THE OPTIMAL STRUCTURE OF INCENTIVES IN THE PRINCIPAL–AGENT PROBLEM UNDER THE SMOOTH AMBIGUITY MODEL

A Master's Thesis

by
FATİH ÖZTÜRK

Department of
Economics
İhsan Doğramacı Bilkent University
Ankara
July 2018

FATİH ÖZTÜRK
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Bilkent University 2018

To my family

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The Graduate School of Economics and Social Sciences
of
İhsan Doğramacı Bilkent University

by

FATİH ÖZTÜRK

In Partial Fulfilment of the Requirements for the Degree of
MASTER OF ARTS IN ECONOMICS

THE DEPARTMENT OF ECONOMICS
İHSAN DOĞRAMACI BİLKENT UNIVERSITY
ANKARA

July 2018

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Assist. Prof. Dr. Nuh Aygün Dalkıran
Supervisor

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Assist. Prof. Dr. Tarık Kara
Examining Committee Member

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Assoc. Prof. Dr. Serkan Küçükşenel
Examining Committee Member

Approval of the Graduate School of Economics and Social Sciences

Prof. Dr. Halime Demirkan
Director

ABSTRACT

THE OPTIMAL STRUCTURE OF INCENTIVES
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Öztürk, Fatih
M.A., Department of Economics
Supervisor: Assist. Prof. Dr. Nuh Aygün Dalkıran

July 2018

We study a principal-agent problem where we model the uncertainty regarding the relationship between the level of effort and the monetary outcome using the smooth ambiguity model (Klibanoff et al., 2005). We provide comparative statics on the optimal wage scheme when the agent has constant absolute ambiguity aversion (CAAA) preferences. Our main result implies that whether the optimal wage scheme should be higher-powered or lower-powered depends on an intuitive measure of ambiguity.

Keywords: Ambiguity Aversion, Incentives, Moral Hazard, Optimal Contract.

ÖZET

PÜRÜZSÜZ MUĞLAKLIK MODELİ ALTINDA ASİL-VEKİL PROBLEMİNDEKİ TEŞVİKLERİN OPTİMAL YAPISI

Öztürk, Fatih
Yüksek Lisans, İktisat Bölümü
Tez Yöneticisi: Dr. Öğr. Üyesi Nuh Aygün Dalkıran

Temmuz 2018

Bu tezde pürüzsüz muğlaklık modeli (Klibanoff et al., 2005) kullanılarak efor ve parasal çıktı arasındaki ilişkiye dair belirsizliğin modellendiği asil-vekil problemi çalışılmıştır. Vekil sabit mutlak muğlaklıktan kaçınma tercihlerine sahipken optimal maaş planı üzerine karşılaştırmalı statik yapılmıştır. Temel çıkarımımız optimal maaş planının yüksek ya da düşük teşvikli oluşunun sezgisel bir muğlaklık ölçüsüne bağlı olduğudur.

Anahtar Kelimeler: Ahlaki Tehlike, Muğlaklıktan Kaçınma, Optimal Kontrat, Teşvikler.

ACKNOWLEDGMENTS

I would like to thank my thesis advisor Assistant Professor Nuh Aygün Dalkıran for his invaluable guidance and continuous support. His assistance helped me overcome many challenges and allowed me to complete this thesis.

I would like also thank to Assistant Professor Tarık Kara as the second reader of this thesis for his valuable comments on this thesis. I want to express my gratitude to Asssociate Professor Serkan Küçükşenel as an examining committee member.

Special thanks to all my precious friends, especially to Ferhat Arberkli, Asu İşbilen, Begüm Özdemir, and Arda Benli, for their support through the process of writing this thesis.

Finally, but by no means least, I would like to express my very great appreciation to my family members, especially my mother for her moral and emotional support in my life. They have provided me with unfailing support and continuous encouragement throughout my years of study.

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INTRODUCTION

1.1 Principal-agent problem

It has been well understood by economists that information asymmetries may lead markets to malfunction. In particular, the presence of moral hazard may lead to inefficient market allocations. Contracts have been an important part of an economy, where moral hazard has been frequently observed. This is why there has been a vast literature analyzing the properties of optimal contracts under moral hazard.

In the classical principal-agent problem that analyzes optimal contracts under moral hazard, there is a principal who is to hire an agent to perform a certain task. The principal cannot observe the level of effort exerted by the agent. However, the principal observes the monetary outcome that is correlated with the exerted level of effort. Therefore, an optimal contract is to provide incentives that leads the agent to exert the level of effort that is best for the principal.

The essence of the incentive problem in principal-agent models is uncertainty. Due to uncertainty, there is an imperfect understanding between the principal and the agent regarding the relationship between the level of effort exerted by the agent and the corresponding monetary outcome of the specific task.

Not surprisingly, most of the literature on principal-agent problems models this uncertainty by employing the standard expected utility model. However, it has been well documented that the standard expected utility model performs poorly in describing how decision makers act under uncertainty.

In this paper, we employ the smooth ambiguity model introduced by Klibanoff et al. (2005) to model the uncertainty regarding the relationship between the level of effort and the corresponding monetary outcome in a principal-agent problem. We believe that the smooth ambiguity model might give better insights than the standard expected utility model since the latter might perform poorly when modeling the uncertainty regarding the relationship between the level of effort and the corresponding monetary outcome.

1.2 Why ambiguity matters/Ellsberg paradox

Daniel Ellsberg (1961) presents two thought-experiments regarding decision making under uncertainty that weakens the validity of *the subjective expected utility hypothesis*. There exist extensive experimental and empirical evidence on attitudes toward ambiguity. For further discussion of the literature, see Machina & Siniscalchi (2014).

Ellsberg's thought-experiment with two urns is known as the *Two-Urn Paradox*. In this thought experiment, the first urn consists of 50 black balls and 50 red balls while the second urn contains 100 balls, each of which is either black or red. The ratio of red balls to black ones in the second urn is unknown. There are four lotteries that is to be considered by a hypothetical decision maker:

L_1 : \$100 if a black ball is drawn from the first urn, \$0 otherwise

L_2 : \$100 if a red ball is drawn from the first urn, \$0 otherwise

L_3 : \$100 if a black ball is drawn from the second urn, \$0 otherwise

L_4 : \$100 if a red ball is drawn from the second urn, \$0 otherwise

Ellsberg notes that a decision maker would prefer L_1 over L_3 and L_2 over L_4. However, this type of preferences violate the subjective expected utility hypothesis since they imply:

$$P(\text{Red}|\text{Urn II}) + P(\text{Black}|\text{Urn II}) < 1$$

Ellsberg describes lotteries L_3 and L_4 as involving *ambiguity*. *Ambiguity aversion* refers to a type of aversion towards subjective uncertainty. This implies that decision makers may display a preference for known probabilities over unknown probabilities.

1.3 Related literature

Even though the agency relationships are ubiquitous in an economy, the origins of the *principal-agent problem* has not been commonly known among economists. The first paper that we are aware of, which used the name "Principal's Problem" in its title, is Ross (1973). In his paper, Ross lays out an analytical framework to study *agency problems*. He explicitly describes the principal's and agent's problems as optimization problems under uncertainty. Since Ross's work there has been a vast literature focusing on principal-agent problems. We briefly review in this section only a few of the seminal papers in this literature – possibly missing out many other important papers.

Two of the of the first papers that studied optimal structure of incentives within an organization and under imperfect monitoring are Mirrlees (1976) and Harris & Raviv (1979). Hölmstrom (1979) extends their work by developing an *informativeness principle* regarding the optimal incentive contracts and shows that inefficiencies may arise when only the monetary outcome is observable to the principal. That is, under imperfect information optimal contracts tends to be *the second best* due to the presence of *moral hazard*. Grossman & Hart (1983) provide sufficient conditions that make optimal compensation scheme monotonic in the outcomes and show that the welfare loss due to *moral hazard* increases when the agent becomes more risk averse –in the case of two outcomes.

All of these papers employ the standard expected utility theory to model the uncertainty regarding the relationship between levels of effort and monetary outcomes. That is, in these papers, it is assumed that any effort level induces a unique and additive (subjective) probability measure over the outcomes. However, it is now well-understood that the standard expected utility theory performs poorly in describing decision making under uncertainty.

The famous Ellsberg (1961) experiment implies that economic agents may fail to be probabilistically sophisticated – weakening the validity of the standard expected utility theory. This is why there has been a vast literature on decision making under uncertainty which allows agents to consider multiple prior probability distributions before they act under uncertainty. This subfield of decision making under uncertainty literature is commonly known as the *ambiguity* literature.

The first paper that incorporates multiple priors into decision making under uncertainty is Gilboa & Schmeidler (1989). Their model, which is also known as *maxmin expected utility*, assumes that decision makers have a distaste for ambiguity, i.e., a distaste for subjective uncertainty. Gilboa & Schmeidler (1989) fully characterize when a decision maker can be modeled as if he holds a set of priors in his mind and whenever he makes a choice he maximizes the expected utility with respect to the worst prior in his mind. However, their model is criticized since evaluating acts by their minimal expected utility seems to be too extreme hence restrictive. Furthermore, maxmin expected utility is analytically less tractable when compared to the standard expected utility model – which makes it relatively hard to use in economic applications to conduct comparative statics.

Klibanoff et al. (2005) instead propose and characterize a smooth model of ambiguity that improves analytical tractability. Their model provides a double expectational form to evaluate a decision maker's preferences towards ambiguity. It also allows the separation of the agent's attitude towards risk and his attitude towards ambiguity. Furthermore, their model allows a separation between ambiguity and ambiguity aversion in terms of subjective beliefs and tastes. Being analytically more tractable and separating the agent's taste for ambiguity, the smooth ambiguity model has been widely used in economic applications involving decision making under (subjective) uncertainty.

To the best of our knowledge, the first paper that studies the principal-agent problem under ambiguity is Ghirardato (1994). Ghirardato analyzes how the principal's profit is affected by ambiguity aversion by employing the Choquet Expected Utility introduced by Schmeidler (1989). Our work is mostly related to that of Kellner (2017) who also studies the principal agent problem under the smooth ambiguity model. Kellner shows that the profit of the principal decreases as he wants to implement a more ambiguous effort level – due to ambiguity aversion of the agent. In the following section, we discuss our motivation and contribution where we provide further details regarding how our results differs/complements that of Kellner (2017).

1.4 Motivation and contribution

Our main objective is to understand the effect of ambiguity (aversion) on the optimal contract in a standard principal-agent setting. To this end, we employ the smooth ambiguity model introduced by Klibanoff et al. (2005). We restrict ourselves to the case where there are only two possible levels of effort, two possible monetary outcomes and two possible probability measures. The smooth ambiguity model, being analytically tractable, allows us in this setup to conduct thorough comparative statics and hence leading us to a better understanding of the effects of ambiguity in principal-agent settings.

As mentioned before, we are not the first to analyze the optimal contract under ambiguity in a principal-agent setting. Ghirardato (1994) analyzes the properties of the optimal contract under ambiguity by employing Choquet Expected utility. Our paper is however more related to Kellner (2017) who analyzes the optimal incentive schemes in a principal-agent problem under the smooth ambiguity model. Even though the setup we consider is very similar, our methodology is different. We reconfirm some of the results Kellner (2017) obtains and provide new results and insight as described below.

Our first result, Theorem 1, considers the first best scenario which analyzes the case where the level of effort is observable to the principal. We show that in the first best scenario the individual rationality constraint (IR) binds under a natural *monotone likelihood ratio property* without the need of restricting the agent to constant absolute ambiguity aversion (CAAA) preferences. Furthermore, we show that the optimal contract in the fisrt best scenario is a *constant wage scheme* – again without restricting the agent's preferences to CAAA preferences.

Our second result, Theorem 2, considers the optimal incentive scheme under moral hazard, i.e., when the level of effort is not observable to the principal. We show that the agent always receives his reservation utility if his preferences belongs to the class of constant absolute ambiguity aversion (CAAA) preferences as in Kellner (2017). We also show that the optimal wage scheme is *monotonic*, i.e., the higher monetary outcome requires a higher compensation to the agent under a natural *monotone likelihood ratio property.* We also show that the incentive compatibility constraint (IC) binds if the agent's preferences are of CAAA type.

Even though we fail to provide a closed form solution for the optimal incentive scheme under moral hazard, we provide thorough comparative statics over the optimal contract under moral hazard (Propositions 1-11). Since we restrict ourselves to the case of two monetary outcomes, *success* and *failure*, we consider the agent's compensation as a fixed wage (utility) plus a a bonus wage (utility).

To understand whether a parameter change makes the optimal contract higher-powered or lower-powered, we focus our attention to the extra utility given to the agent when the principal observes the *success* outcome.[1] A decrease in the bonus utility means a decrease in incentives for the agent to exert the high level of effort –optimal contract becoming lower-powered– whereas an increase in the bonus utility means an increase in the incentives for the agent to exert the high level of effort –optimal contract becoming higher-powered.[2] This is why we believe understanding how the bonus utility changes when a parameter of the principal's problem changes is important. Hence, we provide detailed comparative statics on the bonus utility.

In particular, we observe that an increase in risk aversion always increases the bonus utility if the agent has constant absolute risk aversion (CARA) preferences in which the cost of effort is additively separable as in Grossman & Hart (1983). However, we show that this is not the case for ambiguity aversion. That is, an increase in ambiguity aversion may not increase the bonus utility. We identify

[1]We measure the bonus utility by employing the von Neumann-Morgenstern utility function component, u, of the smooth ambiguity model.

[2](Bolton & Dewatripont, 2005: p.133).

three separate cases in which the bonus utility behaves differently when there is an increase in ambiguity aversion. Furthermore, we show, by an example, that the bonus utility and the principal's profit does not have to be negatively correlated.

The paper is organized as follows. We outline our model in Section 2. In Section 3, we study the optimal contract for when there are two levels of effort and two monetary outcomes where the agent considers two possible probability measures the levels of effort induce over the monetary outcomes. In Section 4, we provide thorough comparative statics on the optimal contract.

CHAPTER 2

THE MODEL

There is a principal who is to hire an agent to perform a specific task. The monetary outcome of this task, denoted by x, is correlated with the level of effort exerted by the agent, denoted by e. The set of all possible monetary outcomes of the task is denoted by X. The set of all possible levels of effort available to the agent is denoted by E. We assume both X and E are finite.

There is a moral hazard problem, i.e., the principal cannot observe the level of effort exerted by the agent. Therefore, the wage scheme offered to the agent –in a contract– can only depend on the monetary outcome of the task. We represent this wage scheme by the function $w : X \to \mathbb{R}$.

Due to uncertainty, the principal and the agent share an imperfect understanding of the relationship between the level of effort exerted by the agent and the corresponding monetary outcome. We model this imperfect understanding as ambiguity as follows: Both the principal and the agent consider a finite set of probability measures, denoted by J, over the set of monetary outcomes given the level of effort exerted by the agent. Both the principal and the agent are uncertain about which probability measure truly describes the likelihoods of possible monetary outcomes when the agent exerts a certain level of effort. Furthermore, they both agree that a set of probability measures (multiple priors) rather than a single probability measure (unique prior) can better represent the relationship between the possible levels of effort and the monetary outcomes.

Furthermore, both the principal and the agent attribute subjective likelihoods to the members of the set of probability measures J. These subjective likelihoods represent their beliefs regarding the likelihoods of various probability measures in J. For any given $e \in E$, the subjective likelihoods, or the second order beliefs, are represented by π^e defined on $\Delta(J)$.[1]

2.1 The preferences under ambiguity

We focus our attention to the smooth ambiguity utility representation proposed and characterized by Klibanoff et al. (2005). Under the smooth ambiguity model, the preferences of the agent can be represented by the following function of the double expectational form:

$$U(e,w) = \int_{\Delta} \phi\left(\int_{S} u(w) - c(e)dp \right) d\pi$$

The function $u : \mathbb{R} \to \mathbb{R}$ is assumed to be strictly increasing, concave, and differentiable, characterizing the agent's attitude towards risk, while the function $\phi : \mathbb{R} \to \mathbb{R}$ is assumed to be strictly increasing, and differentiable, capturing the agent's attitude towards ambiguity. A concave ϕ represents ambiguity averse preferences while an affine ϕ represents ambiguity neutral preferences.

Exerting effort is costly. The effort level e leads the agent to incur an effort cost of $c(e)$. The function $c : E \to \mathbb{R}$ is assumed to be nonnegative, strictly increasing, and convex, representing the disutility coming from exerting effort.

[1] $\Delta(J)$ denotes the set of all probability measures on J.

Given an effort level $e \in E$, the probability of achieving monetary outcome x_i according to the j-th probabilty measure in J is represented by p_{ij}^e, while the subjective likelihood attributed by the agent to this probability measure is represented by π_j^e.

We assume the agent is ambiguity averse in the sense of Klibanoff et al. (2005) and also that the cost of effort is additively separable. Therefore, the agent's payoff can be represented by the utility function U as follows:

$$U(e, w) = \sum_{j=1}^{J} \pi_j^e \phi\left(\sum_{i=1}^{I} p_{ij}^e u(w_i) - c(e) \right).$$

On the other hand, the principal is assumed to be both ambiguity and risk neutral. Hence, the principal's payoff can be represented by the utility function V as follows:

$$V(e, w) = \sum_{j=1}^{J} \pi_j^e \left(\sum_{i=1}^{I} p_{ij}^e (x_i - w_i) \right).$$

2.2 The principal's problem

The principal's goal is to induce the agent to exert the level of effort that is best for himself by choosing a wage scheme $w : X \to \mathbb{R}$. That is, the principal pays the wage $w(x_i)$ to the agent when he observes the monetary outcome $x_i \in X$. The agent accepts such a wage scheme only if he is better off than his outside option, i.e., when an *Individual Rationality* (IR) condition holds. We denote the corresponding utility of the outside option of the agent by $\phi(u_0)$. On the other hand, the agent exerts the level of effort that is best for the principal if this level of effort gives him a better payoff than all other possible levels of effort, i.e., when an *Incentive Compatibility* (IC) condition holds.

2.2.1 The first best

The situation where the principal can observe the level of effort exerted by the agent is referred to as the *first best* scenario. The first best scenario constitutes a *benchmark* for the principal's problem. Under the first best scenario, since the agent's level of effort is *observable*, it is *verifiable/contractible*, hence the principal's problem becomes:

$$\max_{e,w} V(e,w)$$

subject to

$$U(e,w) \geq \phi(u_0) \tag{IR}$$

2.2.2 Moral hazard

The case when the principal cannot observe the agent's effort level is sometimes referred to as the *second best* scenario under moral hazard. Because the agent's level of effort is not observable, it is *not contractible*. Hence, the principal must provide *incentives* to the agent to exert the level of effort that is best for the principal. This induces an extra *incentive compatibility* condition. Therefore, the principal's problem under moral hazard becomes:

$$\max_{w} V(e,w)$$

subject to

$$U(e,w) \geq \phi(u_0) \tag{IR}$$

$$U(e,w) \geq U(e',w) \quad \forall e' \in E \tag{IC}$$

CHAPTER 3

MAIN ANALYSIS: TWO EFFORT LEVELS, TWO MONETARY OUTCOMES, AND TWO PROBABILITY MEASURES

In this section, we study the effects of ambiguity aversion on the principal-agent problem with two probability measures, two effort levels, and two monetary outcomes. Our goal is to analyze the optimal wage scheme under moral hazard. For analytical tractability purposes, we restrict our attention mostly to the case where the agent has preferences that belong to the class of constant absolute ambiguity aversion (CAAA) as in Kellner (2017). We explicitly point out whenever this is the case. Even with such a restriction, we fail to provide the second best closed-form solution of the principal's problem. Yet, we are able to provide detailed comparative statics.

There exist two monetary outcomes, $X = \{x_1, x_2\}$, that the principal can observe. Successful monetary outcome, x_1, is more desirable than the failure monetary outcome, x_2, i.e., $x_1 > x_2$. We will refer to these two possible monetary outcomes as *success* and *failure* respectively. For convenience, we also denote $w(x_1)$ and $w(x_2)$ by w_s and w_f, respectively. Therefore, the wage scheme we consider can be thought as a fixed payment w_f, and a bonus, $w_s - w_f$, if the outcome is a *success*.

The agent is to undertake one of the two levels of effort from the set $E = \{e_1, e_2\}$. We assume $c(e_1) > c(e_2)$. That is e_1 stands for the high level of effort and e_2 stands for the low level of effort. For convenience, we refer to the cost of high level of effort as c_H and the cost of low level of effort as c_L. That is, $c_H = c(e_1)$ and $c_L = c(e_2)$ with $c_H > c_L$.

On top of these, there are two probability measures, $J = \{j_1, j_2\}$, that both the principal and the agent agree on and find relevant to describe the relation between the levels of effort and the monetary outcomes. We recall that given a level of effort, $e \in E$, the probability of the monetary outcome x_i according to the j-th probabilty measure in J is represented by p_{ij}^e while the subjective likelihood attributed by the agent to this probability measure is represented by π_j^e.

For both levels of effort, we assume that the same weights π_1 and π_2 are assigned to the two probability measures, i.e., $\pi_1 = \pi_1^h = \pi_1^l$; $\pi_2 = \pi_2^h = \pi_2^l$. In a sense, π_1 and π_2 are the common second order subjective beliefs between the principal and the agents regarding the levels of effort and the monetary outcomes.[1]

We assume that both the high level of effort and the low level of effort are ambiguous. That is,

$$\forall\ e \in E \text{ and } \forall\ i, j \in \{1, 2\},\ p_{ij}^e > 0$$

Furthermore, we assume a natural *monotone likelihood ratio property*:[2]

$$\forall\ j \in \{1, 2\},\ p_{1j}^h > p_{1j}^l$$

That is to say, we assume that the likelihood of observing *success* is higher when the agent exerts high level of effort – for both of the probability measures.

[1] Note that we have $\pi_1 + \pi_2 = 1$.

[2] We note that in the case of two outcomes *monotone likelihood ratio property* is equivalent to *first order stochastic dominance*.

In the rest of the paper, we simplify the notation on the probability measures as follows:

High effort level	**Low effort level**
$p^h_{s1} = p^{e_1}_{11}, \quad p^h_{f1} = p^{e_1}_{21}$	$p^l_{s1} = p^{e_2}_{11}, \quad p^l_{f1} = p^{e_2}_{21}$
$p^h_{s2} = p^{e_1}_{12}, \quad p^h_{f2} = p^{e_1}_{22}$	$p^l_{s2} = p^{e_2}_{12}, \quad p^l_{f2} = p^{e_2}_{22}$

For example, p^h_{s1} stands for the probability of obtaining *success* according to the 1st probability measure when the agent exerts the high level of effort whereas p^l_{f2} stands for the probability of obtaining *failure* according to the 2nd probability measure when the agent exerts the low level of effort.

Figure 1 below summarizes the agent's payoffs according to each level of effort for both possible probability measures.

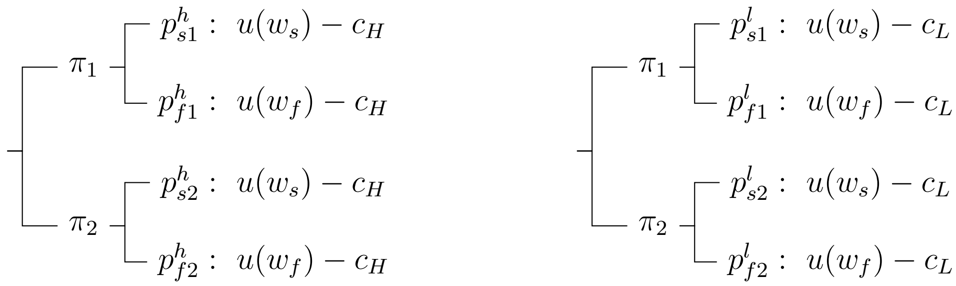

Figure 1: The high effort level versus the low effort level

We are interested in whether the optimal wage scheme becomes higher-powered or lower-powered depending on the parameters of the principal's problem. To answer this question, we focus on how $\Delta := u(w_s) - u(w_f)$ changes as the parameters of the principal's problem change. Δ measures the extra utility that is obtained if the principal observes the monetary outcome *success*, i.e, x_1. This is why we refer to Δ as the bonus utility.

Whenever Δ increases when a parameter changes, we interpret this increase as the optimal wage scheme becoming higher-powered. Similarly, whenever Δ decreases when a parameter changes, we interpret this decrease as the optimal wage scheme becoming lower-powered. That is, as Δ increases (decreases) the agent must be incetivized more (less) to perform the specific task by exerting the level of effort that is best for the principal.

We show that under moral hazard whether a parameter change makes the optimal wage scheme higher-powered or lower-powered depends on an intuitive measure of ambiguity. We also show, by an example, that Δ and the expected profit of the principal does not have to be negatively correlated in general.

3.1 The first best analysis

We start our analysis with the first best scenario. In the first best scenario, since the effort level is contractible, our *monotone likelihood ratio property* assumption guarantees that the principal would ask the agent to exert the high level of effort. Therefore, the principal's problem under the first best scenario can be written as:[3]

$$\max_{w_s,w_f} \ (\pi_1 p^h_{s1} + \pi_2 p^h_{s2})(x_s - w_s) + (\pi_1 p^h_{f1} + \pi_2 p^h_{f2})(x_f - w_f)$$

subject to

$$\pi_1(\phi(p^h_{s1}u(w_s) + p^h_{f1}u(w_f) - c_H)) + \pi_2(\phi(p^h_{s2}u(w_s) + p^h_{f2}u(w_f) - c_H)) \geq \phi(u_0) \quad \text{(IR)}$$

Theorem 1. *Under the first best scenario, i.e., when the effort level is contractible,*

i) the agent's payoff in the optimal wage scheme is the same as his reservation utility, i.e., the IR constraint binds in the optimal wage scheme.

ii) the optimal wage scheme is given by a constant wage, i.e., $w^*_s = w^*_f$.

[3]The Karusch-Kuhn-Tucker Theorem guarantees that a solution exists and it is unique.

Proof. The Lagrangian of the principal's problem is:

$$\mathcal{L}(w_s, w_f, \lambda) = (\pi_1 p^h_{s1} + \pi_2 p^h_{s2})(x_s - w_s) + (\pi_1 p^h_{f1} + \pi_2 p^h_{f2})(x_f - w_f)$$
$$+ \lambda * (\pi_1(\phi(\widehat{w_1})) + \pi_2(\phi(\widehat{w_2})) - \phi(u_0))$$

where $\widehat{w_i} = p^h_{si} u(w_s) + p^h_{fi} u(w_f) - c_H$ and $i \in \{1, 2\}$.

$$\frac{\partial \mathcal{L}(w_s, w_f, \lambda)}{\partial w_s} = -(\pi_1 p^h_{s1} + \pi_2 p^h_{s2})$$
$$+ \lambda^* [\pi_1 \phi'(\widehat{w_1}^*) p^h_{s1} u'(w^*_s) + \pi_2 \phi'(\widehat{w_2}^*) p^h_{s2} u'(w^*_s))] = 0 \quad (3.1)$$

$$\frac{\partial \mathcal{L}(w_s, w_f, \lambda)}{\partial w_f} = -(\pi_1 p^h_{f1} + \pi_2 p^h_{f2})$$
$$+ \lambda^* [\pi_1 \phi'(\widehat{w_1}^*) p^h_{f1} u'(w^*_f) + \pi_2 \phi'(\widehat{w_2}^*) p^h_{f2} u'(w^*_f))] = 0 \quad (3.2)$$

By combining **Equation 3.1** and **Equation 3.2**, we obtain:

$$\lambda^* = \frac{1}{\pi_1 \phi'(\widehat{w_1}^*)[p^h_{s1} u'(w^*_s) + p^h_{f1} u'(w^*_f)] + \pi_2 \phi'(\widehat{w_2}^*)[p^h_{s2} u'(w^*_s) + p^h_{f2} u'(w^*_f)]} > 0 \quad (3.3)$$

That $\lambda^* > 0$ implies that IR binds.

Furthermore, from **Equation 3.1**, we infer that:

$$\lambda = \frac{\pi_1 p^h_{s1} + \pi_2 p^h_{s2}}{\pi_1 \phi'(\widehat{w_1}^*) p^h_{s1} u'(w^*_s) + \pi_2 \phi'(\widehat{w_2}^*) p^h_{s2} u'(w^*_s)} \quad (3.4)$$

Using **Equation 3.3** and **Equation 3.4**, we obtain:

$$\begin{aligned}\pi_1\phi'(\widehat{w_1}^*)p^h_{s1}u'(w^*_s) + \pi_2\phi'(\widehat{w_2}^*)p^h_{s2}u'(w^*_s) &= u'(w^*_s)[\pi_1^2(p^h_{s1})^2 + \pi_2\pi_1p^h_{s1}p^h_{s2}]\phi'(\widehat{w_1}^*)\\ &+ u'(w^*_s)[\pi_1\pi_2p^h_{s1}p^h_{s2} + \pi_2^2(p^h_{s2})^2]\phi'(\widehat{w_2}^*)\\ &+ u'(w^*_f)[\pi_1^2p^h_{s1}p^h_{f1} + \pi_2\pi_1p^h_{s2}p^h_{f1}]\phi'(\widehat{w_1}^*)\\ &+ u'(w^*_f)[\pi_1\pi_2p^h_{s1}p^h_{f2} + \pi_2^2p^h_{s2}p^h_{f2}]\phi'(\widehat{w_2}^*)\end{aligned}$$

$$\begin{aligned}&u'(w^*_s)[(-\pi_1^2(p^h_{s1})^2 - \pi_2\pi_1p^h_{s1}p^h_{s2} + \pi_1p^h_{s1})\phi'(\widehat{w_1}^*) + (-\pi_1\pi_2p^h_{s1}p^h_{s2} - \pi_2^2(p^h_{s2})^2 + \pi_2p^h_{s2})\phi'(\widehat{w_2}^*)]\\ &= u'(w^*_f)[(\pi_1^2p^h_{s1}p^h_{f1} + \pi_2\pi_1p^h_{s2}p^h_{f1})\phi'(\widehat{w_1}^*) + (\pi_1\pi_2p^h_{s1}p^h_{f2} + \pi_2^2p^h_{s2}p^h_{f2})\phi'(\widehat{w_2}^*)]\end{aligned}$$

$$\begin{aligned}&u'(w^*_s)[(\pi_1p^h_{s1}(\pi_1p^h_{f1} + \pi_2p^h_{f2}))\phi'(\widehat{w_1}^*) + (\pi_2p^h_{s2}(\pi_1p^h_{f1} + \pi_2p^h_{f2}))\phi'(\widehat{w_2}^*)]\\ &= u'(w^*_f)[\pi_1p^h_{f1}\pi_1p^h_{s1} + \pi_2p^h_{s2}))\phi'(\widehat{w_1}^*) + (\pi_2p^h_{f2}\pi_1p^h_{s1} + \pi_2p^h_{s2})\phi'(\widehat{w_2}^*)]\end{aligned}$$

$$\frac{u'(w^*_s)}{u'(w^*_f)} = \frac{p^h_s}{1-p^h_s}\left(\frac{\pi_1(1-p^h_{s1})\phi'(\widehat{w_1}^*) + \pi_2(1-p^h_{s2})\phi'(\widehat{w_2}^*)}{\pi_1p^h_{s1}\phi'(\widehat{w_1}^*) + \pi_2p^h_{s2}\phi'(\widehat{w_2}^*)}\right) \tag{3.5}$$

where $p^h_s = \pi_1p^h_{s1} + \pi_2p^h_{s2}$.

Next, we proceed with a proof by contradiction to show that the optimal wage scheme is a constant wage. We have two cases.

Case 1: Suppose $w_s > w_f$. This implies that:

$$\frac{u'(w^*_s)}{u'(w^*_f)} = \frac{p^h_s}{1-p^h_s}\left(\frac{\pi_1(1-p^h_{s1})\phi'(\widehat{w_1}^*) + \pi_2(1-p^h_{s2})\phi'(\widehat{w_2}^*)}{\pi_1p_1\phi'(\widehat{w_1}^*) + \pi_2p_2\phi'(\widehat{w_2}^*)}\right) \leq 1$$

since $\frac{u'(w_s^*)}{u'(w_f^*)} \leq 1$ due to our assumption on u, which is $u' > 0 \geq u''$.

$$\frac{p_s^h}{1-p_s^h}\left(\frac{\pi_1(1-p_{s1}^h)\phi'(\widehat{w_1}^*)+\pi_2(1-p_{s2}^h)\phi'(\widehat{w_2}^*)}{\pi_1 p_1\phi'(\widehat{w_1}^*)+\pi_2 p_2\phi'(\widehat{w_2}^*)}\right) \leq 1$$

$$[p_s^h\pi_1(1-p_{s1}^h)-(1-p_s^h)\pi_1 p_1]\phi'(\widehat{w_1}^*) \leq [-p_s^h\pi_2(1-p_{s2}^h)+(1-p_s^h)\pi_2 p_2]\phi'(\widehat{w_2}^*)$$

$$[p_s^h\pi_1 - p_s^h\pi_1 p_1 - \pi_1 p_1 + p_s^h\pi_1 p_1]\phi'(\widehat{w_1}^*) \leq [-p_s^h\pi_2 + p_s^h\pi_2 p_2 + \pi_2 p_2 - p_s^h\pi_2 p_2]\phi'(\widehat{w_2}^*)$$

$$\pi_1(p_{s1}^h - p_{s2}^h)(\pi_1 - 1)\phi'(\widehat{w_1}^*) \leq \pi_1(p_{s1}^h - p_{s2}^h)(\pi_1 - 1)\phi'(\widehat{w_2}^*)$$

$$\phi'(\widehat{w_1}^*) \geq \phi'(\widehat{w_2}^*) \tag{3.6}$$

Case 2: Now, suppose $w_s < w_f$. This implies that:

$$\frac{u'(w_s^*)}{u'(w_f^*)} = \frac{p_s^h}{1-p_s^h}(\frac{\pi_1(1-p_{s1}^h)\phi'(\widehat{w_1}^*)+\pi_2(1-p_{s2}^h)\phi'(\widehat{w_2}^*)}{\pi_1 p_1\phi'(\widehat{w_1}^*)+\pi_2 p_2\phi'(\widehat{w_2}^*)}) \geq 1$$

since $\frac{u'(w_s^*)}{u'(w_f^*)} \geq 1$ due to our assumption on u, which is $u' > 0 \geq u''$.

$$\frac{p_s^h}{1-p_s^h}(\frac{\pi_1(1-p_{s1}^h)\phi'(\widehat{w_1}^*)+\pi_2(1-p_{s2}^h)\phi'(\widehat{w_2}^*)}{\pi_1 p_1\phi'(\widehat{w_1}^*)+\pi_2 p_2\phi'(\widehat{w_2}^*)}) \geq 1$$

$$[p_s^h\pi_1(1-p_{s1}^h)-(1-p_s^h)\pi_1 p_1]\phi'(\widehat{w_1}^*) \geq [-p_s^h\pi_2(1-p_{s2}^h)+(1-p_s^h)\pi_2 p_2]\phi'(\widehat{w_2}^*)$$

$$[p_s^h\pi_1 - p_s^h\pi_1 p_1 - \pi_1 p_1 + p_s^h\pi_1 p_1]\phi'(\widehat{w_1}^*) \geq [-p_s^h\pi_2 + p_s^h\pi_2 p_2 + \pi_2 p_2 - p_s^h\pi_2 p_2]\phi'(\widehat{w_2}^*)$$

$$\pi_1(p_{s1}^h - p_{s2}^h)(\pi_1 - 1)\phi'(\widehat{w_1}^*) \geq \pi_1(p_{s1}^h - p_{s2}^h)(\pi_1 - 1)\phi'(\widehat{w_2}^*)$$

$$\phi'(\widehat{w_1}^*) \leq \phi'(\widehat{w_2}^*) \tag{3.7}$$

However, neither **Equation 3.6** nor **Equation 3.7** can hold since $\phi' > 0 > \phi''$ and $p_{s1}^h > p_{s2}^h$. Therefore, we must have $w_s^* = w_f^*$. ■

We note that Theorem 1 does not require the assumption of constant absolute ambiguity (CAAA) preferences for the agent. Yet, if the agent has CAAA type preferences we have the following immediate corollary.

Corollary. If the agent has constant absolute ambiguity preferences, i.e., $\phi(u) = -e^{-\alpha u}$, then the optimal wage scheme under the first best scenario is given by, $w_s^* = w_f^* = \overline{w} = u^{-1}(u_0 + c_H)$.

Proof. By Theorem 1, IR binds. Therefore,

$$\begin{aligned} \pi_1 e^{-\alpha(u(\overline{w})-c_H)} + \pi_2 e^{-\alpha(u(\overline{w})-c_H)} &= e^{-\alpha(u_0)} \\ e^{-\alpha(u(\overline{w})-c_H)} &= e^{-\alpha(u_0)} \\ u(\overline{w}) &= u_0 + c_H \\ \overline{w} &= u^{-1}(u_0 + c_H). \end{aligned}$$

■

3.2 Moral hazard

Next, we turn to the analysis of the optimal wage scheme under *moral hazard*, i.e., the case where the principal *cannot* observe the level of effort exerted by the agent. By our *monotone likelihood ratio property*, the principal would like the agent to exert the high level of effort. Yet, since the level of effort is *not contractible*, the principal needs to provide proper *incentives* to the agent to make it incentive compatible for the agent to exert the high level of effort. This induces an extra *incentive compatibility* constraint.

Therefore, the principal's problem under moral hazard can be written as:[4]

$$\max_{w_s, w_f} \; (\pi_1 p^h_{s1} + \pi_2 p^h_{s2})(x_s - w_s) + (\pi_1 p^h_{f1} + \pi_2 p^h_{f2})(x_f - w_f)$$

subject to

$$\pi_1(\phi(p^h_{s1}u(w_s) + p^h_{f1}u(w_f) - c_H)) + \pi_2(\phi(p^h_{s2}u(w_s) + p^h_{f2}u(w_f) - c_H)) \geq \phi(u_0) \quad \text{(IR)}$$

$$\begin{aligned}&\pi_1(\phi(p^h_{s1}u(w_s) + p^h_{f1}u(w_f) - c_H)) + \pi_2(\phi(p^h_{s2}u(w_s) + p^h_{f2}u(w_f) - c_H)) \geq \\ &\quad \pi_1(\phi(p^l_{s1}u(w_s) + p^l_{f1}u(w_f) - c_L)) + \pi_2(\phi(p^l_{s2}u(w_s) + p^l_{f2}u(w_f) - c_L)) \quad \text{(IC)}\end{aligned}$$

Theorem 2. *Suppose the agent has CAAA type of preferences. Under moral hazard, i.e., when the effort level is not contractible,*

- *i) the agent's payoff in the optimal wage scheme is the same as his reservation utility, i.e., the IR constraint binds in the optimal wage scheme,*
- *ii) the optimal wage scheme is monotonic, i.e.,* $w^*_s > w^*_f$
- *iii) the agent is indifferent between exerting the high level of effort and the low level of effort given the optimal wage scheme, i.e., the IC constraint binds in the optimal wage scheme.*

Proof. The Lagrangian of the principal's problem is:

$$\begin{aligned}\mathcal{L}(w_s, w_f, \lambda, \mu) = &(\pi_1 p^h_{s1} + \pi_2 p^h_{s2})(x_s - w_s) + (\pi_1 p^h_{f1} + \pi_2 p^h_{f2})(x_f - w_f) \\ &+ \lambda^*(\pi_1(\phi(\widehat{w_1})) + \pi_2(\phi(\widehat{w_2})) - \phi(u_0)) \\ &+ \mu^*(\pi_1(\phi(\widehat{w_1})) + \pi_2(\phi(\widehat{w_2})) - \pi_1(\phi(\widehat{w_3})) - \pi_2(\phi(\widehat{w_4})))\end{aligned}$$

[4]The Karusch-Kuhn-Tucker Theorem guarantees that a solution exists and it is unique.

where $\widehat{w_j} = p^h_{si}u(w_s) + p^h_{fi}u(w_f) - c_H$ for $j \in \{1,2\}$ and $i \in \{1,2\}$

and $\widehat{w_j} = p^l_{si}u(w_s) + p^l_{fi}u(w_f) - c_L$ for $j \in \{3,4\}$ and $i \in \{1,2\}$

$$\begin{aligned}\frac{\partial \mathcal{L}(w_s, w_f, \lambda, \mu)}{\partial w_s} &= -(\pi_1 p^h_{s1} + \pi_2 p^h_{s2}) \\ &+ \lambda^*[p^h_{s1}\pi_1\phi'(\widehat{w_1}^*)u'(w^*_s) + p^h_{s2}\pi_2\phi'(\widehat{w_2}^*)u'(w^*_s)] \\ &+ \mu^*[p^h_{s1}\pi_1\phi'(\widehat{w_1}^*)u'(w^*_s) + p^h_{s2}\pi_2\phi'(\widehat{w_2}^*)u'(w^*_s) \\ &\quad - p^l_{s1}\pi_1\phi'(\widehat{w_3}^*)u'(w^*_s) - p^l_{s2}\pi_2\phi'(\widehat{w_4}^*)u'(w^*_s))] = 0 \end{aligned} \tag{3.8}$$

$$\begin{aligned}\frac{\partial \mathcal{L}(w_s, w_f, \lambda, \mu)}{\partial w_f} &= -(\pi_1 p^h_{f1} + \pi_2 p^h_{f2}) \\ &+ \lambda^*[p^h_{f1}\pi_1\phi'(\widehat{w_1}^*)u'(w^*_f) + p^h_{f2}\pi_2\phi'(\widehat{w_2}^*)u'(w^*_f)] \\ &+ \mu^*[p^h_{f1}\pi_1\phi'(\widehat{w_1}^*)u'(w^*_f) + p^h_{f2}\pi_2\phi'(\widehat{w_2}^*)u'(w^*_f) \\ &\quad - p^l_{f1}\pi_1\phi'(\widehat{w_3}^*)u'(w^*_f) - p^l_{f2}\pi_2\phi'(\widehat{w_4}^*)u'(w^*_f))] = 0 \end{aligned} \tag{3.9}$$

Equation 3.8 and **Equation 3.9** become **Equation 3.10** and **Equation 3.11** respectively.

$$\begin{aligned}\frac{\pi_1 p^h_{s1} + \pi_2 p^h_{s2}}{u'(w^*_s)} &= \lambda^*[p^h_{s1}\pi_1\phi'(\widehat{w_1}^*) + p^h_{s2}\pi_2\phi'(\widehat{w_2}^*)] \\ &+ \mu^*[p^h_{s1}\pi_1\phi'(\widehat{w_1}^*) + p^h_{s2}\pi_2\phi'(\widehat{w_2}^*) - p^l_{s1}\pi_1\phi'(\widehat{w_3}^*) - p^l_{s2}\pi_2\phi'(\widehat{w_4}^*)]\end{aligned} \tag{3.10}$$

$$\begin{aligned}\frac{\pi_1 p^h_{f1} + \pi_2 p^h_{f2}}{u'(w^*_f)} &= \lambda^*[p^h_{f1}\pi_1\phi'(\widehat{w_1}^*) + p^h_{f2}\pi_2\phi'(\widehat{w_2}^*)] \\ &+ \mu^*[p^h_{f1}\pi_1\phi'(\widehat{w_1}^*) + p^h_{f2}\pi_2\phi'(\widehat{w_2}^*) - p^l_{f1}\pi_1\phi'(\widehat{w_3}^*) - p^l_{f2}\pi_2\phi'(\widehat{w_4}^*)]\end{aligned} \tag{3.11}$$

By combining **Equation 3.10** and **Equation 3.11**, we will obtain:

$$\frac{\pi_1 p_{s1}^h + \pi_2 p_{s2}^h}{u'(w_s^*)} + \frac{\pi_1 p_{f1}^h + \pi_2 p_{f2}^h}{u'(w_f^*)} = \lambda^*[\pi_1 \phi'(\widehat{w_1}^*) + \pi_2 \phi'(\widehat{w_2}^*)]$$
$$+ \mu^*[\pi_1 \phi'(\widehat{w_1}^*) + \pi_2 \phi'(\widehat{w_2}^*) - \pi_1 \phi'(\widehat{w_3}^*) - \pi_2 \phi'(\widehat{w_4}^*)] \tag{3.12}$$

Since $\phi(u) = -e^{-\alpha u}$, i.e., ϕ belongs to constant absolute ambiguity aversion (CAAA) type of preferences, the *complementary slackness* (the necessary Karusch-Kuhn-Tucker condition on the IC constraint) implies that:

$$\mu^*[\pi_1 e^{-\alpha(\widehat{w_1}^*)} + \pi_2 e^{-\alpha(\widehat{w_2}^*)} - \pi_1 e^{-\alpha(\widehat{w_3}^*)} - \pi_2 e^{-\alpha(\widehat{w_4}^*)}] = 0$$

Therefore, we have the following inequality:

$$\lambda^* = \frac{\frac{\pi_1 p_{s1}^h + \pi_2 p_{s2}^h}{\alpha u'(w_s^*)} + \frac{\pi_1 p_{f1}^h + \pi_2 p_{f2}^h}{\alpha u'(w_f^*)}}{\pi_1 e^{-\alpha(\widehat{w_1}^*)} + \pi_2 e^{-\alpha(\widehat{w_2}^*)}} > 0$$

That λ^* is greater than 0 implies that IR binds.

Next, we show that the optimal wage scheme is monotonic, i.e., $w_s^* > w_f^*$.

Observe that under CAAA type of preferences **Equation 3.12** becomes:

$$\frac{\pi_1 p_{s1}^h + \pi_2 p_{s2}^h}{\alpha u'(w_s^*)} + \frac{\pi_1 p_{f1}^h + \pi_2 p_{f2}^h}{\alpha u'(w_f^*)} = \lambda^*[\pi_1 e^{-\alpha(\widehat{w_1}^*)} + \pi_2 e^{-\alpha(\widehat{w_2}^*)}]$$
$$+ \mu^*[\pi_1 e^{-\alpha(\widehat{w_1}^*)} + \pi_2 e^{-\alpha(\widehat{w_2}^*)} - \pi_1 e^{-\alpha(\widehat{w_3}^*)} - \pi_2 e^{-\alpha(\widehat{w_4}^*)}] \tag{3.13}$$

Since u is increasing, to show that the optimal wage scheme is monotonic, it is enough to show that $u(w_s) > u(w_f)$. We proceed by a proof by contradiction.

Suppose $u(w_s) \leq u(w_f)$. This implies that $\frac{u'(w_s)}{u'(w_s)} \geq 1$ since $u' > 0 > u''$.

We will make use of the following Lemma 1.

Lemma 1. If $w_s < w_f$, then we have the following inequality.

$$\frac{\pi_1 p_{s1}^h + \pi_2 p_{s2}^h}{\alpha u'(w_s^*)} < \left(\frac{\pi_1 p_{s1}^h + \pi_2 p_{s2}^h}{\alpha u'(w_s^*)} + \frac{\pi_1 p_{f1}^h + \pi_2 p_{f2}^h}{\alpha u'(w_f^*)} \right) \left(\frac{p_{s1}^h \pi_1 e^{-\alpha(\widehat{w_1}^*)} + p_{s2}^h \pi_2 e^{-\alpha(\widehat{w_2}^*)}}{\pi_1 e^{-\alpha(\widehat{w_1}^*)} + \pi_2 e^{-\alpha(\widehat{w_2}^*)}} \right)$$

Proof of Lemma 1. We make the following observations.
Since $1 > p_{s1}^h, p_{s2}^h, \pi_1, \pi_2 > 0$, $\dfrac{u'(w_s)}{u'(w_s)} \geq 1$ and $p_{s1}^h > p_{s2}^h$,

Observation 1:

$$0 < k := \frac{p_{s1}^h \pi_1 e^{-\alpha(\widehat{w_1}^*)} + p_{s2}^h \pi_2 e^{-\alpha(\widehat{w_2}^*)}}{\pi_1 e^{-\alpha(\widehat{w_1}^*)} + \pi_2 e^{-\alpha(\widehat{w_2}^*)}} < 1$$

Observation 2:

$$0 < \pi_1 p_{s1}^h + \pi_2 p_{s2}^h < 1$$

Observation 3:

$$\pi_1 p_{s1}^h + \pi_2 p_{s2}^h < \frac{p_{s1}^h \pi_1 e^{-\alpha(\widehat{w_1}^*)} + p_{s2}^h \pi_2 e^{-\alpha(\widehat{w_2}^*)}}{\pi_1 e^{-\alpha(\widehat{w_1}^*)} + \pi_2 e^{-\alpha(\widehat{w_2}^*)}} = k$$

Observation 4:

$$\pi_1 p_{s1}^h + \pi_2 p_{s2}^h + (1 - \pi_1 p_{s1}^h - \pi_2 p_{s2}^h) \frac{u'(w_s^*)}{u'(w_f^*)} > 1$$

Consider **Observation 3** and multiply the right hand side of the inequality with $\pi_1 p_{s1}^h + \pi_2 p_{s2}^h + (\pi_1 p_{f1}^h + \pi_2 p_{f2}^h) \dfrac{u'(w_s^*)}{u'(w_f^*)}$. This completes the proof of Lemma 1. ■

Now, we go back to the proof of the monotonicity of the optimal wage scheme.

Lemma 1 implies the following:

$$\mu^*[p^h_{s1}\pi_1 e^{-\alpha(\widehat{w_1}^*)} + p^h_{s2}\pi_2 e^{-\alpha(\widehat{w_2}^*)} - p^l_{s1}\pi_1 e^{-\alpha(\widehat{w_3}^*)} - p^l_{s2}\pi_2 e^{-\alpha(\widehat{w_4}^*)}] < 0$$

Since at the optimal solution $\mu^* > 0$, we need:

$$p^h_{s1}\pi_1 e^{-\alpha(\widehat{w_1}^*)} + p^h_{s2}\pi_2 e^{-\alpha(\widehat{w_2}^*)} - p^l_{s1}\pi_1 e^{-\alpha(\widehat{w_3}^*)} - p^l_{s2}\pi_2 e^{-\alpha(\widehat{w_4}^*)} < 0$$

But, we cannot have the above inequality since $\mu^* > 0$ implies that IC binds. That is to say:

$$\pi_1 e^{-\alpha(\widehat{w_1}^*)} + \pi_2 e^{-\alpha(\widehat{w_2}^*)} = \pi_1 e^{-\alpha(\widehat{w_3}^*)} + \pi_2 e^{-\alpha(\widehat{w_4}^*)}$$

$$\pi_1(e^{-\alpha(\widehat{w_1}^*)} - e^{-\alpha(\widehat{w_3}^*)}) = \pi_2(e^{-\alpha(\widehat{w_4}^*)} - e^{-\alpha(\widehat{w_2}^*)})$$

Suppose $e^{-\alpha(\widehat{w_1}^*)} = e^{-\alpha(\widehat{w_3}^*)} \iff e^{-\alpha(\widehat{w_4}^*)} = e^{-\alpha(\widehat{w_2}^*)}$. However, this implies that: $0 > u(w_s) - u(w_f) = \frac{c_H - c_L}{p^h_{s1} - p^l_{s1}} = \frac{c_H - c_L}{p^h_{s2} - p^l_{s2}}$. This cannot hold since $c_H - c_L, p^h_{s1} - p^l_{s1}$and $p^h_{s2} - p^l_{s2} > 0$.

Now, suppose $e^{-\alpha(\widehat{w_1}^*)} > e^{-\alpha(\widehat{w_3}^*)} \iff e^{-\alpha(\widehat{w_4}^*)} > e^{-\alpha(\widehat{w_2}^*)}$. However, $e^{-\alpha(\widehat{w_4}^*)} \geq e^{-\alpha(\widehat{w_2}^*)}$ since $\widehat{w_4}^* < \widehat{w_2}^*$.

Finally, suppose $e^{-\alpha(\widehat{w_1}^*)} < e^{-\alpha(\widehat{w_3}^*)} \iff e^{-\alpha(\widehat{w_4}^*)} < e^{-\alpha(\widehat{w_2}^*)}$. However, $e^{-\alpha(\widehat{w_1}^*)} \geq e^{-\alpha(\widehat{w_3}^*)}$ since $\widehat{w_1}^* > \widehat{w_3}^*$.

Hence, the following cannot hold at the optimal solution:

$$\mu^*[p^h_{s1}\pi_1 e^{-\alpha(\widehat{w_1}^*)} + p^h_{s2}\pi_2 e^{-\alpha(\widehat{w_2}^*)} - p^l_{s1}\pi_1 e^{-\alpha(\widehat{w_3}^*)} - p^l_{s2}\pi_2 e^{-\alpha(\widehat{w_4}^*)}] < 0$$

This yields a contradiction. Therefore, we conclude that the optimal wage scheme is monotonic.

Next, we show that the IC constraint also binds. First, we substitute λ^* in **Equation 3.10**. Since $\phi(u) = -e^{-\alpha u}$, **Equation 3.10** becomes:

$$\frac{\pi_1 p^h_{s1} + \pi_2 p^h_{s2}}{\alpha u'(w^*_s)} = \lambda^*[p^h_{s1}\pi_1 e^{-\alpha(\widehat{w_1}^*)} + p^h_{s2}\pi_2 e^{-\alpha(\widehat{w_2}^*)}]$$
$$+ \mu^*[p^h_{s1}\pi_1 e^{-\alpha(\widehat{w_1}^*)} + p^h_{s2}\pi_2 e^{-\alpha(\widehat{w_2}^*)} - p^l_{s1}\pi_1 e^{-\alpha(\widehat{w_3}^*)} - p^l_{s2}\pi_2 e^{-\alpha(\widehat{w_4}^*)}]$$

We now use the fact that the optimal wage scheme is monotonic, i.e., $w_s > w_f$ to prove that IC binds. Consider the following observation:

Observation 5:

$$(\pi_1 p^h_{s1} + \pi_2 p^h_{s2})(\pi_1 e^{-\alpha(\widehat{w_1}^*)} + \pi_2 e^{-\alpha(\widehat{w_2}^*)}) > p^h_{s1}\pi_1 e^{-\alpha(\widehat{w_1}^*)} + p^h_{s2}\pi_2 e^{-\alpha(\widehat{w_2}^*)}$$
$$(\pi_1 p^h_{s1} + \pi_2 p^h_{s2} - p^h_{s2})\pi_2 e^{-\alpha(\widehat{w_2}^*)} > (p^h_{s1} - \pi_1 p^h_{s1} - \pi_2 p^h_{s2})\pi_1 e^{-\alpha(\widehat{w_1}^*)}$$
$$(p^h_{s2} + \pi_1(p^h_{s1} - p^h_{s2}) - p^h_{s2})\pi_2 e^{-\alpha(\widehat{w_2}^*)} > (p^h_{s1} - p^h_{s2} - \pi_1(p^h_{s1} - p^h_{s2}))\pi_1 e^{-\alpha(\widehat{w_1}^*)}$$
$$\pi_1\pi_2(p^h_{s1} - p^h_{s2})e^{-\alpha(\widehat{w_2}^*)} > \pi_1(1-\pi_1)(p^h_{s1} - p^h_{s2})e^{-\alpha(\widehat{w_1}^*)}$$
$$\pi_1(1-\pi_1)(p^h_{s1} - p^h_{s2})e^{-\alpha(\widehat{w_2}^*)} > \pi_1(1-\pi_1)(p^h_{s1} - p^h_{s2})e^{-\alpha(\widehat{w_1}^*)}$$

since $\widehat{w_1}^* > \widehat{w_2}^*$ due to $p^h_{s1} > p^h_{s2}$

Next, we show that:

$$\frac{\pi_1 p^h_{s1} + \pi_2 p^h_{s2}}{\alpha u'(w^*_s)} > \left(\frac{\pi_1 p^h_{s1} + \pi_2 p^h_{s2}}{\alpha u'(w^*_s)} + \frac{\pi_1 p^h_{f1} + \pi_2 p^h_{f2}}{\alpha u'(w^*_f)}\right)\left(\frac{p^h_{s1}\pi_1 e^{-\alpha(\widehat{w_1}^*)} + p^h_{s2}\pi_2 e^{-\alpha(\widehat{w_2}^*)}}{\pi_1 e^{-\alpha(\widehat{w_1}^*)} + \pi_2 e^{-\alpha(\widehat{w_2}^*)}}\right)$$
$$\pi_1 p^h_{s1} + \pi_2 p^h_{s2} > \left(\pi_1 p^h_{s1} + \pi_2 p^h_{s2} + \frac{u'(w^*_s)}{u'(w^*_f)}(\pi_1 p^h_{f1} + \pi_2 p^h_{f2})\right)\left(\frac{p^h_{s1}\pi_1 e^{-\alpha(\widehat{w_1}^*)} + p^h_{s2}\pi_2 e^{-\alpha(\widehat{w_2}^*)}}{\pi_1 e^{-\alpha(\widehat{w_1}^*)} + \pi_2 e^{-\alpha(\widehat{w_2}^*)}}\right)$$

RHS of the above inequality is less than $\dfrac{p^h_{s1}\pi_1 e^{-\alpha(\widehat{w_1}^*)} + p^h_{s2}\pi_2 e^{-\alpha(\widehat{w_2}^*)}}{\pi_1 e^{-\alpha(\widehat{w_1}^*)} + \pi_2 e^{-\alpha(\widehat{w_2}^*)}}$ since $\dfrac{u'(w^*_s)}{u'(w^*_f)} < 1$. Therefore, it is sufficient to show that:

$$\pi_1 p^h_{s1} + \pi_2 p^h_{s2} > \frac{p^h_{s1}\pi_1 e^{-\alpha(\widehat{w_1}^*)} + p^h_{s2}\pi_2 e^{-\alpha(\widehat{w_2}^*)}}{\pi_1 e^{-\alpha(\widehat{w_1}^*)} + \pi_2 e^{-\alpha(\widehat{w_2}^*)}}$$

Observation 5 implies that this is true. Hence, we have:

$$\mu^* > 0, \text{ and}$$

$$p^h_{s1}\pi_1 e^{-\alpha(\widehat{w_1}^*)} + p^h_{s2}\pi_2 e^{-\alpha(\widehat{w_2}^*)} - p^l_{s1}\pi_1 e^{-\alpha(\widehat{w_3}^*)} - p^l_{s2}\pi_2 e^{-\alpha(\widehat{w_4}^*)} > 0$$

Therefore, the IC constraint binds at the optimal wage scheme as well. ∎

We summarize the implications of Theorem 2 regarding the optimal wage scheme:

$$\text{IR binds} \iff \pi_1 e^{-\alpha(\widehat{w_1})} + \pi_2 e^{-\alpha(\widehat{w_2})} = e^{-\alpha(u_0)}$$

$$\text{IC binds} \iff \pi_1 e^{-\alpha(\widehat{w_1})} + \pi_2 e^{-\alpha(\widehat{w_2})} = \pi_1 e^{-\alpha(\widehat{w_3})} + \pi_2 e^{-\alpha(\widehat{w_4})}$$

where $\widehat{w_i} = p^h_{sj}u(w_s) + p^h_{fj}u(w_f) - c_H$ for $i \in \{1,2\}$ and $j \in \{1,2\}$
and $\widehat{w_i} = p^l_{sj}u(w_s) + p^l_{fj}u(w_f) - c_L$ for $i \in \{3,4\}$ and $j \in \{1,2\}$

Straightforward simplifications imply that IR and IC constraints can be written as:

$$\pi_1 e^{-\alpha(p^h_{s1}\Delta)} + \pi_2 e^{-\alpha(p^h_{s2}\Delta)} = e^{-\alpha(u_0 + c_H - u(w_f))} \quad \text{(IR)}$$

$$\pi_1 e^{-\alpha(p^h_{s1}\Delta)} + \pi_2 e^{-\alpha(p^h_{s2}\Delta)} = \pi_1 e^{-\alpha(p^l_{s1}\Delta + c)} + \pi_2 e^{-\alpha(p^l_{s2}\Delta + c)} \quad \text{(IC)}$$

where $\Delta = u(w_s) - u(w_f)$ and $c = c_H - c_L$

CHAPTER 4

COMPARATIVE STATICS

In this section, we study how $\Delta = u(w_s) - u(w_f)$, the bonus utility, changes as the parameters of the principal's problem change. As mentioned before, we interpret an increase or decrease in Δ as a result of a parameter change as the optimal wage scheme becoming higher-powered or lower-powered, respectively. That is, as Δ increases (decreases) the agent must be incentivized more (less) to perform the task by exerting the level of effort that is best for the principal.

The bonus utility changes depending on the relative ambiguity of the level of effort that the principal wants to implement. We use the following definition to compare the levels of effort in terms of ambiguity.

Definition. A level of effort is more ambiguous than another level of effort if its variance is greater than that of the other.

We measure the variance of an effort level as a spread from the mean, which is induced by the two probability measures. For example, for the high effort level, the variance is calculated as follows:

$$\sigma_h^2 : \pi_1^h (p_{s1}^h - \mu_h)^2 + \pi_2^h (p_{s2}^h - \mu_h)^2$$

$$\text{where } \mu_h = p^h_{s1}\pi^h_1 + p^h_{s2}\pi^h_2$$

The payoff distributions, means, and variances are as follows:

The high effort level

- μ_{U_H}: $\Delta(\pi_1 p^h_{s1} + \pi_2 p^h_{s2}) + u(w_f) - c_H$
- $\sigma^2_{U_H}$: $\Delta^2(p^h_{s1} - p^h_{s2})^2\pi_1\pi_2$

Figure 2: The payoff mean and variance of the high effort level

The low effort level

- μ_{U_L}: $\Delta(\pi_1 p^l_{s1} + \pi_2 p^l_{s2}) + u(w_f) - c_L$
- $\sigma^2_{U_L}$: $\Delta^2(p^l_{s1} - p^l_{s2})^2\pi_1\pi_2$

Figure 3: The payoff mean and variance of the low effort level

Since there are two probability measures, we can use the difference between two *success* probabilities $(p^e_{s1} - p^e_{s2})$ as spread. For instance, the high effort level is more ambiguous than the low effort level when $p^h_{s1} - p^h_{s2} > p^l_{s1} - p^l_{s2}$. Likewise, the low effort level is more ambiguous than the high effort level when $p^h_{s1} - p^h_{s2} < p^l_{s1} - p^l_{s2}$.

In the following, we provide comparative statics regarding three different sets of parameter. We consider how *ambiguity*, *cost of effort*, and *second order beliefs* change the optimal contract in terms of bonus utility, i.e, whether they make the optimal contract higher-powered or lower-powered.

4.1 Ambiguity

The separation between ambiguity and attitude towards ambiguity is achieved in terms of tastes and beliefs in the smooth ambiguity model. Therefore, there are two different parameters related to ambiguity in the principal's problem. These are

i) the CAAA coefficient, α, and *ii) the spreads,* $p^h_{s1} - p^h_{s2}$ *and* $p^l_{s1} - p^l_{s2}$

Below, we show that these two sources of ambiguity affect Δ in a similar way. However, they have different effects on the expected profit of the principal. In particular, we see that Δ and the expected profit of the principal does not have to be negatively correlated. That is, as Δ increases the expected profit of the principal may increase. We show, by an example, that increasing CAAA coefficient, α, may benefit the principal when the low effort level is more ambiguous. On the other hand, we show that a *uniform increase in the spread*, i.e., increasing the spread for both effort levels, is not beneficial to the principal.

4.1.1 An increase in the CAAA coefficient

An increase in ambiguity aversion coefficient, α, makes the agent more ambiguity averse. Intuitively, the agent becomes more inclined to exert the level of effort that is less ambiguous. Therefore, when the high effort level is more (less) ambiguous, the principal has to provide more (less) incentives that leads to an increase (decrease) in Δ.

Proposition 1. *When the high effort level is more ambiguous,* $p^h_{s1} - p^h_{s2} > p^l_{s1} - p^l_{s2}$*, an increase in the CAAA coefficient* α *makes the wage scheme higher-powered, i.e.,* $\frac{\partial \Delta}{\partial \alpha} > 0$*.*

Proof. By the implicit function theorem we have,

$$\frac{\partial \Delta}{\partial \alpha} = -\frac{\frac{\partial F}{\partial \alpha}}{\frac{\partial F}{\partial \Delta}}$$
$$= \frac{-p^h_{s1}\Delta\pi_1 e^{-\alpha(p^h_{s1}\Delta)} - p^h_{s2}\Delta\pi_2 e^{-\alpha(p^h_{s2}\Delta)} + (p^l_{s1}\Delta + c)\pi_1 e^{-\alpha(p^l_{s1}\Delta+c)} + (p^l_{s2}\Delta + c)\pi_2 e^{-\alpha(p^l_{s2}\Delta+c)}}{\alpha\Delta(p^h_{s1}\pi_1 e^{-\alpha(p^h_{s1}\Delta)} + p^h_{s2}\pi_2 e^{-\alpha(p^h_{s2}\Delta)} - p^l_{s1}\pi_1 e^{-\alpha(p^l_{s1}\Delta+c)} - p^l_{s2}\pi_2 e^{-\alpha(p^l_{s2}\Delta+c)})}$$

where

$$F = \pi_1 e^{-\alpha(p^h_{s1}\Delta)} + \pi_2 e^{-\alpha(p^h_{s2}\Delta)} - \pi_1 e^{-\alpha(p^l_{s1}\Delta+c)} - \pi_2 e^{-\alpha(p^l_{s2}\Delta+c)} \quad \text{(Due to IC binds)}$$

It is easy to see that the denominator of $\frac{\partial\Delta}{\partial\alpha}$ is positive. However, to determine the sign of the numerator we employ the following lemma.

Lemma 2. $\pi_1 x e^{-\alpha x} + \pi_2 y e^{-\alpha y}$ is a decreasing function of x, where $\pi_1 e^{-\alpha x} + \pi_2 e^{-\alpha y} = K,\ \pi_1 + \pi_2 = 1$ and $x > y$.

Proof. We want to show that

$$\frac{\partial(\pi_1 x e^{-\alpha x} + \pi_2 y e^{-\alpha y})}{\partial x} < 0$$

$$\frac{\partial(\pi_1 x e^{-\alpha x} + \pi_2 y e^{-\alpha y})}{\partial x} = \pi_1 e^{-\alpha x}(1 - x\alpha) + \pi_2 e^{-\alpha y}\frac{\partial y}{\partial x}(1 - y\alpha) \tag{4.1}$$

$$-\alpha(\pi_1 e^{-\alpha x} + \pi_2 \frac{\partial y}{\partial x} e^{-\alpha y}) = 0 \text{ since } \pi_1 e^{-\alpha x} + \pi_2 e^{-\alpha y} = K$$

$$\pi_2 \frac{\partial y}{\partial x} e^{-\alpha y} = -\pi_1 e^{-\alpha x} \tag{4.2}$$

If we substitute **Equation 4.1** into **Equation 4.2** we obtain the desired result:

$$\pi_1 e^{-\alpha x}(1 - x\alpha) - \pi_1 e^{-\alpha x}(1 - y\alpha) = \pi_1 \alpha e^{-\alpha x}(y - x) < 0 \text{ since } \alpha > 0 \text{ and } x > y$$

■

We use this lemma to determine the sign of $\frac{\partial F}{\partial \alpha}$ as follows:

$$p^h_{s1}\Delta\pi_1 e^{-\alpha(p^h_{s1}\Delta)} + p^h_{s2}\Delta\pi_2 e^{-\alpha(p^h_{s2}\Delta)} < (p^l_{s1}\Delta+c)\pi_1 e^{-\alpha(p^l_{s1}\Delta+c)} + (p^l_{s2}\Delta+c)\pi_2 e^{-\alpha(p^l_{s2}\Delta+c)}$$

since $\pi_1 m e^{-\alpha m} + \pi_2 n e^{-\alpha n}$ is a decreasing function of m as implied by the lemma and $p^h_{s1}\Delta > p^l_{s1}\Delta + c$. This shows that the nominator is positive as well. Hence, we have $\frac{\partial a}{\partial \alpha} > 0$. ■

Proposition 2. *When the high effort level is less ambiguous, $p^h_{s1} - p^h_{s2} < p^l_{s1} - p^l_{s2}$, an increase in the CAAA coefficient α makes the wage scheme lower-powered, i.e., $\frac{\partial \Delta}{\partial \alpha} < 0$.*

Proof. As before, by the implicit function theorem we have

$$\frac{\partial \Delta}{\partial \alpha} = -\frac{\frac{\partial F}{\partial \alpha}}{\frac{\partial F}{\partial \Delta}}$$
$$= \frac{-p^h_{s1}\Delta\pi_1 e^{-\alpha(p^h_{s1}\Delta)} - p^h_{s2}\Delta\pi_2 e^{-\alpha(p^h_{s2}\Delta)} + (p^l_{s1}\Delta + c)\pi_1 e^{-\alpha(p^l_{s1}\Delta+c)} + (p^l_{s2}\Delta + c)\pi_2 e^{-\alpha(p^l_{s2}\Delta+c)}}{\alpha\Delta(p^h_{s1}\pi_1 e^{-\alpha(p^h_{s1}\Delta)} + p^h_{s2}\pi_2 e^{-\alpha(p^h_{s2}\Delta)} - p^l_{s1}\pi_1 e^{-\alpha(p^l_{s1}\Delta+c)} - p^l_{s2}\pi_2 e^{-\alpha(p^l_{s2}\Delta+c)})}$$

Similar as before, it follows that the numerator is negative and the denominator is positive. Thus, we have $\frac{\partial \Delta}{\partial \alpha} < 0$. ■

Proposition 3. *When the high effort level is as ambiguous as the low effort level, $p^h_{s1} - p^h_{s2} = p^l_{s1} - p^l_{s2}$, an change in the CAAA coefficient α does not have any effect on the bonus utility, i.e., $\frac{\partial \Delta}{\partial \alpha} = 0$.*

Proof. In this case, one obtains

$$\Delta = \frac{c}{p^h_{s1} - p^l_{s1}} = \frac{c}{p^h_{s2} - p^l_{s2}}$$

Hence, we have $\frac{\partial \Delta}{\partial \alpha} = 0$. ■

4.1.2 An increase in the spread

For both effort levels, we consider an increase in the spread by decreasing the success probabilities coming from the second probability measure by the amount of i. Formally,

$$\forall e \in E,\ (p^e_{s2})' = p^e_{s2} - i$$

We refer to this as a *uniform increase in the spread.* Note that such a uniform increase in spread does not change the more ambiguous effort. However, it increases in some sense the ambiguity for both effort levels. Therefore, the agent becomes more inclined to exert the level of effort that is less ambiguous in this case as well. That is, we show that when the high effort level is more (less) ambiguous, the principal has to provide more (less) incentives corresponding an increase (decrease) in Δ.

Proposition 4. *When the high effort level is more ambiguous, $p^h_{s1} - p^h_{s2} > p^l_{s1} - p^l_{s2}$, a uniform increase in the spread makes the optimal wage scheme higher-powered, i.e., $\frac{\partial \Delta}{\partial i} > 0$.*

Proof. By the implicit function theorem,

$$\frac{\partial \Delta}{\partial i} = -\frac{\frac{\partial F}{\partial i}}{\frac{\partial F}{\partial \Delta}}$$
$$= \frac{\Delta \pi_2 \left(e^{-\alpha(p^h_{s2}-i)\Delta} - e^{-\alpha((p^l_{s2}-i)\Delta+c)}\right)}{p^h_{s1}\pi_1 e^{-\alpha(p^h_{s1}\Delta)} + (p^h_{s2}-i)e^{-\alpha(p^h_{s2}-i)\Delta} - p^l_{s1}\pi_1 e^{-\alpha(p^l_{s1}\Delta+c)} - (p^l_{s2}-i)e^{-\alpha((p^l_{s2}-i)\Delta+c)}}$$

The numerator and the denominator are both positive. Therefore, we have $\frac{\partial \Delta}{\partial i} > 0$. ■

Proposition 5. *When the high effort level is less ambiguous, $p^h_{s1} - p^h_{s2} < p^l_{s1} - p^l_{s2}$, a uniform increase in the spread makes the optimal wage scheme lower-powered, i.e., $\frac{\partial \Delta}{\partial i} < 0$.*

Proof. Similar as before,

$$\frac{\partial \Delta}{\partial i} = -\frac{\frac{\partial F}{\partial i}}{\frac{\partial F}{\partial \Delta}}$$
$$= \frac{\Delta \pi_2 \left(e^{-\alpha(p^h_{s2}-i)\Delta} - e^{-\alpha((p^l_{s2}-i)\Delta+c)}\right)}{p^h_{s1}\pi_1 e^{-\alpha(p^h_{s1}\Delta)} + (p^h_{s2}-i)e^{-\alpha(p^h_{s2}-i)\Delta} - p^l_{s1}\pi_1 e^{-\alpha(p^l_{s1}\Delta+c)} - (p^l_{s2}-i)e^{-\alpha((p^l_{s2}-i)\Delta+c)}}$$

The numerator is now negative and the denominator is positive. Thus, we have $\frac{\partial \Delta}{\partial i} < 0$. ■

Proposition 6. *When the high effort level is as ambiguous as the low effort level, $p^h_{s1} - p^h_{s2} = p^l_{s1} - p^l_{s2}$, a uniform increase in the spread does not have any effect on the bonus utility, i.e.,* $\frac{\partial \Delta}{\partial i} = 0$.

Proof.

$$\frac{\partial \Delta}{\partial i} = -\frac{\frac{\partial F}{\partial i}}{\frac{\partial F}{\partial \Delta}}$$
$$= \frac{\Delta \pi_2 (e^{-\alpha(p^h_{s2}-i)\Delta} - e^{-\alpha((p^l_{s2}-i)\Delta+c)})}{p^h_{s1}\pi_1 e^{-\alpha(p^h_{s1}\Delta)} + (p^h_{s2}-i)e^{-\alpha(p^h_{s2}-i)\Delta} - p^l_{s1}\pi_1 e^{-\alpha(p^l_{s1}\Delta+c)} - (p^l_{s2}-i)e^{-\alpha((p^l_{s2}-i)\Delta+c)}}$$

The numerator now becomes zero. Hence, $\frac{\partial \Delta}{\partial i} = 0$. ■

4.2 Cost of effort

Intuitively, an increase in the difference $c_H - c_L$ makes exerting the high level of effort relatively more costly. Therefore, one expects the agent to be incentivized more, i.e. an increase in $c_H - c_L$ leads to an increase in Δ:

Proposition 7. *An increase in the difference between the cost of high level of effort and the cost of low level of effort, $c_H - c_L$, makes the optimal wage scheme higher-powered, i.e.,* $\frac{\partial \Delta}{\partial c} > 0$.

Proof. Again, by the implicit function theorem,

$$\frac{\partial \Delta}{\partial c} = -\frac{\frac{\partial F}{\partial c}}{\frac{\partial F}{\partial \Delta}}$$
$$= -\frac{\alpha(\pi_1 e^{-\alpha(p^l_{s1}\Delta+c)} + \pi_2 e^{-\alpha(p^l_{s2}\Delta+c)})}{-\alpha\Delta(p^h_{s1}\pi_1 e^{-\alpha(p^h_{s1}\Delta)} + p^h_{s2}\pi_2 e^{-\alpha(p^h_{s2}\Delta)} - p^l_{s1}\pi_1 e^{-\alpha(p^l_{s1}\Delta+c)} - p^l_{s2}\pi_2 e^{-\alpha(p^l_{s2}\Delta+c)})}$$
$$= \frac{\pi_1 e^{-\alpha(p^l_{s1}\Delta+c)} + \pi_2 e^{-\alpha(p^l_{s2}\Delta+c)}}{\Delta(p^h_{s1}\pi_1 e^{-\alpha(p^h_{s1}\Delta)} + p^h_{s2}\pi_2 e^{-\alpha(p^h_{s2}\Delta)} - p^l_{s1}\pi_1 e^{-\alpha(p^l_{s1}\Delta+c)} - p^l_{s2}\pi_2 e^{-\alpha(p^l_{s2}\Delta+c)})}$$

The numerator and the denominator are both positive. Thus, we have $\frac{\partial \Delta}{\partial c} > 0$. ■

4.3 Second order beliefs

We now provide comparative statics over second order beliefs, π_1 and π_2.

4.3.1 An increase in π_1

An increase in π_1 can be intrepreted as both the principal and the agent becoming more confident that the true probability measure describing the relation between the level of effort and the corresponding monetary outcome is the 1^{st} probability measure.

We recall that the 1^{st} probability measure assigns a higher likelihood to the *success* outcome compared to the 2^{nd} probability measure for both levels of effort.

However, an increase in π_1 has different effects on Δ depending on which level of effort is more ambiguous. When the high level of effort is more (less) ambiguous, an increase in π_1 leads to a decrease (increase) in Δ.

The intuition behind is as follows: when the high effort level is more (less) ambiguous, the agent must be incentivized less (more) as the principal is becoming more confident that it is now more likely to see the *success* outcome on average.[1]

Proposition 8. *When the high effort level is more ambiguous, $p^h_{s1} - p^h_{s2} > p^l_{s1} - p^l_{s2}$, as π_1 increases the optimal wage scheme becomes lower-powered, i.e, $\frac{\partial \Delta}{\partial \pi_1} < 0$.*

[1] Note that the average *success* probability equals to $\pi_1 p^h_{s1} + \pi_2 p^h_{s2}$.

Proof. By the implicit function theorem,

$$\frac{\partial \Delta}{\partial \pi_1} = -\frac{\frac{\partial F}{\partial \pi_1}}{\frac{\partial F}{\partial \Delta}}$$

$$= -\frac{e^{-\alpha(p^h_{s1}\Delta)} - e^{-\alpha(p^h_{s2}\Delta)} - e^{-\alpha(p^l_{s1}\Delta+c)} + e^{-\alpha(p^l_{s2}\Delta+c)}}{-\alpha\Delta(p^h_{s1}\pi_1 e^{-\alpha(p^h_{s1}\Delta)} + p^h_{s2}\pi_2 e^{-\alpha(p^h_{s2}\Delta)} - p^l_{s1}\pi_1 e^{-\alpha(p^l_{s1}\Delta+c)} - p^l_{s2}\pi_2 e^{-\alpha(p^l_{s2}\Delta+c)})}$$

$$= \frac{e^{-\alpha(p^h_{s1}\Delta)} - e^{-\alpha(p^h_{s2}\Delta)} - e^{-\alpha(p^l_{s1}\Delta+c)} + e^{-\alpha(p^l_{s2}\Delta+c)}}{\alpha\Delta(p^h_{s1}\pi_1 e^{-\alpha(p^h_{s1}\Delta)} + p^h_{s2}\pi_2 e^{-\alpha(p^h_{s2}\Delta)} - p^l_{s1}\pi_1 e^{-\alpha(p^l_{s1}\Delta+c)} - p^l_{s2}\pi_2 e^{-\alpha(p^l_{s2}\Delta+c)})} < 0$$

Due to $p^h_{s1}\Delta > p^l_{s1}\Delta + c$ and $p^h_{s2}\Delta < p^l_{s2}\Delta + c$ we have $\frac{\partial \Delta}{\partial \pi_1} < 0$. ∎

Proposition 9. *When the high effort level is less ambiguous,* $p^h_{s1} - p^h_{s2} < p^l_{s1} - p^l_{s2}$*, as* π_1 *increases the optimal wage scheme becomes higher-powered, i.e,* $\frac{\partial \Delta}{\partial \pi_1} > 0$.

Proof.

$$\frac{\partial \Delta}{\partial \pi_1} = -\frac{\frac{\partial F}{\partial \pi_1}}{\frac{\partial F}{\partial \Delta}}$$

$$= -\frac{e^{-\alpha(p^h_{s1}\Delta)} - e^{-\alpha(p^h_{s2}\Delta)} - e^{-\alpha(p^l_{s1}\Delta+c)} + e^{-\alpha(p^l_{s2}\Delta+c)}}{-\alpha\Delta(p^h_{s1}\pi_1 e^{-\alpha(p^h_{s1}\Delta)} + p^h_{s2}\pi_2 e^{-\alpha(p^h_{s2}\Delta)} - p^l_{s1}\pi_1 e^{-\alpha(p^l_{s1}\Delta+c)} - p^l_{s2}\pi_2 e^{-\alpha(p^l_{s2}\Delta+c)})}$$

$$= \frac{e^{-\alpha(p^h_{s1}\Delta)} - e^{-\alpha(p^h_{s2}\Delta)} - e^{-\alpha(p^l_{s1}\Delta+c)} + e^{-\alpha(p^l_{s2}\Delta+c)}}{\alpha\Delta(p^h_{s1}\pi_1 e^{-\alpha(p^h_{s1}\Delta)} + p^h_{s2}\pi_2 e^{-\alpha(p^h_{s2}\Delta)} - p^l_{s1}\pi_1 e^{-\alpha(p^l_{s1}\Delta+c)} - p^l_{s2}\pi_2 e^{-\alpha(p^l_{s2}\Delta+c)})} > 0$$

Due to $p^h_{s1}\Delta < p^l_{s1}\Delta + c$ and $p^h_{s2}\Delta > p^l_{s2}\Delta + c$ we get $\frac{\partial \Delta}{\partial \pi_1} > 0$. ∎

4.3.2 An increase in π_2

Similarly, one can interpret an increase in π_2 as both the principal and the agent becoming more confident that the true probability measure describing the relation between the level of effort and the corresponding monetary outcome is the 2nd probability measure.

We recall once again that the 1st probability measure assigns higher likelihood to *success* compared to the 2nd probability measure for both effort levels.

As before, an increase in π_2 has different effects on Δ depending on which level of is more ambiguous. When the high effort level is more (less) ambiguous, an increase in π_2 leads to an increase (decrease) in Δ.

The intuition is as follows: when the high effort level is more (less) ambiguous, the agent must be incentivized more (less) as the principal becomes more confident that it is now more likely to see the *failure* outcome on average.[2]

Proposition 10. *When the high effort level is more ambiguous, $p^h_{s1} - p^h_{s2} > p^l_{s1} - p^l_{s2}$, as π_2 increases the optimal wage scheme becomes higher-powered, i.e., $\frac{\partial \Delta}{\partial \pi_2} > 0$.*

Proof. This follows from the fact that $\frac{\partial \Delta}{\partial \pi_1} < 0$ from above and $\pi_1 + \pi_2 = 1$. ■

Proposition 11. *When the high effort level is less ambiguous, $p^h_{s1} - p^h_{s2} < p^l_{s1} - p^l_{s2}$, as π_2 increases the optimal wage scheme becomes lower-powered, i.e., $\frac{\partial \Delta}{\partial \pi_2} < 0$.*

Proof. This follows from the fact that $\frac{\partial \Delta}{\partial \pi_1} > 0$ from above and $\pi_1 + \pi_2 = 1$. ■

Note that we show in **Proposition 3** that when the high effort level and the low effort level are equally ambiguous (i.e. $p^h_{s1} - p^h_{s2} = p^l_{s1} - p^l_{s2}$), Δ is not a function of π_1 and π_2.

[2]Note that the average *failure* probability equals to $\pi_1 p^h_{f1} + \pi_2 p^h_{f2}$.

4.4 The summary of all comparative statics

The following table summarizes all of the comparative statics we obtain above.

Table 1: The summary of all comparative statics

	High effort level and low effort level are equally ambiguous $p^h_{s1} - p^h_{s2} = p^l_{s1} - p^l_{s2}$	High effort level is more ambiguous $p^h_{s1} - p^h_{s2} > p^l_{s1} - p^l_{s2}$	High effort level is less ambiguous $p^h_{s1} - p^h_{s2} < p^l_{s1} - p^l_{s2}$
$\frac{\partial \Delta}{\partial \alpha}$	0	+	-
$\frac{\partial \Delta}{\partial i}$	0	+	-
$\frac{\partial \Delta}{\partial \pi_1}$	0	-	+
$\frac{\partial \Delta}{\partial \pi_2}$	0	+	-
$\frac{\partial \Delta}{\partial c}$	+	+	+

CHAPTER 5

NUMERICAL EXAMPLES

In this section, we provide numerical examples to exemplify some of comparative statics results. **Examples 1-3** analyze the effect of CAAA coefficient, α, on the bonus utility, Δ. On the other hand, **Examples 4-6** focus on the effect of spread on the bonus utility.

We note that the high effort level is less ambiguous in **Examples 1 and 4** while the high effort level is more ambiguous in **Examples 3 and 6**. The high effort level and the low effort level are equally ambiguous in **Examples 2 and 5**.

Example 1. We use the following parameter set to construct the example above: $p_{s1}^h = 0.7,\ p_{s2}^h = 0.62,\ p_{s1}^l = 0.42,\ p_{s2}^l = 0.21,\ \pi_1 = 0.9$ with $\phi = -e^{-\alpha(u(w)-c)}$ where $\alpha = 0.1$ increases by 0.01 in 10 steps, $u(w) = \sqrt{w}$, $u_0 = 15$, and $c = 1$.

Example 2. We use the following parameter set to construct the example above: $p_{s1}^h = 0.72,\ p_{s2}^h = 0.62,\ p_{s1}^l = 0.42,\ p_{s2}^l = 0.32,\ \pi_1 = 0.9$ with $\phi = -e^{-\alpha(u(w)-c)}$ where $\alpha = 0.1$ increases by 0.01 in 10 steps, $u(w) = \sqrt{w}$, $u_0 = 15$, and $c = 1$.

Example 3. We use the following parameter set to construct the example above: $p_{s1}^h = 0.82,\ p_{s2}^h = 0.62,\ p_{s1}^l = 0.42,\ p_{s2}^l = 0.32,\ \pi_1 = 0.9$ with $\phi = -e^{-\alpha(u(w)-c)}$ where $\alpha = 0.1$ increases by 0.01 in 10 steps, $u(w) = \sqrt{w}$, $u_0 = 15$, and $c = 1$.

Example 4. We use the following parameter set to construct the example above: $p^h_{s1} = 0.7$, $p^h_{s2} = 0.62$, $p^l_{s1} = 0.42$, $p^l_{s2} = 0.21$, $\pi_1 = 0.9$ with $\phi = -e^{-\alpha(u(w)-c)}$ where $\alpha = 0.1$, spread increases by 0.01 in 10 steps, $u(w) = \sqrt{w}$, $u_0 = 15$, and $c = 1$.

Example 5. We use the following parameter set to construct the example above: $p^h_{s1} = 0.72$, $p^h_{s2} = 0.62$, $p^l_{s1} = 0.42$, $p^l_{s2} = 0.32$, $\pi_1 = 0.9$ with $\phi = -e^{-\alpha(u(w)-c)}$ where $\alpha = 0.1$, spread increases by 0.01 in 10 steps, $u(w) = \sqrt{w}$, $u_0 = 15$, and $c = 1$.

Example 6. We use the following parameter set to construct the example above: $p^h_{s1} = 0.82$, $p^h_{s2} = 0.62$, $p^l_{s1} = 0.42$, $p^l_{s2} = 0.32$, $\pi_1 = 0.9$ with $\phi = -e^{-\alpha(u(w)-c)}$ where $\alpha = 0.1$, spread increases by 0.01 in 10 steps, $u(w) = \sqrt{w}$, $u_0 = 15$, and $c = 1$.

Appendix contains the graphs related to **Examples 1-6**.

CHAPTER 6

CONCLUSION

We study a principal-agent problem under *moral hazard* where we model the uncertainty regarding the relationship between the level of effort and the monetary outcome using the smooth ambiguity model (Klibanoff et al., 2005). First, we show that the agent receives his reservation utility when the level of effort exerted is *observable* without the agent having constant absolute ambiguity aversion. Second, we demonstrate that under *moral hazard* the agent's payoff in the optimal wage scheme is the same as his reservation utility and the agent is indifferent between exerting the high effort level and the low effort level. Third, we show that the optimal wage scheme is monotonic in the outcomes when we assume a natural *monotone likelihood ratio property.* Fourth, we observe that the bonus utility may not increase as ambiguity aversion increases and identify three separate cases in which the bonus utility behaves differently when there is an increase in ambiguity aversion. This is in contrast to the standard model where an increase in risk aversion increases the bonus utility. Fifth, we show, by an example, that the bonus utility and the principal's profit does not have to be negatively correlated. Finally, we provide thorough comparative statics on the optimal wage scheme.

BIBLIOGRAPHY

Bolton, P., & Dewatripont, M. (2005). *Contract theory*. Massachusetts: MIT Press.

Ellsberg, D. (1961). Risk, ambiguity, and the savage axioms. *The Quarterly Journal of Economics*, 643–669.

Ghirardato, P. (1994). *Agency theory with non-additive uncertainty* (Tech. Rep.). mimeo, http://web. econ. unito. it/gma/paolo/age. pdf.

Gilboa, I., & Schmeidler, D. (1989). Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics*, *18*(2), 141–153.

Grossman, S. J., & Hart, O. D. (1983). An analysis of the principal-agent problem. *Econometrica*, 7–45.

Harris, M., & Raviv, A. (1979). Optimal incentive contracts with imperfect information. *Journal of Economic Theory*, *20*(2), 231–259.

Hölmstrom, B. (1979). Moral hazard and observability. *The Bell Journal of Economics*, 74–91.

Kellner, C. (2017). The principal-agent problem with smooth ambiguity. *Review of Economic Design*, *21*(2), 83–119.

Klibanoff, P., Marinacci, M., & Mukerji, S. (2005). A smooth model of decision making under ambiguity. *Econometrica*, *73*(6), 1849–1892.

Machina, M. J., & Siniscalchi, M. (2014). Ambiguity and ambiguity aversion. In M. J. Machina & K. Viscusi (Eds.), *Handbook of the Economics of Risk and Uncertainty* (Vol. 1, pp. 729–807). North-Holland.

Mirrlees, J. A. (1976). The optimal structure of incentives and authority within an organization. *The Bell Journal of Economics*, 105–131.

Ross, S. A. (1973). The economic theory of agency: The principal's problem. *The American Economic Review*, *63*(2), 134–139.

Schmeidler, D. (1989). Subjective probability and expected utility without additivity. *Econometrica*, 571–587.

APPENDIX

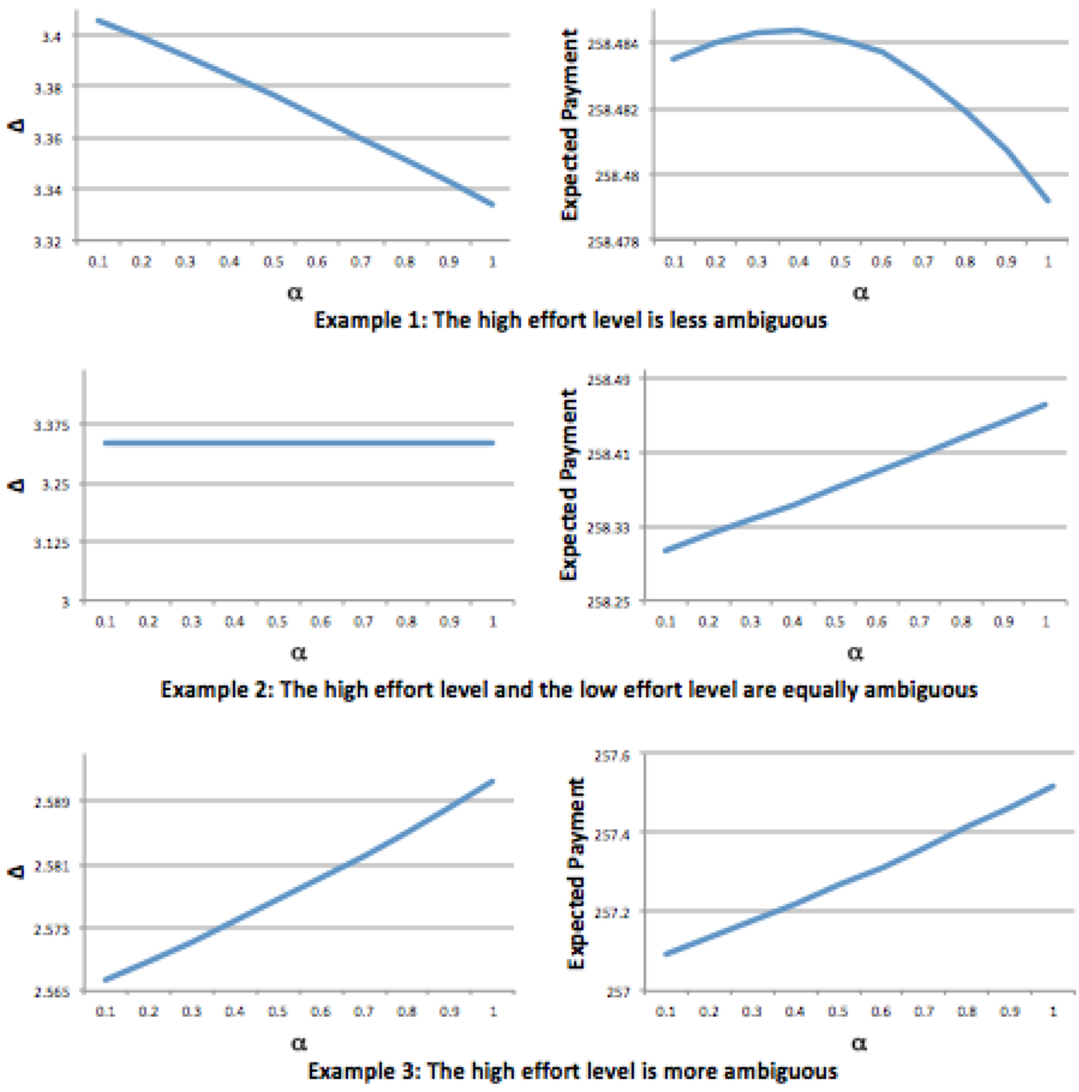

Figure 4: The bonus utility and expected payment in Examples 1-3

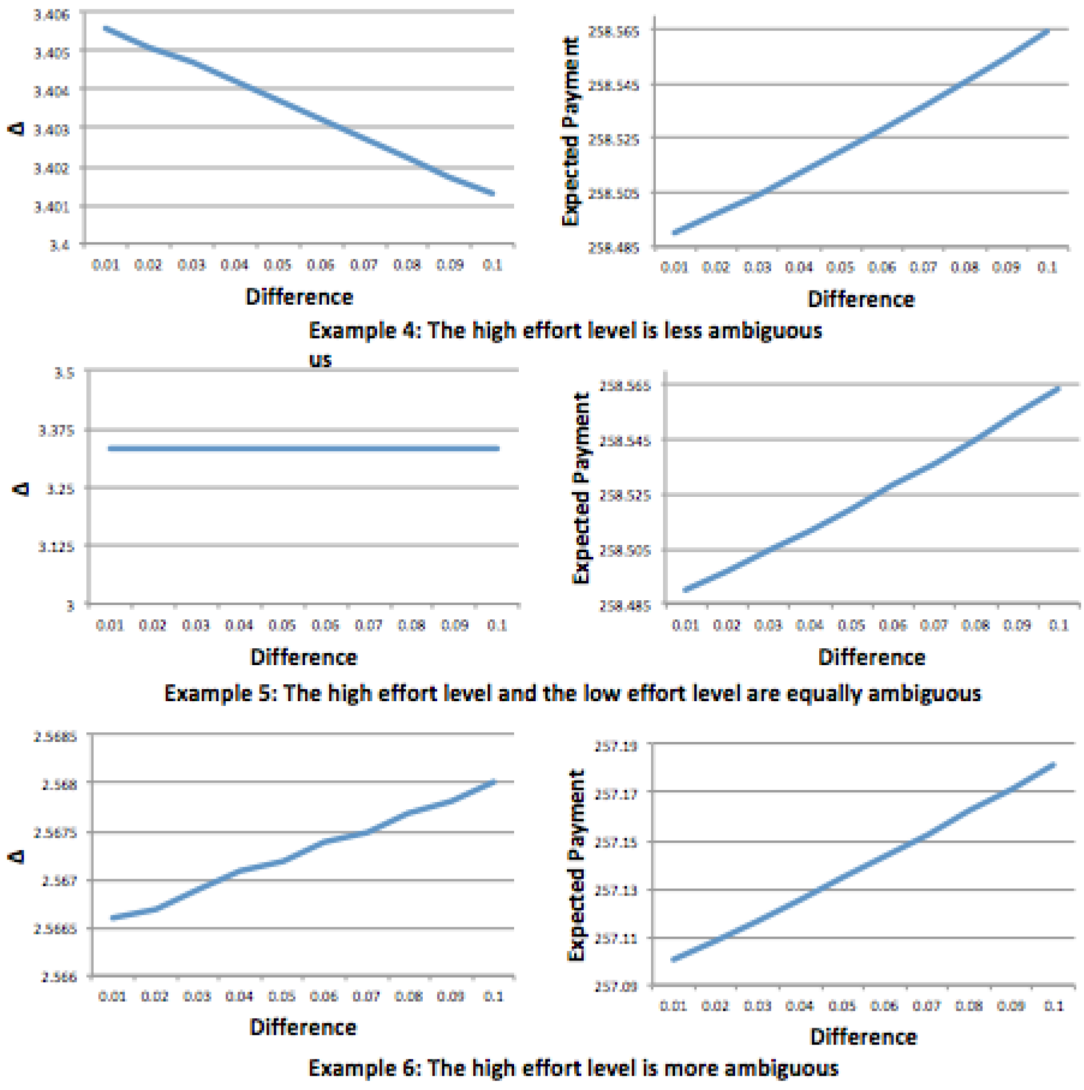

Figure 5: The bonus utility and expected payment in Examples 4-6