \tag{4.81}$$

Let us find out what kind of coordinate transformation that choice corresponds to. We were trying to solve the dynamic part of $\tilde{H}_2$ as in (4.38)

$$\Box\tilde{H}_2^{\mu\nu} = \partial^\mu\Phi_2^\nu + \partial^\nu\Phi_2^\mu - \eta^{\mu\nu}\partial_\lambda\Phi_2^\lambda + \Omega_2^{\mu\nu}[H_1]. \tag{4.82}$$

We know that the solution to this equation is defined in Theorem 5 is

$$\tilde{H}_2^{\mu\nu} = \mathrm{FP}\,\Box_R^{-1}\big(\partial^\mu\Phi_2^\nu + \partial^\nu\Phi_2^\mu - \eta^{\mu\nu}\partial_\lambda\Phi_2^\lambda + \Omega_2^{\mu\nu}[H_1]\big) + q_2^{\mu\nu}, \tag{4.83}$$

where $q_2^{\mu\nu}$ adjusts the divergence coming from the finite part regularization. Thus, if we define $\lambda^\alpha = \mathrm{FP}\,\Box_R^{-1}\Phi_2^\alpha$, then we have

$$\tilde{H}_2^{\mu\nu} = \partial^\mu\lambda^\nu + \partial^\nu\lambda^\mu - \eta^{\mu\nu}\partial_\sigma\lambda^\sigma + \mathrm{FP}\,\Box_R^{-1}\big(\Omega_2^{\mu\nu}[H_1]\big) + q_2^{\mu\nu}. \tag{4.84}$$

However, this is not the general solution, and therefore the resulting metric from this $\tilde{H}_2$ may not match to the initial $h_{can}[M]$. Therefore we must add a homogenous solution to $\tilde{H}_2$ not to lose generality. Therefore, we must consider $\tilde{H}_2^{\mu\nu} - h_{can1}[M']$, which will alter the multipole moments.

After taking care of the $\log r$ terms in $\tilde{H}_2$, in the Theorem 6, we have added $\tilde{H}_2$

$$p^{\mu\nu} = \sum_{l\geq 0} \partial_L \Big(\frac{(-1)^l \tilde{Z}_L^{\mu\nu(-l)}(t-r)}{r} \Big), \tag{4.85}$$

which satisfies $\Box p^{\mu\nu} = 0$ and $\partial_\mu p^{\mu\nu} = 0$, to cancel r^{-2} terms in $\tilde{H}_2$. We can adjust $p^{\mu\nu}$ to cancel the r^{-2} term in $\tilde{H}_2 - h_{can1}[M']$ too since the existence of $p^{\mu\nu}$ requires only the divergencelesness of $h_{can1}[M']$, which is already divergenceless. Thus the homogenous part does not bring any difficulty. From the derivation of (3.33), we find that we can write

$$p^{\mu\nu} = h^{\mu\nu}_{can1}[M''] + \partial^\mu\theta^\nu + \partial^\nu\theta^\mu - \eta^{\mu\nu}\partial_\sigma\theta^\sigma, \tag{4.86}$$

for some θ^μ satisfying $\Box\theta^\mu = 0$. Observe that we then should have $M'' = M''_i = S''_i = 0$ since $p^{\mu\nu}$ is a past zero function. The θ^μ part of it is a coordinate transformation. However, if the canonical part is nonzero, i.e. if any of the multipole moments M''_L and S''_L is nonzero, then this is not a coordinate transformation, and alters the multipole moments. We can write the last metric before the addition of $p^{\mu\nu}$ which we know to represent the same geometry with the canonical metric with multipole moment set M as

$$\begin{aligned} \mathfrak{g}_{can}[M] &\sim \eta + GH_1[M] + G^2(\tilde{H}_2[M] - h_{can1}[M']) + O(G^3) \\ &\sim \eta + GH_1[M] + G^2(H_2[M] - h_{can1}[M'] - p) + O(G^3) \\ &\sim \eta + GH_1[M] + G^2(H_2[M] - h_{can1}[M' + M''] - \partial\theta) + O(G^3) \\ &\sim \eta + GH_1[M - G(M' + M'')] + G^2 H_2[M] + O(G^3). \end{aligned} \tag{4.87}$$

As we have done in the generality theorems, we can change the multipole moment in the $O(G^2)$ term into $M - GM''$ since the change will be at the order of $O(G^3)$. Thus, we find

$$\mathfrak{g}_{can}[M] \sim \mathfrak{g}_{rad}[M - G(M' + M'') + O(G^2)], \tag{4.88}$$

which yields to the following [12]

$$M = M - G(M' + M'') + O(G^2), \tag{4.89}$$

with the coordinate transformation

$$X^\alpha = x^\alpha + G\xi^\alpha + G^2(\lambda^\alpha - \theta^\alpha) + O(G^3). \tag{4.90}$$

As done in [18], those multipole moments can be found from the actual coordinate matching

$$H_1^{\mu\nu} = h^{\mu\nu}_{can1} + \partial^\mu \xi^\nu + \partial^\mu \xi^\nu, \tag{4.91}$$

and

$$\begin{aligned}\tilde{H}_2^{\mu\nu} = h^{\mu\nu}_{can2} + h^{\mu\sigma}_{can1}\partial_\sigma \xi^\nu + h^{\nu\sigma}_{can1}\partial_\sigma \xi^\mu - \xi^\sigma \partial_\sigma h^{\mu\nu}_{can1} \\ + \partial^\mu \lambda^\nu + \partial^\nu \lambda^\mu - \eta^{\mu\nu}\partial_\sigma \lambda^\sigma,\end{aligned} \tag{4.92}$$

where some terms are cancelled by using the definition of ξ^μ directly. By calculating the $1/r$ term of $\tilde{H}_2^{\mu\nu}$ to the leading PN order, the following results have been found

$$\begin{aligned} M_L^{rad}(u) &= M_L(u) + \frac{2GM}{c^3}\int_0^\infty \mathrm{d}\tau \ddot{M}_L(u-\tau)\Big[\log\Big(\frac{c\tau}{2r_0}\Big) + \kappa_l\Big] + O\Big(\frac{1}{c^5}\Big), \\ S_L^{rad}(u) &= S_L(u) + \frac{2GM}{c^3}\int_0^\infty \mathrm{d}\tau \ddot{S}_L(u-\tau)\Big[\log\Big(\frac{c\tau}{2r_0}\Big) + \pi_l\Big] + O\Big(\frac{1}{c^5}\Big), \end{aligned} \tag{4.93}$$

where

$$\begin{aligned} \kappa_l &= \frac{2l^2+5l+4}{l(l+1)(l+2)}\sum_{k=1}^{l-2}\frac{1}{k}, \\ \pi_l &= \frac{l-1}{l(l+1)}\sum_{k=1}^{l-1}\frac{1}{k}, \end{aligned} \tag{4.94}$$

and r_0 is the arbitrary lengthscale we have chosen in the coordinate transformation ξ^μ in (4.31).

4.3 Energy Carried by Gravitational Radiation

In this section, we will calculate how much energy is carried to the null infinity by the gravitational waves. For this, we will define energy as it is defined in [19], which will be derived by using the same idea but in a different way. Then, we will put the metric solution H_n into that definition.

Many ways of defining the gravitational energy momentum tensor, therefore, the total energy momentum tensor has been discussed in [20, 19]. One of the ways is to construct a Lagrangian to generate the linear order wave equation, and then apply the definition of field theoretical energy momentum tensor. But since it is not symmetric, it would not be an appropriate candidate for energy momentum tensor [20]. The other way is to take the Einstein equations (2.25), and expand both sides in the gravitational wave perturbation $\lambda_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$:

$$G_{\mu\nu} = G^{(1)}_{\mu\nu} + G^{(2)}_{\mu\nu} + ... \; , \tag{4.95}$$

where $G^{(1)}_{\mu\nu}$ is the linear part, $G^{(2)}_{\mu\nu}$ is the quadratic part and so on. Then the equations become

$$G^{(1)}_{\mu\nu} = \partial\partial\lambda_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} - G^{(2)}_{\mu\nu} - G^{(3)}_{\mu\nu} ... \; , \tag{4.96}$$

where $\partial\partial$ means a linear quadratic differential operator. Therefore, we can consider $-G^{(2)}_{\mu\nu}$ as the energy momentum tensor of the gravitational wave $\lambda_{\mu\nu}$ in the leading order [19]. A similar approach can be made in our notation too. We will be using radiative coordinates, but since we are not applying any coordinate transformation here, for convenience, we will keep using $(t, \vec{x})$ in this section. We know that (2.25) is equivalent to

$$\Box H^{\mu\nu} = \frac{16\pi G}{c^4} |g| T^{\mu\nu} + \Omega^{\mu\nu} + \partial^\mu \Phi^\nu + \partial^\nu \Phi^\mu - \eta^{\mu\nu} \partial^\lambda \Phi^\lambda, \tag{4.97}$$

where we have $\partial_\mu H^{\mu\nu} = \Phi^\nu$. We know that we have the conservation equation is satisfied

$$\partial_\mu \left(|g| T^{\mu\nu} + \frac{c^4}{16\pi G} \Omega^{\mu\nu} \right) = \partial_\mu \tau^{\mu\nu} = 0. \tag{4.98}$$

Then we can define $\tau^{\mu\nu}$ as the total energy momentum *pseudo* tensor. The therm pseudo should be used since it actually is not a tensor, i.e. it does not transform as a tensor under a general coordinate transformation. However, it will be useful here to develop a concept of energy. Let us define

$$P^\mu = \frac{1}{c} \int_V \mathrm{d}^3 x \tau^{0\mu}, \tag{4.99}$$

as the total energy momentum vector of the region of integration V. Then we can find by using $\partial_t = c\partial_0$

$$\frac{dP^\mu}{dt} = \int_V \mathrm{d}^3 x \partial_0 \tau^{0\mu} = -\int_V \mathrm{d}^3 x \partial_i \tau^{i\mu} = -\int_{\partial V} \mathrm{d}S n^i \tau^{i\mu}, \tag{4.100}$$

where ∂V means the boundary surface of the three volume V, and n^i is the normal vector of the surface. Now, let us take the volume V is as a sphere centered at the origin with radius R. Then we will have $n^i = x^i/r$. We know that $dS \sim R^2$. Then if we take $R \to \infty$ limit by keeping $u = t - r/c$ constant, then we will have $T^{\mu\nu} = 0$ after some value of R, and see that only the $1/r^2$ term in $\Omega^{\mu\nu}$ will give a finite contribution to the energy change. Since we have $H_n^{\mu\nu} = O(1/r^2)$ for $n \geq 2$, this contribution can only come from $\Omega_2^{\mu\nu}$ which is quadratic in $\dot{H}_1^{\mu\nu}$. Then from (4.47), if we restore the factors of c by scaling $k^\mu \to k^\mu/c$ and $\dot{z}^{\mu\nu} \to \dot{z}^{\mu\nu}/c^4$, we find

$$\frac{dP^\mu}{dt} = -\frac{G}{32\pi c^4} \int_{\partial V} \mathrm{d}\Omega n^i \frac{n^i}{c} \frac{k^\mu}{c} \Big(\dot{z}^{\sigma\lambda} \dot{z}_{\sigma\lambda} - \frac{1}{2} (\dot{z}^\lambda_\lambda)^2 \Big), \tag{4.101}$$

where $\dot{z} = dz/dt$. This is the energy-momentum rate that the gravitational radiation takes to infinity. Therefore we can write this as

$$\frac{dP^\mu_{GW}}{dt} = \frac{G}{32\pi c^6} \int_{\partial V} \mathrm{d}\Omega k^\mu \Big(\dot{z}^{\sigma\lambda} \dot{z}_{\sigma\lambda} - \frac{1}{2} (\dot{z}^\lambda_\lambda)^2 \Big). \tag{4.102}$$

Then we find for $E_{GW} = cP^0_{GW}$

$$\frac{dE_{GW}}{dt} = \frac{G}{32\pi c^5} \int_{\partial V} \mathrm{d}\Omega \Big(\dot{z}^{\sigma\lambda} \dot{z}_{\sigma\lambda} - \frac{1}{2} (\dot{z}^\lambda_\lambda)^2 \Big). \tag{4.103}$$

As we can see, the outgoing energy is at the order of 2.5 PN, which is consistent with the Einstein's quadrupole formula. If we plug the c restored version of the definition of $z^{\mu\nu}$ in the equation above (4.43), and using the following formula given in Appendix A of [6]

$$A_{<K>} B_{<L>} \int \mathrm{d}\Omega \hat{n}_K \hat{n}_L = \frac{4\pi k!}{(2k+1)!} \delta_{kl} A_{<K>} B_{<K>}, \tag{4.104}$$

after a lengthy but straightforward calculation, the following result given in [7] can be found

$$\begin{aligned} \frac{dE_{GW}}{dt} = \sum_{l=2}^{\infty} \frac{G}{c^{2l+1}} \Big\{ & \frac{(l+1)(l+2)}{l(l-1)l!(2l+1)!!} M_L^{rad(l+1)} M_L^{rad(l+1)} \\ & - \frac{4l(l+2)}{c^2(l-1)(l+1)!(2l+1)!!} S_L^{rad(l+1)} S_L^{rad(l+1)} \Big\}, \end{aligned} \tag{4.105}$$

where M_L^{rad} and S_L^{rad} are defined in as in the last section.

Another important property of the (4.103) is that it is positive definite [6]:

$$\dot{z}^{\sigma\lambda} \dot{z}_{\sigma\lambda} - \frac{1}{2} (\dot{z}^\lambda_\lambda)^2 = \frac{1}{2} (p_i p_j \dot{z}_{ij} - q_i q_j \dot{z}_{ij})^2 + 2(p_i q_i \dot{z}_{ij})^2, \tag{4.106}$$

where $p_i = \partial n^i / \partial\theta$ and $q_i sin\theta = \partial n^i / \partial\phi$ where θ and ϕ are the common spherical coordinate angles. This positive definiteness shows that the definition (4.103) can actually be considered as the energy of outgoing gravitational waves.

Chapter 5

Post Newtonian Expansion of Metric

In this chater, we will use the gap $a < r < \lambda$ that both PN and PM expansions are valid, and equate them by using the so called matching condition, and calculate multipole moments. After that we will calculate the 1PN correction on the the geodesic equation, and give more accurate 2.5 PN result.

5.1 PN Expansion of Multipole Moments and Matching

We found in Chapter 3 that the metric in the region outside of matter can be represented by using a set of multipole moments, which are called the source multipole moments. Obviously, since the metric should be determined by the energy-momentum tensor $T^{\mu\nu}$ via the d'Alembert equations (3.5), we should be able to determine these multipole moments from $T^{\mu\nu}$. In order to be able to interpret these source multipole moments, in (3.35) and (3.36), we have given the expressions in the really slow moving limit, by assuming that $T^{\mu\nu}$ can be expanded in $1/c$. In this section, under the assumption that the source has a

PN expansion, we will derive more accurate PN expansions for those 6 multipole moments, by using the method in [21].

In order to find the source multipole moments, we should consider some notions we have used: MPM and PN expansions. They are two different types of expansions that we did not prove to be convergent. Therefore, if we show PN expansion of h as $\overline{h}$ an MPM expansion of h as $M(h)$, then the numerical equality

$$\overline{h} = M(h), \tag{5.1}$$

which holds for $r > a$ has no meaning at first, since they both may not be converging series. For simplicity, in this section, we will not put the spacetime indices . In order to extract a meaning out of the equality above, we need to consider MPM and PN expansions of both sides:

$$M(\overline{h}) = \overline{M(h)}, \tag{5.2}$$

and match the corresponding terms. This is somehow a stronger condition than the numerical equality, but this is pretty much the only way to check if both sides are equal since we do not have the exact solution that works in every region.

The equation we want to solve is again (3.5):

$$\Box h = \frac{16\pi G}{c^4}|g|T + \Lambda = \frac{16\pi G}{c^4}\tau. \tag{5.3}$$

Here again, we will divide the problem into static and dynamic part problems. First, let us discuss the dynamic part. Again, for simplicity, we will not put a subscript D to denote the dynamic part, but remember we are dealing with the dynamic part of h until further notice. We have

$$\Box(r^B M(h)) = r^B\Big(\Box M(h) + \frac{2B}{r}\frac{\partial}{\partial r}M(h) + \frac{B(B+1)}{r}M(h)\Big). \tag{5.4}$$

Now, we obviously have $\Box M(h) = M(\Lambda)$, since it is the solution for $r > a$ where no matter source exists. Then, we find

$$r^B M(h) = \Box_R^{-1}\Big\{r^B\Big(M(\Lambda) + \frac{2B}{r}\frac{\partial}{\partial r}M(h) + \frac{B(B+1)}{r}M(h)\Big)\Big\}. \tag{5.5}$$

Taking the finite part of both sides with respect to B, we have

$$M(h) = \underset{B=0}{\mathrm{FP}}\,\Box_R^{-1}\Big\{r^B\Big(M(\Lambda) + \frac{2B}{r}\frac{\partial}{\partial r}M(h) + \frac{B(B+1)}{r}M(h)\Big)\Big\}. \tag{5.6}$$

Let us focus on the second and third terms:

$$M(h) = \mathrm{FP}\,\Box_R^{-1} M(\Lambda) \\ -\frac{1}{4\pi}\,\underset{B=0}{\mathrm{FP}} \int \frac{\mathrm{d}^3 y |\vec{y}|^B}{|\vec{x}-\vec{y}|} B\Big\{\frac{2}{r}\frac{\partial}{\partial r} M(h) + \frac{B+1}{r} M(h)\Big\}\big(\vec{y}, t - \frac{|\vec{x}-\vec{y}|}{c}\big). \tag{5.7}$$

Since the integral is proportional to B, we get

$$M(h) = \mathrm{FP}\,\Box_R^{-1} M(\Lambda) \\ -\frac{1}{4\pi}\,\underset{B=0}{\mathrm{Res}} \int \frac{\mathrm{d}^3 y |\vec{y}|^B}{|\vec{x}-\vec{y}|} \Big\{\frac{2}{r}\frac{\partial}{\partial r} M(h) + \frac{B+1}{r} M(h)\Big\}\big(\vec{y}, t - \frac{|\vec{x}-\vec{y}|}{c}\big). \tag{5.8}$$

Therefore we are looking for a specific part of the integral that diverges when we put $B = 0$. Since we are dealing with the dynamic part, large y means that we are dealing with smaller retarted time, therefore after some point, there is no contribution to the integral. Since we are dealing with the multipole expansions, only source of divergence is $y = 0$ limit. Therefore the region we integrate can be considered as $y < \epsilon$ for some positive real ϵ. This means that components of $\vec{y}$ are really small, then we can make a Taylor expansion on retarded time around $y = 0$. Define

$$K(\vec{y}, t) = \Big\{\frac{2}{r}\frac{\partial}{\partial r} M(h) + \frac{B+1}{r} M(h)\Big\}\big(\vec{y}, t\big). \tag{5.9}$$

Then around $y = 0$

$$\frac{K(\vec{y}, t - \frac{|\vec{x}-\vec{y}|}{c})}{|\vec{x}-\vec{y}|} = \sum_l \frac{(-1)^l}{l!} y_L \partial_L \Big(\frac{K(\vec{y}, t - \frac{r}{c})}{r}\Big). \tag{5.10}$$

Thus the integral would become

$$\begin{aligned} M(h) - \mathrm{FP}\Box_R^{-1} M(\Lambda) &= -\frac{1}{4\pi}\,\underset{B=0}{\mathrm{Res}} \int \frac{\mathrm{d}^3 y |\vec{y}|^B}{|\vec{x}-\vec{y}|} K\big(\vec{y}, t - \frac{|\vec{x}-\vec{y}|}{c}\big) \\ &= -\frac{1}{4\pi}\,\underset{B=0}{\mathrm{Res}} \int \frac{\mathrm{d}^3 y |\vec{y}|^B}{|\vec{x}-\vec{y}|} \sum_l \frac{(-1)^l}{l!} y_L \partial_L \Big(\frac{K(\vec{y}, t - \frac{r}{c})}{r}\Big) \\ &= -\frac{1}{4\pi} \sum_l \frac{(-1)^l}{l!} \partial_L \left(\frac{1}{r}\,\underset{B=0}{\mathrm{Res}} \int \frac{\mathrm{d}^3 y |\vec{y}|^B}{|\vec{x}-\vec{y}|} y_L K\Big(\vec{y}, t - \frac{r}{c}\Big)\right). \end{aligned} \tag{5.11}$$

Thus we have

$$M(h) = \mathrm{FP}\,\Box_R^{-1} M(\Lambda) - \frac{1}{4\pi} \sum_l \frac{(-1)^l}{l!} \partial_L \Big(\frac{K_L(t - \frac{r}{c})}{r}\Big), \tag{5.12}$$

where

$$K_L(u) = \operatorname*{Res}_{B=0} \int_{|\vec{y}|<\epsilon} \mathrm{d}^3 y |\vec{y}|^B y_L \Big(\frac{2}{r}\frac{\partial}{\partial r} M(h) + \frac{B+1}{r} M(h) \Big)(\vec{y}, u). \tag{5.13}$$

Now, let us look at the PN expansion of this. We know that we can expand $M(h)$ on the region we integrate, as

$$\overline{M(h)} = \sum_{lap} \hat{n}^L F_{Lap}(t) r^a \log^p r, \tag{5.14}$$

which is the near zone expansion. Therefore, K_L can be written as a sum of integrals of those terms

$$K_L(u) = \sum_{Qap} \operatorname*{Res}_{B=0} \int_{|\vec{y}|<\epsilon} \mathrm{d}^3 y F_{Qap}(t) n_L \hat{n}^Q y^{a+B+l} \log^p y. \tag{5.15}$$

Now let us focus on the integral. After the angular integration, it will be proportional to

$$\int_0^\epsilon \mathrm{d}y y^{a+B+l+2} \log^p y. \tag{5.16}$$

Now, let us consider the analytical continuation of that integral. This can be done since on (5.5), we have left side analytic on B, therefore we can take analytical continuation of both sides and consider the new version of the equation. Now

$$\begin{aligned} \int_0^\infty \mathrm{d}y y^{a+B+l+2} \log^p y &= \Big(\frac{\partial}{\partial B}\Big)^p \int_0^\infty \mathrm{d}y y^{a+B+l+2} \\ &= \Big(\frac{\partial}{\partial B}\Big)^p \Bigg(\int_0^\epsilon \mathrm{d}y y^{a+B+l+2} + \int_\epsilon^\infty \mathrm{d}y y^{a+B+l+2} \Bigg). \end{aligned} \tag{5.17}$$

The former integral converges for $Re(B) + a + l + 3 > 0$ and the latter one converges for $Re(B) + a + l + 3 < 0$. Taking the analytical continuation of them separately, we find the analytical continuation of (5.16) as

$$\begin{aligned} \int_0^\infty \mathrm{d}y y^{a+B+l+2} \log^p y &= \Big(\frac{\partial}{\partial B}\Big)^p \Bigg(\frac{y^{a+B+l+3}}{a+B+l+3}\Big|_0^\epsilon + \frac{\epsilon^{a+B+l+3}}{a+B+l+3}\Big|_\epsilon^\infty \Bigg) \\ &= \Big(\frac{\partial}{\partial B}\Big)^p \Bigg(\frac{\epsilon^{a+B+l+3}}{a+B+l+3} - 0 + 0 - \frac{\epsilon^{a+B+l+3}}{a+B+l+3} \Bigg) = 0. \end{aligned} \tag{5.18}$$

Therefore we found that, if we consider analytical continuations, we can write

$$\int_0^\epsilon \mathrm{d}y y^{a+B+l+2} \log^p y = -\int_\epsilon^\infty \mathrm{d}y y^{a+B+l+2} \log^p y. \tag{5.19}$$

If we undo the angular integration here and put this on K_L, then we find

$$K_L(u) = -\operatorname*{Res}_{B=0}\int_{|\vec{y}|>\epsilon} \mathrm{d}^3y|\vec{y}|^B y_L\Big(\frac{2}{r}\frac{\partial}{\partial r}\overline{M(h)} + \frac{B+1}{r}\overline{M(h)}\Big)(\vec{y},u). \tag{5.20}$$

If we take ϵ large enough, we would have $\overline{M(h)} = \overline{h}$. Since $\overline{h}$ is regular everywhere and we are interested in with the residue of this integral, then after replacing $\overline{M(h)}$ with $\overline{h}$, we can replace the integral region to the whole space. Then

$$K_L(u) = -\operatorname*{Res}_{B=0}\int \mathrm{d}^3y|\vec{y}|^B y_L\Big(\frac{2}{r}\frac{\partial}{\partial r}\overline{h} + \frac{B+1}{r}\overline{h}\Big)(\vec{y},u). \tag{5.21}$$

Using h instead of $M(h)$ in (5.4)

$$\begin{aligned} K_L(u) &= -\operatorname*{FP}_{B=0}\int \mathrm{d}^3y y_L\big(\Box(r^B\overline{h}) - r^B\Box\overline{h}\big)(\vec{y},u) \\ &= \operatorname*{FP}_{B=0}\int \mathrm{d}^3y y_L\big(\frac{16\pi G}{c^4}r^B\overline{\tau} - \Box(r^B\overline{h})\big)(\vec{y},u). \end{aligned} \tag{5.22}$$

Let us calculate the second term.

$$\begin{aligned} \int \mathrm{d}^3y y_L\big(\Box(r^B\overline{h})\big)(\vec{y},u) &= \int \mathrm{d}^3y y_L\big(\Delta(r^B\overline{h}) - \frac{r^B}{c^2}\frac{\partial^2}{\partial t^2}\overline{h}\big)(\vec{y},u) \\ &= \int \mathrm{d}^3y\big(\Delta y_L\overline{h}(\vec{y},u) - y_L\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\overline{h}(\vec{y},u)\big)y^B \\ &= \int \mathrm{d}^3y\big(l(l-1)y_{(L-2}\delta_{i_{l-1}i_l)}\overline{h}(\vec{y},u) - y_L\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\overline{h}(\vec{y},u)\big)y^B. \end{aligned} \tag{5.23}$$

On the second line, partial integration was taken and the term that has $L-2$ indices is zero for $l<2$. Now, if we put this into the sum in (5.12)

$$\begin{aligned} &\sum_l \frac{(-1)^l}{l!}\partial_L\Big(\frac{1}{r}\int \mathrm{d}^3y\big(l(l-1)y_{(L-2}\delta_{i_{l-1}i_l)}\overline{h}(\vec{y},u) - \frac{y_L}{c^2}\frac{\partial^2}{\partial t^2}\overline{h}(\vec{y},u)\big)y^B\Big) \\ =&\sum_l \frac{(-1)^l}{l!}\int \mathrm{d}^3y\Big\{l(l-1)y_{(L-2}\delta_{i_{l-1}i_l)}\partial_L\Big(\frac{\overline{h}(\vec{y},u)}{r}\Big) - \frac{y_L}{c^2}\frac{\partial^2}{\partial t^2}\partial_L\Big(\frac{\overline{h}(\vec{y},u)}{r}\Big)\Big\}y^B \\ =&\sum_l \frac{(-1)^l}{l!}\int \mathrm{d}^3y\Big\{l(l-1)y_{L-2}\partial_{L-2}\Delta\Big(\frac{\overline{h}(\vec{y},u)}{r}\Big) - \frac{y_L}{c^2}\frac{\partial^2}{\partial t^2}\partial_L\Big(\frac{\overline{h}(\vec{y},u)}{r}\Big)\Big\}y^B \\ =&\sum_l \frac{(-1)^l}{l!}\int \mathrm{d}^3y\Big\{l(l-1)y_{L-2}\partial_{L-2}\Delta\Big(\frac{\overline{h}(\vec{y},u)}{r}\Big) - \frac{y_L}{c^2}\frac{\partial^2}{\partial t^2}\partial_L\Big(\frac{\overline{h}(\vec{y},u)}{r}\Big)\Big\}y^B. \end{aligned} \tag{5.24}$$

Now, for $r>0$, we know that

$$\Box\Big(\frac{\overline{h}(\vec{y},u)}{r}\Big) = \Box\Big(\frac{\overline{h}(\vec{y},t-\frac{r}{c})}{r}\Big) = 0. \tag{5.25}$$

Thus, the sum will become

$$
\begin{aligned}
&=\sum_l \frac{(-1)^l}{l!}\int \mathrm{d}^3y\Big\{l(l-1)\frac{y_{L-2}}{c^2}\frac{\partial^2}{\partial t^2}\partial_{L-2}\Big(\frac{\overline{h}(\vec{y},u)}{r}\Big)-\frac{y_L}{c^2}\frac{\partial^2}{\partial t^2}\partial_L\Big(\frac{\overline{h}(\vec{y},u)}{r}\Big)\Big\}y^B \\
&=\sum_l \int \mathrm{d}^3y\Big\{\frac{(-1)^{l-2}}{(l-2)!}\frac{y_{L-2}}{c^2}\frac{\partial^2}{\partial t^2}\partial_{L-2}\Big(\frac{\overline{h}(\vec{y},u)}{r}\Big)-\frac{(-1)^l}{l!}\frac{y_L}{c^2}\frac{\partial^2}{\partial t^2}\partial_L\Big(\frac{\overline{h}(\vec{y},u)}{r}\Big)\Big\}y^B.
\end{aligned}
\tag{5.26}
$$

Obviously, this is a telescopic sum and will give

$$
=\frac{(-1)^{l_0+1}}{c^2 l_0!}\partial_{L_0}\Big(\frac{1}{r}\int \mathrm{d}^3y y^B y_{L_0}\frac{\partial^2\overline{h}(\vec{y},u)}{\partial t^2}\Big), \tag{5.27}
$$

where l_0 is the maximum value that l can get. For a large l_0, this term can be neglected if we consider the solution for a finite order in $1/r$ and $1/c$. Therefore we find that the second term in (5.22) has no contribution. Therefore we find PN expantion of metric to be

$$
M(h)=\mathrm{FP}\,\Box_R^{-1}M(\Lambda)-\frac{1}{4\pi}\sum_l\frac{(-1)^l}{l!}\partial_L\Big(\frac{K_L(t-\frac{r}{c})}{r}\Big), \tag{5.28}
$$

where K_L moments are given as

$$
K_L(u)=\frac{16\pi G}{c^4}\underset{B=0}{\mathrm{FP}}\int \mathrm{d}^3y y_L y^B \overline{\tau}(\vec{y},u). \tag{5.29}
$$

This was only for the dynamical part. However, the whole calculation can be done for static part in the same way too. Only difference would be solving Laplace equation instead of d'Alembert equation. Instead of Taylor expanding after (5.5), since we are interested in the far region which is $|\vec{x}| \gg a > |\vec{y}|$, we can immediately put the Taylor expansion

$$
\begin{aligned}
\frac{1}{|\vec{x}-\vec{y}|}&=\sum_l \frac{y_L}{l!}\partial^y_L\Big(\frac{1}{|\vec{x}-\vec{y}|}\Big)\Big|_{\vec{y}=0} \\
&=\sum_l \frac{(-1)^l}{l!}y_L\partial^x_L\Big(\frac{1}{r}\Big),
\end{aligned}
\tag{5.30}
$$

to arrive at the same expression as (5.13), which would lead to the same equations as above. Therefore, (5.28) applies not only for dynamic part, but also for the whole h. However, the calculation is not over yet. In (5.28), those K_L moments are not STF tensors. After a few algebraic calculations, one finds for

$$
P_L(u)=\underset{B=0}{\mathrm{FP}}\int \mathrm{d}^3x r^B\hat{x}_L\int_{-1}^{1}\mathrm{d}s\delta_l(s)\overline{\tau}\big(\vec{x},u+\frac{sr}{c}\big), \tag{5.31}
$$

where

$$\delta_l(s) = \frac{(2l+1)!!}{2^{l+1} l!}(1-s^2)^l, \tag{5.32}$$

we would have

$$M(h) = \mathrm{FP}\,\Box_R^{-1} M(\Lambda) - \frac{4G}{c^4}\sum_l \frac{(-1)^l}{l!}\hat{\partial}_L\Big(\frac{P_L(t-\frac{r}{c})}{r}\Big). \tag{5.33}$$

Now, remembering from 3.4 the fact that we can have multipole moments as expansions of G, if we expand both sides of the equation above in G, we would find that the first order term is

$$h_1 = -\frac{4}{c^4}\sum_l \frac{(-1)^l}{l!}\hat{\partial}_L\Big(\frac{P_L(t-\frac{r}{c})}{r}\Big), \tag{5.34}$$

which means, those P_L are the U_L multipole moments we have introduced in (3.25). This can be justified by the fact that for $n > 1$, that expansion generates

$$h_n = \mathrm{FP}\,\Box_R^{-1}\Lambda_n, \tag{5.35}$$

which are precisely the equations hat we solved before. Thus, we can conclude that (5.33) implies that, MPM expansion of the metric is

$$M(h) = h_{part}[M_L, W_L], \tag{5.36}$$

where M_L, S_L and W_L, X_L, Y_L, Z_L are the corresponding irreducible source multipole moments and gauge multipole moments to $P_L^{\mu\nu}$ moments we have found. Thus, as in [21], after some messy algebraic calculations, those multipole moments can be found from (5.31), (3.33), (3.29) and (3.28) as

$$\begin{aligned} M_L(u) &= \mathrm{FP}\int \mathrm{d}^3x \int_{-1}^{1}\mathrm{d}s\Big\{ -\delta_l(s)\hat{x}_L\Sigma - \frac{4(2l+1)\delta_{l+1}(s)}{c^2(l+1)(2l+3)}\hat{x}_{iL}\dot{\Sigma}_i \\ &\qquad + \frac{2(2l+1)\delta_{l+2}(s)}{c^4(l+1)(l+2)(2l+5)}\hat{x}_{ijL}\ddot{\Sigma}_{ij}\Big\}, \\ S_L(u) &= \mathrm{FP}\int \mathrm{d}^3x \int_{-1}^{1}\mathrm{d}s\epsilon_{ab<i_l}\Big\{\delta_l(s)\hat{x}_{L-1>a}\Sigma_b \\ &\qquad - \frac{(2l+1)\delta_{l+1}(s)}{c^2(l+2)(2l+3)}\hat{x}_{L-1>ac}\dot{\Sigma}_{bc}\Big\}, \end{aligned} \tag{5.37}$$

and

$$
\begin{aligned}
W_L(u) &= \mathrm{FP}\int \mathrm{d}^3x \int_{-1}^{1}\mathrm{d}s\Big\{\frac{(2l+1)\delta_{l+1}(s)}{(l+1)(2l+3)}\hat{x}_{iL}\Sigma_i,\\
&\qquad -\frac{(2l+1)\delta_{l+2}(s)}{2c^2(l+1)(l+2)(2l+5)}\hat{x}_{ijL}\dot{\Sigma}_{ij}\Big\}\\
X_L(u) &= \mathrm{FP}\int \mathrm{d}^3x \int_{-1}^{1}\mathrm{d}s\Big\{\frac{(2l+1)\delta_{l+2}(s)}{2(l+1)(l+2)(2l+5)}\hat{x}_{ijL}\Sigma_{ij}\Big\},\\
Y_L(u) &= \mathrm{FP}\int \mathrm{d}^3x \int_{-1}^{1}\mathrm{d}s\Big\{-\delta_l(s)\hat{x}_L\Sigma_{ii}+\frac{3(2l+1)\delta_{l+1}(s)}{(l+1)(2l+3)}\hat{x}_{iL}\dot{\Sigma}_i,\\
&\qquad -\frac{2(2l+1)\delta_{l+2}(s)}{c^2(l+1)(l+2)(2l+5)}\hat{x}_{ijL}\ddot{\Sigma}_{ij}\Big\}\\
Z_L(u) &= \mathrm{FP}\int \mathrm{d}^3x \int_{-1}^{1}\mathrm{d}s\Big\{-\frac{(2l+1)\delta_{l+1}(s)}{(l+2)(2l+3)}\epsilon_{ab<i_l}\hat{x}_{L-1>bc}\Sigma_{ac}\Big\},
\end{aligned}
\tag{5.38}
$$

where in the integral $\dot{\Sigma}=\frac{\partial\Sigma}{\partial t}$, all $\Sigma,\Sigma_i,\Sigma_{ij}$ are function of $\Sigma(\vec{x},u+sr/c)$ and they are defined as

$$
\begin{aligned}
\Sigma &= \frac{\overline{\tau}^{00}+\overline{\tau}^{ii}}{c^2},\\
\Sigma_i &= \frac{\overline{\tau}^{0i}}{c},\\
\Sigma_{ij} &= \overline{\tau}^{ij}.
\end{aligned}
\tag{5.39}
$$

As it can be seen, we have not written these multiple moments just by using $T^{\mu\nu}$, since $\tau^{\mu\nu}$ includes $h^{\mu\nu}$ itself. However, we can expand $h^{\mu\nu}$ and put in those equations to write these multipole moments only in terms of the energy momentum tensor to any order we want either in G or $1/c$. These expansions obviously will include nonlinear terms that can be interpreted as the self interaction energy and angular momentum of the source itself.

In this section, even though we just have calculated the multipole moments, there is a really important result: if $T^{\mu\nu}$ admits a PN expansion, so does the multipole moments, thus h_1 also admits a PN expansion. This is a crucial fact that will be used in next chapters.

5.2 PN Solution

In this section, we will solve the Einstein field equations (3.5) by using a PN expansion for the metric instead of an MPM expansion, by using the methods in [22].

As we know from (3.117), the PN expansion of the metric will involve $\log c$ terms, and cannot be just written by using powers of $1/c$. But, for convenience, we will use an expansion that results from summing up all the logarithmic terms that is multiplied with the same power of $1/c$:

$$\overline{h}(\vec{x},t) = \sum_n \frac{1}{c^n} h_n(\vec{x},t,\log c), \tag{5.40}$$

where spacetime indices were not written for convenience. We will assume a similar expansion for τ

$$\overline{\tau}(\vec{x},t) = \sum_n \frac{1}{c^n} \tau_n(\vec{x},t,\log c). \tag{5.41}$$

Putting these expansions in (3.5), we then get

$$\Delta h_m = \frac{\partial^2 h_{m-2}}{\partial t^2} + 16\pi G \tau_{m-4}, \tag{5.42}$$

for all $m = 2, 3, ...$ where we assume that $h_m = \tau_m = 0$ for $m \leq 1$.

Before solving (5.42), we should observe the fact that the expansion (5.40) is valid only in the nearzone approximation [23]. Because even the most simple expression that solves d'Alembert equation outside the matter source has a PN expansion like

$$\frac{K\left(t-\frac{r}{c}\right)}{r} = \frac{K(t)}{r} - \frac{1}{c}\dot{K}(t) + \frac{1}{2c^2}\ddot{K}(t)r - \frac{1}{6c^3}\dddot{K}(t)r^2 + ... , \tag{5.43}$$

which consists of terms that diverges for $r \to \infty$ limit. And for sufficiently large r, $n+1$th term of the expansion becomes a lot larger than the nth term of the expansion. Therefore, the equation (5.42) is not valid in the whole spacetime, it is valid only in the near zone. Then, if we take the Laplace inverse of both sides with finite part, we then find

$$h_m = \mathrm{FP}\,\Delta^{-1}\frac{\partial^2 h_{m-2}}{\partial t^2} + 16\pi G\,\mathrm{FP}\,\Delta^{-1}\tau_{m-4} + \omega_m(\vec{x},t), \tag{5.44}$$

where ω_m is a well behaving function that satisfies

$$\Delta\omega_m(\vec{x},t) = 0, \tag{5.45}$$

in the near zone. It is not hard to see that the general solution to the equation above is

$$\omega_m(\vec{x},t) = \sum_l B_m^L(t)\hat{x}^L. \tag{5.46}$$

Thus we have

$$h_m = \mathrm{FP}\,\Delta^{-1}\frac{\partial^2 h_{m-2}}{\partial t^2} + 16\pi G\,\mathrm{FP}\,\Delta^{-1}\tau_{m-4} + \sum_l B_m^L(t)\hat{x}^L. \tag{5.47}$$

Let us define $\tilde{\Delta}^{-1} = \mathrm{FP}\,\Delta^{-1}$. Then, we can write

$$h_m = \tilde{\Delta}^{-1}\frac{\partial^2 h_{m-2}}{\partial t^2} + 16\pi G\tilde{\Delta}^{-1}\tau_{m-4} + \sum_l B_m^L(t)\hat{x}^L. \tag{5.48}$$

If we repeat this equality R times, then we get

$$\begin{aligned} h_m = \tilde{\Delta}^{-1}\frac{\partial^{2R+2} h_{m-2R}}{\partial t^{2R+2}} &+ 16\pi G\sum_{k=0}^{R}\tilde{\Delta}^{-1-k}\left\{\frac{\partial^{2k}\tau_{m-4-2k}}{\partial t^{2k}}\right\} \\ &+ \sum_l\sum_{k=0}^{R}\frac{\partial^{2k}B_{m-2k}^L(t)}{\partial t^{2k}}\tilde{\Delta}^{-k}\hat{x}^L, \end{aligned} \tag{5.49}$$

where we have

$$\tilde{\Delta}^{-1-k}\{f\}(\vec{x}) = -\frac{1}{4\pi}\underset{B=0}{\mathrm{FP}}\int \mathrm{d}^3y\frac{|\vec{x}-\vec{y}|^{2k-1}}{(2k)!}y^B f(\vec{y}). \tag{5.50}$$

We know that $h_0 = h_1 = 0$ which means this iteration will truncate at some value of R. Therefore even if we take $R \to \infty$, all the summations in the equation above will actually remain a as a finite sum. Then

$$\begin{aligned} h_m = 16\pi G &\sum_{k=0}^{\infty}\tilde{\Delta}^{-1-k}\left\{\frac{\partial^{2k}\tau_{m-4-2k}}{\partial t^{2k}}\right\} \\ &+ \sum_l\sum_{k=0}^{\infty}\frac{\partial^{2k}B_{m-2k}^L(t)}{\partial t^{2k}}\tilde{\Delta}^{-k}\hat{x}^L. \end{aligned} \tag{5.51}$$

If we calculate h by doing the sum for $m = 2, 3, 4, ...$ as in (5.40), we find

$$\begin{aligned} \overline{h} = \frac{16\pi G}{c^4} &\sum_{k=0}^{\infty}\left\{\frac{\partial}{\partial(ct)}\right\}^{2k}\tilde{\Delta}^{-1-k}\overline{\tau} \\ &+ \sum_l\sum_{k=0}^{\infty}\left\{\frac{\partial}{\partial(ct)}\right\}^{2k}\overline{B}_L(t)\tilde{\Delta}^{-k}\hat{x}^L, \end{aligned} \tag{5.52}$$

where we have defined $\overline{B}_L$ as

$$\overline{B}_L(t) = \sum_m \frac{1}{c^m} B^l_m(t). \tag{5.53}$$

Now, if we carefully look at the first sum, we would see that it is a form of an inverse d'Alembertian. It does not include any odd powers of $1/c$, therefore we can easily conclude that if we define

$$\Box_I^{-1}\overline{\tau} = \sum_{k=0}^{\infty} \left\{\frac{\partial}{\partial(ct)}\right\}^{2k} \tilde{\Delta}^{-1-k}\overline{\tau}, \tag{5.54}$$

where the subscript I stands for instant, then it could be shown that in terms of PN expansions

$$\Box_I^{-1}\overline{\tau} = \frac{1}{2}(\mathrm{FP}\,\overline{\Box_R^{-1}\tau} + \mathrm{FP}\,\overline{\Box_A^{-1}\tau}), \tag{5.55}$$

where R stands for retarded and A stands for advanced.

The second sum has been shown to be an antisymmetric wave solution [22]:

$$\sum_l \sum_{k=0}^{\infty} \left\{\frac{\partial}{\partial(ct)}\right\}^{2k} \overline{B}_L(t)\tilde{\Delta}^{-k}\hat{x}^L = \sum_l \hat{\partial}_L\Big(\frac{\overline{A}_L(t-\frac{r}{c}) - \overline{A}_L(t+\frac{r}{c})}{2r}\Big), \tag{5.56}$$

where we have

$$\overline{B}_L(t) = -\frac{A_L^{(2l+1)}}{c^{2l+1}(2l+1)!!}. \tag{5.57}$$

Thus, we find the PN expansion of h as

$$\overline{h} = \frac{16\pi G}{c^4}\Box_I^{-1}\overline{\tau} + \sum_l \hat{\partial}_L\Big(\frac{\overline{A}_L(t-\frac{r}{c}) - \overline{A}_L(t+\frac{r}{c})}{2r}\Big). \tag{5.58}$$

These A_L functions are called the "radiation reaction functions". At this point, they cannot be calculated without using the matching equation (5.2).

In [22], by doing similar calculations, the following has been found:

$$M(\Box_I^{-1}\overline{\tau}) = \Box_I^{-1}M(\overline{\tau}) - \frac{1}{4\pi}\sum_l \frac{(-1)^l}{l!}\hat{\partial}_L\Big(\frac{\overline{F}_L(t-\frac{r}{c}) + \overline{F}_L(t+\frac{r}{c})}{2r}\Big), \tag{5.59}$$

where

$$F_L(t) = \sum_{j=0}^{\infty} \frac{1}{c^{2j}} \underset{B=0}{\mathrm{FP}} \int \mathrm{d}^3y\,|\vec{y}|^B \tilde{\Delta}^{-j}\hat{y}_L \frac{\partial^{2j}}{\partial t^{2j}}\overline{\tau}(\vec{y},t). \tag{5.60}$$

They also found that we can write

$$\overline{\mathrm{FP}\,\Box_R^{-1} M(\Lambda)} = \Box_I^{-1}\overline{M(\Lambda)} - \frac{4G}{c^4}\sum_l \frac{(-1)^l}{l!}\hat{\partial}_L\Big(\frac{\overline{R}_L(t-\frac{r}{c}) + \overline{R}_L(t+\frac{r}{c})}{2r}\Big), \quad (5.61)$$

where R_L are calculated to be

$$R_L(t) = -2\,\underset{B=0}{\mathrm{FP}}\int \mathrm{d}^3x|\vec{x}|^B \hat{x}_L \int_1^\infty \mathrm{d}z\delta_l(z) M(\tau)\Big(\vec{x}, t - \frac{z|\vec{x}|}{c}\Big), \quad (5.62)$$

where the integral inside is calculated first for the values of l that makes the integral convergent, then the analytical continuation with respect to l is taken. Combining (5.58) and (5.59) we find

$$\begin{aligned} M(\overline{h}) = \frac{16\pi G}{c^4}\Box_I^{-1} M(\overline{\tau}) &+ \sum_l \hat{\partial}_L\Big(\frac{\overline{A}_L(t-\frac{r}{c}) - \overline{A}_L(t+\frac{r}{c})}{2r}\Big) \\ &- \frac{4G}{c^4}\sum_l \frac{(-1)^l}{l!}\hat{\partial}_L\Big(\frac{\overline{F}_L(t-\frac{r}{c}) + \overline{F}_L(t+\frac{r}{c})}{2r}\Big), \end{aligned} \quad (5.63)$$

and by combining (5.33) and (5.61) we find

$$\begin{aligned} \overline{M(h)} = \Box_I^{-1}\overline{M(\Lambda)} &- \frac{4G}{c^4}\sum_l \frac{(-1)^l}{l!}\hat{\partial}_L\Big(\frac{P_L(t-\frac{r}{c})}{r}\Big) \\ &- \frac{4G}{c^4}\sum_l \frac{(-1)^l}{l!}\hat{\partial}_L\Big(\frac{\overline{R}_L(t-\frac{r}{c}) + \overline{R}_L(t+\frac{r}{c})}{2r}\Big). \end{aligned} \quad (5.64)$$

Since we have $\frac{16\pi G}{c^4}M(\tau) = M(\Lambda)$, from (5.2) we directly find

$$P_L(t) = F_L(t), \quad (5.65)$$

and

$$A_L(t) = -\frac{4G}{c^4}\frac{(-1)^l}{l!}(F_L(t) + R_L(t)). \quad (5.66)$$

The former could also be shown by comparing the expressions (5.31) and (5.60). Thus, the radiation reaction multipole moments can also be calculated directly from $T^{\mu\nu}$ itself, and the PN expansion (5.58) is shown to be calculable to any order in $1/c$. In the next section, this expansion will be reconstructed and then calculated to a finite PN order.

5.3 Geodesic Equation to 2.5 PN Order

In this section, we will calculate the 1 PN or $O(1/c^2)$ correction to the geodesic equation by using the method in [24], and give the results given in [7] for higher order terms. To calculate the 1 PN correction, let us find to what order the metric should be calculated. We have the geodesic equation from (2.14) as

$$\frac{du_\alpha}{ds} = \frac{u^\mu u^\nu}{2c}\partial_\alpha g_{\mu\nu}, \tag{5.67}$$

or

$$\frac{du_\alpha}{dt} = \frac{ds}{dt}\frac{u^\mu u^\nu}{2c}\partial_\alpha g_{\mu\nu} = \frac{cu^\mu u^\nu}{2u^0}\partial_\alpha g_{\mu\nu}. \tag{5.68}$$

Now, let us assume that the point particle that follows this geodesic moves slowly. Let us define

$$v^\mu = \frac{dx^\mu}{dt} = c\frac{dx^\mu}{dx^0} = c\frac{u^\mu}{u^0} = (c, v^i). \tag{5.69}$$

We assume $v^i = O(1)$ since we will study the slowly moving particles. It is easy to show that u_μ can be written as

$$u_\mu = \frac{cg_{\mu\alpha}v^\alpha}{\sqrt{-g_{\sigma\lambda}v^\sigma v^\lambda}}. \tag{5.70}$$

If we rewrite the geodesic equation (5.68) by using (5.70), we find

$$\frac{du_\alpha}{dt} = \frac{cv^\mu v^\nu \partial_\alpha g_{\mu\nu}}{2\sqrt{-g_{\sigma\lambda}v^\sigma v^\lambda}}. \tag{5.71}$$

To find the $O(1/c^2)$ term of the right hand side of this equation, it is obvious that we need to know g_{00} to $O(1/c^4)$ terms, g_{0i} to $O(1/c^3)$ terms and g_{ij} to $O(1/c^2)$ terms. Now, from (3.1), we have

$$g^{\mu\nu} = \frac{\eta^{\mu\nu} + h^{\mu\nu}}{\sqrt{-g}}, \tag{5.72}$$

where g is the determinant of the matrix $g_{\mu\nu}$ and $g^{\mu\nu}$ is the inverse of the matrix $g_{\mu\nu}$ as stated before. We know that $h^{00} = O(1/c^2)$, $h^{0i} = O(1/c^3)$ and $h^{ij} = O(1/c^4)$ from (3.33). Therefore, we need to calculate every expansion in h to the quadratic order, since the cubic term is $O(1/c^6)$. The absolute value of the metric determinant can be found from (5.72) as

$$|g| = 1 + h + \frac{1}{2}(h^2 - h_{\mu\nu}h^{\mu\nu}) + O(h^3), \tag{5.73}$$

where $h = \eta_{\mu\nu} h^{\mu\nu}$ and $h_{\mu\nu} = h^{\alpha\beta} \eta_{\mu\alpha} \eta_{\beta\nu}$. The metric can be found from (5.72) as

$$\begin{aligned} g_{\mu\nu} =& \eta_{\mu\nu} - \Big(h_{\mu\nu} - \frac{h}{2} \eta_{\mu\nu} \Big) \\ &+ \Big(h_{\mu\sigma} h^{\sigma}_{\nu} - \frac{h}{2} h_{\mu\nu} + \eta_{\mu\nu} \frac{h^2}{8} - \eta_{\mu\nu} \frac{h^{\alpha\beta} h_{\alpha\beta}}{4} \Big) + O(h^3). \end{aligned} \tag{5.74}$$

The only thing left is to solve $h^{\mu\nu}$ to required orders. We already know the PN expansion of $h^{\mu\nu}$, however, as in [24], we will reconstruct it by defining new gravitational potentials. Since we are dealing with the PN expansion, as stated before, we are solving the metric only in the near zone. Therefore, we will continue to study the metric in harmonic coordinates. In [24], matching between a far zone metric and the solution we will obtain is studied too. This will be unnecessary to calculate the geodesic equations in the near zone, which are needed in the next chapter to calculate the trajectories of the singularities of binary black hole systems, and therefore not will be discussed here.

We will assume that the source is moving slowly, and admits a PN expansion. This fact will be plugged into the calculation as

$$\begin{aligned} T^{00} &= O(c^2), \\ T^{0i} &= O(c), \\ T^{ij} &= O(1). \end{aligned} \tag{5.75}$$

We again will solve (3.5)

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4} |g| T^{\mu\nu} + \Lambda^{\mu\nu}. \tag{5.76}$$

At first iteration, we have

$$\begin{aligned} h^{00} &= O\Big(\frac{1}{c^2}\Big), \\ h^{0i} &= O\Big(\frac{1}{c^3}\Big), \\ h^{ij} &= O\Big(\frac{1}{c^4}\Big). \end{aligned} \tag{5.77}$$

For convenience, as in [24], let us define

$$\begin{aligned} \sigma &= \frac{T^{00} + T^{ii}}{c^2}, \\ \sigma^i &= \frac{T^{0i}}{c}, \\ \sigma^{ij} &= T^{ij}, \end{aligned} \tag{5.78}$$

so that we have all those σ, σ^i, σ^{ij} to be $O(1)$. Then the next order equations are

$$\begin{aligned}
\Box h^{00} &= \frac{16\pi G}{c^2}\Big(\sigma - \frac{\sigma^{ii}}{c^2}\Big) + O\Big(\frac{1}{c^4}\Big) = \frac{16\pi G}{c^2}\sigma + O\Big(\frac{1}{c^4}\Big),\\
\Box h^{0i} &= O\Big(\frac{1}{c^3}\Big),\\
\Box h^{ij} &= O\Big(\frac{1}{c^4}\Big).
\end{aligned} \tag{5.79}$$

Let us define the potential

$$V = G\int \frac{\mathrm{d}^3 y}{|\vec{x}-\vec{y}|}\sigma\Big(\vec{y}, t - \frac{|\vec{x}-\vec{y}|}{c}\Big). \tag{5.80}$$

This integral is convergent since we are dealing with compact sources, i.e. there are no matter for $r > a$. Then we find

$$\begin{aligned}
h^{00} &= \frac{-4V}{c^2} + O\Big(\frac{1}{c^4}\Big),\\
h^{0i} &= O\Big(\frac{1}{c^3}\Big),\\
h^{ij} &= O\Big(\frac{1}{c^4}\Big).
\end{aligned} \tag{5.81}$$

If we plug these in Λ^{00}, Λ^{0i} and Λ^{ij}, we then find

$$\begin{aligned}
\Lambda^{00} &= -\frac{14}{c^4}\partial_k V \partial_k V + O\Big(\frac{1}{c^6}\Big),\\
\Lambda^{0i} &= O\Big(\frac{1}{c^5}\Big),\\
\Lambda^{ij} &= \frac{4}{c^4}\Big(\partial_i V \partial_j V - \frac{1}{2}\delta_{ij}\partial_k V \partial_k V\Big) + O\Big(\frac{1}{c^6}\Big).
\end{aligned} \tag{5.82}$$

For the next order equations, we also need the metric determinant to 1 PN order. We have

$$|g| = 1 + h + O(h^2) = 1 + (-h^{00} + h^{ii}) + O\Big(\frac{1}{c^4}\Big) = 1 + \frac{4V}{c^2} + O\Big(\frac{1}{c^4}\Big). \tag{5.83}$$

Then we have the equations as

$$\begin{aligned}
\Box h^{00} &= \frac{16\pi G}{c^2}\Big(1 + \frac{4V}{c^2}\Big)\Big(\sigma - \frac{\sigma^{ii}}{c^2}\Big) - \frac{14}{c^4}\partial_k V \partial_k V + O\Big(\frac{1}{c^6}\Big),\\
\Box h^{0i} &= \frac{16\pi G}{c^3}\sigma^i + O\Big(\frac{1}{c^5}\Big),\\
\Box h^{ij} &= \frac{16\pi G}{c^4}\sigma^{ij} + \frac{4}{c^4}\Big(\partial_i V \partial_j V - \frac{1}{2}\delta_{ij}\partial_k V \partial_k V\Big) + O\Big(\frac{1}{c^6}\Big).
\end{aligned} \tag{5.84}$$

Second and the third equations are handled by defining new potentials

$$V_i = G \int \frac{\mathrm{d}^3 y}{|\vec{x}-\vec{y}|} \sigma^i \Big(\vec{y}, t - \frac{|\vec{x}-\vec{y}|}{c}\Big), \tag{5.85}$$

and

$$W_{ij} = G \int \frac{\mathrm{d}^3 y}{|\vec{x}-\vec{y}|} \Big(\sigma^{ij} + \frac{1}{4\pi G}\Big(\partial_i V \partial_j V - \frac{1}{2}\delta_{ij}\partial_k V \partial_k V\Big)\Big)(\vec{y}, t_{ret}). \tag{5.86}$$

The integral for V_i is convergent again from the compactness of the source. W_{ij} is also convergent since for $r \to \infty$ we have $V = O(1/r)$, thus $\partial_i V \partial_j V = O(1/r^4)$. Let us put $\Box V = -4\pi G\sigma$ in the first equation, then we find

$$\Box h^{00} = -\frac{4}{c^2}\Box V - \frac{16}{c^4} V \Box V - \frac{16\pi G}{c^4}\sigma^{ii} - \frac{14}{c^4}\partial_k V \partial_k V + O\Big(\frac{1}{c^6}\Big). \tag{5.87}$$

It can be shown that $\Box(V^2) = 2\partial_k V \partial_k V + 2V\Box V$. Then we find

$$\begin{aligned} \Box h^{00} &= -\frac{4}{c^2}\Box V - \frac{8}{c^4}\Box(V^2) - \frac{16\pi G}{c^4}\sigma^{ii} + \frac{2}{c^4}\partial_k V \partial_k V + O\Big(\frac{1}{c^6}\Big) \\ &= -\frac{4}{c^2}\Box V - \frac{8}{c^4}\Box(V^2) + \frac{4}{c^4}\Box W_{ii} + O\Big(\frac{1}{c^6}\Big). \end{aligned} \tag{5.88}$$

Thus, the solution to this iteration will be, as found in [24]

$$\begin{aligned} h^{00} &= \frac{-4V}{c^2} + \frac{4}{c^4}\Big(W_{ii} - 2V^2\Big) + O\Big(\frac{1}{c^4}\Big), \\ h^{0i} &= \frac{-4V_i}{c^3} + O\Big(\frac{1}{c^3}\Big), \\ h^{ij} &= \frac{-4W_{ij}}{c^4} + O\Big(\frac{1}{c^4}\Big). \end{aligned} \tag{5.89}$$

Then by plugging this result in (5.74), the metric tensor can be found as

$$\begin{aligned} g_{00} &= -1 + \frac{2V}{c^2} - \frac{2V^2}{c^4} + O\Big(\frac{1}{c^6}\Big), \\ g_{0i} &= -\frac{4V_i}{c^3} + O\Big(\frac{1}{c^5}\Big), \\ g_{ij} &= \delta_{ij}\Big(1 + \frac{2V}{c^2}\Big) + O\Big(\frac{1}{c^4}\Big). \end{aligned} \tag{5.90}$$

Plugging this metric in (5.70) and (5.71) we find

$$u_i = v^i + \frac{1}{c^2}\Big(\frac{1}{2}v^2 v^i + 3Vv^i - 4V_i\Big) + O\Big(\frac{1}{c^4}\Big), \tag{5.91}$$

and

$$\dot{u}_i = \partial_i V + \frac{1}{c^2}\Big(- V\partial_i V + \frac{3}{2}v^2 \partial_i V - 4\partial_i V_j v^j\Big) + O\Big(\frac{1}{c^4}\Big). \tag{5.92}$$

This calculation can be continued further, as done in [24] and [25]. Here, we will give the results in [25]:

$$
\begin{aligned}
g_{00} &= -1 + \frac{2V}{c^2} - \frac{2V^2}{c^4} + \frac{8}{c^6}\Big(X + V_i V_i + \frac{1}{6}V^3\Big) + O\Big(\frac{1}{c^8}\Big),\\
g_{0i} &= -\frac{4V_i}{c^3} - \frac{8}{c^5}R_i + O\Big(\frac{1}{c^7}\Big),\\
g_{ij} &= \delta_{ij}\Big(1 + \frac{2}{c^2}V + \frac{2}{c^4}V^2\Big) + \frac{4}{c^4}W_{ij} + O\Big(\frac{1}{c^6}\Big),
\end{aligned}
\tag{5.93}
$$

where V, V_i and W_{ij} are as defined before, the other 2 PN potentials are defined as

$$
\begin{aligned}
X &= \Box_R^{-1}\Big[-4\pi G V\sigma_{ii} + W_{ij}\partial_{ij}V + 2V_i\partial_t\partial_i V + V\partial_t^2 V + \frac{3}{2}(\partial_t V)^2 - 2\partial_i V_j \partial_j V_i\Big],\\
R_i &= \Box_R^{-1}\Big[-4\pi G(V\sigma_i - V_i\sigma) - 2\partial_k V\partial_i V_k - \frac{3}{2}\partial_t V\partial_i V\Big],
\end{aligned}
\tag{5.94}
$$

and the equations of motions will be obtained from

$$
\begin{aligned}
u_i = v^i &+ \frac{1}{c^2}\Big(\frac{1}{2}v^2 v^i + 3Vv^i - 4V_i\Big) + \frac{1}{c^4}\Big(\frac{3}{8}v^4 v^i + \frac{7}{2}Vv^2 v^i\\
&- 4V_j v^i v^j - 2V_i v^2 + \frac{9}{2}V^2 v^i - 4VV_i + 4W_{ij}v^j - 8R_i\Big) + O\Big(\frac{1}{c^6}\Big),
\end{aligned}
\tag{5.95}
$$

and

$$
\begin{aligned}
\dot{u}_i = \partial_i V &+ \frac{1}{c^2}\Big(-V\partial_i V + \frac{3}{2}v^2\partial_i V - 4\partial_i V_j v^j\Big) + \frac{1}{c^4}\Big(\frac{7}{8}\partial_i V v^4\\
&- 2\partial_i V_j v^j v^2 + \frac{9}{2}V\partial_i V v^2 + 2\partial_i W_{jk}v^j v^k - 4V_j\partial_i V v^j - 4V\partial_i V_j v^j\\
&- 8\partial_i R_j v^j + \frac{1}{2}V^2\partial_i V + 8V_j\partial_i V_j + 4\partial_i X\Big) + O\Big(\frac{1}{c^6}\Big).
\end{aligned}
\tag{5.96}
$$

In the next chapter, we will calculate these potentials and their derivatives for the energy-momentum tensor of two non-rotating black holes to obtain the trajectories of those black holes from the geodesic equation.

Chapter 6

Binary Black Hole Systems

In this chapter, evolution of the binary black hole systems will be discussed equations of motion of two gravitating particles to 1 PN will be derived.

6.1 Energy Momentum Tensor of a Massive Point Particle

In this section, we will construct the gravitational energy momentum tensor of a point particle from the action. In GR, in the test particle limit, ie. the limit they do not contribute to the gravitational field, they follow the geodesic lines. This is the result of many discussions that can be found in both [10] and [11]. To obtain this result, the obvious choice of action can be made

$$S_{part} = -mc \int \mathrm{d}s, \tag{6.1}$$

so that the minimization will lead to the geodesic equation, where m is the mass of the particle. Now, if we assume that this form of action can actually work for gravitating particles too, as assumed in may other papers, we would find total action to be

$$S = -mc \int \mathrm{d}s - \frac{c^3}{16\pi G} \int \mathrm{d}^4x \sqrt{-g} R, \tag{6.2}$$

where $\mathrm{d}^4x = \mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$. Then, by varying the metric and assuming that this total action is extremized, we then find that the $T^{\mu\nu}$ on the right hand side of the Einstein field equations (2.25) to be

$$T^{\mu\nu} = mc\int \frac{\mathrm{d}s}{\sqrt{-g}}\delta^4(x^\alpha - z^\alpha(s))\frac{dz^\mu}{ds}\frac{dz^\nu}{ds}, \tag{6.3}$$

where the integral region is the trajectory of the particle, and the curve that particle follows is given as $x^\mu = z^\mu(s)$ and the four dimensional Dirac delta function is defined, just like the three dimensional one, as $\delta^4(x^\alpha) = \delta(x^0)\delta(x^1)\delta(x^2)\delta(x^3)$. Additionally, we want our particle not to be in different positions in a given time, ie. we need $z^0(s)$ to be a one to one function. By using these information, if we separate the time part of the four dimensional delta, this integral can easily be done

$$\begin{aligned} T^{\mu\nu}(x^\alpha) &= mc\int \frac{\mathrm{d}s}{\sqrt{-g}}\delta(x^0 - z^0(s))\delta^3(x^i - z^i(s))\frac{dz^\mu}{ds}\frac{dz^\nu}{ds} \\ &= \left(\frac{mc}{\sqrt{-g}}\frac{dz^\mu}{ds}\frac{dz^\nu}{ds}\frac{ds}{dz^0}\delta^3(x^i - z^i(s))\right)_{x^0 - z^0(s)} \\ &= \left(\frac{mc}{\sqrt{-g}}\frac{u^\mu u^\nu}{u^0}\delta^3(x^i - z^i(s))\right)_{x^0 - z^0(s)}, \end{aligned} \tag{6.4}$$

and finally by putting (5.70), we get, as in [10]

$$T^{\mu\nu}(\vec{x}, t) = \frac{mc}{\sqrt{-g}}\frac{v^\mu v^\nu}{\sqrt{-g_{\alpha\beta}v^\alpha v^\beta}}\delta^3(\vec{x} - \vec{z}(t)). \tag{6.5}$$

For a non interacting two particle system, except from gravity of course, the energy momentum tensor will be considered as the summation of energy momentum tensor of each particle, ie.

$$T^{\mu\nu}_{tot}(\vec{x}, t) = \sum_{a=1,2}\frac{m_a c}{\sqrt{-g}}\frac{v^\mu_a v^\nu_a}{\sqrt{-g_{\alpha\beta}v^\alpha_a v^\beta_a}}\delta^3(\vec{x} - \vec{z}_a(t)), \tag{6.6}$$

where a runs over all of two particles. In general, we will not write the index a for convenience, and we will write

$$T^{\mu\nu}_{tot}(\vec{x}, t) = \sum \frac{mc}{\sqrt{-g}}\frac{v^\mu v^\nu}{\sqrt{-g_{\alpha\beta}v^\alpha v^\beta}}\delta^3(\vec{x} - \vec{z}(t)), \tag{6.7}$$

and always remember that this sum is over two particles, and v^μ and z^μ are changing while doing the summation.

6.2 Equations of Motion to 1 PN Order

In this section, we will calculate the equations of motion to 1 PN order as done in [23], and will give further results obtained in [25] at the end.

In [23], instead of using the energy momentum tensor directly in the given form in (6.3), the following form is used:

$$T^{\mu\nu} = mc \int \frac{\mathrm{d}s}{\sqrt{-g}} Z_A(x^\alpha - z^\alpha(s)) \frac{dz^\mu}{ds}\frac{dz^\nu}{ds}. \tag{6.8}$$

A here is a complex number, and the function Z_A is called Riezs' kernel satisfying

$$\lim_{A\to 0} Z_A(x^\alpha) = \delta^4(x^\alpha). \tag{6.9}$$

Exchange of the four dimensional Dirac delta function with Z_A can be shown to be equivalent to exchanging the three dimensional Dirac delta function in (6.5) with the following [18]

$$\delta_A(\vec{x}) = -\frac{1}{4\pi}\Delta(r^{A-1}) = -\frac{A(A-1)}{4\pi} r^{A-3}. \tag{6.10}$$

After plugging this function in the energy momentum tensor, we will calculate the analythical continuation of the acceleration with respect to A, and then take $A \to 0$ limit.

Let us expand the mass term in (6.5) by using (5.90):

$$\tilde{m} = \frac{mc}{\sqrt{-g}\sqrt{-g_{\alpha\beta}v^\alpha v^\beta}} = m\Big(1 - \frac{V}{c^2} + \frac{v^2}{2c^2}\Big) + O\Big(\frac{1}{c^4}\Big). \tag{6.11}$$

If we define

$$\mu = \tilde{m}\Big(1 + \frac{v^2}{c^2}\Big) = m\Big(1 - \frac{V}{c^2} + \frac{3v^2}{2c^2}\Big) + O\Big(\frac{1}{c^4}\Big), \tag{6.12}$$

then we have σ and σ_i of two particle to be (5.78)

$$\begin{aligned} \sigma &= \sum \mu\delta_A(\vec{x} - \vec{z}) + O\Big(\frac{1}{c^4}\Big), \\ \sigma_i &= \sum \tilde{m} v^i \delta_A(\vec{x} - \vec{z}) + O\Big(\frac{1}{c^4}\Big). \end{aligned} \tag{6.13}$$

From (5.91) and (5.92), we see that we need to calculate V to $O(1/c^4)$ and V_i to $O(1/c^2)$. Let us calculate V_i first,

$$\begin{aligned}
V_i &= -4\pi G \Box_R^{-1} \sigma_i \\
&= G \int \frac{d^3y}{|\vec{x}-\vec{y}|} \sigma_i(\vec{y}, t_{ret}) \\
&= G \int \frac{d^3y}{|\vec{x}-\vec{y}|} \Big(\sigma_i - \frac{|\vec{x}-\vec{y}|}{c} \dot{\sigma}_i \Big) + O\Big(\frac{1}{c^2}\Big) \\
&= -4\pi G \Delta^{-1} \sigma_i - \frac{G}{c} \partial_t \int \mathrm{d}^3 y \sigma_i + O\Big(\frac{1}{c^2}\Big) \\
&= \sum \Big(\frac{Gmv^i}{|\vec{x}-\vec{z}|^{1-A}} - \frac{Gm}{c} \partial_t \int \mathrm{d}^3 y \frac{A(1-A)v^i}{4\pi} |\vec{y}-\vec{z}|^{A-3} \Big) + O\Big(\frac{1}{c^2}\Big).
\end{aligned} \tag{6.14}$$

If we shift the integration variable by $\vec{z}$ in the remaining integral, we have an integral of the type $\int_0^\infty dyy^B$, which was shown to be zero after the analytical continuation (5.18). Therefore, we conclude that we have

$$V_i = \sum \frac{Gmv^i}{|\vec{x}-\vec{z}|^{1-A}} + O\Big(\frac{1}{c^2}\Big). \tag{6.15}$$

If we define $\vec{R} = \vec{x} - \vec{z}$, then we can write

$$V_i = \sum Gmv^i R^{A-1} + O\Big(\frac{1}{c^2}\Big). \tag{6.16}$$

Then we have

$$\partial_j V_i = \sum Gm(A-1) v^i R^{A-2} n^j + O\Big(\frac{1}{c^2}\Big), \tag{6.17}$$

where we have $n^i = R^i/R$. At the point $\vec{z}_1$, we find

$$V_i(\vec{z}_1) = (V_i)_1 = Gm_1 v_1^i (0)^{A-1} + Gm_2 v_2^i R_{12}^{A-1} + O\Big(\frac{1}{c^2}\Big), \tag{6.18}$$

where we defined $\vec{R}_{12} = \vec{R}_1 - \vec{R}_2$. Now, $(V_i)_1$ is an analythical function of A for $Re(A) > 1$, and on that region it equals to

$$V_i(\vec{z}_1) = (V_i)_1 = Gm_2 v_2^i R_{12}^{A-1} + O\Big(\frac{1}{c^2}\Big). \tag{6.19}$$

With $A \to 0$ limit

$$V_i(\vec{z}_1) = (V_i)_1 = \frac{Gm_2 v_2^i}{R_{12}} + O\Big(\frac{1}{c^2}\Big). \tag{6.20}$$

We should calculate $\partial_j V_i$ too.

$$(\partial_j V_i)_1 = Gm_2 (A-1) v_2^i R_{12}^{A-2} n_{12}^j + O\Big(\frac{1}{c^2}\Big), \tag{6.21}$$

then taking $A \to 0$ limit

$$(\partial_j V_i)_1 = -\frac{Gm_2 v_2^i}{R_{12}^2} n_{12}^j + O\Big(\frac{1}{c^2}\Big). \tag{6.22}$$

Let us now calculate V. We have

$$\begin{aligned} V &= G\int \frac{d^3y}{|\vec{x}-\vec{y}|}\Big(\sigma - \frac{|\vec{x}-\vec{y}|}{c}\dot{\sigma} + \frac{|\vec{x}-\vec{y}|^2}{2c^2}\ddot{\sigma} - \frac{|\vec{x}-\vec{y}|^3}{6c^3}\dddot{\sigma}\Big) + O\Big(\frac{1}{c^2}\Big) \\ &= \sum G\mu R^{A-1} - \frac{G}{c}\partial_t \int \mathrm{d}^3y\sigma + \frac{G}{2c^2}\partial_t^2 \int \mathrm{d}^3y|\vec{x}-\vec{y}|\sigma \\ &\qquad - \frac{G}{6c^3}\partial_t^3 \int \mathrm{d}^3y|\vec{x}-\vec{y}|^2\sigma + O\Big(\frac{1}{c^4}\Big). \end{aligned} \tag{6.23}$$

The $1/c$ integral equals to zero due to the same reason we had for V_i. In the $1/c^3$ integral, we can write

$$\begin{aligned} \int \mathrm{d}^3y|\vec{x}-\vec{y}|^2|\vec{y}-\vec{z}|^{A-3} &= \int \mathrm{d}^3y|\vec{y}-\vec{R}|^2 y^{A-3} \\ &= \int_0^\infty \mathrm{d}y y^{A-1}\int_{-1}^1 \mathrm{d}u(y^2+R^2-2yRu), \end{aligned} \tag{6.24}$$

then it becomes integral of a polynomial from zero to infinity, thus its analytical continuation is zero too. Analytical continuation of the $1/c^2$ integral is calculated as

$$\int \mathrm{d}^3y|\vec{x}-\vec{y}||\vec{y}-\vec{z}|^{A-3} = -\frac{8\pi R^{A+1}}{(A-1)A(A+1)(A+2)}. \tag{6.25}$$

Then we have

$$V = \sum G\mu R^{A-1} + \frac{G}{c^2}\partial_t^2 \sum\Big(\frac{mR^{A+1}}{(A+1)(A+2)}\Big) + O\Big(\frac{1}{c^4}\Big). \tag{6.26}$$

It is easy to calculate

$$\partial_t^2 R^{A+1} = (A+1)\big(v^2 - R^i a^i + (A-1)(n^i v^i)^2\big)R^{A-1}, \tag{6.27}$$

where $a^i = \dot{v}^i$. Therefore we find

$$\begin{aligned} V &= \sum G\mu R^{A-1} \\ &\quad + \frac{G}{c^2}\sum \frac{mR^{A-1}}{A+2}\Big(v^2 - R^i a^i + (A-1)(n^i v^i)^2\Big) + O\Big(\frac{1}{c^4}\Big). \end{aligned} \tag{6.28}$$

Now, we have μ_1 of the first particle

$$\mu_1 = m_1\Big(1 - \frac{V(\vec{z}_1)}{c^2} + \frac{3v_1^2}{2c^2}\Big) + O\Big(\frac{1}{c^4}\Big). \tag{6.29}$$

The potential V at $\vec{x} = \vec{z}_1$ to Newtonian order is

$$(V)_1 = Gm_2 R_{12}^{A-1} + +O\Big(\frac{1}{c^2}\Big). \tag{6.30}$$

Putting this in (6.29)

$$\mu_1 = m_1\Big(1 - \frac{Gm_2 R_{12}^{A-1}}{c^2} + \frac{3v_1^2}{2c^2}\Big) + O\Big(\frac{1}{c^4}\Big), \tag{6.31}$$

then we finally have

$$\begin{aligned} V &= \sum Gm\Big(1 - \frac{Gm' R_{12}^{A-1}}{c^2} + \frac{3v^2}{2c^2}\Big)R^{A-1} \\ &+ \frac{G}{c^2}\sum \frac{mR^{A-1}}{A+2}\Big(v^2 - R^i a^i + (A-1)(n^i v^i)^2\Big) + O\Big(\frac{1}{c^4}\Big), \end{aligned} \tag{6.32}$$

where m' means use the other mass while summing. Then we find

$$\begin{aligned} \partial_i V &= \sum Gm(A-1)\Big(1 - \frac{Gm' R_{12}^{A-1}}{c^2} + \frac{3v^2}{2c^2}\Big)R^{A-2}n^i \\ &+ \frac{G}{c^2}\Bigg\{\sum \frac{m}{A+2}\Big(v^2 - R^k a^k + (A-1)(n^k v^k)^2\Big)(A-1)R^{A-2}n^i \\ &+ \sum \frac{m}{A+2}R^{A-1}\Big(-a^i + \frac{2(A-1)}{R}(n^k v^k v^i - (n^k v^k)^2 n^i)\Big)\Bigg\} + O\Big(\frac{1}{c^4}\Big). \end{aligned} \tag{6.33}$$

If we calculate it at $\vec{z}_1$

$$\begin{aligned} (\partial_i V)_1 &= Gm_2(A-1)\Big(1 - \frac{Gm_1 R_{12}^{A-1}}{c^2} + \frac{3v_2^2}{2c^2}\Big)R_{12}^{A-2}n_{12}^i \\ &+ \frac{G}{c^2}\frac{m_2}{A+2}\Bigg\{\Big(v_2^2 - R_{12}^k a_2^k + (A-1)(n_{12}^k v_2^k)^2\Big)(A-1)R_{12}^{A-2}n_{12}^i \\ &+ R_{12}^{A-1}\Big(-a_2^i + \frac{2(A-1)}{R_{12}}(n_{12}^k v_2^k v_2^i - (n_{12}^k v_2^k)^2 n_{12}^i)\Big)\Bigg\} + O\Big(\frac{1}{c^4}\Big). \end{aligned} \tag{6.34}$$

Now, if we put $a^i = \partial_i V + O(1/c^2)$ and take $A \to 0$, we find

$$\begin{aligned} (\partial_i V)_1 = -\frac{Gm_2 n_{12}^i}{R_{12}^2} &+ \frac{Gm_2}{R_{12}^2 c^2}n_{12}^i\Big(-2v_2^2 + \frac{3}{2}(n_{12}^k v_2^k)^2 + \frac{Gm_1}{R_{12}}\Big) \\ &- \frac{Gm_2}{R_{12}^2 c^2}(n_{12}^k v_2^k)v_2^i + O\Big(\frac{1}{c^4}\Big). \end{aligned} \tag{6.35}$$

Then by using equations (6.20),(6.22),(6.30) and (6.35), we find

$$(u_i)_1 = v_1^i + \frac{1}{c^2}\Big(\frac{v_1^2}{2}v_1^i + 3\frac{Gm_2}{R_{12}^2}v_1^i - 4\frac{Gm_2 v_2^i}{R_{12}}\Big) + O\Big(\frac{1}{c^4}\Big), \tag{6.36}$$

and

$$
\begin{aligned}
(\dot{u}_i)_1 = a_1^i - \frac{Gm_2}{R_{12}^2 c^2}\Big(n^i \frac{v_1^2}{2} + 4(n_{12}^k v_{12}) v_{12}^i + (n_{12}^k v_2^k) v_1^i\Big) \\
-\frac{G^2 m_2 n_{12}^i}{R_{12}^3 c^2}(3m_2 + 4m_1) + O\Big(\frac{1}{c^4}\Big).
\end{aligned}
\tag{6.37}
$$

Here we defined $v_{12}^i = v_1^i - v_2^i$. Then if we take the time derivative of (6.36) and use the expansion (6.37), we have

$$
\begin{aligned}
a_1^i = &-\frac{Gm_2}{R_{12}^2} n_{12}^i \\
&+ \frac{Gm_2 n_{12}^i}{c^2 R_{12}^2}\Big(- 2v_2^2 - v_1^2 + \frac{3}{2}(n_{12}^k v_2^k)^2 + 4v_1^k v_2^k + \frac{5Gm_1}{R_{12}} + \frac{4Gm_2}{R_{12}}\Big) \\
&+ \frac{Gm_2}{R_{12}^2 c^2} n_{12}^k (4v_1^k - 3v_2^k) v_{12}^i + O\Big(\frac{1}{c^4}\Big).
\end{aligned}
\tag{6.38}
$$

Chapter 7

Conclusion

We have studied the PM expansion of the metric and found a particular solution (3.86) which can be represented by a set of multipole moments $M_L, S_L, ..., Z_L$ which are called the source multipole moments. Then this solution is proved to be the general solution if we consider those multipole moments as PM expansions i.e. taking multipole moments as expansions in G. By assuming that source is moving slowly, the far zone and near zone limits of this solution is discussed, and $\log r$ terms have been found. It has been shown that there exists a coordinate transformation such that in the far zone, the metric can be written as an expansion in $1/r$ only. By using the leading order term of this metric in the far zone limit, radiation multipole moments have been calculated in terms of the source multipole moments. Assuming that the energy momentum tensor is a PN expansion, it has been shown that the metric can also be considered as a PN expansion. We then applied both PM and PN expansions at the same time and used the matching condition (5.2) to relate the source multipole moments to the the energy momentum tensor. In the end, we found the equations of motion of a binary system in inspiralling phase.

With all the connections that is shown in this thesis, it becomes just an algebraic work to calculate first the source multipole moments from the motion of the binary by using the Riezs' kernel and equations (5.37) and (5.38), and then to calculate the radiative multipole moments from the source moments by using equation (4.93). Therefore using methods discussed in this thesis, physical quantities about the inspiralling phase of binary black hole systems including gravitational wave form, energy flux resulting from the gravitational waves, near zone and far zone metric, geodesics of the resulting spacetime and the change of the period of the motion of the binary system in time can be calculated.

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