ZERO SETS OF ANALYTIC FUNCTION SPACES ON THE UNIT DISK

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Berk Bavaş

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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We survey some known results on the zero sets of two families of analytic function spaces and another single space defined on the unit disk in the complex plane. We investigate mostly the basic properties of the zero sets of these spaces that are comparable to those of the Hardy spaces and to each other. The spaces we consider are standard weighted Bergman spaces, the Dirichlet space, and certain Dirichlet-type spaces that are very close to both Hardy spaces and the Bergman spaces.

The completely known zero sets of Hardy spaces are easy to describe, characterized by the Blaschke condition and the same for all the spaces in the family. The zero sets of the other spaces considered have started to be investigated relatively recently and are far from a complete description. Yet it is possible to find conditions similar to the Blaschke condition for the zero sets of Bergman spaces and Dirichlet-type spaces. For the zero sets of the true Dirichlet space, the known results are sporadic and do not form a general theory yet.

Keywords: Zero set, Hardy space, Bergman space, Dirichlet space, Dirichlet-type space, Blaschke condition.

ÖZET

BİRİM DAİREDEKİ ANALİTİK FONKSİYON UZAYLARININ SIFIR KÜMELERİ

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Karmaşık düzlemdeki birim dairede tanımlı iki analitik fonksiyon uzayı ailesinin ve bir tek ayrı uzayın sıfır kümeleri üzerine bilinen bazı sonuçları derledik. Bu uzayların sıfır kümelerinin daha çok Hardy uzaylarının sıfır kümeleriyle ve birbirleriyle karşılaştırılabilen temel özelliklerini inceledik. İncelediğimiz uzaylar, standart ağırlıklı Bergman uzayları, Dirichlet uzayı ve Hardy ile Bergman uzayına çok yakın özellikleri olan bazı Dirichlet türü uzaylardır.

Hardy uzaylarının tam olarak bilinen sıfır kümeleri kolay betimlenir; Blaschke koşulu ile tarif edilir ve ailedeki bütün uzaylar için aynıdır. Diğer baktğımız uzayların sıfır kümeleri nispeten yeni araştırılmaya başlanmıştır ve tam bir betimlemeden uzaktır. Gene de Bergman uzayları ve Dirichlet türü uzayların sıfır kümeleri için Blaschke koşuluna benzer koşullar bulmak mümkündür. Asıl Dirichlet uzayının sıfır kümeleri için bilinen sonuçlar ise ayrı ayrıdır ve henüz genel bir teori oluşturmaz.

Anahtar sözcükler: Sıfır kümesi, Hardy uzayı, Bergman uzayı, Dirichlet uzayı, Dirichlet türü uzay, Blaschke koşulu.

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Contents

Chapter 1

Introduction

The zero set of an arbitrary analytic function on a domain in the complex plane is completely arbitrary except for the restriction that it cannot have any limit point inside the domain. In other words, all possible limit points must be on the boundary of the domain. Hence one is led to consider smaller classes of analytic functions on a given domain to gain more information on the zero sets.

One such family of Banach spaces considered early is the family of Hardy spaces H^p on the unit disk defined by an integral growth condition near the unit cirle. The zero set of a function in H^p is completely characterized by the extremely simple Blaschke condition, which describes how fast the zeros must approach the unit circle. The interesting fact is that this condition is the same no matter what p is with $0 < p \leq \infty$.

There are other families of Banach spaces of analytic functions on the unit disc mostly defined using different kinds of integral growth conditions. The investigation of the zero set of a function in one of these spaces first concentrates on finding a condition that can replace the Blaschke condition.

Bergman spaces A^p have been known for a long time, but the first results on their zero sets were given in mid 1970's by C. Horowitz. Now there is a satisfactory theory of the zero sets of functions in the Bergman spaces and their weighted counterparts A^p_α, $\alpha > -1$.

Research on the zero sets of functions in the Dirichlet space was started in the 1950's by L. Carleson, but progress has been very slow. The results are sporadic and their descriptions are difficult. Results on the zero sets of the family of Dirichlet-type spaces are even newer (2000's and later) and rarer. They rely on the properties of the spaces that are close to those of Bergman or Hardy spaces.

In this thesis, we study some known results on the basic characteristics of the zero sets of the analytic function spaces other than the Hardy spaces. The comparisons we make are summarized at the end of the Chapter 2.

Chapter 2

Preliminaries

Throughout this thesis, C stands for a positive constant changing from line to line. The set of *natural numbers*, denoted $\mathbb{N}$, is $\{0, 1, 2, \dots\}$. The *unit disk* in the complex plane $\mathbb{C}$, denoted $\mathbb{D}$, is the set $\{z \in \mathbb{C} : |z| < 1\}$. The *unit circle*, denoted $\mathbb{T}$, is the set $\{z \in \mathbb{C} : |z| = 1\}$. The *set of all analytic functions* on $\mathbb{D}$ is denoted by $H(\mathbb{D})$.

The following notations are heavily used in this thesis.

Notation 2.0.1. *Let f and g be functions of real numbers. We say "f is big-oh of g as x goes to s" and write*

$$f(x) = O(g(x)), \quad x \to s,$$

if

$$\limsup_{x \to s} \left| \frac{f(x)}{g(x)} \right| < \infty.$$

Note that for a real number s, we have $f(x) = O(g(x))$ as $x \to s$ if and only if there exists $C > 0$ and $\delta > 0$ such that $|f(x)| < C|g(x)|$ whenever $0 < |x - s| < \delta$. We also have $f(x) = O(g(x))$ as $x \to \infty$ if and only if there exists $C > 0$ and $x_0 > 0$ such that $|f(x)| < C|g(x)|$ for all $x > x_0$.

Notation 2.0.2. *Let f and g be functions of real numbers. We say "f is little-oh* *of g as x goes to s" and write*

$$f(x) = o(g(x)), \quad x \to s,$$

if

$$\limsup_{x \to s} \left| \frac{f(x)}{g(x)} \right| = 0.$$

Note that for a real number s, we have $f(x) = o(g(x))$ as $x \to s$ if and only if for all $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x)| < \epsilon |g(x)|$ whenever $0 < |x - s| < \delta$. We also have $f(x) = o(g(x))$ as $x \to \infty$ if and only if for all $\epsilon > 0$ there exists $x_0 > 0$ such that $|f(x)| < \epsilon |g(x)|$ for all $x > x_0$.

Notation 2.0.3. *If $f(x) = O(g(x))$ as $x \to s$ and $g(x) = O(f(x))$ as $x \to s$, then we write $f(x) \asymp g(x)$ as $x \to s$.*

In the three notations above, s may be a real number or ∞. Also the limit in these notations may be one-sided. We omit the expression "$x \to s$" if the context is clear. We sometimes write $f(x) \lesssim g(x)$ instead of $f(x) = O(g(x))$.

Definition 2.0.4. *The gamma function is defined by*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt, \quad \mathrm{Re}(z) > 0$$

and the beta function is defined by

$$\mathrm{B}(z, w) = \int_0^1 t^{z-1} (1-t)^{w-1} \, dt, \quad \mathrm{Re}(z) > 0, \mathrm{Re}(w) > 0.$$

The relation between the gamma and beta functions is given by the following proposition.

Proposition 2.0.5.

$$\mathrm{B}(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \quad \mathrm{Re}(z) > 0, \; \mathrm{Re}(w) > 0.$$

Theorem 2.0.6 (Stirling's formula)**.**

$$\lim_{z \to \infty} \frac{\Gamma(z+1)}{\sqrt{2\pi z} z^z e^{-z}} = 1.$$

Note that Stirling's formula and Proposition 2.0.5 imply that

$$\mathrm{B}(x,y) \asymp \frac{x^{x-1/2}y^{y-1/2}}{(x+y)^{x+y-1/2}}, \quad x \to \infty,\ y \to \infty.$$

So for fixed y, we have $\mathrm{B}(x,y) \asymp x^{-y}$ as $x \to \infty$.

Theorem 2.0.7 (Summation by parts)**.** *Let $\{a_n\}$ and $\{b_n\}$ be two sequences. Set $A_{-1} = 0$ and*

$$A_n = \sum_{k=0}^{n} a_k, \quad n \geq 0.$$

If $0 \leq s \leq t$, then

$$\sum_{n=s}^{t} a_n b_n = \sum_{n=s}^{t-1} A_n(b_n - b_{n+1}) + A_t b_t - A_{s-1} b_s.$$

Definition 2.0.8. *A probability space $(\Omega, \mathcal{F}, P)$ is a measure space such that the measure $P(\Omega)$ of the sample space Ω is 1.*

Definition 2.0.9. *A random variable $X : \Omega \to \mathbb{R}$ is a measurable function from the sample space Ω to the real numbers.*

Definition 2.0.10. *A family of random variables $\{X_i\}_{i\in I}$ is called identically distributed if $P(X_i \leq x) = P(X_j \leq x)$ for all x and $i, j \in I$, where $P(X \leq x) = P(\{\omega \in \Omega : X(\omega) \leq x\})$.*

Definition 2.0.11. *A family of random variables $\{X_i\}_{i\in I}$ is called independent if for every Borel set $\mathcal{B} \in \mathcal{F}$, we have $P(\cap_{i\in I}(X_i \in \mathcal{B})) = \prod_{i\in I} P(X_i \in \mathcal{B})$ for all $i, j \in I$, where $P(X \in \mathcal{B}) = P(\{\omega \in \Omega : X(\omega) \in \mathcal{B}\})$.*

Definition 2.0.12. *If p and p' are positive real numbers such that*

$$\frac{1}{p} + \frac{1}{p'} = 1,$$

then we call p and p' a pair of conjugate exponents.

Note that if p and p' is a pair of conjugate exponents, then $1 < p < \infty$ if and only if $1 < p' < \infty$.

Theorem 2.0.13 (Minkowski's integral inequality)**.** *Let $1 \le p < \infty$. Let (X, μ) and (Y, ν) be σ-finite measure spaces. If $f(x, y)$ is a nonnegative $\nu \times \mu$-measurable function on $X \times Y$, then*

$$\left\{ \int_X \left\{ \int_Y f(x,y)\, d\nu(y) \right\}^p d\mu(x) \right\}^{1/p} \le \int_Y \left\{ \int_X f^p(x,y)\, d\mu(x) \right\}^{1/p} d\nu(y).$$

Theorem 2.0.14 (Hölder's inequality)**.** *Let p and p' be conjugate exponents. Let X be a measure space with measure μ. If f and g are nonnegative measureable functions on X, then*

$$\int_X f(x)g(x)\, d\mu(x) \le \left\{ \int_X f^p(x)\, d\mu(x) \right\}^{1/p} \left\{ \int_X g^{p'}(x)\, d\mu(x) \right\}^{1/p'}.$$

The equality occurs when f^p is multiple of $g^{p'}$ almost everywhere.

Theorem 2.0.15 (Parseval's formula)**.** *Let H be a Hilbert space and $\{x_k\}_{k=0}^{\infty}$ be an orthonormal (Hilbert) basis of H. If $y \in H$, then*

$$\sum_{k=0}^{\infty} |\langle y, x_k \rangle|^2 = \|y\|^2.$$

A typical application of Parseval's formula in this thesis is as follows. Let H be the Hilbert space consisting of Lebesgue square-integrable and complex-valued functions on $[0, 2\pi]$ with the inner product

$$\langle f, g \rangle = \left\{ \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})}\, d\theta \right\}^{1/2}.$$

Let f be an analytic function on $\mathbb{D}$. The Taylor series expansion of f around the point $z = 0$ is

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad z \in \mathbb{D}.$$

Then writing $z = re^{i\theta}$, we find

$$f(re^{i\theta}) = \sum_{k=0}^{\infty} a_k r^k e^{ik\theta}, \quad \theta \in [0, 2\pi], \quad r \in [0, 1).$$

Suppose that for each $r \in (0, 1)$, we have $f_r(e^{i\theta}) \in H$, where $f_r(e^{i\theta}) = f(re^{i\theta})$. Fix $r \in (0, 1)$. Then we apply Parseval's formula and obtain

$$\|f_r\|^2 = \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=0}^{\infty} a_k r^k e^{ik\theta} \right|^2 d\theta = \sum_{k=0}^{\infty} |a_k|^2 r^{2k},$$

since $\{e^{ik\theta}\}_{k=0}^{\infty}$ is a Hilbert basis of H.

Theorem 2.0.16 (Jensen's inequality)**.** [14, Theorem 3.3] *Let* $(\Omega, \mathcal{F}, \mu)$ *be a measure space such that* $\mu(\Omega) = 1$*. If* g *is real-valued function in* $L^1(\mu)$ *and* ϕ *is a convex function on the real line, then*

$$\phi\Big(\int_\Omega g\, d\mu\Big) \le \int_\Omega (\phi \circ g)\, d\mu.$$

When $\phi = \exp$ and $g = \log f$, where the functions exp and log is defined on the positive real line as usual, and f is a positive-valued function, Jensen's inequality becomes

$$\exp\left(\int_\Omega \log f\, d\mu\right) \le \int_\Omega f\, d\mu.$$

which is called the *arithmetic-geometric-mean (agm) inequality.*

Notation 2.0.17. *For* $0 < p < \infty$ *and* $0 \le r < 1$*, we set*

$$M_p(r, f) = \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta\right\}^{1/p}$$

and

$$M_\infty(r, f) = \max_{0 \le \theta < 2\pi} |f(re^{i\theta})|.$$

Theorem 2.0.18 (Hardy's convexity theorem)**.** [3, Theorem 1.5] *Let* $0 < p \le \infty$*. If* $f \in H(\mathbb{D})$*, then* $M_p(r, f)$ *is a nondecreasing function of* r*.*

Let $0 < p < q < \infty$ and $0 < r < 1$. Let $f(z)$ be a non-constant measurable function defined on $\mathbb{D}$. In Hölder's inequality, take f as $|f(re^{i\theta})|^p$ and g as the identity function on $X = [0, 2\pi]$. Then we find

$$M_p^p(r, f) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta < \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^q\, d\theta\right\}^{p/q} = M_q^p(r, f),$$

because $\dfrac{1}{2\pi}d\theta$ is a unit measure on $[0, 2\pi]$.

Theorem 2.0.19. *Let* $0 < r < 1$ *and* f *be a non-constant function. If* $0 < p \le q \le \infty$ *then* $M_p(r, f) \le M_q(r, f)$*.*

Let f be an analytic function defined on $\mathbb{D}$. A point α in $\mathbb{D}$ is called a *zero of* f *of order* m if $f(\alpha) = f'(\alpha) = \cdots = f^{(m-1)}(\alpha) = 0$ and $f^{(m)}(\alpha) \ne 0$. We simply say a zero of f instead of "zero of f of order m" if indicating the order of the zero is not necessary. We say f *vanishes* at a point α if α is a zero of f.

We consider only the analytic functions having infinitely many zeros. The analytic functions having finitely many zeros is out of our interest, because they are easy to describe; an analytic function f defined on $\mathbb{D}$ that has finitely many zeros can be factored as $f(z) = P(z)g(z)$ for all $z \in \mathbb{D}$, where P is a polynomial and g is a nowhere vanishing analytic function on $\mathbb{D}$. Conversely, for a given finite set there exists a polynomial vanishing precisely on it. Since all polynomials are analytic and bounded on the closed unit disk, they belong to the all function spaces that we discuss in this thesis. So there is nothing interesting about the analytic functions that have finitely many zeros.

If f is analytic in $\mathbb{D}$ and its zeros have an accumulation point in $\mathbb{D}$, then f is identically zero on $\mathbb{D}$. In other words, an analytic function that is not identically zero in $\mathbb{D}$ has isolated zeros. A set in the unit disk having uncountably many elements has an accumulation point in the unit disk. So the only analytic function vanishing on an uncountable set is the zero function. Therefore, an analytic function that is not identically zero in $\mathbb{D}$ must have countably many zeros.

Let f be an analytic function that is not identically zero in $\mathbb{D}$. A sequence $\{z_k\}$ is called the *zero set* of f if f vanishes precisely on $\{z_k\}$ and nowhere else. If f has a zero of order m at a point α in $\mathbb{D}$, then the point α occurs m times in the zero set $\{z_k\}$ of f. We say the zero set of an analytic function g is a *subset* of the zero set of another analytic function f if f vanishes wherever g does with at least the same multiplicity. The *union* of two zero sets is the usual set union except it assigns to any common point a multiplicity equal to the sum of two multiplicities.

Definition 2.0.20. *We say a sequence $\{z_k\}$ is a zero set of a space X of analytic functions on the unit disk if there exists a function $0 \not\equiv f \in X$ such that $\{z_k\}$ is the zero set of f.*

We consider only zero sets $\{z_k\}$ with infinitely many elements; hence $|z_k| \to 1^-$ as $k \to \infty$. We order a zero set $\{z_k\}$ so that $0 < |z_1| \leq |z_2| \leq \ldots$, in fact; in most places, we assume $1 - 1/e < |z_1|$ for convenience because if $f(0) = 0$, with multiplicity m, then the results can be applied to $g(z) = f(z)/z^m$.

For $f \in H(\mathbb{D})$ and $0 < r < 1$, we denote the *zero counting function* of f by $n_f(r)$. More explicitly, its value is the number of zeros of f in $|z| < r$ counting multiplicities.

If $f(0) \neq 0$, then we denote the *zero density function* $N_f(r)$ of f by

$$N_f(r) = \int_0^r \frac{n_f(t)}{t}\, dt.$$

We abbreviate $n_f(r)$ and $N_f(r)$ to $n(r)$ and $N(r)$ if the context is clear. Note that $N_f(r)$ is well-defined because the condition $f(0) \neq 0$ assures that there exists $r_0 > 0$ such that $n_f(r) = 0$ for all $r \in [0, r_0]$. Hence we may write

$$N_f(r) = \int_0^r \frac{n_f(t)}{t}\, dt = \int_{r_0}^r \frac{n_f(t)}{t}\, dt$$

and the integral becomes proper.

Proposition 2.0.21. [8, Proposition 4.2]. *If $f \in H(\mathbb{D})$ with $f(0) \neq 0$, then*

$$N(r) = \sum_{k=1}^{n} \log \frac{r}{|z_k|},$$

where $z_1, \ldots, z_n$ are the zeros of f in $|z| < r$ ordered as $|z_1| \leq \ldots \leq |z_n|$, repeated according to multiplicity.

Proof. We have $n(t) = 0$ for $0 < t \leq |z_1|$ because $f(0) \neq 0$. So

$$\begin{aligned} N(r) &= \int_0^r \frac{n(t)}{t}\, dt \\ &= \int_{|z_1|}^r \frac{n(t)}{t}\, dt \\ &= \sum_{k=1}^{n-1} \int_{|z_k|}^{|z_{k+1}|} \frac{n(t)}{t}\, dt + \int_{|z_n|}^r \frac{n(t)}{t}\, dt. \end{aligned}$$

Note that $n(t) = k$ for $|z_k| < t \le |z_{k+1}|$ and $n(t) = n$ for $|z_n| < t \le r$. Hence

$$\begin{aligned} N(r) &= \sum_{k=1}^{n-1} \left\{ k(\log|z_{k+1}| - \log|z_k|) \right\} + n(\log r - \log|z_n|) \\ &= \sum_{k=1}^{n} \log \frac{r}{|z_k|} \end{aligned}$$

as desired. □

Theorem 2.0.22 (Jensen's formula)**.** *Suppose $f \in H(\mathbb{D})$ with $f(0) \neq 0$. Let $0 < r < 1$ and $z_1, \ldots, z_n$ be the ordered zeros of f in the disk $|z| < r$, repeated according to multiplicity. Then*

$$\log|f(0)| + \sum_{k=1}^{n} \log \frac{r}{|z_k|} = \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta.$$

Using Proposition 2.0.21, we may rewrite Jensen's formula as

$$\log|f(0)| + N(r) = \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta.$$

Jensen's formula relates the rate of growth of an analytic function f and the density of the zeros of f.

Definition 2.0.23. *For $0 < p < \infty$, a function $f \in H(\mathbb{D})$ belongs to the Hardy space H^p if $M_p(r, f)$ remains bounded as $r \to 1^-$. A function $f \in H(\mathbb{D})$ belongs to H^∞ if $M_\infty(r, f)$ remains bounded as $r \to 1^-$.*

Proposition 2.0.24. *Let $0 < p \le \infty$. If $f \in H^p$ with $f(0) \neq 0$, then*

$$N(r) = O(1), \quad r \to 1^-$$

and

$$n(r) = O\left(\frac{1}{1-r}\right), \quad r \to 1^-.$$

Proof. Let first $0 < p < \infty$. Without loss of generality we may assume $f(0) = 1$. Fix $0 < r < 1$. Jensen's formula together with Proposition 2.0.21 imply

$$N(r) = \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta.$$

Multiplying the above by p, exponentiating and applying the agm inequality, we obtain

$$e^{pN(r)} \le \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta$$

for all $r \in (0,1)$. Taking limit as $r \to 1^-$, we find

$$\lim_{r\to 1^-} e^{pN(r)} < \infty,$$

because $f \in H^p$. Hence

$$N(r) = O(1), \quad r \to 1^-. \tag{2.1}$$

Note that $n(r)$ is nondecreasing on $(0,1)$. So we have

$$n(r^2)(r-r^2) = \int_{r^2}^{r} n(t)dt \le \int_{r^2}^{r} \frac{n(t)}{t}dt \le \int_0^r \frac{n(t)}{t}dt = N(r)$$

for all $r \in (0,1)$. Using (2.1), we deduce that

$$n(r^2) = O\left(\frac{1}{1-r}\right), \quad r \to 1^-.$$

Note that $1-\sqrt{r} \asymp (1-\sqrt{r})(1+\sqrt{r}) = 1-r$ as $r \to 1^-$. Hence

$$n(r) = O\left(\frac{1}{1-r}\right), \quad r \to 1^-$$

as required. The case $p = \infty$ is similar. □

Theorem 2.0.25. *Let $0 < p \le \infty$. Let $f \in H^p$ with $f(0) \neq 0$ and $\{z_k\}$ be the zero set of f. Then*

$$\sum_{k=1}^{\infty}(1-|z_k|) < \infty. \tag{2.2}$$

Proof. Note that

$$I = \sum_{k=1}^{\infty}(1-|z_k|) = \int_{|z_1|}^{1} (1-r)\, dn(r).$$

We employ integration by parts with $u = 1-r$ and $dv = dn(r)$ to obtain

$$I = (1-r)n(r)\Big|_{|z_1|}^{1} + \int_{|z_1|}^{1} n(r)dr.$$

Now Proposition 2.0.24 implies that the expression

$$(1-r)n(r)\Big|_{|z_1|}^{1} = \lim_{r\to 1^-}(1-r)n(r)$$

is finite. Another integration by parts with $u = r$ and $dv = \dfrac{n(r)}{r}dr$ yields

$$\int_{|z_1|}^{1} n(r)dr = rN(r)\Big|_{|z_1|}^{1} - \int_{|z_1|}^{1} N(r)dr.$$

Again using Proposition 2.0.24, we deduce the expression above is finite. Hence I is finite. □

The statement in (2.2) is called the *Blaschke condition.* A sequence in the unit disk is called a *Blaschke sequence* if it satisfies the Blaschke condition. Basically, Theorem 2.0.25 implies that for a sequence $\{z_k\}$ in the unit disk to be a zero set of H^p, it is necessary that $\{z_k\}$ satisfies the Blaschke condition. This condition also turns out to be sufficient: For a sequence in the unit disk satisfying the Blaschke condition, one can construct an infinite product B, which is given in (2.3) below, so that $B \in H^\infty \subseteq H^p$. To prove this result, we need the following notions.

Definition 2.0.26. *Let $\{a_k\}$ be a sequence of complex numbers such that $a_k \neq 0$ for all $k = 1, 2, \ldots$. Let*

$$P_n = \prod_{k=1}^{n} a_k.$$

If $P = \lim_{n\to\infty} P_n$ exists and is not zero, then we write

$$P = \prod_{k=1}^{\infty} a_k$$

and say that the infinite product P converges. For a sequence $\{a_k\}$ with at most finitely many zero terms, that is $a_k \neq 0$ for all $k \geq m$, we say the infinite product

$$P = \prod_{k=1}^{\infty} a_k$$

converges to zero and set $P = 0$ if

$$\lim_{n\to\infty} \prod_{k=m}^{n} a_k$$

exists and is not zero.

Theorem 2.0.27. [1, p. 242] *Let $\{a_k\}$ be a sequence of positive real numbers. Then the infinite product $\prod_{k=1}^{\infty}(1+a_k)$ converges if and only if the series $\sum_{k=1}^{\infty} a_k$ converges.*

Proof. First note that $1 + x \le e^x$ for all real x. Then

$$a_1 + a_2 + \cdots + a_N \le \prod_{k=1}^{N}(1 + a_k) \le e^{a_1+a_2+\cdots+a_N}$$

and the result follows. □

Corollary 2.0.28. *Let $\{z_k\}$ be the zero set of an analytic function f. Suppose $z_k \neq 0$ for all $k = 1, 2, \ldots$. Then*

$$\sum_{k=1}^{n}(1 - |z_k|) \lesssim \prod_{k=1}^{n} \frac{1}{|z_k|}, \quad n \to \infty.$$

Proof. Set $a_k = (1 - |z_k|)/|z_k|$ so that $a_k > 0$ for all $k = 1, 2, \ldots$. Then we obtain that

$$\sum_{k=1}^{n} \frac{1 - |z_k|}{|z_k|} \le \prod_{k=1}^{n} \frac{1}{|z_k|}$$

for all n, as in the proof of Theorem 2.0.27. We have $0 < c < |z_k| < 1$ for all k, because $\{z_k\}$ is the zero set of an analytic function. Hence

$$\sum_{k=1}^{n}(1 - |z_k|) \lesssim \prod_{k=1}^{n} \frac{1}{|z_k|}, \quad n \to \infty$$

as desired. □

Theorem 2.0.29. *Let $0 < p \le \infty$ and $f \in H^p$ with $f(0) \neq 0$. Let $\{z_k\}$ be the zero set of f. Then*

$$\prod_{k=1}^{\infty} \frac{1}{|z_k|} < \infty.$$

Proof. Let first $0 < p < \infty$. Let also $0 < r < 1$. Without loss of generality we may assume $f(0) = 1$. Let $0 < |z_1| \le |z_2| \le \cdots \le |z_n| < r \le |z_{n+1}| \le \cdots$ be the ordered zeros of f, repeated according to multiplicity. Then

$$\sum_{k=1}^{n} \log \frac{r}{|z_k|} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta$$

by Jensen's formula. Multiplying above by p, exponentiating and applying the agm inequality, we obtain

$$\prod_{k=1}^{n} \frac{r}{|z_k|} \le \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right\}^{1/p} = M_p(r,f).$$

Observe that the inequality above holds for each $r \in (0,1)$ and for arbitrary n. Indeed, the product

$$\prod_{k=1}^{m} \frac{r}{|z_k|}$$

increases as m goes from 1 to n, because $|z_k| < r$ for $k \le n$. It decreases monotonically for $m > n$, since $r \ge |z_k|$ for $k > n$. Let $r_n = 1 - 1/n$ so that we have

$$\left(1 - \frac{1}{n}\right)^n \le \left(1 - \frac{1}{n}\right)^n \prod_{k=1}^{n} \frac{1}{|z_k|} \le M_p(r_n, f)$$

for all n. Now letting $n \to \infty$, we obtain the desired result. The case $p = \infty$ is similar. □

Note that Theorem 2.0.29 together with Corollary 2.0.28 also imply Theorem 2.0.25.

Theorem 2.0.30. [1, Theorem 17.6] *Suppose that $f_k(z)$ is analytic in $\mathbb{D}$ for all $k = 1, 2, \ldots$, and that $\sum_{k=1}^{\infty} |1 - f_k(z)|$ converges uniformly on compact subsets of $\mathbb{D}$. Then the infinite product $\prod_{k=1}^{\infty} f_k(z)$ converges uniformly on compact subsets of $\mathbb{D}$ and represents an analytic function in $\mathbb{D}$.*

Theorem 2.0.31. [14, Theorem 15.21] *Let $\{z_k\}$ be a sequence in $\mathbb{D}$ such that $z_k \neq 0$ for all $k = 1, 2, \ldots$. If*

$$\sum_{k=1}^{\infty} (1 - |z_k|) < \infty$$

and

$$B(z) = \prod_{k=1}^{\infty} b_{z_k}(z), \tag{2.3}$$

where

$$b_{z_k}(z) = \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \overline{z}_k z}, \quad z \in \mathbb{D} \tag{2.4}$$

then $B \in H^\infty$ and $\{z_k\}$ is the zero set of B.

Proof. Theorem 2.0.30 assures that B is analytic if the series

$$\sum_{k=1}^{\infty} |1 - b_{z_k}(z)| \tag{2.5}$$

converges uniformly on compact subsets of $\mathbb{D}$. Let K be a compact set in $\mathbb{D}$. Then there exists $r > 0$ such that $|z| \leq r < 1$ for all $z \in K$. Note that nth term in (2.5) is

$$|1 - b_{z_n}(z)| = \left| \frac{z_n + |z_n| z}{(1 - \overline{z}_n z) z_n} \right| (1 - |z_n|) \leq \frac{1+r}{1-r}(1 - |z_n|)$$

since $|z| \leq r$ and $|z_n| < 1$. Hence we have

$$\sum_{k=1}^{\infty} |1 - b_{z_k}(z)| \leq \frac{1+r}{1-r} \sum_{k=1}^{\infty} (1 - |z_k|) < \infty$$

by hypothesis. So the series in (2.5) converges in K. Since K is an arbitrary compact subset of $\mathbb{D}$, it follows that the series converges uniformly on compact subsets of $\mathbb{D}$. So the Blaschke product B is analytic in $\mathbb{D}$ by Theorem 2.0.30. Note that $|b_{z_k}(z)| < 1$ in $\mathbb{D}$. So B is bounded on $\mathbb{D}$. Thus $B \in H^\infty$. Note also that each Blaschke factor $b_{z_k}(z)$ has a zero at the point $z = z_k$ and has no other zeros in $\mathbb{D}$. So $\{z_k\}$ is the zero set of the infinite product B. □

Remark 2.0.32. *For $0 < p < q < \infty$, we have $H^\infty \subseteq H^q \subseteq H^p$ by Theorem 2.0.19. Hence the zero sets of H^p spaces are independent of the exponent p. In other words, the zero sets of H^p spaces are the same for all p.*

The infinite product B in (2.3) is called the *Blaschke product* and the factors b_{z_k} in (2.4) is called a *Blaschke factor*. Each Blaschke factor is indeed an automorphism of the unit disk. Hence we have $|b_{z_k}(z)| < 1$ for all $z \in \mathbb{D}$ and $|b_{z_k}(z)| = 1$ for all $z \in \mathbb{T}$ by virtue of being an automorphism of the unit disk. In general, we have the following results about the automorphisms of the unit disk. But first we introduce a new notation.

Notation 2.0.33. *Let $w \in \mathbb{D}$. Then we set*

$$\varphi_w(z) = \frac{w - z}{1 - \overline{w} z}, \quad z \in \mathbb{D}.$$

Theorem 2.0.34. *A map $f : \mathbb{D} \to \mathbb{D}$ is an automorphism of the unit disk if and only if there exist $\theta \in [0, 2\pi]$ and $w \in \mathbb{D}$ such that*

$$f(z) = e^{i\theta}\varphi_w(z). \tag{2.6}$$

The map $\varphi_w(z)$ has the following properties.

Proposition 2.0.35. *Let $w \in \mathbb{D}$. Then we have the following statements.*

(i) $z = \varphi_w \circ \varphi_w(z)$ *for all* $z \in \mathbb{D}$*. Hence the map* $\varphi_w(z)$ *is an involution.*

(ii)

$$1 - |\varphi_w(z)|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{w}z|^2}, \quad z \in \mathbb{D}.$$

(iii)

$$\varphi_w'(z) = \frac{|w|^2 - 1}{(1-\overline{w}z)^2}, \quad z \in \mathbb{D}.$$

We denote the *normalized area measure* on $\mathbb{D}$ by dA, that is $A(\mathbb{D}) = 1$. In terms of rectangular and polar coordinates, we have

$$dA(z) = \frac{1}{\pi}\, dx\, dy = \frac{1}{\pi}\, r\, dr\, d\theta, \quad z = x + iy = re^{i\theta}.$$

Notation 2.0.36. *For $0 < p < \infty$ and $-1 < \alpha < \infty$, the weighted Bergman norm of $f \in H(\mathbb{D})$ is denoted by*

$$\|f\|_{p,\alpha} = \left\{ \int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) \right\}^{1/p},$$

where $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$. We abbreviate $\|f\|_{p,0}$ to $\|f\|_p$ when $\alpha = 0$.

Definition 2.0.37. *For $0 < p < \infty$ and $-1 < \alpha < \infty$, the weighted Bergman space A^p_α consists of all functions $f \in H(\mathbb{D})$ with $\|f\|_{p,\alpha} < \infty$. We abbreviate A^p_0 to A^p when $\alpha = 0$.*

Definition 2.0.38. *Let $0 < p < \infty$, $-1 < \alpha < \infty$ and $t \in \mathbb{N}$. The class of functions f satisfying $f^{(t+1)} \in A^p_\alpha$ is denoted by $D^{p,t}_\alpha$. We abbreviate $D^{p,0}_\alpha$ to D^p_α when $t = 0$.*

Remark 2.0.39. *It is known that if $\alpha > p(t+1)-1$, then $D^{p,t}_{\alpha} = A^p_{\alpha-p}$. So the only interesting case is when $\alpha \le p(t+1)-1$, which reduces to $\alpha \le p-1$ when $t = 0$.*

Theorem 2.0.40. *Let $f \in H(\mathbb{D})$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be its Taylor expansion around the point $z = 0$. Then we have the following statements.*

(i) $f \in H^2$ *if and only if* $\sum_{k=0}^{\infty} |a_k|^2 < \infty$.

(ii) $f \in A^2$ *if and only if* $\sum_{k=1}^{\infty} k^{-1}|a_k|^2 < \infty$.

Theorem 2.0.31 shows that every Blaschke sequence is a zero set of H^p. Since $H^p \subset A^p$, it follows that every Blaschke sequence is also a zero set of A^p. However, some zero sets of A^p spaces may not satisfy the Blaschke condition. Hence the Blaschke condition is not necessary for a sequence to be a zero set of A^p. Using the ideas given in [4, p. 94], we shall show that there exists a zero set of A^p that does not satisfy the Blaschke condition. But first we need a technical lemma.

Lemma 2.0.41. [4] *If $\psi(r)$ is continuous and increasing function on the interval $[0,1)$ with $\psi(0) = 1$ and $\psi(r) \to \infty$ as $r \to 1^-$, then there exists an increasing sequence of positive integers $\{n_k\}$ such that*

$$\sum_{k=1}^{\infty} r^{n_k} \le \psi(r), \quad 0 \le r < 1.$$

Proof. See Lemma 2 in [4, Section 3.2]. □

Now we construct an infinite product f that belongs to A^p whose zero set does not satisfy the Blaschke condition. Let $0 < p < \infty$ and $0 < a < 1/2p$. Let also

$$\psi(r) = a \log \frac{1}{1-r} + 1, \quad 0 \le r < 1,$$

so that $\psi(0) = 1$ and $\psi(r) \to \infty$ as $r \to \infty$. Note that $\psi(r)$ is continuous and increasing on $[0,1)$. Then there exists an increasing sequence of positive integers $\{n_k\}$ such that

$$\sum_{k=1}^{\infty} r^{n_k} \le \psi(r), \quad 0 \le r < 1 \tag{2.7}$$

by Lemma 2.0.41. Let

$$f(z) = \prod_{k=1}^{\infty}(1 - 2z^{n_k}), \quad z \in \mathbb{D}.$$

Using Theorem 2.0.30, we shall show that f is analytic in the unit disk. Let K be a compact subset of $\mathbb{D}$. Then there exists $s \in (0,1)$ such that $|z| \leq s < 1$ for all $z \in K$. Note that $k \leq n_k$ for all $k = 1, 2, \ldots$, since $\{n_k\}$ is an increasing sequence of positive integers. So $r^{n_k} \leq r^k$ for all $r \in [0,1)$. Hence

$$\sum_{k=1}^{\infty} |z|^{n_k} \leq \sum_{k=1}^{\infty} s^{n_k} \leq \sum_{k=1}^{\infty} s^k < \infty$$

for all $z \in K$. Thus f is analytic in the unit disk by Theorem 2.0.30. Next note that f has n_k many equally distributed zeros on the circle of radius $2^{-1/n_k}$. Since $e^x \asymp 1 + x$ as $x \to 0$, it follows that

$$2^{-1/n_k} = e^{-\log 2/n_k} \asymp 1 - \frac{\log 2}{n_k}, \quad k \to \infty. \tag{2.8}$$

Let $\{z_k\}$ be the zero set of f. Consider the Blaschke sum

$$\sum_{k=1}^{\infty}(1 - |z_k|) = \sum_{k=1}^{\infty} n_k(1 - 2^{-1/n_k}).$$

The general term $n_k(1-2^{-1/n_k}) \not\to 0$ as $k \to \infty$, because we have $n_k(1-2^{-1/n_k}) \asymp \log 2$ as $k \to \infty$ by (2.8). Hence the Blaschke sum is divergent and the zero set $\{z_k\}$ of f does not satisfy the Blaschke condition. It remains to show $f \in A^p$. Using the the inequality $1 + x \leq e^x$, we obtain

$$|f(z)| \leq \prod_{k=1}^{\infty}(1 + 2r^{n_k}) \leq \exp\left(2\sum_{k=1}^{\infty} r^{n_k}\right) \leq \frac{e^2}{(1-r)^{2a}},$$

where $|z| = r$ and the last inequality is due to (2.7). Hence

$$|f(re^{i\theta})|^p = O\big((1-r)^{-2ap}\big), \quad r \to 1^-.$$

This implies $f \in A^p$ because $0 < a < 1/2p$.

So far we showed that the zero sets of H^p spaces have the following properties.

If $f \in H^p$ and $\{z_k\}$ is the zero set of f, then we have

$$N(r) = O(1), \quad r \to 1^-,$$
$$n(r) = O\left(\frac{1}{1-r}\right), \quad r \to 1^-,$$
$$\prod_{k=1}^{n} \frac{1}{|z_k|} = O(1), \quad n \to \infty.$$

Furthermore, the zero set $\{z_k\}$ of f satisfies the Blaschke condition, that is

$$\sum_{k=1}^{\infty} (1 - |z_k|) < \infty.$$

However the situation is quite different when it comes to the Bergman spaces A^p_α and the Dirichlet-type spaces D^p_{p-1}.

In Chapter 3, we show that for a function $f \in A^p_\alpha$ and its zero set $\{z_k\}$, we have

$$N(r) = O\left(\log\frac{1}{1-r}\right), \quad r \to 1^-,$$
$$n(r) = O\left(\frac{1}{1-r}\log\frac{1}{1-r}\right), \quad r \to 1^-,$$
$$\prod_{k=1}^{n} \frac{1}{|z_k|} = o\big(n^{(\alpha+1)/p}\big), \quad n \to \infty, \tag{2.9}$$
$$\sum_{k=1}^{n} (1 - |z_k|) = O(\log n), \quad n \to \infty.$$

We show in Corollary 3.2.2 that the exponent $(\alpha + 1)/p$ in (2.9) is sharp. We also show that a substitute of the Blaschke condition for the weighted Bergman spaces A^p_α is

$$\sum_{k=1}^{\infty} (1 - |z_k|)\left(\log\frac{1}{1-|z_k|}\right)^{-1-\epsilon} < \infty \tag{2.10}$$

whenever $\epsilon > 0$. We also show in Corollary 3.2.3 that the exponent $-1 - \epsilon$ in (2.10) is sharp; it cannot be replaced by -1. In that chapter, we answer the following three questions. Do the zero sets of A^p_α vary with p and α? In other words, for which values of p and α does there exist a zero set of A^p_α that is not a zero set of A^q_β? Is every subset of a zero set of A^p_α again a zero set of A^p_α? Is the union of two zero sets of A^p_α again a zero set of A^p_α?

In Chapter 4, we study the zero sets of the Dirichlet space D^2, which lies in the H^2 space. In that chapter, we discuss the characteristics of the zero sets of D^2. We show that the Blaschke condition is not sufficient for a sequence to be a zero set of D^2; see Theorem 4.0.12. Interestingly, we also show that there exist two sequences in the unit disk satisfying the Blaschke condition such that one of them is a zero set for D^2 but the other is not. We also present some elementary properties of the zero sets of D^2.

In Chapter 5, we show that for $f \in D^p_{p-1}$ and its zero set $\{z_k\}$, we have

$$\begin{aligned} N(r) &= O\left(\log\log \frac{1}{1-r} \right), \quad r \to 1^-, \\ n(r) &= O\left(\frac{1}{1-r} \log\log \frac{1}{1-r} \right), \quad r \to 1^-, \\ \prod_{k=1}^{n} \frac{1}{|z_k|} &= O\big((\log n)^{1/2-1/p} \big), \quad n \to \infty, \\ \sum_{k=1}^{n} (1-|z_k|) &= O(\log\log n), \quad n \to \infty. \end{aligned}$$

We also show that a substitute of the Blaschke condition for the Dricihlet-type spaces D^p_{p-1} is

$$\sum_{|z_k|>1-1/e} (1-|z_k|) \left(\log\log \frac{1}{1-|z_k|} \right)^{-1-\epsilon} < \infty$$

whenever $\epsilon > 0$.

Chapter 6 is devoted to prove a sufficient condition for a sequence in the unit disk to be a zero set of the weighted Bergman Space A^p_α; see Theorem 6.0.7.

Chapter 3

Zero sets of weighted Bergman spaces

3.1 Preliminary results

Let $0 < p < \infty$ and $-1 < \alpha < \infty$. Recall that a function $f \in H(\mathbb{D})$ is said to be of class A^p_α if the weighted Bergman norm

$$\|f\|_{p,\alpha} = \left\{ \int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) \right\}^{1/p},$$

where $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$, is finite.

In Proposition 3.1.1 and other similiar results below, the assumption $f(0) \neq 0$ causes no harm because if $f(0) = 0$, with multiplicity m, the results can be applied to $g(z) = f(z)/z^m$.

Proposition 3.1.1. [8, Proposition 4.4] *Let $0 < p < \infty$ and $-1 < \alpha < \infty$. If $f \in A^p_\alpha$ with $f(0) \neq 0$, then*

$$\int_0^1 e^{pN(r)}(1-r^2)^\alpha r\, dr < \infty.$$

Proof. Let

$$g(z) = \frac{f(z)}{f(0)}, \quad z \in \mathbb{D}.$$

Then g and f have the same zero sets and $g \in A^p_\alpha$. For $0 < r < 1$, we have

$$N(r) = \frac{1}{2\pi} \int_0^{2\pi} \log |g(re^{i\theta})| \, d\theta$$

by Jensen's formula and Proposition 2.0.21. We multiply the equality above by p and apply the agm inequality to obtain

$$e^{pN(r)} \leq \frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^p \, d\theta.$$

Then we multiply both sides by $(1-r^2)^\alpha r$ and integrate with respect to r to obtain

$$\int_0^1 e^{pN(r)} (1-r^2)^\alpha r \, dr \leq \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} |g(re^{i\theta})|^p \, (1-r^2)^\alpha r \, d\theta \, dr.$$

The expression on the right side is finite because $g \in A^p_\alpha$. This completes the proof. □

Proposition 3.1.2. [8, Proposition 4.6] *Let $0 < p < \infty$ and $-1 < \alpha < \infty$. If $f \in A^p_\alpha$ with $f(0) \neq 0$, then*

$$N(r) = O\left(\log \frac{1}{1-r} \right), \quad r \to 1^-$$

and

$$n(r) = O\left(\frac{1}{1-r} \log \frac{1}{1-r} \right), \quad r \to 1^-.$$

Proof. Write

$$\int_0^1 e^{pN(t)} (1-t)^\alpha \, dt = \int_0^{1/2} e^{pN(t)} (1-t)^\alpha \, dt + \int_{1/2}^1 e^{pN(t)} (1-t)^\alpha \, dt.$$

The integral between 0 and 1/2 is finite because $e^{pN(t)}(1-t)^\alpha$ is bounded for all $t \in [0, 1/2]$. Note that there exists a positive constant c such that $c < (1+t)^\alpha t$ for all $t \in [1/2, 1]$. Hence

$$c \int_{1/2}^1 e^{pN(t)} (1-t)^\alpha \, dt < \int_{1/2}^1 e^{pN(t)} (1-t^2)^\alpha t \, dt < \infty$$

by Proposition 3.1.1. Since $N(t)$ is increasing on $(0,1)$, we have

$$\begin{aligned}\frac{1}{\alpha+1}(1-r)^{\alpha+1}e^{pN(r)} &= e^{pN(r)}\int_r^1 (1-t)^\alpha\,dt \\ &\le \int_r^1 e^{pN(t)}(1-t)^\alpha\,dt < \infty\end{aligned}$$

for all $r \in (0,1)$. Hence

$$N(r) = O\left(\log\frac{1}{1-r}\right), \quad r \to 1^-.$$

Since $n(r)$ is nondecreasing on $(0,1)$, we have

$$n(r^2)(r-r^2) \le \int_{r^2}^r n(t)\,dt \le \int_{r^2}^r \frac{n(t)}{t}\,dt \le N(r) = O\left(\log\frac{1}{1-r}\right), \quad r \to 1^-.$$

So

$$n(r^2) = O\left(\frac{1}{1-r}\log\frac{1}{1-r}\right), \quad r \to 1^-.$$

Note that $1-\sqrt{r} \asymp (1-\sqrt{r})(1+\sqrt{r}) = 1-r$ as $r \to 1^-$. Finally we have

$$n(r) = O\left(\frac{1}{1-r}\log\frac{1}{1-r}\right), \quad r \to 1^-$$

as required. □

Theorem 3.1.3. *Let $0 < p < \infty$ and $-1 < \alpha < \infty$. Let also $f \in A^p_\alpha$ and $\{z_k\}$ be the zero set of f. Then*

$$\sum_{k=1}^n (1-|z_k|) = O(\log n), \quad n \to \infty.$$

Proof. Without loss of generality we may assume $f(0) \ne 0$. Let $0 < r < 1$ and $z_1, z_2, \ldots, z_n$ be the ordered zeros of f in the disk with radius r. Then

$$\sum_{k=1}^n (r-|z_k|) \le \sum_{k=1}^n (1-|z_k|) = \int_{|z_1|}^r (1-t)\,dn(t).$$

Arguing as in the proof of Theorem 2.0.25, we find

$$\int_{|z_1|}^r (1-t)\,dn(t) = (1-r)n(r) + rN(r) - \int_{|z_1|}^r N(t)\,dt$$

so that

$$\sum_{k=1}^{n}(r-|z_k|) \le (1-r)n(r) + rN(r).$$

Observe that the inequality above holds for each $r \in (0,1)$ and for an arbitrary n, since $r - |z_k| > 0$ for all $k \le n$ and $r - |z_k| < 0$ for all $k \ge n+1$. Then taking $r = 1 - 1/n$ and using the estimates for $n(r)$ and $N(r)$ in Proposition 3.1.2, we obtain

$$\sum_{k=1}^{n}(1-|z_k|) \le C \log n$$

as $n \to \infty$. □

Theorem 3.1.4. [8, Theorem 4.7] *Let $0 < p < \infty$ and $-1 < \alpha < \infty$. Let also $\epsilon > 0$. Suppose $f \in A^p_\alpha$ with $f(0) \neq 0$ and $\{z_k\}$ is the zero set of f. Then*

$$\sum_{k=1}^{\infty}(1-|z_k|)\bigg(\log\frac{1}{1-|z_k|}\bigg)^{-1-\epsilon} < \infty.$$

Proof. Note that

$$I = \sum_{k=1}^{\infty}(1-|z_k|)\bigg(\log\frac{1}{1-|z_k|}\bigg)^{-1-\epsilon} = \int_{|z_1|}^{1}(1-r)\bigg(\log\frac{1}{1-r}\bigg)^{-1-\epsilon} dn(r).$$

An integration by parts with $u = (1-r)(-\log(1-r))^{-1-\epsilon}$ and $dv = dn(r)$ gives $I = I_1 + I_2$, where

$$I_1 = (1+\epsilon)\int_{|z_1|}^{1}\bigg(\log\frac{1}{1-r}\bigg)^{-2-\epsilon} n(r)\,dr$$

and

$$I_2 = \int_{|z_1|}^{1}\bigg(\log\frac{1}{1-r}\bigg)^{-1-\epsilon} n(r)\,dr.$$

By Propostion 3.1.2, we have

$$I_1 \le C\int_{|z_1|}^{1}\frac{1}{1-r}\bigg(\log\frac{1}{1-r}\bigg)^{-1-\epsilon} dr = C\int_{b_1}^{\infty} r^{-1-\epsilon}\,dr < \infty,$$

where $b_1 = -\log(1-|z_1|)$. We employ another integration by parts on I_2 with

$$u = r\bigg(\log\frac{1}{1-r}\bigg)^{-1-\epsilon} \quad \text{and} \quad dv = \frac{n(r)}{r}\,dr,$$

which yields

$$I_2 = r\left(\log\frac{1}{1-r}\right)^{-1-\epsilon} N(r)\Bigg|_{|z_1|}^{1} - \int_{|z_1|}^{1}\left(\log\frac{1}{1-r}\right)^{-1-\epsilon} N(r)\,dr$$
$$+ (1+\epsilon)\int_{|z_1|}^{1}\left(\log\frac{1}{1-r}\right)^{-2-\epsilon}\frac{r}{1-r}\,dr.$$

We shall show that each term above is convergent. Note that the first term is finite by Proposition 3.1.2. Again using Proposition 3.1.2, we obtain

$$0 \le \int_{|z_1|}^{1}\left(\log\frac{1}{1-r}\right)^{-1-\epsilon} N(r)\,dr \le C\int_{|z_1|}^{1}\left(\log\frac{1}{1-r}\right)^{-\epsilon} dr < \infty$$

as $\epsilon > 0$. For the last term, we employ a change of variables and find

$$\begin{aligned} 0 \le \int_{|z_1|}^{1}\left(\log\frac{1}{1-r}\right)^{-2-\epsilon}\frac{r}{1-r}\,dr &\le \int_{|z_1|}^{1}\left(\log\frac{1}{1-r}\right)^{-2-\epsilon}\frac{1}{1-r}\,dr \\ &= \int_{u_1}^{\infty} u^{-2-\epsilon}\,du \\ &< \infty, \end{aligned}$$

where $u_1 = -\log(1-|z_1|)$. Hence I_2 is finite. This completes the proof. □

Corollary 3.1.5. *Let $0 < p < \infty$ and $-1 < \alpha < \infty$. Let also $\epsilon > 0$. If $\{z_k\}$ is a zero set of A^p_α, then*

$$\sum_{k=1}^{\infty}(1-|z_k|)^{1+\epsilon} < \infty.$$

Proof. We have

$$\sum_{k=1}^{\infty}(1-|z_k|)\left(\log\frac{1}{1-|z_k|}\right)^{-1-\epsilon} < \infty$$

by Theorem 3.1.4. Let $g(x) = (1-x)^\epsilon - (-\log(1-x))^{-1-\epsilon}$. Since $g(1) = 0$ and $g'(x) > 0$, it follows that $g(x) < 0$ for $0 < x < 1$. Hence for $0 < x < 1$, we have

$$(1-x)^\epsilon \le \left(\log\frac{1}{1-x}\right)^{-1-\epsilon}.$$

So there exists a postive integer N such that

$$\sum_{k=N}^{\infty}(1-|z_k|)^{1+\epsilon} \le \sum_{k=N}^{\infty}(1-|z_k|)\left(\log\frac{1}{1-|z_k|}\right)^{-1-\epsilon} < \infty,$$

which completes the proof. □

Lemma 3.1.6. [4, p. 97] *Let $f \in H(\mathbb{D})$ with $f(0) \neq 0$ and $\{z_k\}$ be the zero set of f. If $0 < p < \infty$ and $0 \leq r < 1$, then*

$$|f(0)| \prod_{k=1}^{N} \frac{r}{|z_k|} \leq M_p(r, f)$$

for all positive integers N.

Proof. Let $0 < r < 1$ and $0 < |z_1| \leq |z_2| \leq \cdots \leq |z_n| < r \leq |z_{n+1}| \leq \cdots$ be the ordered zeros of f, repeated according to multiplicity. Then we have

$$\log |f(0)| + \sum_{k=1}^{n} \log \frac{r}{|z_k|} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta$$

by Jensen's formula. We multiply the equality above by p, exponentiate and apply the agm inequality to obtain

$$|f(0)|^p \prod_{k=1}^{n} \frac{r^p}{|z_k|^p} \leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta = M_p(r, f)^p.$$

Now observe that we have

$$\prod_{k=1}^{N} \frac{r}{|z_k|} \leq \prod_{k=1}^{n} \frac{r}{|z_k|}$$

for all positive integers N, because $|z_k| < r$ for all $k \leq n$ and $|z_k| > r$ for all $k \geq n+1$. In other words, the product on the left side takes its maximum value at $N = n$. Hence

$$|f(0)| \prod_{k=1}^{N} \frac{r}{|z_k|} \leq M_p(r, f)$$

for all positive integers N. □

Theorem 3.1.7. [6, Theorem 1] *Let $0 < p < \infty$ and $-1 < \alpha < \infty$. Let $f \in A^p_\alpha$ with $f(0) \neq 0$ and $\{z_k\}$ be the zero set of f. Then*

$$\prod_{k=1}^{n} \frac{1}{|z_k|} = o\big(n^{(\alpha+1)/p}\big), \quad n \to \infty.$$

Proof. We have

$$\int_0^1 M_p(r, f)^p (1-r)^\alpha \, dr < \infty \tag{3.1}$$

as $f \in A^p_\alpha$. Since $M_p(r,f)$ is an increasing function of r, we have

$$\frac{1}{\alpha+1}(1-r)^{\alpha+1}M_p(r,f)^p \leq \int_r^1 M_p(s,f)^p(1-s)^\alpha\, ds \tag{3.2}$$

for all $r \in (0,1)$. Now (3.1) and (3.2) imply that

$$(1-r)^{(\alpha+1)/p}M_p(r,f) \to 0, \quad r \to 1^-,$$

or

$$M_p(r,f) = o\big((1-r)^{-(\alpha+1)/p}\big), \quad r \to 1^-. \tag{3.3}$$

Let n be a positive integer. Taking $r = 1-1/n$ in Lemma 3.1.6, we obtain

$$\left(1-\frac{1}{n}\right)^n \prod_{k=1}^n \frac{1}{|z_k|} \leq CM_p\left(1-\frac{1}{n}, f\right).$$

Note that we have

$$M_p\left(1-\frac{1}{n}, f\right) = o\big(n^{(\alpha+1)/p}\big), \quad n \to \infty$$

by (3.3). Since $(1-1/n)^n \to 1/e$ as n tends to infinity, it follows that

$$\prod_{k=1}^n \frac{1}{|z_k|} = o\big(n^{(\alpha+1)/p}\big), \quad n \to \infty$$

as required. □

Lemma 3.1.8. [4, p. 81] *Let $2 \leq p < \infty$, $-1 < \alpha < \infty$ and $1/p + 1/p' = 1$. Let $f \in H(\mathbb{D})$ and $f(z) = \sum_{k=0}^\infty a_k z^k$ be its Taylor series expansion around $z = 0$. If*

$$\sum_{k=0}^\infty |a_k|^{p'} k^{(1+\alpha)(1-p')} < \infty,$$

then $f \in A^p_\alpha$.

Proof. For each $r \in (0,1)$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{k=0}^\infty a_k r^k e^{ik\theta}\right|^2 d\theta = \sum_{k=0}^\infty |a_k|^2 r^{2k} \tag{3.4}$$

by Parseval's formula. Then using (3.4) and Tonelli theorem we obtain

$$
\begin{aligned}
\|f\|_{2,\alpha}^2 &= (\alpha+1)\frac{1}{\pi}\int_0^1\int_0^{2\pi} |f(re^{i\theta})|^2\,(1-r^2)^\alpha\,d\theta\,r\,dr\\
&= 2(\alpha+1)\int_0^1 \left\{\sum_{k=0}^\infty |a_k|^2 r^{2k+1}(1-r^2)^\alpha\right\} dr\\
&= 2(\alpha+1)\sum_{k=0}^\infty |a_k|^2 \left\{\int_0^1 r^{2k+1}(1-r^2)^\alpha\,dr\right\}\\
&= (\alpha+1)\sum_{k=0}^\infty |a_k|^2\,\mathrm{B}(k+1,\alpha+1)\\
&= (\alpha+1)\sum_{k=0}^\infty |a_k|^2 \frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(\alpha+k+2)},
\end{aligned}
$$

where B and Γ are the beta and gamma functions. Then Stirling's formula gives

$$
\sum_{k=0}^n |a_k|^2 \frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(\alpha+k+2)} \asymp \sum_{k=0}^n |a_k|^2\,(k+1)^{-\alpha-1}, \quad n\to\infty.
$$

Hence

$$
\|f\|_{2,\alpha}^2 \le C\sum_{k=0}^\infty |a_k|^2\,(k+1)^{-\alpha-1}. \tag{3.5}
$$

Now define a discrete measure $\mu:\mathbb{N}\to\mathbb{R}$ by $\mu(k)=(k+1)^{\alpha+1}$ for each $k\in\mathbb{N}$. Next define a linear operator $\Omega: L^q(\mathbb{N},d\mu)\to L^p(\mathbb{D},dA_\alpha)$ by

$$
\Omega(\mathbf{b}) = \sum_{k=0}^\infty b_k(k+1)^{\alpha+1}z^k,
$$

where $\mathbf{b}=\{b_k\}_{k=0}^\infty \in L^{p'}(\mathbb{N},d\mu)$. We shall show that Ω is bounded for $2\le p\le\infty$. Now if we show that Ω is bounded for $p=2$ and for $p=\infty$, then Riesz-Thorin interpolation theorem assures that Ω is bounded for $2\le p\le\infty$. We first prove Ω is bounded for $p=2$. Let $g(z)=\sum_{k=0}^\infty b_k(k+1)^{\alpha+1}z^k$. Then for $p=2$, (3.5) implies that

$$
\|\Omega(\mathbf{b})\|_{L^2(\mathbb{D},dA_\alpha)}^2 = \|g\|_{2,\alpha}^2 \le C\sum_{k=0}^\infty |b_k|^2\,(k+1)^{\alpha+1} = C\|\mathbf{b}\|_{L^2(\mathbb{N},d\mu)}^2.
$$

Hence $\Omega: L^2(\mathbb{N},d\mu)\to L^2(\mathbb{D},dA_\alpha)$ is bounded. For $p=\infty$, note that the weight function $w(z)=(1-|z|^2)^\alpha$ is integrable on $\mathbb{D}$ for $-1<\alpha<\infty$, so we have

$$
\|\Omega(\mathbf{b})\|_{L^\infty(\mathbb{D},dA_\alpha)} = \|\Omega(\mathbf{b})\|_{L^\infty(\mathbb{D},dA)},
$$

where dA is the unit Lebesgue measure on $\mathbb{D}$. By using above and the trivial fact

$$|g(z)| \le \sum_{k=0}^{\infty} |b_k|(k+1)^{\alpha+1}, \quad z \in \mathbb{D},$$

we conclude that

$$\|\Omega(\mathbf{b})\|_{L^\infty(\mathbb{D},dA_\alpha)} \le \sup_{z\in\mathbb{D}} |g(z)| \le \sum_{k=0}^{\infty} |b_k|(k+1)^{\alpha+1} = \|\mathbf{b}\|_{L^1(\mathbb{N},d\mu)}.$$

Hence $\Omega : L^1(\mathbb{N}, d\mu) \to L^\infty(\mathbb{D}, dA_\alpha)$ is bounded. Thus $\Omega : L^{p'}(\mathbb{N}, d\mu) \to L^p(\mathbb{D}, dA_\alpha)$ is bounded for $2 \le p \le \infty$ by Riesz-Thorin interpolation theorem. Now we prove $f \in A^p_\alpha$. Let $\mathbf{a} = \{a_k(k+1)^{-\alpha-1}\}_{k=0}^{\infty}$. Then $\Omega(\mathbf{a}) = f$. Since Ω is bounded for $2 \le p < \infty$, there exists $K > 0$ such that

$$\|\Omega(\mathbf{a})\|_{L^p(\mathbb{D},dA_\alpha)} = \|f\|_{p,\alpha} \le K\|\mathbf{a}\|_{L^{p'}(\mathbb{N},d\mu)} = K\left\{\sum_{k=0}^{\infty} |a_k|^{p'}(k+1)^{(\alpha+1)(1-p')}\right\}^{1/p'}.$$

But we have

$$\sum_{k=0}^{\infty} |a_k|^{p'}(k+1)^{(\alpha+1)(1-p')} < \infty$$

by hypothesis. Hence $\|f\|_{p,\alpha} < \infty$ and $f \in A^p_\alpha$ as desired. □

Notation 3.1.9. *Let $f \in H(\mathbb{D})$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be its Taylor series expansion around $z = 0$. Then we set*

$$S_n^{(q)}(f) = \sum_{k=0}^{n} |a_k|^q.$$

We abbreviate $S_n^{(q)}(f)$ to $S_n^{(q)}$ if the context is clear.

The following two lemmas appear in [9] without proofs. We provide the details of their proofs.

Lemma 3.1.10. [9, Theorem 6.4] *Let $0 < p < 2$, $-1 < \alpha < \infty$ and $\lambda < (2+2\alpha)/p$. Let $f \in H(\mathbb{D})$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be Taylor series expansion of f around $z = 0$. If*

$$S_n^{(2)}(f) = O(n^\lambda), \quad n \to \infty,$$

then $f \in A^p_\alpha$.

Proof. Hölder's inequality and Parseval's formula give

$$M_p^2(r,f) \le M_2^2(r,f) = \sum_{k=0}^{\infty} |a_k|^2 r^{2k}.$$

A summation by parts, using that $n^\lambda r^{2n} \to 0$ as $n \to \infty$, yields

$$\sum_{k=0}^{\infty} |a_k|^2 r^{2k} = \sum_{n=0}^{\infty} S_n^{(2)}(r^{2n} - r^{2n+2}) = (1-r^2)\sum_{n=0}^{\infty} S_n^{(2)} r^{2n}.$$

There exists $C > 0$ such that

$$M_p^2(r,f) \le (1-r^2)\sum_{n=0}^{\infty} S_n^{(2)} r^{2n} = C(1-r^2)\sum_{n=0}^{\infty}(n+1)^\lambda r^{2n}$$

by hypothesis. Hence

$$M_p^2(r,f) = O((1-r)^{-\lambda}), \quad r \to 1^-,$$

or

$$M_p(r,f) = O((1-r)^{-\lambda/2}), \quad r \to 1^-,$$

which implies that $f \in A_\alpha^p$ because $\lambda/2 < (1+\alpha)/p$. □

Lemma 3.1.11. [9, Theorem 6.6] *Let $2 \le p < \infty$ and $-1 < \alpha < \infty$. Let $1/p + 1/p' = 1$ and $\lambda < (p'-1)(1+\alpha)$. Let $f \in H(\mathbb{D})$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be Taylor series expansion of f around $z = 0$. If*

$$S_n^{(p')}(f) = O(n^\lambda), \quad n \to \infty,$$

then $f \in A_\alpha^p$.

Proof. By Lemma 3.1.8, it suffices to show that

$$\sum_{k=0}^{\infty} |a_k|^{p'}(k+1)^{(\alpha+1)(1-p')} < \infty.$$

A summation by parts, using $n^\lambda (n+1)^{(\alpha+1)(1-p')} \to 0$ as $n \to \infty$, gives

$$\sum_{k=0}^{\infty} |a_k|^{p'}(k+1)^{(\alpha+1)(1-p')} = \sum_{k=0}^{\infty} S_k^{(p')}\big\{(k+1)^{(\alpha+1)(1-p')} - (k+2)^{(\alpha+1)(1-p')}\big\}.$$

Note that $(\alpha+1)(1-p') < 0$ for all $p \in [2, \infty)$ and for all $\alpha \in (-1, \infty)$. Let $h(x) = x^{(\alpha+1)(1-p')}$ for all $x \in (0, \infty)$. Since h' exists on $(0, \infty)$, we can apply the mean value theorem to the function h on $[k+1, k+2]$ for each $k \in \mathbb{Z}^+$. The mean value theorem assures that for each $k \in \mathbb{Z}^+$ there exists $c_k \in (k+1, k+2)$ such that

$$h'(c_k) = \frac{h(k+2) - h(k+1)}{(k+2) - (k+1)} = h(k+2) - h(k+1).$$

Note that we have $h'(k+1) \leq h'(c_k)$ because h' is increasing. So

$$\begin{aligned} -h'(c_k) &= (k+1)^{(\alpha+1)(1-p')} - (k+2)^{(\alpha+1)(1-p')} \\ &\leq -h'(k+1) = (\alpha+1)(p'-1)(k+1)^{(\alpha+1)(1-p')-1}. \end{aligned}$$

Hence

$$\sum_{k=0}^{\infty} S_k^{(p')} \{(k+1)^{(\alpha+1)(1-p')} - (k+2)^{(\alpha+1)(1-p')}\} \leq C \sum_{k=0}^{\infty} S_k^{(p')} (k+1)^{(\alpha+1)(1-p')-1}.$$

Since $S_k^{(p')} = O(k^\lambda)$ as $k \to \infty$, it follows that

$$\sum_{k=0}^{\infty} |a_k|^{p'} (k+1)^{(\alpha+1)(1-p')} \leq C \sum_{k=0}^{\infty} (k+1)^{\lambda+(\alpha+1)(1-p')-1},$$

which is finite beacuse $\lambda + (\alpha+1)(1-p') < 0$. □

3.2 Dependence on p and α

The next result shows that the zero sets of weighted Bergman spaces vary with p and α.

Theorem 3.2.1. [9, Theorem 6.11] *Let $0 < p < q < \infty$ and $\alpha, \beta \in (-1, \infty)$. If $(\beta+1)/q < (\alpha+1)/p$, then there exists a zero set of A^p_α that is not a zero set of A^q_β.*

Proof. Let $0 < p < 2$. We construct an infinite product

$$f(z) = \prod_{v=1}^{\infty} (1 - bz^{m^v}), \quad b > 1, \; m = 2, 3, \ldots,$$

where b and m are chosen such that $f \in A^p_\alpha$ but its zero set $\{z_k\}$ violates the necessary condition of Theorem 3.1.7 for A^q_β. In other words, we show that there exist $b > 1$ and integer $m \geq 2$ such that

1. $f \in H(\mathbb{D})$,
2. $\{z_k\}$ is not a zero set of A^q_β,
3. $\|f\|_{p,\alpha} < \infty$.

We first show $f \in H(\mathbb{D})$. Let K be a compact set in $\mathbb{D}$. Then there exists $r < 1$ such that $|z| \leq r < 1$ for all $z \in K$. Hence if $m \in \{2, 3, \dots\}$ then we have

$$\sum_{v=1}^{\infty} |z|^{m^v} \leq \sum_{v=1}^{\infty} r^{m^v} < \sum_{v=1}^{\infty} r^v = \frac{r}{1-r} < \infty$$

for all $z \in K$. It follows from Theorem 2.0.30 that f is analytic in $\mathbb{D}$ for any choice of $b > 1$ and $m \in \{2, 3, \dots\}$. Now we shall find a condition on $b > 1$ and $m \geq 2$ such that $\{z_k\}$ is not a zero set of A^q_β. Observe that the zeros of f are simple and equally distributed on the circle $|z| = b^{-1/m^v}$ and there are m^v many of them on that circle for each $v = 1, 2, \dots$. Thus the ordered zeros $\{z_k\}$ of f satisfy

$$\begin{aligned} |z_k| &= b^{-1/m}, & 1 &\leq k \leq m, \\ |z_k| &= b^{-1/m^2}, & m+1 &\leq k \leq m^2 + m, \end{aligned}$$

and in general

$$|z_k| = b^{-1/m^v}, \qquad N_{v-1} + 1 \leq k \leq N_v,$$

where $N_v = m + m^2 + \cdots + m^v = \dfrac{m(m^v - 1)}{m-1}$ for $v = 1, 2, \dots$ and $N_0 = 0$. Then we have

$$\begin{aligned} \prod_{k=1}^{N_v} \frac{1}{|z_k|} &= \prod_{k=1}^{m} \frac{1}{|z_k|} \prod_{k=m+1}^{m^2+m} \frac{1}{|z_k|} \cdots \prod_{k=N_{v-1}+1}^{N_v} \frac{1}{|z_k|} \\ &= \left(b^{1/m}\right)^m \left(b^{1/m^2}\right)^{m^2} \cdots \left(b^{1/m^v}\right)^{m^v} \\ &= b^v. \end{aligned}$$

Note that $m^v \le N_v \le 2m^v$ for all $v = 1, 2, \dots$, so we have $N_v = O(m^v)$ as $v \to \infty$. Now if $\{z_k\}$ were a zero set of A^q_β, then Theorem 3.1.7 would imply that

$$\prod_{k=1}^{N_v} \frac{1}{|z_k|} = b^v = o\big(m^{v(\beta+1)/q}\big), \quad v \to \infty.$$

Note that $b^v = m^{\lambda v}$ for $\lambda = \log b / \log m$, hence if $b > 1$ and integer $m \ge 2$ are chosen such that $\lambda > (\beta+1)/q$, then the necessary condition of Theorem 3.1.7 is not satisfied. Thus $\{z_k\}$ is not a zero set of A^q_β if $b > 1$ and integer $m \ge 2$ are chosen such that

$$\frac{\beta+1}{q} < \frac{\log b}{\log m}. \tag{3.6}$$

Next we show that there exist $b > 1$ and integer $m \ge 2$ such that $\|f\|_{p,\alpha} < \infty$ and they satisfy (3.6). We consider the Taylor series expansion of f around $z = 0$, that is,

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

and the summation

$$S_n^{(r)} = \sum_{k=0}^{n} |a_k|^r$$

for $r > 0$. The Taylor coefficients of f are easily computed thanks to the lacunary sequence $\{m^v\}$. In fact the partial product of the infinite product for f is the partial sum of its Taylor expansion. For instance,

$$\prod_{v=1}^{2}(1 - bz^{m^v}) = (1 - bz^m)(1 - bz^{m^2}) = 1 - bz^m - bz^{m^2} - b^2 z^{m+m^2}$$

implies

$$S_{N_2}^{(r)} = \sum_{k=0}^{N_2} |a_k|^r = 1 + b^r + b^r + b^{2r} = (1 + b^r)^2,$$

where $N_2 = m + m^2$. In general we have

$$S_{N_v}^{(r)} = \sum_{k=0}^{n} |a_k|^r = (1 + b^r)^v, \quad r > 0,\ v = 1, 2, \dots,$$

where $N_v = m + m^2 + \cdots + m^v$. Assume $0 < p < 2$. If

$$S_{N_v}^{(2)} = (1 + b^2)^v = O(N_v^\lambda) = O(m^{\lambda v}), \quad v \to \infty$$

for some $\lambda < (2+2\alpha)/p$, then Lemma 3.1.10 assures that $f \in A^p_\alpha$. Note that $1 + b^2 = m^\lambda$ for $\lambda = \log(1+b^2)/\log m$. Hence if $b > 1$ and integer $m \geq 2$ are chosen such that $\lambda < (2+2\alpha)/p$, then $f \in A^p_\alpha$. Thus if the parameters b and m are chosen such that

$$\frac{\beta+1}{q} < \frac{\log b}{\log m} < \frac{\log(1+b^2)}{2\log m} < \frac{\alpha+1}{p},$$

then it will imply that $f \in A^p_\alpha$ and $\{z_k\}$ is not a zero set of A^q_β when $0 < p < 2$. The required inequalities are satisfied with suitable choices of b and m because $\log(1+b^2) \asymp 2\log b$ as $b \to \infty$.

Let $p \geq 2$ and $p' = p/(p-1)$ be the conjugate index of p. Lemma 3.1.11 implies that if

$$S^{(p')}_{N_v} = (1+b^{p'})^v = O(N^\lambda_v) = O(m^{\lambda v}), \quad v \to \infty$$

for some $\lambda < (p'-1)(\alpha+1)$, then $f \in A^p_\alpha$. Now using the equality $1 + b^{p'} = m^\lambda$, where $\lambda = \log(1+b^{p'})/\log m$, we deduce that it suffices to choose $b > 1$ and integer $m \geq 2$ such that $\lambda < (p'-1)(\alpha+1)$. Now the condition

$$\frac{\log(1+b^{p'})}{\log m} < (p'-1)(\alpha+1) \quad \text{or} \quad \frac{\log(1+b^{p'})}{p'\log m} < \frac{\alpha+1}{p}$$

and the inequality $(\beta+1)/q < \log b/\log m$ are satisfied with suitable choices of $b > 1$ and integer $m \geq 2$ because

$$\frac{p-1}{p}\log(1+b^{p'}) \asymp \log b, \quad b \to \infty.$$

This completes the proof. $\square$

The preceding theorem has a useful corollary that guarantees the upper bound in Theorem 3.1.7 is sharp.

Corollary 3.2.2. [9, Corollary 4.11] *Let $0 < p < \infty$, $-1 < \alpha < \infty$ and $0 < s < (\alpha+1)/p$. Then there exists $f \in A^p_\alpha$ with $f(0) \neq 0$, whose ordered zeros $\{z_k\}$ are distinct and have the property that there exists a positive integer N such that*

$$\prod_{k=1}^{n} \frac{1}{|z_k|} \geq Cn^s, \quad \forall n \geq N.$$

Proof. Let first $0 < p < 2$. Define f as in the the proof of Theorem 3.2.1 and choose $b > 1$ and $m = 2, 3, \dots$ such that

$$s < \frac{\log b}{\log m} < \frac{\log(1+b^2)}{2\log m} < \frac{\alpha+1}{p}.$$

Using the arguments presented in the proof of Theorem 3.2.1, we deduce that $f \in A^p_\alpha$ with $f(0) \neq 0$ and the ordered zeros $\{z_k\}$ of f are distinct and have the property that

$$\prod_{k=1}^{N_v} \frac{1}{|z_k|} = b^v = m^{\lambda v} \geq m^{sv}, \quad v = 1, 2, \dots,$$

where $\lambda = \log b / \log m$ and N_v is defined as in the proof of Theorem 3.2.1. Now for any $n \in \mathbb{N}$ with $m^2 \leq n$ choose $v \in \mathbb{N}$ such that $m^{v+1} \leq n < m^{v+2}$. Since $m^v \leq N_v \leq m^{v+1} \leq n < m^{v+2}$, it follows that

$$\prod_{k=1}^{n} \frac{1}{|z_k|} \geq \prod_{k=1}^{N_v} \frac{1}{|z_k|} \geq m^{sv} > m^{-2s} n^s.$$

In other words, there exists $N \in \mathbb{N}$, in fact $N = m^2$, such that

$$\prod_{k=1}^{n} \frac{1}{|z_k|} \geq C n^s, \quad \forall n \geq N.$$

The case $2 \leq p < \infty$ is similiar. $\square$

Corollary 3.2.3 shows indeed that the exponent $-1-\epsilon$ in Theorem 3.1.4 is sharp.

Corollary 3.2.3. [9, Corollary 4.13] *Let $0 < p < \infty$ and $-1 < \alpha < \infty$. Then there exists $f \in A^p_\alpha$ whose zeros $z_k \neq 0$ satisfies*

$$\sum_{k=1}^{\infty} (1 - |z_k|) \left(\log \frac{1}{1 - |z_k|} \right)^{-1} = \infty.$$

Proof. Let f be the function constructed as in the proof of Theorem 3.2.1 and $\{z_k\}$ be the zero set of f. In the proof of Theorem 3.2.1 we show that $f \in A^p_\alpha$ for certain choices of $b > 1$ and integer $m \geq 2$. Note that $z_k \neq 0$ for all k and $|z_k| = b^{-1/m^v}$ for $N_{v-1} + 1 \leq k \leq N_v$. We shall prove that the divergence of series is indeed independent of the choices of $b > 1$ and the integer $m \geq 2$. Since $e^x \asymp 1 + x$ as $x \to 0$, it follows that

$$b^{-1/m^k} = e^{-\log b/m^k} \asymp 1 - \frac{\log b}{m^k}, \quad k \to \infty,$$

equivalently,

$$\frac{\log b}{m^k} \asymp 1 - b^{-1/m^k}, \quad k \to \infty.$$

Thus

$$\begin{aligned} \sum_{k=1}^{N_v} (1 - |z_k|) \left(\log \frac{1}{1 - |z_k|} \right)^{-1} &= \sum_{k=1}^{v} (1 - b^{-1/m^k}) \left(\log \frac{1}{1 - b^{-1/m^k}} \right)^{-1} m^k \\ &\asymp \sum_{k=1}^{v} \frac{1}{k} \\ &\asymp \log v \end{aligned}$$

as $v \to \infty$. Hence

$$\sum_{k=1}^{\infty} (1 - |z_k|) \left(\log \frac{1}{1 - |z_k|} \right)^{-1} = \infty$$

as desired. □

The next result shows that the upper bound $O(\log n)$ in Theorem 3.1.3 is sharp; it cannot be replaced by $o(\log n)$.

Corollary 3.2.4. *Let $0 < p < \infty$ and $-1 < \alpha < \infty$. Then there exists $f \in A^p_\alpha$ whose zero set $\{z_k\}$ satisfies*

$$\sum_{k=1}^{n} (1 - |z_k|) \neq o(\log n), \quad n \to \infty.$$

Proof. We argue as in the proof of Corollary 3.2.3. Let f be the function constructed as in the proof of Theorem 3.2.1 and $\{z_k\}$ be the zero set of f. For any $n = 1, 2, \ldots$ choose $v \in \mathbb{N}$ such that $N_v \leq n < N_{v+1}$. Observe that $v \asymp \log n$ as $n \to \infty$, because $m^v \leq N_v \leq n < N_{v+1} \leq m^{v+2}$. Arguing as in the proof of Corollary 3.2.3, we obtain

$$\sum_{k=1}^{n} (1 - |z_k|) \geq \sum_{k=1}^{N_v} (1 - |z_k|) = \sum_{k=1}^{v} (1 - b^{-1/m^k}) m^k \geq C \log n$$

as $n \to \infty$. This completes the proof. □

3.3 Unions of zero sets

Theorem 3.3.1. [9, Theorem 5.1] *Let* $p, q \in (0, \infty)$ *and* $\alpha, \beta \in (-1, \infty)$. *If* $(\beta+1)/2q < (\alpha+1)/p$, *then there exist two zero sets of* A^p_α *such that their union is not a zero set of* A^q_β.

Proof. Let $(\beta+1)/2q < s < (\alpha+1)/p$. Then there exists $f \in A^p_\alpha$ with $f(0) \neq 0$ and the zero set $\{z_k\}$ of f satisfies that there exists a positive integer N such that

$$\prod_{k=1}^{n} \frac{1}{|z_k|} \geq Cn^s, \quad \forall n \geq N.$$

The existence of such f is guaranteed by Corollary 3.2.2. We choose an angle θ such that $\{e^{i\theta}z_k\}$ is disjoint from $\{z_k\}$. Let $\{w_k\} = \{e^{i\theta}z_k\} \cup \{z_k\}$ and $0 < |w_1| \leq |w_2| \ldots$. Then we have

$$\prod_{k=1}^{2n} \frac{1}{|w_k|} = \prod_{k=1}^{n} \frac{1}{|z_k|^2} \geq C^2 n^{2s}, \quad \forall n \geq N.$$

Hence

$$\prod_{k=1}^{n} \frac{1}{|w_k|} \neq o\big(n^{\frac{\beta+1}{q}}\big), \quad n \to \infty,$$

which implies that $\{w_k\}$ is not a zero set of A^q_β by Theorem 3.1.7. $\square$

3.4 Subsets of zero sets

Recall that for $w \in \mathbb{D}$, a Blaschke factor b_w is defined by

$$b_w(z) = \frac{|w|}{w}\frac{w-z}{1-\overline{w}z}, \quad z \in \mathbb{D}.$$

Proposition 3.4.1. *Let* $\{w_k\}$ *be a sequence of complex numbers in* $\mathbb{D}$. *Then*

$$H(z) = \prod_{k=1}^{\infty} b_{w_k}(z)(2 - b_{w_k}(z)) \tag{3.7}$$

is analytic in $\mathbb{D}$ *if and only if*

$$\sum_{k=1}^{\infty} (1 - |w_k|)^2 < \infty.$$

Proof. Assume that H converges in $\mathbb{D}$. Then we have

$$H(0) = \prod_{k=1}^{\infty} |w_k|(2 - |w_k|) = \prod_{k=1}^{\infty}(1 - (1 - |w_k|)^2) < \infty,$$

which implies

$$\sum_{k=1}^{\infty}(1 - |w_k|)^2 < \infty$$

as desired. Conversely assume that

$$\sum_{k=1}^{\infty}(1 - |w_k|)^2 < \infty.$$

So $|w_k| \to 1^-$ as $k \to \infty$. To show that $H \in H(\mathbb{D})$, it suffices to prove that each factor b_{w_k} is analytic in $\mathbb{D}$ and that

$$\sum_{k=1}^{\infty} |1 - b_{w_k}(z)(2 - b_{w_k}(z))|$$

converges uniformly on every compact set in $\mathbb{D}$. For each k we have $|w_k| < 1$ so b_{w_k} is analytic in $\mathbb{D}$. Let K be a compact set in $\mathbb{D}$. Then there exists $r < 1$ such that $|z| \leq r < 1$ for all $z \in K$. Choose a positive integer N such that $|a_k| \geq \frac{1}{2}$ for all $k \geq N$. Then

$$\begin{aligned}
\left|1 - b_{w_k}(z)(2 - b_{w_k}(z))\right| &= \left|1 - b_{w_k}(z)\right|^2 \\
&= \left|1 - \frac{|w_k|}{w_k}\frac{(w_k - z)}{(1 - \overline{w}_k z)}\right|^2 \\
&= \left|\frac{-|w_k|w_k + z|w_k| + w_k - |w_k|^2 z}{w_k(1 - \overline{w_k} z)}\right|^2 \\
&= \left|\frac{(1 - |w_k|)(w_k + |w_k|z)}{w_k(1 - \overline{w}_k z)}\right|^2 \\
&\leq 16\frac{(1 - |w_k|)^2}{(1 - r)^2}
\end{aligned}$$

for all $k \geq N$ and for all $z \in K$. So

$$\sum_{k=N}^{\infty} |1 - b_{w_k}(z)(2 - b_{w_k}(z))| \leq \frac{16}{(1 - r)^2}\sum_{k=N}^{\infty}(1 - |w_k|)^2 < \infty$$

for all $z \in K$. Since K is an arbitrary compact subset of $\mathbb{D}$, we conclude that

$$\sum_{k=1}^{\infty} |1 - b_{w_k}(z)(2 - b_{w_k}(z))|$$

converges uniformly on every compact set in $\mathbb{D}$. $\square$

The infinite product H in (3.7) is called the *Horowitz product.* Note that the zero set of the Horowitz product H in (3.7) is $\{w_k\}$. We say a Horowitz product H is formed with a sequence $\{w_k\}$ if the zero set of H is $\{w_k\}$. The infinite product H plays a crucial role in proving the statement that every subset of a zero set of A^p_α is again a zero set of A^p_α. Our task would be easy if the Horowitz product H formed with a subset of a zero set of A^p_α were in A^p_α. Because then this would imply that every subset of a zero set of A^p_α is a zero set of A^p_α. But we do not know whether H belongs to A^p_α. However we shall show that if $f \in A^p_\alpha$ and H is the Horowitz product formed with a subset of the zero set of f, then $f/H \in A^p_\alpha$, which serves our purpose.

Proposition 3.4.2. *Let $0 < p < \infty$ and $-1 < \alpha < \infty$. Let also $\{z_k\}$ be a zero set of A^p_α and $\{w_k\}$ be a subsequence of $\{z_k\}$. If H is the Horowitz product formed with the sequence $\{w_k\}$, then H is analytic in $\mathbb{D}$.*

Proof. Corollary 3.1.5 implies

$$\sum_{k=1}^{\infty}(1-|w_k|)^2 < \infty,$$

because $\{w_k\}$ is a subsequence of a zero set $\{z_k\}$ of A^p_α. Then the result follows from Proposition 3.4.1. □

Lemma 3.4.3. [4, p. 32] *If $1 < t < s$ then there exists a constant $C > 0$, depending only on s and t, such that*

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^{t-2}}{|1-zw|^s} dA(w) \le C(1-|z|^2)^{t-s}$$

for all $z \in \mathbb{D}$.

Proof. First note that $(1-|z||w|)^s \le |1-zw|^s$ for all $z, w \in \mathbb{D}$. Hence it suffices to show that there exists a constant $C > 0$, depending only on s and t, such that

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^{t-2}}{(1-\rho|w|)^s} dA(w) \le C(1-\rho^2)^{t-s}$$

for all $\rho \in [0, 1)$. If $\rho \le 1/2$ then we have

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^{t-2}}{(1-\rho|w|)^s} dA(w) \le (1-\rho)^{-s} \int_{\mathbb{D}} (1-|w|^2)^{t-2} dA(w) \le C(1-\rho)^{t-s},$$

where C is a positive constant depending only on t. Here the integral converges because $t-2>-1$. If $\rho>1/2$ then

$$\int_{|w|\le 1/2\rho}\frac{(1-|w|^2)^{t-2}}{(1-\rho|w|)^s}dA(w)\le 2^s\int_{\mathbb{D}}(1-|w|^2)^{t-2}dA(w)\le C(1-\rho)^{t-s},$$

as $1<t<s$. Now we need to estimate

$$\int_{|w|>1/2\rho}\frac{(1-|w|^2)^{t-2}}{(1-\rho|w|)^s}dA(w)=\frac{2}{\pi}\int_{1/2\rho}^{1}\int_0^{\pi}\frac{(1-r^2)^{t-2}}{(1-2\rho r\cos\theta+\rho^2r^2)^{s/2}}\,d\theta\,rdr. \quad (3.8)$$

Since $\sin(\theta/2)\ge\theta/\pi$ for $0\le\theta\le\pi$, it follows that

$$1-2\rho r\cos\theta+\rho^2r^2=(1-\rho r)^2+4\rho r\sin^2(\theta/2)\ge(1-\rho r)^2+4\rho r\theta^2/\pi^2.$$

Hence for $r>1/(2\rho)$, we have

$$\begin{aligned}\int_0^{\pi}\frac{d\theta}{(1-2\rho r\cos\theta+\rho^2r^2)^{s/2}}&\le\frac{1}{(1-\rho r)^s}\int_0^{\pi}\frac{d\theta}{\left(1+\frac{2}{\pi^2}(\frac{\theta}{1-\rho r})^2\right)^{s/2}}\\&\le\frac{1}{(1-\rho r)^{s-1}}\int_0^{\infty}\frac{du}{(1+\frac{2}{\pi^2}u^2)^{s/2}}\\&\le C(1-\rho r)^{1-s},\end{aligned}$$

as $s>1$ assures that the last integral is convergent. So (3.8) is bounded by a constant multiple of

$$\int_0^{\rho}\frac{(1-r^2)^{t-2}}{(1-\rho r)^{s-1}}r\,dr+\int_{\rho}^{1}\frac{(1-r^2)^{t-2}}{(1-\rho r)^{s-1}}r\,dr=I_1+I_2.$$

We first estimate I_1. Note that $0\le r\le\rho$ implies $1-\rho r\le 1-r^2$. If $t<2$ then we have

$$I_1\le\int_0^{\rho}(1-\rho r)^{t-s-1}\,dr\le\frac{2}{s-t}(1-\rho^2)^{t-s}\le C(1-\rho)^{t-s}.$$

If $t\ge 2$ then we can use the inequality $1-r^2\le 2(1-r)\le 2(1-\rho r)$ and obtain

$$I_1\le 2^{t-2}\int_0^{\rho}(1-\rho r)^{t-s-1}\,dr\le C(1-\rho)^{t-s}.$$

We next estimate I_2. Observe that $r\le 1$ implies $1-\rho\le 1-\rho r$. Hence using this inequality we conclude that

$$\begin{aligned}I_2&\le C(1-\rho)^{-s+1}\int_{\rho}^{1}(1-r)^{t-2}\,dr\\&\le C(1-\rho)^{t-s},\end{aligned}$$

where C is a positive constant depending only on s and t. This completes the proof. $\square$

Lemma 3.4.4. [8, Lemma 4.34] *Let* $0 < p < \infty$ *and* $-1 < \alpha < \infty$. *Let* $f \in A^p_\alpha$ *with* $f(0) \neq 0$ *and* $\{z_k\}$ *be the zero set of* f. *Then there exists* $C = C(p, \alpha) > 0$ *such that*

$$|f(0)| \prod_{k=1}^{\infty} \big\{|z_k|(2 - |z_k|)\big\}^{-1} \leq C \, \|f\|_{p,\alpha}.$$

Proof. Let $f \in A^p_\alpha$ with $f(0) \neq 0$. Without loss of generality we may assume $f(0) = 1$. Note that we have

$$n(r) = O\left(\frac{1}{1-r} \log \frac{1}{1-r}\right), \quad r \to 1^- \tag{3.9}$$

by Proposition 3.1.2. Now consider the expression

$$S = -\sum_{k=1}^{\infty} \log \big(|z_k|(2 - |z_k|)\big) = -\int_0^1 \log \big(r(2-r)\big) \, dn(r).$$

By integrating by parts, we obtain

$$\begin{aligned} S &= -\int_0^1 \log \big(r(2-r)\big) \, dn(r) \\ &= 2\int_0^1 \frac{1-r}{r(2-r)} \, n(r) \, dr - n(r) \log \big(r(2-r)\big) \Big|_0^1 \\ &= 2\int_0^1 \frac{1-r}{r(2-r)} \, n(r) \, dr - \lim_{r \to 1^-} n(r) \log \big(r(2-r)\big). \end{aligned}$$

Using the fact that

$$\log \big(r(2-r)\big) = \log \big(1 - (1-r)^2\big) = O\big((1-r)^2\big), \quad r \to 1^-$$

and (3.9), we conclude that

$$\lim_{r \to 1^-} n(r) \log \big(r(2-r)\big) = 0.$$

Hence

$$S = 2\int_0^1 \frac{1-r}{r(2-r)} \, n(r) \, dr.$$

Another integration by parts gives

$$S = 2\int_0^1 \frac{1-r}{r(2-r)}\, n(r)\, dr = 2\int_0^1 \frac{1}{(2-r)^2}\, N(r)\, dr.$$

Jensen's formula and Proposition 2.0.21 imply

$$S = 2\int_0^1 \frac{1}{(2-r)^2}\, N(r)\, dr = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 \frac{\log|f(re^{i\theta})|}{(2-r)^2}\, dr\, d\theta$$

as $f(0) = 1$. Multiplying by p and manipulating the integrand, we obtain

$$\begin{aligned}
Sp &= \frac{1}{\pi}\int_0^{2\pi}\int_0^1 \frac{\log|f(re^{i\theta})|^p}{(2-r)^2}\, dr\, d\theta \\
&= \frac{1}{\pi}\int_0^{2\pi}\int_0^1 \frac{\log\left(|f(re^{i\theta})|^p(1-r^2)^\alpha r\right) - \log((1-r^2)^\alpha) - \log r}{(2-r)^2}\, dr\, d\theta \\
&= \frac{1}{\pi}\int_0^{2\pi}\int_0^1 \frac{\log\left(|f(re^{i\theta})|^p(1-r^2)^\alpha r\right)}{(2-r)^2}\, dr\, d\theta - \frac{1}{\pi}\int_0^{2\pi}\int_0^1 \frac{\alpha\log(1-r^2) + \log r}{(2-r)^2}\, dr\, d\theta \\
&= \frac{1}{\pi}\int_0^{2\pi}\int_0^1 \frac{\log\left(|f(re^{i\theta})|^p(1-r^2)^\alpha r\right)}{(2-r)^2}\, dr\, d\theta - 2\int_0^1 \frac{\alpha\log(1-r^2) + \log r}{(2-r)^2}\, dr.
\end{aligned}$$

Let

$$I = \int_0^1 \frac{\alpha\log(1-r^2) + \log r}{(2-r)^2}\, dr.$$

Note that

$$|I| \le \int_0^1 \frac{|\alpha\log(1-r^2) + \log r|}{|(2-r)^2|}\, dr \le |\alpha|\int_0^1 |\log(1-r^2)|\, dr + \int_0^1 |\log r|\, dr.$$

A direct computation of the integrals yields

$$\int_0^1 |\log(1-r^2)|\, dr = 2 - \log 4$$

and

$$\int_0^1 |\log r|\, dr = 1.$$

So $|I|$ is finite. Exponentiating the expression

$$Sp = \frac{1}{\pi}\int_0^{2\pi}\int_0^1 \frac{\log\left(|f(re^{i\theta})|^p(1-r^2)^\alpha r\right)}{(2-r)^2}\, dr\, d\theta - 2I$$

and applying the agm inequality with respect to the unit measure $\frac{1}{\pi}\frac{1}{(2-r)^2}\, dr\, d\theta$, we obtain

$$\begin{aligned}
e^{Sp} &= \prod_{k=1}^{\infty}\left\{|z_k|(2-|z_k|)\right\}^{-p} \\
&\le C\int_0^{2\pi}\int_0^1 \frac{|f(re^{i\theta})|^p(1-r^2)^\alpha r}{(2-r)^2}\, dr\, d\theta,
\end{aligned}$$

where C is a positive constant depending on α. Since the denominator of the integrand is bounded on $(0,1)$, it follows that

$$\begin{aligned}\prod_{k=1}^{\infty}\left\{|z_k|(2-|z_k|)\right\}^{-p} &\leq C\int_0^{2\pi}\int_0^1 |f(re^{i\theta})|^p(1-r^2)^\alpha r\,dr\,d\theta \\ &= C\int_{\mathbb{D}}|f(z)|^p dA_\alpha(z) \\ &= C\|f\|_{p,\alpha}^p,\end{aligned}$$

which completes the proof. □

Theorem 3.4.5. [9, Theorem 7.9] *Let $0<p<\infty$ and $-1<\alpha<\infty$. If $f\in A_\alpha^p$ and H is the Horowitz product formed with the zero set $Z=\{z_k\}$ of f, then $f/H\in A_\alpha^p$ and $\|f/H\|_{p,\alpha}\leq C\,\|f\|_{p,\alpha}$, where C is a positive constant depending only on p and α.*

Proof. Let H be the Horowitz product formed with Z. For every $w\in\mathbb{D}\setminus Z$, let $f_w=f\circ\phi_w$, where

$$\varphi_w(z)=\frac{w-z}{1-\overline{w}z},\quad z\in\mathbb{D}.$$

We show $f_w\in A_\alpha^p$ as follows. Fix $w\in\mathbb{D}\setminus Z$. Then we employ a change of variables $z=\varphi_w(u)$ and obtain

$$\begin{aligned}\|f_w\|_{p,\alpha}^p &= (1+\alpha)\int_{\mathbb{D}}|f_w(z)|^p(1-|z|^2)^\alpha\,dA(z) \\ &= (1+\alpha)\int_{\mathbb{D}}|f(u)|^p(1-|\varphi_w(u)|^2)^\alpha|\varphi_w'(u)|^2\,dA(u) \\ &= (1+\alpha)\int_{\mathbb{D}}|f(u)|^p\,\frac{(1-|u|^2)^\alpha(1-|w|^2)^{\alpha+2}}{|1-\overline{w}u|^{2\alpha+4}}\,dA(u) \\ &\leq (1+\alpha)\frac{(1-|w|^2)^{\alpha+2}}{(1-|\overline{w}|)^{2\alpha+4}}\int_{\mathbb{D}}|f(u)|^p(1-|u|^2)^\alpha\,dA(u),\end{aligned}$$

where the last line comes from the triangle inequality. So $f_w\in A_\alpha^p$ because $f\in A_\alpha^p$. Note that $\{\varphi_w(z_k)\}$ is the zero set of f_w because $\varphi_w^{-1}=\varphi_w$. Note also that $0\notin\{\varphi_w(z_n)\}$ since $w\notin Z$. Now fix any $\beta>\alpha$. Then $A_\alpha^p\subset A_\beta^p$ so $f_w\in A_\beta^p$. We apply Lemma 3.4.4 to the function f_w and obtain

$$|f(w)|\prod_{k=1}^{\infty}\left\{|\varphi_w(z_k)|(2-|\varphi_w(z_k)|)\right\}^{-1}\leq C\,\|f_w\|_{p,\beta}. \tag{3.10}$$

Note that

$$|\varphi_w(a)|(2-|\phi_w(a)|) \le |b_a(w)(2-b_a(w))| \tag{3.11}$$

for all $w, a \in \mathbb{D}$. Let $g(w) = f(w)/H(w)$. Observe that g is analytic in $\mathbb{D}$ because H is analytic in $\mathbb{D}$ by Proposition 3.4.2 and the zeros of H cancel out the zeros of f. In order to complete the proof, we need to show $\|g\|_{p,\alpha} \le C\|f\|_{p,\alpha}$. Now we use (3.10) and (3.11) to obtain

$$\begin{aligned}|g(w)|^p = \left|\frac{f(w)}{H(w)}\right|^p &= |f(w)|^p \prod_{k=1}^{\infty} \left\{|b_{z_k}(w)(2-b_{z_k}(w))|\right\}^{-p} \\ &\le |f(w)|^p \prod_{k=1}^{\infty} \left\{|\varphi_w(z_k)|(2-|\varphi_w(z_k)|)\right\}^{-p} \\ &\le C^p \, \|f_w\|_{p,\beta}^p.\end{aligned}$$

We employ a change of variables same as above and obtain

$$\begin{aligned}|g(w)|^p &\le C^p \, \|f_w\|_{p,\beta}^p \\ &= C^p \,(1+\beta) \int_{\mathbb{D}} |f_w(z)|^p \,(1-|z|^2)^\beta \, dA(z) \\ &= C^p \,(1+\beta) \int_{\mathbb{D}} |f(z)|^p \,(1-|\varphi_w(z)|^2)^\beta |\varphi_w'(z)|^2 \, dA(z) \\ &= C^p \,(1+\beta) \int_{\mathbb{D}} |f(z)|^p \, \frac{(1-|z|^2)^\beta (1-|w|^2)^{\beta+2}}{|1-\overline{w}z|^{2\beta+4}} \, dA(z)\end{aligned}$$

for all $w \in \mathbb{D} \setminus Z$. In fact, the above holds for all $w \in \mathbb{D}$ since g is continuous. Hence

$$\begin{aligned}\|g\|_{p,\alpha}^p &= (1+\alpha) \int_{\mathbb{D}} |g(w)|^p (1-|w|^2)^\alpha \, dA(w) \\ &\le C^p \,(1+\beta)(1+\alpha) \int_{\mathbb{D}} \int_{\mathbb{D}} |f(z)|^p \, \frac{(1-|z|^2)^\beta (1-|w|^2)^{\alpha+\beta+2}}{|1-\overline{w}z|^{2\beta+4}} \, dA(z) \, dA(w).\end{aligned}$$

By Lemma 3.4.3, we have

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^{\alpha+\beta+2}}{|1-\overline{w}z|^{2\beta+4}} \, dA(w) = (1-|z|^2)^{\alpha-\beta} \tag{3.12}$$

for all $z \in \mathbb{D}$. Then using Fubini's theorem and (3.12), we obtain

$$\begin{aligned}\|g\|_{p,\alpha}^p &\le C^p \,(1+\beta)(1+\alpha) \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\beta (1-|z|^2)^{\alpha-\beta} \, dA(z) \\ &= C \, \|f\|_{p,\alpha}^p\end{aligned}$$

as desired. □

Corollary 3.4.6. [9, Theorem 3] *If $0 < p < \infty$ and $-1 < \alpha < \infty$, then every subset of a zero set of A^p_α is a zero set of A^p_α.*

Proof. Let $f \in A^p_\alpha$ and $\{z_k\}$ be the zero set of f. Let also $\{a_k\}$ be a subset of $\{z_k\}$ and H be the Horowitz product formed with the complement of $\{a_k\}$ in $\{z_k\}$. Then define

$$g(z) = \frac{f(z)}{H(z)}, \quad z \in \mathbb{D}.$$

We deduce that the function g is analytic in $\mathbb{D}$ as in the proof Theorem 3.4.5 and it vanishes precisely on $\{a_k\}$. Note that

$$0 \le (1 - |b_{z_k}(z)|)^2 \le 1$$

or

$$|b_{z_k}(z)|(2 - |b_{z_k}(z)|) \le 1$$

or

$$1 \le \left\{|b_{z_k}(z)|(2 - |b_{z_k}(z)|)\right\}^{-1}$$

for each k and for all $z \in \mathbb{D}$. Hence for all $z \in \mathbb{D}$, we have

$$\begin{aligned} |g(z)| = \left|\frac{f(z)}{H(z)}\right| &= |f(z)| \prod_{k=1}^{\infty} \left\{\left|b_{a_k}(z)(2 - b_{a_k}(z))\right|\right\}^{-1} \\ &\le |f(z)| \prod_{k=1}^{\infty} \left\{|b_{a_k}(z)|\,(2 - |b_{a_k}(z)|)\right\}^{-1} \\ &\le |f(z)| \prod_{k=1}^{\infty} \left\{|b_{z_k}(z)|\,(2 - |b_{z_k}(z)|)\right\}^{-1}, \end{aligned}$$

because $\{a_k\}$ is a subset of $\{z_k\}$. Note that $|b_{z_k}(z)| = |\varphi_{z_k}(z)| = |\varphi_z(z_k)|$ for each k and for all $z \in \mathbb{D}$, where

$$\varphi_{z_k}(z) = \frac{z_k - z}{1 - \overline{z}_k z}, \quad z \in \mathbb{D}.$$

So

$$|g(z)| \le |f(z)| \prod_{k=1}^{\infty} \left\{|\varphi_z(z_k)|(2 - |\varphi_z(z_k)|)\right\}^{-1}$$

for all $z \in \mathbb{D}$. We show $g \in A^p_\alpha$ in the proof of Theorem 3.4.5. Thus $\{a_k\}$ is a zero set of A^p_α. □

Chapter 4

Zero sets for the Dirichlet space

For $f \in H(\mathbb{D})$, the *Dirichlet integral* of f is defined by

$$D^2(f) = \int_{\mathbb{D}} |f'(z)|^2 \, dA(z).$$

The *Dirichlet space* D^2 is the vector space of functions $f \in H(\mathbb{D})$ such that $D^2(f) < \infty$. In this chapter, we shall describe some basic properties of the zero sets of D^2 and investigate how they differ from those of H^2. Our main reference is [5].

Theorem 4.0.1. [5, Theorem 1.1.2] *Let $f \in H(\mathbb{D})$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be its Taylor series expansion around $z = 0$. Then*

$$D^2(f) = \sum_{k=1}^{\infty} k|a_k|^2.$$

Proof. Writing $dA(z)$ in terms of polar coordinates, we have

$$D^2(f) = \int_{\mathbb{D}} \left| \sum_{k=1}^{\infty} k a_k z^{k-1} \right|^2 dA(z) = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \left| \sum_{k=1}^{\infty} k a_k r^{k-1} e^{i(k-1)\theta} \right|^2 d\theta \, r dr.$$

For each $r \in (0, 1)$, we have

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=1}^{\infty} k a_k r^{k-1} e^{i(k-1)\theta} \right|^2 d\theta = \sum_{k=1}^{\infty} k^2 |a_k|^2 r^{2k-2}$$

by Parseval's formula. Hence

$$D^2(f) = 2\int_0^1 \sum_{k=1}^{\infty} k^2 |a_k|^2 r^{2k-1}\, dr = \sum_{k=1}^{\infty} k|a_k|^2$$

as claimed. □

Recall that an analytic function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ belongs to H^2 if and only if $\sum_{k=0}^{\infty} |a_k|^2 < \infty$ and it belongs to A^2 if and only if $\sum_{k=1}^{\infty} k^{-1}|a_k|^2 < \infty$. The preceding theorem shows that $f \in D^2$ if and only if $\sum_{k=1}^{\infty} k|a_k|^2 < \infty$. Hence one can show that $D^2 \subsetneq H^2 \subsetneq A^2$ by choosing appropriate coefficients $\{a_k\}$.

The zero sets of Hardy spaces are characterized by the Blaschke condition. Namely, if a sequence $\{z_k\}$ in the unit disk satisfies the Blaschke condition, that is, $\sum_{k=0}^{\infty}(1 - |z_k|) < \infty$, then there exists a function f in H^∞ such that $\{z_k\}$ is the zero set of f. Recall that $H^\infty \subset H^p$ for $0 < p < \infty$. On the other hand if $f \in H^p$, $0 < p \leq \infty$, then the zero set of f satisfies the Blaschke condition. Since D^2 is contained in H^2, it follows that the Blaschke condition is a necessary condition for the zero sets of D^2. However it is not sufficient for a sequence to be a zero set for D^2; see Theorem 4.0.14.

Lemma 4.0.2. [5, Theorem 1.6.7] *If* $0 < r < 1$, *then*

$$\frac{|1 - z|}{|1 - rz|} \leq \frac{2}{1 + r}, \quad z \in \mathbb{D}.$$

Proof. Consider the Möbius transformation $z \to (1 - z)/(1 - rz)$. It maps the unit circle onto the circle $|w - 1/(1 + r)| = 1/(1 + r)$ which is symmetric with respect to the real axis and passes through 0 and $2/(1 + r)$. So it is the circle with diameter $[0, 2/(1 + r)]$. Then the result follows from the maximum modulus principle. □

Theorem 4.0.3. [5, Theorem 4.1.2] *If* $\{x_k\}$ *is a sequence in* $(0, 1)$ *such that* $\sum_{k=1}^{\infty}(1 - x_k) < \infty$, *then* $\{x_k\}$ *is a zero set for* D^2.

Proof. Let $B(z)$ be the Blaschke product whose zeros are precisely the sequence

$\{x_k\}$. By logarithmic differentiation, we have

$$\frac{B'(z)}{B(z)} = \sum_{n=1}^{\infty} \frac{1 - x_k^2}{(z - x_k)(1 - x_k z)}$$

for all $z \in \mathbb{D} \setminus \{x_k\}_{k=1}^{\infty}$. Now multiplying both sides by $B(z)$, taking absolute values and using Lemma 4.0.2, we obtain

$$|B'(z)| \le \sum_{k=1}^{\infty} \frac{1 - x_k^2}{|1 - x_k z|^2} \le 4 \sum_{k=1}^{\infty} \frac{(1 - x_k)}{|1 - z|^2}$$

for all $z \in \mathbb{D} \setminus \{x_k\}_{k=1}^{\infty}$. Continuity of $|B'(z)|$ implies the inequality above holds for all $z \in \mathbb{D}$. We define $f(z) = (1 - z)^2 B(z)$ for all $z \in \mathbb{D}$. Then f is analytic in the unit disk and f' is bounded as $\sum_{k=1}^{\infty}(1 - x_k) < \infty$ by hypothesis. So $f \in D^2$. Since the zero set of f is the sequence $\{x_k\}$, it follows that $\{x_k\}$ is a zero set for D^2. □

Theorem 4.0.4. [5, Theorem 1.4.1] *Let S_1 and S_2 be domains and let $\phi : S_1 \to S_2$ be a conformal mapping. If $f : S_1 \to \mathbb{C}$ is an analytic function, then*

$$\int_{S_1} |(f \circ \phi)'(z)|^2 dA(z) = \int_{S_2} |f'(w)|^2 dA(w).$$

Proof. Let $w = \phi(z)$. Then a direct computation gives $dA(w) = |\phi'(z)|^2 dA(z)$. Using this we obtain the claimed change of variables formula. □

In particular, if φ is an automorphism of the unit disk, then

$$D^2(f \circ \varphi) = \int_{\mathbb{D}} |(f \circ \varphi)'(z)|^2 dA(z) = \int_{\mathbb{D}} |f'(w)|^2 dA(w) = D^2(f).$$

Hence the Dirichlet integral is Möbius-invariant.

Notation 4.0.5. *For $f \in H(\mathbb{D})$ and $\zeta \in \mathbb{T}$, we write*

$$f^*(\zeta) = \lim_{r \to 1^-} f(r\zeta)$$

whenever this radial limit exists.

Since D^2 lies in H^2, f^* exists almost everywhere on $\mathbb{T}$.

Theorem 4.0.6. [5, Theorem 4.1.3] *Let $f \in H^2$ and B be the Blaschke product whose zero set is $\{z_k\}_{k=1}^n$, where $n = 1, 2, \ldots, \infty$. Then*

$$D^2(Bf) = D^2(f) + \frac{1}{2\pi} \sum_{k=1}^{n} \int_{\mathbb{T}} \frac{1 - |z_k|^2}{|\zeta - z_k|^2} |f^*(\zeta)|^2 |d\zeta|. \tag{4.1}$$

Proof. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Suppose first $B(z) = z$. Using Theorem 4.0.1, we obtain

$$D^2(zf) = \sum_{k=1}^{\infty} (k+1)|a_k|^2 = D^2(f) + \|f\|_{H^2}^2 = D^2(f) + \frac{1}{2\pi} \int_0^{2\pi} |f^*(\zeta)|^2 |d\zeta|,$$

which proves (4.1) when $B(z) = z$.

Now suppose $B(z) = \lambda(z_1 - z)/(1 - \overline{z}_1 z)$, where $0 < |z_1| < 1$ and $|\lambda| = 1$. Note that B is an automorphism of the unit disk. Since the Dirichlet integral is Möbius-invariant, it follows that

$$\begin{aligned} D^2(Bf) &= D^2(z(f \circ B^{-1})) \\ &= D^2(f \circ B^{-1}) + \|f \circ B^{-1}\|_{H^2}^2 \\ &= D^2(f) + \frac{1}{2\pi} \int_{\mathbb{T}} |f^*(\zeta)|^2 |B'(z)| |d\zeta| \\ &= D^2(f) + \frac{1}{2\pi} \int_{\mathbb{T}} |f^*(\zeta)|^2 \frac{1 - |z_1|^2}{|\zeta - z_1|^2} |d\zeta|, \end{aligned}$$

which proves (4.1) in this case.

If B is a Blaschke product with finitely many zeros, then the result follows by induction.

Now we consider the case where $B(z)$ has infinitely many zeros. Let us denote the product of the first n terms of B by b_n and of the remaining terms by B_n. Now we use (4.1) for b_n as it has finitely many zeros. We have

$$D^2(b_n f) = D^2(f) + \frac{1}{2\pi} \sum_{k=1}^{n} \int_{\mathbb{T}} \frac{1 - |z_k|^2}{|\zeta - z_k|^2} |f^*(\zeta)|^2 |d\zeta|.$$

Note that $b_n f \to Bf$ uniformly on compact subsets of $\mathbb{D}$ because b_n is a partial product of B. Then we have $D^2(Bf) \leq \liminf_{n\to\infty} D^2(b_n f)$ by Fatou's lemma.

Hence

$$D^2(Bf) \leq D^2(f) + \frac{1}{2\pi}\sum_{k=1}^{\infty}\int_{\mathbb{T}} \frac{1-|z_k|^2}{|\zeta - z_k|^2}|f^*(\zeta)|^2|d\zeta|. \tag{4.2}$$

Since $|B_n^*(\zeta)| = 1$ a.e. on $\mathbb{T}$, it follows that

$$D^2(Bf) = D^2(b_n B_n f) = D^2(B_n f) + \frac{1}{2\pi}\sum_{k=1}^{n}\int_{\mathbb{T}} \frac{1-|z_k|^2}{|\zeta - z_k|^2}|f^*(\zeta)|^2|d\zeta|, \tag{4.3}$$

as b_n has finitely many zeros. We also have $B_n f \to f$ uniformly on compact subsets of $\mathbb{D}$. Hence $\liminf_{n\to\infty} D^2(B_n f) \geq D(f)$ again by Fatou's lemma. Using (4.3) we deduce that

$$D^2(Bf) - \frac{1}{2\pi}\sum_{k=1}^{\infty}\int_{\mathbb{T}} \frac{1-|z_k|^2}{|\zeta - z_k|^2}|f^*(\zeta)|^2|d\zeta| \geq D^2(f),$$

which together with (4.2) completes the proof. □

Theorem 4.0.7. *Every subset of a zero set of D^2 is again a zero set of D^2.*

Proof. Let $\{z_k\}$ be a zero set of D^2. Then there exists a function $0 \not\equiv f \in D^2$ whose zero set is $\{z_k\}$. Let $\{w_k\}$ be a subset of $\{z_k\}$. Note that $\{w_k\}$ satisfies the Blasckhe condition because $\{z_k\}$ is the zero set of $f \in D^2 \subset H^2$. Let B be the Blaschke product whose zero set is $\{w_k\}$. Set $g = f/B$. Note that $g \in H^2$ and its zero set is the complement of $\{w_k\}$ in $\{z_k\}$. Now we have $D^2(g) \leq D^2(Bg) = D^2(f) < \infty$ by Theorem 4.0.6. So $g \in D^2$. Hence the complement of $\{w_k\}$ in $\{z_k\}$, which is an arbitrary subset of $\{z_k\}$, is a zero set of D^2. □

Definition 4.0.8. *A sequence $\{z_k\}$ in $\mathbb{D}$ is called a uniqueness set for D^2 if the only function $f \in D^2$ satisfying $f(z_k) = 0$ for all $k = 1, 2, \ldots$ is $f \equiv 0$.*

Theorem 4.0.9. [5, Theorem 4.1.4] *A sequence $\{z_k\}$ in $\mathbb{D}$ a zero set for D^2 if and only if it is not a uniqueness set for D^2.*

Proof. If $\{z_k\}$ is a zero set, then there exists a function $0 \not\equiv f \in D^2$ such that $f(z_k) = 0$ for all $k = 1, 2, \ldots$. Clearly $\{z_k\}$ is not a uniqueness set for D^2.

Conversely, suppose that $\{z_k\}$ is not a uniqueness set for D^2. Then there exists $f \in D^2$ such that $f(z_k) = 0$ for all k and $f \not\equiv 0$. The problem is that the function f may have some additional zeros $\{w_k\}$ besides $\{z_k\}$. If $\{w_k\}$ is finite, then we define a polynomial P such that $P(w_k) = 0$ on $\{w_k\}$ so that $g = f/P \in D^2$ and the zero set of g is $\{z_k\}$. If $\{w_k\}$ is an infinite sequence, then $\{w_k\}$ satisfies the Blaschke condition because $\{w_k\}$ is a subset of the zero set of f, which lies in H^2. Let B be the Blaschke product whose zero set is $\{w_k\}$. Let $g = f/B$ so that $g \in H^2$ and the zero set of g is $\{z_k\}$. Using Theorem 4.0.6, we obtain that $D^2(g) \leq D^2(Bg) = D^2(f) < \infty$, which implies $g \in D^2$. $\square$

We write $\log^+ x = \max\{\log x, 0\}$. Note that $\log^+(ab) \leq \log^+ a + \log^+ b$ for all $a, b \geq 0$. Note also that $\log^+ x \leq x$ for all $x \geq 0$. For $\zeta = e^{i\theta} \in \mathbb{T}$, we write $|d\zeta|$ for $d\theta$.

Lemma 4.0.10. *Let $0 < p \leq \infty$. If $f \in H^p$ and $f \not\equiv 0$, then*

$$\log |f(0)| \leq \int_{\mathbb{T}} \log |f^*(\zeta)| \, |d\zeta|.$$

Proof. See Theorem 17.17 in [14]. $\square$

Theorem 4.0.11. [5, Theorem 4.1.5] *If $\{z_k\}$ is a zero set of D^2, then*

$$\int_{\mathbb{T}} \log^+ \left(\sum_{k=1}^{\infty} \frac{1 - |z_k|^2}{|\zeta - z_k|^2} \right) |d\zeta| < \infty. \tag{4.4}$$

Proof. Let $\{z_k\}$ be the zero set of a function $g \in D^2$. Write $g = Bf$ where B is the Blaschke product whose zero set is $\{z_k\}$ and f is a nowhere vanishing analytic function in the unit disk. Note that $f \in H^2$ because $g \in H^2$ and $|B^*(\zeta)| = 1$ on $\mathbb{T}$. Using Theorem 4.0.6, we find

$$\begin{aligned} D^2(Bf) &= D^2(f) + \frac{1}{2\pi} \sum_{k=1}^{\infty} \int_{\mathbb{T}} \frac{1 - |z_k|^2}{|\zeta - z_k|^2} |f^*(\zeta)|^2 |d\zeta| \\ &= D^2(f) + \frac{1}{2\pi} \int_{\mathbb{T}} \left\{ \sum_{k=1}^{\infty} \frac{1 - |z_k|^2}{|\zeta - z_k|^2} |f^*(\zeta)|^2 \right\} |d\zeta| \\ &< \infty, \end{aligned}$$

where interchanging the order of summation and integral is due to Lebesgue's monotone convergence theorem. In particular, we have

$$\int_{\mathbb{T}} \left\{ \sum_{k=1}^{\infty} \frac{1-|z_k|^2}{|\zeta - z_k|^2} |f^*(\zeta)|^2 \right\} |d\zeta| < \infty.$$

Since $\log^+ x \leq x$ for all $x \geq 0$, it follows that

$$\int_{\mathbb{T}} \log^+ \left(\sum_{k=1}^{\infty} \frac{1-|z_k|^2}{|\zeta - z_k|^2} |f^*(\zeta)|^2 \right) |d\zeta| < \infty.$$

Since $f \in H^2$ and $f \not\equiv 0$, it follows from Lemma 4.0.10 that

$$\int_{\mathbb{T}} \log^+ \frac{1}{|f^*(\zeta)|^2} |d\zeta| < \infty.$$

Thus (4.4) follows from the inequality $\log^+(ab) \leq \log^+ a + \log^+ b$ for all $a, b \geq 0$. □

We shall use the inequality $|1 - e^{i\theta}| \leq \theta$ for all $\theta \geq 0$ in the proof of Theorem 4.0.12. To prove this inequality, let $f(\theta) = \theta^2 + 2\cos\theta - 2$. Then $f(0) = 0$ and $f(\theta)$ is nondecreasing for all $\theta \geq 0$, because $0 \leq f'(\theta) = 2\theta - 2\sin\theta$ for all $\theta \geq 0$. So $0 \leq f(\theta)$ for all $\theta \geq 0$. Equivalently, $\sqrt{2(1-\cos\theta)} \leq \theta$ for all $\theta \geq 0$. Hence

$$|1 - e^{i\theta}| = |1 - \cos\theta - i\sin\theta| = \sqrt{2(1-\cos\theta)} \leq \theta$$

for all $\theta \geq 0$.

Theorem 4.0.12. [5, Theorem 4.1.6] *Let $\{\delta_n\}$ be a sequence in $(0,1)$ such that*

$$\sum_{n=1}^{\infty} \delta_n < \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} \delta_n \log(1/\delta_n) = \infty.$$

Set $r_n = 1 - \delta_n$ and $\theta_n = \sum_{k=n}^{\infty} \delta_k$ for each $n = 1, 2, \ldots$. Then $\{r_n e^{i\theta_n}\}$ is a uniqueness set for D^2.

Proof. Without loss of generality we may assume $\sum_{n=1}^{\infty} \delta_n = 2\pi$. If $\theta \in (\theta_{n+1}, \theta_n)$, then we have

$$|e^{i\theta} - r_n e^{i\theta_n}| \leq (\theta_n - \theta) + (1 - r_n) \leq 2\delta_n$$

by the inequality $|1-e^{i\alpha}| \le \alpha$ for all $\alpha \ge 0$. So we have

$$\frac{1-r_n^2}{|e^{i\theta}-r_n e^{i\theta_n}|^2} \ge \frac{\delta_n}{4\delta_n^2} = \frac{1}{4\delta_n}.$$

Using this, we deduce that

$$\begin{aligned}\int_{\theta_{n+1}}^{\theta_n} \log^+\left(\sum_{k=1}^{\infty} \frac{1-r_k^2}{|e^{i\theta}-r_k e^{i\theta_k}|^2}\right) d\theta &\ge \int_{\theta_{n+1}}^{\theta_n} \log^+\left(\frac{1-r_n^2}{|e^{i\theta}-r_n e^{i\theta_n}|^2}\right) d\theta \\ &\ge \delta_n \log^+\left(\frac{1}{4\delta_n}\right)\end{aligned}$$

for each $n = 1, 2, \ldots$. So

$$\int_0^{2\pi} \log^+\left(\sum_{k=1}^{\infty} \frac{1-r_k^2}{|e^{i\theta}-r_k e^{i\theta_k}|^2}\right) d\theta \ge \sum_{n=1}^{\infty} \delta_n \log^+\left(\frac{1}{4\delta_n}\right) = \infty.$$

Hence the sequence $\{r_n e^{i\theta_n}\}$ violates the necessary condition given in Theorem 4.0.11. Thus $\{r_n e^{i\theta_n}\}$ is a uniqueness set for D^2 as claimed. □

A sequence satisfying the hypotheses of Theorem 4.0.12 is $\delta_n = \dfrac{1}{n\log^2 n}$.

Theorem 4.0.13. [5, Theorem 3.5.1] *Let $\kappa > 0$. If $f \in D^2$, then for almost every $\zeta \in \mathbb{T}$, we have $f(z) \to f^*(\zeta)$ as $z \to \zeta$ in the region $\Omega_\kappa(\zeta)$, where*

$$\Omega_\kappa(\zeta) = \left\{ z \in \mathbb{D} : |z-\zeta| < \kappa\left(\log\frac{1}{1-|z|}\right)^{-1} \right\}.$$

Note that the function $f(x) = x\log(1/x)$ defined on $(0,1)$ takes its maximum at $x = 1/e$, which is $f(1/e) = 1/e$. Also it is nonnegative on its domain. So $0 \le f(x) < 1$ for all $x \in (0,1)$. We shall use this fact in the proof of Theorem 4.0.14

Theorem 4.0.14. [5, Theorem 4.2.2] *Let $\{r_k\}$ be a convergent sequence in $(0,1)$ such that*

$$\sum_{k=1}^{\infty}\left(\log\frac{1}{1-r_k}\right)^{-1} = \infty.$$

Then there exists a sequence $\{\theta_k\}$ such that $\{r_k e^{i\theta_k}\}$ is a uniqueness set for D^2.

Proof. Without loss of generality we may assume that $r_k \to 1^-$ as $k \to \infty$, because if $\{r_k\}$ has a limit in $[0,1)$, then the sequence $\{r_k\}$ is a uniqueness set for D^2. To see this, let $f \in D^2$ such that $f(r_k) = 0$ for all $k = 1, 2, \ldots$. Since the zeros of f has a limit in the unit disk and f is analytic, it follows that $f \equiv 0$, which proves $\{r_k\}$ is a uniqueness set for D^2. Hence assuming $r_k \to 1^-$ as $k \to \infty$ causes no harm.

Let $\{I_k\}$ be contiguous arcs in $\mathbb{T}$ such that $|I_k| = (\log(1/1-r_k))^{-1}$ for all $k = 1, 2, \ldots$, where I denotes the length of I. So $\sum_{k=1}^{\infty} |I_k| = \infty$ by hypothesis. Let $e^{i\theta_k}$ be the midpoint of I_k, and set $z_k = r_k e^{i\theta_k}$ for all $k = 1, 2, \ldots$. For each $\zeta \in \mathbb{T}$, let also

$$\Omega(\zeta) = \left\{ z \in \mathbb{D} : |z - \zeta| < 2\left(\log \frac{1}{1-|z|} \right)^{-1} \right\}.$$

Note that for each k, we have $z_k \in \Omega(\zeta)$ for all $\zeta \in I_k$, because

$$\begin{aligned} |z_k - \zeta| &\le |z_k - e^{i\theta_k}| + |e^{i\theta_k} - \zeta| \\ &\le (1 - r_k) + |I_k| \\ &\le 2\left(\log \frac{1}{1-r_k} \right)^{-1} \end{aligned}$$

if $\zeta \in I_k$. For the last inequality we use the inequality

$$(1 - r_k) < \left(\log \frac{1}{1-r_k} \right)^{-1}, \quad k = 1, 2, \ldots.$$

Note that almost every $\zeta \in \mathbb{T}$ lies in infinitely many I_k because $\sum_{k=1}^{\infty} |I_k| = \infty$. So for almost every $\zeta \in \mathbb{T}$, the set $\Omega(\zeta)$ contains infinitely many z_k. Hence for almost every $\zeta \in \mathbb{T}$, we can choose a subsequence $\{I_{n_k}\}$ of $\{I_k\}$ such that $z_{n_k} \to \zeta$ as $k \to \infty$ in $\Omega(\zeta)$. Such a sequence exists because ζ is contained in infinitely many I_k and $r_k \to 1^-$ as $k \to \infty$. To see the existence of such a sequence, suppose $\zeta = e^{i\alpha}$ for some $\alpha \in [0, 2\pi]$. As ζ lies in infinitely many I_k, we have $\zeta \in \cap_{k=1}^{\infty} I_{n_k}$. Observe that

$$|\zeta - z_{n_k}| = |e^{i\alpha} - r_{n_k} e^{\theta_{n_k}}| \le |\alpha - \theta_{n_k}| + |1 - r_{n_k}|$$

and $\theta_{n_k} \to \alpha$ as $k \to \infty$ because $e^{i\theta_{n_k}}$ is the midpoint of I_{n_k} and $\zeta \in \cap_{k=1}^{\infty} I_{n_k}$. Also we have $r_{n_k} \to 1^-$ as $k \to \infty$. Hence $z_{n_k} \to \zeta$ as $k \to \infty$ in $\Omega(\zeta)$. To show

$\{z_k\}$ is a uniqueness set for D^2, suppose that there exists an $f \in D^2$ such that $f(z_k) = 0$ for all $k = 1, 2, \ldots$. Recall that Theorem 4.0.13 assures that if $f \in D^2$, then for almost every $\zeta \in \mathbb{T}$ we have $f(z) \to f^*(\zeta)$ as $z \to \zeta$ in $\Omega(\zeta)$. So for almost every $\zeta \in \mathbb{T}$, we have $f(z_{n_k}) \to f^*(\zeta)$ as $z_{n_k} \to \zeta$ in $\Omega(\zeta)$, which shows that $f^* = 0$ a.e. on $\mathbb{T}$; consequently, $f \equiv 0$ in $\mathbb{D}$. Hence $\{z_k\}$ is a uniqueness set for D^2. □

Theorem 4.0.15 is due to Shapiro and Shields [15]. See also [5, Theorem 4.2.1]. The proof of Theorem 4.0.15 requires the use of Hilbert spaces techniques, so we omit the proof.

Theorem 4.0.15. [15, Theorem 1] *Let $\{z_k\}$ be a sequence in $\mathbb{D}$ such that $z_k \neq 0$ for all $k = 1, 2, \ldots$ and*

$$\sum_{k=1}^{\infty} \left(\log \frac{1}{1 - |z_k|} \right)^{-1} < \infty. \tag{4.5}$$

Then $\{z_k\}$ is a zero set for D^2.

The condition (4.5) is strict in the following sense. If $\sum_{k=1}^{\infty} 1/\log(1/(1-r_k)) = \infty$, then there exists a sequence $\{\theta_k\}$ such that $\{r_k e^{i\theta_k}\}$ is not a zero set of D^2. The existence of such sequence is guaranteed by Theorem 4.0.14.

Basically, Theorem 4.0.15 requires that for a sequence $\{z_k\}$ in $\mathbb{D}$ to be a zero set of D^2, it must approach the boundary $\mathbb{T}$ very fast. Even if we have $|z_k| = 1 - e^{-k}$ for all k so that the sequence $\{z_k\}$ approaches the boundary exponentially, it does not satisfy (4.5). On the other hand, $\{z_k\}$ is a zero set of H^p as it satisfies the Blaschke condition.

Let us briefly discuss what we have done so far in this chapter. Theorem 4.0.12 allows us to construct Blaschke sequences that are uniqueness sets for D^2. Hence it shows that not every zero set of a Hardy space H^p is a zero set of D^2. So the zero sets of H^p spaces differ from those of D^2. But we know that every zero set of D^2 is a zero set of the Hardy space H^p. The preceding discussion shows that the converse is not true. Another way to see this is as follows. Let $\{r_k\}$ be a sequence in $(0,1)$ such that

$$\sum_{k=1}^{\infty}\left(\log\frac{1}{1-r_k}\right)^{-1}=\infty \quad \text{and} \quad \sum_{k=1}^{\infty}(1-r_k)<\infty.$$

(For example $r_k = 1 - 1/k^2$ will do the job.) Then Theorem 4.0.14 assures that there exists a sequence $\{\theta_k\}$ such that $\{r_k e^{i\theta_k}\}$ is not a zero set of D^2. However $\{r_k e^{i\theta_k}\}$ is a zero set of Hardy space H^p as it satisfies the Blaschke condition. Oddly enough, the sequence $\{r_k\}$ is a zero set of D^2 by Theorem 4.0.3. Hence we also deduce that the arguments of the sequences play an important role in determining zero sets of D^2. Nevertheless, if we have a sequence $\{w_k\}$ in the unit disk such that $|w_k| = 1 - e^{-k^2}$ for all k, then Theorem 4.0.15 guarantees that $\{w_k\}$ is zero set for D^2. Theorem 4.0.15 is a useful result, but its assumption of how fast the sequence should approach the boundary is very strong.

Chapter 5

Zero sets of Dirichlet-type spaces

Recall that a function f belongs to $D^{p,t}_\alpha$ if $f^{(t+1)} \in A^p_\alpha$. We abbreviate $D^{p,0}_\alpha$ to D^p_α when $t = 0$.

Lemma 5.0.1. *Let $0 < p < \infty$ and $t \in \mathbb{N}$. If $f \in H(\mathbb{D})$ such that $f^{(t)}(0) = 0$, then*

$$M_p(r, f^{(t)}) \leq \int_0^r M_p(s, f^{(t+1)})\, ds$$

for all $r \in (0,1)$.

Proof. We invoke Minkowski's integral inequality and obtain

$$\begin{aligned} M_p(r, f^{(t)}) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f^{(t)}(re^{i\theta})|^p \, d\theta \right\}^{1/p} &= \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| e^{-i\theta} \int_0^r f^{(t+1)}(se^{i\theta})\, ds \right|^p d\theta \right\}^{1/p} \\ &\leq \int_0^r \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f^{(t+1)}(se^{i\theta})|^p \, d\theta \right\}^{1/p} ds \\ &= \int_0^r M_p(s, f^{(t+1)})\, ds \end{aligned}$$

for all $r \in (0,1)$. □

Proposition 5.0.2. *Let $0 < p < \infty$, $-1 < \alpha < \infty$ and $t \in \mathbb{N}$. Then we have the following statements.*

(i) *If $f \in D^p_{p-1}$, then $M_p(r, f) = o\left(\log \frac{1}{1-r}\right)$, as $r \to 1^-$.*

(ii) If $\alpha < p-1$, then $D^p_\alpha \subseteq H^p$.

(iii) If $\alpha \le p-1$ and $1 \le t$, then $D^{p,t}_\alpha \subseteq H^p$.

Proof. Let $f \in D^{p,t}_\alpha$. Then

$$\int_r^1 M_p^p(s, f^{(t+1)})(1-s)^\alpha \, ds \to 0, \quad r \to 1^-, \tag{5.1}$$

because $f^{(t+1)} \in A^p_\alpha$. Since $M_p^p(r, f^{(t+1)})$ is an increasing function of r, for all $r \in (0,1)$ we have

$$M_p^p(r, f^{(t+1)}) \int_r^1 (1-s)^\alpha \, ds \le \int_r^1 M_p^p(s, f^{(t+1)})(1-s)^\alpha \, ds$$

which together with (5.1) imply

$$M_p(r, f^{(t+1)}) = o\Big((1-r)^{-(\alpha+1)/p}\Big), \quad r \to 1^-. \tag{5.2}$$

In (5.2) take $t = 0$ and $\alpha = p-1$, then (5.2) becomes

$$M_p(r, f') = o\Big(\frac{1}{1-r}\Big), \quad r \to 1^-.$$

Hence for given $\epsilon > 0$, we can choose r_0 such that

$$M_p(r, f') \le \epsilon \frac{1}{1-r}, \quad r_0 < r < 1. \tag{5.3}$$

Without loss of generality we may assume $f(0) = 0$. By using Lemma 5.0.1 and (5.3), we find

$$\begin{aligned} M_p(r, f) &\le \int_0^{r_0} M_p(s, f') \, ds + \int_{r_0}^r M_p(s, f') \, ds \\ &= C + \epsilon \int_{r_0}^r \frac{1}{1-s} \, ds \\ &\le C + \epsilon \log \frac{1}{1-r}. \end{aligned}$$

Now choose r_1 such that

$$C \le \epsilon \log \frac{1}{1-r}, \quad r_1 < r < 1$$

and set $r_2 = \max\{r_0, r_1\}$. Then we have

$$M_p(r, f) \leq 2\epsilon \log \frac{1}{1-r}, \quad r_2 < r < 1.$$

Since ϵ is arbitrary, it follows that

$$M_p(r, f) = o\left(\log \frac{1}{1-r} \right), \quad r \to 1^-.$$

This proves the first statement.

To prove the second statement, let $\alpha < p-1$ and take $f \in D^p_\alpha$. Without loss of generality we may assume $f(0) = 0$. We apply Lemma 5.0.1 to f and obtain

$$M_p(r, f) \leq \int_0^r M_p(s, f^{'})\, ds$$

for all $r \in (0, 1)$. So it suffices to show that

$$\lim_{r \to 1^-} \int_r^1 M_p(s, f^{'})\, ds = 0$$

in order to conclude that $f \in H^p$. We use (5.2) to show this. Take $t = 0$ in (5.2) and for given $\epsilon > 0$ choose r such that

$$M_p(s, f^{'}) \leq \epsilon\, (1-s)^{-(\alpha+1)/p}, \quad r < s < 1.$$

Then

$$\int_r^1 M_p(s, f^{'})\, ds = C\epsilon(1-r)^{1-(\alpha+1)/p} \leq C\epsilon.$$

Since ϵ is arbitrary, it follows that

$$\lim_{r \to 1^-} \int_r^1 M_p(s, f^{'})\, ds = 0,$$

which implies $M_p(r, f) < \infty$ as $r \to 1^-$. So $f \in H^p$.

Now we prove the last statment. Let $\alpha \leq p-1$ and $1 \leq t$ and take $f \in D^{p,t}_\alpha$. Without loss of generality we may assume $f(0) = f^{'}(0) = \cdots = f^{(t)}(0) = 0$. Then applying $t+1$ times Lemma 5.0.1, we obtain

$$\begin{aligned} M_p(r_0, f) &\leq \int_0^{r_0} M_p(r_1, f^{'})\, dr_1 \\ &\leq \int_0^{r_0} \int_0^{r_1} M_p(r_2, f^{''})\, dr_2\, dr_1 \\ &\leq \int_0^{r_0} \int_0^{r_1} \cdots \int_0^{r_t} M_p(r_{t+1}, f^{(t+1)})\, dr_{t+1} \ldots dr_1 \end{aligned}$$

for all $r_0 \in (0,1)$. Now using (5.2) and the arguments used in the proof of second statement and bearing in mind $1 \leq t$, we deduce that $M_p(r_0, f) < \infty$ as $r_0 \to 1^-$. So $f \in H^p$ and the proof is completed. $\square$

Littlewood and Paley show in [10] that H^p lies in D^p_{p-1} if $2 \leq p < \infty$ and D^p_{p-1} lies in H^p if $0 < p \leq 2$. Namely, the space D^p_{p-1} is "big enough" to contain the Hardy space H^p when $2 \leq p < \infty$. Hence trivially, every Blaschke sequence is a zero set of D^p_{p-1} in this case. However the converse is not true; Theorem 5.0.12 shows the existence of the zero sets of D^p_{p-1} that does not satisfy the Blaschke condition. Instead we have a subsitute Blaschke condition for the Dirichlet-type spaces; see Theorem 5.0.10. Furthermore, Theorem 5.0.10 is strict; the exponent $-1-\epsilon$ cannot be replaced by -1 as Theorem 5.0.12 implies.

Proposition 5.0.2 shows how parameters p, α and t are related to the "size" of $D^{p,t}_\alpha$ spaces. A little change in α and t make $D^{p,t}_\alpha$ so small that it lies in the Hardy space H^p, in which case describing the zero sets of $D^{p,t}_\alpha$ to distinguish them from those of H^p is not easy. Because even for the Dirichlet space D^2, one must make use of the Hilbert space techniques to describe the zero sets, which are not ripe enough to give a complete description of the zero sets of D^2. Hence Proposition 5.0.2 suggests that distinguishing the zero sets of $D^{p,t}_\alpha$ from those of H^p when $D^{p,t}_\alpha$ lies in H^p is not possible with the tools presented in this thesis.

Notation 5.0.3. *For $n = 0, 1, \ldots,$ $I(n)$ is the set of integers k such that $2^n \leq k < 2^{n+1}$.*

Lemma 5.0.4. [11, Theorem 1] *Let $0 < p < \infty$ and $0 < \alpha < \infty$. Let $\{a_n\}$ be a sequence of nonnegative numbers and $t_n = \sum_{k \in I(n)} a_n$ for $n = 0, 1, \ldots$. Let also $f(x) = \sum_{n=1}^{\infty} a_n x^{n-1}$ for all $x \in (0,1)$. Then there exists a constant $K(p,\alpha) > 0$ such that*

$$K^{-1} \sum_{n=0}^{\infty} 2^{-n\alpha}\, t_n^p \leq \int_0^1 (1-x)^{\alpha-1} f(x)^p \, dx \leq K \sum_{n=0}^{\infty} 2^{-n\alpha}\, t_n^p.$$

Theorem 5.0.5. [7, Theorem 1.2] *Let $f \in H(\mathbb{D})$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be its Taylor series expansion around $z = 0$. If $2 \leq p < \infty$ and $f \in D^p_{p-1}$, then*

$$\sum_{n=0}^{\infty} \left\{ \sum_{k \in I(n)} |a_k|^2 \right\}^{p/2} < \infty.$$

Proof. Note that $k \asymp 2^n$ if $k \in I(n)$. Hence

$$\sum_{n=0}^{\infty}\bigg\{\sum_{k\in I(n)}|a_k|^2\bigg\}^{p/2} \le \sum_{n=0}^{\infty}\bigg\{\sum_{k\in I(n)}2^{-2n}k^2|a_k|^2\bigg\}^{p/2}$$
$$= \sum_{n=0}^{\infty}2^{-np}\bigg\{\sum_{k\in I(n)}k^2|a_k|^2\bigg\}^{p/2}.$$

Then Lemma 5.0.4 and a change of variable imply that

$$\sum_{n=0}^{\infty}2^{-np}\bigg\{\sum_{k\in I(n)}k^2|a_k|^2\bigg\}^{p/2} \le C\int_0^1(1-t)^{p-1}\bigg\{\sum_{n=1}^{\infty}n^2|a_n|^2t^{n-1}\bigg\}^{p/2}dt$$
$$\le C\int_0^1(1-r^2)^{p-1}\bigg\{\sum_{n=1}^{\infty}n^2|a_n|^2r^{2n-2}\bigg\}^{p/2}dt.$$

Since $M_2(r,f') \le M_p(r,f')$ for $2 \le p < \infty$, we find

$$\int_0^1(1-r^2)^{p-1}\bigg\{\sum_{n=1}^{\infty}n^2|a_n|^2r^{2n-2}\bigg\}^{p/2}dr = \int_0^1(1-r^2)^{p-1}M_2(r,f')^p\,dr$$
$$\le \int_0^1(1-r^2)^{p-1}M_p(r,f')^p\,dr$$

which is finite since $f \in D^p_{p-1}$, where we use Parseval's formula for each $r \in (0,1)$ in the equality. This completes the proof. □

Theorem 5.0.6. [7, Theorem 1.4] *If* $2 < p < \infty$ *and* $f \in D^p_{p-1}$, *then*

$$M_2(r,f) = O\bigg(\Big(\log\frac{1}{1-r}\Big)^{1/2-1/p}\bigg), \quad r \to 1^-.$$

Proof. Without loss of generality we may assume $f(0)=0$. Let $f(z)=\sum_{k=1}^{\infty}a_kz^k$ be the Taylor expansion of f around $z=0$. Applying Parseval's formula for each $r \in (0,1)$, we find

$$M_2^2(r,f) = \sum_{k=1}^{\infty}|a_k|^2r^{2k}.$$

Hölder's inequality with the exponents $p/2$ and $p/(p-2)$ and Theorem 5.0.5 imply

$$\begin{aligned}\sum_{k=1}^{\infty}|a_k|^2 r^{2k} = \sum_{n=0}^{\infty}\sum_{k\in I(n)}|a_k|^2 r^{2k} &\leq \sum_{n=0}^{\infty} r^{2^{n+1}}\Big\{\sum_{k\in I(n)}|a_k|^2\Big\}\\ &\leq \Big\{\sum_{n=0}^{\infty}\Big\{\sum_{k\in I(n)}|a_k|^2\Big\}^{p/2}\Big\}^{2/p}\Big\{\sum_{n=0}^{\infty} r^{2^{n+1}p/(p-2)}\Big\}^{1-2/p}\\ &\leq C\Big(\log\frac{1}{1-r}\Big)^{1-2/p}.\end{aligned}$$

It follows that

$$M_2(r,f) = O\bigg(\Big(\log\frac{1}{1-r}\Big)^{1/2-1/p}\bigg), \quad r\to 1^-$$

as desired. □

Theorem 5.0.7. [7, Theorem 1.6] *Let $2 < p < \infty$ and $f \in D^p_{p-1}$ with $f(0) \neq 0$. Let $\{z_k\}$ be the zero set of f. Then*

$$\prod_{k=1}^{n}\frac{1}{|z_k|} = O\big((\log n)^{1/2-1/p}\big), \quad n\to\infty.$$

Proof. Taking $p = 2$ in Lemma 3.1.6, we find

$$|f(0)|\prod_{k=1}^{n}\frac{r}{|z_k|} \leq M_2(r,f)$$

for all $r \in (0,1)$ and for all positive integers n. By Theorem 5.0.6, there exists $C > 0$ such that

$$|f(0)|\prod_{k=1}^{n}\frac{r}{|z_k|} \leq C\Big(\log\frac{1}{1-r}\Big)^{1/2-1/p}$$

for large enough $r \in (0,1)$ for $2 < p < \infty$, and for all positive integers n. Let $r = 1 - 1/n$. Then

$$\prod_{k=1}^{n}\frac{1}{|z_k|} \leq C(\log n)^{1/2-1/p}$$

for sufficiently large n. Thus

$$\prod_{k=1}^{n}\frac{1}{|z_k|} = O\big((\log n)^{1/2-1/p}\big), \quad n\to\infty$$

as required. □

Proposition 5.0.8. [7, Proposition 4.1] *Let* $2 < p < \infty$ *and* $f \in D^p_{p-1}$ *with* $f(0) \neq 0$. *Then*

$$N(r) = O\left(\log\log\frac{1}{1-r}\right), \quad r \to 1^-$$

and

$$n(r) = O\left(\frac{1}{1-r}\log\log\frac{1}{1-r}\right), \quad r \to 1^-.$$

Proof. Without loss of generality we may assume that $f(0) = 1$. Then we have

$$N(r) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta$$

for all $r \in (0,1)$ by Jensen's formula and Proposition 2.0.21. Multiplying the equality above by p, exponentiating and applying the agm inequality with respect to unit measure $\frac{1}{2\pi}\, d\theta$, we obtain

$$e^{pN(r)} \leq \int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta = M_p^p(r,f).$$

Hence we deduce that

$$e^{pN(r)} \leq \left(\log\frac{1}{1-r}\right)^p$$

for large enough $r \in (0,1)$ by the first statment in Proposition 5.0.2. Thus

$$N(r) = O\left(\log\log\frac{1}{1-r}\right), \quad r \to 1^-.$$

For $n(r)$, observe that we have

$$n(r^2)(r^2 - r) \leq \int_{r^2}^r n(t)\, dt \leq \int_0^r n(t)\, dt \leq N(r) = \int_0^r \frac{n(t)}{t}\, dt$$

for all $r \in (0,1)$. Then using the estimate above for $N(r)$, we conclude that

$$n(r) = O\left(\frac{1}{1-r}\log\log\frac{1}{1-r}\right), \quad r \to 1^-$$

as in the proof of Proposition 3.1.2. □

Theorem 5.0.9. *Let* $2 < p < \infty$. *Let also* $f \in D^p_{p-1}$ *and* $\{z_k\}$ *be the zero set of* f. *Then*

$$\sum_{k=1}^n (1 - |z_k|) = O(\log\log n), \quad n \to \infty.$$

Proof. Without loss of generality we may assume $f(0) \neq 0$. We argue as in the proof of Theorem 3.1.3 and use Proposition 5.0.8 to obtain

$$\sum_{k=1}^{n}(1-|z_k|) \leq C \log\log n$$

as $n \to \infty$. $\square$

A substitute Blaschke condition for D^p_{p-1} spaces is in Theorem 5.0.10. Note that Theorem 5.0.12 implies that the exponent $-1-\epsilon$ is sharp, that is, it cannot be replaced by -1.

Theorem 5.0.10. [7, Theorem 4.2] *Let $2 < p < \infty$ and $\epsilon > 0$. Let also $f \in D^p_{p-1}$ with $f(0) \neq 0$ and $\{z_k\}$ be the zero set of f. Then*

$$\sum_{|z_k|>1-1/e}(1-|z_k|)\left(\log\log\frac{1}{1-|z_k|}\right)^{-1-\epsilon} < \infty.$$

Proof. Without loss of generality we may assume that f has no zeros in the disk $|z| \leq c$ where $1-1/e < c < 1$. The infinite sum in the statement of the theorem is convergent if and only if

$$I = \int_c^1 (1-r)\left(\log\log\frac{1}{1-r}\right)^{-1-\epsilon} dn(r) < \infty.$$

An integration by parts with

$$u = (1-r)\left(\log\log\frac{1}{1-r}\right)^{-1-\epsilon} \quad \text{and} \quad dv = dn(r)$$

gives

$$I = (1-r)\left(\log\log\frac{1}{1-r}\right)^{-1-\epsilon} n(r)\Bigg|_c^1 + I_1 + I_2,$$

where

$$I_1 = \int_c^1 \left(\log\log\frac{1}{1-r}\right)^{-1-\epsilon} n(r)\,dr$$

and

$$I_2 = (1+\epsilon)\int_c^1 \left(\log\log\frac{1}{1-r}\right)^{-2-\epsilon}\left(\log\frac{1}{1-r}\right)^{-1} n(r)\,dr.$$

Now using the estimate of $n(r)$ in Proposition 5.0.8, we deduce that

$$(1-r)\left(\log\log\frac{1}{1-r}\right)^{-1-\epsilon} n(r)\Bigg|_c^1$$

is finite. Next we show that I_2 is convergent. There exists a constant $C > 0$ and $r_0 \in (c, 1)$ such that

$$n(r) \le C\frac{1}{1-r}\log\log\frac{1}{1-r} \tag{5.4}$$

for all $r \in (r_0, 1)$ by Proposition 5.0.8. Clearly, the integrand of I_2 is bounded for all $r \in (c, r_0)$ as $r_0 < 1$ and $1 - 1/e < c$. Now (5.4) and a change of variables with $u = \log\log\frac{1}{1-r}$ imply

$$\begin{aligned} I_3 &\le C\int_{r_0}^{1}\left(\log\log\frac{1}{1-r}\right)^{-1-\epsilon}\left(\log\frac{1}{1-r}\right)^{-1}\frac{1}{1-r}\,dr \\ &= C\int_{u_0}^{\infty} u^{-1-\epsilon}\,du \\ &< \infty, \end{aligned}$$

where $u_0 = \log\log\frac{1}{1-r_0}$. Hence I_2 is convergent. It remains to show that I_1 is convergent. We employ another integration by parts to I_1 with

$$u = r\left(\log\log\frac{1}{1-r}\right)^{-1-\epsilon} \quad \text{and} \quad dv = \frac{n(r)}{r}dr$$

to obtain

$$I_1 = r\left(\log\log\frac{1}{1-r}\right)^{-1-\epsilon} N(r)\Bigg|_c^1 + J_1 + J_2,$$

where

$$J_1 = -\int_c^1\left(\log\log\frac{1}{1-r}\right)^{-1-\epsilon} N(r)\,dr$$

and

$$J_2 = (1+\epsilon)\int_c^1 \frac{r}{1-r}\left(\log\log\frac{1}{1-r}\right)^{-2-\epsilon}\left(\log\frac{1}{1-r}\right)^{-1} N(r)\,dr.$$

The estimation of $N(r)$ in Proposition 5.0.8 implies

$$\lim_{r\to 1^-}\left\{r\left(\log\log\frac{1}{1-r}\right)^{-1-\epsilon} N(r)\right\} = 0$$

as $\epsilon > 0$. The estimation of $N(r)$ in Proposition 5.0.8 also assures that J_1 and J_2 are convergent, so I_1 is convergent and the proof is completed. □

Proposition 5.0.11. [7, Proposition A] *Let $0 < p < \infty$. If $f \in H(\mathbb{D})$ given by a power series with Hadamard gaps, that is, $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ with $n_{k+1} \geq \lambda n_k$ for all k and $\lambda > 1$, then $f \in D^p_{p-1}$ if and only if $\sum_{k=1}^{\infty} |a_k|^p < \infty$.*

Theorem 5.0.12. [7, Theorem 4.3] *If $2 < p < \infty$, then there exists a function $f \in D^p_{p-1}$ with $f(0) \neq 0$ whose zero set $\{z_k\}$ satisfies*

$$\sum_{|z_k|>1-1/e} (1 - |z_k|)\left(\log\log \frac{1}{1-|z_k|}\right)^{-1} = \infty.$$

We introduce the setting of Theorem 5.0.13 given in [13, Theorem 1]. Let $\omega = [0,1]^N$ and $\omega_1, \omega_2, \ldots$ be the coordinate functions $\omega_j : \Omega \to [0,1]$. Let $d\omega$ deonte the product measure on Ω that is derived from the Lebesgue measure on $[0,1]$. Note that $\omega_1, \omega_2, \ldots$ are independent and identically distributed random variables uniformly distributed on $[0,1]$. Note also that $\{e^{2\pi i \omega_j}\}_{j=1}^{\infty}$ is an orthonormal set in $L^2(\Omega)$. Hence if $\sum_{j=1}^{\infty} |a_j|^2 < \infty$, then $\sum_{j=1}^{\infty} a_j e^{2\pi i \omega_j}$ is a well-defined element of $L^2(\Omega)$ with the usual L^2-norm.

Theorem 5.0.13. [13, Theorem 1] *There exists $C > 0$ such that for any sequence of complex numbers $\{a_j\}$ with $\sum_{j=1}^{\infty} |a_j|^2 < \infty$, we have*

$$\exp\left(\int_{\Omega} \log\left|\sum_{j=1}^{\infty} a_j e^{2\pi \omega_j}\right| d\omega\right) \geq C\left\{\sum_{j=1}^{\infty} |a_j|^2\right\}^{1/2}.$$

Theorem 5.0.14. [7, Theorem 1.5] *If $2 < p < \infty$ and $0 < \beta < 1/2 - 1/p$, then there exists a function $f \in D^p_{p-1}$ such that*

$$\exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta\right) \neq o\left(\left(\log\frac{1}{1-r}\right)^{\beta}\right), \quad r \to 1^-. \tag{5.5}$$

Proof. Let $2 < p < \infty$ and $0 < \beta < 1/2 - 1/p$. Set $\epsilon = 1/2 - 1/p - \beta$. Note that $\epsilon > 0$. Define a sequence $\{b_j\}$ as $b_j = j^{-1/p-\epsilon}$ for $j = 1, 2, \ldots$. Then for every $\omega \in \Omega$, we define

$$f_\omega(z) = \sum_{j=1}^{\infty} b_j e^{2\pi \omega_j} z^{2^j} = \sum_{k=1}^{\infty} a_{k,\omega} z^k, \quad z \in \mathbb{D}.$$

Since $\sum_{j=1}^{\infty} |b_j|^p < \infty$, we have $f_\omega \in D_{p-1}^p$ for every $\omega \in \Omega$ by Proposition 5.0.11. Now suppose that (5.5) is false. Then there exists a measurable set $E \subset \Omega$ with positive measure such that for all $\omega \in E$, we have

$$\exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|f_\omega(re^{i\theta})|\,d\theta\right) = o\left(\left(\log\frac{1}{1-r}\right)^\beta\right), \quad r \to 1^-;$$

equivalently,

$$\lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} \log\left(\frac{|f_\omega(re^{i\theta})|}{\big(\log(1/(1-r))\big)^\beta}\right) d\theta = -\infty, \quad \omega \in E. \tag{5.6}$$

On the other hand,

$$\begin{aligned}\left\{\sum_{j=1}^N |b_j|^2\right\}^{1/2} &= \left\{\sum_{j=1}^N \frac{1}{j^{2/p+2\epsilon}}\right\}^{1/2} \\ &\asymp \left\{\int_1^N \frac{1}{x^{2/p+2\epsilon}}\,dx\right\}^{1/2} \asymp N^{1/2-1/p-\epsilon}, \quad N \to \infty.\end{aligned}$$

Hence there exists $C > 0$ and $N_0 > 0$ such that

$$\left\{\sum_{k=1}^N |a_{k,\omega}|^2\right\}^{1/2} \le C(\log N)^{1/2-1/p-\epsilon}, \quad N \ge N_0.$$

This implies that there exists $r_0 \in (0,1)$ such that

$$M_2(r, f_\omega) \le C\left(\log\frac{1}{1-r}\right)^{1/2-1/p-\epsilon}, \quad r_0 < r < 1, \quad \omega \in \Omega.$$

From this we deduce that

$$\exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|f_\omega(re^{i\theta})|\,d\theta\right) \le C\left(\log\frac{1}{1-r}\right)^{1/2-1/p-\epsilon}, \quad r_0 < r < 1, \quad \omega \in \Omega,$$

which implies that there exists $C > 0$ such that

$$\int_0^{2\pi} \log\left(\frac{|f_\omega(re^{i\theta})|}{\big(\log(1/(1-r))\big)^\beta}\right) d\theta \le C, \quad r_0 < r < 1, \quad \omega \in \Omega. \tag{5.7}$$

Since E has positive measure, (5.6) and (5.7) imply

$$\lim_{r\to 1^-} \left\{\int_\Omega \left\{\int_0^{2\pi} \log\left(\frac{|f_\omega(re^{i\theta})|}{\big(\log(1/(1-r))\big)^\beta}\right) d\theta\right\} d\omega\right\} = -\infty. \tag{5.8}$$

Let $\Omega_N = [0,1]^N$ and m_N be the Lebesgue measure on Ω_N. Note that for all N, we have

$$\int_{\Omega_N} \log|f_\omega(re^{i\theta})|\, dm_N(\omega)$$

$$= \int_0^1 \cdots \int_0^1 \log\left|\sum_{j=1}^{N} b_j r^{2^j} e^{i(2\pi\omega_j + 2^j\theta)} + \sum_{j=N}^{\infty} b_j r^{2^j} e^{i(2\pi\omega_j+2^j\theta)}\right| d\omega_1 d\omega_2 \cdots d\omega_N$$

$$= \int_0^1 \cdots \int_0^1 \log\left|\sum_{j=1}^{N} b_j r^{2^j} e^{i2\pi\omega_j} + \sum_{j=N}^{\infty} b_j r^{2^j} e^{i(2\pi\omega_j+2^j\theta)}\right| d\omega_1 d\omega_2 \cdots d\omega_N,$$

where the last equality holds almost everywhere. Since the above is true for arbitrary N, it follows from letting $N \to \infty$ that

$$\int_{\Omega} \log|f_\omega(re^{i\theta})|\, d\omega$$

is independent of θ. Then we use (5.8) and Fubini's theorem to deduce

$$\lim_{r\to 1^-} \left\{ \int_{\Omega} \log\left(\frac{|f_\omega(r)|}{\big(\log(1/(1-r))\big)^{\beta}} \right) d\omega \right\} = -\infty. \tag{5.9}$$

However if we set $r_n = 1 - 1/2^n$, $n = 1, 2, \ldots$, then Theorem 5.0.13 and the inequality

$$e^{-1} \le r_n^{2^n} \le r_n^{2^j}, \quad 1 \le j \le n$$

give

$$\begin{aligned}
\exp\left(\int_{\Omega} \log|f_\omega(r_n)|\, d\omega \right) &= \exp\left(\int_{\Omega} \log\left| \sum_{j=1}^{\infty} b_j e^{2\pi\omega_j} \right| d\omega \right) \\
&\ge C \left\{ \sum_{j=1}^{\infty} |b_j|^2 \{ r_n^{2^j} \}^2 \right\}^{1/2} \\
&= C \left\{ \sum_{j=1}^{n} \frac{1}{j^{2/p+2\epsilon}} \right\}^{1/2} \\
&\ge C n^{1/2-1/p-\epsilon} \ge C \left(\log \frac{1}{1-r_n} \right)^{\beta}
\end{aligned}$$

for all sufficiently large n. This implies

$$\int_{\Omega} \log\left(\frac{|f_\omega(r_n)|}{\big(\log(1/(1-r_n))\big)^{\beta}} \right) d\omega \ge \log C$$

for all sufficiently large n. But this contradicts (5.9). Hence (5.5) is true and the proof is completed. □

Theorem 5.0.15. [7, Theorem 1.7] *Let $2 < p < \infty$ and $0 < \beta < 1/2 - 1/p$. Then there exists a function $f \in D^p_{p-1}$ with $f(0) \neq 0$ such that if $\{z_k\}$ is the zero set of f, then*

$$\prod_{k=1}^{n} \frac{1}{|z_k|} \neq o\big((\log n)^\beta\big), \quad n \to \infty.$$

Proof. Let p and β be as in hypothesis. Take $f \in D^p_{p-1}$ with $f(0) \neq 0$ such that

$$\exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta\right) \neq o\left(\left(\log\frac{1}{1-r}\right)^\beta\right), \quad r \to 1^-. \tag{5.10}$$

Existence of such a function is guaranteed by Theorem 5.0.14. Using (5.10), we can choose a sequence $\{r_j\}_{j=1}^\infty \subset (0,1)$ such that $r_j \uparrow 1^-$ and

$$\exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|f(r_j e^{i\theta})|\,d\theta\right) \geq C\bigg(\log\frac{1}{1-r_j}\bigg)^\beta, \quad j = 1, 2, \ldots,$$

where C is a positive constant independent of j. Using Jensen's formula and above, we deduce that

$$|f(0)| \prod_{k=1}^{n(r_j)} \frac{r_j}{|z_k|} \geq C\bigg(\log\frac{1}{1-r_j}\bigg)^\beta, \quad j = 1, 2, \ldots, \tag{5.11}$$

where $n(r)$ is the number of zeros of f in the disk $|z| < r$. Here observe that $n(r_j) \to \infty$ as $j \to \infty$. On the other hand, Proposition 5.0.8 implies that there exists $C > 0$ such that

$$n(r) \leq C\frac{1}{1-r}\log\log\frac{1}{1-r}, \quad r \to 1^-.$$

This implies

$$\log n(r) \leq C \log\frac{1}{1-r}, \quad r \to 1^-. \tag{5.12}$$

Using (5.11) and (5.12), we conclude that there exists $N \in \mathbb{N}$ and a positive constant $C > 0$ such that

$$|f(0)| \prod_{k=1}^{n(r_j)} \frac{r_j}{|z_k|} \geq C\big(\log n(r_j)\big)^\beta, \quad \forall j \geq N,$$

which finishes the proof. □

Chapter 6

Random zero sets for Bergman spaces

This chapter is devoted to the proof of Theorem 6.0.7, which describes a sufficient condition on a sequence in the unit disk to be a zero set of A^p_α. Our reference is [12].

Notation 6.0.1. *For a sequence $\{z_k\}$ in $\mathbb{D}$, we set*

$$\phi(r, \{z_k\}) = \sum_{|z_k| \le r} (1 - |z_k|), \quad 0 \le r < 1.$$

If there is no element of $\{z_k\}$ such that $|z_k| \le r$, then we set $\phi(r, \{z_k\}) = 0$. We abbreviate $\phi(r, \{z_k\})$ to $\phi(r)$ if the context is clear.

Notation 6.0.2. *For $\epsilon > 0$, we set*

$$h_\epsilon(z) = \left(\log \frac{1}{1 - |z|} \right)^{1+\epsilon}, \quad z \in \mathbb{D}.$$

We abbreviate $h_\epsilon(z)$ to $h(z)$ if the context is clear.

The following three lemmas are used in the proof of Theorem 6.0.6.

Lemma 6.0.3. *Let $0 < p < \infty$ and $-1 < \alpha < \infty$. Let also $\epsilon > 0$. Suppose that $\{z_k\}$ is a sequence in $\mathbb{D}$ with $z_k \neq 0$ and satisfies*

$$\int_0^1 e^{p\phi(r)}\left(\log\frac{1}{1-r}\right)^{(1+\alpha)(1+\epsilon)}(1-r)^\alpha\,dr < \infty.$$

Then

$$\sum_{k=0}^{\infty}(1-|z_k|)h(z_k)^{-1} < \infty.$$

Proof. Observe that the function

$$f(r) = e^{p\phi(r)}\left(\log\frac{1}{1-r}\right)^{(1+\alpha)(1+\epsilon)}$$

is increasing for $r \in (0,1)$. So we have

$$\begin{aligned}\frac{1}{1+\alpha}(1-r)^{1+\alpha}e^{p\phi(r)} &< \frac{1}{1+\alpha}(1-r)^{1+\alpha}f(r)\\ &= f(r)\int_r^1(1-t)^\alpha\,dt\\ &< \int_r^1 f(t)(1-t)^\alpha\,dt\\ &< \infty\end{aligned}$$

for all $r \in (1-1/e, 1)$. So

$$\phi(r) < C\log\frac{1}{1-r}, \qquad r \in (r_0, 1), \tag{6.1}$$

where $1-1/e < r_0 < 1$. Note that

$$\sum_{k=0}^{\infty}(1-|z_k|)h(z_k)^{-1} = \int_0^1\left(\log\frac{1}{1-r}\right)^{-1-\epsilon}d\phi(r).$$

By using the method of integration by parts with $u = (\log(1/(1-r)))^{-1-\epsilon}$ and $dv = d\phi(r)$, we obtain that

$$\begin{aligned}\int_0^1\left(\log\frac{1}{1-r}\right)^{-1-\epsilon}d\phi(r) &= \phi(r)\left(\log\frac{1}{1-r}\right)^{-1-\epsilon}\Bigg|_0^1\\ &\quad + (1+\epsilon)\int_0^1\frac{1}{1-r}\left(\log\frac{1}{1-r}\right)^{-2-\epsilon}\phi(r)\,dr.\end{aligned}$$

Here note that $\phi(r) = 0$ for all $r \in (0, |z_1|)$. Using (6.1) and bearing in mind that $\epsilon > 0$, we obtain

$$\phi(r)\left(\log\frac{1}{1-r}\right)^{-1-\epsilon}\Bigg|_0^1 = 0.$$

Using (6.1) one more time, we obtain that

$$\begin{aligned}\int_0^1 \left(\log\frac{1}{1-r}\right)^{-1-\epsilon} d\phi(r) &= (1+\epsilon)\int_0^1 \frac{1}{1-r}\left(\log\frac{1}{1-r}\right)^{-2-\epsilon}\phi(r)\,dr\\ &\leq C\int_{r_0}^1 \frac{1}{1-r}\left(\log\frac{1}{1-r}\right)^{-1-\epsilon} dr,\end{aligned}$$

which is finite because $\epsilon > 0$. $\square$

Lemma 6.0.4. *Suppose* $1 - 1/e < |\beta| < 1$ *and*

$$B(z) = \frac{|\beta|}{\beta}\frac{\beta - z}{1 - \overline{\beta}z}$$

and that

$$S(z) = \exp\left((1-|\beta|)\left(\frac{\xi+\lambda z}{\xi-\lambda z}\right)\right),$$

where $\xi = \beta/|\beta|$ *and* $\lambda \in (0,1)$ *satisfies the equation* $1 - \lambda = (1-|\beta|)h(\beta)$. *Then for all* $z \in \mathbb{D}$ *such that*

$$\frac{(1-|\beta|)h(\beta)}{1-z} < 1/4$$

we have

$$|\log B(z) + \log S(z)| \leq C\frac{(1-|\beta|)^2 h(\beta)}{|\xi - \lambda z|^2},$$

where $\log B(z)$ *is the branch defined for* $|z| < |\beta|$ *satisfying* $\log|\beta| < 0$ *and* C *is a positive constant independent of* z.

Proof. Define

$$S_1(z) = \exp\left(\left(\frac{\xi+\lambda z}{\xi-\lambda z}\right)\log\frac{1}{|\beta|}\right).$$

Recall that the Taylor expansion of $\log(1-x)$ around $x = 0$ is

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots, \quad 0 < x < 1;$$

equivalently,

$$\log(x) = -(1-x) - \frac{(1-x)^2}{2} - \frac{(1-x)^3}{3} - \cdots, \quad 0 < x < 1.$$

Using the formula above and bearing in mind $1 - 1/e < |\beta| < 1$, we obtain

$$\begin{aligned} |1 - |\beta| + \log|\beta|| &= \left| -\frac{(1-|\beta|)^2}{2} - \frac{(1-|\beta|)^3}{3} - \cdots \right| \\ &\leq \frac{(1-|\beta|)^2}{2} + \frac{(1-|\beta|)^3}{3} + \cdots \\ &\leq C\frac{(1-|\beta|)^2}{2}, \end{aligned}$$

where C is a positive constant independent of $|\beta|$. Hence we have

$$|\log S(z) - \log S_1(z)| = |1-|\beta|+\log|\beta|| \left| \frac{\xi + \lambda z}{\xi - \lambda z} \right| \leq C\frac{(1-|\beta|)^2}{|\xi - \lambda z|} \leq C\frac{(1-|\beta|)^2 h(\beta)}{|\xi - \lambda z|^2},$$

where the last inequality follows from $1/2 < h(\beta)/|\xi - \lambda z|$. So it suffices to estimate $|\log B(z) + \log S_1(z)|$ to finish the proof. An algebraic manipulation yields

$$\log B(z) = \log\frac{1}{|\beta|} + \log(1-\eta_1) - \log(1-\eta_2),$$

where

$$\eta_1 = \frac{\xi - \lambda\beta}{\xi - \lambda z}, \quad \eta_2 = \frac{\xi - \lambda/\overline{\beta}}{\xi - \lambda z}.$$

So

$$\begin{aligned} \log B(z) + \log S_1(z) &= \frac{2\xi\log(1/|\beta|)}{\xi - \lambda z} + \log(1-\eta_1) - \log(1-\eta_2) \\ &= \frac{2\xi\log(1/|\beta|)}{\xi - \lambda z} + \frac{\lambda\beta - \lambda/\overline{\beta}}{\xi - \lambda z} \\ &\quad + \log(1-\eta_1) - \log(1-\eta_2) + \eta_1 - \eta_2. \end{aligned}$$

Using Taylor's theorem for $\log(1/|\beta|)$ as we do above, we obtain

$$\begin{aligned} |\log B(z) + \log S_1(z)| \leq& \left| \frac{2\xi(1-|\beta|)/|\beta| + \lambda\beta - \lambda/\overline{\beta}}{\xi - \lambda z} \right| + C\frac{(1-|\beta|)^2}{|\xi - \lambda z|} \\ &+ |\log(1-\eta_1) - \log(1-\eta_2) + \eta_1 - \eta_2|. \end{aligned}$$

Since $\xi = \beta/|\beta|$ and $\lambda < |\beta|$, we obtain

$$\begin{aligned}
|2\xi(1-|\beta|)/|\beta| + \lambda\beta - \lambda/\overline{\beta}| &= |2(1-|\beta|)/\overline{\beta} + \lambda(\beta - 1/\overline{\beta})| \\
&= |2(1-|\beta|)/\overline{\beta} - \lambda(1-|\beta|)(1+\beta)/\overline{\beta}| \\
&= \left|\frac{1-|\beta|}{\overline{\beta}}(2-\lambda(1+|\beta|))\right| \\
&\leq \frac{1-|\beta|}{|\beta|}|2-\lambda(1+\lambda)| \\
&= \frac{1-|\beta|}{|\beta|}|1-\lambda||2+\lambda|.
\end{aligned}$$

Using this, we deduce that

$$\begin{aligned}
\left|\frac{2\xi(1-|\beta|)/|\beta| + \lambda\beta - \lambda/\overline{\beta}}{\xi - \lambda z}\right| &\leq C\frac{(1-|\beta|)(1-\lambda)}{|\xi - \lambda z|} \\
&= C\frac{(1-|\beta|)^2 h(\beta)}{|\xi - \lambda z|} \leq C\frac{(1-|\beta|)^2 h(\beta)}{|\xi - \lambda z|^2}.
\end{aligned}$$

We also have

$$\frac{(1-|\beta|)^2}{|\xi - \lambda z|} \leq \frac{(1-|\beta|)^2 h(\beta)}{|\xi - \lambda z|}$$

as $1 < h(\beta)$. Hence it remains to estimate $|\log(1-\eta_1) - \log(1-\eta_2) + \eta_1 - \eta_2|$. First observe that

$$|\eta_2| \leq |\eta_1| \leq \frac{2(1-\lambda)}{|\xi - \lambda z|} = \frac{2(1-|\beta|)h(\beta)}{|\xi - \lambda z|} < \frac{2(1-|\beta|)h(\beta)}{1-|z|} < \frac{1}{2},$$

where the first inequality comes from the inequality $1 - \lambda/|\beta| \leq 1 - \lambda|\beta|$. Also observe that we have

$$|\eta_1 - \eta_2| = \left|\frac{-\lambda\beta + \lambda/\overline{\beta}}{\xi - \lambda z}\right| = \frac{\lambda}{|\beta|}\frac{(1-|\beta|^2)}{|\xi - \lambda z|} \leq \frac{2(1-|\beta|)}{|\xi - \lambda z|}.$$

Putting this all together, we conclude that

$$\begin{aligned}
|\log(1-\eta_1) - \log(1-\eta_2) + \eta_1 - \eta_2| &= \left| \sum_{k=2}^{\infty} \left(\frac{\eta_2^k}{k} - \frac{\eta_1^k}{k} \right) \right| \\
&\le \sum_{k=2}^{\infty} \frac{|\eta_1^k - \eta_2^k|}{k} \\
&\le |\eta_1 - \eta_2| \sum_{k=2}^{\infty} |\eta_1|^{k-1} \\
&< |\eta_1 - \eta_2| \frac{|\eta_1|}{1-|\eta_1|} \\
&\le \frac{8(1-|\beta|)^2 h(\beta)}{|\xi - \lambda z|^2}
\end{aligned}$$

which completes the proof. □

Lemma 6.0.5.

$$\int_0^{2\pi} \frac{1}{|e^{i\theta} + \alpha|^2}\, d\theta = \frac{2\pi}{1-|\alpha|^2}, \quad |\alpha| < 1.$$

Proof. Let $\alpha = a + ib$, where $a^2 + b^2 < 1$. Writing $e^{i\theta} = \cos\theta + i\sin\theta$, we obtain

$$\int_0^{2\pi} \frac{1}{|e^{i\theta} + \alpha|^2}\, d\theta = \int_0^{2\pi} \frac{1}{(\cos\theta + a)^2 + (\sin\theta + b)^2}\, d\theta.$$

Writing $e^{i\theta} = z$, we obtain $izd\theta = dz$ and

$$I = \int_0^{2\pi} \frac{1}{(\cos\theta + a)^2 + (\sin\theta + b)^2}\, d\theta = \oint_{|z|=1} \frac{-i}{(1+z\overline{\alpha})(z+\alpha)}\, dz$$

as $\alpha = a + ib$. Using the residue theorem and bearing in mind that $|\alpha| < 1$, we find

$$I = 2\pi i \operatorname{Res}\left\{ \frac{-i}{(1+z\overline{\alpha})(z+\alpha)} \right\}_{z=-\alpha} = \frac{2\pi}{1-|\alpha|^2}$$

because $z = -\alpha$ is the only pole of the integrand within the circle $|z| = 1$. □

Theorem 6.0.6. *Let $0 < p < \infty$ and $\{z_k\}$ be a sequence in $\mathbb{D}$. If for some $\epsilon > 0$ we have*

$$\int_0^1 e^{p\phi(r)} \left(\log \frac{1}{1-r} \right)^{1+\epsilon} dr < \infty, \tag{6.2}$$

then for almost all independent choices of $\{\theta_k\}$, where $\theta_k \in [0, 2\pi]$ for all $k = 1, 2, \ldots,$ the sequence $\{z_k e^{i\theta_k}\}$ is a zero set of A^p.

Proof. Without loss of generality we may assume that $1 - 1/e < |z_k|$ for all $k = 1, 2, \ldots$. For each z_k and $z \in \mathbb{D}$, we define

$$B_\beta(z) = \frac{|\beta|}{\beta} \frac{\beta - z}{1 - \overline{\beta} z}$$

and

$$S_\beta(z) = \exp\left((1 - |\beta|)\left(\frac{\xi + \lambda z}{\xi - \lambda z}\right)\right),$$

where $\xi = \beta/|\beta|$ and $\lambda \in (0, 1)$ satisfying $1 - \lambda = (1 - |\beta|)h(\beta)$. We consider the probability space $\Omega = \prod_{k=1}^{\infty} \Omega_k$, where each Ω_k is the interval $[0, 2\pi]$ with normalized Lebesgue measure. An element $\omega \in \Omega$ is of the form $\omega = (\theta_1, \theta_2, \ldots)$ with $0 \le \theta_k \le 2\pi$ and $\{\theta_k\}$ is a sequence of independent random variables. For a sequence $\Lambda = \{z_k\} \subset \mathbb{D}$, a random set Λ_ω is obtained by a rotation of each z_k through the angle θ_k. We set

$$F_{\Lambda_\omega}(z) = \prod_{k=1}^{\infty} B_{z_k e^{i\theta_k}}(z) S_{z_k e^{i\theta_k}}(z).$$

We shall prove that $F_{\Lambda_\omega}(z)$ is analytic on $\mathbb{D}$ whenever Λ satisfies (6.2). In order to prove $F_{\Lambda_\omega}(z)$ is analytic on $\mathbb{D}$, it suffices to show

$$\sum_{k=1}^{\infty} \left|1 - B_{z_k e^{i\theta_k}}(z) S_{z_k e^{i\theta_k}}(z)\right|$$

converges uniformly on every compact subset of $\mathbb{D}$. Let K be a compact set in $\mathbb{D}$. Then there exists $r \in [0, 1)$ such that for all $z \in K$ we have $|z| \le r$. Bearing in mind that $1 - 1/e < |z_k|$ for all $k = 1, 2, \ldots$, and $|z_k| \to 1^-$ as $k \to \infty$, we estimate the factor $\left|1 - B_{z_k e^{i\theta_k}}(z) S_{z_k e^{i\theta_k}}(z)\right|$ as follows. First choose $N \in \mathbb{N}$ such that $|z_k| > r$ for all $k \ge N$ and $(1 - |z_k|)h(z_k)/(1 - r) < 1/4$ for all $k \ge N$. Then fix $k \ge N$ and choose a branch cut of $\log(B_{z_k e^{i\theta_k}}(z) S_{z_k e^{i\theta_k}}(z))$ such that it is analytic in K. Next we use Taylor's theorem to obtain

$$\begin{aligned}\left|1 - B_{z_k e^{i\theta_k}}(z) S_{z_k e^{i\theta_k}}(z)\right| &= |1 - \exp(\log(B_{z_k e^{i\theta_k}}(z) S_{z_k e^{i\theta_k}}(z)))| \\ &\le C |\log(B_{z_k e^{i\theta_k}}(z) S_{z_k e^{i\theta_k}}(z))|\end{aligned}$$

for all $z \in K$. Applying Lemma 6.0.4, we deduce that

$$\begin{aligned}|\log(B_{z_k e^{i\theta_k}}(z) S_{z_k e^{i\theta_k}}(z))| &\le \frac{(1 - |z_k|)^2 h(z_k)}{|\xi_k - \lambda_k z|^2} \\ &\le \frac{(1 - |z_k|)^2 h(z_k)}{(1 - r)^2}.\end{aligned}$$

Since Λ satisfies (6.2), it follows from Lemma 6.0.3 that

$$\sum_{k=N}^{\infty}(1-|z_k|)^2 h(z_k) \le \sum_{k=N}^{\infty}(1-|z_k|)h(z_k)^{-1} < \infty.$$

Hence

$$\sum_{k=N}^{\infty} |\log(B_{z_k e^{i\theta_k}}(z) S_{z_k e^{i\theta_k}}(z))| < \infty$$

for all $z \in K$. This implies

$$\sum_{k=1}^{\infty} \left|1 - B_{z_k e^{i\theta_k}}(z) S_{z_k e^{i\theta_k}}(z)\right| < \infty$$

for all $z \in K$. Therefore $F_{\Lambda_\omega}(z)$ is analytic on $\mathbb{D}$. We set

$$G(z) = \int_{\Omega} |F_{\Lambda_\omega}(z)|^p \, d\omega. \tag{6.3}$$

Our aim is to show $G \in L^1(\mathbb{D}, dA)$. Then the result will follow from Fubini's theorem, because

$$\int_{\mathbb{D}} G(z)\, dA(z) = \int_{\Omega} \left\{ \int_{\mathbb{D}} |F_{\Lambda_\omega}(z)|^p \, dA(z) \right\} d\omega < \infty$$

implies that for almost all $\omega \in \Omega$ the function $F_{\Lambda_\omega}(z)$ belongs to A^p.

Now we have

$$\begin{aligned}\int_{\Omega} |F_{\Lambda_\omega}(z)|^p \, d\omega &= \int_{\Omega} \left\{ \prod_{k=1}^{\infty} |B_{z_k e^{i\theta_k}}(z) S_{z_k e^{i\theta_k}}(z)|^p \right\} d\omega \\ &= \prod_{k=1}^{\infty} \left\{ \int_0^{2\pi} |B_{z_k e^{i\theta_k}}(z) S_{z_k e^{i\theta_k}}(z)|^p \, \frac{d\theta_k}{2\pi} \right\} \end{aligned} \tag{6.4}$$

by the independence of $\{\theta_1, \theta_2, \ldots\}$. For a fixed $|z| = r$, we define

$$S_r = \{w \in \mathbb{D} : (1-|w|)h(w) \le \frac{1}{4}(1-r)\}.$$

We split the product in the line (6.4) into two parts: the product over $z_k \in S_r$ and the product over $z_k \in S_r^c$.

First, we show that the product over $z_k \in S_r$ is bounded by a constant M which is independent of r. Note that

$$\begin{aligned} I &= \frac{1}{2\pi} \int_0^{2\pi} |B_{z_k e^{i\theta_k}}(z) S_{z_k e^{i\theta_k}}(z)|^p \, d\theta_k \\ &= \frac{1}{2\pi} \int_0^{2\pi} \exp\left(p \operatorname{Re}(\log B_{z_k e^{i\theta_k}}(z) + \log S_{z_k e^{i\theta_k}}(z))\right) d\theta_k. \end{aligned}$$

Now we have

$$I \le 1 + \frac{p}{2\pi}\int_0^{2\pi} \log|B_{z_k e^{i\theta_k}}(z)|\,d\theta_k + \frac{p}{2\pi}\int_0^{2\pi} \operatorname{Re}(\log S_{z_k e^{i\theta_k}}(z))\,d\theta_k$$
$$+ \frac{C}{2\pi}\int_0^{2\pi} |\log B_{z_k e^{i\theta_k}}(z) + \log S_{z_k e^{i\theta_k}}(z)|^2\,d\theta_k$$

by Taylor's theorem. Observe that we have $|z_k| > |z|$ for $z_k \in S_r$. So

$$\frac{1}{2\pi}\int_0^{2\pi} \log|B_{z_k e^{i\theta_k}}(z)|\,d\theta_k = \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{z_k e^{i\theta_k} - z}{1 - z z_k e^{i\theta_k}}\right| d\theta_k = \log|z_k|$$

by Jensen's formula. For the second integral on the right side, the Poisson integral formula gives

$$\frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}(\log S_{z_k e^{i\theta_k}}(z))\,d\theta_k = \frac{1}{2\pi}\int_0^{2\pi} (1-|z_k|)\operatorname{Re}\left(\frac{\xi_k + z\lambda_k e^{-i\theta_k}}{\xi_k - z\lambda_k e^{-i\theta_k}}\right) d\theta_k = 1-|z_k|.$$

We apply Lemma 6.0.4 and obtain

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} |\log B_{z_k e^{i\theta_k}}(z) + \log S_{z_k e^{i\theta_k}}(z)|^2\,d\theta_k &\le \frac{C}{2\pi}\int_0^{2\pi} \frac{(1-|z_k|)^4 h^2(z_k)}{|\xi_k e^{i\theta_k} - z\lambda_k|^4}\,d\theta_k \\
&\le \frac{C}{2\pi}\int_0^{2\pi} \frac{(1-|z_k|)^2}{|\xi_k e^{i\theta_k} - z\lambda_k|^2}\,d\theta_k \\
&= \frac{C(1-|z_k|)^2}{1 - r^2\lambda_k^2} \qquad (6.5) \\
&< \frac{C(1-|z_k|)^2)}{1-\lambda_k} \\
&= C(1-|z_k|)h(z_k)^{-1},
\end{aligned}$$

where the last equality follows from $1 - \lambda_k = (1 - z_k)h(z_k)$ and (6.5) is due to Lemma 6.0.5. Hence

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} |B_{z_k e^{i\theta_k}}(z) S_{z_k e^{i\theta_k}}(z)|^p\,d\theta_k &\le 1 + p(\log|z_k| + 1 - |z_k|) + C(1-|z_k|)h(z_k)^{-1} \\
&< 1 + C(1-|z_k|)h(z_k)^{-1},
\end{aligned}$$

where the last inequality follows from $\log|z_k| < |z_k| - 1$. Now using the estimate above and the inequality $1 + x \le e^x$, we obtain

$$\begin{aligned}\prod_{z_k \in S_r} \frac{1}{2\pi} \int_0^{2\pi} |B_{z_k e^{i\theta_k}}(z) S_{z_k e^{i\theta_k}}(z)|^p \, d\theta_k &< \prod_{z_k \in S_r} \left(1 + C(1 - |z_k|)h(z_k)^{-1}\right) \\ &\le \exp\left(C \sum_{z_k \in S_r} (1 - |z_k|)h(z_k)^{-1}\right) \\ &\le \exp\left(C \sum_{k=1}^{\infty} (1 - |z_k|)h(z_k)^{-1}\right) < \infty\end{aligned}$$

by hypothesis and Lemma 6.0.3. Hence the product over $z_k \in S_r$ is bounded by a constant M. So we have

$$\begin{aligned}G(z) &\le M \prod_{z_k \in S_r^c} \frac{1}{2\pi} \int_0^{2\pi} |B_{z_k e^{i\theta_k}}(z) S_{z_k e^{i\theta_k}}(z)|^p \, d\theta_k \\ &\le M \prod_{z_k \in S_r^c} \frac{1}{2\pi} \int_0^{2\pi} |S_{z_k e^{i\theta_k}}(z)|^p \, d\theta_k,\end{aligned}$$

where the last line follows from the inequality $|B_{z_k e^{i\theta_k}}(z)| \le 1$. Now

$$0 < (1 - |z_k|) \operatorname{Re}\left(\frac{\xi_k e^{i\theta_k} + \lambda_k z}{\xi_k e^{i\theta_k} - \lambda_k z}\right) \le 2$$

implies that

$$J = \frac{1}{2\pi} \int_0^{2\pi} |S_{z_k e^{i\theta_k}}(z)|^p \, d\theta_k = \frac{1}{2\pi} \int_0^{2\pi} \exp\left(p(1 - |z_k|) \operatorname{Re}\left(\frac{\xi_k e^{i\theta_k} + \lambda_k z}{\xi_k e^{i\theta_k} - \lambda_k z}\right)\right) d\theta_k.$$

So we have

$$\begin{aligned}J = \frac{1}{2\pi} \int_0^{2\pi} \Bigg\{ & 1 + p(1 - |z_k|) \operatorname{Re}\left(\frac{\xi_k e^{i\theta_k} + \lambda_k z}{\xi_k e^{i\theta_k} - \lambda_k z}\right) \\ & + O\left((1 - |z_k|)^2 \left(\operatorname{Re}\left(\frac{\xi_k e^{i\theta_k} + \lambda_k z}{\xi_k e^{i\theta_k} - \lambda_k z}\right)\right)^2\right)\Bigg\} \, d\theta_k.\end{aligned}$$

Now observe that

$$\begin{aligned}\frac{1}{2\pi} \int_0^{2\pi} (1 - |z_k|)^2 \left\{\operatorname{Re}\left(\frac{\xi_k e^{i\theta_k} + \lambda_k z}{\xi_k e^{i\theta_k} - \lambda_k z}\right)\right\}^2 d\theta_k &\le \frac{2}{\pi} \int_0^{2\pi} \frac{(1 - |z_k|)^2}{|\xi_k e^{i\theta_k} - \lambda_k z|^2} \, d\theta_k \\ &= \frac{4(1 - |z_k|)^2}{1 - r^2 \lambda_k^2} \qquad (6.6) \\ &\le \frac{4(1 - |z_k|)^2}{1 - \lambda_k} \\ &= 4(1 - |z_k|)h(z_k)^{-1},\end{aligned}$$

where (6.6) is due to Lemma 6.0.5. Thus

$$\frac{1}{2\pi}\int_0^{2\pi} |S_{z_k e^{i\theta_k}}(z)|^p \, d\theta_k \leq 1 + p(1-|z_k|) + C(1-|z_k|)h(z_k)^{-1}.$$

The inequality $1 + x \leq e^x$ and Lemma 6.0.3 imply that

$$\begin{aligned} G(z) &\leq C \prod_{z_k \in S_r^c} \exp\Big(p(1-|z_k|) + (1-|z_k|)h(z_k)^{-1}\Big) \\ &= C \exp\bigg(\sum_{z_k \in S_r^c} \Big\{p(1-|z_k|) + (1-|z_k|)h^{-1}(z_k)\Big\}\bigg) \\ &\leq C \exp\bigg(\sum_{z_k \in S_r^c} p(1-|z_k|)\bigg) \\ &\leq C e^{p\phi(r_1)}, \end{aligned}$$

where r_1 satisfies $(1-r_1)h(r_1) = \frac{1}{4}(1-r)$. Using the change of variable $r_1 = r_1(r)$, we obtain

$$\begin{aligned} \int_{\mathbb{D}} G(z) dA(z) &\leq C \int_0^1 e^{p\phi(r_1)} \, dr \\ &\leq C \int_{r_1(0)}^1 e^{p\phi(r)} \Big\{ h(r) - (1+\epsilon)\Big(\log\frac{1}{1-r}\Big)^{\epsilon}\Big\} \, dr \\ &\leq C \int_0^1 e^{p\phi(r)} \Big(\log\frac{1}{1-r}\Big)^{1+\epsilon} dr \\ &< \infty \end{aligned}$$

by hypothesis. Hence $G \in L^1(\mathbb{D}, dA)$ and the proof is complete. □

Theorem 6.0.7. *Let $0 < p < \infty$, $-1 < \alpha < \infty$ and $\{z_k\}$ be a sequence in $\mathbb{D}$. If for some $\epsilon > 0$ we have*

$$\int_0^1 e^{p\phi(r)} \Big(\log\frac{1}{1-r}\Big)^{(1+\alpha)(1+\epsilon)} (1-r)^{\alpha} \, dr < \infty, \tag{6.7}$$

then for almost all independent choices of $\{\theta_k\}$, where $\theta_k \in [0, 2\pi]$ for all $k = 1, 2, \ldots,$ the sequence $\{z_k e^{i\theta_k}\}$ is a zero set of A^p_α.

Proof. By the discussion in the proof of Theorem 6.0.6, it suffices to prove that G defined as in (6.3) belongs to $L_1(\mathbb{D}, dA_\alpha)$, where $dA_\alpha(z) = (1-|z|)^\alpha dA(z)$. The change of variable $r_1 = r_1(r)$ yields

$$\begin{aligned}\int_{\mathbb{D}} G(z) dA_\alpha(z) &\leq C \int_0^1 e^{p\phi(r_1)}(1-r)^\alpha \, dr \\ &\leq C \int_0^1 e^{p\phi(r)} \biggl\{ h(r) - (1+\epsilon)\biggl(\log \frac{1}{1-r} \biggr)^\epsilon \biggr\} \bigl\{ (1-r)h(r) \bigr\}^\alpha \, dr \\ &\leq C \int_0^1 e^{p\phi(r)} h(r)^{1+\alpha} (1-r)^\alpha \, dr,\end{aligned}$$

which is finite by hypothesis. $\square$

Bibliography

[1] J. Bak and D. J. Newman, *Complex Analysis*, Springer, New York, 2010.

[2] G. Bomash, *A Blaschke-type product and random zero sets for Bergman spaces*, Ark. Mat. **30** (1992), 45–60.

[3] P. Duren, *Theory of H^p spaces*, Pure Appl. Math., vol. 38, Academic Press, New York-London, 1970.

[4] P. Duren and A. Schuster, *Bergman Spaces*, Math. Surveys Monogr., vol. 100, Amer. Math. Soc., Providence, 2004.

[5] O. El-Fallah, K. Kellay, J. Mashreghi and T. Ransford, *A Primer on the Dirichlet Space*, Cambridge University Press, Cambridge, 2014.

[6] D. Girela, M. Nowak and P. Waniurski, *On the zeros of Bloch functions*, Math. Proc. Cambridge Philos. Soc. **129** (2000), 117–128.

[7] D. Girela and J. A. Peláez, *Growth properties and sequences of zeros of analytic functions in spaces of Dirichlet type*, J. Aust. Math. Soc. **80** (2006), 397–418.

[8] H. Hedenmalm, B. Korenblum and K. Zhu, *Theory of Bergman Spaces*, Springer, New York, 2000.

[9] C. Horowitz, *Zeros of functions in the Bergman spaces*, Duke Math. J. **41** (1974), 693–710.

[10] J. E. Littlewood and R. E. A. C. Paley, *Theorems on Fourier series and power series (II)*, Proc. London Math. Soc. **42** (1936), 52–89.

[11] M. Mateljević and M. Pavlović, *L^p-behavior of power series with positive coefficients and Hardy spaces*, Proc. Amer. Math. Soc. **87** (1983), 309–316.

[12] M. Nowak and P. Waniurski, *Random zero sets for Bergman spaces*, Math. Proc. Cambridge Philos. Soc. **134** (2003), 337–345.

[13] D. C. Ullrich, *Khinchin's inequality and zeroes of Bloch functions*, Duke Math. J. **57** (1988), 519–535.

[14] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, 1987.

[15] H. S. Shapiro and A. L. Shields, *On the zeros of functions with finite Dirichlet integral and some related function spaces*, Math. Z. **80** (1962), 217–229.