Factorizations of Matrices Over Projective-free Rings

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January 31, 2018

Abstract

An element of a ring R is called strongly $J^{\#}$-clean provided that it can be written as the sum of an idempotent and an element in $J^{\#}(R)$ that commute. We characterize, in this article, the strongly $J^{\#}$-cleanness of matrices over projective-free rings. These extend many known results on strongly clean matrices over commutative local rings.

2010 Mathematics Subject Classification : 15A13, 15B99, 16L99.
Key words: strongly $J^{\#}$-matrix, characteristic polynomial, projective-free ring.

1 Introduction

Let R be a ring with an identity. We say that $x \in R$ is strongly clean provided that there exists an idempotent $e \in R$ such that $x - e \in U(R)$ and $ex = xe$. A ring R is strongly clean in case every element in R is strongly clean (cf. [9-10]). In [2, Theorem 12], Borooah, Diesl, and Dorsey provide the following characterization: Given a commutative local ring R and a monic polynomial $h \in R[t]$ of degree n, the following are equivalent: (1) h has an SRC factorization in $R[t]$; (2) every $\varphi \in M_n(R)$ which satisfies h is strongly clean. It is demonstrated in [6, Example 3.1.7] that statement (1) of the above can not weakened from SRC factorization to SR factorization. The purpose of this paper is to investigate a subclass of strongly clean rings which behave like such ones but can be characterized by a kind of SR factorizations, and so get more explicit factorizations for many class of matrices over projective-free rings.

Let $J(R)$ be the Jacobson radical of R. Set

$$J^{\#}(R) = \{x \in R \mid \exists\, n \in \mathbb{N} \text{ such that } x^n \in J(R)\}.$$

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For instance, let $R = M_2(\mathbb{Z}_2)$. Then

$$J^{\#}(R) = \{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\},$$

while $J(R) = 0$. Thus, $J^{\#}(R)$ and $J(R)$ are distinct in general. We say that an element $a \in R$ is strongly $J^{\#}$-clean provided that there exists an idempotent $e \in R$ such that $a - e \in J^{\#}(R)$ and $ea = ae$. If R is a commutative ring, then $a \in R$ is strongly $J^{\#}$-clean if and only if $a \in R$ is strongly J-clean (cf. [3]). But they behave different for matrices over commutative rings. A Jordan-Chevalley decomposition of $n \times n$ matrix A over an algebraically closed field (e.g., the field of complex numbers), then A is an expression of it as a sum: $A = E + W$, where E is semisimple, W is nilpotent, and E and W commute. The Jordan-Chevalley decomposition is extensively studied in Lie theory and operator algebra. As a corollary, we will completely determine when an $n \times n$ matrix over a filed is the sum of an idempotent matrix and a nilpotent matrix that commute. Thus, the strongly $J^{\#}$-clean factorizations of matrices over rings is also an analog of that of Jordan-Chevalley decompositions for matrices over fields.

We characterize, in this article, the strongly $J^{\#}$-cleanness of matrices over projective-free rings. Here, a commutative ring R is projective-free provided that every finitely generated projective R-module is free. For instances, every commutative local ring, every commutative semi-local ring, every principal ideal domain, every Bézout domain (e.g., the ring of all algebraic integers) and the ring $R[x]$ of all polynomials over a principal domain R are all projective-free. We will show that strongly $J^{\#}$-clean matrices over projective-free rings are completely determined by a kind of "SC"-factorizations of the characteristic polynomials. These extend many known results on strongly clean matrices to such new factorizations of matrices over projective-free rings (cf. [1-2] and [5]).

Throughout, all rings with an identity and all modules are unitary modules. Let $f(t) \in R[t]$. We say that $f(t)$ is a monic polynomial of degree n if $f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$ where $a_{n-1}, \cdots, a_1, a_0 \in R$. We always use $U(R)$ to denote the set of all units in a ring R. If $\varphi \in M_n(R)$, we use $\chi(\varphi)$ to stand for the characteristic polynomial $det(tI_n - \varphi)$.

2 Full Matrices Over Projective-free Rings

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{Z}_2)$. It is directly verified that $A \in M_2(\mathbb{Z}_2)$ is not strongly $J^{\#}$-clean, though A is strongly clean. It is hard to determine strongly cleanness even for matrices over the integers, but completely different situation is in the strongly $J^{\#}$-clean case. The aim of this section is to characterize a single strongly $J^{\#}$-clean $n \times n$ matrix over projective-free rings. Let M be a left R-module. We denote the endomorphism ring of M by $end(M)$.

Lemma 2.1 *Let M be a left R-module, and let $E = end(M)$, and let $\alpha \in E$. Then the following are equivalent:*

(1) $\alpha \in E$ is strongly $J^{\#}$-clean.

(2) There exists a left R-module decomposition $M = P \oplus Q$ where P and Q are α-invariant, and $\alpha|_P \in J^{\#}\big(end(P)\big)$ and $(1_M - \alpha)|_Q \in J^{\#}\big(end(Q)\big)$.

Proof (1) $\Rightarrow$ (2) Since α is strongly $J^{\#}$-clean in E, there exists an idempotent $\pi \in E$ and a $u \in J^{\#}(E)$ such that $\alpha = (1-\pi)+u$ and $\pi u = u\pi$. Thus, $\pi\alpha = \pi u \in J^{\#}\big(\pi E\pi\big)$. Further, $1-\alpha = \pi + (-u)$, and so $(1-\pi)(1-\alpha) = (1-\pi)(-u) \in J^{\#}\big((1-\pi)E(1-\pi)\big)$. Set $P = M\pi$ and $Q = M(1-\pi)$. Then $M = P \oplus Q$. As $\alpha\pi = \pi\alpha$, we see that P and Q are α-invariant. As $\alpha\pi \in J^{\#}\big(\pi E\pi\big)$, we can find some $t \in \mathbb{N}$ such that $(\alpha\pi)^t \in J\big(\pi E\pi\big)$. Let $\gamma \in end(P)$. For any $x \in M$, it is easy to see that $(x)\pi\big(1_P - \gamma\big(\alpha|_P\big)^t\big) = (x)\pi\big(\pi - (\pi\overline{\gamma}\pi)(\pi\alpha\pi)^t\big)$ where $\overline{\gamma} : M \to M$ given by $(m)\overline{\gamma} = (m)\pi\gamma$ for any $m \in M$. Hence, $1_P - \gamma\big(\alpha|_P\big)^t \in aut(P)$. Hence $\big(\alpha|_P\big)^t \in J\big(end(P)\big)$. This implies that $\alpha|_P \in J^{\#}\big(end(P)\big)$. Likewise, we verify that $(1-\alpha)|_Q \in J^{\#}\big(end(Q)\big)$.

(2) $\Rightarrow$ (1) For any $\lambda \in end(Q)$, we construct an R-homomorphism $\overline{\lambda} \in end(M)$ given by $\big(p+q\big)\overline{\lambda} = (q)\lambda$. By hypothesis, $\alpha|_P \in J^{\#}\big(end(P)\big)$ and $(1_M - \alpha)|_Q \in J^{\#}\big(end(Q)\big)$. Thus, $\alpha = \overline{1_Q} + \overline{\alpha|_P} - \overline{(1_M-\alpha)|_Q}$. As P and Q are α-invariant, we see that $\alpha\overline{1_Q} = \overline{1_Q}\alpha$. In addition, $\overline{1_Q} \in end(M)$ is an idempotent. As $\big(\overline{\alpha|_P}\big)\big(\overline{(1_M-\alpha)|_Q}\big) = 0 = \big(\overline{(1_M-\alpha)|_Q}\big)\big(\overline{\alpha|_P}\big)$, we show that $\overline{\alpha|_P} - \overline{(1_M-\alpha)|_Q} \in J^{\#}\big(end(M)\big)$, as required. □

Lemma 2.2 *Let R be a ring, and let M be a left R-module. Suppose that $x, y, a, b \in end(M)$ such that $xa + yb = 1_M, xy = yx = 0, ay = ya$ and $xb = bx$. Then $M = ker(x) \oplus ker(y)$ as left R-modules.*

Proof Straightforward. (cf. [6, Lemma 3.2.6]). □

Lemma 2.3 *Let R be a commutative ring, and let $\varphi \in M_n(R)$. Then the following are equivalent:*

(1) $\varphi \in J^{\#}\big(M_n(R)\big)$.

(2) $\chi(\varphi) \equiv t^n\big(mod\ J(R)\big)$, i.e., $\chi(\varphi) - t^n \in J(R)[t]$.

(3) There exists a monic polynomial $h \in R[t]$ such that $h \equiv t^{degh}\big(mod\ J(R)\big)$ for which $h(\varphi) = 0$.

Proof (1) $\Rightarrow$ (2) Since $\varphi \in J^{\#}\big(M_n(R)\big)$, there exists some $m \in \mathbb{N}$ such that $\varphi^m \in J\big(M_n(R)\big)$. As $J\big(M_n(R)\big) = M_n\big(J(R)\big)$, we get $\overline{\varphi} \in N\big(M_n(R/J(R))\big)$. In view of [6, Proposition 3.5.4], $\chi\big(\overline{\varphi}\big) \equiv t^n\big(mod\ N(R/J(R))\big)$. Write $\chi(\varphi) = t^n + a_1t^{n-1} + \cdots + a_n$. Then $\chi\big(\overline{\varphi}\big) = t^n + \overline{a_1}t^{n-1} + \cdots + \overline{a_n}$. We infer that each $a_i^{m_i} + J(R) = 0 + J(R)$ where $m_i \in \mathbb{N}$. This implies that $a_i \in J^{\#}(R)$. That is, $\chi(\varphi) \equiv t^n\big(mod\ J^{\#}(R)\big)$. Obviously, $J(R) \subseteq J^{\#}(R)$. For any $x \in J^{\#}(R)$, then there exists some $m \in \mathbb{N}$ such that $x^n \in J(R)$. For any maximal ideal M of R, M is prime, and so $x \in M$. This implies that $x \in J(R)$; hence, $J^{\#}(R) \subseteq J(R)$. Therefore $J^{\#}(R) = J(R)$, as required.

(2) $\Rightarrow$ (3) Choose $h = \chi(\varphi)$. Then $h \equiv t^{degh}\big(mod\ J(R)\big)$. In light of the Cayley-Hamilton Theorem, $h(\varphi) = 0$, as required.

(3) $\Rightarrow$ (1) By hypothesis, there exists a monic polynomial $h \in R[t]$ such that $h \equiv t^{degh}\big(mod\ J(R)\big)$ for which $h(\varphi) = 0$. Write $h = t^n + a_1t^{n-1} + \cdots + a_n$. Choose $\overline{h} = t^n + \overline{a_1}t^{n-1} + \cdots + \overline{a_n} \in \big(R/J(R)\big)[t]$. Then $\overline{h} \equiv t^n\big(mod\ N(R/J(R))\big)$ for which $\overline{h}\big(\overline{\varphi}\big) = 0$. According to [6, Proposition 3.5.4], there exists some $m \in \mathbb{N}$ such that $\big(\overline{\varphi}\big)^m = \overline{0}$ over $R/J(R)$. Therefore $\varphi^m \in M_n\big(J(R)\big)$, and so $\varphi \in J^{\#}\big(M_n(R)\big)$. □

Definition 2.4 *For $r \in R$, define*

$$\mathbb{J}_r = \{f \in R[t] \mid f \text{ monic, and } f \equiv (t-r)^{degf}\big(mod\ J^{\#}(R)\big)\}.$$

Lemma 2.5 *Let R be a projective-free ring, let $\varphi \in M_n(R)$, and let $h \in R[t]$ be a monic polynomial of degree n. If $h(\varphi) = 0$ and there exists a factorization $h = h_0h_1$ such that $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1$, then φ is strongly $J^{\#}$-clean.*

Proof Suppose that $h = h_0h_1$ where $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1$. Write $h_0 = t^p + a_1t^{p-1} + \cdots + a_p$ and $h_1 = (t-1)^q + b_1t^{q-1} + \cdots + b_q$. Then each $a_i, b_j \in J^{\#}(R)$. Since R is commutative, we get each $a_i, b_j \in J(R)$. Thus, $\overline{h_0} = t^p$ and $\overline{h_1} = (t-\overline{1})^q$ in $\big(R/J(R)\big)[t]$. Hence, $\big(\overline{h_0}, \overline{h_1}\big) = \overline{1}$, In virtue of [6, Lemma 3.5.10], we have some $u_0, u_1 \in R[t]$ such that $u_0h_0 + u_1h_1 = 1$. Then $u_0(\varphi)h_0(\varphi) + u_1(\varphi)h_1(\varphi) = 1_{nR}$. By hypothesis, $h(\varphi) = h_0(\varphi)h_1(\varphi) = h_1(\varphi)h_0(\varphi) = 0$. Clearly, $u_0(\varphi)h_1(\varphi) = h_1(\varphi)u_0(\varphi)$ and $h_0(\varphi)u_1(\varphi) = u_1(\varphi)h_0(\varphi)$. In light of Lemma 2.2, $nR = ker\big(h_0(\varphi)\big) \oplus ker\big(h_1(\varphi)\big)$. As $h_0t = th_0$ and $h_1t = th_1$, we see that $h_0(\varphi)\varphi = \varphi h_0(\varphi)$ and $h_1(\varphi)\varphi = \varphi h_1(\varphi)$, and so $ker\big(h_0(\varphi)\big)$ and $ker\big(h_1(\varphi)\big)$ are both φ-invariant. It is easy to verify that $h_0\big(\varphi\mid_{ker(h_0(\varphi))}\big) = 0$. Since $h_0 \in \mathbb{J}_0$, we see that $h_0 \equiv t^{degh_0}\big(mod\ J^{\#}(R)\big)$; hence, $\varphi\mid_{ker(h_0(\varphi))} \in J^{\#}\big(end(kerh_0(\varphi))\big)$.

It is easy to verify that $h_1\big(\varphi\mid_{ker(h_1(\varphi))}\big) = 0$. Set $g(u) = (-1)^{degh_1}h_1(1-u)$. Then $g\big((1-\varphi)\mid_{ker(h_1(\varphi))}\big) = 0$. Since $h_1 \in \mathbb{J}_1$, we see that $h_1 \equiv (t-1)^{degh_1}\big(mod\ J^{\#}(R)\big)$. Hence, $g(u) \equiv (-1)^{degh_1}(-u)^{degg}\big(mod\ J(R)\big)$. This implies that $g \in \mathbb{J}_0$. By virtue of Lemma 2.3, $(1-\varphi)\mid_{ker(h_1(\varphi)}) \in J^{\#}\big(end(ker(h_1(\varphi)))\big)$. According to Lemma 2.1, $\varphi \in M_n(R)$ is strongly $J^{\#}$-clean. □

The matrix

$$\varphi = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix} \in M_n(R)$$

is called the companion matrix C_h of h, where $h = t^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0 \in R[t]$.

Theorem 2.6 *Let R be a projective-free ring and let $h \in R[t]$ be a monic polynomial of degree n. Then the following are equivalent:*

(1) Every $\varphi \in M_n(R)$ with $\chi(\varphi) = h$ is strongly $J^{\#}$-clean.

(2) The companion matrix C_h of h is strongly $J^{\#}$-clean.

(3) There exists a factorization $h = h_0h_1$ such that $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1$.

Proof (1) $\Rightarrow$ (2) Write $h = t^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0 \in R[t]$. Choose

$$C_h = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix} \in M_n(R).$$

Then $\chi(C_h) = h$. By hypothesis, $C_h \in M_n(R)$ is strongly $J^{\#}$-clean.

(2) $\Rightarrow$ (3) In view of Lemma 2.1, there exists a decomposition $nR = A \oplus B$ such that A and B are φ-invariant, $\varphi \mid_A \in J^{\#}\big(end_R(A)\big)$ and $(1-\varphi) \mid_B \in J^{\#}\big(end_R(B)\big)$. Since R is a projective-free ring, there exist $p, q \in \mathbb{N}$ such that $A \cong pR$ and $B \cong qR$. Regarding $end_R(A)$ as $M_p(R)$, we see that $\varphi \mid_A \in J^{\#}\big(M_p(R)\big)$. By virtue of Lemma 2.3, $\chi(\varphi \mid_A) \equiv t^p\big(mod\ J^{\#}(R)\big)$. Thus $\chi(\varphi \mid_A) \in \mathbb{J}_0$. Analogously, $(1-\varphi) \mid_B \in J^{\#}\big(M_q(R)\big)$. It follows from Lemma 2.3 that $\chi\big((1-\varphi) \mid_B\big) \equiv t^q \ \big(\bmod\ J^{\#}(R)\big)$. This implies that $det\big(\lambda I_q - (1-\varphi) \mid_B\big) \equiv \lambda^q \ \big(\bmod\ J^{\#}(R)\big)$. Hence, $det\big((1-\lambda)I_q - \varphi \mid_B\big) \equiv (-\lambda)^q \ \big(\bmod\ J^{\#}(R)\big)$. Set $t = 1-\lambda$. Then $det\big(tI_q - \varphi \mid_B\big) \equiv (t-1)^q \ \big(\bmod\ J^{\#}(R)\big)$. Therefore we get $\chi(\varphi \mid_B) \equiv (t-1)^q \ \big(\bmod\ J^{\#}(R)\big)$. We infer that $\chi(\varphi \mid_B) \in \mathbb{J}_1$. Clearly, $\chi(\varphi) = \chi(\varphi \mid_A)\chi(\varphi \mid_B)$. Choose $h_0 = \chi(\varphi \mid_A)$ and $h_1 = \chi(\varphi \mid_B)$. Then there exists a factorization $h = h_0h_1$ such that $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1$, as desired.

(3) $\Rightarrow$ (1) For every $\varphi \in M_n(R)$ with $\chi(\varphi) = h$, it follows by the Cayley-Hamilton Theorem that $h(\varphi) = 0$. Therefore φ is strongly $J^{\#}$-clean by Lemma 2.5. □

Corollary 2.7 *Let F be a field, and let $A \in M_n(F)$. Then the following are equivalent:*

(1) A is the sum of an idempotent matrix and a nilpotent matrix that commute.

(2) $\chi(A) = t^s(t-1)^t$ *for some* $s, t \geq 0$.

Proof As $J\big(M_n(F)\big) = 0$, we see that a $n \times n$ matrix contains in $J^{\#}\big(M_n(F)\big)$ if and only if A is a nilpotent matrix. So $A \in M_n(F)$ is strongly $J^{\#}$-clean if and only if A is the sum of an idempotent matrix and a nilpotent matrix that commute. By virtue of Theorem 2.6, we see that $A \in M_n(F)$ is the sum of an idempotent matrix and a nilpotent matrix that commute if and only if $\chi(A) = h_0 h_1$, where $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1$. Clearly, $h_0 \in \mathbb{J}_0$ if and only if $h_0 \equiv t^{degh_0}(mod\ J^{\#}(F))$. But $J^{\#}(F) = 0$, and so $h_0 = t^s$, where $s = degh_0$. Likewise, $h_1 = (t-1)^t$, where $t = degh_1$. Therefore we complete the proof. $\square$

For matrices over integers ,we have a similar situation. As $J\big(M_n(\mathbb{Z})\big) = 0$, we see that an $n \times n$ matrix contains in $J^{\#}\big(M_n(\mathbb{Z})\big)$ if and only if it is a nilpotent matrix. Likewise, we show that $A \in M_n(\mathbb{Z})$ is the sum of an idempotent matrix and a nilpotent matrix that commute if and only if $\chi(A) = t^s(t-1)^t$ for some $s, t \geq 0$. For instance, choose $A = \begin{pmatrix} -2 & 2 & -1 \\ -4 & 4 & -2 \\ -1 & 1 & 0 \end{pmatrix} \in M_3(\mathbb{Z})$. Then $\chi(A) = t(t-1)^2$. Thus, A is the sum of an idempotent matrix and an nilpotent matrix that commute. In fact, we have a corresponding factorization $A = \begin{pmatrix} -1 & 1 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 1 & -1 \\ -2 & 2 & -2 \\ -1 & 1 & -1 \end{pmatrix}$.

Corollary 2.8 *Let* R *be a projective-free ring, and let* $\varphi \in M_2(R)$. *Then* φ *is strongly* $J^{\#}$*-clean if and only if*

(1) $\chi(\varphi) \equiv t^2\big(mod\ J(R)\big)$*; or*

(2) $\chi(\varphi) \equiv (t-1)^2\big(mod\ J(R)\big)$*; or*

(3) $\chi(\varphi)$ *has a root in* $J(R)$ *and a root in* $1 + J(R)$.

Proof Suppose that φ is strongly $J^{\#}$-clean. By virtue of Theorem 2.6, there exists a factorization $\chi(\varphi) = h_0 h_1$ such that $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1$.

Case I. $deg(h_0) = 2$ and $deg(h_1) = 0$. Then $h_0 = \chi(\varphi) = t^2 - tr(\varphi)t + det(\varphi)$ and $h_1 = 1$. As $h_0 \in \mathbb{J}_0$, it follows from Lemma 2.3 that $\varphi \in J^{\#}\big(M_2(R)\big)$ or $\chi(\varphi) \equiv t^2\big(mod\ J(R)\big)$.

Case II. $deg(h_0) = 1$ and $deg(h_1) = 1$. Then $h_0 = t - \alpha$ and $h_1 = t - \beta$. Since R is commutative, $J^{\#}(R) = J(R)$. As $h_0 \in \mathbb{J}_0$, we see that $h_0 \equiv t(mod\ J(R))$, and then $\alpha \in J(R)$. As $h_1 \in \mathbb{J}_1$, we see that $h_1 \equiv t - 1(mod\ J(R))$, and then $\beta \in 1 + J(R)$. Therefore $\chi(\varphi)$ has a root in $J(R)$ and a root in $1 + J(R)$.

Case III. $deg(h_0) = 0$ and $deg(h_1) = 2$. Then $h_1(t) = det\big(tI_2 - \varphi\big) \equiv (t-1)^2(mod\ J(R))$. Set $u = 1 - t$. Then $det\big(uI_2 - (I_2 - \varphi)\big) \equiv u^2\big(mod\ J(R)\big)$. According to Lemma 2.3, $I_2 - \varphi \in J^{\#}\big(M_2(R)\big)$ or $\chi(\varphi) \equiv (t-1)^2\big(mod\ J(R)\big)$.

We will suffice to show the converse. If $\chi(\varphi) \equiv t^2\big(mod\ J(R)\big)$ or $\chi(\varphi) \equiv (t-1)^2\big(mod\ J(R)\big)$, then $\varphi \in J^{\#}\big(M_2(R)\big)$ or $I_2 - \varphi \in J^{\#}\big(M_2(R)\big)$. This implies that φ is strongly $J^{\#}$-clean. Otherwise, $\varphi, I_2 - \varphi \notin J\big(M_2(R)\big)$. In addition, $\chi(\varphi)$ has a root in $J(R)$ and a root in $1 + J(R)$. According to [4, Theorem 16.4.31], φ is strongly J-clean, and therefore it is strongly $J^{\#}$-clean. $\square$

Choose $A = \begin{pmatrix} \overline{0} & \overline{2} \\ \overline{1} & \overline{3} \end{pmatrix} \in M_2\big(\mathbb{Z}_4\big)$. It is easy to check that $A, I_2 - A \in M_2\big(\mathbb{Z}_4\big)$ are not nilpotent. But $\chi(A) = t^2 + t + 2$ has a root $\overline{2} \in J(\mathbb{Z}_4)$ and a root $\overline{1} \in 1 + J(\mathbb{Z}_4)$. As $J(\mathbb{Z}_4) = \{\overline{0}, \overline{2}\}$ is nil, we know that every matrix in $J^{\#}\big(M_2(\mathbb{Z}_4)\big)$ is nilpotent. It follows from Corollary 2.8 that A is the sum of an idempotent matrix and a nilpotent matrix that commute. Let $\mathbb{Z}_{(2)} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, 2 \nmid n\}$, and let $A = \begin{pmatrix} 1 & 1 \\ \frac{2}{9} & 0 \end{pmatrix} \in M_2(\mathbb{Z}_{(2)})$. Then $J(\mathbb{Z}_{(2)}) = \{\frac{2m}{n} \mid m, n \in \mathbb{Z}, 2 \nmid n\}$. As $\chi(A) = t^2 - t + \frac{2}{9}$ has a root $\frac{1}{3} \in 1 + J(\mathbb{Z}_{(2)})$ and a root $\frac{2}{3} \in J(\mathbb{Z}_{(2)})$. In light of Corollary 2.8, A is strongly J-clean.

Corollary 2.9 *Let R be a projective-free ring, and let $f(t) = t^2 + at + b \in R[t]$ be degree 2 polynomial with $1 + a \in J(R), b \notin J(R)$. Then the following are equivalent:*

(1) Every $\varphi \in M_2(R)$ with $\chi(\varphi) = f(t)$ is strongly $J^{\#}$-clean.

(2) There exist $r_1 \in J(R)$ and $r_2 \in 1 + J(R)$ such that $f(r_i) = 0$.

(3) There exists $r \in J(R)$ such that $f(r) = 0$.

Proof (1) $\Rightarrow$ (2) Since every $\varphi \in M_2(R)$ with $\chi(\varphi) = f(t)$ is strongly $J^{\#}$-clean, it follows by Corollary 2.8 that $f(t) = (t - r_1)(t - r_2)$ with $r_1 \in J(R), r_2 \in 1 + J(R)$.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) As $r^2 + ar + b = 0$, we see that $f(t) = (t - r)(t + a + r)$. Clearly, $t - r \in \mathbb{J}_0$. As $1 + a + r \in J(R)$, we see that $t + a + r \in \mathbb{J}_1$. According to Theorem 2.6, we complete the proof. $\square$

Let φ be a 3×3 matrix over a commutative ring R. Set $mid(\varphi) = det(I_3 - \varphi) - 1 + tr(\varphi) + det(\varphi)$.

Corollary 2.10 *Let R be a projective-free ring, and let $\varphi \in M_3(R)$. Then φ is strongly $J^{\#}$-clean if and only if*

(1) *$\chi(\varphi) \equiv t^3\big(mod\ J(R)\big)$; or*

(2) *$\chi(\varphi) \equiv (t-1)^3\big(mod\ J(R)\big)$; or*

(3) *$\chi(\varphi)$ has a root in $1 + J(R)$, $tr(\varphi) \in 1 + J(R)$, $mid(\varphi) \in J(R)$, $det(\varphi) \in J(R)$; or*

(4) *$\chi(\varphi)$ has a root in $J(R)$, $tr(\varphi) \in 2 + J(R)$, $mid(\varphi) \in 1 + J(R)$, $det(\varphi) \in J(R)$.*

Proof Suppose that φ is strongly $J^{\#}$-clean. By virtue of Theorem 2.6, there exists a factorization $\chi(\varphi) = h_0h_1$ such that $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1$.

Case I. $deg(h_0) = 3$ and $deg(h_1) = 0$. Then $h_0 = \chi(\varphi)$ and $h_1 = 1$. As $h_0 \in \mathbb{J}_0$, it follows from Lemma 2.3 that $\varphi \in J^{\#}\big(M_3(R)\big)$.

Case II. $deg(h_0) = 0$ and $deg(h_1) = 3$. Then $h_1(t) = det\big(tI_3 - \varphi\big) \equiv (t-1)^3(mod\ J(R))$. Set $u = 1 - t$. Then $det\big(uI_3 - (I_3 - \varphi)\big) \equiv u^3(mod\ J(R))$. According to Lemma 2.3, $I_3 - \varphi \in J^{\#}\big(M_3(R)\big)$.

Case III. $deg(h_0) = 2$ and $deg(h_1) = 1$. Then $h_0 = t^2 + at + b$ and $h_1 = t - \alpha$. As $h_0 \in \mathbb{J}_0$, we see that $h_0 \equiv t^2(mod\ J(R))$; hence, $a, b \in J(R)$. As $h_1 \in \mathbb{J}_1$, we see that $h_1 \equiv t-1(mod\ J(R))$; hence, $\alpha \in 1+J(R)$. We see that $a-\alpha = -tr(\varphi), b-a\alpha = mid(\varphi)$ and $-b\alpha = -det(\varphi)$. Therefore $tr(\varphi) \in 1 + J(R), mid(\varphi) \in J(R)$ and $det(\varphi) \in J(R)$.

Case IV. $deg(h_0) = 1$ and $deg(h_1) = 2$. Then $h_0 = t - \alpha$ and $h_1 = t^2 + at + b$. As $h_0 \in \mathbb{J}_0$, we see that $h_0 \equiv t(mod\ J(R))$; hence, $\alpha \in J(R)$. As $h_1 \in \mathbb{J}_1$, we see that $h_1 \equiv (t-1)^2(mod\ J(R))$, and then $a \in -2 + J(R)$ and $b \in 1 + J(R)$. Obviously, $\chi(\varphi) = t^3 - tr(\varphi)t^2 + mid(\varphi)t - det(\varphi)$, and so $a - \alpha = -tr(\varphi), b - a\alpha = mid(\varphi)$ and $-b\alpha = -det(\varphi)$. Therefore $tr(\varphi) \in 2 + J(R), mid(\varphi) \in 1 + J(R)$ and $det(\varphi) \in J(R)$.

Conversely, if $\chi(\varphi) \equiv t^3\big(mod\ J(R)\big)$ or $\chi(\varphi) \equiv (t-1)^3\big(mod\ J(R)\big)$, then $\varphi \in J^{\#}\big(M_3(R)\big)$ or $I_3 - \varphi \in J^{\#}\big(M_3(R)\big)$. Hence, φ is strongly $J^{\#}$-clean. Suppose $\chi(\varphi)$ has a root $\alpha \in 1+J(R)$ and $tr(\varphi) \in 1+J(R), det(\varphi) \in J(R)$. Then $\chi(\varphi) = (t^2+at+b)(t-\alpha)$ for some $a, b \in R$. This implies that $a - \alpha = -tr(\varphi), -b\alpha = -det(\varphi)$. Hence, $a, b \in J(R)$. Let $h_0 = t^2 + at + b$ and $h_1 = t - \alpha$. Then $\chi(\varphi) = h_0h_1$ where $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1$. According to Theorem 2.6, φ is strongly $J^{\#}$-clean.

Suppose $\chi(\varphi)$ has a root $\alpha \in J(R)$ and $tr(\varphi) \in 2 + J(R), mid(\varphi) \in 1 + J(R)$ and $det(\varphi) \in J(R)$. Then $\chi(\varphi) = (t - \alpha)(t^2 + at + b)$ for some $a, b \in R$. This implies that $a - \alpha = -tr(\varphi), b - a\alpha = mid(\varphi)$. Hence, $a \in -2 + J(R), b \in 1 + J(R)$. Let $h_0 = t - \alpha$ and $h_1 = t^2 + at + b$. Then $\chi(\varphi) = h_0h_1$ where $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1$. According to Theorem 2.6, φ is strongly $J^{\#}$-clean, and we are done. □

3 Matrices Over Power Series Rings

The purpose of this section is to extend the preceding discussion to matrices over power series rings. We use $R[[x]]$ to stand for the ring of all power series over R. Let $A(x) = \big(a_{ij}(x)\big) \in M_n\big(R[[x]]\big)$. We use $A(0)$ to stand for $\big(a_{ij}(0)\big) \in M_n(R)$.

Theorem 3.1 *Let R be a projective-free ring, and let $A(x) \in M_2\big(R[[x]]\big)$. Then the following are equivalent:*

(1) $A(x) \in M_2\big(R[[x]]\big)$ is strongly $J^{\#}$-clean.

(2) $A(0) \in M_2(R)$ is strongly $J^{\#}$-clean.

Proof (1) $\Rightarrow$ (2) Since $A(x)$ is strongly $J^{\#}$-clean in $M_2\big(R[[x]]\big)$, there exists an $E(x) = E^2(x) \in M_2\big(R[[x]]\big)$ and a $U(x) \in J^{\#}\big(M_2(R[[x]])\big)$ such that $A(x) = E(x) + U(x)$ and $E(x)U(x) = U(x)E(x)$. This implies that $A(0) = E(0) + U(0)$ and $E(0)U(0) = U(0)E(0)$ where $E(0) = E^2(0) \in M_2(R)$ and $U(0) \in J^{\#}\big(M_2(R)\big)$. As a result, $A(0)$ is strongly $J^{\#}$-clean in $M_2(R)$.

(2) $\Rightarrow$ (1) Construct a ring morphism $\varphi : R[[x]] \to R, f(x) \mapsto f(0)$. Then $R \cong R[[x]]/kerf$, where $kerf = \{f(x) \mid f(0) = 0\} \subseteq J\big(R[[x]]\big)$. For any finitely generated projective $R[[x]]$-module P, $P \bigotimes\limits_R \big(R[[x]]/kerf\big)$ is a finitely generated projective $R[[x]]/kerf$-module; hence it is free. Write $P \bigotimes\limits_R \big(R[[x]]/kerf\big) \cong \big(R[[x]]/kerf\big)^m$ for some $m\mathbb{N}$. Then $P \bigotimes\limits_R \big(R[[x]]/kerf\big) \cong \big(R[[x]]\big)^m \bigotimes\limits_R \big(R[[x]]/kerf\big)$. That is, $P/P\big(kerf\big) \cong \big(R[[x]]\big)^m/\big(R[[x]]\big)^m\big(kerf\big)$ wit$kerf \subseteq J\big(R[[x]]\big)$. By Nakayama Theorem, $P \cong \big(R[[x]]\big)^m$ is free. Thus, $R[[x]]$ is projective-free. Since $A(0)$ is strongly $J^{\#}$-clean in $M_2(R)$, it follows from Corollary 2.8 that $A(0) \in J^{\#}\big(M_2(R)\big)$, or $I_2 - A(0) \in J^{\#}\big(M_2(R)\big)$, or the characteristic polynomial $\chi\big(A(0)\big) = y^2 + \mu y + \lambda$ has a root $\alpha \in 1+J(R)$ and a root $\beta \in J(R)$. If $A(0) \in J^{\#}\big(M_2(R)\big)$, then $A(x) \in J^{\#}\big(M_2(R[[x]])\big)$. If $I_2 - A(0) \in J^{\#}\big(M_2(R)\big)$, then $I_2 - A(x) \in J^{\#}\big(M_2(R[[x]])\big)$. Otherwise, we write $y = \sum\limits_{i=0}^{\infty} b_i x^i$ and $\chi(A(x)) = y^2 - \mu(x)y - \lambda(x)$. Then $y^2 = \sum\limits_{i=0}^{\infty} c_i x^i$ where $c_i = \sum\limits_{k=0}^{i} b_k b_{i-k}$. Let $\mu(x) = \sum\limits_{i=0}^{\infty} \mu_i x^i, \lambda(x) = \sum\limits_{i=0}^{\infty} \lambda_i x^i \in R[[x]]$ where $\mu_0 = \mu$ and $\lambda_0 = \lambda$. Then, $y^2 - \mu(x)y - \lambda(x) = 0$ holds in $R[[x]]$ if the following equations are satisfied:

$$\begin{gathered} b_0^2 - b_0\mu_0 - \lambda_0 = 0; \\ (b_0b_1 + b_1b_0) - (b_0\mu_1 + b_1\mu_0) - \lambda_1 = 0; \\ (b_0b_2 + b_1^2 + b_2b_0) - (b_0\mu_2 + b_1\mu_1 + b_2\mu_0) - \lambda_2 = 0; \\ \vdots \end{gathered}$$

Obviously, $\mu_0 = \alpha + \beta \in U(R)$ and $\alpha - \beta \in U(R)$. Let $b_0 = \alpha$. Since R is commutative, there exists some $b_1 \in R$ such that

$$b_0b_1 + b_1(b_0 - \mu_0) = \lambda_1 + b_0\mu_1.$$

Further, there exists some $b_2 \in R$ such that

$$b_0b_2 + b_2(b_0 - \mu_0) = \lambda_2 - b_1^2 + b_0\mu_2 + b_1\mu_1.$$

By iteration of this process, we get $b_3, b_4, \cdots$. Then $y^2 - \mu(x)y - \lambda(x) = 0$ has a root $y_0(x) \in 1+J\big(R[[x]]\big)$. If $b_0 = \beta \in J(R)$, analogously, we show that $y^2 - \mu(x)y - \lambda(x) = 0$ has a root $y_1(x) \in J\big(R[[x]]\big)$. In light of Corollary 2.8, the result follows. $\square$

Corollary 3.2 *Let R be a projective-free ring, and let $A(x) \in M_2\big(R[[x]]/(x^m)\big)$ $(m \geq 1)$. Then the following are equivalent:*

(1) $A(x) \in M_2\big(R[[x]]/(x^m)\big)$ is strongly $J^{\#}$-clean.

(2) $A(0) \in M_2(R)$ is strongly $J^{\#}$-clean.

Proof (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1) Let $\psi : R[[x]] \to R[[x]]/(x^m), \psi(f) = \overline{f}$. Then it reduces a surjective ring homomorphism $\psi^* : M_2\big(R[[x]]\big) \to M_2\big(R[[x]]/(x^m)\big)$. Hence, we have a $B \in M_2\big(R[[x]]\big)$ such that $\psi^*\big(B(x)\big) = A(x)$. According to Theorem 3.1, we complete the proof. $\square$

Example 3.3 Let $R = \mathbb{Z}_4[x]/(x^2)$, and let $A(x) = \begin{pmatrix} \overline{2} & \overline{2} + \overline{2}x \\ \overline{2} + x & \overline{3} + \overline{3}x \end{pmatrix} \in M_2(R)$. Obviously, $\mathbb{Z}_4$ is a projective-free ring, and that $R = \mathbb{Z}_4[[x]]/(x^2)$. Since we have the strongly $J^{\#}$-clean decomposition $A(0) = \begin{pmatrix} \overline{0} & \overline{2} \\ \overline{2} & \overline{1} \end{pmatrix} + \begin{pmatrix} \overline{2} & \overline{0} \\ \overline{0} & \overline{2} \end{pmatrix}$ in $M_2(\mathbb{Z}_4)$, it follows by Corollary 3.2 that $A(x) \in M_2(R)$ is strongly $J^{\#}$-clean.

Theorem 3.4 *Let R be a projective-free ring, and let $A(x) \in M_3\big(R[[x]]\big)$. Then the following are equivalent:*

(1) $A(x) \in M_3\big(R[[x]]\big)$ is strongly $J^{\#}$-clean.

(2) $A(x) \in M_3\big(R[[x]]/(x^m)\big)(m \geq 1)$ is strongly $J^{\#}$-clean.

(3) $A(0) \in M_3(R)$ is strongly $J^{\#}$-clean.

Proof (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are clear.

(3) $\Rightarrow$ (1) As $A(0)$ is strongly $J^{\#}$-clean in $M_3(R)$, it follows from Corollary 2.10 that $A(0) \in J^{\#}\big(M_3(R)\big)$, or $I_3 - A(0) \in J^{\#}\big(M_3(R)\big)$, or $\chi\big(A(0)\big)$ has a root in $J(R)$ and $tr\big(A(0)\big) \in 2 + J(R), mid\big(A(0)\big) \in 1 + J(R), det\big(A(0)\big) \in J(R)$, or $\chi\big(A(0)\big)$ has a root in $1 + J(R)$ and $tr\big(A(0)\big) \in 1 + J(R), mid\big(A(0)\big) \in J(R), det\big(A(0)\big) \in J(R)$. If $A(0) \in J^{\#}\big(M_3(R)\big)$ or $I_3 - A(0) \in J^{\#}\big(M_3(R)\big)$, then $A(x) \in J^{\#}\big(M_3(R[[x]])\big)$ or $I_3 - A(x) \in J^{\#}\big(M_3(R[[x]])\big)$. Hence, $A(x) \in M_3\big(R[[x]]\big)$ is strongly $J^{\#}$-clean. Assume that $\chi\big(A(0)\big) = t^3 - \mu t^2 - \lambda t - \gamma$ has a root $\alpha \in J(R)$ and $tr\big(A(0)\big) \in 2 + J(R), mid\big(A(0)\big) \in 1 + J(R), det\big(A(0)\big) \in J(R)$. Write $y = \sum\limits_{i=0}^{\infty} b_i x^i$. Then $y^2 = \sum\limits_{i=0}^{\infty} c_i x^i$ where $c_i = \sum\limits_{k=0}^{i} b_k b_{i-k}$. Further, $y^3 = \sum\limits_{i=0}^{\infty} d_i x^i$ where $d_i = \sum\limits_{k=0}^{i} b_k c_{i-k}$. Let $\mu(x) = \sum\limits_{i=0}^{\infty} \mu_i x^i, \lambda(x) = \sum\limits_{i=0}^{\infty} \lambda_i x^i, \gamma(x) = \sum\limits_{i=0}^{\infty} \gamma_i x^i \in R[[x]]$ where $\mu_0 = \mu, \lambda_0 = \lambda$ and $\gamma_0 = \gamma$. Then, $y^3 - \mu(x)y^2 -$

$\lambda(x)y - \gamma(x) = 0$ holds in $R[[x]]$ if the following equations are satisfied:

$$\begin{gathered} b_0^3 - b_0^2\mu_0 - b_0\lambda_0 - \gamma_0 = 0; \\ (3b_0^2 - 2b_0\mu_0 - \lambda_0)b_1 = \gamma_1 + b_0^2\mu_1 + b_0\lambda_1; \\ (3b_0^2 - 2b_0\mu_0 - \lambda_0)b_2 = \gamma_2 + b_0^2\mu_2 + b_1^2\mu_0 + 2b_0b_1\mu_1 + b_0\lambda_2 + b_1\lambda_0 - 3b_0b_1^2; \\ \vdots \end{gathered}$$

Let $b_0 = \alpha \in J(R)$. Obviously, $\mu_0 = trA(0) \in 2 + J(R)$ and $\lambda_0 = -midA(0) \in U(R)$. Hence, $3b_0^2 - 2b_0\mu_0 - \lambda_0 \in U(R)$. Thus, we see that $b_1 = (3b_0^2 - 2b_0\mu_0 - \lambda_0)^{-1}(\gamma_1 + b_0^2\mu_1 + b_0\lambda_1)$ and $b_2 = (3b_0^2 - 2b_0\mu_0 - \lambda_0)^{-1}(\gamma_2 + b_0^2\mu_2 + b_1^2\mu_0 + 2b_0b_1\mu_1 + b_0\lambda_2 + b_1\lambda_0 - 3b_0b_1^2)$. By iteration of this process, we get $b_3, b_4, \cdots$. Then $y^3 - \mu(x)y^2 - \lambda(x)y - \gamma(x) = 0$ has a root $y_0(x) \in J\big(R[[x]]\big)$. It follows from $trA(0) \in 2 + J(R)$ that $trA(x) \in 2 + J\big(R[[x]]\big)$. Likewise, $midA(x) \in 1 + J\big(R[[x]]\big)$. According to Corollary 2.10, $A(x) \in M_3\big(R[[x]]\big)$ is strongly $J^{\#}$-clean.

Assume that $\chi\big(A(0)\big)$ has a root $1+\alpha \in J(R)$ and $tr\big(A(0)\big) \in 1+J(R), mid\big(A(0)\big) \in J(R), detA(0) \in J(R)$. Then $det\big(I_3 - A(0)\big) = 1 - trA(0) + midA(0) - detA(0) \in J(R)$. Set $B(x) = I_3 - A(x)$. Then $\chi\big(B(0)\big)$ has a root $\alpha \in J(R)$ and $tr\big(B(0)\big) \in 2 + J(R), detB(0) \in J(R)$. This implies that $midB(0) = detA(0) - 1 + trB(0) + detB(0) \in 1 + J(R)$. By the preceding discussion, we see that $B(x) \in M_3\big(R[[x]]\big)$ is strongly $J^{\#}$-clean, and then we are done. □

From this evidence above, we end this paper by asking the following question: Let R be a projective-free ring, and let $A(x) \in M_n\big(R[[x]]\big)(n \geq 4)$. Do the strongly $J^{\#}$-cleanness of $A(x) \in M_3\big(R[[x]]\big)$ and $A(0) \in M_3(R)$ coincide with each other?

Acknowledgements This research was supported by the Scientific and Technological Research Council of Turkey (2221 Visiting Scientists Fellowship Programme) and the Natural Science Foundation of Zhejiang Province (Y6090404).

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