DILATIONS OF DOUBLY INVARIANT KERNELS VALUED IN TOPOLOGICALLY ORDERED $*$-SPACES

A DISSERTATION SUBMITTED TO

THE GRADUATE SCHOOL OF ENGINEERING AND SCIENCE

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR

THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

By

Serdar Ay

June 2018

DILATIONS OF DOUBLY INVARIANT KERNELS VALUED IN TOPOLOGICALLY ORDERED $*$-SPACES

By Serdar Ay

June 2018

We certify that we have read this dissertation and that in our opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

Aurelian B. N. Gheondea E.(Advisor)

Eduard Emelyanov

Hakkı Turgay Kaptanoğlu

Alexandre Goncharov

Uğur Gül

Approved for the Graduate School of Engineering and Science:

Ezhan Karaşan
Director of the Graduate School

ABSTRACT

DILATIONS OF DOUBLY INVARIANT KERNELS VALUED IN TOPOLOGICALLY ORDERED $*$-SPACES

Serdar Ay
Ph.D. in Mathematics
Advisor: Aurelian B. N. Gheondea E.
June 2018

An ordered $*$-space Z is a complex vector space with a conjugate linear involution $*$, and a strict cone $Z+$ consisting of self adjoint elements. A topologically ordered $*$-space is an ordered $*$-space with a locally convex topology compatible with its natural ordering. A VE (Vector Euclidean) space, in the sense of Loynes, is a complex vector space equipped with an inner product taking values in an ordered $*$-space, and a VH (Vector Hilbert) space, in the sense of Loynes, is a VE-space with its inner product valued in a complete topologically ordered $*$-space and such that its induced locally convex topology is complete.

On the other hand, dilation type theorems are important results that often realize a map valued in a certain space as a part of some simpler elements on a bigger space. Dilation results today are of an extraordinary large diversity and it is a natural question whether most of them can be unified under general theorems.

We study dilations of weakly positive semidefinite kernels valued in (topologically) ordered $*$-spaces, which are invariant under left actions of $*$-semigroups and right actions of semigroups, called doubly invariant. We obtain VE and VH-spaces linearisations of such kernels, and on equal foot, their reproducing kernel spaces, and operator representations of the acting semigroups.

The main results are used to unify many of the known dilation theorems for invariant positive semidefinite kernels with operator values, also for kernels valued in certain algebras, as well as to obtain some new dilation type results, in the context of Hilbert C^*-modules, locally Hilbert C^*-modules and VH-spaces.

Keywords: topologically ordered $*$-space, VE-space, VH-space, Hermitian kernel, weakly positive semidefinite kernel, doubly invariant kernel, linearisation, reproducing kernel, $*$-representation, completely positive map.

ÖZET

SIRALI $*$-UZAYI DEĞERLİ ÇİFT DEĞİŞMEZ ÇEKİRDEKLERİN GENLEŞMESİ

Serdar Ay
Matematik, Doktora
Tez Danışmanı: Aurelian B. N. Gheondea E.
Haziran 2018

Eşlenik doğrusal bir involüsyon ve özeşlenik elemanlardan oluşan kesin bir pozitif elemanlar konisi olan kompleks bir vektör uzayına sıralı $*$-uzayı denir. Doğal sıralaması ile uyumlu bir yerel konveks topolojisi olan sıralı $*$-uzayına ise topolojik sıralı $*$-uzayı denir. Loynes anlamında bir VE-uzayı (Vektör Öklid) bir sıralı $*$-uzayında değer alan bir iç çarpıma sahip bir karmaşık vektör uzayıdır. Loynes anlamında bir VH-uzayı (Vektör Hilbert) ise iç çarpımı bir tam topolojik sıralı $*$-uzayında değer alan ve iç çarpımının oluşturduğu yerel konveks topolojisi tam olan bir VE-uzayıdır.

Diğer yandan, genleşme türü teoremler genellikle belli bir uzayda değer alan bir gönderimin daha büyük bir uzayın daha basit elemanlarının bir parçası olarak ifade edilebilmesini sağlayan önemli teoremlerdir. Günümüzde genleşme teoremleri çok fazla çeşitliliktedir ve bu teoremlerin birçoğunun genel teoremler altında birleştirilmesinin mümkün olup olmadığı doğal bir sorudur.

Bu tezde (topolojik) sıralı $*$-uzayı değerli zayıf pozitif yarıtanımlı, $*$-yarıgruplarının sol etkileri altında ve yarıgrupların sağ etkileri altında değişmez, dolayısıyla çift değişmez çekirdeklerin genleşmeleri üzerine çalışacağız. Bu çekirdeklerin VE ve VH-uzayı doğrusallaştırmalarını ve doğuran çekirdek uzaylarını, ayrıca etki eden yarıgrupların operatör temsillerini elde edeceğiz.

Ana sonuçları, Hilbert C^*-modülü, yerel Hilbert C^*-modülü ve VH-uzayı çerçevesindeki operatör veya belli cebir değerli pozitif yarıtanımlı değişmez çekirdeklerin bilinen birçok genleşme teoremlerinin birleştirilmesinde ve bazı yeni genleşme türü sonuçların elde edilmesinde kullanacağız.

Anahtar sözcükler: topolojik sıralı $*$-uzayı, VE-uzayı, VH-uzayı, Hermitsel çekirdek, zayıf pozitif yarıtanımlı çekirdek, çift değişmez çekirdek, doğrusallaştırma, doğuran çekirdek, $*$-temsili, tamamen pozitif gönderim.

Acknowledgement

First and foremost I would like to express my deepest gratitude to my supervisor Prof. Aurelian Gheondea, for his excellent guidance, encouragement, patience and invaluable support.

I would like to thank my mother İzdihar, and my sister Aylin, for their constant support and understanding, and my father Sami, who is deceased, but whose moral support I always feel inside.

I would like to thank to Prof. Eduard Emelyanov and Prof. Hakkı Turgay Kaptanoğlu for being members of the monitoring committee of my Ph.D. studies, as well as Prof. Mefharet Kocatepe, Prof. Alexandre Goncharov and Assoc. Prof. Uğur Gül for accepting to be jury members in my Phd thesis defence.

My friends Abdullah, İsmail, Bekir, Gökalp, Muhammed, Melih from the Mathematics Department, as well as Mert and all other chess friends from Bilkent Chess Society were great company to me. I would like to thank them all.

Contents

Chapter 1

Introduction

Starting with the celebrated Naimark's dilation theorems in [1] and [2], a powerful dilation theory for operator valued maps was obtained through results of B. Sz.-Nagy [3], W.F. Stinespring [4], and their generalisations to VH-spaces (Vector Hilbert spaces) by R.M. Loynes [5], or to Hilbert C^*-modules by G.G. Kasparov [6]. The dilation theory consists today of an extraordinary large diversity of results that may look, at the first glance, as having next to nothing in common, e.g. see N. Aronszajn [7], W.B. Arveson [8], S.D. Barreto et al. [9], D. Gaşpar and P. Gaşpar [10], [11], A. Gheondea and B.E. Uğurcan [12], J. Górniak and A. Weron [13], [14], J. Heo [15], G.G. Kasparov [6], R.M. Loynes [5], G.J. Murphy [16], M. Skeide [17], W.F. Stinespring [4], F.H. Szafraniec [18], [19], B. Sz.-Nagy [3], to cite a few only. Taking into account the importance and the diversity of dilation theorems e.g. see [8], there is a natural question, whether one can unify all, or the most, of these dilation theorems, under one theorem. Such ideas are provided in e.g. [20], [21], [16], [15], and [19], to cite just a few. Attempts to approach this question are made in [22] by using the notions of VE-space over an ordered $*$-space and in [23], [24], [25] by using the notion of VH-space over an admissible space, introduced by R.M. Loynes [5], [26] in 1965. Also see [27]. Following [23], [22], [24], [25] in this thesis it is a primary goal to show that this unifying framework becomes significantly more successful when kernels with values linear operators on VE-spaces and VH-spaces, see [22] and [24] and more generally, kernels with values in (topologically) ordered $*$-spaces are employed.

VE-space (Vector Euclidean space) and VH-space (Vector Hilbert space) are generalisations of the notions of inner product space and of Hilbert space. These are vector spaces on which there are "inner products" with values in certain ordered $*$-spaces, hence "vector valued inner products", see subsections 2.1–2.3 for precise definitions.

On the other hand, special cases of VH-spaces have been later considered independently of the Loynes' articles. Thus, the concept of Hilbert module over a C^*-algebra was introduced in 1973 by W.L. Paschke in [28], following I. Kaplansky [29], and independently by M.A. Rieffel one year later in [30], and these two articles triggered a whole domain of research, see e.g. [31] and [32] and the rich bibliography cited there. Hilbert modules over C^*-algebras are special cases of VH-spaces. Dilation theory plays a very important role in this theory and there are many dilation results of an impressive diversity, but the domain of Hilbert modules over C^*-algebras remained unrelated to that of VH-spaces. Another special case of a VH-space is that of Hilbert modules over H^*-algebras of P.P. Saworotnow [33]. Also, in 1985 A. Mallios [34] and later in 1988 N.C. Phillips [35] introduced and studied the concept of Hilbert module over locally C^*-algebra, which is yet another particular case of VH-space over an admissible space. The theory of Hilbert spaces over locally C^*-algebras is an active domain of research as well, e.g. see [36] and the rich bibliography cited there.

The aim of this thesis is to present a general approach to dilation theory based on weakly positive semidefinite kernels, cf. section 3.1 that are left invariant under actions of $*$-semigroups and right invariant under actions of semigroups and with values (topologically) ordered $*$-spaces. In addition, we show that almost each dilation theorem for such kernels is equivalent to a realisation as a reproducing kernel space with additional properties. Our approach is based on ideas already present under different dilation theorems in [21], [20], [5] [37], [27], [16], [38], [11], [10], [19], [15], [23] and, probably, many others. In this thesis, to a large extent, we make use of the results in our published articles [22], [24], [25] during the Phd studies.

We briefly describe the contents of this thesis. In Chapter 2 we fix some terminology and facts on ordered $*$-spaces, ordered $*$-algebras, VE-spaces over ordered $*$-spaces, and VE-modules over ordered $*$-algebras. On these basic objects, one can build the ordered $*$-algebras of adjointable operators on VE-spaces or VE-modules. We provide many examples that illustrate the richness of this theory, even at the non topological level. Then we study ordered $*$-spaces with a natural topology and VE-spaces with a topology inherited from such ordered $*$-spaces, then VH-spaces and their linear operators. One of the main mathematical objects used in this research is that of Loynes' admissible space, that is, a complete topologically ordered $*$-space. In Lemma 2.3.2 we obtain a first surrogate of the Schwarz inequality, which turns out to be very useful.

Then, in Chapter 3, we consider the main object of investigation which refers to weakly positive semidefinite kernels with values (topologically) ordered $*$-spaces. Here, we draw attention to Lemma 2.3.1 that clarifies the locally convex topology on VH-spaces and to some generic examples that illustrate the unifying potential of the concept of VH-space. Then we briefly show the connection between linearisations and reproducing kernel spaces at this level of generality.

The main results are contained in theorems 3.2.7, 3.2.9, 3.2.10, 3.2.11 and 3.2.13 from which we then show how special cases concerning different kinds of "stronger" positive semidefiniteness can be derived. We consider weakly positive semidefinite kernels not only left invariant under an action of a $*$-semigroup, but also invariant under an action of a semigroup acting on the right. As a result of this, we obtain VE and VH-space linearisations, as well as reproducing kernel VE and VH-spaces, which are left invariant, and are equipped with a right module action which respects their gramian. This right module action is given by a canonical representation of the right acting semigroup in the space of linear operators of the linearisation, and also the linear operators of the reproducing kernel space.

Finally, in Chapters 4 and 5 we show that the main theorems contain the dilation results obtained in many different contexts, including [23], [22], and [24], and hence most of the dilation theory, by explicitly showing how to put the stage in each case.

In Chapter 5 we apply the main theorems to obtain and unify dilation theorems, some known already, in the context of locally C^*-algebras and locally Hilbert modules over them and around different themes of positivity.

Chapter 2

Setting the Stage: Ordered $*$-Spaces, VE-Spaces, VH-Spaces and Their Examples

In this chapter we briefly review most of the definitions and some basic facts on ordered $*$-spaces, VE-spaces over ordered $*$-spaces, and their linear operators, then review and get some facts on VH-spaces over admissible spaces and their linear operators.

2.1 VE-Spaces and Their Linear Operators.

A complex vector space Z is called *ordered $*$-space*, see [39], if:

(a1) Z has an *involution* $*$, that is, a map $Z \ni z \mapsto z^* \in Z$ that is *conjugate linear* ($(sx+ty)^* = \bar{s}x^* + \bar{t}y^*$ for all $s, t \in \mathbb{C}$ and all $x, y \in Z$) and *involutive* ($(z^*)^* = z$ for all $z \in Z$).

(a2) In Z there is a *cone* Z^+ ($sx + ty \in Z^+$ for all numbers $s, t \geq 0$ and all $x, y \in Z^+$), that is *strict* ($Z^+ \cap -Z^+ = \{0\}$), and consisting of *selfadjoint* *elements* only ($z^* = z$ for all $z \in Z^+$). This cone is used to define a *partial order* on the real vector space of all selfadjoint elements in Z: $z_1 \geq z_2$ if $z_1 - z_2 \in Z^+$.

Recall that a $*$-*algebra* $\mathcal{A}$ is a complex algebra onto which there is defined an *involution* $\mathcal{A} \ni a \mapsto a^* \in \mathcal{A}$, that is, $(\lambda a + \mu b)^* = \overline{\lambda} a^* + \overline{\mu} b^*$, $(ab)^* = b^* a^*$, and $(a^*)^* = a$, for all $a, b \in \mathcal{A}$ and all $\lambda, \mu \in \mathbb{C}$.

An *ordered* $*$-*algebra* $\mathcal{A}$ is a $*$-algebra such that it is an ordered $*$-space, more precisely, it has the following property.

(osa1) There exists a strict cone $\mathcal{A}^+$ in $\mathcal{A}$ such that for any $a \in \mathcal{A}^+$ we have $a = a^*$.

Clearly, any ordered $*$-algebra is an ordered $*$-space. In particular, given $a \in \mathcal{A}$, we denote $a \geq 0$ if $a \in \mathcal{A}^+$ and, for $a = a^* \in \mathcal{A}$ and $b = b^* \in \mathcal{A}$, we denote $a \geq b$ if $a - b \geq 0$.

Given a complex linear space $\mathcal{E}$ and an ordered $*$-space space Z, a Z-*gramian*, also called a Z-*valued inner product*, is, by definition, a mapping $\mathcal{E} \times \mathcal{E} \ni (x, y) \mapsto [x, y] \in Z$ subject to the following properties:

(ve1) $[x, x] \geq 0$ for all $x \in \mathcal{E}$, and $[x, x] = 0$ if and only if $x = 0$.

(ve2) $[x, y] = [y, x]^*$ for all $x, y \in \mathcal{E}$.

(ve3) $[x, \alpha y_1 + \beta y_2] = \alpha [x, y_1] + \beta [x, y_2]$ for all $\alpha, \beta \in \mathbb{C}$ and all $x_1, x_2 \in \mathcal{E}$.

A complex linear space $\mathcal{E}$ onto which a Z-gramian $[\cdot, \cdot]$ is specified, for a certain ordered $*$-space Z, is called a *VE-space* (Vector Euclidean space) over Z, cf. [5].

Given a pairing $[\cdot, \cdot] \colon \mathcal{E} \times \mathcal{E} \to Z$, where $\mathcal{E}$ is some vector space and Z is an ordered $*$-space, and assuming that $[\cdot, \cdot]$ satisfies only the axioms (ve2) and (ve3), then a *polarisation formula* holds

$$4[x, y] = \sum_{k=0}^{3} \mathrm{i}^k [x + \mathrm{i}^k y, x + \mathrm{i}^k y], \quad x, y \in \mathcal{E}. \tag{2.1}$$

In particular, this formula holds on a VE-space and it shows that the Z-gramian is perfectly defined by the Z-valued quadratic map $\mathcal{E} \ni x \mapsto [x, x] \in Z$.

A *VE-spaces isomorphism* is, by definition, a linear bijection $U \colon \mathcal{E} \to \mathcal{F}$, for two VE-spaces over the same ordered $*$-space Z, which is *isometric*, that is, $[Ux, Uy]_{\mathcal{F}} = [x, y]_{\mathcal{E}}$ for all $x, y \in \mathcal{E}$.

A useful result for the constructions in the thesis is the following lemma.

Lemma 2.1.1 (Loynes [5]). *Let Z be an ordered $*$-space, $\mathcal{E}$ a complex vector space and $[\cdot,\cdot] \colon \mathcal{E} \times \mathcal{E} \to Z$ a positive semidefinite sesquilinear map, that is, $[\cdot,\cdot]$ is linear in the second variable, conjugate linear in the first variable, and $[x, x] \geq 0$ for all $x \in \mathcal{E}$. If $f \in \mathcal{E}$ is such that $[f, f] = 0$, then $[f, f'] = [f', f] = 0$ for all $f' \in \mathcal{E}$.*

Given two VE-spaces $\mathcal{E}$ and $\mathcal{F}$, over the same ordered $*$-space Z, one can consider the vector space $\mathcal{L}(\mathcal{E}, \mathcal{F})$ of all linear operators $T \colon \mathcal{E} \to \mathcal{F}$. A linear operator $T \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ is called *adjointable* if there exists $T^* \in \mathcal{L}(\mathcal{F}, \mathcal{E})$ such that

$$[Te, f]_{\mathcal{F}} = [e, T^* f]_{\mathcal{E}}, \quad e \in \mathcal{E},\ f \in \mathcal{F}. \tag{2.2}$$

The operator T^*, if it exists, is uniquely determined by T and called its *adjoint*. Since an analog of the Riesz Representation Theorem for VE-spaces may not exist, in general, there may be not so many adjointable operators. Denote by $\mathcal{L}^*(\mathcal{E}, \mathcal{F})$ the vector space of all adjointable operators from $\mathcal{L}(\mathcal{E}, \mathcal{F})$. Note that $\mathcal{L}^*(\mathcal{E}) = \mathcal{L}^*(\mathcal{E}, \mathcal{E})$ is a $*$-algebra with respect to the involution $*$ determined by the operation of taking the adjoint.

An operator $A \in \mathcal{L}(\mathcal{E})$ is called *selfadjoint* if $[Ae, f] = [e, Af]$, for all $e, f \in \mathcal{E}$. Any selfadjoint operator A is adjointable and $A = A^*$. By the polarisation formula (2.1), A is selfadjoint if and only if $[Ae, e] = [e, Ae]$, for all $e \in \mathcal{E}$. An operator $A \in \mathcal{L}(\mathcal{E})$ is *positive* if $[Ae, e] \geq 0$, for all $e \in \mathcal{E}$. Since the cone Z^+ consists of selfadjoint elements only, any positive operator is selfadjoint and hence adjointable. Note that any VE-space isomorphism U is adjointable, invertible, and $U^* = U^{-1}$, hence, equivalently, we can call it *unitary*.

An operator $T \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ is called *bounded* if there exists $C \geq 0$ such that

$$[Te, Te]_{\mathcal{F}} \leq C^2 [e, e]_{\mathcal{E}}, \quad e \in \mathcal{E}. \tag{2.3}$$

Note that the inequality (2.3) is in the sense of the order of Z uniquely determined by the cone Z_+, see the axiom (a2). The infimum of these scalars is denoted by $\|T\|$ and it is called the *operator norm* of T, more precisely,

$$\|T\| = \inf\{C > 0 \mid [Te, Te]_{\mathcal{F}} \leq C^2 [e, e]_{\mathcal{E}}, \text{ for all } e \in \mathcal{E}\}. \tag{2.4}$$

Let $\mathcal{B}(\mathcal{E}, \mathcal{F})$ denote the collection of all bounded linear operators $T \colon \mathcal{E} \to \mathcal{F}$. Then $\mathcal{B}(\mathcal{E}, \mathcal{F})$ is a linear space and $\| \cdot \|$ is a norm on it, cf. Theorem 1 in [26]. In addition, if T and S are bounded linear operators acting between appropriate VE-spaces over the same ordered $*$-space Z, then $\|TS\| \leq \|T\| \|S\|$, in particular TS is bounded. If $\mathcal{E} = \mathcal{F}$ then $\mathcal{B}(\mathcal{E}) = \mathcal{B}(\mathcal{E}, \mathcal{E})$ is a normed algebra, more precisely, the operator norm is submultiplicative.

A *VE-module $\mathcal{E}$ over an ordered $*$-algebra $\mathcal{A}$* is a right $\mathcal{A}$-module on which there exists an $\mathcal{A}$-*gramian* $[\cdot, \cdot]_{\mathcal{E}} \colon \mathcal{E} \times \mathcal{E} \to \mathcal{A}$ with respect to which it is a VE-space, that is, (ve1)-(ve3) hold, and, in addition,

(vem) $[e, fa + gb]_{\mathcal{E}} = [e, f]_{\mathcal{E}} a + [e, g]_{\mathcal{E}} b$ for all $e, f, g \in \mathcal{E}$ and all $a, b \in \mathcal{A}$.

Given an ordered $*$-algebra $\mathcal{A}$ and two VE-modules $\mathcal{E}$ and $\mathcal{F}$ over $\mathcal{A}$, an operator $T \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ is called a *module map* if

$$T(ea) = T(e)a, \quad e \in \mathcal{E}, \ a \in \mathcal{A}.$$

It is easy to see that any operator $T \in \mathcal{L}^*(\mathcal{E}, \mathcal{F})$ is a module map, e.g. see [22].

2.2 Admissible Spaces.

The complex vector space Z is called *topologically ordered $*$-space* if it is an ordered $*$-space, that is, axioms (a1) and (a2) hold and, in addition, (a3) Z is a *Hausdorff locally convex space.*

(a4) The topology of Z is *compatible* with the partial ordering in the sense that there exists a base of the topology, linearly generated by a family of neighbourhoods $\{C\}_{C\in\mathcal{C}_0}$ of the origin that are absolutely convex and *solid*, in the sense that, if $x \in C$ and $y \in Z$ are such that $0 \le y \le x$, then $y \in C$.

Remark 2.2.1. Axiom (a4) is equivalent with the following one:

(a4$'$) There exists a collection of seminorms $\{p_j\}_{j\in\mathcal{J}}$ defining the topology of Z that, for any $j \in \mathcal{J}$, p_j is *increasing*, in the sense that, $0 \le x \le y$ implies $p_j(x) \le p_j(y)$.

To see this, e.g. see Lemma 1.1.1 and Remark 1.1.2 of [40], letting $\mathcal{C}_0$ be a family of open, absolutely convex and solid neighbourhoods of the origin defining the topology of Z, for each $C \in \mathcal{C}_0$, consider the Minkowski seminorm p_C associated to C,

$$p_C(x) = \inf\{\lambda \mid \lambda > 0, \quad x \in \lambda C\}, \quad x \in Z. \tag{2.5}$$

Clearly, $\{p_C \mid C \in \mathcal{C}_0\}$ define the topology of Z. Moreover, p_C is increasing. To see this, for any $\epsilon > 0$, there exists $p_C(x) \le \lambda_\epsilon \le p_C(x) + \epsilon$ such that $x \in \lambda_\epsilon C$. Since C is balanced, $\lambda_\epsilon C \subset (p_C(x) + \epsilon)C$, so $x \in (p_C(x) + \epsilon)C$. As C is also solid, if $0 \le y \le x$, then we have $y \in (p_C(x) + \epsilon)C$, from which we obtain $p_C(y) \le p_C(x) + \epsilon$. Since $\epsilon > 0$ was arbitrary, we have that $p_C(y) \le p_C(x)$.

Conversely, given any increasing continuous seminorm p on Z, the set

$$C_p := \{x \in Z \mid p(x) < 1\}$$

is absolutely convex. Moreover, it is solid since, if $x \in C_p$ with $0 \le y \le x$, then $p(y) \le p(x) < 1$, so $y \in C_p$.

Given a family $\mathcal{C}_0$ of absolutely convex and solid neighbourhoods of the origin that generates the topology of Z, we denote by $S_{\mathcal{C}_0}(Z) = \{p_C \mid C \in \mathcal{C}_0\}$, where p_C is the Minkowski seminorm associated to C as in (2.5). The collection of all continuous increasing seminorms on Z is denoted by $S(Z)$. As a consequence of Remark 2.2.1, $S(Z)$ is in bijective correspondence with the family $\mathcal{C}$ of all open, absolutely convex and solid neighbourhoods of the origin. Note that $S(Z)$ is a directed set: given $p, q \in S(Z)$, consider $r := p + q$. In fact, $S(Z)$ is a cone, i.e. it is closed under all finite linear combinations with positive coefficients.

Z is called an *admissible space*, cf. [5], if, in addition to the axioms (a1)–(a4),

(a5) The cone Z_+ is *closed*, with respect to the specified topology of Z.

(a6) The topology on Z is complete.

Finally, if, in addition to the axioms (a1)–(a6), the space Z satisfies also the following axiom:

(a7) With respect to the specified partial ordering, any bounded monotone sequence is convergent.

then Z is called a *strongly admissible* space [5], also see [41]. A modern treatment of the subject can be found in [42].

Examples 2.2.2. (1) Any C^*-algebra $\mathcal{A}$ is an admissible space, as well as any closed $*$-subspace $\mathcal{S}$ of a C^*-algebra $\mathcal{A}$, with the positive cone $\mathcal{S}^+ = \mathcal{A}^+ \cap \mathcal{S}$ and all other operations (addition, multiplication with scalars, and involution) inherited from $\mathcal{A}$.

(2) Any pre-C^*-algebra is a topologically ordered $*$-space. Any $*$-subspace $\mathcal{S}$ of a pre-C^*-algebra $\mathcal{A}$ is a topologically ordered $*$-space, with the positive cone $\mathcal{S}^+ = \mathcal{A}^+ \cap \mathcal{S}$ and all other operations inherited from $\mathcal{A}$.

(3) Any locally C^*-algebra, cf. [43], [35], (definition is recalled in Chapter 5) is an admissible space. In particular, any closed $*$-subspace $\mathcal{S}$ of a locally C^*-algebra $\mathcal{A}$, with the cone $\mathcal{S}_+ = \mathcal{A}^+ \cap \mathcal{S}$ and all other operations inherited from $\mathcal{A}$, is an admissible space.

(4) Any locally pre-C^*-algebra is a topologically ordered $*$-space. Any $*$-subspace $\mathcal{S}$ of a locally pre-C^*-algebra is a topologically ordered $*$-space, with $\mathcal{S}^+ = \mathcal{A}^+ \cap \mathcal{S}$ and all other operations inherited from $\mathcal{A}$.

(5) Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space and let $\mathcal{C}_1$ be the trace-class ideal, that is, the collection of all linear bounded operators A on $\mathcal{H}$ such that $\operatorname{tr}(|A|) < \infty$. $\mathcal{C}_1$ is a $*$-ideal of $\mathcal{B}(\mathcal{H})$ and complete under the norm $\|A\|_1 = \operatorname{tr}(|A|)$. Positive elements in $\mathcal{C}_1$ are defined in the sense of positivity in $\mathcal{B}(\mathcal{H})$. In addition, the norm $\|\cdot\|_1$ is increasing, since $0 \leq A \leq B$ implies $\operatorname{tr}(A) \leq \operatorname{tr}(B)$, hence $\mathcal{C}_1$ is a normed admissible space.

(6) Let V be a complex Banach space and let V' be its conjugate dual space. On the vector space $\mathcal{B}(V, V')$ of all bounded linear operators $T \colon V \to V'$, a natural notion of positive operator can be defined: T is *positive* if $(Tv)(v) \geq 0$ for all $v \in V$. Let $\mathcal{B}(V, V')^+$ be the collection of all positive operators and note that it is a strict cone that is closed with respect to the weak operator topology. The involution $*$ in $\mathcal{B}(V, V')$ is defined in the following way: for any $T \in \mathcal{B}(V, V')$, $T^* = T'|V$, that is, the restriction to V of the dual operator $T' \colon V'' \to V'$. With respect to the weak operator topology, the cone $\mathcal{B}(V, V')^+$, and the involution $*$ just defined, $\mathcal{B}(V, V')$ becomes an admissible space. See A. Weron [44], as well as D. Gaşpar and P. Gaşpar [38].

(7) Let X be a nonempty set and denote by $\mathcal{K}(X)$ the collection of all complex valued kernels on X, that is, $\mathcal{K}(X) = \{k \mid k \colon X \times X \to \mathbb{C}\}$, considered as a complex vector space with the operations of addition and multiplication of scalars defined elementwise. An involution $*$ can be defined on $\mathcal{K}(X)$ as follows: $k^*(x, y) = \overline{k(y, x)}$, for all $x, y \in X$ and all $k \in \mathcal{K}(X)$. The cone $\mathcal{K}(X)^+$ consists of all *positive semidefinite* kernels, that is, those kernels $k \in \mathcal{K}(X)$ with the property that, for any $n \in \mathbb{N}$ and any $x_1, \ldots, x_n \in X$, the complex matrix $[k(x_i, x_j)]_{i,j=1}^n$ is positive semidefinite. Then $\mathcal{K}(X)$ is an ordered $*$-space.

Further, consider the set $\mathcal{P}_0(X)$ of all finite subsets of X. For each $A \in \mathcal{P}_0(X)$, let $A = \{x_1, \ldots, x_n\}$ and define the seminorm $p_A \colon \mathcal{K}(X) \to \mathbb{R}$ by

$$p_A(k) = \|[k(x_i, x_j)]_{i,j=1}^n\|, \quad k \in \mathcal{K}(X),$$

the norm being the operator norm of the $n \times n$ matrix $[k(x_i,x_j)]_{i,j=1}^n$. Since a reordering of the elements $x_1,\ldots,x_n$ produces a unitary equivalent matrix, the definition of p_A does not depend on which order of the elements of the set A is considered. It is easy to see that each seminorm p_A is increasing and that, with the locally convex topology defined by $\{p_A\}_{A\in\mathcal{P}_0(X)}$, $\mathcal{K}(X)$ is an admissible space.

(8) Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-algebras. Recall that, in this case, the specified strict cone $\mathcal{A}^+$ linearly generates $\mathcal{A}$. On $\mathcal{L}(\mathcal{A},\mathcal{B})$, the vector space of all linear maps $\varphi\colon \mathcal{A}\to\mathcal{B}$, we define an involution: $\varphi^*(a)=\varphi(a^*)^*$, for all $a\in\mathcal{A}$. A linear map $\varphi\in\mathcal{L}(\mathcal{A},\mathcal{B})$ is called positive if $\varphi(\mathcal{A}^+)\subseteq\mathcal{B}^+$. It is easy to see that $\mathcal{L}(\mathcal{A},\mathcal{B})^+$, the collection of all positive maps from $\mathcal{L}(\mathcal{A},\mathcal{B})$, is a cone, and that it is strict because $\mathcal{A}^+$ linearly generates $\mathcal{A}$. In addition, any $\varphi\in\mathcal{L}(\mathcal{A},\mathcal{B})^+$ is selfadjoint, again due to the fact that $\mathcal{A}^+$ linearly generates $\mathcal{A}$. Consequently, $\mathcal{L}(\mathcal{A},\mathcal{B})$ has a natural structure of ordered $*$-space.

On $\mathcal{L}(\mathcal{A},\mathcal{B})$ we consider the collection of seminorms $\{p_a\}_{a\in\mathcal{A}^+}$ defined by $p_a(\varphi)=\|\varphi(a)\|$, for all $\varphi\in\mathcal{L}(\mathcal{A},\mathcal{B})$. All these seminorms are increasing and the topology generated by $\{p_a\}_{a\in\mathcal{A}^+}$ is Hausdorff and complete. Consequently, $\mathcal{L}(\mathcal{A},\mathcal{B})$ is an admissible space.

With a slightly more involved topology, it can be shown that the same conclusion holds for the case when $\mathcal{A}$ and $\mathcal{B}$ are locally C^*-algebras.

(9) Let $\{Z_\alpha\}_{\alpha\in A}$ be a family of admissible spaces such that, for each $\alpha\in A$, Z_α^+ is the specified strict cone of positive elements in Z_α, and the topology of Z_α is generated by the family of increasing seminorms $\{p_{\alpha,j}\}_{j\in\mathcal{J}_\alpha}$. On the product space $Z=\prod_{\alpha\in A}Z_\alpha$ let $Z^+=\prod_{\alpha\in A}Z_\alpha^+$ and observe that Z^+ is a strict cone. Letting the involution $*$ on Z be defined elementwise, it follows that Z^+ consists on selfadjoint elements only. In this way, Z is an ordered $*$-space.

For each $\beta\in A$ and each $j\in\mathcal{J}_\beta$, let

$$q_j^{(\beta)}((z_\alpha)_{\alpha\in A})=p_j^{(\beta)}(z_\beta),\quad (z_\alpha)_{\alpha\in A}\in Z. \tag{2.6}$$

It is easy to show that $q_j^{(\beta)}$ is an increasing seminorm on Z and that, with the topology generated by the family of increasing seminorms $\{q_j^{(\beta)}\}_{\substack{\beta\in A\\ j\in\mathcal{J}_\beta}}$, Z becomes an admissible space.

2.3 Vector Hilbert Spaces and Their Linear Operators.

If Z is a topologically ordered $*$-space, any VE-space $\mathcal{E}$ over Z can be made in a natural way into a Hausdorff locally convex space by considering the topology $\tau_{\mathcal{E}}$, the weakest topology on $\mathcal{E}$ that makes the quadratic map $Q : E \ni h \mapsto [h, h] \in Z$ continuous. More precisely, letting $\mathcal{C}_0$ be a collection of open, absolutely convex and solid neighbourhoods of the origin in Z, that generates the topology of Z as in axiom (a5), the collection of sets

$$D_C = \{x \in \mathcal{E} \mid [x, x] \in C\}, \quad C \in \mathcal{C}_0, \tag{2.7}$$

is a topological base of open and absolutely convex neighbourhoods of the origin of $\mathcal{E}$ that linearly generates $\tau_{\mathcal{E}}$, cf. [5]. We are interested in explicitly defining the topology $\tau_{\mathcal{E}}$ in terms of seminorms.

Lemma 2.3.1. *Let Z be a topologically ordered $*$-space and $\mathcal{E}$ a VE-space over Z.*

(1) $(\mathcal{E}; \tau_{\mathcal{E}})$ is a Hausdorf locally convex space.

(2) For every continuous increasing seminorm p on Z

$$\tilde{p}(h) = p([h, h])^{1/2}, \quad h \in \mathcal{E}, \tag{2.8}$$

is a continuous seminorm on $(\mathcal{E}; \tau_{\mathcal{E}})$.

(3) Let $\{p_j\}_{j\in\mathcal{J}}$ be a family of increasing seminorms defining the topology of Z as in axiom (a4′). Then, with the definition (2.8), the family of seminorms $\{\tilde{p}_j\}_{j\in\mathcal{J}}$ generates $\tau_{\mathcal{E}}$.

(4) The gramian $[\cdot, \cdot] : \mathcal{E} \times \mathcal{E} \to Z$ is jointly continuous.

Statements (1) and (4) are proven in Theorem 1 in [5]. Statement (2) is claimed in Proposition 1.1.1 in [40] but, unfortunately, the proof provided there is irremediably flawed, so we provide full details.

Proof of Lemma 2.3.1. We first prove that, if p is a continuous and increasing seminorm on Z, $\tilde{p}$ is a quasi seminorm on $\mathcal{E}$. Indeed, for any $\lambda \in \mathbb{C}$ and any $h \in \mathcal{E}$

$$\tilde{p}(\lambda h) = p([\lambda h, \lambda h])^{1/2} = |\lambda| p([h,h])^{1/2} = |\lambda| \tilde{p}(h),$$

hence $\tilde{p}$ is positively homogeneous.

For arbitrary $h, k \in \mathcal{E}$ we have

$$[h \pm k, h \pm k] = [h,h] + [k,k] \pm [h,k] \pm [k,h] \geq 0,$$

in particular,

$$[h,k] + [k,h] \leq [h,h] + [k,k]. \tag{2.9}$$

and

$$0 \leq [h \pm k, h \pm k] \leq [h-k, h-k] + [h+k, h+k] = 2([h,h] + [k,k]). \tag{2.10}$$

Since p is increasing, it follows that

$$\begin{aligned} \tilde{p}(h+k) &= \big(p([h+k, h+k])\big)^{1/2} \leq \sqrt{2}(p([h,h]) + p([k,k])^{1/2} \\ &\leq \sqrt{2}\big(p([h,h])^{1/2} + p([k,k])^{1/2}\big) = \sqrt{2}\big(\tilde{p}(h) + \tilde{p}(k)\big). \end{aligned}$$

This concludes the proof that $\tilde{p}$ is a quasi seminorm.

Also, since $\tilde{p}$ is the composition of the square root function $\sqrt{\ }$, a homeomorphism of $\mathbb{R}_+$ onto itself, with p and the quadratic map $\mathcal{E} \ni x \mapsto [x,x] \in Z$, clearly $\tilde{p}$ is continuous with respect to the topology $\tau_{\mathcal{E}}$. This observation shows that, if $\{p_j\}_{j \in \mathcal{J}}$ is a family of increasing seminorms generating the topology of Z, then $\{\tilde{p}_j\}_{j \in \mathcal{J}}$ is a family of quasi seminorms generating $\tau_{\mathcal{E}}$. In particular, $(\mathcal{E}; \tau_{\mathcal{E}})$ is a topological vector space.

We prove now that $\tilde{p}$ satisfies the triangle inequality, hence it is a seminorm. To see this, consider the unit quasi ball

$$U_{\tilde{p}} = \{h \in \mathcal{E} \mid \tilde{p}(h) < 1\}.$$

Since $\tilde{p}$ is continuous, $U_{\tilde{p}}$ is open, hence absorbing for each of its points. Since $\tilde{p}$ is positively homogeneous, $U_{\tilde{p}}$ is balanced. We prove that $U_{\tilde{p}}$ is convex as well. Let $h, k \in U_{\tilde{p}}$ and $0 \leq t \leq 1$ arbitrary. Then,

$$0 \leq [th + (1-t)k, th + (1-t)k] = t^2[h,h] + (1-t)^2[k,k] + t(1-t)\big([h,k] + [k,h]\big)$$

and then using (2.9),

$$\begin{aligned} &\leq t^2[h,h] + (1-t)^2[k,k] + t(1-t)\big([h,h] + [k,k]\big) \\ &= t[h,h] + (1-t)[k,k], \end{aligned}$$

hence, since p is increasing, it follows

$$\tilde{p}(th+(1-t)k) = p\big([th+(1-t)k, th+(1-t)k]\big)^{1/2} \leq \big(tp([h,h])+(1-t)p([k,k])\big)^{1/2} < 1,$$

hence $th + (1-t)k \in U_{\tilde{p}}$.

It is a routine exercise to show that $\tilde{p}$ is the gauge of $U_{\tilde{p}}$

$$\tilde{p}(h) = \inf\{t > 0 \mid h \in tU_{\tilde{p}}\},$$

hence, by Proposition IV.1.14 in [45], it follows that $\tilde{p}$ is a seminorm.

Statement (4) is a consequence of the polarisation formula (2.1). □

From now on, any time we have a VE-space $\mathcal{E}$ over a topologically ordered $*$-space Z, we consider on $\mathcal{E}$ the topology $\tau_{\mathcal{E}}$ defined as in Lemma 2.3.1. With respect to this topology, we call $\mathcal{E}$ a *topological VE-space* over Z. Denote

$$S(\mathcal{E}) := S_{\mathcal{C}}(\mathcal{E}) = \{\tilde{p}_C \mid C \in \mathcal{C}\}, \tag{2.11}$$

where $\mathcal{C}$ is the collection of all open, absolutely convex and solid neighbourhoods of the origin of Z as in (2.7). Note that $S(\mathcal{E})$ is directed, more precisely, given $\tilde{p}_C, \tilde{p}_D \in S(\mathcal{E})$ consider $S(Z) \ni q := p_C + p_D$ and define $\tilde{q}(h) := q([h,h]_{\mathcal{E}})^{1/2}$. Also note that $S(\mathcal{E})$ is closed under positive scalar multiplication.

If Z is an admissible space and $\mathcal{E}$ is a topological VE-space whose locally convex topology is complete, then $\mathcal{E}$ is called a *VH-space* (Vector Hilbert space). Any topological VE-space $\mathcal{E}$ on an admissible space Z can be embedded as a dense subspace of a VH-space $\mathcal{H}$ over Z, uniquely determined up to an isomorphism, cf. Theorem 2 in [5].

We now prove a surrogate of Schwarz Inequality.

Lemma 2.3.2. *Let $\mathcal{E}$ be a topological VE-space over the topologically ordered $*$-space Z and $p \in S(Z)$. Then*

$$p([e,f]) \leq 4p([e,e])^{1/2}p([f,f])^{1/2} = 4\tilde{p}(e)\tilde{p}(f), \quad e,f \in \mathcal{E}. \tag{2.12}$$

Proof. For arbitrary $h,k \in \mathcal{E}$ we have

$$[h \pm k, h \pm k] = [h,h] + [k,k] \pm [h,k] \pm [k,h] \geq 0,$$

in particular,

$$[h,k] + [k,h] \leq [h,h] + [k,k],$$

and

$$0 \leq [h+k,h+k] \leq [h-k,h-k] + [h+k,h+k] = 2([h,h]+[k,k]). \tag{2.13}$$

Taking into account that $p \in S(Z)$ is increasing, from (2.13) it follows that

$$p([h+k,h+k]) \leq 2\big(p([h,h]) + p([k,k])\big). \tag{2.14}$$

Let now $e,f \in \mathcal{E}$ be arbitrary. By the polarisation formula (2.1) and (2.14), we have

$$\begin{aligned} p([e,f]) &= p\Big(\frac{1}{4}\sum_{k=0}^{3} \mathrm{i}^k[e+\mathrm{i}^k f, e+\mathrm{i}^k f]\Big) \leq \frac{1}{4}\sum_{k=0}^{3} p([e+\mathrm{i}^k f, e+\mathrm{i}^k f]) \\ &\leq \frac{2}{4}\sum_{k=0}^{3}\big(p([e,e]) + p([\mathrm{i}^k f, \mathrm{i}^k f])\big) = 2\big(p([e,e]) + p([f,f])\big). \end{aligned}$$

Letting $\lambda > 0$ be arbitrary and changing e with $\sqrt{\lambda}e$ and f with $f/\sqrt{\lambda}$ in the previous inequality, we get

$$p([e,f]) \leq 2\big(\lambda p([e,e]) + \lambda^{-1}p([f,f])\big),$$

hence, since the left hand side does not depend on λ, it follows

$$p([e,f]) \leq \inf_{\lambda>0} 2\big(\lambda p([e,e]) + \lambda^{-1}p([f,f])\big) = 4p([e,e])^{1/2}p([f,f])^{1/2}. \qquad \square$$

Examples 2.3.3. (1) Any Hilbert module $\mathcal{H}$ over a C^*-algebra $\mathcal{A}$, e.g. see [31], [32], can be viewed as a VH-space $\mathcal{H}$ over the admissible space $\mathcal{A}$, see Example 2.2.2.(1). In particular, any closed subspace $\mathcal{S}$ of $\mathcal{H}$ is a VH-space over the admissible space $\mathcal{A}$.

(2) Any Hilbert module $\mathcal{H}$ over a locally C^*-algebra $\mathcal{A}$, e.g. see [43], [35], can be viewed as a VH-space $\mathcal{H}$ over the admissible space $\mathcal{A}$, see Example 2.2.2.(2). In particular, any closed subspace $\mathcal{S}$ of $\mathcal{H}$ is a VH-space over the admissible space $\mathcal{A}$.

(3) With notation as in Example 2.2.2.(5), consider $\mathcal{C}_2$ the ideal of Hilbert-Schmidt operators on $\mathcal{H}$. Then $[A, B] = A^*B$, for all $A, B \in \mathcal{C}_2$, is a gramian with values in the admissible space $\mathcal{C}_1$ with respect to which $\mathcal{C}_2$ becomes a VH-space. Observe that, since $\mathcal{C}_1$ is a normed admissible space, by Lemma 2.3.1 it follows that $\mathcal{C}_2$ is a normed VH-space, with norm $\|A\|_2 = \operatorname{tr}(|A|^2)^{1/2}$, for all $A \in \mathcal{C}_2$. More abstract versions of this example have been considered by Saworotnow in [33].

(4) Let $\{\mathcal{E}_\alpha\}_{\alpha\in A}$ be a family of VH-spaces such that, for each $\alpha \in A$, $\mathcal{E}_\alpha$ is a VH-space over the admissible space Z_α. As in Example 2.2.2, consider the admissible space $Z = \prod_{\alpha\in A} Z_\alpha$ and the vector space $\mathcal{E} = \prod_{\alpha\in A} \mathcal{E}_\alpha$ on which we define

$$[(e_\alpha)_{\alpha\in A}, (f_\alpha)_{\alpha\in A}] = ([e_\alpha, f_\alpha])_{\alpha\in A} \in Z, \quad (e_\alpha)_{\alpha\in A}, (f_\alpha)_{\alpha\in A} \in \mathcal{E}.$$

Then $\mathcal{E}$ is a VE-space over Z. On Z consider the topology generated by the family of increasing seminorms $\{q_j^{(\beta)}\}_{\substack{\beta\in A\\ j\in\mathcal{J}_\beta}}$ defined at (2.6), with respect to which Z becomes an admissible space. For each $\beta \in A$ and each $j \in \mathcal{J}_\beta$, in view of Lemma 2.3.1, consider the seminorm

$$\tilde{q}_j^{(\beta)}((e_\alpha)_{\alpha\in A}) = p_j^{(\beta)}([e_\alpha, e_\alpha])^{1/2}, \quad (e_\alpha)_{\alpha\in A} \in \mathcal{E}.$$

The family of seminorms $\{\tilde{q}_j^{(\beta)}\}_{\substack{\beta\in A\\ j\in\mathcal{J}_\beta}}$ generates on $\mathcal{E}$ the topology with respect to which it is a VH-space over Z.

(5) Let Z be an admissible space and $\mathcal{E}_1, \ldots, \mathcal{E}_n$ VH-spaces over Z. On $\mathcal{E} =$

$\prod_{j=1}^{n} \mathcal{E}_j$ define

$$[(e_j)_{j=1}^n, (f_j)_{j=1}^n]_{\mathcal{E}} = \sum_{j=1}^{n} [e_j, f_j]_{\mathcal{E}_j}, \quad (e_j)_{j=1}^n, (f_j)_{j=1}^n \in \mathcal{E}, \tag{2.15}$$

and observe that $(\mathcal{E}; [\cdot, \cdot]_{\mathcal{E}})$ is a VE-space over Z. In addition, for any $p \in S(Z)$ letting $\widetilde{p}\colon \mathcal{E} \to \mathbb{R}_+$ be defined as in (2.8), $\widetilde{p}(e) = p([e,e]_{\mathcal{E}})^{1/2}$, for all $e \in \mathcal{E}$, it is easy to see that $\mathcal{E}$ is a VH-space over Z. It is clear that we can denote this VH-space by $\bigoplus_{j=1}^{n} \mathcal{E}_j$ and call it the *direct sum VH-space* of the VH-spaces $\mathcal{E}_1, \ldots, \mathcal{E}_n$.

(6) Let $\mathcal{H}$ be a Hilbert space and $\mathcal{E}$ a VH-space over the admissible space Z. On the algebraic tensor product $\mathcal{H} \otimes \mathcal{E}$ define a gramian by

$$[h \otimes e, l \otimes f]_{\mathcal{H} \otimes \mathcal{E}} = \langle h, l \rangle_{\mathcal{H}} [e, f]_{\mathcal{E}} \in Z, \quad h, l \in \mathcal{H},\ e, f \in \mathcal{E},$$

and then extend it to $\mathcal{H} \otimes \mathcal{E}$ by linearity. It can be proven that, in this way, $\mathcal{H} \otimes \mathcal{E}$ is a VE-space over Z. Since Z is an admissible space, $\mathcal{H} \otimes \mathcal{E}$ can be topologised as in Lemma 2.3.1 and then completed to a VH-space $\mathcal{H} \widetilde{\otimes} \mathcal{E}$ over Z.

If $\mathcal{H} = \mathbb{C}^n$ for some $n \in \mathbb{N}$ then, with notation as in item (5), it is clear that $\mathbb{C}^n \otimes \mathcal{E}$ is isomorphic with $\bigoplus_{j=1}^{n} \mathcal{E}_j$, with $\mathcal{E}_j = \mathcal{E}$ for all $j = 1, \ldots, n$.

Remark 2.3.4. If $\mathcal{E}$ and $\mathcal{F}$ are two VH-spaces over the same admissible space Z, by $\mathcal{L}_c(\mathcal{E}, \mathcal{F})$ we denote the space of all continuous operators from $\mathcal{E}$ to $\mathcal{F}$. Let $\mathcal{C}_0$ be a system of open and absolutely convex neighbourhoods of the origin defining the topology of Z. Since $S(\mathcal{E})$ is directed and it is closed under positive scalar multiplication, the continuity of a linear operator $T \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ is equivalent with: for any $p \in S_{\mathcal{C}_0}(\mathcal{F})$, there exists $q \in S(\mathcal{E})$ and a constant $c \geq 0$ such that $p(Th) \leq c\, q(h)$ for all $h \in \mathcal{E}$. We will use this fact frequently in this article.

For $\mathcal{E}$ and $\mathcal{F}$ two VH-spaces over the same admissible space Z, we denote by $\mathcal{L}_c^*(\mathcal{E}, \mathcal{F})$ the subspace of $\mathcal{L}^*(\mathcal{E}, \mathcal{F})$ consisting of all continuous and continuously adjointable operators. Note that $\mathcal{L}_c^*(\mathcal{E}) = \mathcal{L}_c^*(\mathcal{E}, \mathcal{E})$ is an ordered $*$-subalgebra of $\mathcal{L}^*(\mathcal{E})$.

A subspace $\mathcal{M}$ of a VH-space $\mathcal{H}$ is *orthocomplemented*, or *accessible* [5], if every element $h \in \mathcal{H}$ can be written as $h = g + k$ where g is in $\mathcal{M}$ and k is such that $[l, k] = 0$ for all $l \in \mathcal{M}$, that is, k is in the *orthogonal companion* $\mathcal{M}^{\perp}$ of $\mathcal{M}$. Observe that if such a decomposition exists it is unique and hence the *orthogonal projection* $P_{\mathcal{M}}$ onto $\mathcal{M}$ can be defined by $P_{\mathcal{M}} h = g$. Any orthogonal projection P is selfadjoint and idempotent, in particular we have $[Ph, k] = [Ph, Pk]$ for all $j, k \in \mathcal{H}$, hence P is positive and contractive, in the sense $[Ph, Ph] \leq [h, h]$ for all $h \in \mathcal{H}$, hence P is continuous. Conversely, any selfadjoint idempotent operator is an orthogonal projection onto its range subspace. Any orthocomplemented subspace is closed.

Chapter 3

The Main Theorems: Dilations of Doubly Invariant Kernels Valued in Ordered $*$-Spaces

In this chapter we are going to state and prove the main theorems of this thesis, see Theorems 3.2.7, 3.2.9, 3.2.11, 3.2.13 as well as 3.2.15. Theorems 3.2.7 and 3.2.9 appear in our article [25].

3.1 Hermitian Kernels.

Let X be a nonempty set and Z an ordered $*$-space. A map $\mathbf{k}\colon X \times X \to Z$ is called a *Z-valued kernel on X*. If no confusion may arise we also say simply that $\mathbf{k}$ is a kernel. The *adjoint* kernel $\mathbf{k}^*\colon X \times X \to Z$ is defined by $\mathbf{k}^*(x,y) = \mathbf{k}(y,x)^*$, for $x, y \in X$. The kernel $\mathbf{k}$ is called *Hermitian* if $\mathbf{k}^* = \mathbf{k}$.

Consider $\mathbb{C}^X$ the complex vector space of all functions $f\colon X \to \mathbb{C}$, as well as its subspace $\mathbb{C}_0^X$ consisting of those functions $f \in \mathbb{C}^X$ with finite support. Given a Z-valued kernel $\mathbf{k}$ on X, a pairing $[\cdot,\cdot]_{\mathbf{k}}\colon \mathbb{C}_0^X \times \mathbb{C}_0^X \to Z$ can be defined

$$[f,g]_{\mathbf{k}} = \sum_{x,y\in X} \overline{f(x)} g(y)\mathbf{k}(x,y), \quad f,g \in \mathbb{C}_0^X. \tag{3.1}$$

The pairing $[\cdot,\cdot]_{\mathbf{k}}$ is linear in the second variable and conjugate linear in the first variable. If, in addition, $\mathbf{k} = \mathbf{k}^*$, then the pairing $[\cdot,\cdot]_{\mathbf{k}}$ is *Hermitian*, that is,

$$[f,g]_{\mathbf{k}} = [g,f]_{\mathbf{k}}^*, \quad f,g \in \mathbb{C}_0^X. \tag{3.2}$$

Conversely, if the pairing $[\cdot,\cdot]_{\mathbf{k}}$ is Hermitian then $\mathbf{k} = \mathbf{k}^*$.

A *convolution* operator $K\colon \mathbb{C}_0^X \to Z^X$, where Z^X is the complex vector space of all functions $g\colon X \to Z$, can be associated to the Z-kernel $\mathbf{k}$ by

$$(Kg)(x) = \sum_{y\in X} g(y)\mathbf{k}(x,y), \quad f \in \mathbb{C}_0^X. \tag{3.3}$$

Clearly, K is a linear operator. A natural relation exists between the paring $[\cdot,\cdot]_{\mathbf{k}}$ and the convolution operator K, more precisely,

$$[f,g]_{\mathbf{k}} = \sum_{x\in X} \overline{f(x)}(Kg)(x), \quad f,g \in \mathbb{C}_0^X. \tag{3.4}$$

Therefore, it is easy to see from here that the kernel $\mathbf{k}$ is Hermitian if and only if the pairing $[\cdot,\cdot]_{\mathbf{k}}$ is Hermitian.

Given a natural number n, a Z-valued kernel $\mathbf{k}$ is called *weakly n-positive* if for all $x_1,\ldots,x_n \in X$ and all $t_1,\ldots,t_n \in \mathbb{C}$ we have

$$\sum_{j,k=1}^{n} \overline{t_k} t_j \mathbf{k}(x_k,x_j) \geq 0. \tag{3.5}$$

The kernel $\mathbf{k}$ is called *weakly positive semidefinite* if it is n-positive for all $n \in \mathbb{N}$.

Lemma 3.1.1. *Let the Z-kernel $\mathbf{k}$ on X be weakly 2-positive. Then:*

(1) $\mathbf{k}$ is Hermitian.

(2) If, for some $x \in X$, $\mathbf{k}(x,x) = 0$, then $\mathbf{k}(x,y) = 0$ for all $y \in X$.

(3) There exists a unique decomposition $X = X_0 \cup X_1$, $X_0 \cap X_1 = \emptyset$, such that $\mathbf{k}(x,y) = 0$ for all $x,y \in X_0$ and $\mathbf{k}(x,x) \neq 0$ for all $x \in X_1$.

Proof. (1) Clearly, weak 2-positivity implies weak 1-positivity, hence $\mathbf{k}(x,x) \geq 0$ for all $x \in X$. Let $x, y \in X$ be arbitrary. Since $\mathbf{k}$ is weakly 2-positive, for any $s, t \in \mathbb{C}$ we have

$$|s|^2\mathbf{k}(x,x) + |t|^2\mathbf{k}(y,y) + \bar{s}t\mathbf{k}(x,y) + s\bar{t}\mathbf{k}(y,x) \geq 0. \tag{3.6}$$

Since the sum of the first two terms in (3.6) is in Z_+ and taking into account that Z_+ consists of selfadjoint elements only, it follows that the sum of the last two terms in (3.6) is selfadjoint, that is,

$$\bar{s}t\mathbf{k}(x,y) + s\bar{t}\mathbf{k}(y,x) = \bar{t}s\mathbf{k}(x,y)^* + \bar{s}t\mathbf{k}(y,x)^*.$$

Letting $s = t = 1$ and then $s = 1$ and $t = \mathrm{i}$, it follows that $\mathbf{k}(y,x) = \mathbf{k}(x,y)^*$.

(2) Assume that $\mathbf{k}(x,x) = 0$ and let $y \in X$ be arbitrary. From (3.6) it follows that for all $s, t \in \mathbb{C}$ we have

$$\bar{s}t\mathbf{k}(x,y) + s\bar{t}\mathbf{k}(y,x) \geq -|t|^2\mathbf{k}(y,y). \tag{3.7}$$

We claim that for all $s, t \in \mathbb{C}$ we have

$$\bar{s}t\mathbf{k}(x,y) + s\bar{t}\mathbf{k}(y,x) = 0. \tag{3.8}$$

To prove this, note that for $t = 0$ the equality (3.8) it trivially true. If $t \in \mathbb{C}\setminus\{0\}$, note that we can distinguish two cases: first, if $\mathbf{k}(y,y) = 0$, then from (3.7) it follows $\bar{s}t\mathbf{k}(x,y)+s\bar{t}\mathbf{k}(y,x) \geq 0$ and then, changing t to $-t$ the opposite inequality holds, hence (3.8). The second case is $\mathbf{k}(y,y) \neq 0$ when we observe that the right hand side in (3.7) does not depend on s hence, replacing s by ns, $n \in \mathbb{Z}$, a routine reasoning shows that (3.8) must hold as well.

Finally, in (3.8) we first let $s = 1 = t$ and then $s = 1$ and $t = \mathrm{i}$ and solve for $\mathbf{k}(x,y)$ which should be 0.

(3) Denote $X_0 = \{x \in X \mid \mathbf{k}(x,x) = 0\}$ and let $X_1 = X \setminus X_0$. Then use (2) in order to obtain $\mathbf{k}(x,y) = 0$ for all $x, y \in X_0$. □

3.1.1 Weak Linearisations

Given an ordered $*$-space Z and a Z-valued kernel $\mathbf{k}$ on a nonempty set X, a *weak VE-space linearisation*, or *weak Kolmogorov decomposition* of $\mathbf{k}$ is, by definition, a pair $(\mathcal{E}; V)$, subject to the following conditions:

(vel1) $\mathcal{E}$ is a VE-space over the ordered $*$-space Z.

(vel2) $V\colon X \to \mathcal{E}$ satisfies $\mathbf{k}(x,y) = [V(x), V(y)]_{\mathcal{E}}$ for all $x, y \in X$.

If, in addition, the following condition holds

(vel3) $\operatorname{Lin} V(X) = \mathcal{E}$,

then the weak VE-space linearisation $(\mathcal{E}; V)$ is called *minimal*.

Two weak VE-space linearisations $(V; \mathcal{E})$ and $(V'; \mathcal{E}')$ of the same kernel $\mathbf{k}$ are called *unitarily equivalent* if there exists a unitary operator $U\colon \mathcal{E} \to \mathcal{E}'$ such that $UV(x) = V'(x)$ for all $x \in X$.

Remarks 3.1.2. (1) Note that any two minimal weak VE-space linearisations $(\mathcal{E}; V)$ and $(\mathcal{E}'; V')$ of the same Z-kernel $\mathbf{k}$ are unitarily equivalent. The proof follows in the usual way: if $(\mathcal{E}'; V')$ is another minimal weak VE-space linearisation of $\mathbf{k}$, for arbitrary $x_1, \ldots, x_m, y_1, \ldots, y_n \in X$ and arbitrary $t_1, \ldots, t_m, s_1, \ldots, s_n \in \mathbb{C}$, we have

$$[\sum_{j=1}^m t_j V(x_j), \sum_{k=1}^m s_k V(y_k)]_{\mathcal{E}} = \sum_{j=1}^m \sum_{k=1}^n s_k \overline{t_j} [V(x_j), V(y_k)]_{\mathcal{E}} = \sum_{k=1}^n \sum_{j=1}^m s_k \overline{t_j} \mathbf{k}(x_j, y_k)$$
$$= \sum_{j=1}^m \sum_{k=1}^n s_k \overline{t_j} [V'(x_j), V'(y_k)]_{\mathcal{E}'} = [\sum_{j=1}^m t_j V'(x_j), \sum_{k=1}^n s_k V'(y_k)]_{\mathcal{E}'},$$

hence $U\colon \operatorname{Lin} V(X) \to \operatorname{Lin} V'(X)$ defined by

$$\sum_{j=1}^m t_j V(x_j) \mapsto \sum_{j=1}^m t_j V'(x_j), \quad x_1, \ldots, x_m \in X,\ t_1, \ldots, t_m \in \mathbb{C},\ m \in \mathbb{N}, \tag{3.9}$$

is a correctly defined linear operator, isometric, everywhere defined, and onto. Thus, U is a VE-space isomorphism $U\colon \mathcal{E} \to \mathcal{E}'$ and $UV(x) = V'(x)$ for all $x \in X$, by construction.

(2) From any weak VE-space linearisation $(\mathcal{E}; V)$ of $\mathbf{k}$ one can make a minimal one in a canonical way, more precisely, letting $\mathcal{E}_0 = \operatorname{Lin} V(X)$ and $V_0\colon X \to \mathcal{E}_0$ defined by $V_0(x) = V(x)$, $x \in X$, it follows that $(\mathcal{E}_0; V_0)$ is a minimal weak VE-space linearisation of $\mathbf{k}$.

Let us assume now that Z is an admissible space and $\mathbf{k}$ is a Z-kernel on a set X. A *weak VH-space linearisation* of $\mathbf{k}$ is a linearisation $(\mathcal{H}; V)$ of $\mathbf{k}$ such that $\mathcal{H}$ is a VH-space. The weak VH-space linearisation $(\mathcal{H}; V)$ is called *topologically minimal* if

(vhl3) $\operatorname{Lin} V(X)$ is dense in $\mathcal{H}$.

Two weak VH-space linearisations $(\mathcal{H}; V)$ and $(\mathcal{H}'; V')$ of the same Z-kernel $\mathbf{k}$ are called *unitary equivalent* if there exists a unitary operator $U \in \mathcal{B}^*(\mathcal{H}, \mathcal{H}')$ such that $UV(x) = V'(x)$ for all $x \in X$.

Remarks 3.1.3. (a) Any two topologically minimal weak VH-space linearisations of the same Z-kernel are unitarily equivalent. Indeed, letting $(\mathcal{H}; V)$ and $(\mathcal{H}'; V')$ be two minimal weak VH-space linearisations of the Z-kernel $\mathbf{k}$, we proceed as in Remark 3.1.2.(a) and define $U\colon \operatorname{Lin} V(X) \to \operatorname{Lin} V'(X)$ as in (3.9). Since U is isometric, it is bounded in the sense of (2.3), hence continuous, and then U can be uniquely extended to an isometric operator $U\colon \mathcal{H} \to \mathcal{H}'$. Since $\operatorname{Lin} V'(X)$ is dense in $\mathcal{H}'$ and U has closed range, it follows that U is surjective, hence $U \in \mathcal{B}^*(\mathcal{H}, \mathcal{H}')$ is unitary and, by its definition, see (3.9), we have $UV(x) = V'(x)$ for all $x \in X$.

(b) From any weak VH-space linearisation $(\mathcal{H}; V)$ of $\mathbf{k}$ one can make, in a canonical way, a topologically minimal weak VH-space linearisation $(\mathcal{H}_0; V_0)$ by letting $\mathcal{H}_0 = \overline{\operatorname{Lin} V(X)}$ and $V_0(x) = V(x)$ for all $x \in X$.

Theorem 3.1.4. (a) *Given an ordered $*$-space Z and a Z-valued kernel $\mathbf{k}$ on a nonempty set X, the following assertions are equivalent:*

(1) $\mathbf{k}$ *is positive semidefinite.*

(2) $\mathbf{k}$ *admits a weak VE-space linearisation* $(\mathcal{E}; V)$*.*

Moreover, if exists, a weak VE-space linearisation $(\mathcal{E}; V)$ *can always be chosen such that* $\mathcal{E} \subseteq Z^X$*, that is, consisting of functions* $f\colon X \to Z$ *only, and minimal.*

(b) *If, in addition,* Z *is an admissible space and* $\mathbf{k}\colon X \times X \to Z$ *is a kernel, then any of the assertions* (1) *and* (2) *is equivalent with:*

(3) $\mathbf{k}$ *admits a weak VH-space linearisation* $(\mathcal{H}; V)$*.*

Moreover, if exists, a weak VH-space linearisation $(\mathcal{H}; V)$ *can always be chosen such that* $\mathcal{H} \subseteq Z^X$ *and topologically minimal.*

Proof. (1)⇒(2). Assuming that $\mathbf{k}$ is positive semidefinite, by Lemma 3.1.1.(1) it follows that $\mathbf{k}$ is Hermitian, that is, $\mathbf{k}(x,y)^* = \mathbf{k}(y,x)$ for all $x, y \in X$. With notation as in Subsection 3.1, we consider the convolution operator $K\colon \mathbb{C}_0^X \to Z^X$ and let Z_K^X be its range, more precisely,

$$\begin{aligned} Z_K^X &= \{f \in Z^X \mid f = Kg \text{ for some } g \in \mathbb{C}_0^X\} \\ &= \{f \in \mathcal{F} \mid f(x) = \sum_{y \in X} g(y)\mathbf{k}(x,y) \text{ for some } g \in \mathbb{C}_0^X \text{ and all } y \in X\}. \end{aligned} \tag{3.10}$$

A pairing $[\cdot,\cdot]_{\mathcal{E}}\colon Z_K^X \times Z_K^X \to Z$ can be defined by

$$[e,f]_{\mathcal{E}} = [g,h]_{\mathbf{k}} = \sum_{x,y \in X} \overline{g(x)} h(y) \mathbf{k}(x,y), \tag{3.11}$$

where $f = Kh$ and $e = Kg$ for some $g, h \in \mathbb{C}_0^X$. We observe that

$$\begin{aligned} [e,f]_{\mathcal{E}} = \sum_{x \in X} \overline{g(x)} f(x) &= \sum_{x,y \in X} \overline{g(x)} \mathbf{k}(x,y) h(y) \\ &= \sum_{x,y \in X} h(y) \overline{g(x)} \mathbf{k}(y,x)^* = \sum_{x \in X} h(y) e(y)^*, \end{aligned}$$

which shows that the definition in (3.11) is correct, that is, independent of g and h such that $e = Kg$ and $f = Kh$.

We claim that $[\cdot,\cdot]_{\mathcal{E}}$ is a Z-valued inner product, that is, it satisfies all the requirements (ve1)–(ve3). The only fact that needs a proof is $[f,f]_{\mathcal{E}} = 0$ implies $f = 0$. To see this, we use Lemma 2.1.1 and first get that $[f,f']_{\mathcal{E}} = 0$ for all $f' \in Z_K^X$. For each $x \in X$, let $\delta_x \in \mathbb{C}_0^X$ denote the δ-function with support $\{x\}$,

$$\delta_x(y) = \begin{cases} 1, & \text{if } y = x, \\ 0, & \text{if } y \neq x. \end{cases} \tag{3.12}$$

Letting $f' = K\delta_x$ we have

$$0 = [f,f']_{\mathcal{E}} = \sum_{y\in X} \delta_x f(y) = f(x),$$

hence, since $x \in X$ are arbitrary, it follows that $f = 0$.

Thus, $(Z_K^X; [\cdot,\cdot]_{\mathcal{E}})$ is a VE-space. For each $x \in X$ we define $V(x) \in Z_K^X \subseteq \mathcal{E}$ by

$$V(x) = K\delta_x. \tag{3.13}$$

Actually, there is an even more explicit way of expressing $V(x)$, namely,

$$(V(x))(y) = (K\delta_x)(y) = \sum_{z\in X} \delta_x(z)\mathbf{k}(y,z) = \mathbf{k}(y,x), \quad x \in X. \tag{3.14}$$

On the other hand, for any $x, y \in X$, by (3.13) and (3.14), we have

$$[V(x), V(y)]_{\mathcal{E}} = (V(y))(x) = \mathbf{k}(x,y),$$

hence $(\mathcal{E}; V)$ is a linearisation of $\mathbf{k}$. We prove that it is minimal as well. To see this, note that for any $g \in \mathbb{C}_0^X$, with notation as in (3.12), we have

$$g = \sum_{x\in \operatorname{supp}(g)} g(x)\delta_x,$$

hence, by (3.13), the linear span of $V(X)$ equals Z_K^X.

(2)⇒(1). This is proven exactly as in the classical case:

$$\sum_{j,k=1}^{n} t_j\overline{t_k}\mathbf{k}(x_k,x_j) = \sum_{j,k=1}^{n} t_j\overline{t_k}[V(x_k),V(x_j)]_{\mathcal{E}} = [\sum_{j=1}^{n} t_kV(x_k), \sum_{j=1}^{n} t_jV(x_j)]_{\mathcal{E}} \geq 0,$$

for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$, and $t_1, \ldots, t_n \in \mathcal{H}$.

(3)⇒(2). Clear.

(1)⇒(3). Assuming that Z is an admissible space, let $\mathbf{k}$ be positive semidefinite, let $(\mathcal{E}; V)$ be the weak VE-space linearisation of $\mathbf{k}$. Then, $\mathcal{E}$ is naturally equipped with a Hausdorff locally convex topology, see Subsection 2.3, and then completed to a VH- space $\mathcal{H}$. Thus, $(\mathcal{H}; V)$ is a weak VH-space linearisation of $\mathbf{k}$ and it is easy to see that it is topologically minimal. The fact that this completion can be made within Z^X will follow from Proposition 3.1.8. □

3.1.2 Reproducing Kernel Spaces

Let Z be an ordered $*$-space and let X be a nonempty set. As in Subsection 3.1, we consider the complex vector space Z^X of all functions $f\colon X \to Z$. A VE-space $\mathcal{R}$ over the ordered $*$-space Z is called a *weak Z-reproducing kernel VE-space on X* if there exists a Hermitian kernel $\mathbf{k}\colon X \times X \to Z$ such that the following axioms are satisfied:

(rk1) $\mathcal{R}$ is a subspace of Z^X, with all algebraic operations.

(rk2) For all $x \in X$, the Z-valued map $\mathbf{k}_x = \mathbf{k}(\cdot, x)\colon X \to Z$ belongs to $\mathcal{R}$.

(rk3) For all $f \in \mathcal{R}$ we have $f(x) = [\mathbf{k}_x, f]_{\mathcal{R}}$, for all $x \in X$.

The axiom (rk3) is called the *reproducing property* and note that, as a consequence, we have

$$\mathbf{k}(x, y) = \mathbf{k}_y(x) = [\mathbf{k}_x, \mathbf{k}_y]_{\mathcal{R}}, \quad x, y \in X. \tag{3.15}$$

A weak Z-reproducing kernel VE-space $\mathbf{k}$ on X is called *minimal* if

(rk4) $\operatorname{Lin}\{\mathbf{k}_x \mid x \in X\} = \mathcal{R}$.

If Z is an admissible space, a weak Z-reproducing kernel VE-space $\mathcal{R}$ that is a VH-space is called a *weak Z-reproducing kernel VH-space.* Such an $\mathcal{R}$ is called *topologically minimal* if

(rk4)′ $\operatorname{Lin}\{\mathbf{k}_x \mid x \in X\}$ is dense in $\mathcal{R}$.

Remark 3.1.5. Let $\mathcal{R}$ be a weak Z-reproducing kernel VH-space with respect to some admissible space Z. In general, the closed subspace $\overline{\operatorname{Lin}\{\mathbf{k}_x \mid x \in X\}} \subseteq \mathcal{R}$ may or may not be orthocomplemented in $\mathcal{R}$, see Subsection 2.3. This anomaly makes some differences when compared with the classical theory of reproducing kernel spaces, as is the case in closely related situations as in [22] and [24] as well.

Proposition 3.1.6. *A weak Z-reproducing kernel VH-space $\mathcal{R}$ with respect to some admissible space Z is topologically minimal if and only if the closed subspace $\overline{\operatorname{Lin}\{\mathbf{k}_x \mid x \in X\}}$ is orthocomplemented in $\mathcal{R}$.*

Proof. If $\mathcal{M} := \overline{\operatorname{Lin}\{\mathbf{k}_x \mid x \in X\}}$ is orthocomplemented then, as a consequence of (rk3), $\mathcal{R}$ is topologically minimal, in the sense of (rk4)′. Indeed, let $f \in \mathcal{R}$ be arbitrary. Since $\mathcal{M}$ is orthocomplemented, there exists $f_1 \in \mathcal{M}$ and $f_2 \in \mathcal{M}^\perp$ with $f = f_1 + f_2$. By (rk3) we obtain that $0 = [\mathbf{k}_x, f_2] = f_2(x)$ for all $x \in \mathcal{R}$, and that $f_2 = 0$. It follows that $f \in \mathcal{M}$ and $\mathcal{M} = \mathcal{R}$, i.e. $\operatorname{Lin}\{\mathbf{k}_x \mid x \in X\}$ is dense in $\mathcal{R}$. The converse implication is trivial. □

We first consider the relation between weak Z-reproducing kernel VE/VH-spaces and their reproducing kernels.

Proposition 3.1.7. (a) *Let $\mathcal{R}$ be a weak Z-reproducing kernel VE-space on X, with respect to some ordered $*$-space Z and with kernel $\mathbf{k}$. Then:*

(i) $\mathbf{k}$ *is positive semidefinite and uniquely determined by $\mathcal{R}$.*

(ii) $\mathcal{R}_0 = \operatorname{Lin}\{\mathbf{k}_x \mid x \in X\} \subseteq \mathcal{R}$ *is a minimal weak Z-reproducing kernel VE-space on X and uniquely determined by* $\mathbf{k}$ *with this property.*

(iii) The gramian $[\cdot,\cdot]_{\mathcal{R}}$ is uniquely determined by $\mathbf{k}$ *on $\mathcal{R}_0$.*

(b) *Assume that Z is admissible and $\mathcal{R}$ is a weak Z-reproducing kernel VH-space. Then:*

(i) $\overline{\mathcal{R}}_0$ is a topologically minimal Z-reproducing kernel VH-space in $\mathcal{R}$.

(ii) The gramian $[\cdot,\cdot]_{\mathcal{R}}$ is uniquely determined by $\mathbf{k}$ on $\overline{\mathcal{R}}_0 \subseteq \mathcal{R}$.

(iii) If $\mathcal{R}$ is topologically minimal then it is unique with this property.

Proof. (a) Let $t_1, \ldots, t_n \in \mathbb{C}$ and $x_1, \ldots, x_n \in X$ be arbitrary. Using (3.15) it follows

$$\sum_{j,k=1}^{n} \overline{t_j} t_k \mathbf{k}(x_j, x_k) = \sum_{j,k=1}^{n} \overline{t_j} t_k [\mathbf{k}_{x_j}, \mathbf{k}_{x_k}]_{\mathcal{R}} = [\sum_{j=1}^{n} t_j \mathbf{k}_{x_j}, \sum_{k=1}^{n} t_k \mathbf{k}_{x_k}]_{\mathcal{R}} \geq 0$$

hence $\mathbf{k}$ is positive semidefinite. On the other hand, by (rk3) it follows that for all $x \in X$ the functions $\mathbf{k}_x$ are uniquely determined by $(\mathcal{R}; [\cdot,\cdot]_{\mathcal{R}})$, hence $\mathbf{k}(y,x) = \mathbf{k}_x(y)$, $x, y \in X$, are uniquely determined. Hence assertion (i) is proven. Assertion (ii) is clear by inspecting the definitions. Assertion (iii) is now clear by (rk3), see (3.15).

(b) The subspace $\overline{\mathcal{R}}_0$ of $\mathcal{R}$ is a topologically minimal Z-reproducing kernel VH-space, by definition. Using the assertion at item (a).(ii) and the continuity of the gramian $[\cdot,\cdot]_{\mathcal{R}}$, it follows that it is uniquely determined by $\mathbf{k}$ on $\overline{\mathcal{R}}_0$.

Assume that $\mathcal{R}$ is topologically minimal and let $\mathcal{R}'$ be another topologically minimal weak Z-reproducing kernel VH-space on X with the same kernel $\mathbf{k}$. By axiom (rk2) and the property (rk4), $\mathcal{R}_0 = \operatorname{Lin}\{\mathbf{k}_x \mid x \in X\}$ is a linear space that lies and is dense in both of $\mathcal{R}$ and $\mathcal{R}'$. By axiom (rk3), the Z-valued inner products $[\cdot,\cdot]_{\mathcal{R}}$ and $[\cdot,\cdot]_{\mathcal{R}'}$ coincide on $\mathcal{R}_0$ and then, due to the special way in which the topologies on VH-spaces are defined, see (2.7) and (2.8), it follows that $\mathcal{R}$ and $\mathcal{R}'$ induce the same topology on $\mathcal{R}_0$ hence, taking into account the density of $\mathcal{R}_0$ in both $\mathcal{R}$ and $\mathcal{R}'$, we actually have $\mathcal{R} = \mathcal{R}'$ as VH-spaces. □

Consequently, given $\mathcal{R}$ a weak Z-reproducing kernel VE-space on X, without any ambiguity we can talk about *the* Z-reproducing kernel $\mathbf{k}$ corresponding to

$\mathcal{R}$.

As a consequence of Proposition 3.1.7, weakly positive semidefiniteness is an intrinsic property of the reproducing kernel of any weak reproducing kernel VE-space. In the following we clarify in an explicit fashion the relation between weak VE/VH-linearisations and weak reproducing kernel VE/VH-spaces associated to positive semidefinite kernels.

Proposition 3.1.8. *Let* $\mathbf{k}$ *be a weakly positive semidefinite kernel on* X *and with values in the ordered* $*$*-space* Z*.*

(a) *Any weak reproducing kernel VE-space* $\mathcal{R}$ *associated to* $\mathbf{k}$ *gives rise to a weak VE-space linearisation* $(\mathcal{E}; V)$ *of* $\mathbf{k}$*, where* $\mathcal{E} = \mathcal{R}$ *and*

$$V(x) = \mathbf{k}_x, \quad x \in X. \tag{3.16}$$

If $\mathcal{R}$ *is minimal, then* $(\mathcal{E}; V)$ *is minimal.*

(b) *Any minimal weak VE-space linearisation* $(\mathcal{E}; V)$ *of* $\mathbf{k}$ *gives rise to the minimal weak reproducing kernel VE-space* $\mathcal{R}$*, where*

$$\mathcal{R} = \{[V(\cdot), h]_{\mathcal{E}} \mid h \in \mathcal{E}\}, \tag{3.17}$$

that is, $\mathcal{R}$ *consists of all functions* $X \ni x \mapsto [V(x), e]_{\mathcal{K}} \in Z$*, for all* $e \in \mathcal{E}$*, in particular,* $\mathcal{R} \subseteq Z^X$ *and* $\mathcal{R}$ *is endowed with the algebraic operations inherited from the complex vector space* Z^X*.*

Proof. (a) Assume that $(\mathcal{R}; [\cdot,\cdot]_{\mathcal{R}})$ is a weak Z-reproducing kernel VE-space on X, with reproducing kernel $\mathbf{k}$. We let $\mathcal{E} = \mathcal{R}$ and define V as in (3.16). Note that $V(x) \in \mathcal{E}$ for all $x \in X$. Also, by (3.15) we have

$$[V(x), V(y)]_{\mathcal{E}} = \mathbf{k}(x, y), \quad x, y \in X.$$

Thus, $(\mathcal{E}; V)$ is a weak VE-space linearisation of $\mathbf{k}$.

(b) Let $(\mathcal{E}; V)$ be a minimal weak VE-space linearisation of $\mathbf{k}$. Let $\mathcal{R}$ be defined by (3.17), that is, $\mathcal{R}$ consists of all functions $X \ni x \mapsto [V(x), h]_{\mathcal{E}} \in Z$, in particular $\mathcal{R} \subseteq Z^X$ with all algebraic operations inherited from the complex vector space Z^X.

The correspondence

$$\mathcal{E} \ni h \mapsto Uh = [V(\cdot), h]_{\mathcal{E}} \in \mathcal{R} \tag{3.18}$$

is clearly surjective. In order to verify that it is injective as well, let $h, g \in \mathcal{E}$ be such that $[V(\cdot), h]_{\mathcal{E}} = [V(\cdot), g]_{\mathcal{E}}$. Then, for all $x \in X$ we have

$$[V(x), h]_{\mathcal{E}} = [V(x), g]_{\mathcal{E}},$$

equivalently,

$$[V(x), h - g]_{\mathcal{E}} = 0, \quad x \in X. \tag{3.19}$$

By the minimality of the linearisation $(\mathcal{E}; V)$ it follows that $g = h$. Thus, U is a bijection.

Clearly, the bijective map U defined at (3.18) is linear, hence a linear isomorphism of complex vector spaces $\mathcal{E} \to \mathcal{R}$. On $\mathcal{R}$ we introduce a Z-valued pairing

$$[Uf, Ug]_{\mathcal{R}} = [f, g]_{\mathcal{E}}, \quad f, g \in \mathcal{E}. \tag{3.20}$$

Since $(\mathcal{E}; [\cdot,\cdot]_{\mathcal{E}})$ is a VE-space over Z, it follows that $(\mathcal{R}; [\cdot,\cdot]_{\mathcal{R}})$ is a VE-space over Z. Indeed, this follows from the observation that, by (3.20), we transported the Z-gramian from $\mathcal{E}$ to $\mathcal{R}$ or, in other words, we have defined on $\mathcal{R}$ the Z-gramian that makes the linear isomorphism U a unitary operator between the VE-spaces $\mathcal{E}$ and $\mathcal{R}$.

We show that $(\mathcal{R}; [\cdot,\cdot]_{\mathcal{R}})$ is a weak Z-reproducing kernel VE-space with corresponding reproducing kernel $\mathbf{k}$. By definition, $\mathcal{R} \subseteq Z^X$. On the other hand, since

$$\mathbf{k}_x(y) = \mathbf{k}(y, x) = [V(y), V(x)]_{\mathcal{E}}, \text{ for all } x, y \in X,$$

taking into account that $V(x) \in \mathcal{E}$, by (3.17) it follows that $\mathbf{k}_x \in \mathcal{R}$ for all $x \in X$. Further, for all $f \in \mathcal{R}$ and all $x \in X$ we have

$$[\mathbf{k}_x, f]_{\mathcal{R}} = [\mathbf{k}_x, [V(\cdot), g]_{\mathcal{E}}]_{\mathcal{R}} = [V(x), g]_{\mathcal{E}},$$

where $g \in \mathcal{E}$ is the unique vector such that $[V(\cdot), g]_{\mathcal{E}} = f$, which shows that $\mathcal{R}$ satisfies the reproducing axiom as well. Finally, taking into account the minimality of the linearisation $(\mathcal{E}; V)$ and the definition (3.17), it follows that $\text{Lin}\{\mathbf{k}_x \mid x \in X\} = \mathcal{R}$. Thus, $(\mathcal{R}; [\cdot, \cdot]_{\mathcal{R}})$ is a minimal weak Z-reproducing kernel VE-space with reproducing kernel $\mathbf{k}$. □

Proposition 3.1.9. *Let* $\mathbf{k}$ *be a weakly positive semidefinite kernel on* X *and valued in the admissible space* Z*.*

(a) *Any weak reproducing kernel VH-space* $\mathcal{R}$ *associated to* $\mathbf{k}$ *gives rise to a weak VH-space linearisation* $(\mathcal{H}; V)$ *of* $\mathbf{k}$*, where* $\mathcal{H} = \mathcal{R}$ *and*

$$V(x) = \mathbf{k}_x, \quad x \in X. \tag{3.21}$$

If $\mathcal{R}$ *is topologically minimal then* $(\mathcal{H}; V)$ *is topologically minimal.*

(b) *Any topologically minimal weak VH-space linearisation* $(\mathcal{H}; V)$ *of* $\mathbf{k}$ *gives rise to the topologically minimal weak reproducing kernel VH-space* $\mathcal{R}$*, where*

$$\mathcal{R} = \{[V(\cdot), h]_{\mathcal{H}} \mid h \in \mathcal{H}\}, \tag{3.22}$$

that is, $\mathcal{R}$ *consists of all functions* $X \ni x \mapsto [V(x), e]_{\mathcal{K}} \in Z$*, for all* $e \in \mathcal{H}$*, in particular,* $\mathcal{R} \subseteq Z^X$ *and* $\mathcal{R}$ *is endowed with the algebraic operations inherited from the complex vector space* Z^X*.*

Proof. (a) The argument is similar to that used to prove assertion (a) of Proposition 3.1.8.

(b) Let $(\mathcal{H}; V)$ be a topologically minimal weak VH-space linearisation of $\mathbf{k}$ and let $\mathcal{R}$ be defined as in (3.22). The correspondence

$$\mathcal{H} \ni h \mapsto Uh = [V(\cdot), h]_{\mathcal{H}} \in \mathcal{R} \tag{3.23}$$

is a linear bijection $U \colon \mathcal{H} \to \mathcal{R}$. The argument to support this claim is similar with that used during the proof of item (b) in Proposition 3.1.8, with the difference that from (3.19) we the topological minimality of the linearisation $(\mathcal{H}; V)$ in order to conclude that $g = h$. Thus, U is a bijection.

On $\mathcal{R}$ we introduce a Z-valued pairing as in (3.20) Since $(\mathcal{H}; [\cdot,\cdot]_{\mathcal{H}})$ is a VH-space over Z, it follows that $(\mathcal{R}; [\cdot,\cdot]_{\mathcal{R}})$ is a VH-space over Z. This follows from the observation that, by (3.20), we transported the Z-gramian from $\mathcal{H}$ to $\mathcal{R}$ or, in other words, we have defined on $\mathcal{R}$ the Z-gramian that makes the linear isomorphism U a unitary operator between the VH-spaces $\mathcal{H}$ and $\mathcal{R}$.

Finally, $(\mathcal{R}; [\cdot,\cdot]_{\mathcal{R}})$ is the topologically minimal weak Z-reproducing kernel VH-space with corresponding reproducing kernel $\mathbf{k}$. The argument is again similar with that used in the proof of item (b) in Proposition 3.1.8, with the difference that here we use the topological minimality. □

The following theorem adds one more characterisation of positive semidefinite kernels, when compared to Theorem 3.1.4, in terms of reproducing kernel spaces. It's proof is a direct consequence of Proposition 3.1.8, Proposition 3.1.9, and Theorem 3.1.4.

Theorem 3.1.10. (a) *Let Z be an ordered $*$-space, X a nonempty set, and $\mathbf{k}\colon X \times X \to Z$ a Hermitian kernel. The following assertions are equivalent:*

(1) $\mathbf{k}$ is weakly positive semidefinite.

(2) $\mathbf{k}$ is the Z-valued reproducing kernel of a VE-space $\mathcal{R}$ in Z^X.

(b) *If, in addition, Z is an admissible space then assertions (1) and (2) are equivalent with*

(3) $\mathbf{k}$ is the Z-valued reproducing kernel of a VH-space $\mathcal{R}$ in Z^X.

In particular, any weakly positive semidefinite Z-valued kernel $\mathbf{k}$ has a topologically minimal weak Z-reproducing kernel VH-space $\mathcal{R}$, uniquely determined by $\mathbf{k}$.

As a consequence of the last assertion of Theorem 3.1.10, given $\mathbf{k}\colon X \times X \to Z$ a positive semidefinite kernel for an admissible space Z, we can denote, without any ambiguity, by $\mathcal{R}_{\mathbf{k}}$ the unique topologically minimal weak Z-reproducing kernel VH-space on X associated to $\mathbf{k}$.

3.2 Invariant Weakly Positive Semidefinite Kernels

Let X be a nonempty set equipped with the action of a (multiplicative) semigroup Γ denoted by $\xi \cdot x$, for all $\xi \in \Gamma$ and all $x \in X$. By definition, we have

$$\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x \text{ for all } \alpha, \beta \in \Gamma \text{ and all } x \in X. \tag{3.24}$$

In case the semigroup Γ has a unit ϵ, the action is called *unital* if $\epsilon \cdot x = x$ for all $x \in X$, equivalently, $\epsilon \cdot = \mathrm{Id}_X$.

We assume further that Γ is a $*$-semigroup, that is, there is an *involution* $*$ on Γ; this means that $(\xi\eta)^* = \eta^*\xi^*$ and $(\xi^*)^* = \xi$ for all $\xi, \eta \in \Gamma$. Note that, in case Γ has a unit ϵ then $\epsilon^* = \epsilon$.

3.2.1 Doubly Invariant Kernels

Let X be a nonempty set and Δ be a (multiplicative) semigroup acting on X on the right, where the action is denoted by $x \cdot a$ for all $x \in X$ and $a \in \Delta$. By definition, we have

$$x \cdot (ab) = (x \cdot a) \cdot b \text{ for all } a, b \in \Delta \text{ and for all } x \in X.$$

Let $\mathbf{k} : X \times X \to Z$ be a kernel. Let Δ be a (multiplicative) semigroup acting on the right on the nonempty set X and on the ordered $*$-space Z, subject to the following conditions:

(rik1) For every $x, y \in X$ and $\gamma \in \Delta$ the equality

$$\mathbf{k}(x, y \cdot \gamma) = \mathbf{k}(x, y) \cdot \gamma$$

holds.

(rik2) For any $n \in \mathbb{N}$, $x, \{y_i\}_{i=1}^n \in X$, $\gamma \in \Delta$ and $\{s_i\}_{i=1}^n \in \mathbb{C}$ the equality

$$(\sum_{i=1}^n s_i \mathbf{k}(x, y_i)) \cdot \gamma = \sum_{i=1}^n s_i(\mathbf{k}(x, y_i) \cdot \gamma)$$

holds.

Remark 3.2.1. Clearly, (rik2) is automatically satisfied if the action of Δ on the ordered $*$-space Z is linear, i.e. the following hold:

$$s(z \cdot a) = (sz) \cdot a$$

for all $s \in \mathbb{C}$, $z \in Z$ and $a \in \Delta$, and,

$$(z_1 + z_2) \cdot a = z_1 \cdot a + z_2 \cdot a$$

for all $z_1, z_2 \in Z$ and $a \in \Delta$.

A kernel $\mathbf{k} : X \times X \to Z$ satisfying (rik1) and (rik2) is called a *right invariant kernel*.

Now let a $*$-semigroup Γ act on the nonempty set X from the left and a semigroup Δ act on X from the right. Assume further that Δ acts on the ordered $*$-space Z from the right. If a kernel $\mathbf{k}\colon X \times X \to Z$ is left invariant, that is,

$$\mathbf{k}(y, \xi \cdot x) = \mathbf{k}(\xi^* \cdot y, x) \text{ for all } x, y \in X \text{ and all } \xi \in \Gamma. \tag{3.25}$$

holds, and it is also right invariant, then it is called a *doubly invariant kernel* under the actions of Γ and Δ.

Remark 3.2.2. Note that a right invariant kernel $\mathbf{k}\colon X \times X \to Z$ can always be considered doubly invariant by taking Γ to be the trivial $*$-semigroup with its trivial left action on the set X. Similarly, a left invariant kernel is always a doubly invariant kernel.

Remark 3.2.3. Notice that we do not assume that the actions of Γ and Δ on the set X are compatible. The following shows that, the left and right invariance, or the double invariance of the kernel implies a weaker compatibility of the two actions:

$$\mathbf{k}(y,(\alpha\cdot x)\cdot a)=\mathbf{k}(\alpha^* y,x)\cdot a=\mathbf{k}(y,\alpha\cdot(x\cdot a)),\ \text{for all}\ x\in X,\ \alpha\in\Gamma\ \text{and}\ a\in\Delta.$$

This compatibility is natural and strong enough for the applications, see Theorems 3.2.11 and 3.2.13 below.

Let $\mathcal{E}$ be a VE-space over an ordered $*$-space Z and recall that $\mathcal{L}(\mathcal{E})$ denotes the algebra of all linear operators $T\colon \mathcal{E}\to\mathcal{E}$, and $\mathcal{L}^*(\mathcal{E})$ denotes the $*$-algebra of all adjointable linear operators $T\colon \mathcal{E}\to\mathcal{E}$, see subsection 2.2.

A triple $(\mathcal{E};\pi;V)$ is called a *left invariant weak VE-space linearisation* of the Z-valued kernel $\mathbf{k}$ and the action of Γ on X, see [25], if:

(ivel1) $(\mathcal{E};V)$ is a weak VE-space linearisation of the kernel $\mathbf{k}$.

(ivel2) $\pi\colon\Gamma\to\mathcal{L}^*(\mathcal{E})$ is a $*$-representation, that is, a multiplicative $*$-morphism.

(ivel3) V and π are related by the formula: $V(\xi\cdot x)=\pi(\xi)V(x)$, for all $x\in X$, $\xi\in\Gamma$.

A quadruple $(\mathcal{E};V;\pi;\tau)$ is called a *doubly invariant VE-space linearisation* of the Z-valued kernel $\mathbf{k}\colon X\times X\to Z$ and actions of Γ on X and Δ on X and Z if we have

(divel1) The triple $(\mathcal{E};V;\pi)$ is a left invariant VE-space linearisation.

(divel2) $\tau\colon\Delta\to\mathcal{L}(\mathcal{E})$ is a representation of the semigroup Δ on $\mathcal{L}(\mathcal{E})$, $\mathcal{E}$ is a right module under the action of $\operatorname{Lin}\tau(\Delta)$, and the right module action respects the gramian of $\mathcal{E}$ in the following sense:

$$[k,l\tau(\gamma)]_{\mathcal{E}}=[k,l]_{\mathcal{K}}\cdot\gamma$$

for all $k,l\in\mathcal{E}$ and $\gamma\in\Delta$.

(divel3) V and τ are related by the formula $V(x \cdot \gamma) = V(x)\tau(\gamma)$.

If, in addition, $(\mathcal{E}; V)$ is minimal, that is,

(divel4) $\operatorname{Lin} V(X) = \mathcal{E}$,

holds, then we call $(\mathcal{E}; V; \pi; \tau)$ a *doubly invariant minimal VE-space linearisation* of the kernel $\mathbf{k}$ and the actions of Γ and Δ.

Remark 3.2.4. Take Γ to be the trivial $*$-semigroup with the trivial left action on X (Which we can always do when a left action is not given). Then in the above we obtain a triple $(\mathcal{E}; V; \tau)$ which is called a *right invariant VE-space linearisation* of the kernel $\mathbf{k}$ and the action of Δ, and in the case the underlying VE-space linearisation is minimal, a *right invariant minimal VE-space linearisation.*

Let $(\mathcal{E}; \pi; V)$ be an invariant weak VE-space linearisation of the kernel $\mathbf{k}$. Since $(\mathcal{E}; V)$ is a weak linearisation and taking into account the axiom (ivel3), for all $x, y \in X$ and all $\xi \in \Gamma$, we have

$$\begin{aligned} \mathbf{k}(y, \xi \cdot x) &= [V(y), V(\xi \cdot x)]_{\mathcal{E}} = [V(y), \pi(\xi)V(x)]_{\mathcal{E}} \\ &= [\pi(\xi^*)V(y), V(x)]_{\mathcal{E}} = [V(\xi^* \cdot y), V(x)]_{\mathcal{E}} = \mathbf{k}(\xi^* \cdot y, x), \end{aligned} \tag{3.26}$$

hence $\mathbf{k}$ is invariant under the action of Γ on X. The same statement holds for a doubly invariant weak VE-space linearisation $(\mathcal{E}; V; \pi; \tau)$ of a kernel $\mathbf{k}$.

Now we make the definitions for topological invariant linearisations. Let Z be an admissible space and X be a nonempty set. A triple $(\mathcal{K}; \pi; V)$ is called a *left invariant weak VH-space linearisation* of the kernel $\mathbf{k}\colon X \times X \to Z$ and the action of Γ on X, see [25], if:

(ivel1) $(\mathcal{K}; V)$ is a weak VH-space linearisation of the kernel $\mathbf{k}$.

(ivel2) $\pi\colon \Gamma \to \mathcal{L}_c^*(\mathcal{K})$ is a $*$-representation, that is, a multiplicative $*$-morphism.

(ivel3) V and π are related by the formula: $V(\xi \cdot x) = \pi(\xi)V(x)$, for all $x \in X$, $\xi \in \Gamma$.

Now a quadruple $(\mathcal{K}; V; \pi; \tau)$ is called a doubly invariant VH-space linearisation of the Z-valued kernel $\mathbf{k}$ and actions of Γ on X and Δ on X and Z if we have

(divhl1) The triple $(\mathcal{K}; V; \pi)$ is a left invariant VH-space linearisation.

(divhl2) $\tau\colon \Delta \to \mathcal{L}_c(\mathcal{K})$ is a representation of the semigroup Δ on $\mathcal{L}_c(\mathcal{K})$, $\mathcal{K}$ is a right module under the action of the algebra $\operatorname{Lin}\tau(\Delta)$, and the right module action respects the gramian of $\mathcal{K}$ in the following sense:

$$[k, l\tau(\gamma)]_{\mathcal{K}} = [k, l]_{\mathcal{K}} \cdot \gamma$$

for all $k, l \in \mathcal{K}$ and $\gamma \in \Delta$.

(divhl3) V and τ are related by the formula $V(x \cdot \gamma) = V(x)\tau(\gamma)$.

If, in addition, $(\mathcal{K}; V)$ is minimal, that is,

(divhl4) $\operatorname{Lin} V(X)$ is dense in $\mathcal{K}$,

then we call $(\mathcal{K}; V; \pi; \tau)$ a doubly invariant minimal VH-space linearisation of the kernel $\mathbf{k}$ and the actions of Γ and Δ.

As in Remark 3.2.4, we have the corresponding notions of a *right invariant VH-space linearisation* and a *right invariant minimal VH-space linearisation.*

As usually [3], minimal left invariant VE-space linearisations preserve linearity.

Proposition 3.2.5. *Assume that, given an ordered $*$-space Z valued kernel $\mathbf{k}$, invariant under the action of the $*$-semigroup Γ on X, for some fixed $\alpha, \beta, \gamma \in \Gamma$ we have $\mathbf{k}(y, \alpha\cdot x) + \mathbf{k}(y, \beta\cdot x) = \mathbf{k}(y, \gamma\cdot x)$ for all $x, y \in X$. Then for any minimal* *weak invariant VE-space linearisation* $(\mathcal{E};\pi;V)$ *of* $\mathbf{k}$*, the representation satisfies* $\pi(\alpha)+\pi(\beta)=\pi(\gamma)$*.*

The same conclusion holds when Z *is an admissible space and* $(\mathcal{K};V;\tau)$ *is a topologically minimal right invariant VH-space linearisation of* $\mathbf{k}$

Proof. For any $x, y \in X$ we have

$$\begin{aligned}[(\pi(\alpha)+\pi(\beta))V(x),V(y)]_{\mathcal{E}} &= [\pi(\alpha)V(x)+\pi(\beta)V(x),V(y)]_{\mathcal{E}} \\ &= \mathbf{k}(\alpha\cdot x,y)+\mathbf{k}(\beta\cdot x,y) \\ &= \mathbf{k}(\gamma\cdot x,y) = [\pi(\gamma)V(x),V(y)]_{\mathcal{E}}\end{aligned}$$

hence, since $V(X)$ linearly spans $\mathcal{E}$, it follows that $\pi(\alpha)+\pi(\beta)=\pi(\gamma)$.

When Z is an admissible space and $(\mathcal{K};V;\tau)$ is a topologically minimal right invariant VH-space linearisation of $\mathbf{k}$, the same argument applies with the small difference that we use the topological minimality. □

For minimal right invariant linearisations a similar result holds, see the Proposition below. Hence minimal invariant linearisations also preserve linearity.

Proposition 3.2.6. *Assume that, given a kernel* $\mathbf{k}\colon X\times X\to Z$ *for an ordered* $*$*-space* Z *and a nonempty set* X*, right invariant under the action of the semigroup* Δ *on* X *and* Z*, for some fixed* $a,b,c\in\Delta$ *we have* $\mathbf{k}(y,x\cdot a)+\mathbf{k}(y,x\cdot b)=\mathbf{k}(y,x\cdot c)$ *for all* $x,y\in X$*. Then for any minimal right invariant VE-space linearisation* $(\mathcal{E};V;\tau)$ *of* $\mathbf{k}$*, the representation* τ *satisfies* $\tau(a)+\tau(b)=\tau(c)$*.*

The same conclusion holds when Z *is an admissible space and* $(\mathcal{K};V;\tau)$ *is a topologically minimal right invariant VH-space linearisation of* $\mathbf{k}$*.*

Proof. This follows by the same arguments as in the proof of Proposition 3.2.5 with obvious modifications. □

Let us now define doubly invariant reproducing kernel VE-spaces. A triple $(\mathcal{R};\rho;\sigma)$ is called a *doubly invariant reproducing kernel VE-space* of the kernel

$\mathbf{k}\colon X \times X \to Z$ and the actions of a $*$-semigroup Γ and semigroup Δ if

(dirkve1) $\mathcal{R}$ is a reproducing kernel VE-space of the kernel $\mathbf{k}$, see 3.1.2.

(dirkve2) $\rho\colon \Gamma \to \mathcal{L}^*(\mathcal{R})$ is a $*$-representation of the $*$-semigroup Γ, such that $\rho(\xi)\mathbf{k}_x = \mathbf{k}_{\xi\cdot x}$ for all $\xi \in \Gamma$ and $x \in X$.

(dirkve3) $\sigma\colon \Delta \to \mathcal{L}(\mathcal{R})$ is a representation of the semigroup Δ, $\mathcal{R}$ is a right module under the action of $\operatorname{Lin}\sigma(\Delta)$ such that

$$[q, r\sigma(a)]_{\mathcal{R}} = [q, r]_{\mathcal{R}} \cdot a \text{ holds for all } q, r \in \mathcal{R} \text{ and } a \in \Delta.$$

If in addition the underlying VE-space is minimal; i.e. we have

(dirkve4) $\mathcal{R} = \operatorname{Lin}\{\mathbf{k}_x \mid x \in X\}$

then we call $(\mathcal{R};\rho;\sigma)$ the *minimal doubly invariant reproducing kernel VE-space* of the kernel $\mathbf{k}\colon X \times X \to Z$ and the actions of a $*$-semigroup Γ and semigroup Δ.

As in Remark 3.2.4, if we consider the trivial $*$-semigroup with the trivial left action on the set X, we get $(\mathcal{R};\sigma)$ which we call *right invariant reproducing kernel VE-spaces*.

Similar to the case of linearisations, we now define doubly invariant reproducing kernel VH-spaces. A triple $(\mathcal{R};\rho;\sigma)$ is called a *doubly invariant reproducing kernel VH-space* of the kernel $\mathbf{k}\colon X \times X \to Z$ and the actions of Γ and Δ if

(dirkvh1) $\mathcal{R}$ is a reproducing kernel VH-space of the kernel $\mathbf{k}$, see subsection 3.1.2.

(dirkvh2) $\rho\colon \Gamma \to \mathcal{L}_c^*(\mathcal{R})$ is a $*$-representation of the $*$-semigroup Γ, such that $\rho(\xi)\mathbf{k}_x = \mathbf{k}_{\xi\cdot x}$ for all $\xi \in \Gamma$ and $x \in X$.

(dirkvh3) $\sigma\colon \Delta \to \mathcal{L}_c(\mathcal{R})$ is a representation of the semigroup Δ, $\mathcal{R}$ is a right module under the action of $\operatorname{Lin}\sigma(\Delta)$ such that

$$[q, r\sigma(a)]_{\mathcal{R}} = [q, r]_{\mathcal{R}} \cdot a \text{ holds for all } q, r \in \mathcal{R} \text{ and } a \in \Delta.$$

If in addition we have

(dirkvh4) The set $\mathrm{Lin}\{\mathbf{k}_x \mid x \in X\}$ is dense in $\mathcal{R}$

then we call $(\mathcal{R};\rho;\sigma)$ the *minimal doubly invariant reproducing kernel VH-space* of the kernel $\mathbf{k}\colon X\times X\to Z$ and the actions of a $*$-semigroup Γ and semigroup Δ.

Two doubly invariant weak VE-space linearisations $(\mathcal{E};\pi;V;\tau)$ and $(\mathcal{E}';\pi';V';\tau')$, of the same Hermitian kernel $\mathbf{k}$, are called *unitarily equivalent* if there exists a unitary operator $U\colon\mathcal{E}\to\mathcal{E}'$ such that $U\tau(\alpha)=\tau'(\alpha)U$ for all $\alpha\in\tau$, $U\pi(\xi)=\pi'(\xi)U$ for all $\xi\in\Gamma$, and $UV(x)=V'(x)$ for all $x\in X$. Let us note that, in case both of these invariant weak VE-space linearisations are minimal, then this is equivalent with the requirement that the weak VE-space linearisations $(\mathcal{E};V)$ and $(\mathcal{E}';V')$ are unitary equivalent.

The following theorem is the first main theorem of this thesis in which invariant weakly positive semidefinite kernels are characterised by invariant weak VE-space linearisations and by certain $*$-representations on weak Z-reproducing kernel VE-spaces.

Theorem 3.2.7. *Let Γ be a $*$-semigroup that acts on the nonempty set X, and let $\mathbf{k}\colon X\times X\to Z$ be a Z-valued kernel for some ordered $*$-space Z. The following assertions are equivalent:*

(1) $\mathbf{k}$ *satisfies the following conditions:*

(a) $\mathbf{k}$ *is weakly positive semidefinite.*

(b) $\mathbf{k}$ *is invariant under the action of Γ on X, that is,* (3.25) *holds.*

(2) $\mathbf{k}$ *has an invariant weak VE-space linearisation $(\mathcal{E};\pi;V)$.*

(3) $\mathbf{k}$ *admits a weak Z-reproducing kernel VE-space $\mathcal{R}$ and there exists a $*$-representation $\rho\colon\Gamma\to\mathcal{L}^*(\mathcal{R})$ such that $\rho(\xi)\mathbf{k}_x=\mathbf{k}_{\xi\cdot x}$ for all $\xi\in\Gamma$, $x\in X$.*

Moreover, in case any of the assertions (1), (2), *or* (3) *holds, then a minimal invariant weak VE-space linearisation of* $\mathbf{k}$ *can be constructed and a minimal weak* Z*-reproducing kernel* $\mathcal{R}$ *as in* (3) *can constructed as well.*

Proof. (1)⇒(2). We consider the notation and the minimal weak VE-space linearisation $(\mathcal{E}; V)$ constructed as in the proof of the implication (1)⇒(2) of Theorem 3.1.4. For each $\xi \in \Gamma$ we let $\pi(\xi)\colon Z^X \to Z^X$ be defined by

$$(\pi(\xi)f)(y) = f(\xi^* \cdot y), \quad f \in Z^X,\ y \in X,\ \xi \in \Gamma. \tag{3.27}$$

We claim that $\pi(\xi)$ leaves Z^X_K invariant, where K is the convolution operator defined at (3.3) and $Z^X_K \subseteq Z^X$ denotes its range. To see this, let $f \in Z^X_K$, that is, $f = Kg$ for some $g \in \mathbb{C}^X_0$ or, even more explicitly, by (3.10),

$$f(y) = \sum_{x \in X} g(x)\mathbf{k}(x, y), \quad y \in X. \tag{3.28}$$

Then,

$$f(\xi^* \cdot y) = \sum_{x\in X} g(x)\mathbf{k}(x, \xi^* \cdot y) = \sum_{x\in X} g(x)\mathbf{k}(\xi \cdot x, y) = \sum_{z\in X} g_\xi(z)\mathbf{k}(z, y), \tag{3.29}$$

where,

$$g_\xi(z) = \begin{cases} 0, & \text{if } \xi \cdot x = z \text{ has no solution } x \in X,\\ \sum\limits_{\xi\cdot x = z} g(x), & \text{otherwise.}\end{cases}$$

Since clearly $g_\xi \in \mathbb{C}^X_0$, that is, g_ξ has finite support, it follows that $\pi(\xi)$ leaves Z^X_K invariant. In the following we denote by the same symbol $\pi(\xi)$ the map $\pi(\xi)\colon Z^X_K \to Z^X_K$.

In the following we prove that π is a representation of the semigroup Γ on the complex vector space Z^X_K, that is,

$$\pi(\alpha\beta)f = \pi(\alpha)\pi(\beta)f, \quad \alpha, \beta \in \Gamma,\ f \in Z^X_K. \tag{3.30}$$

To see this, let $f \in Z^X_K$ be fixed and denote $h = \pi(\beta)f$, that is, $h(y) = f(\beta^* \cdot y)$ for all $y \in X$. Then $\pi(\alpha)\pi(\beta)f = \pi(\alpha)h$, that is, $(\pi(\alpha)h)(y) = h(\alpha^* \cdot y) = f(\beta^*\alpha^* \cdot y) = f((\alpha\beta)^* \cdot y) = (\pi(\alpha\beta))(y)$, for all $y \in X$, which proves (3.30).

Next we show that π is actually a $*$-representation, that is,

$$[\pi(\xi)f, f']_{\mathcal{E}} = [f, \pi(\xi^*)f']_{\mathcal{E}}, \quad f, f' \in Z_K^X. \tag{3.31}$$

To see this, let $f = Kg$ and $f' = Kg'$ for some $g, g' \in \mathbb{C}_0^X$. Then, by (3.11) and (3.29),

$$\begin{aligned}[\pi(\xi)f, f']_{\mathcal{E}} &= \sum_{y\in X} g'(y)f(\xi^* \cdot y) = \sum_{x,y\in X} g'(y)\overline{g(x)}\mathbf{k}(\xi^* \cdot y, x) \\ &= \sum_{x,y\in X} g'(y)\overline{g(x)}\mathbf{k}(y, \xi \cdot x) = \sum_{x\in X} \overline{g(x)} f'(\xi \cdot x)^* = [f, \pi(\xi^*)f']_{\mathcal{E}},\end{aligned}$$

and hence the formula (3.31) is proven.

In order to show that the axiom (vel3) holds as well, we use (3.14) and (3.27). Thus, for all $\xi \in \Gamma$, $x, y \in X$ and taking into account that $\mathbf{k}$ is invariant under the action of Γ on X, we have

$$\begin{aligned}(V(\xi \cdot x))(y) &= \mathbf{k}(\xi \cdot x, y) = \mathbf{k}(x, \xi^* \cdot y) \\ &= (V(x))(\xi^* \cdot y) = (\pi(\xi)V(x))(y),\end{aligned} \tag{3.32}$$

which proves (vel3). Thus, $(\mathcal{E}; \pi; V)$, here constructed, is an invariant weak VE-space linearisation of the Hermitian kernel $\mathbf{k}$. Note that $(\mathcal{E}; \pi; V)$ is minimal, that is, the axiom (vel4) holds, since the linearisation $(\mathcal{E}; V)$ is minimal, by the proof of Theorem 3.1.4.

In order to prove the uniqueness of the minimal weak invariant linearisation, let $(\mathcal{K}'; \pi'; V')$ be another minimal invariant weak VE-space linearisation of $\mathbf{k}$. We consider the unitary operator $U : \mathcal{K} \to \mathcal{K}'$ defined as in (3.9) and we already know that $UV(x) = V'(x)$ for all $x \in X$. Since, for any $\xi \in \Gamma$, $x \in X$ we have

$$U\pi(\xi)V(x) = UV(\xi \cdot x) = V'(\xi \cdot x) = \pi'(\xi)V'(x) = \pi'(\xi)UV(x),$$

and taking into account the minimality, it follows that $U\pi(\xi) = \pi'(\xi)U$ for all $\xi \in \Gamma$.

(2)$\Rightarrow$(1). Let $(\mathcal{E}; \pi; V)$ be an invariant weak VE-space linearisation of the kernel $\mathbf{k}$. We already know from the proof of Theorem 3.1.4 that $\mathbf{k}$ is positive semidefinite and it was shown in (3.26) that $\mathbf{k}$ is invariant under the action of Γ on X.

(2)⇒(3). Let $(\mathcal{E};\pi;V)$ be an invariant weak VE-space linearisation of the kernel $\mathbf{k}$ and the action of Γ on X. Without loss of generality, we can assume that it is minimal. Indeed, since we have already proven the implication (2)⇒(1), we observe that during the proof of the implication (1)⇒(2), we obtained a minimal invariant weak VE-space linearisation of $\mathbf{k}$.

We use the notation and the facts established during the proof of the implication (2)⇒(3) in Theorem 3.1.10. Then, for all $x, y \in X$ we have

$$\mathbf{k}_{\xi\cdot x}(y) = \mathbf{k}(y, \xi\cdot x) = [V(y), V(\xi\cdot x)]_{\mathcal{K}} = [V(y), \pi(\xi)V(x)]_{\mathcal{K}},$$

hence, letting $\rho(\xi) = U\pi(\xi)U^{-1}$, where $U\colon \mathcal{K}\to\mathcal{R}$ is the unitary operator defined as in (3.18), we obtain a $*$-representation of Γ on the VE-space $\mathcal{R}$ such that $\mathbf{k}_{\xi\cdot x} = \rho(\xi)\mathbf{k}_x$ for all $\xi\in\Gamma$ and $x\in X$.

(3)⇒(2). Let $(\mathcal{R};\rho)$, where $\mathcal{R}$ is a weak Z-reproducing kernel VE-space of $\mathbf{k}$ and $\rho\colon\Gamma\to\mathcal{L}^*(\mathcal{R})$ is a $*$-representation such that $\mathbf{k}_{\xi\cdot x} = \rho(\xi)\mathbf{k}_x$ for all $\xi\in\Gamma$ and $x\in X$. As in the proof of the implication (3)⇒(2) in Theorem 3.1.10, we show that $(\mathcal{R};V)$, where V is defined as in (3.16), is a minimal linearisation of $\mathbf{k}$. Letting $\pi=\rho$, it is then easy to see that $(\mathcal{R};\pi;V)$ is an invariant weak VE-space linearisation of the kernel $\mathbf{k}$ and the action of Γ on X. □

3.2.2 Boundedly Adjointable Invariant Weak VH-Space Linearisations.

Let us assume now that Z is an admissible space and $\mathbf{k}\colon X\times X\to Z$ is a kernel. A triple $(\mathcal{K};\pi;V)$ is called a *boundedly adjointable invariant weak VH-space linearisation* of the Z-valued kernel $\mathbf{k}$ and the action of Γ on X, if:

(ivhl1) $(\mathcal{K};V)$ is a weak VH-space linearisation of the kernel $\mathbf{k}$.

(ivhl2) $\pi\colon \Gamma \to \mathcal{B}^*(\mathcal{K})$ is a $*$-representation, that is, a multiplicative $*$-morphism.

(ivhl3) V and π are related by the formula: $V(\xi \cdot x) = \pi(\xi)V(x)$, for all $x \in X$, $\xi \in \Gamma$.

Let $(\mathcal{K};\pi;V)$ be a boundedly adjointable invariant weak VH-space linearisation of the kernel $\mathbf{k}$. As in (3.26), it follows that $\mathbf{k}$ is invariant under the action of Γ on X.

If, in addition to the axioms (ivhl1), (ivhl2), and (ivhl3), the triple $(\mathcal{K};\pi;V)$ has also the property

(ivhl4) $\operatorname{Lin} V(X)$ is dense in $\mathcal{K}$,

that is, the weak VH-space linearisation $(\mathcal{H};V)$ is topologically minimal, then $(\mathcal{K};\pi;V)$ is called *topologically minimal*. An observation can be made: in case Γ has a unit then (ivhl4) is equivalent with saying $\operatorname{Lin} \pi(\Gamma)V(X)$ is dense in $\mathcal{K}$ but, in general the apparently more candidate $\operatorname{Lin} \pi(\Gamma)V(X)$ is too small to provide a suitable topological minimality condition.

Two boundedly adjointable invariant weak VH-space linearisations $(\mathcal{K};\pi;V)$ and $(\mathcal{K}';\pi';V')$ of the same kernel $\mathbf{k}$ are *unitarily invariant* if there exists a unitary $U \in \mathcal{B}^*(\mathcal{K},\mathcal{K}')$ such that $U\pi(\xi) = \pi'(\xi)U$ for all $\xi \in \Gamma$ and $UV(x) = V'(x)$ for all $x \in X$. Let us note that, in case both of these boundedly adjointable invariant weak VH-space linearisations are topologically minimal then they are unitarily equivalent.

The analog of Proposition 3.2.5 for topologically minimal invariant weak VH-space linearisations holds as well.

Proposition 3.2.8. *Assume that, given an admissible space Z and a Z-valued kernel $\mathbf{k}$, invariant under the action of the $*$-semigroup Γ on X, for some fixed $\alpha,\beta,\gamma \in \Gamma$ we have $\mathbf{k}(y,\alpha\cdot x) + \mathbf{k}(y,\beta\cdot x) = \mathbf{k}(y,\gamma\cdot x)$ for all $x,y \in X$. Then, for any topologically minimal boundedly adjointable invariant weak VH-space linearisation $(\mathcal{K};\pi;V)$ of $\mathbf{k}$, the representation satisfies $\pi(\alpha)+\pi(\beta) = \pi(\gamma)$.*

Proof. The same argument as in the proof of Proposition 3.2.5 applies with the small difference that we use the topological minimality and get the same conclusion. $\square$

Here we have the second main theorem of this thesis in which invariant weakly positive semidefinite kernels are characterised by boundedly adjointable invariant weak VE-space linearisations and by certain $*$-representations with boundedly adjointable operators on weak Z-reproducing kernel VE-spaces. This is the first topological analogue of Theorem 3.2.7.

Theorem 3.2.9. *Let Γ be a $*$-semigroup that acts on the nonempty set X, and let $\mathbf{k}\colon X \times X \to Z$ be a Z-valued kernel for some admissible space Z. The following assertions are equivalent:*

(1) $\mathbf{k}$ *satisfies the following conditions:*

(a) $\mathbf{k}$ *is weakly positive semidefinite.*

(b) $\mathbf{k}$ *is invariant under the action of Γ on X, that is,* (3.25) *holds.*

(c) For any $\alpha \in \Gamma$ there exists $c(\alpha) \geq 0$ such that

$$\sum_{j,k=1}^{n} t_j \overline{t_k} \mathbf{k}(\alpha \cdot x_k, \alpha \cdot x_j) \leq c(\alpha)^2 \sum_{j,k=1}^{n} t_j \overline{t_k} \mathbf{k}(x_k, x_j), \tag{3.33}$$

for $n \in \mathbb{N}$, all $x_1, \ldots, x_n \in X$, and all $t_1, \ldots, t_n \in \mathbb{C}$.

(2) $\mathbf{k}$ *has a boundedly adjointable invariant weak VH-space linearisation $(\mathcal{K}; \pi; V)$.*

(3) $\mathbf{k}$ *admits a weak Z-reproducing kernel VH-space $\mathcal{R}$ and there exists a $*$-representation $\rho\colon \Gamma \to \mathcal{B}^*(\mathcal{R})$ such that $\rho(\xi)\mathbf{k}_x = \mathbf{k}_{\xi \cdot x}$ for all $\xi \in \Gamma$, $x \in X$.*

Moreover, in case any of the assertions (1), (2), *or* (3) *holds, then a topologically minimal boundedly adjointable invariant weak VH-space linearisation can be constructed and a topologically minimal weak Z-reproducing kernel VH-space $\mathcal{R}$ as in assertion* (3) *can be constructed as well.*

Proof. (1)⇒(2). We consider the notation and the minimal invariant weak VE-space linearisation $(\mathcal{E};\pi;V)$ constructed as in the proof of the implication (1)⇒(2) of Theorem 3.2.7. Considering Z_K^X as a VE-space with Z-gramian $[\cdot,\cdot]_{\mathcal{E}}$, we consider its natural topology as in Subsection 2.3 and we prove now that $\pi(\xi)$ is bounded for all $\xi \in \Gamma$. Indeed, let $f = Kg$ for some $g \in \mathbb{C}_0^X$. Using the definition of $\pi(\xi)$ and the boundedness condition (c), we have

$$\begin{aligned}[]
[\pi(\xi)f, \pi(\xi)f]_{\mathcal{K}} &= [\pi(\xi^*)\pi(\xi)f, f]_{\mathcal{K}} = [\pi(\xi^*\xi)f, f]_{\mathcal{K}} \\
&= \sum_{x,y\in X} g(y)\overline{g(x)}\mathbf{k}(\xi^*\xi\cdot y, x) = \sum_{x,y\in X} g(y)\overline{g(x)}\mathbf{k}(\xi\cdot y, \xi\cdot x) \\
&\leq c(\xi)^2 \sum_{x,y\in X} g(y)\overline{g(x)}\mathbf{k}(y,x) = c(\xi)^2[f,f]_{\mathcal{K}},
\end{aligned}$$

and hence the boundedness of $\pi(\xi)$ is proven. This implies that $\pi(\xi)$ can be uniquely extended by continuity to an operator $\pi(\xi) \in \mathcal{B}(\mathcal{K})$. In addition, since $\pi(\xi^*)$ also extends by continuity to an operator $\pi(\xi^*) \in \mathcal{B}(\mathcal{K})$ and taking into account (3.31), it follows that $\pi(\xi)$ is adjointable and $\pi(\xi^*) = \pi(\xi)^*$. We conclude that π is a $*$-representation of Γ in $\mathcal{B}^*(\mathcal{K})$, that is, the axiom (ivhl2) holds.

The uniqueness of the topologically minimal boundedly adjointable invariant weak VH-space linearisation follows as usually.

(2)⇒(1). Let $(\mathcal{K};\pi;V)$ be a boundedly adjointable invariant weak VH-space linearisation of the kernel $\mathbf{k}$. We already know from the proof of Theorem 3.1.4 that $\mathbf{k}$ is positive semidefinite and it was shown in (3.26) that $\mathbf{k}$ is invariant under the action of Γ on X. In order to show that the boundedness condition (c) holds as well, let $\alpha \in \Gamma$, $n \in \mathbb{N}$, $x_1,\ldots,x_n \in X$, and $t_1,\ldots,t_n \in \mathbb{C}$ be arbitrary. Then

$$\begin{aligned}
\sum_{j,k=1}^n \overline{t_k}t_j\mathbf{k}(\alpha\cdot x_k, \alpha\cdot x_j) &= \sum_{j,k=1}^n \overline{t_k}t_j[\pi(\alpha^*)\pi(\alpha)V(x_k), V(x_j)]_{\mathcal{K}} \\
&= \sum_{j,k=1}^n \overline{t_k}t_j[\pi(\alpha)V(x_k), \pi(\alpha)V(x_j)]_{\mathcal{K}}
\end{aligned}$$

$$
\begin{aligned}
&= [\pi(\alpha)\sum_{k=1}^{n} t_k V(x_k), \pi(\alpha)\sum_{j=1}^{n} t_j V(x_j)]_{\mathcal{K}} \\
&\leq \|\pi(\alpha)\|^2 [\sum_{k=1}^{n} t_k V(x_k), \sum_{j=1}^{n} t_j V(x_j)]_{\mathcal{K}} \\
&= \|\pi(\alpha)\|^2 \sum_{j,k=1}^{n} \overline{t_k} t_j \mathbf{k}(x_k, x_j),
\end{aligned}
$$

and hence (c) holds with $c(\alpha) = \|\pi(\alpha)\| \geq 0$.

(2)⇒(3). Let $(\mathcal{K}; \pi; V)$ be a boundedly adjointable invariant weak VH-space linearisation of the kernel $\mathbf{k}$ with respect to the action of Γ on X. Without loss of generality we can assume that it is topologically minimal. Indeed, since we have already proven the implication (2)⇒(1), we observe that during the proof of the implication (1)⇒(2), we obtained a topologically minimal invariant weak VH-space linearisation of $\mathbf{k}$.

We use the notation and the facts established during the proof of the implication (2)⇒(3) in Theorem 3.1.10. Then, for all $x, y \in X$ we have

$$
\mathbf{k}_{\xi \cdot x}(y) = \mathbf{k}(y, \xi \cdot x) = [V(y), V(\xi \cdot x)]_{\mathcal{K}} = [V(y), \pi(\xi)V(x)]_{\mathcal{K}},
$$

hence, letting $\rho(\xi) = U\pi(\xi)U^{-1}$, where $U \colon \mathcal{K} \to \mathcal{R}$ is the unitary operator defined as in (3.18), we obtain a $*$-representation of Γ on the VH-space $\mathcal{R}$ such that $\mathbf{k}_{\xi \cdot x} = \rho(\xi)\mathbf{k}_x$ for all $\xi \in \Gamma$ and $x \in X$.

(3)⇒(2). Let $(\mathcal{R}; \rho)$, where $\mathcal{R} = \mathcal{R}(\mathbf{k})$ is the weak reproducing kernel VH-space of $\mathbf{k}$ and $\rho \colon \Gamma \to \mathcal{B}^*(\mathcal{R})$ is a $*$-representation such that $\mathbf{k}_{\xi \cdot x} = \rho(\xi)\mathbf{k}_x$ for all $\xi \in \Gamma$ and $x \in X$. As in the proof of the implication (3)⇒(2) in Theorem 3.1.10, we show that $(\mathcal{R}; V)$, where V is defined as in (3.16), is a minimal weak linearisation of $\mathbf{k}$. Letting $\pi = \rho$, it is then easy to see that $(\mathcal{R}; \pi; V)$ is a boundedly adjointable invariant weak VH-space linearisation of the kernel $\mathbf{k}$ with respect to the action of Γ on X. □

3.2.3 Continuously Adjointable Invariant Weak VH-Space Linearisations.

Let Z be an admissible space. A triple $(\mathcal{K};\pi;V)$ is called a *continuously adjointable invariant weak VH-space linearisation* of the Z-valued kernel $\mathbf{k}$ and the action of Γ on X, if the requirements (ivhl1) and (ivhl2) holds and, instead of (ihvl2), it satisfies

(ivhl2)′ $\pi\colon \Gamma \to \mathcal{L}_{\mathrm{c}}^*(\mathcal{K})$ is a $*$-representation, that is, a multiplicative $*$-morphism.

Clearly, for any continuously adjointable invariant weak VH-space linearisation $(\mathcal{K};\pi;V)$ of the kernel $\mathbf{k}$, it follows that $\mathbf{k}$ is invariant under the action of Γ on X.

If, in addition to the axioms (ivhl1), (ivhl2)′, and (ivhl3), the triple $(\mathcal{K};\pi;V)$ has also the property (ivhl4), that is, the weak VH-space linearisation $(\mathcal{H};V)$ is topologically minimal, then $(\mathcal{K};\pi;V)$ is called a *topologically minimal continuously adjointable invariant weak VH-space linearisation* of the Z-kernel $\mathbf{k}$ with respect to the action of Γ on X.

The unitary equivalence of two continuously adjointable invariant weak VH-space linearisations $(\mathcal{K};\pi;V)$ and $(\mathcal{K}';\pi';V')$ of the same kernel $\mathbf{k}$ is defined as in the case of boundedly adjointable invariant weak VH-space linearisations and their topological minimality implies their unitary equivalence.

The analog of Proposition 3.2.5 for topologically minimal continuously adjointable invariant weak VH-space linearisations holds as well.

The next theorem is the analogue of Theorem 3.2.9 for continuously adjointable invariant weak VH-space linearisations in which the boundedness condition 1.(c) of Theorem 3.2.9 is replaced with a weaker one.

Theorem 3.2.10. *Let Γ be a $*$-semigroup that acts on the nonempty set X, and let $\mathbf{k}\colon X\times X\to Z$ be a Z-valued kernel for some admissible space Z. The* *following assertions are equivalent:*

(1) $\mathbf{k}$ *satisfies the following conditions:*

(a) $\mathbf{k}$ *is weakly positive semidefinite.*

(b) $\mathbf{k}$ *is invariant under the action of* Γ *on* X*, that is,* (3.25) *holds.*

(c) For any $\alpha \in \Gamma$ *and any seminorm* $p \in S(Z)$*, there exists a seminorm* $q \in S(Z)$ *and a constant* $c(\alpha) \geq 0$ *such that*

$$p(\sum_{j,k=1}^{n} t_j \overline{t_k} \mathbf{k}(\alpha \cdot x_k, \alpha \cdot x_j)) \leq c(\alpha)^2 q(\sum_{j,k=1}^{n} t_j \overline{t_k} \mathbf{k}(x_k, x_j)), \tag{3.34}$$

for $n \in \mathbb{N}$*, all* $x_1, \ldots, x_n \in X$*, and all* $t_1, \ldots, t_n \in \mathbb{C}$*.*

(2) $\mathbf{k}$ *has a continuously adjointable invariant weak VH-space linearisation* $(\mathcal{K}; \pi; V)$*.*

(3) $\mathbf{k}$ *admits a weak* Z*-reproducing kernel VH-space* $\mathcal{R}$ *and there exists a* $*$*-representation* $\rho \colon \Gamma \to \mathcal{L}_{\mathrm{c}}^*(\mathcal{R})$ *such that* $\rho(\xi)\mathbf{k}_x = \mathbf{k}_{\xi \cdot x}$ *for all* $\xi \in \Gamma$*,* $x \in X$*.*

Moreover, in case any of the assertions (1), (2), *or* (3) *holds, then a topologically minimal continuously adjointable invariant VH-space linearisation can be constructed and a topologically minimal weak* Z*-reproducing kernel VH-space* $\mathcal{R}$ *as in assertion* (3) *can be constructed as well.*

Proof. (1)⇒(2). We consider the notation and constructions as in the proof of the implication (1)⇒(2) of Theorem 3.2.7, and follow the same idea as in the proof of the implication (1)⇒(2) of Theorem 3.2.9, with the difference that the weak boundedness condition 1.(c) is used. For any $\xi \in \Gamma$, $f = Kg$ and $p \in S(Z)$ there exist $q \in S(Z)$ and $c(\xi) \geq 0$ such that

$$\begin{aligned} p([\pi(\xi)f, \pi(\xi)f]_{\mathcal{K}}) &= p(\sum_{x,y \in X} g(y)\overline{g(x)} \mathbf{k}(\xi \cdot y, \xi \cdot x)) \\ &\leq c(\xi)^2 q(\sum_{x,y \in X} g(y)\overline{g(x)} \mathbf{k}(y, x)) = c(\xi)^2 q([f, f]_{\mathcal{K}}), \end{aligned}$$

hence the continuity of $\pi(\xi)$ is proven. This implies that $\pi(\xi)$ can be uniquely extended by continuity to an operator $\pi(\xi) \in \mathcal{L}_{\mathrm{c}}(\mathcal{K})$. In addition, since $\pi(\xi^*)$ also extends by continuity to an operator $\pi(\xi^*) \in \mathcal{L}_{\mathrm{c}}(\mathcal{K})$ and taking into account (3.31), it follows that $\pi(\xi)$ is adjointable and $\pi(\xi^*) = \pi(\xi)^*$. We conclude that π is a $*$-representation of Γ in $\mathcal{L}_{\mathrm{c}}^*(\mathcal{K})$.

The uniqueness of the topologically minimal continuously adjointable invariant VH-space linearisation follows as usually.

(2)⇒(1). By the proof of the implication (2)⇒(1) of Theorem 3.2.9, we only have to show that the boundedness condition (c) holds. Let $\alpha \in \Gamma$, $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$, and $t_1, \ldots, t_n \in \mathbb{C}$ be arbitrary. Then, due to the continuity of $\pi(\alpha)$ and taking into account the $S(Z)$ is directed, there exist $q \in S(Z)$ and $c(\alpha) \geq 0$ such that

$$\begin{aligned} p(\sum_{j,k=1}^{n} \overline{t_k} t_j \mathbf{k}(\alpha \cdot x_k, \alpha \cdot x_j)) &= p([\pi(\alpha)\sum_{k=1}^{n} t_k V(x_k), \pi(\alpha)\sum_{j=1}^{n} t_j V(x_j)]_{\mathcal{K}}) \\ &\leq c(\alpha)^2 q([\sum_{k=1}^{n} t_k V(x_k), \sum_{j=1}^{n} t_j V(x_j)]_{\mathcal{K}}) \\ &= c(\alpha)^2 q(\sum_{j,k=1}^{n} \overline{t_k} t_j \mathbf{k}(x_k, x_j)). \end{aligned}$$

(2)⇒(3). Let $(\mathcal{K}; V; \pi)$ be a continuously adjointable weak VH-space linearisation of the kernel $\mathbf{k}$ with respect to the action of Γ on X. Using exactly the same ideas in the proof of the implication (2)⇒(1) of Theorem 3.2.9, we obtain a continuous $*$-representation of Γ on the VH-space $\mathcal{R}$ defined by $\rho(\xi) = U\pi(\xi)U^{-1}$, where $U \colon \mathcal{K} \to \mathcal{R}$ is the unitary operator defined as in (3.18).

(3)⇒(2). Let $(\mathcal{R}; \rho)$, where $\mathcal{R} = \mathcal{R}(\mathbf{k})$ is the weak reproducing kernel VH-space of $\mathbf{k}$ and $\rho \colon \Gamma \to \mathcal{L}_{\mathrm{c}}^*(\mathcal{R})$ is a $*$-representation such that $\mathbf{k}_{\xi \cdot x} = \rho(\xi)\mathbf{k}_x$ for all $\xi \in \Gamma$ and $x \in X$. As in the proof of the implication (3)⇒(2) in Theorem 3.2.9, letting $\pi = \rho$, it is then easy to see that $(\mathcal{R}; \pi; V)$ is a weak VH-space linearisation of the kernel $\mathbf{k}$ and π satisfies the required properties. $\square$

The following theorem is one of the main theorems of this thesis. It is a doubly invariant kernel version of Theorem 3.2.7 and it gives characterizations of doubly invariant weakly positive semidefinite kernels by doubly invariant VE-space linearisations and reproducing kernel VE-spaces with certain representations of semigroups.

Theorem 3.2.11. *Let X be a nonempty set and Z be an ordered $*$-space. Let Γ be a $*$-semigroup that acts on X on the left, Δ be a semigroup that acts X on the right and Z on the right and let $\mathbf{k}\colon X \times X \to Z$ be a kernel. The following assertions are equivalent:*

(1) $\mathbf{k}$ *satisfies the following conditions:*

  (a) $\mathbf{k}$ *is positive semidefinite.*

  (b) $\mathbf{k}$ *is doubly invariant under the left action of Γ on X, and the right action of Δ on X and on Z.*

(2) $\mathbf{k}$ *has a doubly invariant VE-space linearisation $(\mathcal{E}; V; \pi; \tau)$.*

(3) $\mathbf{k}$ *admits a doubly invariant reproducing kernel VE-space $(\mathcal{R}; \rho; \sigma)$.*

Moreover, in case any of the assertions (1), (2), *or* (3) *holds, then a minimal doubly invariant VE-space linearisation can be constructed, any minimal doubly invariant linearisation is unique up to unitary equivalence, and the triple $(\mathcal{R}; \rho; \sigma)$ in assertion* (3) *is uniquely determined by* $\mathbf{k}$ *as well.*

Proof. (1)⇒(2). By Theorem 3.2.7, there exists a minimal left invariant linearisation $(\mathcal{E}; V; \pi)$ of the kernel $\mathbf{k}$.

Referring to the constructions as in the proof of Theorem 3.2.7, define $\tau\colon \Delta \to \mathbb{C}^X$ by

$$(f\tau(a))(x) := \sum_{y \in X} g(y)\mathbf{k}(x, y \cdot a) \tag{3.35}$$

where $f = Kg$ as in (3.3) and $a \in \Delta$. We show that $\tau(a)$, for each $a \in \Delta$ is well defined on $\mathcal{E}$. Let $\mathcal{E} \ni f = Kg = Kh$. Then we have

$$\begin{aligned}(f\tau(a))(x) &= \sum_{y\in X} g(y)\mathbf{k}(x, y \cdot a) = (\sum_{y\in X} g(y)\mathbf{k}(x, y)) \cdot a \\ &= (\sum_{y\in X} h(y)\mathbf{k}(x, y)) \cdot a = \sum_{y\in X} h(y)\mathbf{k}(x, y \cdot a)\end{aligned}$$

for all $a \in \Delta$ and $x \in X$, where we used that the kernel $\mathbf{k}$ is doubly invariant. Therefore $\tau(a)$ on $\mathcal{E}$ is well defined, and it follows by the doubly invariance of the kernel that it is linear.

We now show that this action leaves Z_K^X invariant. Let $f = Kg$. Then

$$(f\tau(a))(x) = \sum_{y\in X} g(y)\mathbf{k}(x, y \cdot a) = \sum_{z\in X} g_a(z)\mathbf{k}(x, z)$$

where

$$g_a(z) = \begin{cases} 0, & \text{if } y \cdot a = z \text{ has no solution } y \in X, \\ \sum_{y \cdot a = z} g(y), & \text{otherwise.} \end{cases} \tag{3.36}$$

Since clearly $g_a \in \mathbb{C}_0^X$, that is, g_a has finite support, it follows that $\tau(a)$ leaves Z_K^X invariant. Hence $\tau(a) \in \mathcal{L}(\mathcal{E})$ for all $a \in \Delta$.

To see that τ is a representation of the semigroup Δ on $\mathcal{L}(\mathcal{E})$, let $a, b \in \Delta$ and $\mathcal{E} \ni f = Kg$. Then

$$\begin{aligned}(f\tau(ab))(x) &= \sum_{y\in X} g(y)\mathbf{k}(x, y \cdot \ ab) = (\sum_{z\in X} g_a(z)\mathbf{k}(x, z)) \cdot b \\ &= \sum_{z\in X} g_a(z)\mathbf{k}(x, z \cdot b) = ((f\tau(a))\tau(b))(x)\end{aligned}$$

and since $(f\tau(a))\tau(b) = f(\tau(a)\tau(b))$, it follows that $\tau(ab) = \tau(a)\tau(b)$.

Consider the algebra $\operatorname{Lin}\tau(\Delta)$ generated by $\tau(\Delta)$ in $\mathcal{L}(\mathcal{E})$

$$\operatorname{Lin}\tau(\Delta) = \{\sum_{j=1}^{n} \lambda_j \tau(\alpha_j) \mid n \in \mathbb{N}, \lambda_j \in \mathbb{C}, \alpha_j \in \Delta\} \subseteq \mathcal{L}(\mathcal{E}).$$

Linearly extending the action of $\tau(\Delta)$ on $\mathcal{E}$ to $\operatorname{Lin}\tau(\Delta)$, it follows that $\mathcal{E}$ is turned into a right $\operatorname{Lin}\tau(\Delta)$-module.

Next, we show that $V(x \cdot \alpha) = V(x)\tau(\alpha)$ for all $x \in X$ and $\alpha \in \Delta$. Using equation (3.14) and that $\mathbf{k}$ is doubly invariant, we obtain

$$(V(x \cdot \alpha))(y) = \mathbf{k}(y, x\alpha) = \mathbf{k}(y, x)\alpha = (V(x)\tau(\alpha))(y)$$

for all $x, y \in X$ and $\alpha \in \Delta$.

We show that the right action of $\tau(\alpha)$ respects the gramian. Let $e = Kg$, $f = Kh$ and $\alpha \in \Delta$. Then

$$[e, f\tau(\alpha)]_{\mathcal{E}} = \sum_{x,y \in X} \overline{g(x)} h(y)\mathbf{k}(x, y\alpha) = (\sum_{x,y \in X} \overline{g(x)} h(y)\mathbf{k}(x, y)) \cdot \alpha = [e, f]_{\mathcal{E}} \cdot \alpha.$$

In order to prove the uniqueness of the minimal doubly invariant linearisation, let $(\mathcal{E}'; V'; \pi'; \tau')$ be another minimal doubly invariant linearisation of $\mathbf{k}$. By uniqueness of the left invariant linearisation $(\mathcal{E}; V; \pi)$ by the proof of Theorem 3.2.7, there exists a unitary operator $U \colon \mathcal{E} \to \mathcal{E}'$ with $UV(x) = V'(x)$ for all $x \in X$. Since, for any $\alpha \in \Delta$, $x \in X$ we have

$$UV(x)\tau(\alpha) = UV(x \cdot \alpha) = V'(x \cdot \alpha) = V'(x)\tau'(\alpha) = UV(x)\tau'(\alpha),$$

and taking into account the minimality, we obtain uniqueness.

(2)⇒(1). Let $(\mathcal{E}; V; \pi; \tau)$ be a doubly invariant linearisation of the kernel $\mathbf{k}$. By te proof of Theorem 3.2.7 we only have to show that $\mathbf{k}$ is invariant under the right action of Δ on X and Z. Let $x, y \in X$ and $\alpha \in \Delta$. We have

$$\mathbf{k}(x, y\alpha) = [V(x), V(y\alpha)]_{\mathcal{E}} = [V(x), V(y)\tau(\alpha)]_{\mathcal{E}} = [V(x), V(y)]_{\mathcal{E}} \cdot \alpha = \mathbf{k}(x, y) \cdot \alpha$$

and the first condition holds. We also have

$$\begin{aligned}\sum_{i=1}^{n} s_i(\mathbf{k}(x, y_i \cdot \gamma)) &= \sum_{i=1}^{n} s_i([V(x), V(y_i)\tau(\gamma)]_{\mathcal{E}}) = [V(x), (\sum_{i=1}^{n} s_i V(y_i))\tau(\gamma)]_{\mathcal{E}} \\ &= (\sum_{i=1}^{n} s_i [V(x), V(y_i)]_{\mathcal{E}}) \cdot \gamma = (\sum_{i=1}^{n} s_i(\mathbf{k}(x, y_i))) \cdot \gamma\end{aligned}$$

for all $n \in \mathbb{N}$, $\{s_i\}_{i=1}^{n} \in \mathbb{C}$, $x, \{y_i\}_{i=1}^{n}$, $\gamma \in \Delta$ and the second condition is shown.

(2)⇒(3). Let $(\mathcal{E}; V; \pi; \tau)$ be a doubly invariant VE-space linearisation of the kernel $\mathbf{k}$ and the actions of Γ on X and Δ on X and Z. Without loss of generality we can assume that $\mathcal{E}$ is minimal. By the proof of Theorem 3.2.7 there exists a minimal reproducing kernel VE-space $(\mathcal{R};\rho)$. Define $\sigma(\gamma)$ for any $\gamma \in \Delta$ the following way: $\mathbf{k}_x\sigma(\gamma) = U((U^{-1}\mathbf{k}_x)\tau(\gamma))$ where $U : \mathcal{E} \to \mathcal{R}$ is the unitary operator in (2)⇒(3) of the proof of Theorem 3.2.7. It is easy to see that $\sigma(\gamma)$ gives a right module action on $\mathcal{R}$. Moreover, we have

$$\begin{aligned}[][\mathbf{k}_y, \mathbf{k}_x\sigma(\gamma)]_{\mathcal{R}} &= [\mathbf{k}_y, U((U^{-1}\mathbf{k}_x)\tau(\gamma))]_{\mathcal{R}} = [V(y), V(x)\tau(\gamma)]_{\mathcal{E}} \\ &= [V(y), V(x\cdot\gamma)]_{\mathcal{E}} = [\mathbf{k}_y, \mathbf{k}_{x\cdot\gamma}]_{\mathcal{R}}\end{aligned}$$

for all $x, y \in X$ and $\gamma \in \Delta$, hence, by minimality, it follows that $\mathbf{k}_x\sigma(\gamma) = \mathbf{k}_{x\cdot\gamma}$ for all $x \in X$ and $\gamma \in \Delta$.

(3)⇒(2). Let $(\mathcal{R};\rho;\sigma)$ be a Z-reproducing kernel VE-space $\mathcal{R}$ of $\mathbf{k}$ and $\rho\colon \Gamma \to \mathcal{L}^*(\mathcal{R})$ a $*$-representation such that $\rho(\xi)\mathbf{k}_x = \mathbf{k}_{\xi\cdot x}$ for all $\xi \in \Gamma$, $x \in X$ and a representation $\sigma\colon \Delta \to \mathcal{L}(\mathcal{R})$ turning $\mathcal{R}$ into a right module, and respecting the gramian, i.e.

$$[q, r\sigma(\gamma)]_{\mathcal{R}} = [q, r]_{\mathcal{R}} \cdot \gamma$$

for all $q, r \in \mathcal{R}$ and $\gamma \in \Delta$. By (2)⇒(3) of the proof of Theorem 3.2.7, $(\mathcal{R}; V; \pi)$ where $\pi = \rho$ is a minimal invariant linearisation of $\mathbf{k}$. Letting $\tau = \sigma$, it is easy to see that we obtain a minimal doubly invariant VE-space linearisation $(\mathcal{R}; V; \pi; \tau)$ of $\mathbf{k}$. □

Remark 3.2.12. By considering the right action of the trivial semigroup τ on the set X in Theorem 3.2.11 above we obtain Theorem 3.2.7.

Let Δ be a semigroup acting on the right to the topologically ordered $*$-space Z. By definition, the right action of Δ on Z is continuous if for any convergent net $(z_i)_{i\in I} \in Z$ with $\lim_i z_i = z$ and $a \in \Delta$ we have $\lim_i(z_i \cdot a) = z \cdot a$.

We now state and prove a topological version of Theorem 3.2.11, which is a generalization of Theorem 3.2.9 as it allows right actions, similar to the algebraic case as in Remark 3.2.12.

Theorem 3.2.13. *Assume that Γ is a $*$-semigroup that acts on the nonempty set X on the left, Δ is a semigroup that acts on the set X on the right, and also* *on an admissible space Z on the right, and that the right action of Δ on Z is continuous. Let $\mathbf{k}\colon X \times X \to Z$ be a Z-valued kernel. The following assertions are equivalent:*

(1) $\mathbf{k}$ *satisfies the following conditions:*

(a) $\mathbf{k}$ *is positive semidefinite.*

(b) $\mathbf{k}$ *is doubly invariant under the actions of Γ and Δ on X.*

(c) For any $\alpha \in \Gamma$ and any seminorm $p \in S(Z)$, there exists a seminorm $q \in S(Z)$ and a constant $c_p(\alpha) \geq 0$ such that there exists $c(\alpha) \geq 0$ such that

$$p(\sum_{j,k=1}^{n} t_j\overline{t_k}\mathbf{k}(\alpha \cdot x_k, \alpha \cdot x_j)) \leq c_p(\alpha)^2 q(\sum_{j,k=1}^{n} t_j\overline{t_k}\mathbf{k}(x_k, x_j)), \tag{3.37}$$

for $n \in \mathbb{N}$, all $x_1, \ldots, x_n \in X$, and all $t_1, \ldots, t_n \in \mathbb{C}$.

(d) For any $a \in \Delta$ and any seminorm $p \in S(Z)$, there exists a seminorm $q \in S(Z)$ and a constant $c_p(a) \geq 0$ such that

$$p(\sum_{j,k=1}^{n} t_j\overline{t_k}\mathbf{k}(x_k \cdot a, x_j \cdot a)) \leq c_p(a)^2 q(\sum_{j,k=1}^{n} t_j\overline{t_k}\mathbf{k}(x_k, x_j)), \tag{3.38}$$

for $n \in \mathbb{N}$, all $x_1, \ldots, x_n \in X$, and all $t_1, \ldots, t_n \in \mathbb{C}$.

(2) $\mathbf{k}$ *has a doubly invariant VH-space linearisation $(\mathcal{K}; V; \pi; \tau)$.*

(3) $\mathbf{k}$ *admits a doubly invariant reproducing kernel VH-space $(\mathcal{R}; \rho; \sigma)$.*

Moreover, in case any of the assertions (1), (2), *or* (3) *holds, then a topologically minimal doubly invariant linearisation can be constructed, any minimal doubly invariant linearisation is unique up to unitary equivalence, and the triple $(\mathcal{R}; \rho; \sigma)$ as in assertion* (3) *is uniquely determined by* $\mathbf{k}$ *as well.*

Proof. (1)$\Rightarrow$(2). By the proof of Theorem 3.2.10, there exists a left invariant VH-space linearisation $(\mathcal{K}; V; \pi)$, where the main constructions are, with notation as in Subsection 3.1 and in the proof of Theorem 3.2.11, as follows: The VH-space $\mathcal{K}$

is the VH-space completion of the topological VE-space $\mathcal{E}$ as defined by equations (3.10) and (3.11). The consructions of $V(x) \in \mathcal{K}$ for any $x \in X$ and $\pi(\alpha) \in \mathcal{L}_c^*(\mathcal{K})$ are similar, with the difference that using the boundedness condition 1.(c), an operator $\pi(\alpha)$ is uniquely extended from $\mathcal{E}$ to $\mathcal{K}$, to a continuous $*$-representation of the semigroup Γ.

We prove that the right action of $\tau(a)$ on the space $\mathcal{E}$ is continuous for all $a \in \Delta$. Let $f = Kg$ for some $g \in \mathbb{C}_0^X$. By definition of $\tau(a)$ and condition (d), we have

$$\begin{aligned} p([f\tau(a), f\tau(a)]_{\mathcal{E}}) &= p(\sum_{x,y \in X} \overline{g(x)} g(y) \mathbf{k}(x \cdot a, y \cdot a)) \\ &\leq c_p(a)^2 q(\sum_{x,y \in X} \overline{g(x)} g(y) \mathbf{k}(x, y)) \\ &= c_p(a)^2 q([f, f]_{\mathcal{E}}) \end{aligned}$$

where $p, q \in S(Z)$ and the continuity of the right action of $\tau(\gamma)$ is proven. Therefore the right action of $\tau(\gamma)$ can be uniquely extended to the right action of an operator $\mathcal{L}_c(\mathcal{K})$ on $\mathcal{K}$. By uniqueness, it follows that $\tau(a_1 a_2) = \tau(a_1)\tau(a_2)$ for all $a_1, a_2 \in \Delta$. Hence τ is a representation of the semigroup Δ on $\mathcal{L}_c(\mathcal{K})$.

To prove that the right action of $\tau(a)$ for each $a \in \Delta$ respects the gramian, let $k, l \in \mathcal{K}$ and $a \in \Delta$ be arbitrary elements and let $(k_i)_{i \in I}$ and $(l_j)_{j \in J}$ be nets in $\mathcal{E}$ such that $k_i \to k$ and $l_j \to l$. Then by the continuity of the right action of Δ on Z and the continuity of the gramian we have

$$\begin{aligned} [k, l\tau(a)] =& \lim_i [k_i, l\tau(a)] = \lim_i (\lim_j [k_i, l_j \tau(a)]_{\mathcal{E}}) \\ &= (\lim_i (\lim_j [k_i, l_j]_{\mathcal{E}})) \cdot a = \lim_i [k_i, l]_{\mathcal{K}} \cdot a = [k, l]_{\mathcal{K}} \cdot a \end{aligned}$$

and by uniqueness of limits we have $[k, l\tau(a)] = [k, l] \cdot a$ for any $k, l \in \mathcal{K}$ and $a \in \Delta$.

The uniqueness of minimal doubly invariant VH-space linearisation follows as usually.

(2)⇒(1). Let $(\mathcal{K}; V; \pi; \tau)$ be a doubly invariant VH-space linearisation of the kernel $\mathbf{k}$. By the proof of Theorem 3.2.9 and Theorem 3.2.11 we only need to show that the boundedness condition (d) holds. Let $a \in \Delta$, $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$, $t_1, \ldots, t_n \in \mathbb{C}$, $p \in S(Z)$ be arbitrary. Then

$$\begin{aligned} p(\sum_{j,k=1}^{n} \overline{t_k} t_j \mathbf{k}(x_k \cdot a, x_j \cdot a)) &= p(\sum_{j,k=1}^{n} \overline{t_k} t_j [V(x_k)\tau(a), V(x_j)\tau(a)]_{\mathcal{K}}) \\ &= p([\sum_{k=1}^{n} t_k V(x_k)\tau(a), \sum_{j=1}^{n} t_j V(x_j)\tau(a)]_{\mathcal{K}}) \\ &\leq c_p(a)^2 q([\sum_{k=1}^{n} t_k V(x_k), \sum_{j=1}^{n} t_j V(x_j)]_{\mathcal{K}}) \\ &= c_p(a)^2 q(\sum_{j,k=1}^{n} \overline{t_k} t_j \mathbf{k}(x_k, x_j)), \end{aligned}$$

for some $q \in S(Z)$ by the continuity of $\tau(a)$ and (d) holds with $c_p(a) \geq 0$.

(2)⇒(3). Let $(\mathcal{K}; V; \pi; \tau)$ be a doubly invariant VH-space linearisation of the kernel $\mathbf{k}$ and the actions of Γ on X and Δ on X and Z. Without loss of generality we can assume that it is minimal.

Define $\mathbf{k}_x \sigma(a) = U(U^{-1}\mathbf{k}_x)\tau(a)$ for any $a \in \Delta$ where $U : \mathcal{K} \to \mathcal{R}$ is the unitary operator in (2)⇒(3) of the proof of Theorem 3.2.9. A similar calculation as in (2)⇒(3) of the proof of Theorem 3.2.11 and minimality shows that $\mathbf{k}_x \sigma(a) = \mathbf{k}_{x \cdot a}$ for all $x \in X$ and $a \in \Delta$. Moreover, σ is a representation of Δ in $\mathcal{L}_c(\mathcal{K})$. It is easy to see that σ gives a right module action on $\mathcal{R}$.

(3)⇒(2). Let $(\mathcal{R}; \rho; \sigma)$ be a doubly invariant Z-reproducing kernel VH-space $\mathcal{R}$ of $\mathbf{k}$ and the actions of Γ and Δ. By (3)⇒(2) of the proof of Theorem 3.2.9, $(\mathcal{R}; \pi)$, where $\pi = \rho$ is a minimal left invariant linearisation of $\mathbf{k}$. Letting $\tau = \sigma$, it is easy to see that we obtain a minimal doubly invariant VH-space linearisation $(\mathcal{R}; \pi; \tau)$ of $\mathbf{k}$. □

Remark 3.2.14. By considering the right action of the trivial semigroup τ on the set X and the admissible space Z in Theorem 3.2.13 above we recover Theorem 3.2.9.

The following theorem is a version of Theorem 3.2.11 for kernels valued in topologically ordered $*$-spaces. Its proof is similar to the proof of Theorem 3.2.11.

Theorem 3.2.15. *Assume that Γ is a $*$-semigroup that acts on the nonempty set X on the left, Δ is a semigroup that acts on the set X on the right, and also on a topologically ordered $*$ space Z on the right. Let $\mathbf{k}\colon X \times X \to Z$ be a Z-valued kernel. The following assertions are equivalent:*

(1) $\mathbf{k}$ *satisfies the following conditions:*

(a) $\mathbf{k}$ *is positive semidefinite.*

(b) $\mathbf{k}$ *is doubly invariant under the actions of Γ and Δ on X.*

(c) For any $\alpha \in \Gamma$ and any seminorm $p \in S(Z)$, there exists a seminorm $q \in S(Z)$ and a constant $c_p(\alpha) \geq 0$ such that there exists $c(\alpha) \geq 0$ such that

$$p(\sum_{j,k=1}^{n} t_j \overline{t_k} \mathbf{k}(\alpha \cdot x_k, \alpha \cdot x_j)) \leq c_p(\alpha)^2 q(\sum_{j,k=1}^{n} t_j \overline{t_k} \mathbf{k}(x_k, x_j)), \tag{3.39}$$

for $n \in \mathbb{N}$, all $x_1, \ldots, x_n \in X$, and all $t_1, \ldots, t_n \in \mathbb{C}$.

(d) For any $a \in \Delta$ and any seminorm $p \in S(Z)$, there exists a seminorm $q \in S(Z)$ and a constant $c_p(a) \geq 0$ such that

$$p(\sum_{j,k=1}^{n} t_j \overline{t_k} \mathbf{k}(x_k \cdot a, x_j \cdot a)) \leq c_p(a)^2 q(\sum_{j,k=1}^{n} t_j \overline{t_k} \mathbf{k}(x_k, x_j)), \tag{3.40}$$

for $n \in \mathbb{N}$, all $x_1, \ldots, x_n \in X$, and all $t_1, \ldots, t_n \in \mathbb{C}$.

(2) $\mathbf{k}$ *has a doubly invariant topological VE-space linearisation $(\mathcal{K}; V; \pi; \tau)$ where $\pi\colon \Gamma \to \mathcal{L}_c^*(\mathcal{K})$ and $\tau\colon \Delta \to \mathcal{L}_c(\mathcal{K})$.*

(3) $\mathbf{k}$ *admits a doubly invariant reproducing kernel topological VE-space $(\mathcal{R}; \rho; \sigma)$ where $\rho\colon \Gamma \to \mathcal{L}_c^*(\mathcal{R})$ and $\tau\colon \Delta \to \mathcal{L}_c(\mathcal{R})$.*

Moreover, in case any of the assertions (1), (2), *or* (3) *holds, then a topologically minimal doubly invariant linearisation can be constructed, any minimal doubly invariant linearisation is unique up to unitary equivalence, and the triple $(\mathcal{R}; \rho; \sigma)$ as in assertion* (3) *is uniquely determined by* $\mathbf{k}$ *as well.*

Chapter 4

Applications of the Main Dilation Theorems I: Dilations of Operator Valued Left Invariant Positive Semidefinite Kernels

4.1 Invariant Kernels with Values Adjointable Operators.

We show that Theorem 2.8 in [22] can be seen as a special case of Theorem 3.2.7. We first recall necessary definitions from [22].

In this subsection we will consider a kernel on a nonempty set X and taking values in $\mathcal{L}^*(\mathcal{H})$, for a VE-space $\mathcal{H}$ over an ordered $*$-space Z, that is, a map $\mathbf{l}\colon X \times X \to \mathcal{L}^*(\mathcal{H})$.

A kernel $\mathbf{l}\colon X \times X \to \mathcal{L}^*(\mathcal{H})$ is called *positive semidefinite* if for all $n \in \mathbb{N}$,

$x_1, x_2, \cdots, x_n \in X$, and $h_1, h_2, \cdots, h_n \in \mathcal{H}$, we have

$$\sum_{i,j=1}^{n} [\mathbf{l}(x_i, x_j)h_j, h_i]_{\mathcal{H}} \geq 0. \tag{4.1}$$

An *invariant* $\mathcal{L}^*(\mathcal{H})$*-valued VE-space linearisation* of a kernel $\mathbf{l}$ and an action of a $*$-semigroup Γ on X is, by definition, a triple $(\tilde{\mathcal{E}}; \tilde{\pi}; \tilde{V})$ such that

(hvel1) $\tilde{\mathcal{E}}$ is a VE-space over the same ordered $*$-space Z,

(hvel2) $\tilde{\pi} \colon \Gamma \to \mathcal{L}^*(\tilde{\mathcal{E}})$ is a $*$-representation,

(hvel3) $\tilde{V} \colon X \to \mathcal{L}^*(\mathcal{H}, \tilde{\mathcal{E}})$, satisfying $\mathbf{k}(x, y) = \tilde{V}(x)^* \tilde{V}(y)$ for all $x, y \in X$ and $\tilde{V}(\xi \cdot x) = \tilde{\pi}(\xi)\tilde{V}(x)$ for all $x \in X$, $\xi \in \Gamma$.

If an invariant $\mathcal{L}^*(\mathcal{H})$-valued VE-space linearisation has the property that $\operatorname{Lin} V(X)\mathcal{H} = \tilde{\mathcal{E}}$, then it is called *minimal*. Two invariant $\mathcal{L}^*(\mathcal{H})$-VE-space linearisations $(\tilde{\mathcal{E}}; \tilde{\pi}; \tilde{V})$ and $(\tilde{\mathcal{F}}; \tilde{\rho}; \tilde{W})$ of the same kernel $\mathbf{l}$ are called *unitarily equivalent* if there exists a unitary operator $U : \tilde{\mathcal{E}} \to \tilde{\mathcal{F}}$ such that $U\tilde{\pi}(\gamma) = \tilde{\rho}(\gamma)U$ for all $\gamma \in \Gamma$ and $U\tilde{V}(x) = \tilde{W}(x)$ for all $x \in X$.

Let $\mathcal{H}^X$ be the vector space of all maps $f \colon X \to \mathcal{H}$, for a nonempty set X and a VE-space $\mathcal{H}$ over the ordered $*$-space Z. A VE-space $\tilde{\mathcal{R}}$ over the same ordered $*$-space Z is called a $\mathcal{L}^*(\mathcal{H})$*-reproducing kernel VE-space* on X of the kernel $\mathbf{l}$ if

(hrk1) $\tilde{\mathcal{R}}$ is a vector subspace of $\mathcal{H}^X$.

(hrk2) For all $x \in X$ and $h \in \mathcal{H}$, the $\mathcal{H}$-valued function $\mathbf{l}_x h := \mathbf{l}(\cdot, x)h$ belongs to $\tilde{\mathcal{R}}$.

(hrk3) For all $f \in \tilde{\mathcal{R}}$ we have $[f(x), h]_{\mathcal{H}} = [f, \mathbf{l}_x h]_{\tilde{\mathcal{R}}}$ for all $x \in X$ and $h \in \mathcal{H}$.

The space $\tilde{\mathcal{R}}$ is *minimal* if $\tilde{\mathcal{R}} = \operatorname{Lin}\{\mathbf{l}_x h \mid x \in X,\ h \in \mathcal{H}\}$.

Theorem 4.1.1 (Theorem 2.8 in [22]). *Let Γ be a $*$-semigroup acting on a nonempty set X, $\mathcal{H}$ be a VE-space on an ordered $*$-space Z, and $\mathbf{l} \colon X \times X \to \mathcal{L}^*(\mathcal{H})$ be a kernel. The following assertions are equivalent:*

(1) $\mathbf{l}$ *satisfies the following properties:*

(a) $\mathbf{l}$ *is positive semidefinite.*

(b) $\mathbf{l}$ *is invariant under the action of* Γ *on* X*.*

(2) $\mathbf{l}$ *has an invariant* $\mathcal{L}^*(\mathcal{H})$*-valued VE-space linearisation* $(\tilde{\mathcal{E}};\tilde{\pi};\tilde{V})$*.*

(3) $\mathbf{l}$ *admits a* $\mathcal{L}^*(\mathcal{H})$*-reproducing kernel VE-space* $\tilde{\mathcal{R}}$ *and there exists a* $*$*-representation* $\tilde{\rho}:\Gamma\to\mathcal{L}^*(\tilde{\mathcal{R}})$ *such that* $\tilde{\rho}(\xi)\mathbf{l}_x h=\mathbf{l}_{\xi\cdot x}h$ *for all* $\xi\in\Gamma$*,* $x\in X$*,* $h\in\mathcal{H}$*.*

Moreover, in case any of the assertions (1), (2) or (3) holds, a minimal invariant $\mathcal{L}^*(\mathcal{H})$*-VE-space linearisation can be constructed, and a pair* $(\tilde{\mathcal{R}};\tilde{\rho})$ *as in (3) with* $\tilde{\mathcal{R}}$ *can be always obtained as well.*

Proof. (1)⇒(2). Define a kernel $\mathbf{k}\colon (X\times\mathcal{H})\times(X\times\mathcal{H})\to Z$ by

$$\mathbf{k}((x,h),(y,g)) := [\mathbf{l}(y,x)h,g]_{\mathcal{H}},\quad x,y\in X,\ h,g\in\mathcal{H}.$$

Since $\mathbf{l}$ is semipositive definite in the sense of (4.1), $\mathbf{k}$ is weakly positive semidefinite:

$$\sum_{k,j=1}^{n}\overline{t_k}t_j\mathbf{k}((x_k,h_k),(x_j,h_j))=\sum_{k,j=1}^{n}\overline{t_k}t_j[\mathbf{l}(x_j,x_k)h_k,h_j]=\sum_{k,j=1}^{n}[\mathbf{l}(x_j,x_k)t_kh_k,t_jh_j]\geq 0$$

for all $n\in\mathbb{N}$, $\{x_j\}_{j=1}^n\in X$, $\{h_j\}_{j=1}^n\in\mathcal{H}$ and $\{t_j\}_{j=1}^n\in\mathbb{C}$.

Define an action of Γ on $(X\times\mathcal{H})$ in the following way: $\xi\cdot(x,h)=(\xi\cdot x,h)$ for all $\xi\in\Gamma$, $x\in X$ and $h\in\mathcal{H}$. Using the Γ invariance of $\mathbf{l}$ it follows that $\mathbf{k}$ is Γ invariant: letting $\xi\in\Gamma$, $x,y\in X$ and $g,h\in\mathcal{H}$ we have

$$\mathbf{k}(\xi\cdot(x,h),(y,g))=[\mathbf{l}(y,\xi\cdot x)h,g]=[\mathbf{l}(\xi^*\cdot y,x)h,g]=\mathbf{k}((x,h),\xi^*(y,g)).$$

By Theorem 3.2.7, there exists a minimal weak VE-space linearisation $(\mathcal{E};\pi;V)$ of $\mathbf{k}$ and the action of Γ on $(X\times\mathcal{H})$. By construction, see (3.14), it is clear that $V(x,h)$ depends linearly on $h\in\mathcal{H}$, therefore, for each $x\in X$ a linear operator of VE-spaces $\tilde{V}(x)\colon\mathcal{H}\to\mathcal{E}$ can be defined by $\tilde{V}(x)h=V(x,h)$.

We now have $[\tilde{V}(x)h, \tilde{V}(y)g]_{\mathcal{E}} = \mathbf{k}((x,h),(y,g)) = [\mathbf{l}(y,x)h, g]_{\mathcal{H}}$ for all $x, y \in X$ and $h, g \in \mathcal{H}$. By the minimality of $\mathcal{E}$, it follows that $\tilde{V}(x)$ is an adjointable operator with $\tilde{V}(y)^*\tilde{V}(x) = \mathbf{l}(y,x)$ for all $x, y \in X$.

On the other hand, we have $\pi(\xi)V(x,h) = V(\xi \cdot x, h) = \tilde{V}(\xi \cdot x)h$ for all $h \in \mathcal{H}$ and hence $\pi(\xi)\tilde{V}(x) = \tilde{V}(\xi \cdot x)$ for all $\xi \in \Gamma$ and $x \in X$, showing that $(\mathcal{E}; \pi; \tilde{V})$ is a minimal invariant $\mathcal{L}^*(\mathcal{H})$-valued VE-space linearisation of the kernel $\mathbf{l}$ and the action of Γ on X.

(2)⇒(3). Let $(\tilde{\mathcal{E}}; \tilde{\pi}; \tilde{V})$ be an invariant $\mathcal{L}^*(\mathcal{H})$-valued VE-space linearisation of the kernel $\mathbf{l}$, hence $\mathbf{l}(x,y) = \tilde{V}(x)^*\tilde{V}(y)$ for all $x, y \in X$. Define $V\colon (X \times \mathcal{H}) \to \tilde{\mathcal{E}}$ by

$$V(x,h) = \tilde{V}(x)h, \quad x \in X,\ h \in \mathcal{H}. \tag{4.2}$$

We also have

$$\tilde{\pi}(\xi)V(x,h) = \tilde{\pi}(\xi)\tilde{V}(x)h = \tilde{V}(\xi\cdot x)h = V(\xi\cdot x, h), \quad \xi \in \Gamma,\ x \in X,\ h \in \mathcal{H}, \tag{4.3}$$

hence $\tilde{\pi}(\xi)$ leaves $\tilde{\mathcal{E}}_0 = \operatorname{Lin} V(X, \mathcal{H})$ invariant for all $\xi \in \Gamma$. In the following, we denote by the same symbol $\tilde{\pi}\colon \Gamma \to \mathcal{L}^*(\tilde{\mathcal{E}}_0)$, the $*$-representation viewed as $\tilde{\pi}(\gamma)\colon \tilde{\mathcal{E}}_0 \to \tilde{\mathcal{E}}_0$ for all $\gamma \in \Gamma$. Then $(\tilde{\mathcal{E}}_0; \tilde{\pi}; V)$ is a minimal invariant weak VE-space linearisation for the kernel $\mathbf{k}\colon (X \times \mathcal{H}) \times (X \times \mathcal{H}) \to Z$ defined by

$$\begin{aligned}
\mathbf{k}((x,h),(y,g)) &= [V(x,h), V(y,g)]_{\tilde{\mathcal{E}}} \\
&= [\tilde{V}(x)h, \tilde{V}(y)g]_{\tilde{\mathcal{E}}} = [h, \tilde{V}(x)^*\tilde{V}(y)g]_{\tilde{\mathcal{E}}} \\
&= [h, \mathbf{l}(x,y)g]_{\mathcal{H}}, \quad x, y \in X,\ h, g \in \mathcal{H},
\end{aligned}$$

and the action of Γ on $(X \times \mathcal{H})$ given by

$$\xi \cdot (x,h) = (\xi \cdot x, h), \quad \xi \in \Gamma,\ x \in X,\ h \in \mathcal{H}. \tag{4.4}$$

By Theorem 3.2.7, there exists a minimal weak Z-reproducing kernel VE-space $\mathcal{R} \subseteq Z^{X \times \mathcal{H}}$, with reproducing kernel $\mathbf{k}$, and a $*$-representation $\rho\colon \Gamma \to \mathcal{L}^*(\mathcal{R})$ such that $\rho(\xi)\mathbf{k}_{(x,h)} = \mathbf{k}_{\xi\cdot(x,h)}$ for all $\xi \in \Gamma$, $x \in X$, $h \in \mathcal{H}$. As the proof of Theorem 3.2.7 shows, without loss of generality we can assume that $\mathcal{R}$ is the collection of all maps $X \times \mathcal{H} \to Z$ defined by $X \times \mathcal{H} \ni (x,h) \mapsto [\tilde{V}(x)h, f]_{\tilde{\mathcal{E}}}$, where $f \in \tilde{\mathcal{E}}_0$, which provides an identification of $\mathcal{R}$ with $\tilde{\mathcal{E}}_0$ by the formula

$$f(x,h) = [V(x,h), f]_{\mathcal{R}} = [\tilde{V}(x)h, f]_{\tilde{\mathcal{E}}} = [h, \tilde{V}(x)^* f]_{\mathcal{H}}, \quad h \in \mathcal{H}. \tag{4.5}$$

Consequently, for each $f \in \mathcal{R}$ and $x \in X$, there exists a unique vector $\tilde{f}(x) = \tilde{V}(x)^* f \in \mathcal{H}$ such that

$$f(x,h) = [h, \tilde{f}(x)]_{\mathcal{H}}, \quad h \in \mathcal{H}, \tag{4.6}$$

which gives rise to a map $\mathcal{R} \ni f \mapsto \tilde{f} \in \mathcal{H}^X$. Let $\tilde{\mathcal{R}}$ be the vector space of all $\tilde{f}$, for $f \in \mathcal{R}$. Since, by the reproducing property of the kernel $\mathbf{k}$ and (4.6) we have

$$[\mathbf{k}_{(x,h)}, \mathbf{k}_{(y,g)}]_{\mathcal{R}} = \mathbf{k}_{(y,g)}(x,h) = [h, \widetilde{\mathbf{k}_{(y,g)}}(x)]_{\mathcal{H}}, \quad h, g \in \mathcal{H},\ x, y \in X,$$

taking into account the reproducing property of the kernel $\mathbf{l}$, it follows that $\mathbf{l}_x h = \widetilde{\mathbf{k}_{(x,h)}} \in \tilde{\mathcal{R}}$ for all $x \in X$ and $h \in \mathcal{H}$.

It is easy to see that the map $U \colon \mathcal{R} \ni f \to \tilde{f} \in \tilde{\mathcal{R}}$ is linear, one-to-one, and onto. Therefore, defining $[\tilde{f}, \tilde{g}]_{\tilde{\mathcal{R}}} := [f, g]_{\mathcal{R}}$ makes $\tilde{\mathcal{R}}$ a VE-space, and U becomes a unitary operator of VE-spaces. Defining $\tilde{\rho} := U \rho U^*$, the pair $(\tilde{\mathcal{R}}, \tilde{\rho})$ has all the required properties.

(3)⇒(1). Assume that $(\tilde{\mathcal{R}}; \tilde{\rho})$ is a pair consisting of an $\mathcal{L}^*(\mathcal{H})$-reproducing kernel VE-space of $\mathbf{l}$ and a $*$-representation $\tilde{\rho} : \Gamma \to \mathcal{L}^*(\tilde{\mathcal{R}})$ such that $\rho(\xi)\mathbf{l}_x h = \mathbf{l}_{\xi \cdot x} h$ for all $\xi \in \Gamma$, $x \in X$, $h \in \mathcal{H}$. We have

$$\begin{aligned} \sum_{i,j=1}^{n} [\mathbf{l}(x_i, x_j) h_j, h_i] &= \sum_{i,j=1} [\mathbf{l}_{x_j} h_j(x_i), h_i] = \sum_{i,j=1}^{n} [\mathbf{l}_{x_j} h_j, \mathbf{l}_{x_i} h_i] \\ &= [\sum_{j=1} \mathbf{l}_{x_j} h_j, \sum_{i=1} \mathbf{l}_{x_i} h_i] \geq 0 \end{aligned}$$

for all $n \in \mathbb{N}$, $\{x_i\}_{i=1}^n \in X$, $\{h_i\}_{i=1}^n \in \mathcal{H}$. Therefore $\mathbf{l}$ is positive semidefinite in the sense of (4.1). Moreover, by (hrk3)

$$\begin{aligned} [\mathbf{l}(x, \xi \cdot y) h, g] &= [\mathbf{l}_{\xi \cdot y} h(x), g] = [\tilde{\rho}(\xi) \mathbf{l}_y h(x), g] \\ &= [\tilde{\rho}(\xi) \mathbf{l}_y h, \mathbf{l}_x g] = [\mathbf{l}_y h, \tilde{\rho}(\xi^*) \mathbf{l}_x g] = [\mathbf{l}(\xi^* x, y) h, g], \end{aligned}$$

for all $x, y \in X$ and $g, h \in \mathcal{H}$, and the invariance of the kernel $\mathbf{l}$ is proven. □

Remark 4.1.2. The crucial point in the proof of the implication (2)⇒(3) of Theorem 4.1.1 is the proof of (4.6) which we obtained as a consequence of the identification of $\mathcal{R}$ with $\tilde{\mathcal{E}}_0$. In the following we show that there is a direct proof of (4.6), without using this identification.

By minimality, $\mathcal{R} = \mathrm{Lin}\{\mathbf{k}_{(x,h)} \mid x \in X,\ h \in \mathcal{H}\}$ so let $f = \sum_{j=1}^n \alpha_j \mathbf{k}_{(y_j,g_j)}$ for some $n \in \mathbb{N}$, $y_1, \ldots, y_n \in X$, $g_1, \ldots, g_n \in \mathcal{H}$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, be an arbitrary element $f \in \mathcal{R}$. Then, for any $x \in X$ and $h \in \mathcal{H}$, we have

$$
\begin{aligned}
f(x,h) &= \sum_{j=1}^n \alpha_j \mathbf{k}_{(y_j,g_j)}(x,h) = \sum_{j=1}^n \alpha_j \mathbf{k}((x,h),(y_j,g_j)) = \sum_{j=1}^n \alpha_j [V(x,h), V(y_j,g_j)]_{\tilde{\mathcal{E}}} \\
&= \sum_{j=1}^n \alpha_j [\tilde{V}(x)h, \tilde{V}(y_j)g_j]_{\tilde{E}} = \sum_{j=1}^n \alpha_j [h, \tilde{V}(x)^* \tilde{V}(y_j)g_j]_{\mathcal{H}} \\
&= [h, \tilde{V}(x)^* \sum_{j=1}^n \alpha_j \tilde{V}(y_j)g_j]_{\mathcal{H}},
\end{aligned}
$$

hence, letting $\tilde{f}(x) = V(x)^* \sum_{j=1}^n \alpha_j \tilde{V}(y_j)g_j$, (4.6) holds.

4.1.1 Invariant Kernels with Values Continuously Adjointable Operators.

In this subsection we show that Theorem 2.10 in [24] can be recovered as a special case of Theorem 3.2.10. We first review definitions in [24] that we will use in this subsection.

Let X be a nonempty set and let $\mathcal{H}$ be a VH-space over an admissible space Z. In this subsection we will consider kernels $\mathbf{k}\colon X \times X \to \mathcal{L}_{\mathrm{c}}^*(\mathcal{H})$. Such a kernel k is called *positive semidefinite* if it is n-positive for all natural numbers n, in the sense of (4.1).

A $\mathcal{L}_{\mathrm{c}}^*(\mathcal{H})$*-valued VH-space linearisation* of $\mathbf{k}$, or $\mathcal{L}_{\mathrm{c}}^*(\mathcal{H})$*-valued VH-space Kolmogorov decomposition* of $\mathbf{k}$, is a pair $(\mathcal{K}; V)$, subject to the following conditions:

(vhl1) $\mathcal{K}$ is a VH-space over Z.

(vhl2) $V\colon X \to \mathcal{L}_{\mathrm{c}}^*(\mathcal{H}, \mathcal{K})$ satisfies $\mathbf{k}(x, y) = V(x)^* V(y)$ for all $x, y \in X$.

$(\mathcal{K}; V)$ is called *topologically minimal* if

(vhl3) $\operatorname{Lin} V(X)\mathcal{H}$ is dense in $\mathcal{K}$.

We call $\mathbf{k}$ Γ-*invariant* if

$$\mathbf{k}(\xi \cdot x, y) = \mathbf{k}(x, \xi^* \cdot y), \quad \xi \in \Gamma,\ x, y \in X. \tag{4.7}$$

A triple $(\mathcal{K}; \pi; V)$ is called a Γ-*invariant* $\mathcal{L}_{\mathrm{c}}^*(\mathcal{H})$-*valued VH-space linearisation* for $\mathbf{k}$ if

(ihl1) $(\mathcal{K}; V)$ is an $\mathcal{L}_{\mathrm{c}}^*(\mathcal{H})$-valued VH-space linearisation of $\mathbf{k}$.

(ihl2) $\pi\colon \Gamma \to \mathcal{L}_{\mathrm{c}}^*(\mathcal{K})$ is a $*$-representation.

(ihl3) $V(\xi \cdot x) = \pi(\xi) V(x)$ for all $\xi \in \Gamma$ and all $x \in X$.

Also, $(\mathcal{K}; \pi; V)$ is *topologically minimal* if the $\mathcal{L}_{\mathrm{c}}^*(\mathcal{H})$-VH-space linearisation $(\mathcal{K}; V)$ is topologically minimal, that is, $\mathcal{K}$ is the closure of the linear span of $V(X)\mathcal{H}$.

A VH-space $\mathcal{R}$ over the ordered $*$-space Z is called a $\mathcal{L}_{\mathrm{c}}^*(\mathcal{H})$-*reproducing kernel VH-space on* X if there exists a Hermitian kernel $\mathbf{k}\colon X \times X \to \mathcal{L}_{\mathrm{c}}^*(\mathcal{H})$ such that the following axioms are satisfied:

(rkh1) $\mathcal{R}$ is a subspace of $\mathcal{H}^X$, with all algebraic operations.

(rkh2) For all $x \in X$ and all $h \in \mathcal{H}$, the $\mathcal{H}$-valued function $\mathbf{k}_x h = \mathbf{k}(\cdot, x) h \in \mathcal{R}$.

(rkh3) For all $f \in \mathcal{R}$ we have $[f(x), h]_{\mathcal{H}} = [f, \mathbf{k}_x h]_{\mathcal{R}}$, for all $x \in X$ and $h \in \mathcal{H}$.

(rkh4) For all $x \in X$ the evaluation operator $\mathcal{R} \ni f \mapsto f(x) \in \mathcal{H}$ is continuous.

In this operator valued setting, let us note the appearance of the axiom (rkh4) which makes a difference with classical cases, see [24] for some results pointing out its significance.

Theorem 4.1.3 (Theorem 2.10 in [24])**.** *Let Γ be a $*$-semigroup that acts on the nonempty set X and let $\mathbf{l}\colon X \times X \to \mathcal{L}_{\mathrm{c}}^*(\mathcal{H})$ be a kernel, for some VH-space $\mathcal{H}$ over an admissible space Z. Then the following assertions are equivalent:*

(1) $\mathbf{l}$ has the following properties:

(a) $\mathbf{l}$ is positive semidefinite, in the sense of (4.1), and invariant under the action of Γ on X, that is, (3.25) holds.

(b) For any $\xi \in \Gamma$ and any seminorm $p \in S(Z)$, there exists a seminorm $q \in S(Z)$ and a constant $c_p(\xi) \geq 0$ such that for all $n \in \mathbb{N}$, $\{h_i\}_{i=1}^n \in \mathcal{H}$, $\{x_i\}_{i=1}^n \in X$ we have

$$p(\sum_{i,j=1}^{n} [\mathbf{l}(\xi \cdot x_i, \xi \cdot x_j)h_j, h_i]_{\mathcal{H}}) \leq c_p(\xi)\, q(\sum_{i,j=1}^{n} [\mathbf{l}(x_i, x_j)h_j, h_i]_{\mathcal{H}}).$$

(c) For any $x \in X$ and any seminorm $p \in S(Z)$, there exists a seminorm $q \in S(Z)$ and a constant $c_p(x) \geq 0$ such that for all $n \in \mathbb{N}$, $\{y_i\}_{i=1}^n \in X$, $\{h_i\}_{i=1}^n \in \mathcal{H}$ we have

$$p(\sum_{i,j=1}^{n} [\mathbf{l}(x, y_i)h_i, \mathbf{l}(x, y_j)h_j]_{\mathcal{H}}) \leq c_p(x)\, q(\sum_{i,j=1}^{n} [\mathbf{l}(y_j, y_i)h_i, h_j]_{\mathcal{H}}).$$

(2) $\mathbf{l}$ has a Γ-invariant $\mathcal{L}_{\mathrm{c}}^*(\mathcal{H})$-valued VH-space linearisation $(\mathcal{K};\pi;V)$.

(3) $\mathbf{l}$ admits an $\mathcal{L}_{\mathrm{c}}^*(\mathcal{H})$-reproducing kernel VH-space $\mathcal{R}$ and there exists a $*$-representation $\rho\colon \Gamma \to \mathcal{L}_{\mathrm{c}}^*(\mathcal{R})$ such that $\rho(\xi)\mathbf{l}_x h = \mathbf{l}_{\xi\cdot x} h$ for all $\xi \in \Gamma$, $x \in X$, $h \in \mathcal{H}$.

In addition, in case any of the assertions (1), (2), or (3) holds, then a minimal Γ-invariant $\mathcal{L}_{\mathrm{c}}^*(\mathcal{H})$-valued VH-space linearisation of $\mathbf{l}$ can be constructed, and the pair $(\mathcal{R};\rho)$ as in assertion (3) can be chosen with $\mathcal{R}$ topologically minimal as well.

Proof. (1)⇒(2). Define the kernel $\mathbf{k}\colon (X\times\mathcal{H})\times(X\times\mathcal{H})\to Z$ by

$$\mathbf{k}((x,h),(y,g)) := [\mathbf{l}(y,x)h,g]_{\mathcal{H}},\quad x,y\in X,\ h,g\in\mathcal{H}.$$

As in the proof of the implication (1)⇒(2) of Theorem 4.1.4, $\mathbf{k}$ is weakly positive semidefinite and invariant under the action of Γ on $X\times\mathcal{H}$ given by $\xi\cdot(x,h) = (\xi\cdot x,h)$ for all $\xi\in\Gamma$, $x\in X$, $h\in\mathcal{H}$ In order to see that this kernel satisfies the property 1.(c) of Theorem 3.2.9, observe that for all $n\in\mathbb{N}$, $\{t_i\}_{i=1}^n\subset\mathbb{C}$, $\alpha\in\Gamma$, and $p\in S(Z)$, by assumption, see property 1.(b), there exists $q\in S(Z)$ and $c(\alpha)\geq 0$, we have

$$\begin{aligned}
p(\sum_{j,k=1}^n t_j\overline{t_k}\mathbf{k}(\alpha\cdot(x_k,h_k),\alpha(x_j,h_j))) &= p(\sum_{j,k=1}^n t_j\overline{t_k}[\mathbf{l}(\alpha\cdot x_j,\alpha\cdot x_k)h_k,h_j]) \\
&= p(\sum_{j,k=1}^n [\mathbf{l}(\alpha\cdot x_j,\alpha\cdot x_k)t_kh_k,t_jh_j]) \\
&\leq c(\alpha)^2 q(\sum_{j,k=1}^n [\mathbf{l}(x_j,x_k)t_kh_k,t_jh_j]) \\
&= c(\alpha)^2 q(\sum_{j,k=1}^n t_j\overline{t_k}\mathbf{k}((x_k,h_k),(x_j,h_j))).
\end{aligned}$$

By Theorem 3.2.9, there exists a minimal weak VH-space linearisation $(\mathcal{K};\pi;V)$ of $\mathbf{k}$ and the action of Γ on $(X\times\mathcal{H})$. Same arguments as in the proof of the implication (1)⇒(2) of Theorem 4.1.4 show that, for any $x\in X$, there exists an adjointable operator of VE-spaces $\tilde{V}(x)\colon\mathcal{H}\to\mathcal{K}_0$, given by $\tilde{V}(x)h\colon = V(x,h)$ for $x\in X$ and $h\in\mathcal{H}$, where $\mathcal{K}_0\colon = \operatorname{Lin}V(X)\mathcal{H}$, with the property that $\tilde{V}(x)^*\tilde{V}(y) = \mathbf{l}(x,y)$ for all $x,y\in X$. Arguing as in the proof of the implication (1)⇒(2) of Theorem 2.10 of [24], it follows that $\tilde{V}(x)\in\mathcal{L}^*_{\mathrm{c}}(\mathcal{H},\mathcal{K}_0)$. Now using the boundedness condition (c), for any $p\in S(Z)$ there exist $q\in S(Z)$ and $c_p(x)\geq 0$ such that, for all $\sum_{i=1}^n V(y_i)h_i\in\mathcal{K}_0$ we have

$$\begin{aligned}
p([V(x)^*(\sum_{i=1}^n V(y_i)h_i),V(x)^*(\sum_{i=1}^n V(y_i)h_i)]_{\mathcal{H}}) &= p([\sum_{i=1}^n \mathbf{l}(x,y_i)h_i,\sum_{i=1}^n \mathbf{l}(x,y_i)h_i]_{\mathcal{H}}) \\
&\leq c_p(x)q(\sum_{i,j=1}^n [\mathbf{l}(y_j,y_i)h_i,h_j]_{\mathcal{H}}) \\
&= c_p(x)\,q([\sum_{i=1}^n V(y_i)h_i,\sum_{i=1}^n V(y_i)h_i]_{\mathcal{K}_0})
\end{aligned}$$

hence $\tilde{V}(x)^* \in \mathcal{L}_c^*(\mathcal{K}_0, \mathcal{H})$ for any $x \in X$. Consequently, $\tilde{V}(x)^*$ extends uniquely to an operator $\tilde{V}(x)^* \in \mathcal{L}_c^*(\mathcal{K}, \mathcal{H})$ for each $x \in X$. It follows that $(\mathcal{K}; \pi; \tilde{V})$ is an invariant $\mathcal{L}_c^*(\mathcal{H})$-valued VH-space linearisation of the kernel $\mathbf{l}$ and the action of Γ on X.

(2)⇒(3). Let $(\tilde{\mathcal{K}}; \tilde{\pi}; \tilde{V})$ be an invariant $\mathcal{L}_c^*(\mathcal{H})$-valued VH-space linearisation of the kernel $\mathbf{l}$. In order to avoid repetition, we use some facts obtained during the proof of the implication (2)⇒(3) of Theorem 4.1.1. Define $V\colon (X \times \mathcal{H}) \to \tilde{\mathcal{K}}$ by $V(x,h) = \tilde{V}(x)h$ for all $x \in X$ and $h \in \mathcal{H}$. Letting $\tilde{\mathcal{K}}_0 = \overline{\operatorname{Lin} V(X, \mathcal{H})} \subseteq \tilde{\mathcal{K}}$, similarly we see that $(\overline{\operatorname{Lin} V(X, \mathcal{H})}; \tilde{\pi}_0; V)$ is a topological minimal invariant weak VH-space linearisation for the kernel $\mathbf{k}\colon (X \times \mathcal{H}) \times (X \times \mathcal{H}) \to Z$ defined by $\mathbf{k}((x,h),(y,g)) = [V(x,h), V(y,g)]$ for all $x, y \in X$ and $h, g \in \mathcal{H}$ and the action of Γ on $(X \times \mathcal{H})$ defined by $\xi \cdot (x,h) = (\xi \cdot x, h)$ for all $\xi \in \Gamma$, $x \in X$ and $h \in \mathcal{H}$.

By Theorem 3.2.9 there exists a topologically minimal weak Z-reproducing kernel VH-space $\mathcal{R}$ and a $*$-representation $\rho\colon \Gamma \to \mathcal{L}_c^*(\mathcal{R})$ such that $\rho(\xi)\mathbf{k}_{(x,h)} = \mathbf{k}_{\xi\cdot(x,h)}$ for all $\xi \in \Gamma$, $x \in X$, $h \in \mathcal{H}$. The rest of the proof is similar with the end of the proof of the implication (2)⇒(3) as in Theorem 4.1.1. We show that, for each $f \in \mathcal{R}$ and $x \in X$ there exists a unique element $\tilde{f}(x)$ such that (4.6) holds and, consequently, this gives rise to a map $\mathcal{R} \ni f \mapsto Uf = \tilde{f} \in \mathcal{H}^X$, which is linear and bijective between $\mathcal{R}$ and its range $\tilde{\mathcal{R}} \subseteq \mathcal{H}^X$. Letting $[\tilde{f}, \tilde{g}]_{\tilde{\mathcal{R}}} = [f, g]_{\mathcal{R}}$ for all $f, g \in \mathcal{R}$, $\tilde{\mathcal{R}}$ becomes an $\mathcal{H}$-valued reproducing kernel VH-space with kernel $\mathbf{l}$, and then letting $\tilde{\rho}\colon = U\rho U^*$, $(\tilde{\mathcal{R}}, \tilde{\rho})$ is a pair having all the required properties.

(3)⇒(1). Assume that the pair $(\tilde{\mathcal{R}}; \tilde{\rho})$ consists of an $\mathcal{L}_c^*(\mathcal{H})$-valued reproducing kernel VH-space of $\mathbf{l}$ and a $*$-representation $\tilde{\rho}$ of Γ on $\mathcal{L}_c^*(\tilde{\mathcal{R}})$ such that $\rho(\xi)\mathbf{l}_x h = \mathbf{l}_{\xi\cdot x} h$ for all $\xi \in \Gamma$, $x \in X$, $h \in \mathcal{H}$. Similarly as in the proof of the implication (3)⇒(1) of Theorem 4.1.1, the kernel $\mathbf{l}$ is shown to be positive semidefinite and invariant under the action of Γ on X. On the other hand, the inequalities (b) and (c) are obtained from the continuity of the operator $\rho(\xi)\colon \mathcal{R} \to \mathcal{R}$, for any $\xi \in \Gamma$ and, respectively, from the continuity of the evaluation operator $E_x\colon \mathcal{R} \to \mathcal{H}$, for any $x \in X$. □

4.1.2 Invariant Kernels with Values Boundedly Adjointable Operators.

We show that Theorem 4.2 in [23] is a special case of Theorem 3.2.9. We review necessary definitions in [23].

Given a $\mathcal{B}^*(\mathcal{H})$-valued kernel $\mathbf{l}$ on a nonempty set X, where $\mathcal{H}$ is a VH-space over the admissible space Z, a *$\mathcal{B}^*(\mathcal{H})$-valued VH-space linearisation* of $\mathbf{l}$ is a pair $(\tilde{\mathcal{K}};\tilde{V})$ with

(hvhl1) $\tilde{\mathcal{K}}$ is a VH-space over Z.

(hvhl2) $\tilde{V}\colon X \to \mathcal{B}^*(\mathcal{H};\mathcal{K})$ satisfies $\mathbf{l}(x,y) = \tilde{V}(x)^*\tilde{V}(y)$ for all $x, y \in X$.

If Γ is a $*$-semigroup acting on X, $(\tilde{\mathcal{K}};\tilde{\pi};\tilde{V})$ is called an *invariant $\mathcal{B}^*(\mathcal{H})$-valued VH-space linearisation* of the kernel $\mathbf{l}$ and the action of Γ on X, if, in addition to (hvhl1) and (hvhl2), we have,

(hvhl3) $\tilde{\pi}\colon \Gamma \to \mathcal{B}^*(\tilde{\mathcal{K}})$ is a $*$-representation.

(hvhl4) $\tilde{V}(\xi \cdot x) = \tilde{\pi}(\xi)V(x)$ for every $\xi \in \Gamma$, $x \in X$.

If we have

(hvhl5) $\mathrm{Lin}\tilde{V}(X)\mathcal{H}$ is dense in $\tilde{\mathcal{K}}$,

then $(\tilde{\mathcal{K}};\tilde{\pi};\tilde{V})$ is called *topologically minimal*.

Given a nonempty set X and a VH-space $\mathcal{H}$ over the admissible space Z, a VH-space $\tilde{\mathcal{R}}$ over Z is called a *$\mathcal{B}^*(\mathcal{H})$-valued reproducing kernel VH-space* on X if there exists a kernel $\mathbf{l}\colon X \times X \to \mathcal{B}^*(\mathcal{H})$ such that

(hrk1) $\tilde{\mathcal{R}}$ is a subspace of $\mathcal{H}^X$ with all algebraic operations.

(hrk2) $\mathbf{l}_x h = \mathbf{l}(\cdot, x)h \in \tilde{\mathcal{R}}$ for all $x \in X$, $h \in \mathcal{H}$.

(hrk3) $[f(x), h]_{\mathcal{H}} = [f, \mathbf{l}_x h]_{\tilde{\mathcal{R}}}$ holds for all $f \in \tilde{\mathcal{R}}$, $x \in X$ and $h \in \mathcal{H}$.

$\tilde{\mathcal{R}}$ is called *topologically minimal* if

(hrk4) $\mathrm{Lin}\{\mathbf{l}_x h \mid x \in X, h \in \mathcal{H}\}$ is dense in $\tilde{\mathcal{R}}$,

and, in this case, $\tilde{\mathcal{R}}$ is uniquely determined by the kernel $\mathbf{l}$.

Theorem 4.1.4 (Theorem 4.2 in [23]). *Let Γ be a $*$-semigroup acting on a nonempty set X, $\mathcal{H}$ be a VH-space on an admissible space Z, and $\mathbf{l} : X \times X \to \mathcal{B}^*(\mathcal{H})$ be a kernel. Then the following are equivalent:*

(1) $\mathbf{l}$ has the following properties:

(a) $\mathbf{l}$ is positive semidefinite.

(b) $\mathbf{l}$ is invariant under the action of Γ on X.

(c) For any $\alpha \in \Gamma$ there exists $c(\alpha) \geq 0$ such that, for all $n \in \mathbb{N}$, $x_1, x_2, \cdots x_n \in X$, $h_1, h_2, \cdots, h_n \in \mathcal{H}$, we have

$$\sum_{i,j=1}^{n} [\mathbf{l}(\alpha \cdot x_i, \alpha \cdot x_j)h_j, h_i]_{\mathcal{H}} \leq c(\alpha)^2 [\mathbf{l}(x_i, x_j)h_j, h_i]_{\mathcal{H}}. \tag{4.8}$$

(2) $\mathbf{l}$ has an invariant $\mathcal{B}^*(\mathcal{H})$-valued VH-space linearisation $(\tilde{\mathcal{E}}; \tilde{\pi}; \tilde{V})$.

(3) $\mathbf{l}$ admits a $\mathcal{B}^*(\mathcal{H})$-reproducing kernel VH-space $\tilde{\mathcal{R}}$ and there exists a $*$-representation $\tilde{\rho} : \Gamma \to \mathcal{B}^*(\tilde{\mathcal{R}})$ such that $\tilde{\rho}(\xi)\mathbf{l}_x h = \mathbf{l}_{\xi \cdot x} h$ for all $\xi \in \Gamma$, $x \in X$, $h \in \mathcal{H}$.

Moreover, in case any of the assertions (1), (2) or (3) holds, a topologically minimal invariant $\mathcal{B}^*(\mathcal{H})$-valued VH-space linearisation can be constructed.

Proof. (1)⇒(2). Define the kernel $\mathbf{k}\colon (X \times \mathcal{H}) \times (X \times \mathcal{H}) \to Z$ by

$$\mathbf{k}((x,h),(y,g)) := [\mathbf{l}(y,x)h, g]_{\mathcal{H}}$$

for all $x, y \in X$ and $h, g \in \mathcal{H}$. Then $\mathbf{k}$ is weakly positive semidefinite and invariant under the action of Γ on $X \times \mathcal{H}$ given by $\xi \cdot (x,h) = (\xi \cdot x, h)$ for all $\xi \in \Gamma$, $x \in X$, $h \in \mathcal{H}$, as in the proof of (1)⇒(2) of Theorem 4.1.1. To see that this kernel satisfies condition 1.(c) of Theorem 3.2.9, we use the assumption (c) and get that, for any $\alpha \in \Gamma$ there exists $c(\alpha) \geq 0$ such that, for all $n \in \mathbb{N}$, $\{t_i\}_{i=1}^n \subset \mathbb{C}$, we have

$$\begin{aligned}
\sum_{j,k=1}^{n} t_j \overline{t_k} \mathbf{k}(\alpha \cdot (x_k, h_k), \alpha(x_j, h_j)) &= \sum_{j,k=1}^{n} t_j \overline{t_k} [\mathbf{l}(\alpha \cdot x_j, \alpha \cdot x_k) h_k, h_j] \\
&= \sum_{j,k=1}^{n} [\mathbf{l}(\alpha \cdot x_j, \alpha \cdot x_k) t_k h_k, t_j h_j] \\
&\leq c(\alpha)^2 \sum_{j,k=1}^{n} [\mathbf{l}(x_j, x_k) t_k h_k, t_j h_j] \\
&= c(\alpha)^2 \sum_{j,k=1}^{n} t_j \overline{t_k} \mathbf{k}((x_k, h_k), (x_j, h_j)).
\end{aligned}$$

By Theorem 3.2.9, there exists a minimal weak VH-space linearisation $(\mathcal{K}; V; \pi)$ of $\mathbf{k}$ and the action of Γ on $(X \times \mathcal{H})$. Same arguments as in proof of (1)⇒(2) of Theorem 4.1.1 gives an adjointable operator of VE-spaces $\tilde{V}(x)\colon \mathcal{H} \to \mathcal{K}_0$, given by $\tilde{V}(x)h\colon = V(x,h)$ for $x \in X$ and $h \in \mathcal{H}$, where $\mathcal{K}_0\colon = \mathrm{Lin} V(X)\mathcal{H}$, with the property that $\tilde{V}(x)^*\tilde{V}(y) = \mathbf{l}(x,y)$ for all $x, y \in X$. Arguing as in the proof of Theorem 3.3 of [23], it follows that $\tilde{V}(x) \in \mathcal{B}^*(\mathcal{H}, \mathcal{K}_0)$ and $\tilde{V}(x)^* \in \mathcal{B}^*(\mathcal{K}_0, \mathcal{H})$. Hence $\tilde{V}(x)^*$ extends uniquely to an operator $V(x)^* \in \mathcal{B}^*(\mathcal{K}, \mathcal{H})$ for each $x \in X$. It follows that $(\mathcal{K}; \pi; \tilde{V})$ is a topologically minimal invariant $\mathcal{B}^*(\mathcal{H})$-valued VH-space linearisation of the kernel $\mathbf{l}$ and the action of Γ on X.

(2)⇒(3). Let $(\tilde{\mathcal{K}}; \tilde{\pi}; \tilde{V})$ be an invariant $\mathcal{B}^*(\mathcal{H})$ VH-space linearisation of the kernel $\mathbf{l}$. We essentially use Theorem 3.2.9 with details very close to the proof of the implication (2)⇒(3) of Theorem 4.1.3, with the difference that we obtain bounded adjointable operators instead of continuously adjointable operators. Define $V\colon (X \times \mathcal{H}) \to \tilde{\mathcal{K}}$ by $V(x,h) = \tilde{V}(x)h$ for all $x \in X$ and $h \in \mathcal{H}$.

We also have $\tilde{\pi}(\xi)\tilde{V}(x)h = \tilde{V}(\xi \cdot x)h = V(\xi \cdot x, h)$ for all $\xi \in \Gamma$, $x \in X$, $h \in \mathcal{H}$. Then $(\overline{\mathrm{Lin}(V(X,\mathcal{H})}; V; \tilde{\pi})$ is a topologically minimal weak invariant VH-space linearisation for the kernel $\mathbf{k}\colon (X \times \mathcal{H}) \times (X \times \mathcal{H}) \to Z$ defined by $\mathbf{k}((x,h),(y,g)) = [V(x,h), V(y,g)]$, for all $x, y \in X$ and $h, g \in \mathcal{H}$, and the action of Γ on $(X \times \mathcal{H})$ defined by $\xi \cdot (x,h) = (\xi \cdot x, h)$, for all $\xi \in \Gamma$, $x \in X$, and $h \in \mathcal{H}$.

By Theorem 3.2.9 there exists a weak Z-reproducing kernel VH-space $\mathcal{R}$ and a $*$-representation $\rho\colon \Gamma \to \mathcal{B}^*(\mathcal{R})$ such that $\rho(\xi)\mathbf{k}_{(x,h)} = \mathbf{k}_{\xi\cdot(x,h)}$ for all $\xi \in \Gamma$, $x \in X$, $h \in \mathcal{H}$. Define $\tilde{f}\colon X \to \mathcal{H}$ as follows: for each $x \in X$ let $\tilde{f}(x) \in \mathcal{H}$ be the unique element satisfying $[\tilde{f}(x), h]_{\mathcal{H}} = f(x,h)$ for all $h \in \mathcal{H}$ and let $\tilde{\mathcal{R}}$ be the vector space of all $\tilde{f}$, when $f \in \mathcal{R}$. Since we have

$$[\mathbf{k}_{(x,h)}, \mathbf{k}_{(y,g)}] = \mathbf{k}_{(x,h)}(y,g) = [\widetilde{\mathbf{k}_{(x,h)}}(y), g], \quad x, y \in X,\ h, g \in \mathcal{H},$$

it follows that $\mathbf{l}_x h = \widetilde{\mathbf{k}_{(x,h)}} \in \tilde{\mathcal{R}}$ for all $x \in X$ and $h \in \mathcal{H}$.

It is easy to check that the map $U\colon \mathcal{R} \ni f \to \tilde{f} \in \tilde{\mathcal{R}}$ is linear, one-to-one, and onto. Therefore, defining $[\tilde{f}, \tilde{g}]_{\tilde{\mathcal{R}}} := [f,g]_{\mathcal{R}}$ makes $\tilde{\mathcal{R}}$ a $\mathcal{B}^*(\mathcal{H})$-reproducing kernel VH-space with reproducing kernel $\mathbf{l}$, and U becomes a unitary operator of VH-spaces. Defining $\tilde{\rho} := U\rho U^*$, the pair $(\tilde{\mathcal{R}}, \tilde{\rho})$ has all the required properties.

(3)⇒(1). Assume that $(\tilde{\mathcal{R}}; \tilde{\rho})$ is a $\mathcal{B}^*(\mathcal{H})$-reproducing kernel VH-space of $\mathbf{l}$ with a representation $\tilde{\rho}$ of Γ on $\mathcal{B}^*(\tilde{\mathcal{R}})$ such that $\rho(\xi)\mathbf{l}_x h = \mathbf{l}_{\xi\cdot x} h$ for all $\xi \in \Gamma$, $x \in X$, $h \in \mathcal{H}$. Similarly as in proof of the implication (3)⇒(1) of Theorem 4.1.1, the kernel $\mathbf{l}$ is shown to be positive semidefinite and invariant under the action of Γ on X. On the other hand, using the fact that the linear operator $\tilde{\rho}(\xi)\colon \mathcal{H} \to \mathcal{H}$ is bounded for all $\xi \in \Gamma$, it follows that, for any $\xi \in \Gamma$, there exists $c(\xi) \geq 0$ such that, for all $\xi \in \Gamma$, $n \in \mathbb{N}$, $\{x_i\}_{i=1}^n \in X$ and $\{h_i\}_{i=1}^n \in \mathcal{H}$, we have

$$\sum_{j,k=1}^n [\tilde{\mathbf{l}}(\xi \cdot x_j, \xi \cdot x_k)h_k, h_j]_{\mathcal{H}} = \sum_{j,k=1}^n [\tilde{\mathbf{l}}_{\xi\cdot x_k} h_k(\xi \cdot x_j), h_j]_{\mathcal{H}}$$

$$= \sum_{j,k=1}^n [\tilde{\mathbf{l}}_{\xi\cdot x_k} h_k, \tilde{\mathbf{l}}_{\xi\cdot x_j} h_j]_{\tilde{\mathcal{R}}} = [\tilde{\rho}(\sum_{k=1}^n \tilde{\mathbf{l}}_{x_k} h_k), \tilde{\rho}(\sum_{j=1}^n \tilde{\mathbf{l}}_{x_j} h_j)]_{\tilde{\mathcal{R}}}$$

$$\leq c(\xi)^2 [\sum_{k=1}^n \tilde{\mathbf{l}}_{x_k} h_k, \sum_{j=1}^n \tilde{\mathbf{l}}_{x_j} h_j]_{\tilde{\mathcal{R}}} = c(\xi)^2 \sum_{j,k=1}^n [\tilde{\mathbf{l}}(x_j, x_k)h_k, h_j]_{\mathcal{H}},$$

hence $\mathbf{l}$ has the property (c). □

4.2 Positive Semidefinite $\mathcal{L}(\mathcal{X}, \mathcal{X}'_Z)$ Valued Maps on $*$-Semigroups.

In this subsection we obtain stronger versions of Theorem 3.1 and Theorem 4.2 in [46] as applications of Theorem 3.2.7 and, respectively, of Theorem 3.2.9. We first reorganise some definitions from [46] and [13].

Let $\mathcal{X}$ be a vector space, and Z be an ordered $*$-space. By $\mathcal{X}'_Z$ we denote the space of all conjugate linear functions from $\mathcal{X}$ to Z and call it the *algebraic conjugate Z-dual space*. Let $\mathcal{L}(\mathcal{X}, \mathcal{X}'_Z)$ denote the vector space of all linear operators $T\colon \mathcal{X} \to \mathcal{X}'_Z$. For any VE-space $\mathcal{E}$ over Z and any linear operator $A\colon \mathcal{X} \to \mathcal{E}$, we define a linear operator $A'\colon \mathcal{E} \to \mathcal{X}'_Z$, called the *algebraic Z-adjoint operator*, by

$$(A'f)(x) = [Ax, f]_{\mathcal{E}}, \quad f \in \mathcal{E},\ x \in \mathcal{X}. \tag{4.9}$$

If Γ is a $*$-semigroup, a map $T\colon \Gamma \to \mathcal{L}(\mathcal{X}, \mathcal{X}'_Z)$ is called *$\mathcal{L}(\mathcal{X}, \mathcal{X}'_Z)$-valued n-positive* if

$$\sum_{i,j=1} (T_{s_i^* s_j} x_j)(x_i) \geq 0_Z \text{ for all } (s_i)_{i=1}^n \in \Gamma \text{ and all } (x_j)_{j=1}^n \in \mathcal{X}. \tag{4.10}$$

If T is n-positive for all $n \in \mathbb{N}$ then it is called *$\mathcal{L}(\mathcal{X}, \mathcal{X}'_Z)$-valued positive semidefinite*.

Remarks 4.2.1. With notation as before, let $T\colon \Gamma \to \mathcal{L}(\mathcal{X}, \mathcal{X}'_Z)$.

(1) We define a kernel $\mathbf{k}\colon (\Gamma \times \mathcal{X}) \times (\Gamma \times \mathcal{X}) \to Z$ by

$$\mathbf{k}((s, x), (t, y)) = (T_{s^*t} y)x, \quad s, t \in \Gamma,\ x, y \in \mathcal{X}. \tag{4.11}$$

Then for all $n \in \mathbb{N}$, all $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$, and all $(s_i, x_i)_{i=1}^n \in (S \times X)$ we have

$$\sum_{i,j=1}^{n} \overline{\alpha}_i \alpha_j \mathbf{k}((s_i, x_i), (s_j, x_j)) = \sum_{i,j=1} \overline{\alpha}_i \alpha_j (T_{s_i^* s_j} x_j) x_i = \sum_{i,j=1} (T_{s_i^* s_j} \alpha_j x_j)(\alpha_i x_i).$$

This shows that, for $n \in \mathbb{N}$, the map T is n-positive if and only if the kernel $\mathbf{k}$ is weakly n-positive. In particular, T is positive semidefinite if and only if the kernel $\mathbf{k}$ is weakly positive semidefinite.

(2) Recall that, see (3.2), the kernel $\mathbf{k}$ is Hermitian if $\mathbf{k}((s,x),(t,y)) = \mathbf{k}((t,y),(s,x))^*$ for all $s,t \in \Gamma$ and all $x,y \in \mathcal{X}$. From (4.11) it follows that $\mathbf{k}$ is Hermitian if and only if

$$(T_{s^*t}y)x = ((T_{t^*s}x)y)^*, \quad \text{for all } s,t \in \Gamma, \text{ and all } x,y \in \mathcal{X}. \tag{4.12}$$

Consequently, by Lemma 3.1.1 it follows that, if T is 2-positive, then (4.12) holds.

In addition, if Γ has a unit $e = e^*$, then (4.11) is equivalent with

$$(T_{s^*}y)x = ((T_s x)y)^*, \quad \text{for all } s \in \Gamma, \text{ and all } x,y \in \mathcal{X}. \tag{4.13}$$

(3) We define a left action of Γ on $(\Gamma \times \mathcal{X})$ by

$$u \cdot (s,x) = (us,x), \text{ for all } u,s \in \Gamma, \text{ and all } x \in \mathcal{X}. \tag{4.14}$$

For all $u \in \Gamma$ and all $(s,x) \in \Gamma \times \mathcal{X}$ we have

$$\mathbf{k}((s,x), u \cdot (t,y)) = (T_{s^*ut}y)x = (T_{(u^*s)^*ty})x = \mathbf{k}(u^*(s,x),(t,y)),$$

hence the kernel $\mathbf{k}$ is invariant under the left action of Γ on $\Gamma \times \mathcal{X}$ defined as in (4.14).

Theorem 4.2.2. *Let Z be an ordered $*$-space, let $\mathcal{X}$ be complex vector space with algebraic conjugate Z-dual space $\mathcal{X}_Z'$, and consider $T\colon \Gamma \to \mathcal{L}(\mathcal{X}, \mathcal{X}_Z')$, for some $*$-semigroup Γ with unit. The following assertions are equivalent:*

(i) T is positive semidefinite, in the sense of (4.10).

(ii) There exist a VE-space $\mathcal{E}$ over Z, a unital $*$-representation $\pi\colon \Gamma \to \mathcal{L}^*(\mathcal{E})$, and an operator $A \in \mathcal{L}(\mathcal{X}, \mathcal{E})$, such that

$$T_t = A'\pi(t)A, \quad t \in \Gamma. \tag{4.15}$$

If any of the conditions (i) and (ii) holds, then the VE-space $\mathcal{E}$ can be chosen minimal in the sense that it coincides with the linear span of $\pi(\Gamma)A\mathcal{X}$ and, in this case, it is unique modulo a unitary equivalence.

Proof. (i)⇒(ii). We consider the kernel $\mathbf{k}\colon \Gamma \times \mathcal{X} \to Z$ as in (4.11) and the left action of Γ on $\Gamma \times \mathcal{X}$ as in (4.14). By Remark 4.2.1.(1) and Remark 4.2.1.(2), $\mathbf{k}$ is a Z-valued weakly positive semidefinite kernel invariant under the action of Γ as in (4.14) hence, by Theorem 3.2.7, there exists a minimal invariant weak VE-space linearisation $(\mathcal{E}, \pi, V)$ of $\mathbf{k}$. Since

$$[V(s,x), V(t,y)]_{\mathcal{E}} = \mathbf{k}((s,x),(t,y)) = (T_{s^*t}y)x, \quad s,t \in \Gamma,\ x,y \in \mathcal{X},$$

it follows that $\mathcal{X} \ni x \mapsto V(s,x) \in \mathcal{E}$ is linear, for all $s \in \Gamma$. This shows that, we can define $\tilde{V}\colon \Gamma \to \mathcal{L}(\mathcal{X}, \mathcal{E})$ by $\tilde{V}(s)x = V(s,x)$, for all $s \in \Gamma$ and all $x \in \mathcal{X}$. Taking into account (4.9) it follows that

$$(\tilde{V}(s)' f)x = [\tilde{V}(s)x, f]_{\mathcal{E}}, \quad s \in \Gamma,\ x \in \mathcal{X},\ f \in \mathcal{E},$$

hence, letting $A = \tilde{V}(e) \in \mathcal{L}(\mathcal{X}, \mathcal{E})$ it follows that, for all $s \in \Gamma$ and all $x, y \in \mathcal{X}$ we have

$$\begin{aligned}(A'\pi(s)Ax)y &= (\tilde{V}(e)'\pi(s)\tilde{V}(e)x)y = [\tilde{V}(e)y, \pi(s)\tilde{V}(e)x]_{\mathcal{E}} \\ &= [V(e,y), \pi(s)V(e,x)]_{\mathcal{E}} = [V(e,y), V(s,x)]_{\mathcal{E}} = \mathbf{k}((e,y),(s,x)) = (T_s y)x,\end{aligned}$$

and hence (4.15) is proven. The minimality and the uniqueness property follow by standard arguments that we omit.

(ii)⇒(i). This follows by a standard argument that we omit. □

Theorem 4.2.2 is stronger than Theorem 3.1 in [46] since, in addition to positive semidefiniteness of T they require the condition (4.13) as well. As we have seen in Remark 4.2.1.(3), this condition is a consequence of the positive semidefiniteness of T. Also, the ordered $*$-space Z need not be admissible, actually, the topology of Z does not play any role.

From now on we assume that Z is a topologically ordered $*$-space and that $\mathcal{X}$ is a locally bounded topological vector space, that is, in $\mathcal{X}$ there exists a bounded neighbourhood of 0. By $\mathcal{X}_Z^*$ we denote the subspace of $\mathcal{X}_Z'$ of all continuous conjugate linear functions from $\mathcal{X}$ to Z and call it the *topological conjugate Z-dual space*. The space $\mathcal{X}_Z^*$ is considered with the topology of uniform convergence on bounded sets, that is, a net $(f_i)_{i\in\mathcal{I}} \in \mathcal{X}_Z^*$ converges to 0 if for any bounded subset $B \subset \mathcal{X}$ the Z-valued net $(f_i(y))_{i\in\mathcal{I}}$ converges to 0 uniformly with respect to $y \in B$, equivalently, for any bounded set $B \subset \mathcal{X}$, any $p \in S(Z)$ and any $\epsilon > 0$, there exists $i_0 \in \mathcal{I}$ such that $i \geq i_0$ implies $p(f_i(y)) < \epsilon$ for all $y \in B$. Let $\mathcal{L}_{\mathrm{c}}(\mathcal{X}, \mathcal{X}_Z^*)$ be the space of all continuous linear operators from $\mathcal{X}$ to $\mathcal{X}_Z^*$.

Let $\mathcal{E}$ be a VE-space over Z, with topology defined as in Subsection 2.3. Following [46] and [14], for any $A \in \mathcal{L}_{\mathrm{c}}(\mathcal{X}, \mathcal{E})$ the *topological Z-adjoint operator* of A is, by definition, the operator $A^*\colon \mathcal{E} \to \mathcal{X}_Z^*$ defined by

$$(A^* f)x = [Ax, f]_{\mathcal{E}}, \quad f \in \mathcal{E},\ x \in \mathcal{X}. \tag{4.16}$$

By Lemma 2.3.2 the definition of A^* is correct.

Theorem 4.2.3. *Let Γ be a $*$-semigroup with unit e and $\mathcal{X}$ be a locally bounded topological vector space with topological conjugate Z-dual space $\mathcal{X}_Z^*$ for an admissible space Z. Let $T\colon \Gamma \to \mathcal{L}_{\mathrm{c}}(\mathcal{X}, \mathcal{X}_Z')$ subject to the following properties:*

(a) T is an $\mathcal{L}(\mathcal{X}, \mathcal{X}_Z')$-valued positive semidefinite map.

(b) For all $u \in \Gamma$, there is a constant $c(u) \geq 0$ such that for all $n \in \mathbb{N}$, all $s_1, \ldots, s_n \in \Gamma$, and all $x_1, \ldots, x_n \in \mathcal{X}$, we have

$$\sum_{i,j=1}^{n} (T_{s_i^* u^* u s_j} x_j)(x_i) \leq c(u)^2 \sum_{i,j=1}^{n} (T_{s_i^* s_j} x_j)(x_i). \tag{4.17}$$

(c) $T(e) \in \mathcal{L}_{\mathrm{c}}(\mathcal{X}, \mathcal{X}_Z^*)$.

Then:

(i) There exist a VH-space $\mathcal{K}$ over Z, a $*$-representation $\pi\colon \Gamma \to \mathcal{B}^*(\mathcal{K})$ and an operator $A \in \mathcal{L}_{\mathrm{c}}(\mathcal{X}, \mathcal{K})$, such that $T_s = A^* \pi(s) A$ for any $s \in \Gamma$.

(ii) $T_s \in \mathcal{L}_{\mathrm{c}}(\mathcal{X}, \mathcal{X}_Z^*)$ for all $s \in \Gamma$.

(iii) If $(u_l)_{l\in\mathcal{L}}$ is a net in Γ with $\sup_{l\in\mathcal{L}} c(u_l) < \infty$ and $(T_{su_lt})_{l\in\mathcal{L}}$ converges to T_{sut}, for some $u \in \Gamma$ and any $s, t \in \Gamma$, in the weak topology of $\mathcal{L}_{\mathrm{c}}(\mathcal{X}, \mathcal{X}_Z^*)$, then $(\pi(u_l))_{l\in\mathcal{L}}$ converges to $\pi(u)$ in the weak topology of $\mathcal{B}^*(\mathcal{K})$.

Proof. Define the kernel $\mathbf{k}\colon (\Gamma\times\mathcal{X})\times(\Gamma\times\mathcal{X})\to Z$ as in (4.11). By Remark 4.2.1, it follows that $\mathbf{k}$ is a Z-valued weakly positive semidefinite kernel. Next, consider the left action of Γ on $(\Gamma\times\mathcal{X})$ as in (4.14) and by Remark 4.2.1 it follows that $\mathbf{k}$ is invariant under this action. In order to show that the property 1.(c) of Theorem 3.2.9 holds, let $u\in\Gamma$, $n\in\mathbb{N}$, $(s_i,x_i)_{i=1}^n\in(\Gamma\times\mathcal{X})$. Then, using (4.17) it follows that

$$\sum_{i,j=1}^n \mathbf{k}(u\cdot(s_i,x_i),u\cdot(s_j,x_j)) = \sum_{i,j=1}^n (T_{s_i^*u^*us_j}x_j)x_i$$
$$\leq c(u)^2\sum_{i,j=1}^n (T_{s_i^*s_j}x_j)(x_i) = c(u)^2\sum_{i,j=1}^n \mathbf{k}((s_i,x_i),(s_j,x_j)).$$

By Theorem 3.2.9, there exists a topologically minimal invariant weak VH-space linearisation $(\mathcal{K};\pi;V)$ of the kernel $\mathbf{k}$. Since $[V(s,x),V(t,y)]_{\mathcal{K}} = \mathbf{k}((s,x),(t,y)) = (T_{s^*t}y)(x)$ for all $s,t\in\Gamma$ and $x,y\in\mathcal{X}$, we observe that $V(s,x)$ depends linearly on $x\in\mathcal{X}$ for each $s\in\Gamma$. As a consequence, letting $\tilde{V}(s)x = V(s,x)$, for all $x\in\mathcal{X}$, we obtain a linear operator $\tilde{V}(s):\mathcal{X}\to\mathcal{K}$ for each $s\in\Gamma$. To see that $\tilde{V}(s)$ is continuous for each $s\in\Gamma$, let $(x_l)_{l\in\mathcal{L}}$ be a net in $\mathcal{X}$ converging to 0. Since $\mathcal{X}$ is locally bounded, there exists $B\subset\mathcal{X}$ a bounded neighbourhood of 0 and then there exists $l_1\in\mathcal{L}$ such that $(x_l)_{l\geq l_1}$ is contained in B. Since $T_e\in\mathcal{L}_{\mathrm{c}}(\mathcal{X},\mathcal{X}_Z^*)$, taking into account the topology of $\mathcal{X}_Z^*$, given any $\varepsilon>0$ and any $p\in S(Z)$ we can find $l_2\in\mathcal{L}$ such that $\mathcal{L}\ni l\geq l_2$ implies $p((T_ex_l)y)<\varepsilon$ for all $y\in B$. Since $\mathcal{L}$ is directed, there exists $l_0\in\mathcal{L}$ with $l_0\geq l_1$ and $l_0\geq l_2$. Then, for any $l\geq l_0$, by (4.17) and taking into account how the topology of $\mathcal{K}$ is defined, see Subsection 2.3, we have

$$\begin{aligned} p([\tilde{V}(s)x_l,\tilde{V}(s)x_l]_{\mathcal{K}}) &= p(\mathbf{k}((s,x_l),(s,x_l))\\ &= p(\mathbf{k}(s\cdot(e,x_l),s\cdot(e,x_l))\leq c(s)^2p(\mathbf{k}((e,x_l),(e,x_l)))\\ &= c(s)^2p((T_ex_l)x_l)\leq c(s)^2\sup_{y\in B}p((T_ex_l)y)\leq c(s)^2\varepsilon, \end{aligned}$$

hence $\tilde{V}(s)\in\mathcal{L}_{\mathrm{c}}(\mathcal{X},\mathcal{K})$, for any $s\in\Gamma$. In addition, for each $s\in\Gamma$ the operator $\tilde{V}(s)^*\in\mathcal{L}(\mathcal{K},\mathcal{X}_Z^*)$ is defined as in (4.16).

Letting $A := \tilde{V}(e)$ we have

$$(A^*\pi(s)Ax)y = (\tilde{V}(e)^*\pi(s)\tilde{V}(e)x)(y) = [V(e,y), V(s,x)]_{\mathcal{K}}$$
$$= \mathbf{k}((e,y),(s,x)) = (T_s x)y \text{ for all } s \in \Gamma \text{ and } x, y \in \mathcal{X}.$$

Therefore $A^*\pi(s)A = T_s \in \mathcal{L}_{\mathrm{c}}(\mathcal{X}, \mathcal{X}_Z^*)$, for all $s \in \Gamma$.

The rest of the proof, which shows that $\pi(u_l)_{l\in\mathcal{L}}$ converges to $\pi(u)$ in the weak topology of $\mathcal{B}^*(\mathcal{K})$, as in the second part of the conclusion, uses standard arguments and is the same with that in [46]. For completeness, we present it here. Let $\mathcal{K}_0 := \mathrm{Lin} V(\Gamma \times \mathcal{X})$. By minimality, $\mathcal{K}_0$ is dense in $\mathcal{K}$. Let $e, f \in \mathcal{K}_0$, with $e = \sum_{i=1}^n \alpha_i V(s_i, x_i)$ and $f = \sum_{j=1}^m \beta_j V(r_j, y_j)$. We have

$$[e, \pi(u_l)f]_{\mathcal{K}} = [(\sum_{i=1}^n \alpha_i V(s_i, x_i)), \pi(u_l) \sum_{j=1}^m \beta_j V(r_j, y_j)]_{\mathcal{K}}$$
$$= \sum_{i=1}^n \sum_{j=1}^m \overline{\alpha}_i \beta_j (T_{s_i^* u r_j} y_j)(x_i) \underset{l}{\rightarrow} \sum_{i=1}^n \sum_{j=1}^m \overline{\alpha}_i \beta_j (T_{s_i^* u r_j} y_j) x_i = [e, \pi(u)f]_{\mathcal{K}}$$

by the assumption that $(T_{s u_l t})_{l\in\mathcal{L}}$ converges to T_{sut} for any $s, t \in \Gamma$. Now let $g, h \in \mathcal{K}$ and let $(g_i)_{i\in\mathcal{I}} \in \mathcal{K}_0$ be a net converging to g. For any $p \in S(Z)$, $j \in \mathcal{I}$ and $l \in \mathcal{L}$ we have

$$p([(\pi(u) - \pi(u_l))g, h]_{\mathcal{K}}) \leq p([\pi(u)(g - g_j), h]_{\mathcal{K}}) + p([(\pi(u) - \pi(u_l))g_j, h]_{\mathcal{K}})$$
$$+ p([\pi(u_l)(g_j - g), h]_{\mathcal{K}})$$
$$\leq 4\tilde{p}(h)\big(\tilde{p}(\pi(u)(g - g_j)) + \tilde{p}(\pi(u_l)(g_j - g)) + \tilde{p}((\pi(u) - \pi(u_l))g_j)\big)$$
$$\leq 4\tilde{p}(h)\big(c^2\tilde{p}(g - g_j) + \tilde{p}((\pi(u) - \pi(u_l))g_j)\big)$$

for some constant c, where the second inequality follows by the Schwarz type inequality (2.12) and the third inequality by the fact that $\pi(u), \pi(u_l) \in \mathcal{B}^*(\mathcal{K})$ and the assumption that $\sup_{l\in\mathcal{L}} c(u_l) < \infty$. Now that weak convergence was shown in $\mathcal{K}_0$, a standard argument finishes the proof. □

Remarks 4.2.4. (1) Theorem 4.2.3 is stronger than Theorem 4.2 of [46], see also the correction in [47], with respect to two aspects: firstly, since they have the additional assumption that (4.13) holds, which is actually a consequence of positive semidefiniteness, as Remark 4.2.1.(2) shows, and secondly since their assumption $T_s \in \mathcal{L}_{\mathrm{c}}(\mathcal{X}, \mathcal{X}_Z^*)$, for all $s \in \Gamma$, is actually a consequence of the weaker one $T_e \in \mathcal{L}_{\mathrm{c}}(\mathcal{X}, \mathcal{X}_Z^*)$, as the proof of Theorem 4.2.3 shows.

(2) It is easy to see that, there is a "converse" to Theorem 4.2.3 in the sense that, if assertion (i) is assumed, then assertions (a), (b), (c), and (ii) are obtained as consequences.

Chapter 5

Applications of the Main Dilation Theorems II: Unification of Various Dilation Theorems in the Context of Locally C^*-Algebras

5.1 Locally C^*-Algebras, Locally Hilbert Modules and Locally Hilbert C^*-Correspondences

5.1.1 Hilbert Modules over Locally C^*-Algebras

A $*$-algebra $\mathcal{A}$ that has a complete Hausdorff topology induced by a family of C^**-seminorms*, that is, seminorms p on $\mathcal{A}$ that satisfy the C^**-condition* $p(a^*a) = p(a)^2$ for all $a \in \mathcal{A}$, is called a *locally* C^**-algebra* [43] (equivalent names are *(Locally Multiplicatively Convex) LMC***-algebras* [48], [34], or b^**-algebra* [49], [50], or *pro* C^**-algebra* [51]), [35]. Note that, any C^*-seminorm is *submultiplicative*, $p(ab) \leq p(a)p(b)$ for all $a, b \in \mathcal{A}$, cf. [52], and $*$*-invariant*, $p(a^*) = p(a)$ for all $a \in \mathcal{A}$. Denote the collection of all continuous C^*-seminorms by $S_*(\mathcal{A})$.

Then $S_*(\mathcal{A})$ is a directed set under pointwise maximum seminorm, namely, given $p, q \in S_*(\mathcal{A})$, letting $r(a) := \max\{p(a), q(a)\}$ for all $a \in \mathcal{A}$, then r is a continuous C^*-seminorm and $p, q \leq r$. Locally C^*-algebras were studied in [49], [50], [43], [48], [35], and [53], to cite a few.

Any locally C^*-algebra is, in particular, an admissible space, see [24].

A *pre-Hilbert module over a pre-locally C^*-algebra* $\mathcal{A}$, or a *pre-Hilbert $\mathcal{A}$-module* is a topological VE-module $\mathcal{H}$ over $\mathcal{A}$. Note that the topology on the pre-Hilbert $\mathcal{A}$-module $\mathcal{H}$ is given by the family of seminorms $\{\tilde{p}\}_{p \in S_*(\mathcal{A})}$, where $\tilde{p}(h) = p([h,h])^{1/2}$ for all $p \in S_*(\mathcal{A})$ and all $h \in \mathcal{H}$. A pre-Hilbert $\mathcal{A}$-module $\mathcal{H}$ is called a *Hilbert $\mathcal{A}$-module* if it is complete, e.g. see [35].

For a Hilbert $\mathcal{A}$-module $\mathcal{H}$ and $p \in S_*(\mathcal{A})$, denote $\mathcal{I}_p^{\mathcal{A}} := \{a \in \mathcal{A} \mid p(a) = 0\}$, or simply $\mathcal{I}_p$ when there will be no danger of confusion on the ambient locally C^*-algebra, and $\tilde{\mathcal{I}}_p^{\mathcal{H}} := \{x \in \mathcal{H} \mid [x,x] \in \mathcal{I}_p\}$, or simply $\tilde{\mathcal{I}}_p$. Then $\mathcal{I}_p$ is a closed $*$-ideal in $\mathcal{A}$ and it is known, cf. [50], that the quotient $\mathcal{A}_p := \mathcal{A}/\mathcal{I}_p$ is a C^*-algebra with C^*-norm $\|a + \mathcal{I}_p\|_{\mathcal{A}_p} := p(a)$ for $a \in \mathcal{A}$. Also, $\tilde{\mathcal{I}}_p$ is a closed $\mathcal{A}$-submodule in $\mathcal{H}$ and the quotient module $\mathcal{H}_p := \mathcal{H}/\tilde{\mathcal{I}}_p$ is a Hilbert module over the C^*-algebra $\mathcal{A}_p$, with module action given by

$$(h + \tilde{\mathcal{I}}_p)(a + \mathcal{I}_p) := ha + \tilde{\mathcal{I}}_p, \quad h \in \mathcal{H},\ a \in \mathcal{A},$$

and gramian given by

$$[h + \tilde{\mathcal{I}}_p, k + \tilde{\mathcal{I}}_p]_{\mathcal{H}_p} := [h,k]_{\mathcal{H}} + \mathcal{I}_p, \quad h,k \in \mathcal{H},\ a \in \mathcal{A}.$$

On the other hand, when $\mathcal{H}$ and $\mathcal{K}$ are Hilbert modules over the same locally C^*-algebra $\mathcal{A}$, the space of all adjointable linear operators $T \colon \mathcal{H} \to \mathcal{K}$, denoted by $\mathcal{L}^*(\mathcal{H}, \mathcal{K})$, has some additional properties, when compared to VH-spaces. Any operator $T \in \mathcal{L}^*(\mathcal{H}, \mathcal{K})$ is automatically a module map and continuous, cf. [54] or Lemma 3.2 in [53], in particular, $T(h \cdot a) = T(h) \cdot a$ for all $h \in \mathcal{H}$, $a \in \mathcal{A}$ and $\mathcal{L}^*(\mathcal{H}, \mathcal{K}) = \mathcal{L}_c^*(\mathcal{H}, \mathcal{K})$, see Subsection 2.3 for notation.

For fixed $p \in S_*(\mathcal{A})$, any operator $T \in \mathcal{L}^*(\mathcal{H}, \mathcal{K})$ induces an adjointable, hence a continuous module map operator T_p from the Hilbert $\mathcal{A}_p$-module $\mathcal{H}_p$ to the

Hilbert $\mathcal{A}_p$-module $\mathcal{K}_p$, via

$$T_p(h + \tilde{\mathcal{I}}_p^{\mathcal{H}}) := Th + \tilde{\mathcal{I}}_p^{\mathcal{K}}, \quad h \in \mathcal{H}, \tag{5.1}$$

with adjoint

$$T_p^*(k + \tilde{\mathcal{I}}_p^{\mathcal{K}}) := T^*k + \tilde{\mathcal{I}}_p^{\mathcal{H}}, \quad k \in \mathcal{K}. \tag{5.2}$$

Moreover, there is a constant $C \geq 0$ such that

$$\tilde{p}_{\mathcal{K}}(Th) \leq C\, \tilde{p}_{\mathcal{H}}(h), \quad h \in \mathcal{H}, \tag{5.3}$$

see [35] and [53].

A topology on $\mathcal{L}^*(\mathcal{H}, \mathcal{K})$ can be defined via the collection of seminorms $\{\overline{p}_{\mathcal{H},\mathcal{K}}\}_{p \in S_*(\mathcal{A})}$: for arbitrary $p \in S_*(\mathcal{A})$,

$$\overline{p}_{\mathcal{H},\mathcal{K}}(T) := \|T_p\|, \quad T \in \mathcal{L}^*(\mathcal{H}, \mathcal{K}), \tag{5.4}$$

where $\|\cdot\|$ denotes the operator norm in $\mathcal{L}^*(\mathcal{H}_p, \mathcal{K}_p)$, equivalently, $\|T_p\|$ is the infimum of all $C \geq 0$ satisfying inequality (5.3). For the case $\mathcal{H} = \mathcal{K}$, these seminorms become C^*-seminorms and they turn $\mathcal{L}^*(\mathcal{H})$ into a locally C^*-algebra, c.f. [35] and [53].

A triple $(\mathcal{H}, \mathcal{A}, \mathcal{B})$ where $\mathcal{H}$ is a locally Hilbert $\mathcal{B}$-module and $\mathcal{A}$ is a locally C^*-algebra with a continuous left $*$-action on $\mathcal{H}$, i.e. a continuous $*$-representation of $\mathcal{A}$ on $\mathcal{L}^*(\mathcal{H})$ is called a *locally Hilbert $\mathcal{A}$-$\mathcal{B}$ correspondence.* In the rest of the article, when needed we will use $\pi(a)h$ to denote the left action of $\mathcal{A}$ on $\mathcal{H}$, where π denotes the $*$-representation of $\mathcal{A}$ on $\mathcal{L}^*(\mathcal{H})$. If this is not needed then we use just ah, $a \in \mathcal{A}$, $h \in \mathcal{H}$ to keep the notation simple.

For a locally C^*-algebra $\mathcal{A}$, let $M_n(\mathcal{A})$ denote the $*$-algebra of all $n \times n$ matrices over $\mathcal{A}$. $M_n(\mathcal{A})$ becomes a locally C^*-algebra considered with the topology generated by the C^*-seminorms

$$p_n([a_{ij}]_{i,j=1}^n) := \|[a_{ij} + \mathcal{I}_p]_{i,j=1}^n\|_{M_n(\mathcal{A}_p)}, \quad [a_{ij}]_{i,j=1}^n \in M_n(\mathcal{A}),$$

where $\|\cdot\|_{M_n(\mathcal{A}_p)}$ is the C^*-norm on the C^*-algebra $M_n(\mathcal{A}_p)$.

Given a locally Hilbert $\mathcal{A}$-$\mathcal{B}$ correspondence $\mathcal{H}$, consider n-fold the algebraic direct sum $\mathcal{H}^{(n)}$ for some $n \in \mathbb{N}$. Then $\mathcal{H}^{(n)}$ has a natural structure of an $M_n(\mathcal{A})$-$\mathcal{B}$ correspondence. The continuous $*$-representation of $M_n(\mathcal{A})$ on $\mathcal{L}^*(\mathcal{H}^n)$ is $\pi^n([a_{i,j}]_{i,j=1}^n) := [\pi(a_{i,j})]_{i,j=1}^n$ for $[a_{i,j}]_{i,j=1}^n \in M_n(\mathcal{A})$ acting on $\mathcal{H}^n$ by $[\pi(a_{i,j})]_{i,j=1}^n [h_i]_{i=1}^n := [a_{i,j}h_j]_{i=1}^n$ and the inner product is $\langle [h_i]_{i=1}^n, [g_i]_{i=1}^n \rangle_{\mathcal{H}^n} := \sum_{i=1}^n \langle h_i, g_i \rangle_{\mathcal{H}}$.

In what follows we state and prove a series of lemmas on locally C^*-algebras and locally Hilbert modules that will have a technical role in the next section, where we apply theorems 3.2.11 and 3.2.13 in the context of locally C^*-algebras and locally Hilbert C^*-modules to obtain various dilation theorems. Lemmas 5.1.1 and 5.1.2 below are known, see e.g. [35]. Lemma 5.1.3 is stated mainly for the lack of an explicit reference and it should be known as well. Lemmas 5.1.8, 5.1.5 and 5.1.7 are variations of Lemma 3.7. in [24] and they will be used to prove boundedness of some kernels under certain left actions in the next section. We provide proofs for all of the lemmas for the sake of completeness.

The proof of the lemma below uses an idea from [55].

Lemma 5.1.1. *Let $\mathcal{A}$ be a locally C^*-algebra. Then for any $a \in \mathcal{A}$ we have*

$$\operatorname{sp}(a) = \bigcup_{p \in S^*(\mathcal{A})} \operatorname{sp}(a_p)$$

Proof. By passing to unitization of $\mathcal{A}$ if necessary, we can assume without loss of generality that $\mathcal{A}$ is unital.

"$\operatorname{sp}(a) \supseteq \bigcup_{p \in S^*(\mathcal{A})} \operatorname{sp}(a_p)$": Assume there exists $\lambda \in \bigcup_{p \in S^*(\mathcal{A})} \operatorname{sp}(a_p)$, but such that $\lambda \notin \operatorname{sp}(a)$. Then $\lambda e - a$ is invertible, so by passing to quotients, $(\lambda e - a)_p = \lambda e_p - a_p$ is invertible for each $p \in S^*(\mathcal{A})$ and consequently $\lambda \notin \bigcup_{p \in S^*(\mathcal{A})} \operatorname{sp}(a_p)$, a contradiction.

"$\operatorname{sp}(a) \subseteq \bigcup_{p \in S^*(\mathcal{A})} \operatorname{sp}(a_p)$": Assume to the contrary that we have $\lambda \in \operatorname{sp}(a)$, but there is no $p \in S^*(\mathcal{A})$ such that $\lambda \in \operatorname{sp}(a_p)$. Then $(\lambda e_p - a_p) = (\lambda e - a)_p$ is invertible in the C^*-algebra $\mathcal{A}_p$ for any fixed $p \in S^*(\mathcal{A})$, so there exists a net of elements $(b^{(p)})_{p\in S^*(\mathcal{A})}$ in $\mathcal{A}$ such that $b_p^{(p)}(\lambda e - a)_p = e_p$, or equivalently, $p(b^{(p)}(\lambda e - a) - e) = 0$.

We claim that the net $(b^{(p)})_{p\in S^*(\mathcal{A})}$ is a Cauchy net in $\mathcal{A}$. Given any $q \in S^*(\mathcal{A})$, and $\epsilon > 0$, fix q as an index. Then given any $r_1, r_2 \in S^*(\mathcal{A})$ with $r_1, r_2 \geq q$ we have

$$q(b^{(r_i)}(\lambda e - a) - e) \leq r_i(b^{(r_i)}(\lambda e - a) - e) = 0$$

for $i = 1, 2$. By uniqueness of inverse in a C^*-algebra we have $b_q^{(r_1)} = b_q^{(r_2)}$, consequently $q(b^{(r_1)} - b^{(r_2)}) = 0 < \epsilon$ and the claim is proven.

Now let the limit of the net $(b^{(p)})_{p\in S^*(\mathcal{A})}$ be b. Given $p \in S^*(\mathcal{A})$ and $\delta > 0$, choose $s \in S^*(\mathcal{A})$ such that $p(b^{(s)} - b) < \delta/M$, where $M > 1$ is a number such that $M > p(\lambda e - a)$. By subadditivity and submultiplicativity of any $p \in S^*(\mathcal{A})$ we have

$$p(b(\lambda e - a) - e) \leq p((b^{(s)})(\lambda e - a) - e) + p((b - b^{(s)})(\lambda e - a)) \leq 0 + \delta = \delta$$

so $p(b(\lambda e - a) - e) = 0$ for all $p \in S^*(\mathcal{A})$. Since $\mathcal{A}$ is Hausdorff separated, $b(\lambda e - a) - e = 0$ and therefore $\lambda \notin \mathrm{sp}(a)$, a contradiction which finishes the proof. □

Lemma 5.1.2. *Let $\mathcal{A}$ be a locally C^*-algebra. Let $a \in \mathcal{A}$ be self adjoint. Then a is positive if and only if a_p is positive in the quotient C^*-algebra $\mathcal{A}_p$ for all $p \in S^*(\mathcal{A})$.*

Proof. " $\Rightarrow$ ": Assume a is positive in $\mathcal{A}$. By definition we have $\mathrm{sp}(a) \subseteq \mathbb{R}+$. By Lemma 5.1.1 $\mathrm{sp}(a) = \bigcup_{p\in S^*(\mathcal{A})} \mathrm{sp}(a_p)$ so we have $\mathrm{sp}(a_p) \subseteq \mathbb{R}+$ for any $p \in S^*(\mathcal{A})$. We also have $a_p^* = a_p$ for any $p \in S^*(\mathcal{A})$, hence a_p is positive in $\mathcal{A}_p$ for any $p \in S^*(\mathcal{A})$.

" $\Leftarrow$ ": Assume a_p is positive in $\mathcal{A}_p$ for any $p \in S^*(\mathcal{A})$. Since $a_p^* = a_p$, we have $p(a^* - a) = 0$ for all $p \in S^*(\mathcal{A})$ and since $\mathcal{A}$ is Hausdorff separated, it follows that $a^* = a$. We also have $\mathrm{sp}(a_p) \subseteq \mathbb{R}+$ for any $p \in S^*(\mathcal{A})$ and consequently $\bigcup_{p\in S^*(\mathcal{A})} \mathrm{sp}(a_p) \subseteq \mathbb{R}+$, whence by Lemma 5.1.1 it follows that $\mathrm{sp}(a) \subseteq \mathbb{R}+$, so a is positive. □

Lemma 5.1.3. *Let $\mathcal{A}$ be a locally C^*-algebra, $n \in \mathbb{N}$ and $A := [a_{i,j}]_{i,j=1}^n \in M_n(\mathcal{A})$ be a matrix. Then A is positive in the locally C^*-algebra $M_n(\mathcal{A})$ if and only if we have*

$$\sum_{i,j=1}^n b_i^* a_{i,j} b_j \geq 0$$

for any elements $\{b_k\}_{k=1}^n \subseteq \mathcal{A}$.

Proof. Using the definitions it is straightforward to check that the two C^*-algebras $M_n(\mathcal{A}_p)$ and $(M_n(\mathcal{A}))_p$ for any $p \in S_*(\mathcal{A})$ can be identified isometrically $*$-isomorphically by $\iota_p \colon (M_n(\mathcal{A}))_p \to M_n(\mathcal{A}_p)$, $\iota_p(R_p) := [r_{i,j_p}]_{i,j=1}^n$ where $R := [r_{i,j}]_{i,j=1}^n$. For all purposes in this article we can therefore assume that $M_n(\mathcal{A}_p) = (M_n(\mathcal{A}))_p$ for all $p \in S_*(\mathcal{A})$.

" $\Rightarrow$ ": Assume that a matrix $A := [a_{i,j}]_{i,j=1}^n \in M_n(\mathcal{A})$ is positive. By Lemma 5.1.2, the matrix $A_p \in M_n(\mathcal{A}_p)$ is positive for any $p \in S_*(\mathcal{A})$. By a well known characterization of positivity in the C^*-algebra $M_n(\mathcal{A}_p)$, see [28], we have

$$\sum_{i,j=1}^n b_{i_p}^* a_{i,j_p} b_{j_p} = (\sum_{i,j=1}^n b_i^* a_{i,j} b_j)_p \geq 0$$

for any $\{b_i\}_{i=1}^n \in \mathcal{A}$. By Lemma 5.1.2 again, $\sum_{i,j=1}^n b_i^* a_{i,j} b_j \geq 0$.

" $\Leftarrow$ ": We observe that all the steps in " $\Rightarrow$ " above can be reversed and the proof is finished. □

Lemma 5.1.4. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be C^*-algebras and $(\mathcal{E};\pi)$ be an $\mathcal{A}-\mathcal{B}$ correspondence and $(\mathcal{F};\rho)$ be a $\mathcal{B}-\mathcal{C}$ correspondence. Then for any $a \in \mathcal{A}$ there exists a constant $c(a) \geq 0$ with $c(a) \leq \|\pi(a)\|$such that for any $n \in \mathbb{N}$, $\{\alpha_i\}_{i=1}^n \in \mathbb{C}$, $\{x_i\}_{i=1}^n \in \mathcal{E}$, $\{y_i\}_{i=1}^n \in \mathcal{F}$ we have*

$$\sum_{i,j=1}^n \overline{\alpha_i}\alpha_j \langle y_i, \langle ax_i, ax_j\rangle_{\mathcal{E}} y_j\rangle_{\mathcal{F}} \leq c(a)^2 \sum_{i,j=1}^n \overline{\alpha_i}\alpha_j \langle y_i, \langle x_i, x_j\rangle_{\mathcal{E}} y_j\rangle_{\mathcal{F}}.$$

Consequently, we have

$$\| \sum_{i,j=1}^n \overline{\alpha_i}\alpha_j \langle y_i, \langle ax_i, ax_j\rangle_{\mathcal{E}} y_j\rangle_{\mathcal{F}} \|_{\mathcal{C}} \leq c(a)^2 \| \sum_{i,j=1}^n \overline{\alpha_i}\alpha_j \langle y_i, \langle x_i, x_j\rangle_{\mathcal{E}} y_j\rangle_{\mathcal{F}} \|_{\mathcal{C}}.$$

Proof. Let $a \in \mathcal{A}$ be such that $\|\pi(a)\| < 1$, by dividing by $\|\pi(a)\| + \epsilon$ for some $\epsilon > 0$ if necessary. Hence $\|\pi(a^*a)\| < 1$. Let $I_\mathcal{E}$ be the identity operator on $\mathcal{E}$. Then $I_\mathcal{E} - \pi(a^*a)$ is positive in the C^*-algebra $\mathcal{L}^*(\mathcal{E})$, so there exists a positive element $y \in \mathcal{L}^*(\mathcal{E})$ such that $I_\mathcal{E} - \pi(a^*a) = y^2$. We then have

$$\begin{aligned} &\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j\langle y_i, \langle x_i, x_j\rangle y_j\rangle - \sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j\langle y_i, \langle ax_i, ax_j\rangle y_j\rangle \\ =&\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j\langle y_i, \langle (I_\mathcal{E} - \pi(a^*a)x_i, x_j\rangle y_j\rangle \\ =&\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j\langle y_i, \langle yx_i, yx_j\rangle y_j\rangle \\ =&\langle [\alpha_i y_i]_{i=1}^n, [\langle yx_i, yx_j\rangle]_{i,j=1}^n [\alpha_j y_j]_{j=1}^n\rangle_{\mathcal{F}^n} \geq 0 \end{aligned}$$

for any $n \in \mathbb{N}$, $\{\alpha_i\}_{i=1}^n \in \mathbb{C}$, $(x_i, y_i)_{i=1}^n \in \mathcal{E} \times \mathcal{F}$ where the operator $[\langle yx_i, yx_j\rangle]_{i,j=1}^n$ is positive by the same argument in (5.2.2). The first inequality now follows with some $c(a) \geq 0$ with $c(a) \leq \|\pi(a)\|$, while the second inequality follows by taking norms of both sides of the first. □

Lemma 5.1.5. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be locally C^*-algebras and $(\mathcal{E};\pi)$ be an $\mathcal{A} - \mathcal{B}$ correspondence and $(\mathcal{F};\rho)$ be a $\mathcal{B} - \mathcal{C}$ correspondence. Then for any $a \in \mathcal{A}$ and $p \in S_*(\mathcal{C})$ there exists a constant $c_p(a) \geq 0$ such that for any $n \in \mathbb{N}$, $\{\alpha_i\}_{i=1}^n \in \mathbb{C}$, $\{x_i\}_{i=1}^n \in \mathcal{E}$, $\{y_i\}_{i=1}^n \in \mathcal{F}$ we have*

$$p(\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j\langle y_i, \langle ax_i, ax_j\rangle_\mathcal{E} y_j\rangle_\mathcal{F}) \leq c_p(a)^2 p(\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j\langle y_i, \langle x_i, x_j\rangle_\mathcal{E} y_j\rangle_\mathcal{F}).$$

Moreover, c_p can be chosen such that there exists a seminorm $r \in S_*(\mathcal{A})$ with the property that $c_p(a) \leq r(a)$ for all $a \in \mathcal{A}$.

Proof. Fix any $p \in S_*(\mathcal{C})$. Since $S_*(\mathcal{B})$ is directed and ρ is continuous, there exists $q \in S_*(\mathcal{B})$ and a constant $d_{p,q} \geq 0$ such that $\overline{p}(\rho(x)) \leq d_{p,q}q(x)$ for any $x \in \mathcal{B}$. So if $q(x) = 0$ for some $x \in \mathcal{B}$, then $\overline{p}(\rho(x)) = 0$. Consequently, the map $\rho_p \colon \mathcal{B}_q \to \mathcal{L}^*(\mathcal{F}_p)$ given by $\rho_p(b + I_q^\mathcal{B}) := \rho(b)_p$, $b \in \mathcal{B}$ is well defined.

Similarly, since $S_*(\mathcal{A})$ is directed and π is continuous, there exists $r \in S_*(\mathcal{A})$ and a constant $e_{q,r} \geq 0$ such that $\overline{q}(\pi(y)) \leq e_{q,r}r(y)$ for any $y \in \mathcal{B}$. So if $r(y) = 0$

for some $y \in \mathcal{B}$, then $\overline{q}(\pi(y)) = 0$. Consequently, the map $\pi_q \colon \mathcal{B}_r \to \mathcal{L}^*(\mathcal{E}_q)$ given by $\pi_q(a + I_r^{\mathcal{A}}) := \pi(a)_q$, $a \in \mathcal{A}$ is well defined.

Now for any $n \in \mathbb{N}$, $\{\alpha_i\}_{i=1}^n \in \mathbb{C}$, $\{x_i\}_{i=1}^n \in \mathcal{E}$, $\{y_i\}_{i=1}^n \in \mathcal{F}$ we have

$$p(\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j \langle y_i, \rho(\langle \pi(a)x_i, \pi(a)x_j\rangle)_{\mathcal{E}} y_j\rangle_{\mathcal{F}})$$
$$= \| \sum_{i,j=1}^{n} \langle y_{i_p}, \rho_p(\langle \pi_q(a_r)(x_{i_q}), \pi_q(a_r)(x_{j_q})\rangle_{\mathcal{E}_q}) y_{j_p}\rangle_{\mathcal{F}_p} \|_{\mathcal{C}_p}$$
$$\leq c_{p,q,r}(a_r)^2 \| \sum_{i,j=1}^{n} \langle y_{i_p}, \rho_p(\langle x_{i_q}, x_{j_q}\rangle_{\mathcal{E}_q}) y_{j_p}\rangle_{\mathcal{F}_p} \|_{\mathcal{C}_p}$$
$$= c_{p,q,r}(a_r)^2 p(\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j \langle y_i, \rho(\langle x_i, x_j\rangle)_{\mathcal{E}} y_j\rangle_{\mathcal{F}})$$

where the inequality follows by Lemma 5.1.4 with $c_{p,q,r}(a_r) \leq \|\pi_q(a_r)\|_q = q(\pi(a))$. Hence we have $c_{p,q,r}(a_r) \leq r(a)$. Since once $p \in S_*(\mathcal{C})$ is fixed, $q \in S_*(\mathcal{B})$ and $r \in S_*(\mathcal{A})$ are also fixed, we can write $c_{p,q,r}(a_r) = c_p(a)$ and the proof is finished. □

For the next lemma, given two pre-C^*-algebras A and B, let $\mathcal{B}(A, B)$ be the set of all bounded linear mappings. The positivity assumption that appears in the lemma is due to [9], see subsection 5.2.4 for more details.

Lemma 5.1.6. *Let A and B be two unital pre-C^*-algebras, X be a nonempty set and $\mathbf{k} \colon X \times X \to \mathcal{B}(A, B)$ be a kernel satisfying*

$$\sum_{i,j} b_i^* \mathbf{k}(x_i, x_j)(a_i^* a_j) b_j \geq 0 \tag{5.5}$$

for all $n \in \mathbb{N}$, $\{x_i\}_{i=1}^n \in X$, $\{a_i\}_{i=1}^n \in A$, $\{b_i\}_{i=1}^n \in B$. Then for any $a \in \mathcal{A}$ there exists a constant $c(a) \geq 0$ with $c(a) \leq \|a\|$ such that we have

$$\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j b_i^* \mathbf{k}(x_i, x_j)(a_i^* a^* a a_j) b_j \leq c(a)^2 \sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j b_i^* \mathbf{k}(x_i, x_j)(a_i^* a_j) b_j$$

for all $n \in \mathbb{N}$, $\{\alpha_i\}_{i=1}^n \in \mathbb{C}$, $\{x_i\}_{i=1}^n$, $\{a_i\}_{i=1}^n \in \mathcal{A}$, $\{b_i\}_{i=1}^n \in \mathcal{B}$.

Consequently, for any $a \in \mathcal{A}$ there exists a constant $c(a) \geq 0$ with $c(a) \leq \|a\|$ such that

$$\| \sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j b_i^* \mathbf{k}(x_i, x_j)(a_i^* a^* a a_j) b_j \| \leq c(a)^2 \| \sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j b_i^* \mathbf{k}(x_i, x_j)(a_i^* a_j) b_j \|$$

for all $n \in \mathbb{N}$, $\{\alpha_i\}_{i=1}^n \in \mathbb{C}$, $\{x_i\}_{i=1}^n$, $\{a_i\}_{i=1}^n \in \mathcal{A}$, $\{b_i\}_{i=1}^n \in \mathcal{B}$.

Proof. Since $\mathbf{k}(x, y)$, $x, y \in X$ is a bounded linear mapping, it has unique linear bounded extension to the C^*-algebra completion of $\mathcal{A}$ and valued in the C^*-algebra completion of $\mathcal{B}$, which we denote by the same symbol. By continuity arguments and closedness of positive elements in a C^*-algebra, the positivity property remains true for the extended kernel as well.

Given $a \in A$ by dividing by $\|a\| + \epsilon$ for some $\epsilon > 0$ if necessary, we can assume that $\|a\| < 1$. We have, for all $n \in \mathbb{N}$, $\{\alpha_i\}_{i=1}^n \in \mathbb{C}$, $\{x_i\}_{i=1}^n$, $\{a_i\}_{i=1}^n \in A$, $\{b_i\}_{i=1}^n \in B$,

$$\begin{aligned} &\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j b_i^* \mathbf{k}(x_i, x_j)(a_i^* a_j) b_j - \sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j b_i^* \mathbf{k}(x_i, x_j)(a_i^* a^* a a_j) b_j \\ &= \sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j b_i^* \mathbf{k}(x_i, x_j)(a_i^* a_j (e_{\mathcal{A}} - a^* a)) \geq 0 \end{aligned}$$

where the inequality follows by arguing similar to the proof of Lemma 5.1.4. It follows that the constant $c(a) \geq 0$ can be chosen to satisfy $c(a) \leq \|a\|$ as well.

The second inequality in the lemma follows by the first by taking norms of both sides. □

For the next lemma, given two pre-locally C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, let $\mathcal{L}_c(\mathcal{A}, \mathcal{B})$ be the set of all linear and continuous mappings from $\mathcal{A}$ to $\mathcal{B}$. See subsection 5.2.4 for the positivity assumption that appears in the lemma.

Lemma 5.1.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be two unital pre-locally C^*-algebras, X be a nonempty set and $\mathbf{k}\colon X \times X \to \mathcal{L}_c(\mathcal{A}, \mathcal{B})$ be a kernel satisfying*

$$\sum_{i,j} b_i^* \mathbf{k}(x_i, x_j)(a_i^* a_j) b_j \geq 0 \tag{5.6}$$

for all $n \in \mathbb{N}$, $\{x_i\}_{i=1}^n \in X$, $\{a_i\}_{i=1}^n \in A$, $\{b_i\}_{i=1}^n \in B$. *In addition, assume that we have the following uniform continuity condition that, for any* $p \in S_*(\mathcal{B})$ *there exists* $q \in S_*(\mathcal{A})$ *such that* $p(\mathbf{k}(x,y)(a)) \leq q(a)$ *holds for all* $x, y \in X$ *and* $a \in \mathcal{A}$.

Then for any $p \in S_*(\mathcal{B})$ *and* $a \in \mathcal{A}$ *there exists a constant* $c_p(a) \geq 0$ *such that*

$$p(\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j b_i^* \mathbf{k}(x_i, x_j)(a_i^* a^* a a_j) b_j) \leq c_p(a)^2 p(\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j b_i^* \mathbf{k}(x_i, x_j)(a_i^* a_j) b_j)$$

holds for all $n \in \mathbb{N}$, $\{\alpha_i\}_{i=1}^n \in \mathbb{C}$, $\{x_i\}_{i=1}^n$, $\{a_i\}_{i=1}^n \in \mathcal{A}$, $\{b_i\}_{i=1}^n \in \mathcal{B}$. *Moreover,* c_p *can be chosen such that there exists a seminorm* $q \in S_*(\mathcal{A})$ *with the property that* $c_p(a) \leq q(a)$ *for all* $a \in \mathcal{A}$.

Proof. Arguing similar to the proof of Lemma 5.1.6, a map $\mathbf{k}(x,y)$, $x, y \in X$ extends uniquely to a linear continuous mapping from the locally C^*-algebra completion of $\mathcal{A}$ to locally C^*-algebra completion of $\mathcal{B}$, which we denote with the same symbol.

Using the uniform boundedness condition and similarly to the proof of Lemma 5.1.5, the maps $\mathbf{k}(x,y)_p \colon \mathcal{A}_q \to \mathcal{B}_p$ given by $\mathbf{k}(x,y)_p(a + I_q^{\mathcal{A}}) := \mathbf{k}(x,y)(a) + I_p^{\mathcal{B}}$ are well defined for any $x, y \in X$; moreover, they are linear and bounded. Now define kernel $\mathbf{k}_p \colon X \times X \to \mathcal{B}(\mathcal{A}_q, \mathcal{B}_p)$ by $\mathbf{k}_p(x,y)(a_q) := (\mathbf{k}(x,y)(a))_p$. It follows by the positivity assumption (5.6) and Lemma 5.1.2, that the positivity condition (5.5) of Lemma 5.1.6 is satisfied, that is,

$$\sum_{i,j=1}^{n} b_{i\,p}^* \mathbf{k}_p(x_i, x_j)(a_{i\,q}^* a_{j\,q}) b_{j\,p} \geq 0$$

for p and q as above and any $n \in \mathbb{N}$, $\{a_i\}_{i=1}^n \in \mathcal{A}$ and $\{b_i\}_{i=1}^n \in \mathcal{B}$.

Now, similar to the proof of Lemma 5.1.5 again, for p as above and any $a \in \mathcal{A}$

we have

$$p(\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j b_i^* \mathbf{k}(x_i, x_j)(a_i^* a^* a a_j) b_j)$$

$$= \| \sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j {b_i^*}_p \mathbf{k}_p(x_i, x_j)({a_i^*}_q {a^*}_q a_q {a_j}_q) {b_j}_p \|_p$$

$$\leq c_p(a_q)^2 \| \sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j {b_i^*}_p \mathbf{k}_p(x_i, x_j)({a_i^*}_q {a_j}_q) {b_j}_p \|_p$$

$$= c_p(a_q)^2 p(\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j b_i^* \mathbf{k}(x_i, x_j)(a_i^* a_j) b_j)$$

where the inequality follows by Lemma 5.1.6 with $c_p(a_q) \leq \|a_q\|_q = q(a)$. Since once $p \in S_*(\mathcal{B})$ is fixed, $q \in S_*(\mathcal{A})$ is also fixed, we can write $c_p(a_q) = c_p(a)$ and the proof is finished. □

Lemma 5.1.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be locally C^*-algebras and $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a continuous and completely positive map. Then for any $a \in \mathcal{A}$ and any $p \in S_*(\mathcal{B})$ there exists a constant $c_p(a) \geq 0$ such that, for all $n \in \mathbb{N}$, $\{\alpha_i\}_{i=1}^n \in \mathbb{C}$, $\{a_i\}_{i=1}^n \in \mathcal{A}$, $\{b_i\}_{i=1}^n \in \mathcal{B}$ we have*

$$p(\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j b_i^* \varphi(a_i^* a^* a a_j) b_j) \leq c_p(a)^2 p(\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j b_i^* \varphi(a_i^* a_j) b_j).$$

Moreover, c_p can be chosen such that there exists a seminorm $q \in S_*(\mathcal{A})$ with the property that $c_p(a) \leq q(a)$ for all $a \in \mathcal{A}$

Proof. Let $X := \{x\}$ be a one point set and define kernel $\mathbf{k}\colon X \times X \to \mathcal{L}_c(\mathcal{A}, \mathcal{B})$ by $\mathbf{k}(x, x) := \varphi$. The positivity condition of Lemma 5.1.7 is characterized by the complete positivity of φ, using Lemma 5.1.3 and the fact that positive matrix in $M_n(\mathcal{A})$ is a sum matrices of the form $[a_i^* a_j]_{i,j=1}^n$, where this fact is proven with the same way as in C^*-algebra case. The uniform boundedness condition of Lemma 5.1.7 is satisfied by the continuity of φ, and then an application of Lemma 5.1.7 produces Lemma 5.1.8. □

5.2 Unification of Some Dilation Theorems

In this section, we prove some dilation theorems from applications of Theorems 3.2.11 and 3.2.13.

5.2.1 Paschke's Theorem for Completely Positive Maps

Paschke's construction of Hilbert C^*-correspondences from completely positive maps of C^*-algebras is well known. A similar construction for completely positive maps of locally C^*-algebras can be obtained from the more general Theorem 4.6. of [56]. It can also be obtained from Theorem 5.2.4 below, see Remark 5.2.5. In the following, we obtain this theorem directly from Theorem 3.2.13.

Theorem 5.2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital locally C^*-algebras and $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a continuous completely positive map. Then there exists an $\mathcal{A}$-$\mathcal{B}$ correspondence $\mathcal{K}$ and an element $\zeta \in \mathcal{K}$ such that $\varphi(a) = \langle a\zeta, \zeta\rangle$ for all $a \in \mathcal{A}$ and the set $\operatorname{Lin}\{a\zeta b \mid a \in \mathcal{A}, b \in \mathcal{B}\}$ is dense in $\mathcal{K}$.*

Proof. Consider the set $\mathcal{A}\times\mathcal{B}$ with a left action of $\mathcal{A}$, regarded as a $*$-semigroup, as follows: $a\cdot(a_1, b_1) = (a_1, b_1)$ for all $a \in \mathcal{A}$ and $(a_1, b_1) \in (\mathcal{A}\times\mathcal{B})$. Also define a right action of $\mathcal{B}$, regarded as a semigroup, on $\mathcal{A}\times\mathcal{B}$ by $(a_1, b_1)\cdot b \colon = (a_1, b_1 b)$ for all $b \in \mathcal{B}$ and $(a_1, b_1) \in (\mathcal{A}\times\mathcal{B})$, and also consider the right action of $\mathcal{B}$ on itself. Since any locally C^*-algebra is, in particular, an admissible space, see e.g. [24], we can define the admissible space valued kernel $\mathbf{k}\colon (\mathcal{A}\times\mathcal{B})\times(\mathcal{A}\times\mathcal{B}) \to \mathcal{B}$ by

$$\mathbf{k}((a_1, b_1), (a_2, b_2))\colon = b_1^*\varphi(a_1^* a_2)b_2$$

for all $a_1, a_2 \in \mathcal{A}$ and $b_1, b_2 \in \mathcal{B}$. rIn the following we show that this kernel is a weakly positive semidefinite doubly invariant kernel satisfying boundedness conditions (1)(c)-(d) of Theorem 3.2.13.

In order to verify that $\mathbf{k}$ is weakly positive semidefinite, we check that

$$\begin{aligned}\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j \mathbf{k}((a_i,b_i),(a_j,b_j)) &= \sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j b_i^* \varphi(a_i^* a_j) b_j \\ &= \sum_{i,j=1}^{n} (\alpha_i b_i)^* \varphi(a_i^* a_j)(\alpha_j b_j) \geq 0\end{aligned}$$

for all $n \in \mathbb{N}$, $(\alpha_i)_{i=1}^n \in \mathbb{C}$, $(a_i)_{i=1}^n \in \mathcal{A}$ and $(b_i)_{i=1}^n \in \mathcal{B}$, by the complete positivity of φ and Lemma 5.1.3. Therefore $\mathbf{k}$ is weakly positive semidefinite.

We have $\mathbf{k}((a_1,b_1), a\cdot(a_2,b_2)) = b_1^*\varphi(a_2^* a^* a_1)b_2^* = \mathbf{k}(a^*\cdot(a_1,b_1),(a_2,b_2))$ for all $a \in \mathcal{A}$ and $(a_1,b_1),(a_2,b_2) \in (\mathcal{A}\times\mathcal{B})$, so $\mathbf{k}$ is left invariant under the action of $\mathcal{A}$. We also have

$$\begin{aligned}\mathbf{k}((a_1,b_1),(a_2,b_2)\cdot b) &= b_1^*\varphi(a_2^* a_1)b_2 b \\ &= \mathbf{k}((a_1,b_1),(a_2,b_2))b\end{aligned}$$

for all $b \in \mathcal{B}$, $(a_1,b_1),(a_2,b_2) \in \mathcal{A}\times\mathcal{B}$, and clearly the action of $\mathcal{B}$ on itself is linear. Therefore $\mathbf{k}$ is a doubly invariant kernel under the actions of $\mathcal{A}$ and $\mathcal{B}$.

For the boundedness condition 1(c) of Theorem 3.2.13, given any $p \in S_*(\mathcal{B})$ and $a \in \mathcal{A}$ we have

$$\begin{aligned}p(\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j \mathbf{k}(a(a_i,b_i),a(a_j,b_j)) &= p(\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j b_i^*\varphi(a_i^* a^* a a_j)b_j) \\ \leq c(a)^2 p(\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j b_i^*\varphi(a_i^* a_j)b_j) &= p(\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j \mathbf{k}((a_i,b_i),(a_j,b_j)))\end{aligned}$$

for all $n \in \mathbb{N}$, $\{\alpha_i\}_{i=1}^n \in \mathbb{C}$, $\{(a_i,b_i)\}_{i=1}^n \in \mathcal{A}\times\mathcal{B}$, where the inequality follows by Lemma 5.1.8. Hence 1(c) holds.

For boundedness condition 1(d) of Theorem 3.2.13, given any $p \in S_*(\mathcal{B})$ and $b \in \mathcal{B}$ we have

$$\begin{aligned}p(\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j \mathbf{k}((a_i,b_i)\cdot b,(a_j,b_j)\cdot b)) &= p(b^*(\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j b_i^*\varphi(a_j^* a_i)b_j)b) \\ &\leq p(b)^2 p(\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j \mathbf{k}((a_i,b_i),(a_j,b_j)))\end{aligned}$$

for all $n \in \mathbb{N}$, $\{\alpha_i\}_{i=1}^n \in \mathbb{C}$, $b \in \mathcal{B}$, $\{(a_i, b_i)\}_{i,j=1}^n \in (\mathcal{A} \times \mathcal{B})$ by the submultiplicativity of p and condition 1(d) holds.

By theorem 3.2.13, there exists a minimal doubly invariant VH-space linearisation $(\mathcal{K}; V; \pi; \tau)$, which, here is an $\mathcal{A}$-$\mathcal{B}$ correspondence. Letting $\zeta := V(e_{\mathcal{A}}, e_{\mathcal{B}})$, we have

$$\begin{aligned}\langle \pi(a)\zeta, \zeta\rangle_{\mathcal{K}} &= \langle V(a, e_{\mathcal{B}}), V(e_{\mathcal{A}}, e_{\mathcal{B}})\rangle_{\mathcal{K}} \\ &= \mathbf{k}((a, e_{\mathcal{B}}), (e_{\mathcal{A}}, e_{\mathcal{B}})) = e_{\mathcal{B}}^*\varphi(e_{\mathcal{A}}^* a)e_{\mathcal{B}} = \varphi(a)\end{aligned}$$

for all $a \in \mathcal{A}$. Finally, since $\pi(a)(\zeta\tau(b)) = V(a, b)$ for all $a \in \mathcal{A}$, $b \in \mathcal{B}$, and $\mathcal{K}$ is minimal,

$$\begin{aligned}\operatorname{Lin}\{\pi(a)(\zeta\tau(b)) \mid a \in \mathcal{A}, b \in \mathcal{B}\} &= \operatorname{Lin}\{(\pi(a)\zeta)\tau(b) \mid a \in \mathcal{A}, b \in \mathcal{B}\} \\ &= \operatorname{Lin}\{V(a, b) \mid a \in \mathcal{A}, b \in \mathcal{B}\}\end{aligned}$$

is dense in $\mathcal{K}$. □

5.2.2 Internal Tensor Product of Two C^*-Correspondences

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be locally C^*-algebras and $\mathcal{E}$ be an $\mathcal{A}$-$\mathcal{B}$ correspondence, and $\mathcal{F}$ be a $\mathcal{B}$-$\mathcal{C}$ correspondence. In this subsection, as an application of Theorem 3.2.13, we will obtain the internal tensor product $\mathcal{E} \odot \mathcal{F}$, as an $\mathcal{A}$-$\mathcal{C}$ correspondence.

Theorem 5.2.2. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be locally C^*-algebras and $\mathcal{E}$ be an $\mathcal{A} - \mathcal{B}$ correspondence, and $\mathcal{F}$ be a $\mathcal{B} - \mathcal{C}$ correspondence. Then there exists an $\mathcal{A} - \mathcal{C}$ correspondence $\mathcal{E} \odot \mathcal{F}$ in which the simple tensors $x \odot y$ satisfy $\langle x_1 \odot y_1, x_2 \odot y_2\rangle = \langle y_1, \langle x_1, x_2\rangle y_2\rangle$ and the simple tensors span a dense subspace of $\mathcal{E} \odot \mathcal{F}$, and we have $a(x \odot y) = (ax) \odot y$ and $x \odot by = (xb \odot y)$ for all $a \in \mathcal{A}$, $b \in \mathcal{B}$, $x \in \mathcal{E}$, $y \in \mathcal{F}$. Such an $\mathcal{A}$-$\mathcal{C}$ correspondence is unique up to unitary equivalence.*

Proof. Consider the set $\mathcal{E} \times \mathcal{F}$ with a left action of $\mathcal{A}$ defined by $a \cdot (x, y) :\,= (ax, y)$ for all $a \in \mathcal{A}$, $x \in \mathcal{E}$, $y \in \mathcal{F}$, and a right action of $\mathcal{C}$ defined by $(x, y) \cdot c := (x, yc)$ for all $c \in \mathcal{C}$, $(x, y) \in (\mathcal{E} \times \mathcal{F})$.

Define a kernel $\mathbf{k}\colon (\mathcal{E} \times \mathcal{F}) \times (\mathcal{E} \times \mathcal{F}) \to \mathcal{C}$ by

$$\mathbf{k}((x_1, y_1), (x_2, y_2))\colon = \langle y_1, \langle x_1, x_2 \rangle y_2 \rangle_{\mathcal{F}}.$$

We show that this kernel satisfies 1(a)–(d) of Theorem 3.2.13.

Let $\mathcal{F}^{(n)}$ be the direct sum of $\mathcal{F}$ with itself n times, for any $n \in \mathbb{N}$. $\mathbf{k}$ is weakly positive semidefinite, for we have

$$\begin{aligned}\sum_{i,j=1}^{n} \overline{\alpha_i} \alpha_j \mathbf{k}((x_i, y_i), (x_j, y_j)) &= \sum_{i,j=1}^{n} \langle \alpha_i y_i, \langle x_i, x_j \rangle \alpha_j y_j \rangle \\ &= \langle [\alpha_i y_i]_{i=1}^{n}, [\langle x_i, x_j \rangle]_{i,j=1}^{n} [\alpha_j y_j]_{j=1}^{n} \rangle_{\mathcal{F}^{(n)}} \geq 0\end{aligned}$$

for all $\{\alpha_i\}_{i=1}^{n} \in \mathbb{C}$, $(x_i, y_i)_{i=1}^{n} \in \mathcal{E} \times \mathcal{F}$. The inequality follows since the matrix $[\langle x_i, x_j \rangle]_{i,j=1}^{n} \in \mathrm{M}_n(\mathcal{B})$ is positive by

$$\sum_{i,j=1}^{n} b_i^* \langle x_i, x_j \rangle b_j = \langle \sum_{i=1}^{n} x_i b_i, \sum_{j=1}^{n} x_j b_j \rangle \geq 0$$

for all $\{b_i\}_{i=1}^{n} \in \mathcal{B}$, and by Lemma 5.1.3.

We show that $\mathbf{k}$ is invariant under the left action of $\mathcal{A}$:

$$\begin{aligned}\mathbf{k}((x_1, y_1), a \cdot (x_2, y_2)) &= \langle y_1, \langle x_1, a x_2 \rangle y_2 \rangle \\ &= \langle y_1, \langle a^* x_1, x_2 \rangle y_2 \rangle \\ &= \mathbf{k}(a^* \cdot (x_1, y_1), (x_2, y_2))\end{aligned}$$

for all $a \in \mathcal{A}$ and $(x_1, y_1), (x_2, y_2) \in \mathcal{E} \times \mathcal{F}$, and left invariance is shown. We also have

$$\begin{aligned}\mathbf{k}((x_1, y_1), (x_2, y_2) c) &= \langle y_1, \langle x_1, x_2 \rangle y_2 c \rangle \\ &= \langle y_1, \langle x_1, x_2 \rangle y_2 \rangle c \\ &= \mathbf{k}((x_1, y_1), (x_2, y_2)) c\end{aligned}$$

for all $c \in \mathcal{C}$, $(x_1, y_1), (x_2, y_2) \in \mathcal{E} \times \mathcal{F}$, and $\mathbf{k}$ is right invariant under the action $\mathcal{C}$. Therefore, $\mathbf{k}$ is a doubly invariant kernel under the actions of $\mathcal{A}$ and $\mathcal{C}$.

To verify that condition 1(c) of Theorem 3.2.13 holds, given any $p \in S_*(\mathcal{C})$ and $a \in \mathcal{A}$ we check that

$$p(\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j \mathbf{k}(a \cdot (x_i, y_i), a \cdot (x_j, y_j))) = p(\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j \langle y_i, \langle ax_i, ax_j \rangle y_j \rangle$$
$$\leq c(a)^2 p(\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j \langle y_i, \langle x_i, x_j \rangle y_j \rangle) = p(\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j \mathbf{k}((x_i, y_i), (x_j, y_j)))$$

for all $n \in \mathbb{N}$, $\{\alpha_i\}_{i=1}^n \in \mathbb{C}$, $(x_i, y_i)_{i=1}^n \in \mathcal{E} \times \mathcal{F}$, where the inequality follows by Lemma 5.1.5 for some constant $c(a) \geq 0$. Hence condition 1(c) holds.

In order to see that 1(d) of Theorem 3.2.13 holds, given any $p \in S_*(\mathcal{C})$ and $c \in \mathcal{C}$ we check that

$$p(\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j \mathbf{k}((x_i, y_i) \cdot c, (x_j, y_j) \cdot c)) = p(c^* \sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j \langle y_i, \langle x_i, x_j \rangle y_j \rangle c)$$
$$\leq p(c)^2 p(\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j \mathbf{k}((x_i, y_i), (x_j, y_j)))$$

for all $c \in \mathcal{C}$, $n \in \mathbb{N}$, $(x_i, y_i)_{i=1}^n \in (\mathcal{E} \times \mathcal{F})$, $(\alpha_i)_{i=1}^n \in \mathbb{C}$, by the submultiplicativity of p and condition 1(d) holds.

By theorem 3.2.13, there exists a minimal doubly invariant VH-space linearisation $(\mathcal{K}; V; \pi; \tau)$, which is clearly an $\mathcal{A}$-$\mathcal{C}$ correspondence. By minimality of $\mathcal{K}$, $\mathrm{Lin}V(\mathcal{E} \times \mathcal{F})$ is dense in $\mathcal{K}$. We check that we have $\pi(a)V(x, y) = V(ax, y)$, $V(x, y)\tau(c) = V(x, yc)$ for all $a \in \mathcal{A}$, $c \in \mathcal{C}$, $x \in \mathcal{E}$, $y \in \mathcal{F}$. We also have

$$\begin{aligned} \langle V(xb, y), V(z, t) \rangle_{\mathcal{K}} &= \langle y, \langle xb, z \rangle t \rangle \\ &= \langle by, \langle x, z \rangle t \rangle \\ &= \langle V(x, by), V(z, t) \rangle_{\mathcal{K}} \end{aligned}$$

for all $b \in \mathcal{B}$, $x, z \in \mathcal{E}$ and $y, t \in \mathcal{F}$. By minimality, $V(xb, y) = V(x, by)$ for all $b \in \mathcal{B}$, $x \in \mathcal{E}$ and $y \in \mathcal{F}$. Hence, defining $x \odot y := V(x, y)$, $\mathcal{K} =: \mathcal{E} \odot \mathcal{F}$ is the required space. Finally, uniqueness up to unitary equivalence follows as well. □

5.2.3 A characterization of φ-maps through dilations

For this subsection, following [17], we make the following definitions. Let E and F be locally Hilbert modules over locally C^*-algebras $\mathcal{B}$ and $\mathcal{C}$, respectively. Let φ be a map from $\mathcal{B}$ to $\mathcal{C}$. A linear map $T : E \to F$ is called a *φ-map* if

$$\langle Tx, Ty \rangle = \varphi(\langle x, y \rangle)$$

for all $x, y \in E$.

The following theorem is locally Hilbert module analogue of the Theorem in [17]. A slightly differently stated version of it was proved in Lemma 3.2 of [57]. In this article we will prove it as an application of our Theorem 3.2.13. The tensor product notation $\odot$ is used for the internal tensor product, see Theorem 5.2.2.

Theorem 5.2.3. *Let E and F be locally Hilbert modules over unital locally C^*-algebras $\mathcal{B}$ and $\mathcal{C}$, respectively. Then for every linear map $T : E \to F$ the following are equivalent:*

(i) T is a φ-map for some completely positive map $\varphi : \mathcal{B} \to \mathcal{C}$.

(ii) There exists a pair $(\mathcal{F}, \zeta)$ of locally C^*-correspondence $\mathcal{F}$ from $\mathcal{B}$ to $\mathcal{C}$ and a vector $\zeta \in \mathcal{F}$, and there exists an isometry $v : E \odot \mathcal{F} \to F$ such that

$$T = v(\mathrm{id}_E \odot \zeta) : x \mapsto v(x \odot \zeta).$$

Proof. (ii)$\Rightarrow$(i). $\varphi\colon \mathcal{B} \to \mathcal{C}$ defined by $\varphi(b) := \langle \zeta, b\zeta \rangle$ is such a map. Taking into account Lemma 5.1.3, this follows with the same arguments as in the C^*-algebra case.

(i)$\Rightarrow$(ii). By Paschke's Theorem for locally C^*-algebras, see Theorem 5.2.1, there exists a $\mathcal{B}$-$\mathcal{C}$ correspondence $\mathcal{F}$ and a vector $\zeta \in \mathcal{F}$ such that $\langle \zeta, b\zeta \rangle = \varphi(b)$ for all $b \in \mathcal{B}$ and $\mathcal{F} = \overline{\mathrm{Lin}\mathcal{B}\zeta\mathcal{C}}$.

Consider the kernel $\mathbf{k}\colon \mathcal{E} \times \mathcal{E} \to \mathcal{C}$ given by $\mathbf{k}(x, y) = \varphi(\langle x, y \rangle)$. Since T is a φ-map, it follows that $\overline{\mathrm{Lin}T(E)} \subseteq F$ with linearisation T is a minimal VH-space linearisation of $\mathbf{k}$. On the other hand, since we have

$$\langle x \odot \zeta, y \odot \zeta \rangle = \langle \zeta, \langle x, y \rangle \zeta \rangle = \varphi(\langle x, y \rangle)$$

for all $x, y \in \mathcal{E}$, $\overline{\operatorname{Lin}(E \odot \zeta)} \subseteq E \odot \mathcal{F}$ with linearisation $x \odot \zeta$ is also a minimal VH-space linearisation of $\mathbf{k}$.

By Theorem 3.2.13, there exists a unitary operator $U\colon \overline{\operatorname{Lin}(E \odot \zeta)} \to \overline{\operatorname{Lin} T(E)}$ such that $U(x \odot \zeta) = Tx$ for all $x \in \mathcal{E}$. Since $\mathcal{F} = \overline{\operatorname{Lin} \mathcal{B} \zeta \mathcal{C}}$ and $x \odot b\zeta = xb \odot \zeta$ for all $x \in \mathcal{E}$ and $b \in \mathcal{B}$, U extends uniquely to a unitary operator, which we still denote with U, $U\colon \mathcal{E} \odot \mathcal{F} \to \overline{\operatorname{Lin} T(\mathcal{E})\mathcal{C}}$, which can be considered an isometry $v\colon \mathcal{E} \odot \mathcal{F} \to F$. □

5.2.4 Dilations of Continuous Mapping Valued Completely Positive Definite Kernels

Following [9], we define completely positive definite kernels in the context of locally C^*-algebras. Let X be a nonempty set and A and B be pre-locally C^*-algebras. Let $\mathcal{L}_c(A, B)$ be the set of all linear continuous mappings from A to B. Let $\mathbf{k} : X \times X \to \mathcal{L}_c(A, B)$ be a kernel. Then $\mathbf{k}$ is called *completely positive definite* if it satisfies

$$\sum_{i,j} b_i^* \mathbf{k}(x_i, x_j)(a_i^* a_j) b_j \geq 0 \tag{5.7}$$

for all $n \in \mathbb{N}$, $\{x_i\}_{i=1}^n \in X$, $\{a_i\}_{i=1}^n \in A$, $\{b_i\}_{i=1}^n \in B$.

The following theorem is a locally C^*-algebra version of Theorem 3.2.3 in [9].

Theorem 5.2.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital pre-locally C^*-algebras, X a nonempty set, $\mathbf{k}\colon X \times X \to \mathcal{L}_c(\mathcal{A}, \mathcal{B})$ be a kernel satisfying the hypothesis of Lemma 5.1.7, that is, a completely positive definite kernel having the property that for any $p \in S_*(\mathcal{B})$ and $a \in \mathcal{A}$ there exists $q \in S_*(\mathcal{A})$ such that $p(\mathbf{k}(x, y)(a)) \leq q(a)$ holds for all $x, y \in X$. Then there exists a pre-locally Hilbert $\mathcal{A}$-$\mathcal{B}$-module $\mathcal{E}$ with the representations of $\mathcal{A}$ and $\mathcal{B}$ both continuous and a mapping $i : X \to \mathcal{E}$ such that*

$$\mathbf{k}(x, y)(a) = \langle i(x), a i(y) \rangle \tag{5.8}$$

for all $x, y \in X$ and $a \in \mathcal{A}$ and $\mathcal{E}$ is minimal in the sense that it is the linear span of $\mathcal{A}i(X)\mathcal{B}$. Moreover, $(\mathcal{E}, i)$ is unique up to unitary equivalence.

Conversely, if $\mathcal{E}$ is a pre-locally Hilbert $\mathcal{A}$-$\mathcal{B}$ module with both representations continuous and $X \subseteq \mathcal{E}$, then the kernel $\mathbf{k}(x,y)(a) := \langle x, ay \rangle$, $x, y \in X$ and $a \in \mathcal{A}$ is completely positive definite.

Proof. Consider the set $\mathcal{A} \times X \times \mathcal{B}$, with a left action of $\mathcal{A}$ by $a \cdot (a_1, x_1, b_1) = (aa_1, x_1, b_1)$ for all $a \in \mathcal{A}$ and $(a_1, x_1, b_1) \in (\mathcal{A} \times X \times \mathcal{B})$ and a right action of $\mathcal{B}$ by $(a_1, x_1, b_1) \cdot b := (a_1, x_1, b_1 b)$ for all $b \in \mathcal{B}$ and $(a_1, x_1, b_1) \in (\mathcal{A} \times X \times \mathcal{B})$. Define kernel $\mathbf{l}$: $(\mathcal{A} \times X \times \mathcal{B}) \times (\mathcal{A} \times X \times \mathcal{B}) \to \mathcal{B}$, where the pre C^*-algebra $\mathcal{B}$ is regarded as a topological VE-space, see [24], by

$$\mathbf{l}((a_1, x_1, b_1), (a_2, x_2, b_2)) := b_1^* \mathbf{k}(x_1, x_2)(a_1^* a_2) b_2$$

for all $(a_1, x_1, b_1), (a_2, x_2, b_2) \in (\mathcal{A} \times X \times \mathcal{B})$. We check that $\mathbf{l}$ is weakly positive semidefinite. We have

$$\sum_{i,j=1}^{n} \overline{\alpha_i} \alpha_j \mathbf{l}((a_i, x_i, b_i), (a_j, x_j, b_j)) = \sum_{i,j=1}^{n} (\alpha_i b_i)^* \mathbf{k}(x_i, x_j)(a_i^* a_j)(\alpha_j b_j) \geq 0$$

for all $n \in \mathbb{N}$, $(\alpha_i)_{i=1}^n \in \mathbb{C}$, $(a_i, x_i, b_i)_{i=1}^n \in (\mathcal{A} \times X \times \mathcal{B})$ and $\mathbf{l}$ is weakly positive semidefinite.

We show that $\mathbf{l}$ is left invariant under the left action of $\mathcal{A}$.

$$\begin{aligned}\mathbf{l}((a_, x_1, b_1), a \cdot (a_2, x_2, b_2)) &= b_1^* \mathbf{k}(x_1, x_2)(a_1^* a a_2) b_2 \\ &= \mathbf{l}(a^*(a_1, x_1, b_1), (a_2, x_2, b_2))\end{aligned}$$

for all $a \in \mathcal{A}$ and $(a_1, x_1, b_1), (a_2, x_2, b_2) \in (\mathcal{A} \times X \times \mathcal{B})$ holds, so left invariance of the kernel $\mathbf{l}$ is shown.

To see the right invariance of $\mathbf{l}$ under the right action of $\mathcal{B}$, note that

$$\begin{aligned}\mathbf{l}((a_1, x_1, b_1), (a_2, x_2, b_2) \cdot b) &= (b_1^* \mathbf{k}(x_1, x_2)(a_1^* a_2) b_2) b \\ &= \mathbf{l}((a_1, x_1, b_1), (a_2, x_2, b_2)) \cdot b\end{aligned}$$

for all $b \in \mathcal{B}$ and $(a_1, x_1, b_1), (a_2, x_2, b_2) \in (\mathcal{A} \times X \times \mathcal{B})$. Hence $\mathbf{l}$ is right invariant as well. We conclude that $\mathbf{l}$ is a doubly invariant kernel under the actions of $\mathcal{A}$ and $\mathcal{B}$.

For the boundedness condition 1(c) of Theorem 3.2.15, note that given any $p \in S_*(\mathcal{B})$ and $a \in \mathcal{A}$, by Lemma 5.1.7 we have

$$
\begin{aligned}
&p(\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j \mathbf{l}(a(a_i, x_i, b_i), a(a_j, x_j, b_j))) \\
&\qquad = p(\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j b_i^* \mathbf{k}(x_i, x_j)(a_i^* a^* a a_j) b_j) \\
&\leq c(a)^2 p(\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j b_i^* \mathbf{k}(x_i, x_j)(a_i^* a_j) b_j) \\
&\qquad = c(a)^2 p(\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j \mathbf{l}((a_i, x_i, b_i), (a_j, x_j, b_j)))
\end{aligned}
$$

for all $n \in \mathbb{N}$, $(\alpha_i)_{i=1}^n \in \mathbb{C}$, $(a_i, x_i, b_i)_{i=1}^n \in (\mathcal{A} \times X \times \mathcal{B})$, where for some $q \in S_*(\mathcal{A})$ we have $c(a) \leq q(a)$ for all $a \in \mathcal{A}$. So condition 1(c) of Theorem 3.2.15 holds.

For the boundedness condition 1(d) of Theorem 3.2.15, note that given any $p \in S_*(\mathcal{B})$ and $b \in \mathcal{B}$ we have

$$
\begin{aligned}
&p(\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j \mathbf{l}((a_i, x_i, b_i) \cdot b, (a_j, x_j, b_j) \cdot b)) \\
&\qquad = p(\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j b^* b_i^* \mathbf{k}(x_i, x_j)(a_i^* a_j) b_j b) \\
&\leq p(b)^2 p(\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j b_i^* \mathbf{k}(x_i, x_j)(a_i^* a_j) b_j) \\
&\qquad = p(b)^2 p(\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j \mathbf{l}((a_i, x_i, b_i) \cdot b, (a_j, x_j, b_j) \cdot b))
\end{aligned}
$$

since p is submultiplicative and $p(b^*) = p(b)$. Hence 1(d) of Theorem 3.2.15 holds with $d(b) \leq p(b)$.

By Theorem 3.2.15 there exists a minimal doubly invariant topological VE-space linearisation $(\mathcal{E}; V; \pi; \tau)$ with $\pi \colon \mathcal{A} \to \mathcal{L}_{c,p}^*(\mathcal{E})$ and $\tau \colon \mathcal{B} \to \mathcal{L}_{c,p}(\mathcal{E})$, which turns out to be a pre-Hilbert $\mathcal{A}$-$\mathcal{B}$ module. Define $i : X \to \mathcal{E}$ by $i(x) := V(e_{\mathcal{A}}, x, e_{\mathcal{B}})$ where $e_{\mathcal{A}}$ and $e_{\mathcal{B}}$ are the units of $\mathcal{A}$ and $\mathcal{B}$, respectively. Then we have

$$\mathbf{k}(x_1, x_2)(a) = \mathbf{l}((e_{\mathcal{A}}, x_1, e_{\mathcal{B}}), (ae_{\mathcal{A}}, x_2, e_{\mathcal{B}})) = \langle i(x_1), \pi(a)i(x_2)\rangle$$

for all $x_1, x_2 \in X$ and $a \in \mathcal{A}$. We also have $\mathcal{E} = \text{Lin}V(\mathcal{A} \times X \times \mathcal{B}) = \text{Lin}\pi(\mathcal{A})i(X)\tau(\mathcal{B})$, by minimality. A pair $\mathcal{E}, i$ as in (5.8) produces a minimal doubly invariant topological VE-space linearisation of the kernel $\mathbf{l}$, so it should be unique up to unitary equivalence by Theorem 3.2.15. Finally, since we have $c(a) \leq q(a)$ and $d(b) \leq p(b)$, the representations π and τ are continuous.

Converse statement is straightforward and is left to the reader. □

Remark 5.2.5. As in the proof of Lemma 5.1.8, putting $\mathbf{k}(x, x) := \varphi$ where φ is as in the statement of Theorem 5.2.1 and an application of Theorem 5.2.4 produces Theorem 5.2.1. The details are left to the reader.

Bibliography

[1] M. A. Naimark, "On the representations of additive operator set functions," *C.R. (Doklady) Acad. Sci. USSR*, vol. 41, pp. 359–361, 1943.

[2] M. A. Naimark, "Positive definite operator functions on a commutative group," *Bull. (Izv.) Acad. Sci. USSR*, vol. 7, pp. 237–244, 1943.

[3] B. Sz.-Nagy, "Prolongement des transformations de l'espace de hilbert qui sortent de cet espace, par f. riesz et b. sz.-nagy," *Appendice au livre "Leçons d'analyse fonctionnelle" par F. Riesz et B. Sz.-Nagy*, pp. 439–573, 1955.

[4] W. F. Stinespring, "Positive functions on C^*-algebras," *Proc. Amer. Math. Soc.*, vol. 6, pp. 211–216, 1955.

[5] R. M. Loynes, "On generalized positive-definite function," *Proc. London Math. Soc. III. Ser.*, vol. 15, pp. 373–384, 1965.

[6] G. G. Kasparov, "Hilbert C^*-modules: theorems of stinespring and voiculescu," *J. Operator Theory*, vol. 4, pp. 133–150, 1980.

[7] N. Aronszajn, "Theory of reproducing kernels," *Trans. Amer. Math. Soc.*, vol. 68, pp. 337 – 404, 1950.

[8] W. B. Arveson, "Dilation theory yesterday and today, in *A glimpse at Hilbert space operators*," *Oper. Theory Adv. Appl.*, vol. 207, pp. 99 – 123, 2010.

[9] S. D. Barreto, B. V. R. Bhat, V. Liebscher, and M. Skeide, "Type i product systems of Hilbert modules," *J. Func. Anal.*, vol. 212, pp. 121 – 181, 2004.

[10] D. Gaşpar and P. Gaşpar, "Reproducing kernel Hilbert modules over locally C^* algebras," *Analele Univ. de Vest Timişoara Ser. Mat.-Inform.*, vol. 45, pp. 245 – 252, 2007.

[11] D. Gaşpar and P. Gaşpar, "Reproducing kernel Hilbert $B(X)$-modules," *Analele Univ. de Vest Timişoara Ser. Mat.-Inform.*, vol. 43, pp. 47 – 71, 2005.

[12] A. Gheondea and B. E. Uğurcan, "On two equivalent dilation theorems in VH-spaces," *Complex Analysis and Operator Theory*, vol. 6, pp. 625 – 650, 2012.

[13] J. Górniak and A. Weron, "An analogue of Sz.-Nagy's dilation theorem," *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, vol. 24, pp. 867 – 872, 1976.

[14] J. Górniak and A. Weron, "Aronszajn–Kolmogorov type theorems for positive definite kernels in locally convex spaces," *Studia Math.*, vol. 69, pp. 235 – 246, 1981.

[15] J. Heo, "Reproducing kernel Hilbert C^*-modules and kernels associated to cocycles," *J. Mathematical Physics*, vol. 49, p. 103507, 2008.

[16] G. J. Murphy, "Positive definite kernels and Hilbert C^*-modules," *Proc. Edinburgh Math. Soc.*, vol. 40, pp. 367–374, 1997.

[17] M. Skeide, "A factorization theorem for φ-maps," *J. Opr. Theory*, vol. 68:2, pp. 543–547, 2012.

[18] F. H. Szafraniec, "Dilations of linear and nonlinear maps, in *Functions, series, operators*, vol. i, ii," *Colloq. Math. Soc. Janos Bolyai*, vol. 35, pp. 1165–1169, 1983.

[19] F. H. Szafraniec, "Murphy's positive definite kernels and Hilbert C^*-modules reorganized," *Banach Center Publications*, vol. 89, pp. 275–295, 2010.

[20] D. E. Evans and J. T. Lewis, "Dilations of irreducible evolutions in algebraic quantum theory," *Comm. Dublin Inst. Adv. Studies Ser. A*, vol. 24, 1977.

[21] K. R. Parthasaraty and K. Schmidt, *Positive-Definite Kernels, Continous Tensor Products and Central Limit Theorems of Probability Theory, Lecture Notes in Mathematics, Vol.* **272**. Springer-Verlag, Berlin, 1972.

[22] S. Ay and A. Gheondea, "Representations of $*$-semigroups associated to invariant kernels with values adjointable operators," *Linear Algebra Appl.*, vol. 486, pp. 361 – 388, 2015.

[23] A. Gheondea, "Dilations of some VH-spaces operator valued kernels," *Integral Equations and Operator Theory*, vol. 74, pp. 451 – 479, 2012.

[24] S. Ay and A. Gheondea, "Representations of $*$-semigroups associated to invariant kernels with values continuously adjointable operators," *Integral Equations and Operator Theory*, vol. 87, pp. 263 – 307, 2017.

[25] S. Ay and A. Gheondea, "Invariant weakly positive semidefinite kernels with values in topologically ordered $*$-spaces," *Studia Mathematica, to appear.*

[26] R. M. Loynes, "Linear operators in VH-spaces," *Trans. Amer. Math. Soc.*, vol. 116, pp. 167–180, 1965.

[27] T. Constantinescu and A. Gheondea, "Representations of hermitian kernels by means of Krein spaces. ii. invariant kernels.," *Comm. Math. Phys.*, vol. 216, pp. 409 – 430, 2001.

[28] W. A. Paschke, "Inner product modules over B^*-algebras," *Trans. American Math. Soc.*, vol. 182, pp. 443–468, 1973.

[29] I. Kaplansky, "Modules over operator algebras," *Amer. J. Math.*, vol. 75, pp. 839–853, 1953.

[30] M. A. Rieffel, "Induced representations of C^*-algebras," *Adv. Math.*, vol. 13, pp. 176–257, 1974.

[31] E. Lance, *Hilbert C^*-Modules. A toolkit for operator algebraists.* London Mathematical Society Lecture Note Series, 210. Cambridge University Press, Cambridge 1995, 1995.

[32] V. Manuilov and E. Troitsky, *Hilbert C^*- Modules.* Amer. Math. Soc., Providence R.I., 2005.

[33] P. P. Saworotnow, "Linear spaces with an H^*-algebra-valued inner product," *Trans. Amer. Math. Soc.*, vol. 262, pp. 543–549, 1980.

[34] A. Mallios, "Hermitian k-theory over topological $*$-algebras," *J. Math. Anal. Appl.*, vol. 106, pp. 454–539, 1985.

[35] N. C. Phillips, "Inverse limits of C^*-algebras," *J. Operator Theory*, vol. 19, pp. 159–195, 1988.

[36] M. Joiţa, "Hilbert modules over locally C^*-algebras," *Editura Universitatii Bucureşti*, 2006.

[37] T. Constantinescu and A. Gheondea, "Representations of hermitian kernels by means of Krein spaces," *Publ. RIMS. Kyoto Univ.*, vol. 33, pp. 917 – 951, 1997.

[38] D. Gaşpar and P. Gaşpar, "An operatorial model for hilbert $B(X)$-modules," *Analele Univ. de Vest Timişoara Ser. Mat.-Inform.*, vol. 40, pp. 15 – 29, 2002.

[39] V. R. Paulsen and M. Tomforde, "Vector spaces with an order unit," *Indiana Univ. Math. J.*, vol. 58, pp. 1319–1359, 2009.

[40] L. Ciurdariu, "Classes of linear operators on pseudo-Hilbert spaces and applications, part i," *Monografii Matematice, Tipografia Universităţii de Vest din Timişoara*, vol. 79, 2006.

[41] I. Namioka, *Partially Ordered Linear Topological Spaces, Memoirs of the AMS, Number:24.* American Mathematical Society, 1957.

[42] C. Aleprantis and R. Tourky, *Cones and Duality*, vol. 84. American Mathematical Society, 2007.

[43] A. Inoue, "Locally C^*-algebras," *Mem. Fac. Sci. Kyushu Univ. Ser. A*, vol. 25, pp. 197–235, 1971.

[44] A. Weron, “Prediction theory in Banach spaces,” *Proceedings of the Winter School of Probability, Karpacz*, pp. 207–228, 1975.

[45] J. B. Conway, *A Course in Functional Analysis*. Springer Verlag, 1985.

[46] F. Pater and T. Bînzar, “On some dilation theorems for positive definite operator valued functions,” *Studia Mathematica*, vol. 228(2), pp. 109–122, 2015.

[47] F. Pater and T. Bînzar, “Correction to ’on some dilation theorems for positive definite operator valued functions’,” *Studia Mathematica, to appear.*

[48] K. Schmüdgen, “Über LMC^*-algebren,” *Math. Nachr.*, vol. 68, pp. 167–182, 1975.

[49] G. R. Allan, “On a class of locally convex algebras,” *Proc. London Math. Soc.*, vol. 15, pp. 399 – 421, 1965.

[50] C. Apostol, “b^*-algebras and their representations,” *J. London Math. Soc.*, vol. 33, pp. 30 – 38, 1971.

[51] D. Voiculescu, “Dual algebraic structures on operator algebras related to free products,” *J. Operator Theory*, vol. 17, pp. 85–98, 1987.

[52] Z. Sebestyen, “Every C^*-seminorm is automatically submultiplicative,” *Period. Math. Hun.*, vol. 10, pp. 1–8, 1979.

[53] F. S. Yu. I. Zhuraev, “Hilbert modules over locally C^*-algebras, preprint,” *arXiv:math/0011053v3 [math.OA]*, 2001.

[54] J. Weidner, “Topological invariants for generalized operator algebras, phd disseration,” *J. Operator Theory*, 1987.

[55] R. Arens, “A generalization of normed rings,” *Pacific J. Math.*, vol. 2, pp. 455 – 471, 1952.

[56] M. Joiţa, “Strict completely positive maps between locally C^*-algebras and representations on Hilbert modules,” *J. London Mat. Soc.*, vol. 66, pp. 421–432, 2002.

[57] K. Karimi and K. Sharifi, "Completely positive maps on Hilbert modules over pro-C^*-algebras," *Bull. Math. Soc. Sci. Math. Roumanie*, vol. 60(108), pp. 181–193, 2017.