Stability Analysis of Human–Adaptive Controller Interactions

Tansel Yucelen∗, Yildiray Yildiz†, Rifat Sipahi‡, Ehsan Yousefi§ and Nhan Nguyen¶

In this paper, stability of human-in-the-loop model reference adaptive control architectures is analyzed. For a general class of linear human models with time-delay, a fundamental stability limit of these architectures is established, which depends on the parameters of this human model as well as the reference model parameters of the adaptive controller. It is shown that when the given set of human model and reference model parameters satisfy this stability limit, the closed-loop system trajectories are guaranteed to be stable.

I. Introduction

Adaptive control design approaches1–4 are important candidates for uncertain dynamical systems, since they can effectively cope with the effects of system uncertainties online and require less modeling information than fixed-gain robust control design approaches5,6. Motivated from this fact, the results of this paper builds on a well-known and important class of adaptive controllers; namely, model reference adaptive controllers7,8, where their architecture includes a reference model, a parameter adjustment mechanism, and a controller.

Specifically, a desired closed-loop dynamical system behavior is captured by the reference model, where its output (respectively, state) is compared with the output (respectively, state) of the uncertain dynamical system. This comparison yields to a system error signal, which is used to drive the parameter adjustment mechanism online. Then, the controller adapts feedback gains to minimize this error signal using the information received from the parameter adjustment mechanism. As a consequence, the output (respectively, state) of the uncertain dynamical system behaves as the output (respectively, state) of the reference model asymptotically or approximately in time, and hence, guarantees system stability and achieves a level of desired closed-loop dynamical system behavior.

From this standpoint, model reference adaptive controllers offer mathematical tools to effectively cope with system uncertainties arising from ideal assumptions, linearization, model order reduction, exogenous disturbances, and degraded modes of operations. However, in certain applications when humans are in the loop9–15, they can lead to unstable system trajectories. The contribution of this paper is to analyze stability of human in the loop model reference adaptive control architectures. For a general class of linear human...
models with time-delay, a fundamental stability limit of these architectures is established, which depends on the parameters of this human model as well as the reference model parameters of the adaptive controller. It is shown that when the given set of human model and reference model parameters satisfy this stability limit, the closed-loop system trajectories are guaranteed to be stable.

II. Problem Formulation

For representing human in the loop model reference adaptive controllers, we consider the block diagram configuration given by Figure 1, where the outer loop architecture includes the reference that is fed into the human dynamics to generate a command for the inner loop architecture in response to the variations resulting from the uncertain dynamical system and the inner loop architecture includes the uncertain dynamical system and the model reference adaptive controller components (i.e., the reference model, the parameter adjustment mechanism, and the controller). Specifically, at the other loop architecture, we consider a class of human models with time-delay given by

\[
\dot{\xi}(t) = A_h \xi(t) + B_h \theta(t - \tau), \quad \xi(0) = \xi_0, \quad (1)
\]

\[
c(t) = C_h \xi(t) + D_h \theta(t - \tau), \quad (2)
\]

where \(\xi(t) \in \mathbb{R}^{n_x}\) is the internal human state vector, \(c(t) \in \mathbb{R}^{n_c}\) is the command representing the output of the outer loop architecture in Figure 1,

\[
\theta(t) \triangleq r(t) - E_h x(t), \quad (3)
\]

\(\theta(t) \in \mathbb{R}^{n_r}\), with \(r(t) \in \mathbb{R}^{n_r}\) being the bounded reference and \(x(t) \in \mathbb{R}^{n}\) being the state vector to be defined that is received from the inner loop architecture, \(\tau \in \mathbb{R}_+\) is the internal human time-delay, and \(A_h \in \mathbb{R}^{n_x \times n_x}\), \(B_h \in \mathbb{R}^{n_x \times n_r}\), \(C_h \in \mathbb{R}^{n_c \times n_x}\), \(D_h \in \mathbb{R}^{n_c \times n_r}\), and \(E_h \in \mathbb{R}^{n_r \times n}\). Note that the dynamics given by (1), (2), and (3) is general enough to capture, for example, widely studied linear time-invariant human models with time-delay including Neal-Smith model and its extensions.

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aExtended version of this paper is currently under review [16].
Next, at the inner loop architecture, we consider the uncertain dynamical system given by

\[ \dot{x}_p(t) = A_p x_p(t) + B_p \lambda u(t) + B_p \delta_p(x_p(t)), \quad x_p(0) = x_{p0}, \]  

(4)

where \( x_p(t) \in \mathbb{R}^{n_p} \) is the accessible state vector, \( u(t) \in \mathbb{R}^m \) is the control input, \( \delta_p : \mathbb{R}^{n_p} \rightarrow \mathbb{R}^m \) is an uncertainty, \( A_p \in \mathbb{R}^{n_p \times n_p} \) is a known system matrix, \( B_p \in \mathbb{R}^{n_p \times m} \) is a known control input matrix, and \( \Lambda \in \mathbb{R}^{n_p \times n_p} \cap D^{m \times m} \) is an unknown control effectiveness matrix. Furthermore, we assume that the pair \((A_p, B_p)\) is controllable and the uncertainty is parameterized as

\[ \delta_p(x_p) = W_p^T \sigma_p(x_p), \quad x_p \in \mathbb{R}^{n_p}, \]  

(5)

where \( W_p \in \mathbb{R}^{s \times m} \) is an unknown weight matrix and \( \sigma_p : \mathbb{R}^{n_p} \rightarrow \mathbb{R}^s \) is a known basis function of the form \( \sigma_p(x_p) = [\sigma_{p1}(x_p), \sigma_{p2}(x_p), \ldots, \sigma_{pn}(x_p)]^T \). Note for the case where the basis function \( \sigma_p(x_p) \) is unknown that the parameterization in (5) can be relaxed without significantly changing the results of this paper, for example, by considering^20,21

\[ \delta_p(x_p) = W_p^T \sigma_p^{mn}(V_p^T x_p) + \varepsilon_p^{nn}(x_p), \quad x_p \in D_{x_p}, \]  

(6)

where \( W_p \in \mathbb{R}^{s \times m} \) and \( V_p \in \mathbb{R}^{n_p \times s} \) are unknown weight matrices, \( \sigma_p^{mn} : D_{x_p} \rightarrow \mathbb{R}^m \) is a known basis composed of neural networks function approximators, \( \varepsilon_p^{nn} : D_{x_p} \rightarrow \mathbb{R}^m \) is an unknown residual error, and \( D_{x_p} \) is a compact subset of \( \mathbb{R}^{n_p} \).

To address command following at the inner loop architecture, let \( x_c(t) \in \mathbb{R}^{n_c} \) be the integrator state satisfying

\[ \dot{x}_c(t) = E_p x_p(t) - c(t), \quad x_c(0) = x_{c0}, \]  

(7)

where \( E_p \in \mathbb{R}^{n_c \times n_p} \) allows to choose a subset of \( x_u(t) \) to be followed by \( c(t) \). Now, (4) can be augmented with (7) as

\[ \dot{x}(t) = A x(t) + B A \lambda u(t) + B W_p^T \sigma_p(x_p(t)) + B_c c(t), \quad x(0) = x_0, \]  

(8)

where \( x(t) \triangleq [x_p^T(t), x_c^T(t)]^T \in \mathbb{R}^n, n = n_p + n_c \), is the (augmented) state vector, \( x_0 \triangleq [x_{p0}^T, x_{c0}^T]^T \in \mathbb{R}^n \),

\[ A \triangleq \begin{bmatrix} A_p & 0_{n_p \times n_c} \\ E_p & 0_{n_c \times n_c} \end{bmatrix} \in \mathbb{R}^{n \times n}, \]  

(9)

\[ B \triangleq \begin{bmatrix} B_p^T \\ 0_{n_c \times n_c}^T \end{bmatrix} \in \mathbb{R}^{n \times m}, \]  

(10)

\[ B_c \triangleq \begin{bmatrix} 0_{n_p \times n_c}^T \\ -I_{n_c \times n_c} \end{bmatrix} \in \mathbb{R}^{n \times n_c}, \]  

(11)

In this inner loop architecture setting, it is reasonable to set \( E_{hp} = [E_{hp}, 0_{n_c \times n_c}] \), \( E_{hp} \in \mathbb{R}^{n_c \times n_p} \), in (3) without loss of generality since a subset of the accessible state vector is usually available and/or sensed by the human at the outer loop (not the states of the integrator).
Finally, consider the feedback control law at the inner loop architecture given by

\[ u(t) = u_n(t) + u_a(t), \]  

(12)

where \( u_n(t) \in \mathbb{R}^m \) and \( u_a(t) \in \mathbb{R}^m \) are the nominal and adaptive control laws, respectively. Furthermore, let the nominal control law be

\[ u_n(t) = -Kx(t), \]  

(13)

\( K \in \mathbb{R}^{m \times n} \), such that \( A_r \triangleq A - BK \) is Hurwitz. Using (12) and (13) in (8) yields

\[ \dot{x}(t) = A_r x(t) + B_r c(t) + BA [u_n(t) + \bar{W}^T \sigma(x(t))], \]  

(14)

where \( \bar{W}^T \triangleq [\Lambda^{-1} W_p^T, (\Lambda^{-1} - I_{m \times m}) K] \in \mathbb{R}^{(s+n) \times m} \) is an unknown (aggregated) weight matrix and \( \sigma^T(x(t)) \triangleq [\sigma_p^T(x_p(t)), x^T(t)] \in \mathbb{R}^{s+n} \) is a known (aggregated) basis function. Considering (14), let the adaptive control law be

\[ u_a(t) = -\hat{\bar{W}}^T(t) \sigma(x(t)), \]  

(15)

where \( \hat{\bar{W}}(t) \in \mathbb{R}^{(s+n) \times m} \) be the estimate of \( \bar{W} \) satisfying the parameter adjustment mechanism

\[ \dot{\hat{\bar{W}}}(t) = \gamma \sigma(x(t)) e^T(t) P B, \quad \hat{\bar{W}}(0) = \hat{\bar{W}}_0, \]  

(16)

where \( \gamma \in \mathbb{R}_+ \) is the learning rate, \( e(t) \triangleq x(t) - x_r(t) \) is the system error with \( x_r(t) \in \mathbb{R}^n \) being the reference state vector satisfying the reference system

\[ \dot{x}_r(t) = A_r x_r(t) + B_r c(t), \quad x_r(0) = x_{r0}, \]  

(17)

and \( P \in \mathbb{R}^{n \times n} \cap \mathbb{S}^{n \times n} \) is a solution of the Lyapunov equation \( 0 = A_r^T P + PA_r + R \) with \( R \in \mathbb{R}_+^{n \times n} \cap \mathbb{S}^{n \times n} \).

Although we consider a specific yet widely studied parameter adjustment mechanism given by (16), one can also consider other types of parameter adjustment mechanisms\(^{22-35}\) and still use the approach presented in this paper.

Based on the given problem formulation, the next section analyzes the stability of the coupled inner and outer loop architectures depicted in Figure 1 in order to establish a fundamental stability limit (to guarantee the closed-loop system stability when this limit is satisfied by the given human model at the outer loop and the given adaptive controller at the inner loop).

### III. Stability Analysis

For stability analysis purposes, we now write the system error dynamics using (14), (15), and (17) as

\[ \dot{e}(t) = A_r e(t) - B \bar{W}^T(t) \sigma(x(t)), \quad e(0) = e_0, \]  

(18)
where \( \tilde{W}(t) \triangleq \tilde{W}(t) - W \in \mathbb{R}^{(s+n)\times m} \) is the weight error and \( \epsilon_0 \triangleq x_0 - x_{ro} \). In addition, we write the weight error dynamics using (16) as

\[
\dot{\tilde{W}}(t) = \gamma \sigma(x(t))e^T(t)PB, \quad \tilde{W}(0) = \tilde{W}_0,
\]

where \( \tilde{W}_0 \triangleq \tilde{W}(0) - W \). The following lemma is now immediate.

**Lemma 1**. Consider the uncertain dynamical system given by (4) subject to (5), the reference model given by (17), and the feedback control law given by (12), (13), (15), and (16). Then, the solution \((e(t), \tilde{W}(t))\) is Lyapunov stable for all \((e_0, \tilde{W}_0) \in \mathbb{R}^n \times \mathbb{R}^{(s+n)\times m} \) and \(t \in \mathbb{R}_+\).

Since the solution \((e(t), \tilde{W}(t))\) is Lyapunov stable for all \((e_0, \tilde{W}_0) \in \mathbb{R}^n \times \mathbb{R}^{(s+n)\times m} \) and \(t \in \mathbb{R}_+\) from Lemma 1, this trivially implies that \(e(t) \in L_\infty\) and \(\tilde{W}(t) \in L_\infty\). At this stage, it should be noted that one cannot use the Barbalat’s lemma to conclude \(\lim_{t \to \infty} e(t) = 0\), since the boundedness of the reference model needs to be assured first. From this standpoint, we next show the boundedness of the reference model, which also reveals the fundamental stability limit for guaranteeing the closed-loop system stability.

Using (2) in (17), we first write

\[
\dot{x}_r(t) = A_r x_r(t) + B_r (C_h \xi(t) + D_h \theta(t - \tau)),
\]

\[
= A_r x_r(t) - B_r D_h E_h x_r(t - \tau) + B_r C_h \xi(t) - B_r D_h E_h e(t - \tau) + B_r D_h r(t - \tau).
\]

Next, it follows from (1) that

\[
\dot{\xi}(t) = A_h \xi(t) - B_h E_h x_r(t - \tau) - B_h E_h e(t - \tau) + B_h r(t - \tau).
\]

Finally, by letting \(\phi(t) \triangleq [x_r^T(t), \xi^T(t)]^T\), one can write

\[
\dot{\phi}(t) = A_0 \phi(t) + A_r \phi(t - \tau) + \varphi(\cdot), \quad \phi(0) = \phi_0,
\]

using (20) and (21), where

\[
A_0 \triangleq \begin{bmatrix} A_r & B_r C_h \\ 0_{n_x \times n} & A_h \end{bmatrix} \in \mathbb{R}^{(n + n_x) \times (n + n_x)},
\]

\[
A_r \triangleq \begin{bmatrix} -B_r D_h E_h & 0_{n \times n_x} \\ -B_h E_h & 0_{n_x \times n_x} \end{bmatrix} \in \mathbb{R}^{(n + n_x) \times (n + n_x)},
\]

\[
\varphi(\cdot) \triangleq \begin{bmatrix} -B_r D_h E_h e(t - \tau) + B_r D_h r(t - \tau) \\ -B_h E_h e(t - \tau) + B_h r(t - \tau) \end{bmatrix} \in \mathbb{R}^{n + n_x}.
\]

As a consequence of Lemma 1 and the boundedness of the reference, one can conclude that \(\varphi(\cdot) \in L_\infty\).

To reveal the fundamental stability limit, an approach would be to employ frequency domain tools, where one studies the eigenvalues of the corresponding linear time invariant system with time delay. Moreover, since the time delay in human dynamics can in general be known in practice for certain applications, at least within a certain range, it is possible to utilize the delay information in the stability analysis as stated in the following theorem.
Theorem 2. Consider the uncertain dynamical system given by (4) subject to (5), the reference model given by (17), the feedback control law given by (12), (13), (15), and (16), and the human dynamics given by (1), (2), and (3). Then, $e(t) \in \mathcal{L}_\infty$ and $\tilde{W}(t) \in \mathcal{L}_\infty$. If, in addition, the real parts of all the infinitely many roots of the following characteristic equation

$$\det\left(sI - (A_0 + A_\tau e^{-\tau s})\right) = 0,$$

where $I \in \mathbb{R}^{(n+n_\xi) \times (n+n_\xi)}$ is the identity matrix, have strictly negative real parts, then $x_r(t) \in \mathcal{L}_\infty$, $\xi(t) \in \mathcal{L}_\infty$, and $\lim_{t \to \infty} e(t) = 0$.

IV. Numerical Example

In order to demonstrate the stability criterion stated in Theorem 2, an illustrative numerical example is provided here to show the effect of various incorporated parameters. Specifically, consider the human transfer function given by

$$k_p \frac{(\tau_z s + 1)}{\tau_p s + 1} e^{-\tau s}.$$

Table 1 shows numerical values used in this section. By definition, $A_r$ (of the reference model) is $A - BK$. The optimal $K$ is found using the linear quadratic regulator (LQR) method, which is achieved by minimizing the performance index

$$J(x, u) = \int_0^\infty (x^T Q x + u^T R u) dt,$$

(see, for example, Ref. 38). Here, $R$ matrix is considered to be of the form $\mu I$, where $I$ is the identity matrix, and the stability is checked for various values of $\mu$. In the following subsections, the effect of various time delays and various $Q$ matrices on the stability of the system are analyzed.

A. Effect of various time delays

In this section, the effect of time delay $\tau$ on the stability is analyzed. Figure 2 depicts the real part of the right most pole (RMP) of the system versus $\mu$ for various time delays. From this figure and for the selected range of $\mu$, we have the following information:

- for $0 \leq \tau < 0.386$, the system is always stable as indicated with $RMP < 0$;
- for $0.386 \leq \tau < 0.551$, the system is stable for smaller values of $\mu$; unstable for larger $\mu$, and stable once again for even larger $\mu$ values;

<table>
<thead>
<tr>
<th>$A_p$</th>
<th>$[0.003, 0.39, 0, -0.322; -0.065, -0.319, 7.740, 0; 0.020, -0.101, -0.429, 0; 0, 0, 1, 0]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_p$</td>
<td>$[0.010; -0.180; -1.160; 0]$</td>
</tr>
<tr>
<td>$E_p$</td>
<td>$[0, 0, 0, 1]$</td>
</tr>
<tr>
<td>$E_h$</td>
<td>$[0, 0, 0, 1, 0]$</td>
</tr>
<tr>
<td>$k_p = 5$, $\tau_z = 1$, $\tau_p = 5$, $\tau = 0.5$</td>
<td></td>
</tr>
</tbody>
</table>

Table 1. Numerical data
Figure 2. Right Most Pole (RMP) vs $\mu$ for various time delays $\tau$

- for $\tau \geq 0.551$, the system is unstable for smaller values of $\mu$ and stable for larger $\mu$.

B. Effect of various $Q$ matrices

Next, the effect of various $Q$ matrices on the performance index of LQR method is analyzed. In order to satisfy the criteria of the LQR method, the $Q$ matrix is considered to be a diagonal positive definite matrix, $Q = \text{diag}\{Q_{1,1}, \ldots, Q_{k,k}, \ldots, Q_{5,5}\}$. Since we focus on the pitch attitude tracking task (similar to Ref. 9) in this example, the considered state for the performance index is pitch angle $\theta(t)$. Therefore, $Q_{1,1}$, $Q_{2,2}$, and $Q_{3,3}$, which are associated with the out of purpose states of the plant are considered to be zero, and $Q_{4,4}$ and $Q_{5,5}$, which are associated with fourth state of the plant, i.e., $\theta(t)$, and the state of the integrator $x_c(t)$, are important as they relate to the coupling effects of inner and outer loops and are considered to be 1 and 2.5, respectively.

Figure 3 depicts RMP of the system versus $\mu$ for various fourth diagonal element of the $Q$ matrix ($Q_{4,4}$), which penalizes the fourth state of the plant. Entry $Q_{5,5}$ is kept the same as its original value 2.5. Accordingly, for the selected range of $\mu$ it could be noted that

- for $Q_{4,4} < 0.817$, the system is unstable for smaller $\mu$ and stable for larger $\mu$;

- for $0.817 \leq Q_{4,4} \leq 1.697$, the system is stable for smaller $\mu$ then becomes unstable and stable once again as $\mu$ is increased;

- for $Q_{4,4} > 1.697$, the system is always stable.
Next the effects of $Q_{5,5}$ is analyzed. In Figure 4, RMP of the system is depicted with respect to $\mu$ while keeping $Q_{4,4} = 1$. Accordingly, it could be noted that

- for $Q_{5,5} \leq 1.475$, the system is always stable;
- for $1.475 \leq Q_{5,5} < 4.182$, the system is stable for smaller $\mu$ but then switches to unstable and then to stable configuration as $\mu$ is increased;
- for $Q_{5,5} \geq 4.182$, the system is unstable for smaller $\mu$ values but then recovers stability for larger $\mu$ values.

V. Conclusion

Human in the loop model reference adaptive control architectures were analyzed and a fundamental stability limit was presented in this paper. This limit resulted from the coupling between outer and inner loop architectures, where the outer loop portion includes the human dynamics modeled as linear time-invariant systems with time delay and the inner loop portion includes the uncertain dynamical system, the reference model, the parameter adjustment mechanism, and the controller. A case study based on a pilot model is presented next to demonstrate the approach.

References

Figure 4. Right Most Pole (RMP) vs $\mu$ for various $Q_{5,5}$ elements. Here $Q_{4,4} = 1$. 


