OBSERVABILITY AND OBSERVERS FOR NONLINEAR AND SWITCHING SYSTEMS

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DOCTOR OF PHILOSOPHY

By

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August 2001

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ABSTRACT

OBSERVABILITY AND OBSERVERS FOR NONLINEAR AND SWITCHING SYSTEMS

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One of the fundamental problems in control theory is the observability and observer construction for nonlinear systems. In this thesis we study the observability conditions for some classes of nonlinear and switching systems and related observer design methods.

We adapt and modify high-gain observers to suit their application to observer-based chaos synchronization, control and tracking. We show that stringent global conditions on the system structure can be relaxed for chaotic systems. Using the close relation between chaos synchronization and observers, we provide chaotic message encoding schemes that incorporate exponential observers with robust message reconstruction properties. We also provide an eigenvalue assignment procedure to reduce gains and transient overshoots in high-gain observers for systems in Brunowsky canonical form. For the observability of discrete-time switching systems, we give an equivalence relation for the case of periodic switching, which in turn is used in observer construction. In the more general setting, we formulate the observability of switching systems as a feasibility problem and provide a recursive algorithm to determine the observability of a linear switching system. The algorithm relies on the recursive reduction of indistinguishable sets. We also prove some

sufficiency conditions on the stability of switching systems, extending the stable classes provided by some existing results in the field. Relation between stability for arbitrary switching and the existence of norms defined by non-trivial convex sets is also shown.

Keywords: Observability, Observers, Chaos synchronization, Chaotic encoding, Switching systems, Periodic switching.

ÖZET

DOĞRUSAL OLMAYAN VE ANAHTARLAMALI SİSTEMLER İÇİN GÖZLENEBİLİRLİK VE GÖZLEYİCİLER

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Kontrol teorisindeki temel problemlerden birisi doğrusal olmayan sistemlerin gözlenebilirliği ve gözleyici tasarımıdır. Biz bu tezde bir kısım doğrusal olmayan ve anahtarlamalı sistem sınıflarının gözlenebilirliklerini ve buna bağlı gözleyici tasarımlarını çalıştık.

Yüksek kazançlı gözleyicileri, kaos eşzamanlamasına, denetimine ve izlemesine uygun olacak şekilde uyarladık. Kaotik sistemler için sistem yapısına dair global sıkı koşulların gevşetilebileceğini gösterdik. Kaos eşzamanlaması ile gözleyiciler arasındaki yakın ilişkiden faydalanarak, üstel gözleyici kullanan gürbüz kestirim özellikli kaotik mesaj kodlama yöntemleri önerdik. Ayrıca Brunowsky kanonik şeklindeki sistemlerin yüksek kazanç gözleyicileri için kazançları ve geçici aşmaları düşüren özdeğer atama yöntemlerini gösterdik. Anahtarlamalı sistemlerin gözlenebilirliği konusunda, peryodik anahtarlama durumu için eşdeğer bir rank koşulu verdik ve bunu gözleyici tasarımında kullandık. Daha genel anahtarlama durumunda, anahtarlamalı sistemin gözlenebilirliğini bir fizibilite problemi olarak formüle ettik ve gözlenebilirliği belirlemek amacıyla tekrarlamalı bir algoritma önerdik. Algoritma ayırdedilemez kümelerin tekrarlamalı olarak küçültülmesine dayalıdır. Ayrıca anahtarlamalı sistemlerin kararlılığı üzerine bu alandaki kararlı sistem sınıflarını genişleten birkaç yeterli koşul ıspatladık. Rasgele anahtarlama altında kararlılık ile dışbükey kümelerle tanımlanan normlar arasındaki ilişkiyi gösterdik.

Anahtar Kelimeler: Gözlenebilirlik, Gözleyiciler, Kaos eşzamanlaması, Kaotik kodlama, Anahtarlamalı sistemler, Peryodik anahtarlama.

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Contents

List of Figures

List of Tables

Chapter 1

Introduction

Over the last three decades observability and observer design problems for dynamical systems have received a rather considerable attention from control community. Foremost among different motivations for the study of observers is the fact that state feedback control schemes are a superset of output feedback methods. Indeed, as the output of a dynamical system is an algebraic function of its states, any control objective realized using output feedback can also be achieved using state feedback provided that the output function can be implemented in the controller structure. However, in most practical cases, not all of the states are available nor can be made available through measurements. As a simple example compare the simplicity of measuring joint angles of a robot arm with cheap sensors to complexity involved in measuring their velocities.

If the designer of a plant and the designer of its controller are the same person, one would expect to have features in the system that anticipate and facilitate the control design. Unfortunately this is rarely the case. In general control designer is expected to take the system structure as it is, with actuator and sensor constraints, and provide a control configuration which satisfies certain objectives, like stability.

An observer is an auxiliary system which, when fed with input/output of a dynamical system, yields estimates of its internal states. In this respect, observers are employed to overcome the limitations imposed by the existing sensor topology of a dynamical system. When the state estimates are to be used in a state feedback configuration, the natural question to be asked is how good the controller performs when furnished with estimates rather than actual measurements. Related is the question whether it is possible to play with the inputs to improve (or deteriorate) state estimates. These two problems closely tie observability to inputs.

1.1 Inputs and Observability

Roughly speaking observability is the property that the internal states of a dynamical system can be determined from its input/output behavior and an observer is a dynamical system that achieves that. When using an observer in the control loop, one often implicitly assumes that a state feedback controller using the observed (estimated) states instead of true states (which are not available through direct measurements) will still satisfy the control objectives like stability and robustness. While this assumption proves valid in the case of linear systems controlled by linear observer-controller structure, lack of strong separability results for nonlinear systems often prevents the designer from analytically verifying the stability of the closed loop when the state estimates are used in the controller, [1].

Apart from the difficulty of using a working observer in the closed loop, building the observer itself for a nonlinear system is a challenging problem. Intricacies involved in nonlinear observer design start from the definition of observability.

Contrary to the case of linear systems, the observability of a nonlinear system is dependent on inputs in the general case. While some inputs may distinguish every state from every other state, in general input functions should be carefully chosen not to render the system unobservable. This is another way of seeing the difficulty in separating controller and observer design for nonlinear systems. For instance, if an observer is shown to work for a certain open set of inputs then controller should be designed so as to yield control inputs remaining strictly inside this set. Since this stringent requirement brings about further restrictions on the controller design, research has generally focused on uniformly observable systems, [2,3] or, totally disregarding inputs, on unforced systems. In fact the latter class of systems trivially generalizes to the cases where the input is injected into the system dynamics only additively, for, any additive input term can be exactly replicated in the observer dynamics thus cancelling in the error equation.

As the usual trade-off between generality and usefulness necessitates, in all areas of engineering one has to restrict the class of systems for which a working solution is proposed. In the context of this thesis all the dynamical systems for which observers are proposed are either unforced or the input appears additively in the system dynamics.

1.2 Observer Structure

After one establishes that a nonlinear dynamical system satisfies certain observability criteria for a range of operating regions, in order to design an observer to estimate unknown system state, one basically has to exploit the error between measured quantities, namely the outputs of the system and the observer. As in the estimation using a Kalman filter [4], generally the output error is used to obtain a direction of modification for the estimate. The new estimate is in turn reflected in the output error, which is, in turn, used in a still newer estimate. This is the well-known "estimate - predict - modify - estimate" cycle. Obviously this cycle is implemented in a differential setting for continuous-time systems while it becomes iteration for discrete-time systems.

Many existing observer design schemes can be considered as variations of the above approach and they differ only in the way the output error is injected into the estimation dynamics.

1.3 Applications of Observers

Besides the obvious use of observers in a control loop, some other interesting engineering problems can either be formulated as observer design problems or facilitated by the use of observers.

1.3.1 Fault Detection

One common application of observers has been the *fault detection*. Loosely speaking, the fault detection is the determination of occurences of faults in the structure of a dynamical system by comparing its actual behavior with an anticipated behavior based on the nominal model without faults, [5]. The subject of fault detection is itself a large area of research with vast amounts of related literature, for a review of the subject see [6] and the references cited therein. Observer-based fault diagnosis and detection methods basically employ a modification of classical observers to yield residual signals which, in turn, can be used in various decision algorithms to detect and localize the system faults, [7, 8]. As they become available, many of the observer design techniques have been modified and ported to fault detection problems, [9].

1.3.2 System Identification

Generally speaking *system identification* is the estimation of the unknown parameters in the model of a dynamical system through the observation of inputs, outputs and possibly states, [10]. Certain system identification problems can also be posed as observer design problems. Indeed, if a dynamical system can be given a structured model with a set of unknown but fixed parameters, identification of those parameters turns out to be equivalent to finding a functional observer. The equivalence can easily be seen when one includes the unknown parameters in the state vector and assigns their derivatives in the system description to zero (or assign the next iteration to the previous one for discrete-time systems). Similar techniques have successfully been applied to several identification problems, [11].

1.3.3 Chaos Synchronization

More recently *chaos synchronization* has been receiving a growing attention from diverse research communities. What sparked this interest were the seminal works [12, 13]. Although these early works lack the mathematical rigor and sound proofs, by demonstrating (by examples) that two chaotic systems can be coupled to have their uncoupled signals to synchronize, they were enough to create a new and fruitful area of research drawing on the efforts of electrical engineers, mathematicians and physicists, which is clearly evidenced by the appearance of special journal issues devoted to the subject, [14, 15].

Chaos synchronization has been shown to be reducible to an observer design problem by designating one of the chaotic systems as an observer for the other one and synchronizing signal as the output, [16, 17]. Doing so puts many ad hoc chaos synchronization schemes under the same analytical framework. Moreover, many existing nonlinear observer design methods can be modified to improve state estimation by incorporating some of the basic properties of chaotic systems.

Also some interesting problems are brought about by employing nonlinear observers in chaos synchronization. In control theoretical framework, when designing observers output function is assumed to be fixed due to system construction and availability of sensors. However, for the purpose of chaos synchronization one can tailor a suitable output function (therefore a synchronization signal that is to be sent over the communication channel) which will facilitate observer design.

Another challenge is to compensate for the filtering effects of the communication channel over the synchronization signal. Apart from the omnipresence of noise, wideband chaotic signals are adversely modified through a low-pass channel which may induce large estimation errors in the observer dynamics.

1.4 Observability of Switching Systems

A switching system is one which switches among distinct dynamics. Dynamic behavior of switching systems are described by a finite set of dynamical models and associated rules specifying the switching pattern among these models. Relays and hysteresis elements are familiar blocks that appear in switching systems, [18]. Also of interest is the study of the stability properties of systems that are controlled by switching among multiple controllers, [19]. Certain unstable systems for which no stabilizing smooth control laws exists can be stabilized using switching controllers, [20].

Switching system descriptions also arise when one approximates a nonlinear dynamical system by a collection of (possibly affine) linear systems. In this case switching rule can be specified by partitioning the state space into disjoint regions and assigning to each a linear system that is a sufficiently close approximation to the original when the state vector lies in that region.

Although undecidability results impose limits on the achievable practical necessary and sufficient conditions and resulting algorithms for the problems related to switching systems, [21], it did not hinder an increase in the research efforts in the study of certain restricted classes of switching systems, [22–24]. Stability, verification and simulation of switching systems are just a few of active research areas on the subject. The problems related to the observability of switching systems still remain unexplored. The present work also addresses some observability and stability issues encountered in the field.

1.5 Organization of the Thesis

Chapters 2 and 3 introduce the basic notions related to the observability and observers in a general context. The intricacy of the observability of nonlinear systems is illustrated through some examples. A few common observer design schemes for some classes of linear and nonlinear systems are recalled with special emphasis on observers for systems in the Brunowsky canonical form.

In Chapter 4 we show how the chaos synchronization can be formulated as a nonlinear observer design problem. Using the property that chaotic systems evolve in a bounded region of the state space, we adapt and modify the observer design for systems in the Brunowsky canonical form to obtain synchronized chaotic systems. Then we employ exponential observers in the chaotic modulation of messages with possible application to the secure transmission. We give two distinct message encoding schemes and illustrate them with simulations.

In Chapter 5 we mainly study the gain selection for observers of systems in the Brunowsky form. Exploiting the structure of the Vandermonde matrix we give a gain selection procedure that reduces the gains through eigenvalue assignment on a common circle. We provide some numerical optimization results and compare with the case when the eigenvalues are constrained to be real. We also study the output selection problem which is particularly meaningful when the observer is used in chaos synchronization.

In Chapter 6 we apply the gain selection results we obtained to observer based chaos control. We show that, under mild assumptions, forced chaotic oscillators can be controlled to have a desired dynamical behavior or even track a reference trajectory.

In Chapter 7 we study the observability of periodically switching systems and obtain equivalent observability rank conditions. We then use the result to design hybrid observers and prove their convergence.

In Chapter 8 the observability of more general switching systems is considered. We give the complete characterization of the observability for planar bimodal systems. We also show that the observability of switching systems can be formulated as feasibility problems and for bimodal systems give recursive procedure to analyze the observability. As a side issue we also address the stability of linear switching systems and find some stable classes of switching systems that are not covered by the existing sufficient conditions on stability.

The contributions of the thesis can be briefly summarized as follows; (*i*) formulation of the chaos synchronization problem as a nonlinear observer design problem and its subsequent application in chaotic message modulation and recovery (Chapter 4), (*ii*) a novel gain selection procedure to minimize gain magnitudes in observers for systems in Brunowsky canonical form and an attempt at parameterization of output maps that yield observable dynamical systems (Chapter 5), (*iii*) application of nonlinear observers to the control and tracking problems for forced chaotic oscillators (Chapter 6), (*iv*) a rank condition for the observability of periodically switching discrete-time systems and an observer design that employs this condition, a novel hybrid observer for periodic continuous-time systems (Chapter 7) (*v*) the complete characterization of the observability of bimodal discrete-time switching systems in the planar case and a novel recursive procedure to analyze the observability in higher dimensions, proof of the stability for arbitrary switching of systems switching among commonly symmetrizable matices, derivation of a known result as a corollary of this, necessary and sufficient conditions on the common symmetrizability of two matrices and its characterization as eigenvector inlerlacing for planar systems, a procedure to construct a common matrix norm for the set of matrices of systems stable for arbitrary switching (Chapter 8).

Chapter 2

Observability

Observability of a dynamical system is a measure of the transparency of the state information through input-output measurements. Thus observability is a property associated with the state-space description of the system rather than possible operator description from the space of inputs to the space of outputs. In the context of observability, system states are viewed as internal variables (not necessarily corresponding to physical quantities) some of which are not available through direct measurements. All the measurable functions of the states can be bunched together in the output map and observability becomes equivalent to the property that enables one to invert the relation from the states to the outputs.

An observer is a dynamical system that implements this operation of inversion in the sense that it dynamically yields at its output the state information of the original system while it takes as inputs the measurable quantities of the original system.

What follows is a brief overview of the basic definitions and results pertaining to the observability of dynamical systems and, in the next chapter, their implementations in observers.

2.1 Definition of Observability

Consider the state-space description of a time-invariant dynamical system

$$\dot{x} = f(x, u), \tag{2.1}$$

$$y = h(x), \tag{2.2}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$, $y \in \mathbf{R}^p$ are, respectively, state, input and output vectors, $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ and $h : \mathbf{R}^n \to \mathbf{R}^p$. Note that the respective sets for the state, input and output are taken to be the whole Euclidean spaces of appropriate dimensions. This assumption can be modified to include systems evolving on manifolds or more general abstract spaces (see, e.g., [25]) but this approach is not followed in the present work.

To introduce a proper observability definition for the system (2.1), (2.2) we first need to define distinguishability of states. Let us denote by $y(\cdot, x_0, u_0(\cdot))$ the output function (of time) corresponding to a solution of the system starting from a particular initial state x_0 and forced by a particular input function $u_0(\cdot)$.

Definition 2.1 *For the system described by (2.1), (2.2), two states x_1, x_2 are said to be* **distinguishable** *if there exists an input function $u_0(\cdot)$ such that two instances of system evolution starting from x_1 and x_2 yield different output functions, namely, $y(\cdot, x_1, u_0(\cdot)) \neq y(\cdot, x_2, u_0(\cdot))$. The system (2.1), (2.2) is said to be* **locally observable at $\mathbf{x_1}$** *if there is a neighborhood Ω around x_1 such that every other state in Ω is distinguishable from x_1 and simply* **locally observable** *if it is locally observable at every state. Further, a system is said to be* **globally observable** *if every state is distinguishable from every other state in the whole state space. Conversely, a system is said to be* **unobservable** *if it is not observable, [18, p. 414], [26].*

We sometimes refer to the observability of the system (2.1), (2.2) also as the **observability of the pair** (h, f). When the system is linear, that is, $\dot{x} = Ax$ and $y = Cx$, we also talk about the observability of the pair (C, A).

The input function $u(\cdot)$ in Definition 2.1 is sometimes referred to as a distinguishing input and in general depends on the states to be distinguished. In some cases a particular input function may distinguish every pair of states. Such inputs are known as **universal inputs**, [27]. Depending on the system description, the distinguishability property of some states may be independent of the input applied. Such is the case when the system is linear, [18, p. 415]. Input independent observability is an important property because it enables one to completely separate the observer design problem from the control synthesis problem, [2]. Otherwise, when the observer states are used in conjunction with a state feedback controller, special care has to be taken to ensure that the controller drives the system with distinguishing inputs, which can become quite cumbersome.

A related concept in control theory is the **detectability** which is a bit more relaxed condition than observability and more closely related to the conditions of the existence of an observer, [28–31]. Basically, a system is detectable if its unobservable dynamics are stable in some sense. Detectability property is especially important when an observer is used in conjunction with a state feedback controller. For a thorough discussion of nonlinear detectability see, e.g., [31].

For many classes of systems the observability definition has been translated into easily checkable algebraic conditions. The following section elaborates on these conditions that have appeared in the relevant literature.

2.2 Observability Conditions

2.2.1 Linear Systems

Observability of a linear system can be expressed in neat necessary and sufficient conditions leading to global decomposition of the system dynamics into observable and unobservable parts. The next theorem summarizes these results after a few preliminary definitions.

Consider a linear time invariant system described by

$$\dot{x} = Ax + Bu, \tag{2.3}$$

$$y = Cx, \tag{2.4}$$

where $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $C \in \mathbf{R}^{p\times n}$ and x, u and y are respectively the state, the input and the output vectors of appropriate dimensions. For this system the **observability matrix** $\mathbf{O} \in \mathbf{R}^{pn\times n}$ and the **observability grammian** $G_o(t_0, t_f) \in \mathbf{R}^{n\times n}$ are defined as

$$\mathbf{O} \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}, \qquad G_o(t_0, t_f) \triangleq \int_{t_0}^{t_f} e^{A^T\tau} C^T C e^{A\tau} d\tau \tag{2.5}$$

Theorem 2.1 *The following are equivalent.*

i. The system (2.3), (2.4) is locally observable,

ii. the system (2.3), (2.4) is globally observable,

iii. the columns of Ce^{At} are linearly independent over $\mathbf{R}$ for all t,

iv. the observability matrix $\mathbf{O}$ has full column rank,

v. the observability grammian is nonsingular for all $t_f > t_0$,

vi. the matrix $\begin{bmatrix} \lambda I - A \\ C \end{bmatrix}$ has full column rank for all eigenvalues λ_i of A.

vii. for any self-conjugate[1] set of n complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, there exists a gain matrix $K \in \mathbf{R}^{n\times p}$ such that λ_i's are eigenvalues of the matrix $A_c = A - KC$.

[1] *A set is self-conjugate if it is closed under conjugation operation.*

Proof: Proofs of these well-known facts can be found in any standard textbook on linear system theory, see, e.g., [32]. □

Besides giving checkable algebraic conditions for the observability, Theorem 2.1 underlines the fact that, for linear systems, observability is independent of inputs; if two states are distinguishable for a particular input then they are also distinguishable for every other input including zero input. Resulting separability property enables the design of controller to be independent of the observer.

For linear systems it is possible to find global transformations of the states which isolate the unobservable states (in the new coordinates) of the system, [32]. If the rank of the observability matrix $\mathbf{O}$ is $r < n$, then there is a nonsingular matrix T defining the state transformation $z = T^{-1}x$ such that in the z coordinates the system (2.3), (2.4) is expressed as

$$\dot{z}_1 = \bar{A}_{11}z_1 + \bar{B}_1 u, \tag{2.6}$$

$$\dot{z}_2 = \bar{A}_{21}z_1 + \bar{A}_{22}z_2 + \bar{B}_2 u, \tag{2.7}$$

$$y = \bar{C}_1 z_1, \tag{2.8}$$

where $z_1 \in \mathbf{R}^r$, $z_2 \in \mathbf{R}^{n-r}$ and the pair $(\bar{C}_1, \bar{A}_{11})$ is observable. Note that in the new coordinates z_2 represent the unobservable dynamics of the system which has no effect on the output neither through direct addition nor through injection into the dynamics of z_1.

Existence of unobservable dynamics implies that the state space system description (2.3), (2.4) is not minimal[2] and the transfer function of the system (A, B, C) is the same as the transfer function of the system $(\bar{A}_{11}, \bar{B}_1, \bar{C}_1)$, namely, $C(sI_n - A)^{-1}B = \bar{C}_1(sI_r - \bar{A}_{11})^{-1}\bar{B}_1$, see, e.g., [32].

Detectability condition is more apparent when the system is transformed into the

[2]A state space description of an input/output behavior is minimal if no other description of the same behavior has a smaller dimension.

canonical form (2.6)–(2.8). A system in this canonical form is detectable if and only if A_{22} has all its eigenvalues on the open left half-plane. Equivalent conditions of detectability are obtained if the last two statements in the Theorem 2.1 is modified. Namely, detectability of the system (2.3), (2.4) is equivalent to any of the following two conditions;

- the matrix $\begin{bmatrix} \lambda I - A \\ C \end{bmatrix}$ has full column rank for all *unstable* eigenvalues λ_i of A.
- there exists a gain matrix $K \in \mathbf{R}^{n \times p}$ such that eigenvalues of the matrix $A_c = A - KC$ lie in the open left half complex plane.

2.2.2 Nonlinear Systems

As mentioned earlier, the two main difficulties involved in the observability of nonlinear systems are the dependence on the particular input of the distinguishability of the states and the locality of any dynamical behavior. The first difficulty is usually circumvented by searching for the system structures where the observability is independent of the inputs. However, the latter is not specific to observability and one seldom arrives at global results for most classes of nonlinear systems.

Some of the early attempts at the observability problems for nonlinear systems started with the observer design problem with certain restrictions and along the way introduced some conditions simplifying the design of observers, [33–38]. While this useful approach is still encountered in the literature, the conditions obtained are usually different than observability conditions. In regard of the definition of nonlinear detectability as the existence of an (asymptotic) observer, [18], these conditions can be viewed as various sufficient conditions for nonlinear detectability.

The first comprehensive definitions and unifying sufficient conditions of nonlinear observability (and dually controllability) that do not depend on the particular structure of an observer are given in [26]. Subsequently their differential algebraic approach became the standard tool in tackling nonlinear observability problems, see, e.g., [25].

The following discussion is borrowed from [18] with minor modifications and simplifications. For the sake of simplicity of the exposition we consider single-input, single-output systems.

Consider a SISO nonlinear system given by

$$\dot{x} = f(x) + g(x)u, \tag{2.9}$$

$$y = h(x), \tag{2.10}$$

where y and u are scalar functions of time and the functions f, g and h are mappings from and to spaces of appropriate dimensions. Before introducing the details of the observability conditions a few mathematical definitions are in order.

For a vector field $f : \mathbf{R}^n \to \mathbf{R}^n$ and a scalar function $h : \mathbf{R}^n \to \mathbf{R}$, **Lie derivative** (also referred to as directional derivative in more familiar contexts) $L_f h : \mathbf{R}^n \to \mathbf{R}$ of h along f is defined as $L_f h(x) = (\mathrm{d}h(x))f(x)$, where $\mathrm{d}h$ denotes the differential (seen as a row vector) of the function $h(x)$. Higher order Lie derivatives are recursively defined as

$$\begin{aligned} L_f^0 h(x) &= h(x), \\ L_f^k h(x) &= L_f(L_f^{k-1} h(x)). \end{aligned}$$

If, as in a linear system, $f(x) = Ax$ and $h(x) = Cx$ then $L_f^k h(x) = CA^k x$. Also we denote, as a convention, by $L_{f_1 f_2 \ldots f_k} h(x)$ the function $L_{f_1} L_{f_2} \ldots L_{f_k} h(x)$.

Related is the **Lie bracket** operation. Given two vector fields f and g the Lie bracket of the two is another vector field and is defined as

$$\frac{\partial g}{\partial x} f - \frac{\partial h}{\partial x} g.$$

The Lie bracket is also denoted by $ad_f g$ and higher order Lie brackets are recursively defined as

$$ad_f^0 g = g$$

$$ad_f^k g \quad = \quad ad_f(ad_f^{k-1} g).$$

For the system (2.9), (2.10), assuming that f, h and u and therefore x and y are smooth functions of their arguments, successive differentiation of the measured output $y(t)$ with respect to time yields more and more number of relations giving information about the system state x. Indeed, writing out a few terms we see that, at a particular time instant t,

$$\begin{aligned}
y(t) &= L_f^0 h(x), \\
\dot{y}(t) &= L_f h(x) + u(t) L_g h(x), \\
\ddot{y}(t) &= L_f^2 h(x) + u(t)[L_{fg}(x) + L_{gf}(x)] + \dot{u}(t) L_g h(x) + u^2(t) L_g^2 h(x), \\
\vdots \quad & \quad \vdots
\end{aligned}$$

In these expressions the derivatives of all orders of y and u are assumed to be available through measurements and we are searching for the conditions to guarantee the existence of a particular input function $u(\cdot)$ which would enable us to extract the state information from these expressions. Note that a derivative of order k of the output y is a linear combination of terms of the form $L_{z_s \ldots z_1} h(x),\ 1 \leq s \leq k$, where each vector field z_i is an element in the set $\{f, g\}$. For instance, $\ddot{y}(t)$ is a linear combination of the functions $\{L_f^2 h(x), L_{fg}(x), L_{gf}(x), L_g h(x), L_g^2 h(x)\}$. Considering the set of all such functions appearing in the expressions for derivatives of all orders of y we obtain a natural linear space structure. Next is given a formal definition for this space.

Definition 2.2 *Given the system described by (2.9), (2.10) the* **observation space** $\mathcal{O}$ *of the system is the linear space of functions over the field* $\mathbf{R}$ *spanned by all functions of the form* $L_{z_s \ldots z_1} h(x),\ s \geq 0,\ z_i \in \{f, g\}$, *namely,*

$$\mathcal{O} \stackrel{\triangle}{=} \mathrm{span}\{\alpha(x) \mid \alpha(x) = L_{z_s \ldots z_1} h(x),\ s \geq 0,\ z_i \in \{f, g\}\}. \tag{2.11}$$

Also we denote by $\mathrm{d}\mathcal{O}(x)$ the infinite matrix whose rows are the differentials of the functions $\alpha(x)$ in the definition of the observation space. The span of these rows are also referred to as the **observability codistribution**. Following is a sufficient condition for the local observability of the system (2.9), (2.10).

Theorem 2.2 *The system (2.9), (2.10) is locally observable around a point* x_0 *if the forms* $\mathrm{d}L_{z_s \dots z_1} h(x_0)$, $s \geq 0$, $z_i \in \{f, g\}$, *contain* n *linearly independent row vectors.*

Proof: We refer to [18] for the proof of this theorem which is technically a bit involved but mainly relies on the application of inverse function theorem. □

Remark 2.2.1 *When the system is linear the observation space becomes the span of the set* $\{Cx, CAx, \dots, CA^{n-1}x\}$. *Higher order Lie derivatives are not needed as, by Cayley-Hamilton theorem, they linearly depend on the first* n *functions. Moreover, taking differentials of these functions yields the usual observability matrix* $\mathbf{O}$.

Remark 2.2.2 *When the nonlinear system is an unforced one* ($u = 0$) *then the observation space is spanned by the set of functions* $\{h(x), L_f h(x), \dots, L_f^{n-1} h(x), \dots\}$. *Note that to establish the observability of the system in general one may have to use Lie derivatives of order higher than* $n - 1$.

Example 2.1 *To illustrate the last remark consider the following smooth dynamical system, [39];*

$$\begin{aligned} \dot{x} &= x, \\ y &= x^3 - x. \end{aligned}$$

Obviously, $\mathrm{d}L_f^0 h(0) = 0$ *but* $\mathrm{d}L_f h(0) = -1$, *therefore while the system dimension is 1, to establish the local observability at* $x = 0$ *by Theorem 2.2 we need to use the first derivative of the output.*

When the highest order of the derivatives of the output in the definition of the observation space is restricted to be $n-1$ and when the system is unforced the rank condition in Theorem 2.2 is used to establish **uniform observability**, as defined in [2]. As illustrated by the previous example, uniform observability is a stronger property than observability.

Although Theorem 2.2 gives only a sufficient rank condition for local observability, the observation space can be used to give a necessary and sufficient condition (albeit not easily checkable) for analytic[3] systems, [40]. Consider an unforced nonlinear dynamical system,

$$\dot{x} = f(x), \tag{2.12}$$

$$y = h(x), \tag{2.13}$$

where both f and h are analytic functions of their arguments in their domains of interest. Then we have the following result.

Theorem 2.3 *An analytic system (2.12), (2.13) is observable on a domain $M \subset \mathbf{R}^n$ if and only if the observation space $\mathcal{O}$ distinguishes the points of M, that is, for any $x_1, x_2 \in M,\ x_1 \neq x_2$, there is a function $\alpha(x) \in \mathcal{O}$ such that $\alpha(x_1) \neq \alpha(x_2)$.*

Proof: See [40]. □

Example 2.2 *Consider the following dynamical system, [40],*

$$\dot{x} = x,$$

$$y = x^3.$$

This system fails the condition of Theorem 2.2 at $x = 0$ as $\mathrm{d}\mathcal{O}(0)$ vanishes. Yet, by Theorem 2.3 it is everywhere observable. Indeed an exact estimate of the state can be obtained from the output simply by $\hat{x} = \sqrt[3]{y}$.

[3] A function is analytic at a particular point if its Taylor expansion converges to the value of the function at that point.

Chapter 3

Observers

Given a dynamical system with its state space description, an observer is an auxiliary system that estimates the states of the original system using the measurable quantities as its input. Thus the input to the observer are the inputs and outputs of the original system and its output is a subset of the states of the original system. Taken in its largest sense, while an observer can be another dynamical system, it can even be a computer program with suitable interfaces for conversion to and from continuous and digital domains. Figure 3.1 illustrates this configuration for the case when both the original system and the observer have continuous dynamics.

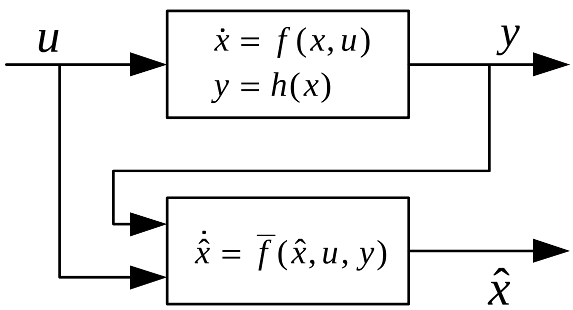

Figure 3.1: Observer configuration

In general, a system-observer configuration with only continuous dynamics is usually expressed as follows;

$$\begin{aligned} \dot{x} &= f(x,u), &(3.1)\\ y &= h(x), &(3.2)\end{aligned}$$

$$\begin{aligned} \dot{\hat{x}} &= \bar{f}(\hat{x},u,y), &(3.3)\\ \omega &= T(\hat{x},y), &(3.4)\end{aligned}$$

where $x \in \mathbf{R}^n$, $\hat{x} \in \mathbf{R}^r$, $r \leq n$, $y \in \mathbf{R}^p$, $\omega \in \mathbf{R}^s$, $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$, $h : \mathbf{R}^n \to \mathbf{R}^p$, $T : \mathbf{R}^r \times \mathbf{R}^p \to \mathbf{R}^s$, $s \leq n$ and the functions $f, h, \bar{f}, T$ are sufficiently smooth functions of their respective arguments. The input $u : \mathbf{R}_+ \to \mathbf{R}^m$ is a sufficiently smooth function of time. In this general setting the dimension r of the observer system may be smaller than the system dimension n. The actual estimate is provided with the vector $\omega \in \mathbf{R}^s$.

The error between the system state and the estimate obtained through the observer is defined as

$$\varepsilon = Px - \omega, \tag{3.5}$$

where $P \in \mathbf{R}^{s\times n}$. Here P is introduced to incorporate functional observers in the framework. The observation requirement is that the error system must be stable in some sense.

With the definition of the observation error in (3.5), its dynamics are obtained by (3.1)-(3.4) as

$$\begin{aligned} \dot{\varepsilon} &= P\dot{x} - \dot{\omega},\\ &= Pf(x,u) - \frac{\partial T}{\partial \hat{x}}(\hat{x},y)\bar{f}(\hat{x},u,y) - \frac{\partial T}{\partial y}(\hat{x},y)[L_f h_1(x) \ldots L_f h_p(x)]^T, &(3.6)\end{aligned}$$

where the output map h is written as $h(x) = [h_1(x)\ h_2(x)\ \ldots\ h_p(x)]^T$ with $h_i(x) : \mathbf{R}^n \to \mathbf{R}$, $i = 1 \ldots p$.

If we have $r = s = n$ and $T(\hat{x}, y) = \hat{x}$ then we have a **full-order observer** and in the observer error definition we have $P = I_n$. When $r < n$ and $P = I_n$ the observer (3.3), (3.4) is a **reduced-order observer** for the system (3.1), (3.2). What is distinctive in reduced-order observers is that the observer dynamics have a dimension smaller than that of the original system. On the other hand, it is possible to use a full-order observer to estimate a strict subset or a set of linear functionals the states. In this case, $P \neq I_n$ and we have **functional observer**, which is particularly useful when the observer is used in a state feedback configuration.

Apart from the dimension of the underlying dynamics, observers are also classified according to the stability properties of the error system. First requirement of an observer is that $\varepsilon = 0$ must be an equilibrium point of the error system (3.6), that is, the following identity must be satisfied;

$$Pf(x,u) - \frac{\partial T}{\partial x}(x,y)\bar{f}(x,u,y) - \frac{\partial T}{\partial y}(x,y)[L_f h_1(x) \ldots L_f h_p(x)]^T = 0 \tag{3.7}$$

This guarantees that if the estimate is exact at a particular time instant, it remains so subsequently. If we have $\varepsilon(t) \to 0$ as $t \to \infty$ then the observer (3.3), (3.4) is an **asymptotic observer** and if, further, there are real numbers $M > 0$ and $\alpha > 0$ such that $\|\varepsilon(t)\| \leq \|\varepsilon(0)\| e^{-\alpha t},\ t \geq 0$, then (3.3), (3.4) is **an exponential observer**. If these convergence requirements are satisfied when the initial observation error lies in an open neighborhood $\mathcal{N}$ around the origin of 0 then we have a **local (exponential) observer**. If $\mathcal{N} = \mathbf{R}^n$ the observer is **global.** Note that here the locality is different than the one encountered in linearization contexts where the system (3.1), (3.2) is linearized around its equilibrium point. Here the neighborhood of interest is not around a particular fixed point of the original system (3.1), (3.2) but rather around the equilibrium point of the error system (3.6). Now we analyze in detail several observer construction schemes and their stability properties for different classes of system.

3.1 Observers for Linear Systems

The theory of observers for linear systems has matured and found many applications through a continuous interest beginning with the seminal work of Luenberger, [41]. The following treatment of the subject is adapted with adaptations from [32,42]. The system under consideration is given by (2.3), (2.4) which, for the sake of completeness, are repeated here;

$$\dot{x} = Ax + Bu, \tag{3.8}$$

$$y = Cx. \tag{3.9}$$

3.1.1 Full-order Observer

A full-order observer for the system (3.8), (3.9) can be constructed as

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x}), \tag{3.10}$$

with $\hat{x} \in \mathbf{R}^n$. Referring to (3.3), (3.4) we have $\omega = \hat{x}$ and since the observer is full-order we use $P = I_n$ to write the error equation as $\varepsilon = x - \hat{x}$. Subtracting (3.10) from (3.8) we obtain the error dynamics as

$$\dot{\varepsilon} = (A - KC)\varepsilon. \tag{3.11}$$

The system given by (3.11) is linear time-invariant. It is exponentially stable if and only if all the eigenvalues of the matrix $A_c = A - KC$ lie on the open left-half complex plane. The only freedom in constructing the observer is the choice of the gain matrix K. Using the modification of the last item of Theorem 2.1, there exists a gain K which makes A_c exponentially stable if and only if the pair (C, A) is detectable. If, further, the pair (C, A) is observable then the eigenvalues of A_c can be assigned at arbitrary locations by the choice of K. Note that since the error system (3.11) is linear its stability is global.

3.1.2 Reduced-order Observer

Basically a reduced-order observer uses the algebraic relation between the states and the output to reduce the dimension of the observer dynamics. Consider again the linear system (3.8) (3.9). Assume without loss of generality that the matrix C has rank p (if not, we can drop extra rows which can be expressed as linear combinations of others). By appending a matrix $\bar{C} \in \mathbf{R}^{(n-p)\times n}$ to C we form a nonsingular matrix $M \in \mathbf{R}^{n\times n}$, namely,

$$M = \begin{bmatrix} C \\ \bar{C} \end{bmatrix}^{-1},$$

and partition the transformed matrices $\bar{A} = M^{-1}AM,\ \bar{B} = M^{-1}B$ as

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix}.$$

If the pair (C, A) is observable then there exists a gain $K \in \mathbf{R}^{(n-p)\times p}$ such that the following is an exponential reduced-order observer for (3.8) (3.9);

$$\begin{aligned} \dot{\hat{x}} &= (\bar{A}_{22} - K\bar{A}_{12})\hat{x} + (\bar{A}_{21} + \bar{A}_{22}K - K\bar{A}_{12}K - K\bar{A}_{11})y \\ &\quad +(\bar{B}_2 - K\bar{B}_1)u, \end{aligned} \tag{3.12}$$

$$\omega = M \begin{bmatrix} 0 & I_p \\ I_{n-p} & K \end{bmatrix} \begin{bmatrix} \hat{x} \\ y \end{bmatrix}. \tag{3.13}$$

Note that $\hat{x} \in \mathbf{R}^{n-p}$, namely, the dimension of the observer is rank(C) less than the system dimension n.

The proof of the above result is quite straightforward and basically relies on the fact that if the pair (C, A) is observable so is the pair $(\bar{A}_{12}, \bar{A}_{22})$. For details see [42].

3.1.3 Functional Observer

The following discussion is borrowed from the recent work [43]. P in (3.5) can, in general, be different than identity matrix. Obviously the cases where P is nonsingular can trivially be considered in full-order or reduced-order observers. Once one has an estimate of the state vector x, an estimate of the vector Px can be obtained by simple multiplication, but the reverse is not true in general. Therefore, the observability and observer design conditions for functional observers are milder than those for full (reduced)-order observers.

Assume, without loss of generality, that C has rank p and $P \in \mathbf{R}^{r\times n}$ has rank r. A functional observer has the following general form;

$$\dot{\hat{x}} = D\hat{x} + Ey + Fu, \tag{3.14}$$

$$\omega = \hat{x} + Gy \tag{3.15}$$

and the error equation is again $\varepsilon = Px - \omega$. Hence the problem is the existence (and of course actual finding) of matrices $D,\ E,\ F,\ G$ that makes the error system exponentially stable. The following theorem and its corollaries provide precisely these.

Theorem 3.1 *There exists a functional observer in the form (3.14), (3.15) with exponentially stable error dynamics if and only if we have*

$$\operatorname{rank}\begin{bmatrix} PA \\ CA \\ C \\ P \end{bmatrix} = \operatorname{rank}\begin{bmatrix} CA \\ C \\ P \end{bmatrix}$$

and

$$\operatorname{rank}\begin{bmatrix} sP - PA \\ CA \\ C \end{bmatrix} = \operatorname{rank}\begin{bmatrix} CA \\ C \\ P \end{bmatrix},\ \forall s \in \mathbf{C},\ \Re(s) \geq 0.$$

Proof: See [43]. $\square$

3.1.4 Unknown Input Observer

For all of the linear observers analyzed so far all of the inputs to the system are assumed to be available through measurements and used in the observer construction. However, as with disturbances, some of the inputs might not be available. Observer design for such a case is discussed next. First let us introduce the unknown inputs to the model (3.8) as

$$\dot{x} = Ax + Bu + Jv, \tag{3.16}$$

where $v \in \mathbf{R}^q$ is the unknown input and $J \in \mathbf{R}^{q\times n}$ is assumed to have rank q. Unknown input observer dynamics for this system has the expression

$$\begin{aligned} \dot{\hat{x}} &= D\hat{x} + Ey + Fu, && (3.17) \\ \omega &= \hat{x} + Gy, && (3.18) \end{aligned}$$

with $P = I_n$. This is the same expression for a functional observer. However, the stability and decoupling requirements (that the error dynamics should be independent of the unknown input v) yield different necessary and sufficient conditions for the existence of an observer and consequently the matrices D, E, F, G are different than the ones in (3.14), (3.15). The following result is borrowed from [44], but similar or equivalent results can also be found in [45, 46].

Theorem 3.2 *An unknown input, full-order, exponential observer in the form (3.17), (3.18) for the linear system (3.16) and (3.9) exists if and only if*

- $\mathrm{rank}(CJ) = \mathrm{rank}(J) = q$,
- $\mathrm{rank}\begin{bmatrix} sH - HA \\ C \end{bmatrix} = n \;\forall s \in \mathbf{C},\; \Re(s) \geq 0,$ *where* $H = (I_n + YC)(I_n - J(CJ)^+C)$ *and* Y *is an arbitrary matrix of appropriate dimensions.* $(\cdot)^+$ *denotes the generalized inverse.*

Proof: See [44]. □

3.1.5 Further Problems in Linear Observers

While the linear observers examined so far have found applications in many areas, it is still possible to pose different interesting problems in the field. One may generalize the above results to time-varying or infinite-dimensional linear systems, [47]. As in the design of a single controller for multiple systems, [48], one may search for an observer structure to simultaneously observe the states of multiple systems, [49]. Needless to say, the counterparts of these problems for discrete-time linear systems have been addressed in the vast literature. A good starting point to the literature on the theory of linear observers are [50, 51] and the references therein.

3.2 Observers for Nonlinear Systems

It might not be too wrong to state that there are no truly nonlinear observers. Most of the observers proposed for nonlinear systems basically rely on linear observer design methods and their extensions, without making use of the nonlinear nature of the problem. Rather than exploiting the functional form of the system vector fields and the output functions, existing observer design schemes employ certain bounds on the system nonlinearities, therefore yielding somewhat conservative results.

Many nonlinear observer design schemes decompose a nonlinear system into linear and nonlinear parts and make structural and bounding assumptions on these parts, which facilitate the determination of the observer dynamics. Once one finds sufficient conditions for the existence of an observer for a system having a special structure, the results naturally generalize to the class of systems that can be transformed into this structure by a sufficiently well-behaved state transformation. In this section, we review some of the well-known nonlinear observer structures, starting from the ones with easy-to-prove stability properties.

3.2.1 Observers for Linear Systems up to Output Injection

Consider an unforced nonlinear system having the following dynamics.

$$\dot{x} = Ax + \alpha(u, y), \tag{3.19}$$

$$y = Cx, \tag{3.20}$$

with (C, A) being an observable pair and $\alpha : \mathbf{R}^m \times \mathbf{R}^p \to \mathbf{R}^n$ a nonlinear function. Note that in this system structure the nonlinearity is a function of the measurable quantities only, namely the input and the output. An observer can easily be designed for such systems as

$$\dot{\hat{x}} = A\hat{x} + \alpha(u, y) + K(y - C\hat{x}), \tag{3.21}$$

with K chosen to make $A - KC$ stable. Indeed, defining the observation error as $\varepsilon = x - \hat{x}$, its dynamics are given by

$$\dot{\varepsilon} = (A - KC)\varepsilon,$$

which is an exponentially stable system. This is an obvious extension of the full-order observer for a linear system when the nonlinearity is available through measurements and thus can be exactly replicated in the observer to cancel in the error dynamics. A particular example to this class of systems is a Lure' type system, [18, p. 219], given in Figure 3.2.

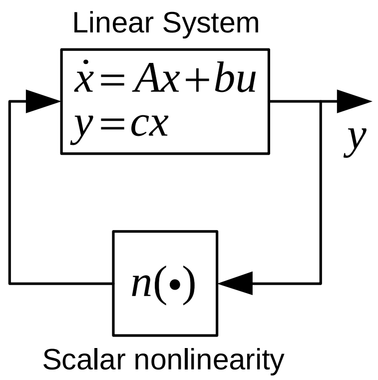

Figure 3.2: Lure' type system.

3.2.2 Transformation to input-output injection form

Having shown how a simple observer can be constructed for systems having the special structure (3.19), (3.20), next question is when a general nonlinear system can be transformed to this form. The next theorem establishes necessary and sufficient conditions on the existence of the transformation. Consider the following single-output nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x,u) && (3.22) \\ y &= h(x), && (3.23) \end{aligned}$$

where $f,\ g$ and h are smooth functions of their arguments with $g(x,0) = 0$ and $h(0) = 0$. The transformed system is a linear system up to input-output injection with its linear part having an observer canonical form, namely, under the transformation $z = \tau(x)$, the system in z coordinates is given by

$$\dot{z} = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} z + \begin{bmatrix} \alpha_1(u,y) \\ \alpha_2(u,y) \\ \vdots \\ \alpha_n(u,y) \end{bmatrix} \triangleq A_o z + \alpha(u,y), \tag{3.24}$$

$$y = [0\ 0\ \dots\ 0\ 1]z \triangleq c_o z. \tag{3.25}$$

Theorem 3.3 *There exists a local diffeomorphism*[1] τ *in a neighborhood around the origin with* $z = \tau(x)$ *and* $\tau(0) = 0$ *transforming (3.22), (3.23) into (3.24), (3.25) if and only if in a neighborhood* $\mathcal{N}$ *around the origin*

i. $\operatorname{rank}\left\{dh, d(L_f h), \dots, d(L_f^{n-1}h)\right\} = n,$

ii. $[ad_f^i, ad_f^j r] = 0,\ 0 \le i,j \le n-1,$

iii. $[g, ad_f^j r] = 0,\ 0 \le n-2,\ \forall u \in \mathbf{R}^m$

[1] *A transformation is a diffeomorphism if it is smooth and has a smooth inverse.*

where r is the vector field solution of

$$\begin{bmatrix} \langle dh, r \rangle \\ \vdots \\ \left\langle d(L_f^{n-1}h), r \right\rangle \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

The transformation τ is global if and only if $\mathcal{N} = \mathbf{R}^n$ and additionally we have

iv. $ad_f^i r,\ 0 \leq i \leq n-1,$ are complete [2] vector fields.

Proof: See [52]. □

Theorem 3.3 relates the existence conditions for the transformation to the functional forms of f, g and h. Yet, the observer design is still within the linear framework. Moreover, computing the transformation may become quite difficult as it is found through the solution of a set of partial differential equations, [53].

Such exact transformation approaches to observer design date back to the works [54–57] and have become standard topics in recent textbooks on nonlinear control. The main difference among nonlinear transformations considered lie in the assumptions about the form of the map τ. In general one can choose the dimension of the new state vector z higher than the dimension n of x. This yields a system immersion rather than a diffeomorphism, [58]. In [59–61] necessary and sufficient conditions were examined for the existence of a state transformation in the form $z = \tau(x, u, \dot{u}, \ldots, u^{(n-1)})$, thus including the input and its derivatives in the transformation. The works [62–64] relax the transformation to be time-varying and obtain conditions on its existence. The counter-part of Theorem 3.3 for discrete-time systems is given in [65].

[2] *A vector field $r(x)$ is complete if the solution of $\dot{x} = r(x)$ is defined for all times $t \in \mathbf{R}$.*

3.2.3 Construction of Observers through Linear Approximation

Another observer design scheme is to decompose the system into linear and nonlinear parts and basically design an observer for the linear part. The basic difference with the above approaches is the fact that the error system is not linear. Stability is ensured by making use of the structure of the nonlinear part and by choosing the observer gains to satisfy certain criteria other than eigenvalue placement. Naturally, once an observer is designed for a system having a special structure, same can be done for the class of systems that can be transformed to this particular structure under a sufficiently well-behaved transformation.

To illustrate the classical linearization approach, consider the following unforced system decomposed into linear and nonlinear parts and having a linear output function;

$$\dot{x} = Ax + f(x), \tag{3.26}$$

$$y = Cx, \tag{3.27}$$

where the pair (C, A) is assumed to be detectable and $f(0) = 0$. In the following observer structure, output error is injected to the observer dynamics through the constant gain vector K. Hence the observer is given by

$$\dot{\hat{x}} = A\hat{x} + f(\hat{x}) + K(y - C\hat{x}). \tag{3.28}$$

Subtracting (3.28) from (3.26) dynamics for the error $\varepsilon = x - \hat{x}$ is obtained as

$$\begin{aligned} \dot{\varepsilon} &= (A - KC)\varepsilon + f(x) - f(\hat{x}), && (3.29) \\ &= (A - KC)\varepsilon - f(x - \varepsilon) + f(x). && (3.30) \end{aligned}$$

The stability of the error system can be established using Lyapunov functions. The following result from [33] uses a quadratic Lyapunov function to derive sufficient conditions for the exponential decay of the observation error. Similar results were also obtained by [66].

Theorem 3.4 *Assume that the function f in (3.28) is Lipschitz continuous in a neighborhood $\mathcal{N}$ around the origin, with Lipschitz constant $\gamma > 0$ and a suitable norm $\|\cdot\|$, i.e.*

$$\|f(x_1) - f(x_2)\| \leq \gamma \|x_1 - x_2\|, \ \forall x_1, x_2 \in \mathcal{N} \subset \mathbf{R}^n. \tag{3.31}$$

Then the system (3.28) is a local exponential observer for the system (3.26), (3.27) if there exists a gain vector K rendering $A - KC$ stable and a symmetric positive definite matrix Q such that the unique solution P of the Lyapunov equation

$$(A - KC)^T P + P(A - KC) = -Q$$

satisfies the inequality

$$\frac{\lambda_{min}(Q)}{2\lambda_{max}(P)} - \gamma > 0, \tag{3.32}$$

The observer is global if the function f is globally Lipschitz continuous.

Proof: See [33]. □

Remark 3.4.1 *The sufficient conditions for the existence of the observer do not include the observability conditions of the system. Considering the definition of detectability as the existence of an observer, these conditions can be thought of as sufficient conditions for the detectability of the system (3.26), (3.27).*

Remark 3.4.2 *The ratio in the inequality (3.32) is maximized when $Q = I$. Therefore the design is accomplished when a gain K is chosen such that*

$$\gamma < \frac{1}{2\lambda_{max}(P)}.$$

However, no straightforward method to choose K to satisfy this inequality exists.

Note that the assumptions in Theorem 3.4 are rather mild. In fact the detectability of the pair (C, A) (when they are the linear parts of the corresponding vector field and output map) is a necessary condition for the existence of a smooth observer in the form (3.28), [28].

A more tractable sufficient condition for the existence of a gain K to satisfy (3.32) and a constructive method for its choice is given in [67] using the concept of distance to unobservability, [68]. The condition is that the distance to unobservability of the pair (C, A) is larger than the Lipschitz constant γ. Obviously such a method naturally generalizes to the class of systems that can be transformed to a nonlinear system whose linear part is sufficiently far from unobservability. However, determination of the conditions under which such a transformation exists is an open problem in the general case. When further assumptions are made regarding the structure of the pair (C, A) and function f, a parameterization can be provided.

3.2.4 Observers for nonlinear systems in Brunowsky canonical form

Consider a single-output, unforced nonlinear system

$$\begin{aligned} \dot{x} &= Ax + B\alpha(x), \qquad (3.33) \\ y &= Cx, \qquad (3.34) \end{aligned}$$

where $\alpha : \mathbf{R}^n \to \mathbf{R}$ is a scalar nonlinearity and (A, B, C) has the following Brunowsky canonical form;

$$A = \begin{bmatrix} 0 & 1 & 0\dots & & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 & 1 \\ 0 & \dots & \dots & \dots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad C = [1\ 0\ \dots\ 0]. \qquad (3.35)$$

Theorem 3.5 *Given a nonlinear system (3.33), (3.34) in Brunowsky canonical form where the function $\alpha(x)$ is globally Lipschitz with Lipschitz constant γ, there exists a gain vector $K \in \mathbf{R}^n$ such that the following is a globally exponential observer;*

$$\dot{\hat{x}} = A\hat{x} + B\alpha(\hat{x}) + K(y - C\hat{x}). \tag{3.36}$$

Proof: A lengthy proof of this statement can be found in [69]. Here we provide a shorter proof with Lyapunov functions. First, let us introduce the Vandermonde matrix for a given self-conjugate set of n distinct complex numbers $\lambda = \{\lambda_1, \ldots, \lambda_n\}$ as

$$V(\lambda) = \begin{bmatrix} \lambda_1^{n-1} & \lambda_1^{n-2} & \ldots & \lambda_1 & 1 \\ \lambda_2^{n-1} & \lambda_2^{n-2} & \ldots & \lambda_2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_n^{n-1} & \lambda_n^{n-2} & \ldots & \lambda_n & 1 \end{bmatrix}.$$

When the observable pair (C, A) is in the form given by (3.36) and K is chosen such that the eigenvalues of $A - KC$ correspond to a set $\lambda \subset \mathbf{C}_-$, we have $A - KC = V^{-1}(\lambda)\Lambda V(\lambda)$, where $\Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_n\}$ with $\Re(\lambda_1) \geq \Re(\lambda_2) \geq \ldots \geq \Re(\lambda_n)$. Choose the Lyapunov function candidate $L(\varepsilon) = \varepsilon^T V^T(\lambda) V(\lambda)\varepsilon$ to establish the stability of the error dynamics

$$\dot{\varepsilon} = (A - KC)\varepsilon + B(\alpha(x) - \alpha(\hat{x})).$$

Then we have

$$\begin{aligned} \dot{L} &= 2\varepsilon^T (A - KC)^T V^T(\lambda) V(\lambda)\varepsilon + 2\varepsilon^T V^T(\lambda) V(\lambda) B(\alpha(x) - \alpha(\hat{x})), \\ &= 2\varepsilon^T V^T(\lambda)\Lambda V(\lambda)\varepsilon + 2\varepsilon^T V^T(\lambda) V(\lambda) B(\alpha(x) - \alpha(\hat{x})), \\ &\leq 2\Re(\lambda_1)\|V(\lambda)\varepsilon\|^2 + 2\|V(\lambda)\varepsilon\|\sqrt{n}\gamma\|\varepsilon\|, \\ &\leq 2\Re(\lambda_1)\|V(\lambda)\varepsilon\|^2 + 2\|V\varepsilon\|^2\sqrt{n}\gamma\|V^{-1}(\lambda)\|, \\ &= 2\beta\|V(\lambda)\varepsilon\|^2 = 2\beta L, \end{aligned}$$

where the norm used is the Euclidean norm and

$$\beta = \Re(\lambda_1) + \sqrt{n}\gamma \|V^{-1}(\lambda)\|. \tag{3.37}$$

It is possible to show that the set λ can be assigned to get $\beta < 0$, (in Chapter 5 we will examine in detail some eigenvalue assignment procedures to make the exponent negative). Then we have the bound on the observation error as $\|\varepsilon(t)\| \leq Me^{\beta t}\|\varepsilon(0)\|$, which shows the globally exponential convergence of the observer (3.36). $\square$

3.2.5 Transformation to Brunowsky Form

As opposed to the case with observer design through classical linearization, observability related sufficient conditions can be given for a nonlinear system to be diffeomorphic to a system in Brunowsky canonical form. Given a nonlinear system

$$\begin{aligned} \dot{x} &= f(x), \\ y &= h(x), \end{aligned}$$

let us define as the new coordinates the output and its time derivatives up to order $n-1$, namely, $z_1 = y,\ z_2 = \dot{y}, \ldots, z_n = y^{(n-1)}$. Using the definition of Lie derivative given in the previous chapter, these relations can be written as

$$z = \Phi(x) \triangleq \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix}. \tag{3.38}$$

After the transformation Φ, the system in the new coordinates is in Brunowsky canonical form with $\alpha(x) = L_f^n h(\Phi^{-1}(z))$ Therefore for a system to admit an exponential observer in the form (3.36) following two conditions are sufficient.

i. The transformation $\Phi(x)$ is a diffeomorphism in the state-space.

ii. The function $L_f^n h(\Phi^{-1}(z))$ is globally Lipschitz.

The first condition can easily be verified by checking whether Jacobian of the transformation Φ has full rank throughout the state-space. Note also that this condition is sufficient to establish observability of the original nonlinear system, see Theorem 2.2. Verification of the second condition is a bit involved and often one ends up with a conservative Lipschitz constant.

3.2.6 Observers for Triangular Systems

From the foregoing discussion of nonlinear observers, there appears a distinctive pattern of attack towards nonlinear observer design problems; first, find a suitable system form which facilitate the observer design (possibly using well-known linear techniques), then parameterize the class of nonlinear systems which can be smoothly transformed into this particular form. Observers for triangular systems are no different. Basic novelty is that they incorporate forced case into the framework of observers for systems in Brunowsky canonical form. The observability properties triangular systems were analyzed in [2].

This time we consider a multi-input, single-output nonlinear system given by

$$\begin{aligned} \dot{x} &= Ax + B\alpha(x) + \sum_{i=1}^{m} g_i(x)u_i, && (3.39)\\ y &= Cx, && (3.40) \end{aligned}$$

where (A, B, C) has the Brunowsky canonical form and each g_i, $1 \leq i \leq m$ is triangular, that is,

$$g_i(x_1, \ldots, x_n) = \begin{bmatrix} g_{i1}(x_1) \\ g_{i2}(x_1, x_2) \\ \vdots \\ g_{in}(x_1, x_2, \ldots, x_n) \end{bmatrix}.$$

The observer structure for (3.39), (3.40) is very similar to (3.36),

$$\dot{\hat{x}} = A\hat{x} + B\alpha(\hat{x}) + \sum_{i=1}^{m} g_i(\hat{x})u_i + K(y - C\hat{x}). \tag{3.41}$$

Sufficient conditions to guarantee the existence of a gain vector K that makes the error system stable have been analyzed and various construction procedures have been proposed in the literature, see e.g., [3, 69–73]. Here we give a theorem that states sufficient conditions for the existence of the gain vector K. Further conditions on the existence of a change of coordinates to transform a given nonlinear system into the form (3.39), (3.40) can be found in [3, 69].

Theorem 3.6 *For the system (3.39), (3.40), assume that the following holds;*

i. The functions $\alpha(x)$ and $g_i(x)$, $1 \leq i \leq n$, are globally Lipschitz.

ii. The set of inputs is compact.

Then there exists a gain vector K such that (3.41) is a globally exponential observer for the system (3.39), (3.40).

Proof: See [3]. □

Chapter 4

Chaos Synchronization and Observers

During the last decade, there has been a steady increase of interest in chaos synchronization and its possible applications to telecommunications since the conception of the topic by the seminal works [12,13]. The underlying motivation for searching methods to obtain synchronized chaotic systems is its possible application to secure communication that is immune to eavesdropping. Although no full-fledged chaotically synchronized communication systems are in use today, promising small-scale experiments and implementations have been reported in the literature, [74–76].

A set of chaotic systems are said to be synchronized if a subset of their signals are the same. This is achieved by implementing a coupling among the set of chaotic systems. The properties of synchronization are determined by the particular form of the coupling. Obviously one can not except any two systems, let alone chaotic ones, to robustly synchronize when there is no interaction between the two.

What distinguishes the synchronization of chaotic systems from others is the existence of dynamical behavior that are unique to chaotic systems. In this respect, *synchronized* and *chaotic* appear as contrasting adjectives, having connotations of order and irregularity. In fact, it is the coexistence of these two opposing properties that gives the synchronized chaotic systems the potential to find application in secure communications. In a successful message transmission scheme using chaotic synchronization, the intended receiver of the message can synchronize to extract the useful information from what an interceptor sees as noise-like gibber. Apart from the peculiar mathematical properties of a chaotic system, this communication scheme is not basically different than an encryption-decryption setting. Figure 4.1 depicts a general communication setting employing chaotic synchronization.

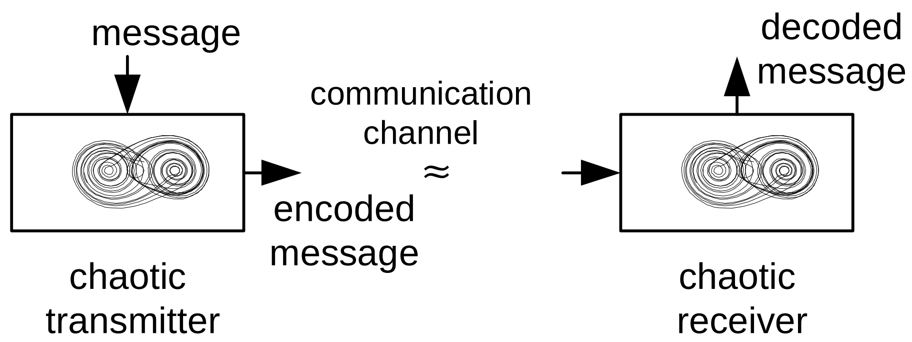

Figure 4.1: A chaotic transmission scheme

Apart from the problem of finding a suitable method to inject a message into chaotic dynamics, synchronization has a nice interpretation in the control framework. In the chaos synchronization jargon that originated from [12], the chaotic system at the transmitter end is referred to as the **drive system**, the one at the receiver end as the **response system** and the coupling between them as the **synchronization signal**. The aim is to choose the form of the synchronization signal and the response system so that a subset of its states converge to the corresponding states from the drive system. Stripped from its jargon, this actually describes a modified observer design problem. Drive system is the original system, response system is a (possibly reduced-order) observer and synchronization signal is nothing other than the output, After our work [17], where the attention is drawn to the observer based synchronization, many existing observer design methods have been ported, with modifications to exploit the nature of chaos, to the field of chaos synchronization. Also many ad hoc synchronization schemes have been noticed to be applications of nonlinear observers. Yet there are examples of synchronized chaotic systems that use truly nonlinear interactions which have no obvious counterpart in the nonlinear observer theory.

This chapter is an extended exposition of this parallel between chaos synchronization and nonlinear observers and draws mainly on our work that started with [17, 77]. To make the terms clear, we first briefly define chaotic systems.

4.1 A Tiny Brief on Chaos

Contrary to wide-spread usage of the term "chaos" that associates it with disorder and randomness, chaos has a very precise mathematical definition and is a property of deterministic nonlinear systems. We start with defining a few mathematical notions related to the solutions of a dynamical system

$$\dot{x} = f(x), \quad x \in \Omega \subset \mathbf{R}^n. \tag{4.1}$$

Let us denote by $\phi(t, x_0)$ the solution of 4.1 at time t starting from the initial condition $x(0) = x_0$. Similarly denote by $\phi(t, \Omega_1)$ the set of points that is reached at time t starting from the initial conditions in the set $\Omega_1 \subset \Omega$, that is, $\phi(t, \Omega_1) = \{x_t \mid x_t = \phi(t, x_0),\ x_0 \in \Omega_1\}$.

Definition 4.1 *The dynamical system 4.1 is said to be* **topologically transitive** *on* Ω *if for any pair of open sets* $\Omega_1, \Omega_2 \subset \Omega$ *there exists a time* $t > 0$ *such that* $\phi(t, \Omega_1) \cap \Omega_2 \neq \emptyset$.

Definition 4.2 *The dynamical system 4.1 has* **sensitive dependence on initial conditions** *if there exists* $\delta > 0$ *such that, for any* $x \in \Omega$ *and any neighborhood* $\mathcal{N}$ *of* x, *there exists* $z \in \mathcal{N}$ *and* $t > 0$ *such that* $\|\phi(t, x) - \phi(t, z)\| > \delta$

Definition 4.3 *The dynamical system 4.1 is said to be* **chaotic** *if*

i. it has sensitive dependence on initial conditions,

ii. it is topologically transitive on Ω,

iii. its periodic solutions are dense [1] *in* Ω.

Put in other words, a chaotic system is unpredictable by its sensitive dependence on initial conditions, its state space cannot be decomposed into disjoint invariant sets by transitivity and it has an underlying regular structure by the density of periodic solutions, [78]. These conditions are simple to express, yet may be difficult to verify mathematically for high-order dynamical systems.

4.2 Observer Based Synchronization of Chaotic Systems

In the context of chaos synchronization as it was initiated with [12], to synchronize two chaotic systems a state of one system (drive) is directly used in the dynamic equation of the other (response). To illustrate the method, consider the Lorentz chaotic attractor, one of the earliest systems considered for the purpose of synchronization, given by

$$\dot{x}_1 \quad = \quad \sigma(x_2 - x_1),$$

[1] *A set* Ω_1 *is dense in another set* $\Omega \supset \Omega_1$ *if for any point* x *in* Ω *there is a point* z *in* Ω_1 *that is arbitrarily close to* x.

$$\dot{x}_2 = -x_1x_3 + \rho x_1 - x_2,$$
$$\dot{x}_3 = x_1x_2 - \beta x_3,$$

where the positive constants σ, β and ρ are chosen to guarantee a chaotic behaviour.

The response system is constructed replicating the dynamic equations for x_2 and x_3 and using x_1 as the synchronization signal that is injected into the response dynamics. Namely,

$$\dot{\hat{x}}_2 = -x_1\hat{x}_3 + \rho x_1 - \hat{x}_2, \tag{4.2}$$
$$\dot{\hat{x}}_2 = x_1\hat{x}_2 - \beta\hat{x}_3. \tag{4.3}$$

As we assume x_1 is already available through measurements (it is used in the response dynamics), the synchronization is proved by showing the stability of the error dynamics for the last two estimates. Indeed, this can easily be done by choosing the simple Lyapunov function $L = (\varepsilon_2^2 + \varepsilon_3^2)/2$, where $\varepsilon_i = (x_i - \hat{x}_i),\ i = 2, 3$. We then have,

$$\begin{aligned}\dot{L} &= \dot{\varepsilon}_2\varepsilon_2 + \dot{\varepsilon}_3\varepsilon_3,\\ &= -\varepsilon_2^2 - \beta\varepsilon_3^2,\\ &\leq -cL,\ \text{if } \beta > 1,\end{aligned}$$

where c is a positive constant. Therefore we have $|L(t)| \leq Me^{-ct}|L(0)|$. This, in turn, implies that both $\varepsilon_2(t)$ and $\varepsilon_3(t)$ converge to zero exponentially fast. Hence the synchronization is achieved for all initial conditions of the response subsystem.

Now we will show how this synchronization occurs as a particular case of an observer design scheme. Consider a dynamical system in the form

$$\dot{x} = A_1x + A_2(y)x, \tag{4.4}$$
$$y = Cx. \tag{4.5}$$

The following theorem gives a sufficient condition for the convergence of a Luenberger-like observer for the system (4.4), (4.5).

Theorem 4.1 *For the system (4.4), (4.5) assume that the pair* (C, A_1) *is detectable and that there exists a gain vector* $K \in \mathbf{R}^n$ *such that the solution* P *of the Lyapunov equation*

$$(A_1 - KC)^T P + P(A - KC) = -Q, \tag{4.6}$$

for some symmetric positive definite matrix Q *satisfies*

$$A_2^T(y)P + PA_2(y) = 0, \ \forall y. \tag{4.7}$$

Then the following is a global observer for (4.4), (4.5);

$$\dot{\hat{x}} = A_1\hat{x} + A_2(y)\hat{x} + K(y - C\hat{x}). \tag{4.8}$$

Proof: Defining the error as $\varepsilon = x - \hat{x}$ its dynamics are given by

$$\dot{\varepsilon} = (A_1 - KC)\varepsilon + A_2(y)\varepsilon. \tag{4.9}$$

Now to establish the stability of (4.9), choose the Lyapunov function candidate $L = \varepsilon^T P\varepsilon$. Then its derivative along the error trajectories is given as

$$\begin{aligned} \dot{L} &= \dot{\varepsilon}^T P\varepsilon + \varepsilon^T P\dot{\varepsilon}, \\ &= \varepsilon^T[(A_1 - KC)^T P + P(A_1 - KC) + A_2^T(y)P + PA_2^T(y)]\varepsilon, \\ &= -\varepsilon^T Q\varepsilon, \quad \text{by (4.6) and (4.7).} \end{aligned}$$

Therefore the error system (4.9) is globally exponentially stable. □

Remark 4.1.1 *For the Lorentz system we have*

$$A_1 = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{bmatrix}, \quad A_2(y) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -y \\ 0 & y & 0 \end{bmatrix}, \quad C = [1\ 0\ 0].$$

Note that the pair (C, A_1) *is detectable though not observable. Therefore a gain vector* K *can be found to render* $A_1 - KC$ *stable. In particular,* $K = [0\ \rho\ 0]^T$ *would do. For* *this choice of K and $P = \mathrm{diag}[\gamma\ \beta\ \beta]$, (4.6), (4.7) are satisfied if $4\beta > \gamma\sigma$. Indeed when the gain vector is so chosen, the last two equations in the observer dynamics give the response system (4.2), (4.3). Another advantage of using an observer structure as the response system is apparent when considers the extra design freedom from the ability to choose the output function. Since the pair (C, A_1) is only detectable and the stable undetectable mode of the linear system decays with rate β, the dynamics of the observer cannot be made to converge faster than that. However, when C is chosen to make (C, A_1) observable then we can achieve arbitrarily fast synchronization. We will have more to say on this issue in Chapter 5.*

In the foregoing application of a nonlinear observer to chaos synchronization we did not make use of the fact that the systems to be synchronized are chaotic. The observer (4.8) work for all the systems, whether chaotic or not, having the form (4.4), (4.5) and satisfying the detectability assumption. Indeed, it is very difficult to characterize the chaoticity of a nonlinear system by its structure in state-space form only.

Next we make use of the fact that trajectories of a chaotic system remain in a bounded region of the state-space to construct a global exponential nonlinear observer. Again, the boundedness of the trajectories is not peculiar to the chaotic systems. For example, the same observer design scheme can be used for periodic nonlinear systems whose set of limit cycles is bounded. Such an approach will be followed in the observer-based control of chaos.

4.3 Observers for Nonlinear Systems with Bounded Attractors

An important property of the chaotic systems is that their state-space have certain attractor sets with non-conventional properties like the fractional dimension, [78]. As the name implies, the trajectories starting close to an attractor are eventually absorbed into the attractor and furthermore an attractor has the positive-invariance property, that is, the trajectories starting inside the attractor remain so for all times. Thus all the interesting behavior of a chaotic system occur in a bounded region of the whole state-space. Therefore, we are led to seek nonlinear observer construction schemes which exploit this property.

To this end we modify the observer for systems in the Brunowsky observer form given in Section 3.2.4. An assumption made about the structure of the system was that the nonlinearity admitted a *global* Lipschitz bound, which is the main restriction. However, for many chaotic systems this assumption fails. Still, it is possible to use the fact that states remain in a compact region to rewrite the system equation in a way to satisfy the global Lipschitz bound. Consider a chaotic system in Brunowsky form

$$\dot{x} = Ax + B\alpha(x), \tag{4.10}$$

$$y = Cx, \tag{4.11}$$

where (A, B, C) are in the form (3.35). Also assume that all the trajectories of (4.10), (4.11) remain in a compact region $\Omega \subset \mathbf{R}^n$. Consider the same system with the nonlinear function $\alpha(\cdot)$ replaced by its extension $\bar{\alpha}(\cdot)$, namely,

$$\dot{x} = Ax + B\bar{\alpha}(x), \tag{4.12}$$

$$y = Cx, \tag{4.13}$$

where $\bar{\alpha}(\cdot)$ satisfies

$$\bar{\alpha}(x) = \alpha(x), \ \forall x \in \Omega. \tag{4.14}$$

Let us denote a trajectory of (4.10), (4.11) by $\phi_\alpha(\cdot, x_0)$ and that of (4.12), (4.13) by $\phi_{\bar{\alpha}}(\cdot, x_0)$. Then the following fact trivially follows.

Fact 4.1 *The trajectories of the systems (4.10) and (4.12) initiating from the set Ω are* *identical, that is,*

$$\phi_\alpha(t, x_0) = \phi_{\bar{\alpha}}(t, x_0), \ \forall t \geq 0.$$

However simple, this fact enables us to replace the dynamical equation of a chaotic system with another one that satisfies the assumption of global Lipschitz continuity without changing the solutions that are of interest. In fact, smooth extension of vector fields can be applied not only to systems in the Brunowsky canonical form but to any system with a bounded set of trajectories. Now we give its application to synchronization by an observer for chaotic systems that do not necessarily have globally Lipschitzian vector fields.

Theorem 4.2 *Consider a chaotic system in the form (4.10), (4.11) whose trajectories remain in a compact region* $\Omega \subset \mathbf{R}^n$. *Assume that the nonlinearity* $\alpha(\cdot)$ *is a smooth function of its arguments. Then there exists a smooth, globally Lipschitz function* $\bar{\alpha} : \mathbf{R}^n \to \mathbf{R}$ *satisfying* $\bar{\alpha}(x) = \alpha(x), \ \forall x \in \Omega$ *and a gain vector* K *such that the following is a globally exponential observer for (4.10), (4.11) for all its trajectories starting inside* Ω.

$$\dot{\hat{x}} = A\hat{x} + B\bar{\alpha}(\hat{x}) + K(y - C\hat{x}). \tag{4.15}$$

Proof: The main element of the proof is the existence of the extension $\bar{\alpha}$ which is globally Lipschitz. This is provided by the classical extension theorems in real analysis which basically state that "if a function is Lipschitz continuous in a compact domain $\Omega \subset \mathbf{R}^n$ then it can be extended to whole $\mathbf{R}^n$, still preserving the same Lipschitz bound", [79, Theorem 6.1.1]. Actually for the purpose of observer design, it is enough that extension can be achieved with any Lipschitz bound.

Using (4.10) and (4.11) we can write the error dynamics as

$$\begin{aligned} \dot{\varepsilon} &= (A - KC)\varepsilon + B(\alpha(x) - \bar{\alpha}(\hat{x})), \\ &= (A - KC)\varepsilon + B(\bar{\alpha}(x) - \bar{\alpha}(\hat{x})), \ \text{since } \alpha(x) = \bar{\alpha}(x), \ \forall x \in \Omega. \end{aligned}$$

Using the fact that $\bar{\alpha}$ is globally Lipschitz and Theorem 3.5, global exponential convergence of the error system is established. $\square$

Remark 4.2.1 *Theorem 4.2 can be extended to include classes of systems which can be transformed into the Brunowsky canonical form under a diffeomorphic change of coordinates. A particular class of chaotic systems for which the above observer works without a need for transformation is the class of forced oscillators given in* n^{th} *order nonlinear differential equation form as*

$$z^{(n)} + F(z,\ \dot{z}, \ldots, z^{(n-1)}) = h(t),$$

where F *is a sufficiently smooth function of its arguments and* h *is a known forcing term that is available through measurements. By the usual assignment of the states as* $x_i = z^{(i)}$ *and choosing the first state* z_1 *as the output, the state-space description of the forced oscillator admits the form (4.10), (4.11) with* $\alpha(x) = -F(x),\ x = [x_1\ x_2\ \ldots\ x_n]^T$*. The extra input term* $h(t)$ *can be ignored as it can be exactly replicated in the observer dynamics thus cancelling out in the error equation. Moreover since the trajectories remain in a compact convex region* Ω*, a Lipschitz constant* γ *for* F *can be calculated as*

$$\gamma = \sup_{x \in \Omega} \|dF(x)\|.$$

Remark 4.2.2 *The gain vector* K *that renders the error system globally exponentially stable is quite large. In fact, it can be shown that the last component* k_n *of the gain satisfies* $|k_n| \geq n^n \gamma^n$*, where* γ *is the Lipschitz constant for* $\bar{\alpha}$*. Such large gains result in extremely large overshoots before the error converges to zero. A constructive procedure to find the gain vector and an eigenvalue assignment scheme to reduce its magnitude will be given in the next chapter. We also leave it to the next chapters to illustrate the synchronization properties with simulated examples.*

Remark 4.2.3 *When the nonlinearity in the mathematical description of a chaotic system (not necessarily in the Brunowsky form) is a function of the system output only,* *e.g. Lure' type systems, the observer design can be considerably simplified by choosing an observer in the form (3.21). Thus the dynamics of the observation error can be made linear, which also reduces the observer gains.*

Remark 4.2.4 *When a globally exponential observer is used in synchronizing chaotic systems, the synchronization is robust to corruption of the synchronizing signal by noise. This does not mean that noise is rejected completely but rather the synchronization error eventually remains in a ball whose radius is proportional to noise amplitude, [77]. Hence chaotically synchronized transmitter and receiver systems can tolerate a certain amount of channel noise. A similar result holds for robustness of the synchronization to the perturbation of parameters, namely, even when the observer parameters deviate from those of the original system, the synchronization is not totally lost. Such perturbation results are actually corollaries of the exponential stability of the error system, see, e.g. [80].*

4.4 Message Transmission Using Observers

In the previous section we established how an observer can be used for the purpose of the chaos synchronization. While the study of synchronizing chaotic systems is an interesting topic in itself, its starting motivation is the possibility of its application to the secure message transmission. Therefore in this section we propose two chaotic message transmission schemes based on nonlinear observers and illustrate them with simulations.

Before the details of these procedures let us briefly mention another message encoding scheme, *parameter modulation*, which received a considerable attention in the literature. In the parameter modulation, the message signal is used to induce time-variations in a set of constant parameters of the chaotic system. For examples of this method and message reconstruction systems see, e.g., [81].

4.4.1 Algebraic Encoding

In the algebraic encoding method the chaotic dynamics is not affected by the message signal. The message is encoded using a function of the states that can be reconstructed at the receiver end. Hence we have, at the transmitter end, a structure like the following.

$$\dot{x} = f(x), \tag{4.16}$$

$$y = h(x), \tag{4.17}$$

$$s = \sigma(x, m), \tag{4.18}$$

where m is the message signal and the function $\sigma(x, \cdot)$ is uniformly continuous in x and admits an inverse that is uniformly continuous in x, that is,

$$m = \sigma^{-1}(x, s), \;\; \forall x \in \mathbf{R}^n. \tag{4.19}$$

We then communicate the signals s and y. At the receiver end we have an observer which asymptotically estimates the state x. The inverse of the coding function is used to reconstruct the message signal m. Using the estimated states $\hat{x}$ we can get an estimate of the message as

$$\hat{m} = \sigma^{-1}(\hat{x}, s), \tag{4.20}$$

$$= \sigma^{-1}(\hat{x}, \sigma(x, m)). \tag{4.21}$$

Defining the error between the message and its estimate as $\varepsilon_m(t) = m(t) - \hat{m}(t)$, we establish its asymptotic convergence to zero by using the uniform continuity of σ and σ^{-1}.

As a simple example suppose the output of a chaotic system is chosen to be x_1 and the message is additively encoded using another state, say, x_2 as $s = \sigma(x, m) = x_2 + m$. When the observer at the transmitter end yields an exponential estimate $\hat{x}_2$ then the message can be exponentially reconstructed using $\hat{m} = x_2 + m - \hat{x}_2 = m + \varepsilon$.

The obvious disadvantage of the algebraic encoding method is the need to send two signals, the output y and the encoded signal s. As we will see in the next section this can be remedied using a different method of encoding.

4.4.2 Encoding by Injection

In this method the message is injected as input into the chaotic dynamics at the transmitter end changing the structure of the system. Therefore one has to make sure to keep the message amplitude within appropriate bounds not to destroy the chaotic dynamics.

Again consider a single-output chaotic system given by (4.16), (4.17) and suppose that there exists an asymptotic observer for this system in the form

$$\dot{\hat{x}} = f(\hat{x}) + \kappa(y, \hat{x}). \tag{4.22}$$

Namely, the error system

$$\dot{\varepsilon} = f(x) - f(\hat{x}) - \kappa(y, \hat{x}) \tag{4.23}$$

is globally asymptotically stable.

Also assume the message signal to be encoded is a scalar function and there is a function $\sigma_1 : \mathbf{R} \to \mathbf{R}^n$ and an uniformly continuous invertible function $\sigma_2 : \mathbf{R} \to \mathbf{R}$ with uniformly continuous inverse which satisfy the following;

$$\sigma_1(m) - \kappa(h(x) + \sigma_2(m), \hat{x}) = -\kappa(h(x), \hat{x}), \ \forall \hat{x} \in \mathbf{R}^n. \tag{4.24}$$

Then to inject the message we modify the chaotic system (4.16), (4.17) as

$$\begin{aligned} \dot{x} &= f(x) + \sigma_1(m), && (4.25) \\ y &= h(x) + \sigma_2(m), && (4.26) \end{aligned}$$

and use (4.22) as the observer at the receiver. Using (4.25), (4.26), (4.22) and the fact that (4.24) holds we obtain the error dynamics (4.23). Therefore $[x(t) - \hat{x}(t)] \to 0$. The message is reconstructed using

$$\hat{m} = \sigma_2^{-1}(y - h(\hat{x})).$$

Convergence of this estimate to the message sent is easily shown using the uniform continuity properties of h and σ_2^{-1}.

We note that when in the observer the output error is linearly injected, that is, $\kappa(y, \hat{x}) = K(y - C\hat{x})$, choice of $\sigma_1(m) = Km$ and $\sigma_2(m) = m$ satisfies (4.24).

4.4.3 Comparison and Simulations

To illustrate the two methods we choose a 3-dimensional chaotic system in Lure' form, [82]. This particular system is chosen for the simplicity of the observer design for these type of systems, see Section 3.2.1. The system is given as

$$\dot{x}_1 = x_2, \tag{4.27}$$

$$\dot{x}_2 = x_3, \tag{4.28}$$

$$\dot{x}_3 = -1.25x_2 - x_3 + \alpha(y), \tag{4.29}$$

$$y = x_1, \tag{4.30}$$

where the nonlinear function $\alpha : \mathbf{R} \to \mathbf{R}$ is given as

$$\alpha(y) = \begin{cases} -1.8y & |y| \leq 1, \\ 3.6y - 3.6 \operatorname{sign}(y) & 1 \leq |y| \leq 3, \\ 5.4 \operatorname{sign}(y) & |y| \geq 3. \end{cases}$$

A globally exponential observer for this system can be constructed by choosing the gain vector $K = [2.9\ 0.55\ \ -2.375]^T$ for the output error injection. Since the nonlinearity, being a function of output only, can be reconstructed at the receiver end, the observer structure in Section 3.2.1 can be used. Then the observer dynamics are given by

$$\dot{\hat{x}}_1 = \hat{x}_2 + 2.9(y - \hat{x}_1), \tag{4.31}$$

$$\dot{\hat{x}}_2 = \hat{x}_3 + 0.55(y - \hat{x}_1), \tag{4.32}$$

$$\dot{\hat{x}}_3 = -1.25\hat{x}_2 - \hat{x}_3 + \alpha(y) - 2.375(y - \hat{x}_1). \tag{4.33}$$

A typical trajectory of this system is shown Figure 4.2.

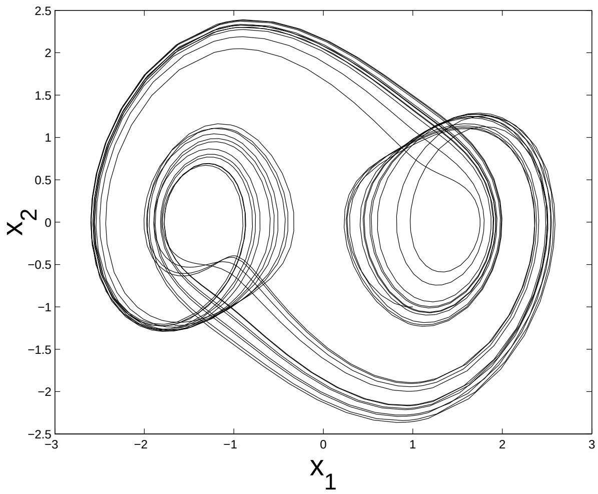

Figure 4.2: A phase portrait of Brockett's chaotic system

Algebraic Encoding

To keep the exposition simple, we take $s = \sigma(x, m) = x_3 + m$ and, at the receiver end, restore the message using the estimate $\hat{x}_3$ as $\hat{m} = s - \hat{x}_3$. The message is chosen to be a periodic square wave of period π secs. In Figure 4.3 is shown the error in message reconstruction.

Encoding by Message Injection

With the same observer structure, the chaotic system (4.27) - (4.29) is modified as

$$\begin{aligned}
\dot{x}_1 &= x_2 + 2.9m(t), \\
\dot{x}_2 &= x_3 + 0.55m(t), \\
\dot{x}_3 &= -1.25x_2 - x_3 + \alpha(y) - 2.375m(t),
\end{aligned}$$

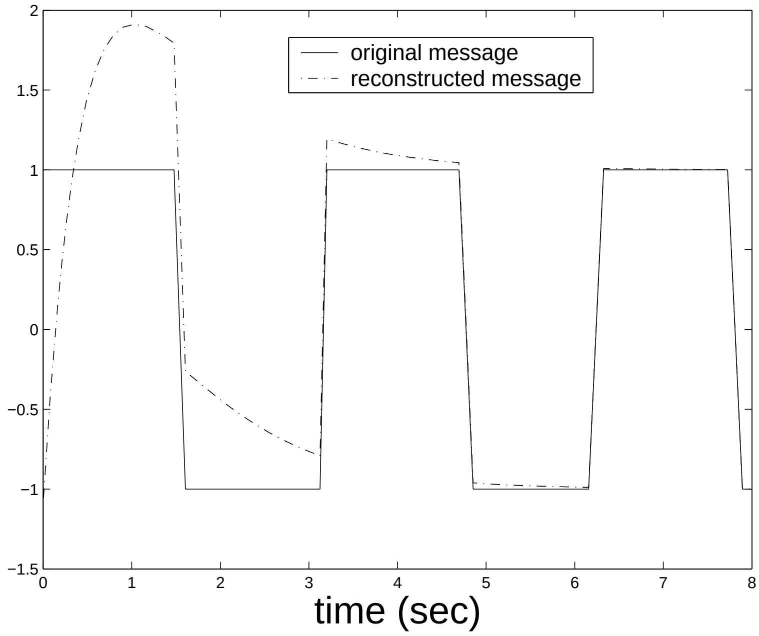

Figure 4.3: Message and its reconstruction using algebraic encoding.

$$y = x_1 + m(t).$$

The message is reconstructed using $\hat{m} = y - \hat{x}_1$. Figure 4.4 shows the error in message reconstruction.

As it is also apparent from the simulation results, there are no basic differences between the two schemes in terms of their convergence properties. This is expected since the error dynamics for the state estimation are the same. Yet, the first scheme requires two signals to be sent to the receiver end. Note that in the second scheme the transmitter has the exact knowledge of the observer gains used in the receiver end. However, this is not a serious disadvantage of the method as the gains are kept constant and their values can be transmitted once and used subsequently.

We also note that the described message transmission schemes are robust to parameter mismatch and noise. This robustness is a benefit of using exponential observers, which guarantee that the error in message reconstruction remains within bounds that are directly proportional to mismatch and noise magnitudes.

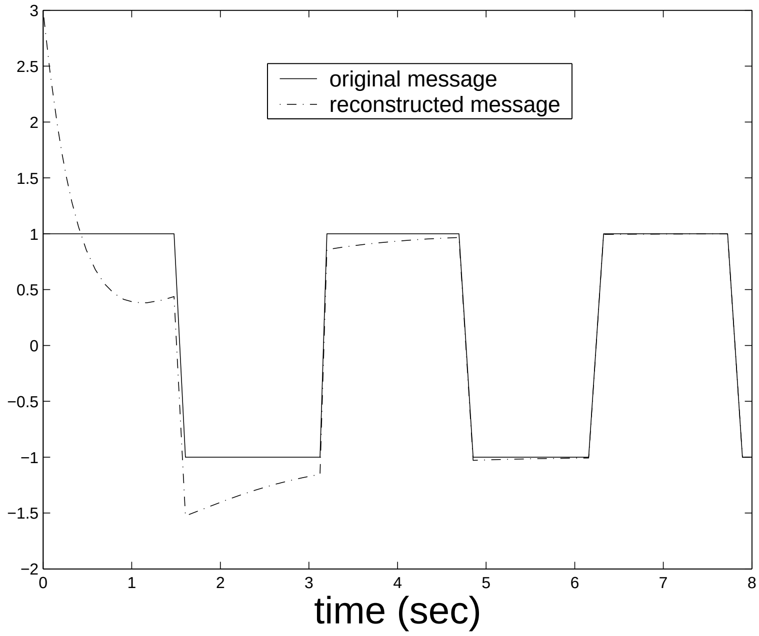

Figure 4.4: Message and its reconstruction using message injection.

We note that other interesting message encoding schemes are possible depending on the system and observer structures. For example, for synchronization using Lur'e type systems, having sufficiently band-limited messages facilitate its reconstruction. Similar simplifications are obtained when the message is digital. Another method is to pre-filter the message before injection to improve the reconstruction. We followed such approaches in [83].

Chapter 5

Gain and Output Selection for High-Gain Observers

In the last chapter we employed observers for the purpose of chaos synchronization with the assumption that for a given chaotic system a sufficiently well-behaving observer exists. This assumption still needs to be justified by an actual method of constructing the observer. The basic element in the construction is the choice of the function through which the measured output error is injected into the observer dynamics. If, as in most cases, the output error is injected through a gain vector then the basic problem is the determination of the gain vector. When the chaotic system is linear up to output injection as in Lure' type systems, gain K is selected to render a matrix $A - KC$ stable for a detectable pair (C, A). However, for other types of systems, this simple assignment of eigenvalues to left-half complex plane is not enough to guarantee the stability of the error system.

In this chapter we provide a gain selection procedure for high-gain observers for systems in the Brunowsky canonical form. Another concern is the minimization of gain magnitudes as high gains result in excessive overshoots before the error finally decays to zero. To this end, we also provide a modification of the gain assignment procedure to reduce the gain magnitudes close to their theoretical limits imposed by the particular stability proof.

An extra freedom in the observer design for chaos synchronization is brought about by the ability to choose the output map, or the synchronization signal. At least theoretically, one can exploit this freedom to facilitate the observer design. For example, when the nonlinearity in the system description is a function of one state only, then that state can be chosen as the output, thus, bringing the system in a form that is linear up to output injection. The second part of this chapter is devoted to the problem of output choice.

5.1 Gain Selection

Theorem 3.5 gives a sufficient condition for a system in the Brunowsky canonical form (3.33), (3.34) to admit a Luenberger-like observer in the form (3.36). The proof of the theorem relies on the fact that there exists a self-conjugate set of n distinct complex numbers $\lambda = \{\lambda_1, \ldots, \lambda_n\}$, $\Re(\lambda_1) \geq \Re(\lambda_2) \geq \ldots \geq \Re(\lambda_n)$, on the left-half plane which satisfy the following inequality;

$$\beta \triangleq \Re(\lambda_1) + \sqrt{n}\gamma \|V^{-1}(\lambda)\|_2 < 0. \tag{5.1}$$

The constant $\sqrt{n}$ in (5.1) is due to the usage of induced 2-norm. Since the calculation of ∞-norm is easier than the calculation of singular values, we modify (5.1) using the fact that $\|M\|_2 \leq \sqrt{n}\|M\|_\infty$, $\forall M \in \mathbf{R}^{n\times n}$, to obtain

$$\beta \leq \bar{\beta} = \Re(\lambda_1) + n\gamma \|V^{-1}(\lambda)\|_\infty. \tag{5.2}$$

Therefore if we can assign the eigenvalues to make $\bar{\beta}$ negative then the exponential stability of the error system is guaranteed. The following theorem, which was part of our previous work, establishes this.

Theorem 5.1 *Let $V(\lambda)$ be the Vandermonde matrix constructed with the set of eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ satisfying $\lambda_i = r^{i-1}\eta$, $r > 1$ and $\eta < 0$. Then for sufficiently large $|\eta|$ and r, we have $\|V^{-1}(\lambda)\|_\infty$ independent of η and*

$$\lim_{r\to\infty} \|V^{-1}(\lambda)\|_\infty = 1. \tag{5.3}$$

Proof: See [84]. □

Corollary 5.1.1 *By Theorem 5.1, as we increase $|\eta|$, $\bar{\beta}$ decreases, eventually becoming negative. Therefore there exists a set of eigenvalues for which the error system is exponentially stable.*

After the assignment of the eigenvalues that make $\bar{\beta}$ negative, the gain vector $K = [k_1 \; k_2 \; \ldots \; k_n]$ is calculated from the identity

$$s^n + \sum_{i=1}^{n} k_i s^{n-i} = \prod_{i=1}^{n} (s - \lambda_i). \tag{5.4}$$

Note that the gain largest in magnitude is k_n and satisfies

$$|k_n| = |\lambda_1 \lambda_2 \ldots \lambda_n|.$$

To prove that there is a lower bound on the gain element k_n, we first give a preliminary lemma about the ∞-norm of the Vandermonde matrix.

Lemma 5.1 *Let $V(\lambda)$ be the Vandermonde matrix of n distinct complex numbers, $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then*

$$\|V^{-1}(\lambda)\|_\infty \geq 1. \tag{5.5}$$

Proof: Let us denote $V = V(\lambda)$ and $\bar{V} = V^{-1}(\lambda)$. Then $\bar{V}V = I_n$, which implies in particular, $1 = \sum_{j=1}^{n} \bar{v}_{nj} v_{jn}$. Since $v_{jn} = 1,\ j = 1, \ldots, n$ we have $1 = \sum_{j=1}^{n} \bar{v}_{nj} = |\sum_{j=1}^{n} \bar{v}_{nj}| \leq \sum_{j=1}^{n} |\bar{v}_{nj}|$. Recalling the expression of ∞-norm, $\|\bar{V}\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |\bar{v}_{ij}|$, the result follows. $\square$

Theorem 5.2 *For an eigenvalue assignment that satisfy (5.2) and (5.4) we have*

$$|k_n| \geq n^n \gamma^n.$$

Proof: Starting from the fact $|k_n| = |\lambda_1 \lambda_2 \ldots \lambda_n|$ we write

$$\begin{aligned}
|k_n| &= |\lambda_1||\lambda_2| \ldots |\lambda_n|, \\
&\geq |\Re(\lambda_1)||\Re(\lambda_2)| \ldots |\Re(\lambda_n)|, \ \text{by } |\lambda_i| \geq |\Re(\lambda_i)|, \\
&\geq |\Re(\lambda_1)|^n, \ \text{by } \Re(\lambda_1) \leq \Re(\lambda_i), \ 2 \leq i \leq n, \\
&\geq (n\gamma)^n = n^n \gamma^n, \ \text{by (5.2) and Lemma 5.1.} \qquad \square
\end{aligned}$$

Remark 5.2.1 *It is important to note that the lower bound provided by Theorem 5.2 depends on the particular Lyapunov function used to prove the stability of the error system. However, using the functional form of* α, *it may be possible to use gains that fail the lower bound in Theorem 5.2. As an example, consider the case* $\alpha(x) = x_1 + 2x_2$ *which has Lipschitz constant* $\gamma = 3$. *Theorem 5.2 gives the lower bound as* $|k_2| \geq 36$. *However, since* α *is linear, choosing* $k_1 = 5,\ k_2 = 13$ *yields a linear error system which is globally exponentially stable.*

Theorem 5.2 imposes a limit on the achievable minimum of the largest component of the gain that makes the exponent $\bar{\beta}$ negative. Many eigenvalue assignment schemes, including the one provided by Theorem 5.1 well exceed this limit. The basic failure of these schemes is due to the fact that they constrain the eigenvalues to be real. Following observation gives a clue to why we can expect higher gains when eigenvalues are assigned real; as two eigenvalues get close to each other (measured with respect to their magnitudes) $V(\lambda)$ approaches singularity and $\|V^{-1}(\lambda)\|_\infty$ becomes larger. However to make $\bar{\beta}$ negative, λ_1, the eigenvalues that is closest to the imaginary axis, must be pushed farther into the negative real axis, increasing the gains. When the eigenvalues are allowed to take complex values, two eigenvalues can be distanced from one another without increasing their magnitudes. Consequently we are led to seek a method to optimally place the eigenvalues on the complex plane, in particular on the same circle with center at the origin. Obviously, placing the eigenvalues on a common circle is still sub-optimal however it yields gains that are smaller than the case of real eigenvalues.

Now we illustrate the method for planar systems.

Example 5.1 *Let $n = 2$. Then we have*

$$\|V^{-1}(\lambda)\|_\infty = \max\{\frac{2}{|\lambda_1 - \lambda_2|}, \frac{|\lambda_1| + |\lambda_2|}{|\lambda_1 - \lambda_2|}\}.$$

Suppose $\gamma > 1$. For stability we require that $\Re(\lambda_1) + n\gamma\|V^{-1}(\lambda)\|_\infty < 0$. Substituting $n = 2$ and using Lemma 5.1, we obtain $|\lambda_1| > 2$. Therefore

$$\|V^{-1}(\lambda)\|_\infty = \frac{|\lambda_1| + |\lambda_2|}{|\lambda_1 - \lambda_2|}$$

Choosing λ_1 and λ_2 on the same circle with center at the origin, that is, $\lambda_1 = re^{j\theta}$, $\lambda_2 = re^{-j\theta}$, $r > 0$, $\pi/2 < \theta < \pi$, we have

$$\|V^{-1}(\lambda)\|_\infty = \frac{1}{\sin\theta}.$$

To make the exponent $\bar{\beta}$ zero (we can then make it negative by slightly increasing the radius r) we need to choose r and θ to satisfy

$$r\cos\theta + \frac{2\gamma}{\sin\theta} = 0$$

while minimizing the largest gain $k_n = r^2$. This problem can be easily solved using standard Lagrange multipliers technique of constrained optimization, [85], to get

$$\theta_0 = \frac{3\pi}{4}, \quad r_0 = 4\gamma.$$

Thus the gains are found as $k_1 = 4\sqrt{2}\gamma$ *and* $k_2 = 16\gamma^2$.

To compare it with the case where the eigenvalues are assigned real, we can calculate the optimal gain values as $k_1 = (8+4\sqrt{5})\gamma$ *and* $k_2 = (22+10\sqrt{5})\gamma^2$*, which are larger than the ones we found before.*

For planar systems we could find analytical expression for optimum gain values without recourse to numerical optimization. However, for $n \geq 3$ the expression for $\|V^{-1}(\lambda)\|_\infty$ becomes progressively nastier, which prevents one from obtaining analytical solutions. The following lemma provides a useful result that facilitates possible numerical optimization.

Lemma 5.2 *Consider a Vandermonde matrix* $V(\lambda_1, \ldots, \lambda_n)$ *where* λ_i*'s lie on a common circle with center at the origin, that is,* $\lambda_i = re^{j\theta_i}$*. Then for* r *large enough,* $\|V^{-1}(\lambda)\|_\infty$ *is independent of* r*.*

Proof: When $\lambda_i = re^{j\theta_i}$ we have $V(\lambda_1, \ldots, \lambda_n) = V(e^{j\theta_1}, \ldots, e^{j\theta_n})R$, where $R = \mathrm{diag}[r^{n-1}\ r^{n-2}\ \ldots\ r\ 1]$. Hence $V^{-1}(\lambda_1, \ldots, \lambda_n) = R^{-1}V^{-1}(e^{j\theta_1}, \ldots, e^{j\theta_n})$. Since multiplication from the left by R^{-1} scales the rows of $V(e^{j\theta_1}, \ldots, e^{j\theta_n})$, as r gets larger first $n-1$ rows of $V^{-1}(\lambda_1, \ldots, \lambda_n)$ diminishes while its last row remains unchanged and independent of r. Therefore the ∞-norm becomes the absolute sum of the last row, from which the result follows. $\square$

Corollary 5.2.1 *When the eigenvalues are assigned on a common circle, the exponent in the error equation becomes*

$$\bar{\beta} = r\cos\theta_1 + n\gamma\|V^{-1}(e^{j\theta_1}, \ldots, e^{j\theta_n})\|_\infty, \tag{5.6}$$

where $\pi/2 < \theta_1 < \ldots < \theta_n < 3\pi/2$.

In choosing eigenvalue locations for the purpose of gain minimization, (5.6) provides a useful simplification. Our aim is to choose θ_i and r such that $\bar{\beta} = 0$ is satisfied while minimizing $k_n = r^n$. Since $k_n(r) = r^n$ is a strictly monotone increasing function we can instead minimize r. By (5.6), we have $r = -n\gamma \| V^{-1}(e^{j\theta_1}, \ldots, e^{j\theta_n}) \|_\infty / \cos\theta_1$. Therefore we end up with the following constrained minimization problem.

Optimization Problem: Determine a set of angles $\theta_1, \ldots, \theta_n$ to minimize the objective function

$$J(\theta_1, \ldots, \theta_n) = \frac{\| V^{-1}(e^{j\theta_1}, \ldots, e^{j\theta_n}) \|_\infty}{\cos\theta_1}$$

subject to

$$\begin{aligned} &\frac{\pi}{2} < \theta_1 < \ldots < \theta_n < \frac{3\pi}{2}, \\ &\theta_i = -\theta_{n+1-i}, \; i = 1 \ldots n. \end{aligned}$$

One can use a nonlinear optimization software to solve for optimum angles. Here we tabulate the results we obtained for dimensions up to $n = 9$. Table 5.1 shows only the angles on the third quadrant. The remaining angles are given by their negative. Note that when n is odd, we have a real eigenvalue whose angle is not explicitly shown on the table.

n	θ_1	θ_2	θ_3	θ_4
2	134.97			
3	114.92			
4	109.19	145.57		
5	106.32	129.82		
6	103.46	119.79	157.60	
7	102.03	114.06	143.28	
8	100.59	109.76	133.25	163.91
9	99.16	106.90	126.09	151.02

Table 5.1: Optimum angles for eigenvalues assignment on a circle.

5.2 Output Selection

Most control problems related to the observability of dynamical systems assume that the output map is fixed. Indeed, most of the time the output map is determined by the location of sensors in a dynamic system and constrained by the availability of sensors for the quantities to be measured. However, depending on the use, the ability to choose the output map may prove beneficial.

One particular application is in the synchronization of chaotic systems, where the synchronization signal can be seen as the output of the transmitter chaotic system. Despite their sophisticated dynamic behavior, the structure of chaotic systems are generally simple and one can afford to employ sensors for all system states. Since the choice of receiver system for chaos synchronization can essentially be seen as an observer design problem, an interesting problem is to determine the set of output maps which yield a globally observable system.

This problem is quite intractable for general nonlinear systems but a parametric solution can be given for linear systems and some numerical approaches are possible for planar nonlinear systems.

5.2.1 Output Selection for Linear Systems

Consider the following linear system

$$\dot{x} = Ax, \tag{5.7}$$

where $x \in \mathbf{R}^n$, $A \in \mathbf{R}^{n\times n}$. The output choice problem is the parameterization of the set of matrices $C \in \mathbf{R}^{1\times n}$ (i.e., the output is scalar) such that the pair (C, A) is observable. The following theorem, which expresses a known result in a slightly different form, provides the fist step in such a parameterization. We also give its proof as it illustrates the method to choose the row vector C.

Theorem 5.3 *For the linear system given by (5.7), there exists a row vector C which makes the pair (C, A) observable if and only if, in the Jordan form of A, there is exactly one Jordan block corresponding to each distinct eigenvalue of A.*

Proof: Since observability is invariant under similarity transformations, without loss of generality we can assume A is a matrix in Jordan form.
(**only if**) Assume the contrary; Jordan form J contains two Jordan blocks for an eigenvalue. Then with suitable ordering, J has the form,

$$J = \mathrm{diag}[J_a\ J_b], \quad J_a = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \text{ or } J_a = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

Now partition C as $C = [C_a\ C_b]$, where the sizes of C_a and C_b are compatible with the sizes of J_a and J_b and C_a is given as either $C_a = [c_{a1}\ c_{a2}\ c_{a3}]$ or $C_a = [c_{a1}\ c_{a2}]$, depending on the size of J_a. Then the observability matrix $\mathbf{O}(C, J)$ of the pair (C, J) can be written in two column partitions as

$$\mathbf{O}(C, J) = \begin{bmatrix} C_a & C_b \\ C_a J_a & C_b J_b \\ C_a J_a^2 & C_b J_b^2 \\ \vdots & \vdots \\ C_a J_a^{n-1} & C_b J_b^{n-1} \end{bmatrix}$$

The observability matrix of the pair (C_a, J_a) is calculated as

$$\mathbf{O}(C_a, J_a) = \begin{bmatrix} c_{a1} & c_{a2} & c_{a3} \\ c_{a1}\lambda & c_{a1} + c_{a2}\lambda & c_{a3}\lambda \\ c_{a1}\lambda^2 & 2c_{a1}\lambda + c_{a2}\lambda^2 & c_{a3}\lambda^2 \end{bmatrix} \text{ or } \mathbf{O}(C_a, J_a) = \begin{bmatrix} c_{a1} & c_{a2} \\ c_{a1}\lambda & c_{a2}\lambda \end{bmatrix}.$$

A simple calculation shows that (C_a, J_a) is rank-deficient in both cases. By Cayley-Hamilton theorem, the rows $C_a J_a^k$ linearly depend on the rows of $\mathbf{O}(C_a, J_a)$. Therefore the first partition of $\mathbf{O}(C, J)$ is also rank-deficient, which implies the singularity of $\mathbf{O}(C, J)$.

Hence (C, J) is not observable for any C.

(**if**) Assume that the Jordan form J has the desired property, i.e. has the form $J = \mathrm{diag}[J_{\lambda_1}\ J_{\lambda_2}\ \ldots\ J_{\lambda_m}]$ with distinct λ_i's, $J_{\lambda_i} \in \mathbf{R}^{n_i \times n_i}$ and $n_1 + n_2 + \ldots + n_m = n$. Each J_{λ_i} has the form

$$J_{\lambda_i} = \begin{bmatrix} \lambda_i & 1 & 0 & \ldots & 0 \\ 0 & \lambda_i & 1 & \ldots & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & & \ldots & \lambda_i & 1 \\ 0 & & \ldots & 0 & \lambda_i \end{bmatrix}.$$

Choose an output matrix $C = [C_1,\ C_2, \ldots, C_m]$, $C_i = [1\ 0\ \ldots\ 0] \in \mathbf{R}^{1 \times n_i}$. Then a simple calculation shows that the observability matrix $\mathbf{O}(C, J)$ is the generalized Vandermonde matrix formed by the complex numbers $\lambda_1, \ldots, \lambda_m$ with multiplicities $n_1, \ldots, n_m$. Using the formula given in [86, p. 400], the determinant is given by

$$\det(\mathbf{O}(C, J)) = \prod_{i=1}^{m} \prod_{l=0}^{n_i - 1} l! \prod_{i>j} (\lambda_i - \lambda_j)^{n_i n_j}.$$

Since λ_i's are distinct, the determinant is nonzero and therefore, for this choice of C, the pair (C, J) is observable. $\square$

Remark 5.3.1 *The condition on the Jordan form of A can be replaced by the condition that the characteristic and minimal polynomials of A are the same, [87] or the condition that A is* cyclic*, [32].*

Corollary 5.3.1 *Assume that the condition in Theorem 5.3 is satisfied. Then the set of row vectors C that make (C, A) observable is given by*

$$C = [C_1\ C_2 \ldots C_k] M^{-1},\ C_i \in \mathbf{R}^{1 \times n_i},\ C_i = [C_{i1}\ C_{i2} \ldots C_{in_i}],\ C_{i1} \neq 0,\ i = 1 \ldots k, \quad (5.8)$$

where n_i is the size of i^{th} Jordan block of A and M is a modal matrix that satisfies $M^{-1} A M = J$.

This parameterization is especially useful when one seeks optimum sensor locations without sacrificing observability. In particular, one would want to retain observability while measuring as few states as possible. This corresponds to maximizing the number of zeros in C. With the above formulation this problem can be solved very easily. Indeed, writing (5.8) as

$$CM = [C_1 \; C_2 \dots C_k], \;\; C_{i1} \neq 0,$$

minimum number of non-zero entries in C is the number of rows of the largest submatrix of M which have non-zero entries in places corresponding to C_{i1}'s.

5.2.2 Nonlinear Systems

The situation is far more complicated than it is for linear systems, for, a workable generalization of modal decomposition for nonlinear systems is not obvious. Consider a nonlinear system

$$\begin{aligned} \dot{x} &= f(x), & (5.9) \\ y &= h(x), & (5.10) \end{aligned}$$

where f and h are sufficiently smooth functions. This system is globally observable if the map $\Phi : \mathbf{R}^n \to \mathbf{R}^n$, $\Phi(x) = [h(x) \; L_f(x) \dots L_f^{n-1} h(x)]^T$ is full rank for every $x \in \mathbf{R}$, (see Section 2.2.2). The last condition is, in turn, equivalent to having the Jacobian matrix

$$Q(x) = \frac{\partial \Phi(x)}{\partial x} \tag{5.11}$$

nonsingular for every $x \in \mathbf{R}^n$

Since the dependence of the matrix Q to the functional form of h is highly nonlinear, it is difficult to parameterize the set of output maps h that render the system (5.9), (5.10) globally observable. However when h is constrained to be linear and when f is linearized around the equilibrium as $f(x) = Ax + \bar{f}(x)$, the result obtained in the previous section might prove useful because for global observability of smooth systems it is necessary to have the linearized system observable, [28]. Hence if a linear output map C is to be chosen, it should be chosen so as to make (C, A) observable. Yet this is not sufficient as the observability may be destroyed by the nonlinear part. The result given in the previous section can thus be used as a first step to parameterize C matrices that satisfy this necessary condition.

Chapter 6

Observer Based Control of Chaos

When for a dynamical system one has an observer with sufficiently fast estimation dynamics, the natural application is using this observer in a control loop to modify the behavior of the dynamical system. In fact the need for an observer stems to a large extent from the superiority of state feedback control schemes to output feedback control. It is also possible to see observer-controller configuration as dynamic output feedback. This is most apparent through a factorization approach to control problems, see e.g., [48].

In this chapter we apply observer design schemes for chaotic systems that we proposed in the last two chapters to related control problems. First problem we consider is the determination of appropriate feedback control structures to drive a forced chaotic oscillator among different operating regimes. Then we give an observer based solution to reference tracking problem for forced chaotic oscillators. Also in this chapter we illustrate the gain selection scheme we proposed in Section 5.1. The material in this chapter is based on our works, [88, 89].

6.1 Control of Forced Chaotic Oscillators

We consider the class of chaotic systems given in the form

$$y^{(n)} + F(y,\ \dot{y}, \ldots, y^{(n-1)}) = h(t) + r(t), \tag{6.1}$$

where $y^{(i)}$ represents the ith time derivative of y, for $i = 1, 2, \ldots, n-1$, $h(t)$ is a known forcing function and $r(t)$ is the control input to be determined. We assume that F is differentiable with respect to its arguments. This class of systems covers a wide range of chaotic oscillators, e.g. Duffing equation, Van der Pol oscillator, etc.

We will first explain the control method we use. Assume that the signal y as well as its time derivatives $y^{(i)}$, $i = 1, \ldots, n-1$ are all available. Choose the control input r as

$$r(t) = \hat{h}(t) - h(t) + F(y, \ldots, y^{(n-1)}) - \hat{F}(y, \ldots, y^{(n-1)}), \tag{6.2}$$

and substitute in (6.1) to get

$$y^{(n)} + \hat{F}(y,\ \dot{y}, \ldots, y^{(n-1)}) = \hat{h}(t). \tag{6.3}$$

Hence by the application of this particular control input (6.2) we can convert the dynamics of the forced oscillator (6.1) to a different desired dynamics given by (6.3). A particular application of this idea is the following. Assume that F in (6.1) has the following form :

$$F(y, \ldots, y^{(n-1)}) = \sum_{i=1}^{p} \alpha_i F_i(y, \ldots, y^{(n-1)}) + F_r(y, \ldots, y^{(n-1)}), \tag{6.4}$$

where for $i = 1, 2, \ldots, p$, F_i are known and differentiable functions, and α_i are real and constant parameters. Assume that the dynamical behavior of (6.1) depends on the parameters α_i, (e.g. a bifurcation diagram in terms of parameters α_i is known). Then, by choosing the control input $r(t)$ as

$$r(t) = \sum_{i}^{p} (\alpha_i - \hat{\alpha}_i) F_i + \hat{h}(t) - h(t), \tag{6.5}$$

(6.1) is reduced to

$$x^{(n)} + \sum_{i}^{p} \hat{\alpha}_i F_i + F_r = \hat{h}(t), \tag{6.6}$$

hence any behavior in the bifurcation diagram of (6.1) can be obtained with appropriate choice of the parameters $\hat{\alpha}_i$.

The basic problem in the scheme presented above is the unavailability of the signals $y^{(i)}(t)$, for $i = 1, \dots, n-1$. We assume that $y(t)$ is measurable, however obtaining the derivatives by numerical derivation is not desirable since inevitably this operation is adversely affected by the presence of noise in measurements. However, to dynamically estimate the derivatives one can use a nonlinear observer which, in the limit, acts as an output differentiator. Hence in the control laws given by (6.2) or (6.5) we use the estimates $\hat{y}^{(i)}$, $i = 1, \dots, n-1$ instead of actual derivatives. This brings a perturbation term $\epsilon(t)$ in the right hand sides of (6.3) and (6.6). Assuming that the signals are bounded and that $\epsilon(t)$ decays exponentially to zero, one would expect to observe a dynamical behavior that asymptotically converge to those of (6.3) and (6.6). Indeed this asymptotic behavioral equivalence can be satisfied with the assumption that the behavior of (6.3) and (6.6) (e.g. chaos, limit cycle, etc.) are structurally stable. To make this point clearer consider the perturbed system

$$y^{(n)} + \hat{F}(y,\ \dot{y}, \dots, y^{(n-1)}) = \hat{h}(t) + n(t, y, \dot{y}, \dots, y^{n-1}), \tag{6.7}$$

where $n : \mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{R}$ is an arbitrary smooth function. A dynamical system is structurally stable if there exists an $\epsilon_0 > 0$ such that the solutions of the perturbed system (6.7) are topologically equivalent[1] to those of the unperturbed system (6.3) for all perturbations satisfying $n < \epsilon_0$.

To find a suitable control law, we first transform (6.1) into the state space form by defining the states

$$x_1 = y, \quad x_{i+1} = y^{(i)}, \quad i = 1,\ 2, \dots,\ n-1.$$

[1]Two dynamical systems are said to be topologically equivalent if their trajectories can be deformed to each other by a continuous and invertible function, see e.g. [90].

Also define $f(x) = F(x_1, x_2, \ldots, x_{n-1})$. With these definitions, state space representation of (6.1) is of the form (3.33), (3.34) with $\alpha(x)$ in (3.33) replaced by $-f(x)$. Furthermore by the fact that F is sufficiently smooth and that the trajectories remain in a bounded region Ω, a global Lipschitz bound on f can readily be found. Therefore the observer structure (3.36) can be used to estimate the state x.

Next we choose the control input r as (cf. (6.2))

$$r(t) = \hat{h}(t) - h(t) + f(\hat{x}) - \hat{f}(\hat{x}). \tag{6.8}$$

Although we do not have x, $\hat{x}$ is available, hence the control law (6.8) is implementable. Substituting this control law in (6.1) we obtain the expression of the controlled system as

$$y^{(n)} + \hat{F}(y, \dot{y}, \ldots, y^{(n-1)}) = \hat{h}(t) + \epsilon(t), \tag{6.9}$$

where

$$\epsilon(t) = [\hat{f}(x) - \hat{f}(\hat{x})] + [f(\hat{x}) - f(x)]. \tag{6.10}$$

Since $\hat{f}$ is also Lipschitz continuous with Lipschitz constant, say, $\hat{\gamma}$, it follows from Theorem 3.5 that an observer in the form (3.36) can be constructed to satisfy

$$|\epsilon(t)| \leq (\gamma + \hat{\gamma}) M e^{-\beta t} \|\varepsilon(0)\|, \tag{6.11}$$

where $\varepsilon(0)$ is the error between the initial states of (6.1) and the observer. Although we do not know $\varepsilon(0)$, by assigning the observer initial state inside Ω we can put an upper bound on its magnitude. Indeed, if $\hat{x}(0) \in \Omega$ then we have $\|\varepsilon(0)\| \leq d(\Omega)$, where $d(\Omega)$ is the diameter of Ω and is defined as

$$d(\Omega) = \sup_{z_1, z_2 \in \Omega} \|z_1 - z_2\|.$$

Let $\epsilon_0 > 0$ be the bound on the perturbation term in (6.7) mentioned above. From (6.11) it follows easily that for $t > T = \frac{1}{\beta} \ln \frac{(\gamma + \hat{\gamma}) M d}{\epsilon_0}$ we have $\epsilon(t) < \epsilon_0$. Hence we can apply $r(t)$ given by (6.8) for $t > T$, then by structural stability assumption the solutions of (6.3) and (6.9) will be topologically equivalent. Moreover, since $\epsilon(t)$ decays to zero exponentially fast, it is reasonable to expect the behaviors of (6.3) and (6.9) to be the same.

Example 6.1 *Consider the following forced Van der Pol oscillator :*

$$\ddot{y} + d(y^2 - 1)\dot{y} + y = a\cos\omega t + r(t). \tag{6.12}$$

It was shown in [91] that for various values of d, a *and* ω, *this oscillator exhibits a large variety of nonlinear phenomena, including chaos. This system is in the form given by (6.1) with*

$$F(y, \dot{y}) = d(y^2 - 1)\dot{y} + y. \tag{6.13}$$

We first write (6.12) in state space form by defining the states as $x_1 = y$, $x_2 = \dot{y}$ *and obtain*

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -d(x_1^2 - 1)x_2 - x_1 + a\cos\omega t + r(t), \\ y &= x_1. \end{aligned}$$

Then the observer has the following form

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + k_1(x - \hat{x}_1), \\ \dot{\hat{x}}_2 &= -d(\hat{x}_1^2 - 1)\hat{x}_2 - \hat{x}_1 + k_2(x - \hat{x}_1) + a\cos\omega t + r(t). \end{aligned}$$

The gains k_1 *and* k_2 *can be chosen using the result of Example 5.1 as* $k_1 = 4\sqrt{2}\gamma$ *and* $k_2 = 16\gamma^2$. *The Lipschitz constant* γ *is found from*

$$\gamma = \sup_{x_1, x_2 \in \Omega} |1 + 2dx_1x_2 + x_1^2|.$$

For the set of parameters a, d *and* ω *used in this example the states satisfy* $x_1 \in (-3, 3)$ *and* $x_2 \in (-10, 10)$, *therefore the Lipschitz constant can be chosen as* $\gamma = 10 + 60d$.

Now the control input is chosen as

$$r(t) = -a\cos\omega t + \hat{a}\cos\hat{\omega}t + (d-\hat{d})(\hat{x}_1^2 - 1)\hat{x}_2$$

to obtain a system behavior corresponding to bifurcation parameters $\hat{a}, \hat{\omega}$ *and* $\hat{d}$. *For the simulations we consider the following two sets of parameters corresponding to the indicated behavior, [91].*

Set 1 = $(a = 2.5,\ d = 6,\ \omega = 3)$*, chaos,*

Set 2 = $(a = 2.5,\ d = 0.5,\ \omega = 3)$*, limit cycle.*

To switch the chaotic behavior of the system which corresponds to the parameter Set 1 to the limit cycle corresponding to the parameter values of Set 2, we choose the control law as $r(t) = 5.5(\hat{x}_1^2 - 1)\hat{x}_2$. *Also using the larger* d *value the Lipschitz constant* γ *can be assigned as* 370, *yielding the observer gains* $k_1 = 2094$ *and* $k_2 = 2190400$. *The simulation results are shown in the figure below.*

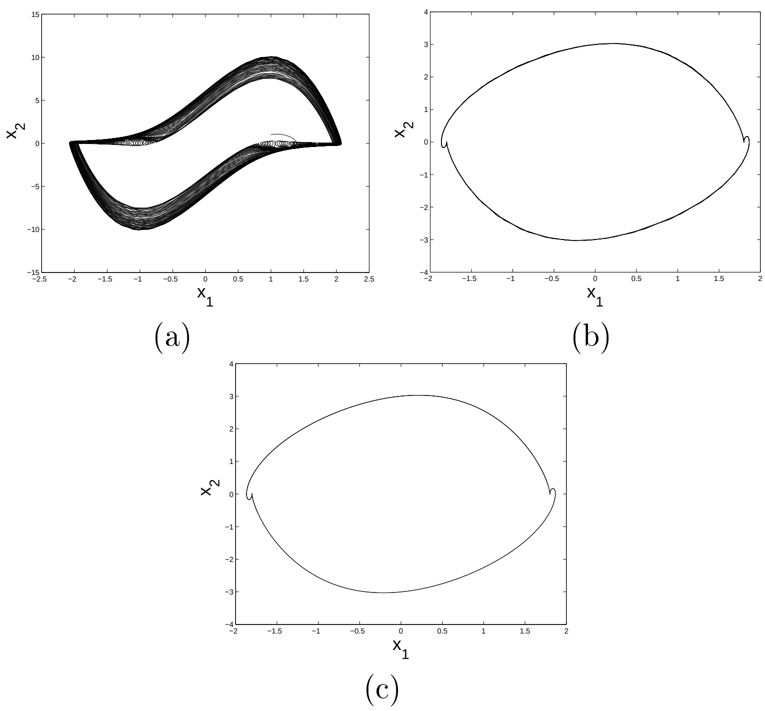

Figure 6.1: (a) A typical trajectory of the forced Van der Pol oscillator corresponding to Set 1, (b) limit cycle corresponding to Set 2, (c) limit of the behavior when the control $r(t) = 5.5(\hat{x}_1^2 - 1)\hat{x}_2$ is applied.

6.2 Tracking

The problem of choosing a control input to force a chaotic oscillator to track a certain reference trajectory is a bit different than controlling it to change its dynamic behavior. In the latter problem we need not make any assumptions on the structural stability of the system. This time what we need is not only the reference trajectory but also its derivatives.

The objective is to force the chaotic oscillator (6.1) to track a sufficiently smooth reference trajectory $y_d(t)$. Let $\hat{x} = [\hat{x}_1 \; \hat{x}_2 \; \ldots \; \hat{x}_{n-1}]^T$ be the estimate of $x = [x_1 \; x_2 \; \ldots \; x_{n-1}]^T$ provided by the observer. We choose the control law $r(t)$ as follows

$$r(t) = x_d^{(n)}(t) - l_1(\hat{x} - x_d) - l_2(\dot{\hat{x}} - \dot{x}_d) \ldots - l_n(\hat{x}^{(n-1)} - x_d^{(n-1)}) - h(t) + F(\hat{x}). \quad (6.14)$$

Define the tracking error $\bar{\varepsilon} = y - y_d$ and use (6.14) in (6.1) with the identity $\hat{x}^{(i)} - x_d^{(i)} = \bar{\varepsilon}^{(i)} + \hat{x}^{(i)} - x^{(i)}$, to obtain

$$\bar{\varepsilon}^{(n)} + l_n \bar{\varepsilon}^{(n-1)} + \ldots + l_1 \bar{\varepsilon} = \delta(t), \quad (6.15)$$

where $\delta(t)$ is an exponentially decaying term by the exponential convergence of observer error dynamics and the Lipschitz property of the functions involved. Let us choose the controller gains l_i such that the roots of the polynomial $p(s) = s^n + l_n s^{n-1} + \ldots + l_1$ have all negative real parts. It easily follows that the solution of (6.15) decays exponentially to zero. Hence we will have $y^{(i)}(t) \to y_d^{(i)}(t), \; i = 1, 2, \ldots, n-1$. Moreover the convergence is exponential.

Note that in the control law (6.14) we use the derivatives of the observer states. Actually there is no derivative operation involved. Since the observer dynamics are functions of its own states and the measured inputs and outputs of (6.1), the derivatives of the observer states are readily available, (cf. (3.36)).

Example 6.2 *As an example of the tracking problem, let us consider Van der Pol oscillator (6.12) with the parameters in Set 1 given above. As a reference trajectory we take*

$x_d(t) = \sin(5t)$, *and assign the control signal using (6.14) as*

$$
\begin{aligned}
r(t) &= -25\sin(5t) - l_1(\hat{x}_1 - \sin(5t)) \\
&\quad -l_2(\hat{x}_2 + k_1(x_1 - \hat{x}_1) - 5\cos(5t)) - a\cos(wt) + d(\hat{x}_1^2 - 1)\hat{x}_2 + \hat{x}_1.
\end{aligned}
$$

We simulate the resulting closed loop system using the same observer gains as before. Also we choose the coefficients l_i *in (6.15) as* $l_1 = 15$ *and* $l_2 = 8$. *The resulting trajectory is plotted in the figure below.*

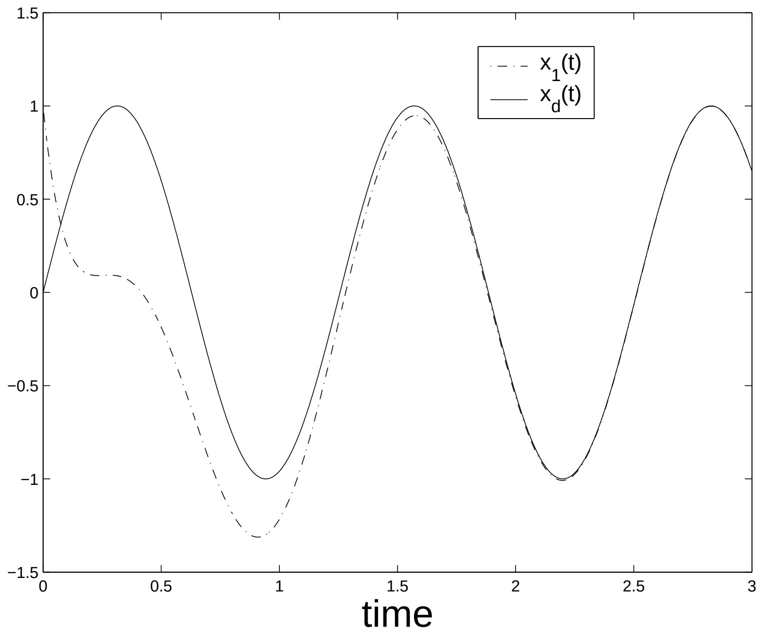

Figure 6.2: Observer based reference tracking with Van der Pol oscillator.

Chapter 7

Observability of Periodic Systems and Observers

Loosely speaking, a hybrid system is one whose dynamics are combinations of continuous valued vector fields and a set of decision rules. As an obvious class, hybrid systems include switching linear systems, where the system switches among a finite set of linear vector fields. The switching rule may depend on the condition that the state hits a boundary in the state space or the switch may be set externally. In this chapter we analyze the observability properties of discrete-time switching linear systems where the switching is periodic and the period is known.

It is also possible to look at the properties of periodic systems in the framework of Floquet theory, [42]. However, we follow, with modifications, the approach taken in [92].

The material in this chapter was developed as part of our work on the observability of more general switching systems, which is the topic of the next chapter. A compact version of this chapter has appeared as [93].

7.1 Observability of Periodic Discrete-Time Systems

The class of systems we consider are given by

$$x_{k+1} = F_k x_k \tag{7.1}$$

$$y_k = G_k x_k, \tag{7.2}$$

where $x_k \in \mathbf{R}^n$, $y \in \mathbf{R}$ and the matrices F_k and G_k satisfy

$$F_k \in \{A_1, A_2, \ldots, A_N\},$$

$$G_k \in \{C_0, C_1, C_2, \ldots, C_N\}$$

and

$$F_k = A_{i+1}, \qquad \sum_{j=1}^{i} P_j \leq k \pmod P < \sum_{j=1}^{i+1} P_j,$$

$$G_k = C_{i+1}, \qquad \sum_{j=1}^{i} P_j \leq (k-1) \pmod P < \sum_{j=1}^{i+1} P_j,$$

$$G_0 = C_0.$$

Here P is the period of the system and P_j, $j = 1 \ldots N$ is the number of steps the system is iterated with the j^{th} matrices A_j and C_j. Hence the overall period of the system is given by

$$P = \sum_{j=1}^{N} P_j.$$

Namely, beginning from the initial state x_0 the system is iterated P_1 times with A_1 and P_2 times with A_2 and so on. After the last iteration by A_N the system is again governed by A_1.

Definition 7.1 *The system (7.1), (7.2) is said to be observable if the initial state $x(0)$ of the system can be determined by using the output measurements only.*

Next we recall a condition of the observability of (7.1), (7.2) in the same lines of Theorem 2.3.

Fact 7.1 *The system (7.1), (7.2) is globally observable if and only if the map* $H : \mathbf{R}^n \to \mathbf{R}^\infty$, $H(x_0) = \{y_0, y_1, y_2, \ldots\}$ *is invertible.*

To see the form of the map H, consider a system with $N = 2$, $P_1 = 2$ and $P_2 = 3$. Its output sequence is given by

$$\{C_0x_0, C_1A_1x_0, C_1A_1^2x_0, C_2A_2A_1^2x_0, C_2A_2^2A_1^2x_0, C_2A_2^3A_1^2x_0, C_1A_1A_2^3A_1^2x_0, \ldots\}$$

Therefore this system is observable if and only if the following infinite-row matrix has n independent rows

$$\begin{bmatrix} C_0 \\ C_1A_1 \\ C_1A_1^2 \\ C_2A_2A_1^2 \\ C_2A_2^2A_1^2 \\ C_2A_2^3A_1^2 \\ C_1A_1A_2^3A_1^2 \\ \vdots \end{bmatrix}.$$

Using the periodicity of the system, the rank test of an infinite matrix can be reduced to the rank test of a finite one. For this we first define the observability matrix of order P_i for a pair (C_i, A_i) as

$$\mathbf{O}_i \triangleq \begin{bmatrix} C_iA_i \\ C_iA_i^2 \\ \vdots \\ C_iA_i^{P_i} \end{bmatrix}. \tag{7.3}$$

Note that $\mathbf{O}_i$ so defined is different from the one in (2.5) in two ways; first row is C_iA_i instead of C_i and also it has P_i rows rather than as many as the system dimension.

Theorem 7.1 *Using the system (7.1), (7.2) define another system as*

$$z_{k+1} = \tilde{A}z_k, \tag{7.4}$$

$$\omega_k = \tilde{C} z_k, \tag{7.5}$$

$$\tilde{A} = A_N^{P_N} \dots A_2^{P_2} A_1^{P_1},$$

$$\tilde{C} = \begin{bmatrix} C_0 \\ \mathbf{O}_1 \\ \mathbf{O}_2 A_1^{P_1} \\ \mathbf{O}_3 A_2^{P_2} A_1^{P_1} \\ \vdots \\ \mathbf{O}_N A_{N-1}^{P_{N-1}} \dots A_2^{P_2} A_1^{P_1} \end{bmatrix}.$$

Then the system (7.1), (7.2) is observable if and only if the system (7.4), (7.5) is observable.

Proof: It can easily be seen that when started from the same initial state, both systems produce the same sequence of outputs. Using Fact 7.1 the result immediately follows. Indeed we have $z_k = x_{kP}$ and $\omega_k = [y_{kP} \; y_{kP+1} \; \dots \; y_{(k+1)P-1}]^T$. □

Remark 7.1.1 *The usual observability matrix* $\mathbf{O}$ *of the pair* $(\tilde{C}, \tilde{A})$ *has dimension* $(1 + P) \times n$*. Therefore if the system (7.4), (7.5) is observable, at most* $P + 1$ *consecutive output measurements are enough to determine its initial state.*

7.2 Observers for Periodic Discrete-Time Systems

The equivalence provided by Theorem 7.1 naturally leads to the construction of an observer with exponentially decaying error dynamics. In fact, the observer is constructed for the system (7.4), (7.5) which describes nothing other than the evolution of the system (7.1), (7.2) between period beginnings and augmenting its outputs throughout the period in the extended vector ω_k.

Theorem 7.2 *If the system (7.1), (7.2) is observable then there exists a gain vector K such that the following system is an exponential observer for (7.1), (7.2).*

$$\begin{aligned}
\hat{x}_{(j+1)P} &= \tilde{A}\hat{x}_{jP} + K(\tilde{C}x_{jP} - \tilde{C}\hat{x}_{jP}), && (7.6)\\
\hat{x}_k &= A_s^l A_{s-1}^{P_{s-1}} \ldots A_2^{P_2} A_1^{P_1} \hat{x}_{jP}, && (7.7)\\
j &= S(\frac{k}{P}),\\
l &= k \ (\mathrm{mod}\ P) - \sum_{r=1}^{s-1} P_r,
\end{aligned}$$

where l is an integer smaller than P_s and the function $S : \mathbf{R} \to \mathbf{Z}$ is defined such that for a real number h, $S(h)$ is the largest integer that is not larger than h.

Proof: Defining the estimation error as $\varepsilon_k = x_k - \hat{x}_k$, for the error at period beginnings we have

$$\begin{aligned}
\varepsilon_{(j+1)P} &= x_{(j+1)P} - \hat{x}_{(j+1)P},\\
&= \tilde{A}x_{jP} - \tilde{A}\hat{x}_{jP} - K(\tilde{C}x_{jP} - \tilde{C}\hat{x}_{jP}),\\
&= (\tilde{A} - K\tilde{C})\varepsilon_{jP}.
\end{aligned}$$

Since the pair $(\tilde{C}, \tilde{A})$ is observable, there exists a gain vector K such that the eigenvalues of the matrix $\tilde{A} - K\tilde{C}$ are inside the unit circle. Therefore, for this choice of K the error at period beginnings converge exponentially to zero. Using (7.1) and (7.7), for the other errors inside the period we can write

$$\varepsilon_k = A_s^l A_{s-1}^{P_{s-1}} \ldots A_2^{P_2} A_1^{P_1} \varepsilon_{jP}.$$

Since for all $s \in \{1, \ldots, N\}$ and for all $l \in \{1, \ldots, P_s\}$ the operator $A_s^l A_{s-1}^{P_{s-1}} \ldots A_2^{P_2} A_1^{P_1}$ is bounded, ε_k exponentially decays to zero as ε_{jP} does. $\square$

Example 7.1 *For $N = 2$ and $P_1 = P_2 = 2$ let the system matrices be given as*

$$C_0 = [1\ 0], C_1 = [1\ 0], C_2 = [0\ 1],$$

$$A_1 = \begin{bmatrix} 3 & 0 \\ 2 & 3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}.$$

Then we calculate $\tilde{C},\ \tilde{A}$ *as*

$$\tilde{C} = \begin{bmatrix} 1 & 0 \\ 3 & 0 \\ 9 & 0 \\ 24 & 18 \\ 48 & 36 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} 132 & 72 \\ 48 & 36 \end{bmatrix}.$$

Assigning the eigenvalues of $\tilde{A} - K\tilde{C}$ *as* $\{-0.4,\ 0.4\}$, K *is obtained as*

$$K = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 3.9556 & -0.0593 \\ 0 & 0 \\ 2.0000 & 1.0111 \end{bmatrix}.$$

Below are given the simulation results for this value of gain and an initial observation error $\varepsilon(0) = [1\ 1]^T$.

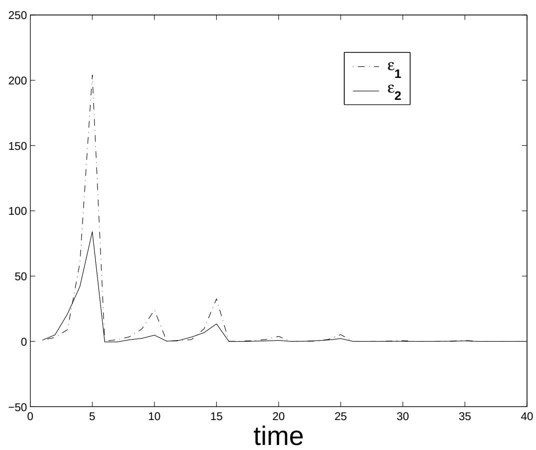

Figure 7.1: Estimation error convergence when eigenvalues are assigned at $0.4,\ -0.4$.

7.3 Relation to Lifting

The observability equivalence given in Theorem 7.1 can also be interpreted in the context of **lifting**. Given a periodically time-varying discrete-time system with period P

$$x_{k+1} = A_k x_k, \tag{7.8}$$

$$y_k = C_k x_k, \tag{7.9}$$

the **P−lifting** of (7.8), (7.9) is the time-invariant system

$$z_{k+1} = A z_k, \tag{7.10}$$

$$\omega_k = C z_k, \tag{7.11}$$

where A is the transition matrix over one period, i.e.,

$$A = A_{P-1}A_{P-2}\dots A_1 A_0$$

and

$$C = \begin{bmatrix} C_0 \\ C_1 A_0 \\ C_2 A_1 A_2 \\ \vdots \\ C_{P-1}A_{P-1}\dots A_1 A_0 \end{bmatrix},$$

see e.g., [94] and the references therein. Comparing with (7.4), (7.5) we see that $C = \tilde{C}$ and $A = \tilde{A}$.

Since every time-invariant system is 1-periodic, it is possible to define its $M-$lifting for arbitrary $M > 1$ to obtain a periodic system. Such an approach is taken in [94, 95] to design periodic controllers for time-invariant plants. In our case, the observer (7.6), (7.7) is in fact designed for the $P-$lifted system (7.4), (7.5).

7.4 A Hybrid Observer for Continuous-Time Periodic Systems

The observer design method proposed in the last section can be ported to continuous-time periodic systems with simple modifications. Consider a system in the form

$$\dot{x} = A(t)x, \tag{7.12}$$

$$y = C(t)x, \tag{7.13}$$

where $A(t), C(t)$ are piecewise continuous periodic functions of time satisfying $A(t) = A(t+T)$, $C(t) = C(t+T)$ $\forall t \in \mathbf{R}$ for some $T > 0$. The transition matrix of (7.12), (7.13) is given by

$$\Phi(t_2, t_1) = P(t_2)e^{R(t_2-t_1)}P^{-1}(t_1), \tag{7.14}$$

where R is a constant matrix and $P(t)$ is continuous and invertible at each t. Moreover $P(t)$ has period T, see [42, p. 81]. Therefore, for $t_1 > t_0$ we have $x(t_1) = \Phi(t_1, t_0)x(t_0)$.

Consider the case $t_1 = t_0 + T$. Then we have

$$\begin{aligned} x(t_0+T) &= \Phi(t_0+T, t_0)x(t_0) = P(t_0+T)e^{RT}P^{-1}(t_0)x(t_0), \\ &= P(t_0)e^{RT}P^{-1}(t_0)x(t_0), \text{ by the periodicity of } P(t). \end{aligned}$$

Let us define $\Phi_T(t_0) \triangleq P(t_0)e^{RT}P^{-1}(t_0)$ which is the transition matrix over one period. Since $P(t)$ is periodic with period T, so is $\Phi_T(t_0)$, i.e. $\Phi_T(t_0+kT) = \Phi_T(t_0), \ \forall k \geq 0$. Then we have

$$\begin{aligned} x(t_0+kT) &= \Phi(t_0+kT, t_0+(k-1)T)x(t_0+(k-1)T) \\ &= \Phi_T(t_0+(k-1)T)x(t_0+(k-1)T), \\ &= \Phi_T(t_0)x(t_0+(k-1)T). \end{aligned}$$

Hence the recursion so defined yields

$$x(t_0+kT) = \Phi_T^k(t_0)x(t_0).$$

Assume that there exists a time $t_0 \in [0, T)$ such that the pair $(C(t_0), \Phi_T(t_0))$ is observable. If we sample (7.12), (7.13) at times $t = t_0 + kT$, $k \geq 0$ we obtain the discrete-time system

$$z_{k+1} = \Phi_T(t_0) z_k, \quad z_0 = x(t_0), \tag{7.15}$$

$$\omega_k = C(t_0) z_k. \tag{7.16}$$

Using this sampled system we can construct a hybrid observer for the system (7.12), (7.13) as

$$\hat{z}_{k+1} = \Phi_T(t_0)\hat{z}_k + K(\omega_k - C(t_0)\hat{z}_k), \tag{7.17}$$

$$\hat{x}(t) = \hat{z}_k \quad \text{if } t = t_0 + kT, \tag{7.18}$$

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t), \tag{7.19}$$

where K is chosen to assign the eigenvalues of $\Phi_T(t_0) - KC(t_0)$ inside the unit circle. It follows that the error between $\hat{z}_k$ and z_k decays exponentially to zero. Since $A(t)$ is a bounded matrix the same convergence result holds for the error between $x(t)$ and $\hat{x}(t)$. The basic disadvantage of this observer is that due to its hybrid nature, the estimate is reset every period. However, to estimate the state it uses just a fraction of the knowledge about the original system.

Example 7.2 *Assume we are given a 2π-periodic system with*

$$A(t) = \begin{bmatrix} \frac{-1}{2\pi} & 0 \\ 0 & \frac{-2}{2\pi} \end{bmatrix} + \begin{bmatrix} \ln(1.1)+1 & 0 \\ 0 & \ln(1.1)+2 \end{bmatrix} \frac{|\sin(t)|}{4},$$

$$C(t) = [c_1(t) \ c_2(t)],$$

where $c_1(t)$ and $c_2(t)$ are given in the Figure 7.2 for the duration of one period. Using the method given in [42, p. 94], one can calculate $\Phi_{2\pi}(t_0)$ as

$$\begin{bmatrix} 1.1 & 0 \\ 0 & 1.2 \end{bmatrix}.$$

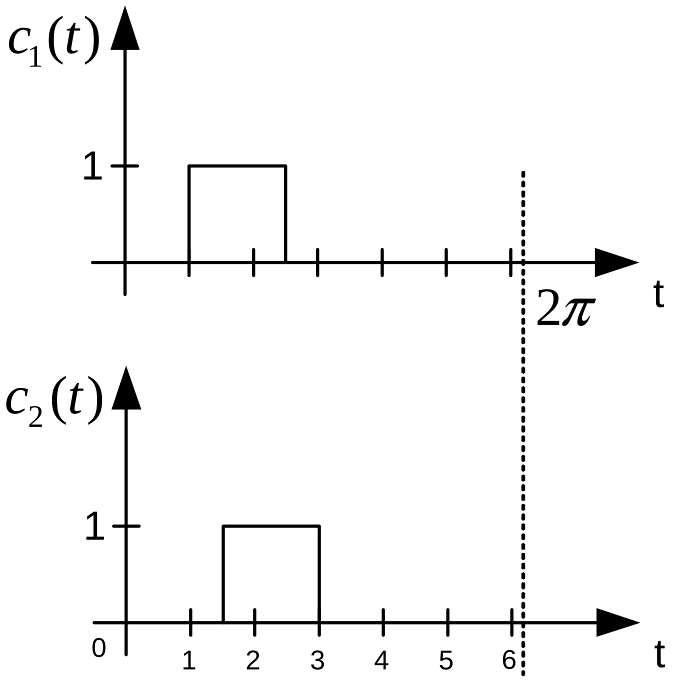

Figure 7.2: $c_1(t)$ and $c_2(t)$.

Note that $\Phi_{2\pi}(t_0)$ is constant. Comparing with $C(t)$, we see that $(C(2), \Phi_{2\pi}(2))$ is observable while it is not so for most values of $C(t)$. Choosing $K = [-9.6\ 11.9]^T$ assigns the eigenvalues of $\Phi_{2\pi}(2) - KC(2)$ at $0.5, \ -0.5$. Then the observer is constructed as in (7.17)-(7.19). The simulations are provided in Figure 7.3. Note that the original system is unstable.

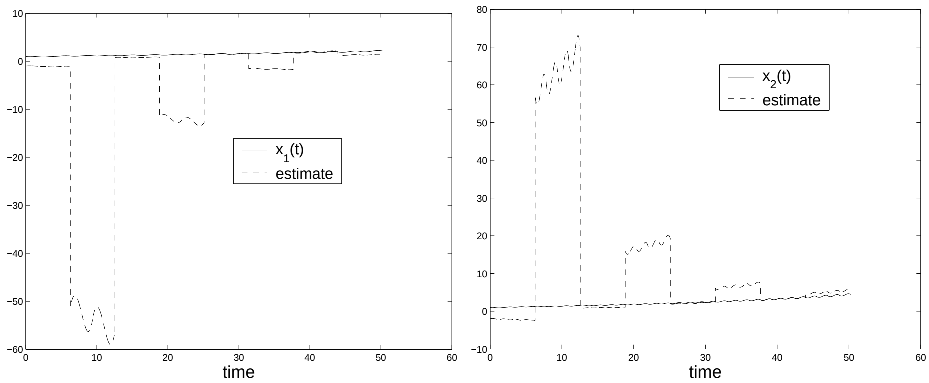

Figure 7.3: Original states and their estimates for a periodic system.

Chapter 8

Observability and Stability of Hybrid Systems

Hybrid systems are composites of digital and analog dynamics with additional rules and dynamics specifying the interaction between the two. Hence in a hybrid system, on the one hand, we have analog dynamics which can be described in terms of differential/difference equations, on the other hand, we have digital dynamics that are usually described as finite-state machines or automata, [96]. In this respect the study of hybrid systems incorporate and extend the set of problems in the theory of smooth dynamical systems and automata. Naturally, the interest in hybrid systems is fueled by the ever increasing use of digital computers in almost every area of technology and created a rich interdisciplinary field populated largely by researchers from electrical and computer engineering, [23, 24].

Despite this variety and wealth of recent literature on hybrid systems, there is still a need to knit together this diversity into a mature theory. Even starting from the modeling stage, there are considerably different approaches to hybrid systems mostly dictated by the class of problems under study, [96]. As it is the case with all modeling issues, a proper model of a hybrid system should be general enough to encompass large variety of physical phenomena, yet be structured enough to present a workable model through which interesting problems can be mathematically framed.

Thus, we begin this chapter by specifying a simple yet powerful model of a hybrid system. We will further restrict the model as we consider observability issues.

8.1 Hybrid System Model

We use the following hybrid system model throughout.

$$\dot{x} = f_i(x), \tag{8.1}$$

$$i^+ = q(i^-, x), \tag{8.2}$$

where the set of vector fields $f_i : \mathbf{R}^n \to \mathbf{R}^n$, $i \in \{1, \ldots, N\}$ is a finite set of digital states with each element satisfying usual regularity and smoothness assumptions as necessary to yield meaningful solutions and $q : \{1, \ldots, N\} \times \mathbf{R}^n \to \{1, \ldots, N\}$ specifies the switching rule among vector fields f_i. A system given by (8.1), (8.2) is also referred to as a **switching system** due to the fact that the system dynamics switch between different vector fields. The function q is usually expressed as a collection of *IF* ... *THEN* ... clauses rather than analytical expressions. The latter can be done if one bunches together conditions using discontinuous functions like unit step and sign, but in most cases this does not bring anything more than compactness of mathematical expressions.

Thus starting from an initial condition $x(t_0)$ and a vector field f_{i_0}, the system (8.1), (8.2) evolves with $\dot{x} = f_{i_0}(x)$ until time t_1 at which $i_1 = q(i_0, x(t_1))$ is satisfied for some i_1. Then the system evolves with $\dot{x} = f_{i_1}(x)$ starting from the initial condition $x(t_1)$ and the cycle repeats itself with vector field switching whenever a switching condition specified in q holds.

Similarly, when the analog part of a hybrid system is described by a difference equation in the form,

$$x(k+1) = f_i(x(k)), \tag{8.3}$$

instead of the one given by (8.1), we have a discrete-time analog system interacting with digital dynamics given by (8.2). We emphasize that, what distinguishes a digital system is not the discreteness of time but rather the finiteness of the possible values of states. In (8.1) and (8.3), the state x take values in $\mathbf{R}^n$, while in (8.2), the digital state i takes values from the finite set $\{1, \ldots, N\}$.

Naturally many other interesting cases are left out by choosing such a model. Inputs and time variation can be incorporated in the model to allow for externally induced switches as in a car transmission model. It is also possible to modify the switching function q to set the continuous state at switching times. This is the case when one models a moving particle in a confined area which allows for collisions. Immediately after the collision, the momentum of the particle undergoes an instantaneous jump. For different types of models and interesting examples of different switching systems, see [96] and the references therein.

In our work on the observability of hybrid systems, we restrict ourselves to discrete-time switching systems. This choice is appropriate as it gives a sufficiently regular framework to strip the observability problem from issues like well-posedness and uniqueness of solutions. For stability, we provide some interesting results on both continuous and discrete time systems.

8.2 Observability of Discrete-Time Switching Systems

Consider a discrete-time switching system given by the following discrete dynamics and a set of switching rules;

$$
\begin{aligned}
x(k+1) &= f_{i(k)}(x(k)), && (8.4)\\
y(k) &= h_{i(k)}(x(k)), && (8.5)\\
i(k) &= j \;\text{ if }\; x \in \Omega_j, && (8.6)\\
\mathbf{R}^n &= \bigcup_{j=1}^{N} \Omega_j \;\text{ and }\; \Omega_{j_1} \cap \Omega_{j_2} \neq \emptyset \;\text{ if }\; j_1 \neq j_2. && (8.7)
\end{aligned}
$$

In other words, we have a collection of N systems (h_i, f_i), $i = 1, \ldots, N$. Each system is active in its region Ω_i and the collection of regions form a non-overlapping covering of the state space $\mathbf{R}^n$. The question is whether every pair of initial states $(x_0, \bar{x}_0)$ can be distinguished from each other by observing their respective output sequences. This is the question of global observability of the system (8.4)-(8.7).

One important difference between the observability of smooth nonlinear systems and the observability of (8.4)-(8.7) is that the output sequence of the latter system does not, in general, contain any information about the switching sequence of the system. Therefore a part of the observability problem is the extraction of this information from the output sequence, [97, 98]. Now we give a formal definition of the observability for this kind of systems. The definitions are adapted from the works, [99, 100].

Definition 8.1 *A pair of states $(x_0, \bar{x}_0)$ are said to be* **distinguishable** *if the respective output sequences of the system (8.4)-(8.7) starting from the initial states x_0 and $\bar{x}_0$ are different. The system is said to be* **globally observable** *if for every distinct pair of initial states $(x_0, \bar{x}_0)$ are distinguishable.*

While global observability as we have defined it in Definition 8.1 relates to the full infinite length of output sequences, it is possible to define a finite-time observability considering truncated output sequences.

Definition 8.2 *A pair of states* $(x_0, \bar{x}_0)$ *are said to be* **L-step distinguishable** *if the respective output sequences of the system (8.4)-(8.7) starting from the initial states* x_0 *and* $\bar{x}_0$ *differ in their first L entries. The system is said to be* **globally L-step observable** *if every distinct pair is L-step distinguishable.*

The L-step observability is a stronger condition than the observability because in the L-step observability, to distinguish the initial states, only a finite portion of their outputs are used. However this definition still proves useful, as it can be used to recursively test global observability.

8.2.1 Bimodal Linear Systems

When there are only two vector fields and a partition of the whole state space into two distinct regions we have a **bimodal system**, [101]. The system we consider switches between linear vector fields and have the form

$$x(k+1) = \begin{cases} A_1 x(k), & Cx(k) \geq 0, \\ A_2 x(k), & Cx(k) < 0, \end{cases} \tag{8.8}$$

$$y(k) = Cx(k). \tag{8.9}$$

Thus we have $\Omega_1 = \{x \in \mathbf{R}^n \mid Cx \geq 0\}$ and $\Omega_2 = \{x \in \mathbf{R}^n \mid Cx < 0\}$. We especially picked a system in this form to illustrate how complex the observability characterization can be, even for a simple class such as the one given by (8.8), (8.9). Note that with this description of the system, the boundary between two regions corresponds to the hyperplane given by $Cx = 0$. The observability of this system is connected to the observability properties of each mode in an interesting way.

Theorem 8.1 *The switching system (8.8), (8.9) is globally observable only if the pair (C, A_1) is observable.*

Proof: Suppose the pair (C, A_1) is not observable. Then by Theorem 2.1 there exists a non-zero vector $x_0 \in \mathbf{R}^n$ which is in the kernel of the observability matrix, i.e.,

$$\begin{bmatrix} C \\ CA_1 \\ \vdots \\ CA_1^{n-1} \end{bmatrix} x_0 = 0.$$

Also by Cayley-Hamilton theorem we have $CA_1^k x_0 = 0, \ \forall k \geq n$. Assume we start the system from the initial condition $x(0) = x_0$. We have $CA_1^k x_0 = 0, \ \forall k \geq 0$. Therefore the system trajectory is always in the region Ω_1, and the output is identically zero. There is the trivial initial state $x(0) = 0$ which gives the same output sequence. Hence the initial states $x(0) = x_0$ and $\bar{x}(0) = 0$ are indistinguishable, which proves that the system (8.8), (8.9) is not globally observable. $\square$

Remark 8.1.1 *By the near symmetry of the system one might be tempted to expect the same result to hold for the pair (C, A_2). Interestingly, this is not the case as illustrated by the following example.*

Example 8.1 *Consider the system (8.8), (8.9) with the following matrices.*

$$C = [1 \ 0], \quad A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Obviously, (C, A_1) is observable while (C, A_2) is not. Consider an initial state $x(0) = x_0 \in \Omega_1$, i.e., $Cx_0 \geq 0$. Then we have $y(0) = Cx_0$ and $y(1) = CA_1 x_0$. Since (C, A_1) is an observable pair, x_0 can be uniquely determined by

$$x_0 = \begin{bmatrix} C \\ CA_1 \end{bmatrix}^{-1} \begin{bmatrix} y(0) \\ y(1) \end{bmatrix} \quad \text{if} \quad y(0) \geq 0.$$

Now consider an initial state in Ω_2, that is, $x(0) = x_0$ such that $Cx_0 < 0$. Hence we have $y(0) = Cx_0$ and $y(1) = CA_2x_0$. However, since A_2 maps all the states in region Ω_2 to Ω_1, we have $x(1) \in \Omega_1$, which implies $y(2) = CA_1A_2x_0$. By the observability of (C, A) we can calculate $x(1)$ using $y(1)$ and $y(2)$. Since $x(1) = A_2x_0$ and A_2 is nonsingular, we can uniquely determine x_0 once we have $x(1)$. Consequently, we have

$$x_0 = A_2^{-1} \begin{bmatrix} C \\ CA_1 \end{bmatrix}^{-1} \begin{bmatrix} y(1) \\ y(2) \end{bmatrix} \quad \text{if} \quad y(0) < 0.$$

The foregoing discussion also shows that for this particular system, we can determine the initial state after at most two iterations.

In fact, it is possible to completely characterize the observability of the bimodal planar systems in the form (8.8), (8.9) by exhaustively examining all the possible cases. Assume $A_1, A_2 \in \mathbf{R}^{2\times 2}$ and $C \in \mathbf{R}^{1\times 2}$. By Theorem 8.1 if (C, A_1) is unobservable then the system (8.8), (8.9) is unobservable. Therefore assume (C, A_1) is observable.

Fact 8.1 *Assume $A_1, A_2 \in \mathbf{R}^{2\times 2}$ and $C \in \mathbf{R}^{1\times 2}$. If both (C, A_1) and (C, A_2) are observable then the system (8.8), (8.9) is observable.*

Proof: Assume the initial state $x(0) \in \Omega_1$, i.e., $Cx(0) \geq 0$. Then $y(1) = CA_1x(0)$ and we have

$$\begin{bmatrix} y(0) \\ y(1) \end{bmatrix} = \begin{bmatrix} C \\ CA_1 \end{bmatrix} x(0).$$

By the observability of (C, A_1), we have

$$x(0) = \begin{bmatrix} C \\ CA_1 \end{bmatrix}^{-1} \begin{bmatrix} y(0) \\ y(1) \end{bmatrix}.$$

Now assume $x(0) \in \Omega_2$, i.e., $Cx(0) < 0$. Then $y(1) = CA_2x(0)$. Imitating the preceding argument and using the observability of (C, A_2) we obtain

$$x(0) = \begin{bmatrix} C \\ CA_2 \end{bmatrix}^{-1} \begin{bmatrix} y(0) \\ y(1) \end{bmatrix}.$$

Hence every initial state can be determined after one iteration, which proves the claim. □

Therefore when (C, A_1) is observable, the system (8.8), (8.9) can be unobservable only if (C, A_2) is unobservable. But in this case there exists an α such that $CA_2 = \alpha C$. We examine the cases $\alpha > 0$ and $\alpha \leq 0$ separately. Also note that since (C, A_1) is observable and A_1 is active in Ω_1, only the initial states $x(0) \in \Omega_2$ can cause the observability to fail. So we need to consider only those to check observability in the remaining cases.

$\alpha > 0$: We have $x(0) \in \Omega_2$, hence $y(0) = Cx(0) < 0$. Also $y(1) = CA_2x(0) = \alpha Cx(0) < 0$. By the same token, $y(k) = \alpha^k Cx(0) < 0$. Therefore $x(k) \in \Omega_2,\ \forall k \geq 0$. Any other initial state $\bar{x}(0) = x(0) + r,\ r \in \ker C$ is in Ω_2 and yield the same output sequence, hence is indistinguishable from $x(0)$. The system is not observable.

$\alpha \leq 0$: Again start with $x(0) \in \Omega_2$. Then we have $y(0) = Cx(0) < 0$, $y(1) = Cx(1) = CA_2x(0) = \alpha Cx(0) \geq 0$. Hence $y(2) = CA_1x(1)$. By the observability of (C, A_1) we can determine $x(1)$. Further iterations do not bring any extra information on $x(0)$. To determine $x(0)$ we have only two relations; $y(0) = Cx(0)$ and $x(1) = A_2x(0)$, which can be written as

$$\begin{bmatrix} y(0) \\ x(1) \end{bmatrix} = \begin{bmatrix} C \\ A_2 \end{bmatrix} x(0).$$

Hence $x(0)$ can be uniquely determined if and only if

$$\operatorname{rank} \begin{bmatrix} C \\ A_2 \end{bmatrix} = 2.$$

The foregoing discussion is summarized in the following fact.

Fact 8.2 *The system (8.8), (8.9) with $A_1, A_2 \in \mathbf{R}^{2\times 2}$ and $C \in \mathbf{R}^{1\times 2}$ is globally observable if and only if (C, A_1) is observable and one of the following holds;*

i. (C, A_2) is observable,

ii. (C, A_2) *is not observable, the solution* α *of* $CA_2 = \alpha C$ *is nonpositive and*

$$\operatorname{rank}\begin{bmatrix} C \\ A_2 \end{bmatrix} = 2.$$

Proof: Follows from the preceding discussion. □

We next formulate the observability of bimodal switching systems as a feasibility problem which amounts to the condition that the set defined by an infinite number of constraints is a fixed hyperplane.

8.2.2 Observability as a Feasibility Problem

Our formulation of the global observability problem for a discrete-time bimodal system starts with characterizing the indistinguishable states. Let x_0 and $\bar{x}_0$ be two initial states, not necessarily distinct. Also let $x(k)$ and $\bar{x}(k)$ be the respective solutions at time k of the system (8.8), (8.9) starting from x_0 and $\bar{x}_0$.

Suppose that these initial states yield the same output sequence, i.e. $Cx(k) = C\bar{x}(k), \forall k \geq 0$. Since the switching condition depends on the output, we have $x(k), \bar{x}(k) \in \Omega_{i(k)}$, $i(k) = 1, 2$, $\forall k \geq 0$. Namely, at each step both trajectories are iterated with the same system matrix. Therefore we have, for all $k \geq 1$,

$$\begin{aligned} x(0) &= x_0, \\ \bar{x}(0) &= \bar{x}_0, \\ x(k) &= A_{i(k-1)}A_{i(k-2)} \ldots A_{i(1)}A_{i(0)}x_0, \\ \bar{x}(k) &= A_{i(k-1)}A_{i(k-2)} \ldots A_{i(1)}A_{i(0)}\bar{x}_0. \end{aligned}$$

Knowing this, we can write the condition that the first k outputs corresponding to x_0 and $\bar{x}_0$ are the same in the form

$$C(x_0 - \bar{x}_0) = 0,$$

$$
\begin{aligned}
CA_{i(0)}(x_0 - \bar{x}_0) &= 0, \\
CA_{i(1)}A_{i(0)}(x_0 - \bar{x}_0) &= 0, \\
\vdots \quad & \quad \vdots \\
CA_{i(k-1)}A_{i(k-2)} \dots A_{i(1)}A_{i(0)}(x_0 - \bar{x}_0) &= 0.
\end{aligned} \tag{8.10}
$$

Also, since the switching sequence is the same for both trajectories starting from x_0 and $\bar{x}_0$, the following inclusions hold;

$$
\begin{aligned}
x_0,\ \bar{x}_0 &\in \Omega_{i(0)}, \\
A_{i(0)}x_0,\ A_{i(0)}\bar{x}_0 &\in \Omega_{i(1)}, \\
A_{i(1)}A_{i(0)}x_0,\ A_{i(1)}A_{i(0)}\bar{x}_0 &\in \Omega_{i(2)}, \\
\vdots \quad & \quad \vdots \\
A_{i(k-2)}A_{i(k-3)} \dots A_{i(1)}A_{i(0)}x_0,\ A_{i(k-2)}A_{i(k-3)} \dots A_{i(1)}A_{i(0)}\bar{x}_0 &\in \Omega_{i(k)}.
\end{aligned} \tag{8.11}
$$

Given the truncation s^k of a particular switching sequence s as $s^k = \{i(0), i(1), \dots, i(k-1)\}$, the inclusions in (8.11) can be written as linear inequalities. Hence every truncation of length k yields $2k$ inequalities and $k+1$ equalities in terms of initial states x_0 and $\bar{x}_0$. By convention we take s^0 to be the empty sequence; in this case we do not have any inequalities in (8.11) and have only the equality $C(x_0 - \bar{x}_0) = 0$ in (8.10). To put the problem into the form of linear matrix inequalities, let us define the following matrices that depend on s^k.

$$
N_{s^k} \triangleq \begin{bmatrix} C \\ CA_{i(0)} \\ CA_{i(1)}A_{i(0)} \\ \vdots \\ CA_{i(k-1)}A_{i(k-2)} \dots A_{i(1)}A_{i(0)} \end{bmatrix},
$$

$$M_{s^k}^1 \triangleq \begin{bmatrix} CA_{i(r_1-1)}A_{i(r_1-2)}\dots A_{i(1)}A_{i(0)} \\ CA_{i(r_2-1)}A_{i(r_2-2)}\dots A_{i(1)}A_{i(0)} \\ \vdots \\ CA_{i(r_\mu-1)}A_{i(r_\mu-2)}\dots A_{i(1)}A_{i(0)} \end{bmatrix}, \quad M_{s^k}^2 \triangleq \begin{bmatrix} CA_{i(p_1-1)}A_{i(p_1-2)}\dots A_{i(1)}A_{i(0)} \\ CA_{i(p_2-1)}A_{i(p_2-2)}\dots A_{i(1)}A_{i(0)} \\ \vdots \\ CA_{i(p_\nu-1)}A_{i(p_\nu-2)}\dots A_{i(1)}A_{i(0)} \end{bmatrix},$$

where $r_\mu > \ldots > r_2 > r_1 \geq 0$, $p_\nu > \ldots > p_2 > p_1 \geq 0$ are k nonnegative integers (i.e., $\mu + \nu = k$) that satisfy $i(r_j) = 1$ and $i(p_j) = 2$. Also as a convention we take the first row of $M_{s^k}^1$ to be C whenever $r_1 = 0$. Likewise, when $p_1 = 0$, we assign the first row of $M_{s^k}^2$ as C. Note that only one of these conditions hold for a particular choice of s^k. Also when $k = 0$ we do not have matrices $M_{s^0}^1$ and $M_{s^0}^2$. Then the equalities given by (8.10) and the inclusions given by (8.11) are written in matrix form as

$$N_{s^k}(x_0 - \bar{x}_0) = 0, \tag{8.12}$$

$$M_{s^k}^1 x_0 \geq 0, \tag{8.13}$$

$$M_{s^k}^1 \bar{x}_0 \geq 0, \tag{8.14}$$

$$M_{s^k}^2 x_0 < 0, \tag{8.15}$$

$$M_{s^k}^2 \bar{x}_0 < 0, \tag{8.16}$$

where the inequalities are taken entry-wise.

Example 8.2 *Consider the truncation of a switching sequence as* $s^4 = \{1, 2, 1, 1\}$. *Therefore the state evolution is* $\{x_0, A_1x_0, A_2A_1x_0, A_1A_2A_1x_0, A_1A_1A_2A_1x_0\}$. *By looking at the states that are iterated by* A_1, *we determine* $r_1 = 0,\ r_2 = 2,\ r_3 = 3$. *Likewise since* $i(1) = 2$ *we find* $p_1 = 1$. *Consequently we construct the matrices* $N_{s^4},\ M_{s^4}^1,\ M_{s^4}^2$ *as*

$$N_{s^4} = \begin{bmatrix} C \\ CA_1 \\ CA_2A_1 \\ CA_1A_2A_1 \\ CA_1A_1A_2A_1 \end{bmatrix}, \quad M_{s^4}^1 = \begin{bmatrix} C \\ CA_2A_1 \\ CA_1A_2A_1 \end{bmatrix}, \quad M_{s^4}^2 = \begin{bmatrix} CA_1 \end{bmatrix}.$$

Note that if we set $x_0 = \bar{x}_0$, there exists a sequence s_0 such that (8.12)-(8.16) are satisfied for every truncation s^k. Indeed, we only need to choose s_0 as the switching sequence which corresponds to the solution of the system when started from $x(0) = x_0$. This is not surprising since every state is indistinguishable from itself. We are rather interested if this remains the only solution for any switching sequence as $k \to \infty$. If so, then the global observability of the system (8.9), (8.9) follows.

The L-step observability can be nicely related to the foregoing formulation in terms of matrix equalities and inequalities. First let us define the following set of pairs of states.

$$\mathcal{I}_k \triangleq \{(x_0, \bar{x}_0) \in \mathbf{R}^n \times \mathbf{R}^n \mid (x_0, \bar{x}_0) \text{ solves } (8.12) - (8.16) \text{ for some } s^k\}.$$

In fact $\mathcal{I}_k$ is the set of pairs of states that are not k-step distinguishable. With this definition a few properties of $\mathcal{I}_k$ are in order.

Lemma 8.1

$$\{(x_0, x_0) \mid x_0 \in \mathbf{R}^n\} \subset \mathcal{I}_k.$$

Proof: Since $\mathcal{I}_k$ is the set of pairs that are not k-distinguishable, the result follows from the fact that every state is indistinguishable from itself for any number of steps. □

Lemma 8.2

$$\mathcal{I}_{k_1} \subset \mathcal{I}_{k_2} \text{ if } k_1 > k_2.$$

Proof: Assume $(x_0, \bar{x}_0) \in \mathcal{I}_{k_1}$. Then by definition there exists a truncated sequence of length k_1, $s^{k_1} = \{i(0), \ldots, i(k_2 - 1), \ldots, i(k_1 - 1)\}$ such that $(x_0, \bar{x}_0)$ yield the same output sequence until time $k_1 - 1$. Therefore the same pair of initial states with $s^{k_2} =$

$\{i(0), \ldots, i(k_2 - 1)\} \subset s^{k_1}$ yield the same output sequence until time $k_2 - 1$. Hence $(x_0, \bar{x}_0) \in \mathcal{I}_{k_2}$. $\square$

Our aim in formulating the observability problem in terms of linear matrix equalities and inequalities is the possibility of using the above facts to develop a recursive algorithm to calculate the sets $\mathcal{I}_k$ and see if at any point k we have $\{(x_0, x_0) \mid x_0 \in \mathbf{R}^n\} = \mathcal{I}_k$. If so then the bimodal system is globally observable. Moreover by Lemma 8.2, $\mathcal{I}_k$ is a nonincreasing sequence of nested sets hence to find $\mathcal{I}_{k+1}$ one needs to use only the state pairs that are already in $\mathcal{I}_k$. Yet, whether such an algorithm stops at trivially indistinguishable pairs, or stabilizes with $\mathcal{I}_k = \mathcal{I}_{k+1} = \ldots$ are interesting open problems. However, an algorithm to test k-step distinguishability stops because there are only 2^k different truncated sequences of length k. Moreover, some of these sequences can be discarded because of the assumed observability of the pair (C, A_1), (otherwise the system is not observable by Theorem 8.1). Indeed, if a sequence begins with $(n-1)$ 1's, then N_{s^n} has full column rank. Therefore the only solution is trivially $x_0 = \bar{x}_0$ and we need not bother with matrices $M^1_{s^n}, M^2_{s^n}$.

Example 8.3 *Consider the planar system in Example 8.1. Let us establish its observability using the algorithm by finding $\mathcal{I}_k$. Let us consider two initial states $x, \bar{x} \in \mathbf{R}^2$.*

$\mathcal{I}_0$*: We do not have $M^1_{s^0}$ and $M^2_{s^0}$ since switching sequence is empty. However we have $N_{s^0} = [1\ 0]$. By (8.12) we have $[1\ 0](x - \bar{x}) = 0$, or $x_1 = \bar{x}_1$. Therefore we have*

$$\mathcal{I}_0 = \{(x, \bar{x}) \in \mathbf{R}^2 \times \mathbf{R}^2 \mid x_1 = \bar{x}_1\}$$

and the system is not 0-step observable since $([1\ 2]^T, [1\ 0]^T) \in \mathcal{I}_0$. In fact, since we are considering planar systems 0-step observability would mean the nonsingularity of C which is ruled out by our choice of scalar output.

$\mathcal{I}_1$*: There are two possible truncated switching sequences of length 1, namely*

$$s^1_1 = \{1\}, \quad s^1_2 = \{2\}.$$

We can discard s_1^1 by the observability of (C, A_1). For s_2^1 we have

$$N_{s_1^1} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \quad M^2_{s_2^1} = [1\ 0].$$

By (8.12) we have

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ x_2 - \bar{x}_2 \end{bmatrix} = 0,$$

which is satisfied for all $x_2, \bar{x}_2 \in \mathbf{R}$. Also by (8.15) we have

$$[1\ 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} < 0, \quad [1\ 0] \begin{bmatrix} x_1 \\ \bar{x}_2 \end{bmatrix} < 0.$$

Hence $x_1 < 0$. Note that we implicitly used Lemma 8.2 in the form $x_1 = \bar{x}_1$. Therefore we have

$$\mathcal{I}_1 = \{(x, \bar{x}) \in \mathbf{R}^2 \times \mathbf{R}^2 \mid x_1 = \bar{x}_1 < 0\}$$

and the system is not 1-step observable since $([-1\ 2]^T, [-1\ 0]^T) \in \mathcal{I}_1$.

$\mathcal{I}_2$: We have four truncated switching sequences of length 2;

$$s_1^2 = \{1, 1\}, \quad s_2^2 = \{1, 2\}, \quad s_3^2 = \{2, 1\}, \quad s_4^2 = \{2, 2\}.$$

We can discard s_1^2 and s_2^2 by the observability of (C, A_1). For s_3^2 we have

$$N_{s_3^2} = \begin{bmatrix} -1 & 0 \\ -1 & 0 \\ -1 & 1 \end{bmatrix},$$

which has full column rank, hence by (8.12) we have only the trivial solution or no solution at all. Therefore we do not need to consider the matrices $M^1_{s_3^2}$ and $M^2_{s_3^2}$. For s_4^2 we have

$$M^2_{s_4^2} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}.$$

Substituting this in (8.15) we see that it does not have a solution. Therefore we have

$$\mathcal{I}_2 = \{(x, x) \mid x \in \mathbf{R}^2\}.$$

Hence the system is 2-step observable, which is exactly the same result we have found before.

The machinery in the previous example can easily be implemented using widely available linear programming software packages like MATLAB and cplex which include efficient algorithms for solutions of linear matrix inequalities. However, the ease of computation should not give the impression that a recursive algorithm like the one we provided can always decide on the observability (or unobservability) of a given discrete-time bimodal system in the form (8.9), (8.9). One obvious obstacle is the lack of certainty on whether such an algorithm stops. This issue is also known as the problem of **decidability**. Decidability of a property of a class of systems, is the question of whether there exists an algorithm which, when input with the system specification, decides, in finite time, if the system has the property, [102]. Of crucial interest in the study of switching systems is whether properties like stability, observability, reachability are decidable. Unfortunately even for simple classes of hybrid systems, these problems are either undecidable or computationally complex. For a survey on computational issues in hybrid systems, see [21, 103–106].

8.3 Stability of Switching Systems

The stability of switching systems have received a far greater attention compared to the observability problem. As it is with the observability, the stability of a switching system depends on its constituent subsystems (or modes) in a not-so-obvious way. In general, stability of a switching system cannot be deduced from the stability of vector fields among which the system switches, [107]. Still, seeking conditions under which this can be done is an active area of research.

Stability of a switching system depends primarily on the switching sequence. It is quite possible to have a switching sequence between two stable vector fields to render the overall system unstable, [107]. Conversely, by an appropriate switching sequence one can stabilize a switching system composed of two unstable modes. In these cases, switching sequence acts as feedback control, [108].

One interesting problem in the stability of switching systems is the question of stability that is independent of the switching sequence. Our work in the sequel is confined to this case.

First we briefly review some basic facts on the stability for arbitrary switching and its relation to the existence of common Lyapunov functions. In the rest of the section we give our contributions built on and extending these facts.

8.3.1 Stability for Arbitrary Switching

The class of switching systems we consider have the form

$$\dot{x} = A_{\sigma(t)}x, \quad x(0) = x_0, \tag{8.17}$$

where $\sigma : \mathbf{R}_+ \to \{1, \ldots, N\}$ is a switching path function corresponding to the sequence of switching times $\tau = \{t_0, t_1, \ldots\}$ that extend to infinity. Hence given a particular switching sequence, we have

$$\sigma(t) = i, \text{ for } t_k \leq t < t_{k+1}, \; i \in \{1, \ldots, N\}$$

and $A_i \in \Sigma = \{A_1, \ldots, A_N\}$. The solution $x(t)$ is then given by

$$x(t) = e^{A_{\sigma(t_k)}(t-t_k)}e^{A_{\sigma(t_{k-1})}(t_k-t_{k-1})} \ldots e^{A_{\sigma(t_1)}(t_2-t_1)}e^{A_{\sigma(t_0)}(t_1-t_0)}x_0.$$

The system (8.17) is exponentially stable for a particular switching path if $x(0)$ converges to zero exponentially fast. In our work we are mainly interested in the stability for arbitrary switching, namely, we are looking for conditions that guarantee the exponential stability of (8.17) for all $\sigma(t)$. Note that since $\sigma(t) = i,\ \forall t \geq 0$ is a valid switching path, to have stability for arbitrary switching it is necessary that $A_i,\ i = 1, \ldots, N$ are stable. However it is not sufficient as illustrated by the following example, [107].

Example 8.4 *Consider a switching system with two matrices*

$$A_1 = \begin{bmatrix} -1 & 10 \\ -100 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 100 \\ -10 & -1 \end{bmatrix}.$$

Each of the matrices A_i are stable as can easily be verified. However, if the switching path σ is chosen such that $\sigma(t) = 1$ when the state is in the second or the fourth quadrants and $\sigma(t) = 2$ when the state is in the first or the third quadrants, then the switching system becomes unstable. Figure 8.1 illustrates how such a switching yields unbounded trajectories. The thick-lined trajectory corresponds to A_1 and the thin-lined corresponds to A_2.

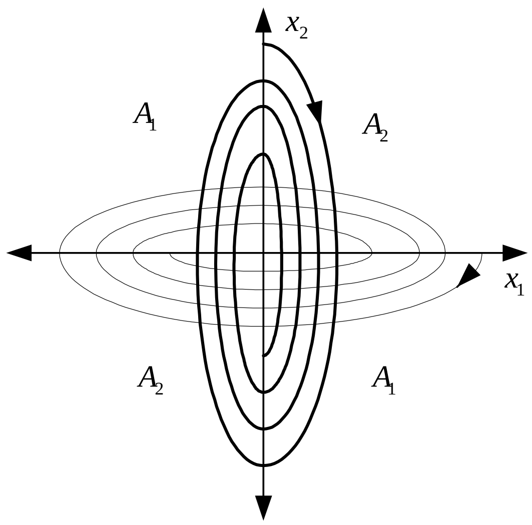

Figure 8.1: Typical trajectories of individual systems corresponding to A_1 and A_2.

Lemma 8.3 *The switching system (8.17) is exponentially stable if there exists a positive definite matrix P such that $V(x) = x^T Px$ is a common quadratic Lyapunov function for all $A_i \in \Sigma$, i.e.,*

$$A_i^T P + PA_i = -Q_i, \tag{8.18}$$

for some positive matrices Q_i.

Proof: See [109]. $\square$

Next three results specify some classes of matrix sets which satisfy the condition of this lemma.

Theorem 8.2 *If there exists a nonsingular matrix $T \in \mathbf{R}^{n\times n}$ such that $\tilde{A}_i = TA_iT^{-1}$, $i = 1, \ldots, N$, are upper-triangular then there exists a positive definite matrix P such that (8.18) holds.*

Proof: See [110]. $\square$

Theorem 8.3 *If the matrices A_i, $i = 1, \ldots, N$, pairwise commute, that is, $A_iA_j = A_jA_i$, $1 \le i, j \le N$, then they admit a common quadratic Lyapunov function.*

Proof: See [109]. $\square$

Theorem 8.4 *Assume that $N = 2$ and $A_1, A_2 \in \mathbf{R}^{2\times 2}$. Then they have a common quadratic Lyapunov function if and only if the matrices $\alpha A_1 + (1-\alpha)A_2$ and $\alpha A_1 + (1-\alpha)A_2^{-1}$ are stable for all $\alpha \in [0, 1]$.*

Proof: *See [111].* $\square$

8.3.2 Further Sufficient Conditions on Stability for Arbitrary Switching

Here we state and prove some results that came out of our elaborations on the facts given in the previous section. Since they are only sufficient conditions, they prove the stability of a particular class of switching systems.

Theorem 8.5 *Assume the stable matrices* $A_1, \ldots, A_N \in \mathbf{R}^{n\times n}$ *are commonly symmetrizable, i.e., there exists a common similarity transformation under which every* A_i *is similar to a symmetric matrix. Then the system (8.17) is exponentially stable for arbitrary switching.*

Proof: By assumption there exists a nonsingular matrix T such that

$$TA_iT^{-1} = S_i, \quad S_i = S_i^T, \quad i = 1, \ldots, N. \tag{8.19}$$

Since each A_i is stable and the eigenvalues of a matrix are invariant under similarity transformation we have S_i negative definite and consequently,

$$x^T S_i x \leq \lambda_{\max}(S_i)\|x\|^2, \ \forall x \in \mathbf{R}^n. \tag{8.20}$$

Choose the Lyapunov function candidate $V(x) = x^T T^T T x/2 = \|Tx\|^2/2$. Then we have

$$\begin{aligned}
\dot{V} &= \frac{1}{2}(x^T A_{\sigma(t)}^T T^T T x + x^T T^T T A_{\sigma(t)} x) \\
&= x^T T^T S_{\sigma(t)} T x, \text{ by } TA_{\sigma(t)} = S_{\sigma(t)}T, \\
&\leq \beta\|Tx\|^2, \text{ by (8.20)}.
\end{aligned}$$

where $\beta = \max_{1\leq i\leq N} \lambda_{\max}(S_i)$ which is negative. Therefore $\dot{V} \leq 2\beta V$ and consequently $\|x(t)\| \leq Me^{\beta t}\|x_0\|, \ M > 0$. Hence the switching system is exponentially stable. $\square$

Note that commonly symmetrizable matrices do not necessarily commute. Therefore the class of systems that are shown to be exponentially stable for arbitrary switching by Theorem 8.5 is not wholly covered by the one provided by Theorem 8.3. Indeed the following two matrices are symmetric negative definite, yet do not commute as a simple calculation shows;

$$A_1 = \begin{bmatrix} -10 & 1 \\ 1 & -9 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -11 & 2 \\ 2 & -10 \end{bmatrix}.$$

We now show that switching stability of commonly symmetrizable matrices are robust to small perturbations. Namely, perturbing a commonly symmetrizable set of matrices a little does not destroy the stability of the switching system.

Theorem 8.6 *Consider a set of stable matrices $A_1, \ldots, A_N \in \mathbf{R}^{n\times n}$, which are nearly commonly symmetrizable, i.e., there exists a nonsingular matrix T such that*

$$TA_iT^{-1} = S_i + E_i, \quad S_i = S_i^T, \quad \|E_i\| < \epsilon, \quad \epsilon > 0. \tag{8.21}$$

Then if ϵ is sufficiently small, then the switching system is exponentially stable.

Proof: Again consider the Lyapunov function candidate $V(x) = \|Tx\|^2/2$. Then we have

$$\begin{aligned} \dot{V} &= \frac{1}{2}(x^T A_{\sigma(t)}^T T^T Tx + x^T T^T T A_{\sigma(t)} x) && (8.22)\\ &= x^T T^T S_{\sigma(t)} Tx + x^T T^T (\frac{E_{\sigma(t)}^T + E_{\sigma(t)}}{2}) Tx, \text{ by } TA_{\sigma(t)} = S_{\sigma(t)}T + E_{\sigma(t)}T, && (8.23)\\ &\leq 2(\beta_m + \gamma_m)V, && (8.24) \end{aligned}$$

where $\beta_m = \max_{1\leq i\leq N} \lambda_{\max}(S_i)$ and $\gamma_m = \|E_i + E_i^T\|/2$. Since S_i and A_i are not similar we can not directly conclude $\lambda_{\max}(S_i)$ to be negative. However we know that eigenvalues of a matrix continuously depend on its elements, [87]. For each A_i we have

$$A_i = T^{-1}S_iT + T^{-1}E_iT.$$

Hence, for $\|E_i\|$ sufficiently small, the eigenvalues of A_i and the eigenvalues of $T^{-1}S_iT$ are very close. In particular, β_m is close to $\lambda_{\max}(A_i)$ which is negative. Hence, for sufficiently small ϵ we have $\beta_m < 0$ and γ_m very small. Consequently $\beta + \gamma_m < 0$. Exponential stability of the switching system then follows. $\Box$

Using this last result we arrive at Theorem 8.2 in a different way. We first need the following lemma.

Lemma 8.4 *If a set of matrices $A_1, \ldots, A_N \in \mathbf{R}^{n\times n}$, are commonly upper-triangulizable then they are commonly nearly symmetrizable with arbitrary small perturbation.*

Proof: By assumption we have a nonsingular T such that

$$TA_iT^{-1} = U_i, \quad U_i \text{ upper} - \text{triangular}$$

For the diagonal matrix $D(\epsilon) = \text{diag}[1 \ \epsilon \ \epsilon^2 \ldots \ \epsilon^{n-1}]$ consider the matrix $\tilde{U}_i = D^{-1}(\epsilon)U_iD(\epsilon)$ which obviously has the form

$$\tilde{U}_i = \begin{bmatrix} u_{11} & \epsilon u_{12} & \epsilon^2 u_{13} & \ldots & \ldots \\ 0 & u_{22} & \epsilon u_{23} & \epsilon^2 u_{24} & \ldots \\ & & \ddots & \ddots & \ddots \\ & & & & u_{nn} \end{bmatrix}.$$

Hence as $\epsilon \to 0$, $\tilde{U}_i$ approaches a diagonal matrix which is obviously symmetric. So, choosing the new transformation matrix as $\tilde{T} = D(\epsilon)T$ transforms the commonly upper-triangulizable set to a nearly symmetric set with arbitrary small perturbation. $\Box$

Corollary 8.6.1 *If the set $A_1, \ldots, A_N$ is commonly upper-triangulizable then the switching system is exponentially stable for arbitrary switching.*

Proof: By Lemma 8.4 the set is nearly commonly symmetrizable up to any perturbation. Then result follows using Theorem 8.6. $\Box$

Determining easily checkable conditions under which a set is commonly symmetrizable is not very easy. In the next section we simplify this problem for bimodal systems and give a visual characterization for planar systems.

8.3.3 Common Symmetrizability of Bimodal Systems

Suppose we have two stable matrices A_1 and A_2. If A_i is symmetrizable then we have for a nonsingular matrix T,

$$T^{-1}A_iT = S_i, \quad S_i = S_i^T$$

By the property of symmetric matrices there exists a real orthonormal matrix $U_i \in \mathbf{R}^{n\times n}$ and a diagonal matrix Λ_i such that

$$T^{-1}A_iT = U_i^T\Lambda_iU_i, \quad U_i^TU_i = I_n. \tag{8.25}$$

For simplicity let us assume that A_i has distinct eigenvalues. Therefore we can write $V_i^{-1}A_iV_i = \Lambda_i$, because necessarily we have $\Lambda_i = \mathrm{diag}[\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{in}]$ for eigenvalues λ_{ij} of A_i. Here V_i denotes the matrix of eigenvectors of A_i. Hence using (8.25) we have

$$\begin{aligned} T^{-1}V_i\Lambda_iV_i^{-1}T &= U_i^T\Lambda_iU_i, \\ \text{or } \; U_iT^{-1}V_i\Lambda_iV_i^{-1}TU_i^T &= \Lambda_i. \end{aligned}$$

Define $D_i \triangleq V_i^{-1}TU_i^T$. Hence we have $D_i^{-1}\Lambda_iD_i = \Lambda_i$ or $\Lambda_iD_i = D_i\Lambda_i$. Therefore D_i is the matrix of eigenvectors of the diagonal matrix Λ_i with distinct entries on the diagonal. Hence D_i has to be a diagonal matrix. Therefore $U_iT^{-1}V_i = D_i$ for a diagonal matrix D_i. Consequently, we can parameterize the symmetrizing transformations T of a matrix A_i as

$$T = V_iD_iU_i.$$

But by scaling the columns of the eigenvector matrix V_i we can cancel D_i to get

$$T = V_iU_i. \tag{8.26}$$

Hence a matrix$T$ symmetrizes A_i if and only if T is a matrix obtained by rotating the rows of the eigenvector matrix of A_i.

This parameterization of symmetrizing transformations of a matrix is useful in deciding whether two matrices are simultaneously symmetrizable. Namely, two matrices

A_1 and A_2 are simultaneously symmetrizable if and only if for two orthonormal matrices U_1 and U_2 and for *some* eigenvector matrices V_i of A_i we have

$$V_1U_1 = V_2U_2.$$

or by rearranging this we get

$$V_1^{-1}V_2 = U, \tag{8.27}$$

where U is an orthonormal matrix.

A couple of remarks are in order. First the equation (8.27) is invariant under eigenvector permutation but is not invariant under eigenvector scaling. Namely, if (8.27) is satisfied and we have $V_1^{-1}V_2$ orthonormal then permuting the columns of V_i with a permutation matrix P_i to get $\tilde{V}_i = V_iP_i$ we still have $\tilde{V}_1^{-1}\tilde{V}_2$ an orthonormal matrix. The proof of this fact is actually very simple and uses the properties of permutation matrices, see, e.g., [87].

However when we scale the eigenvectors to get $\tilde{V}_i = V_iK_i$, K_i being a diagonal matrix, $\tilde{V}_1^{-1}\tilde{V}_2$ may fail to be orthonormal. This can easily be seen by using (8.27) to write

$$\tilde{V}_1^{-1}\tilde{V}_2 = K_1UK_2^{-1}$$

and obviously $K_1UK_2^{-1}$ may fail to be orthonormal even if U is orthonormal.

Investigation of these two invariance properties are important because if we have a particular eigenvector matrix, all the other eigenvector matrices can be obtained by permutation and scaling.

The arguments so far can be summarized in the following fact.

Fact 8.3 *Two matrices A_1, A_2 with distinct eigenvalues are commonly symmetrizable if and only if for any two eigenvector matrices V_1 and V_2 there exist diagonal matrices K_1 and K_2 such that $K_1V_1^{-1}V_2K_2$ is orthogonal.*

Proof: Obvious from the preceding discussion. □

To see how to calculate K_1 and K_2 let us define the matrix X as $X = V_1^{-1}V_2$ and denote the entries of the matrix X as x_{ij}. Also let $K_i = \text{diag}[k_{i1}, k_{i2}, \ldots, k_{in}]$. Then the ij^{th} entry u_{ij} of the matrix $U = K_1V_1^{-1}V_2K_2$ is given by $u_{ij} = x_{ij}k_{1i}k_{2j}$.

By definition, U is orthonormal if and only if its entries satisfy the following equations;

$$\begin{aligned} \forall j,\quad 1 \le j \le n,\quad \sum_{i=1}^{n} x_{ij}^2 k_{1i}^2 k_{2j}^2 &= 1, \\ \forall j_1, j_2,\quad j_1 \ne j_2,\quad 1 \le j_1, j_2 \le n,\quad \sum_{i=1}^{n} x_{ij_1}k_{1i}k_{2j_1}x_{ij_2}k_{1i}k_{2j_2} &= 0, \end{aligned}$$

which can be simplified to

$$\forall j,\quad 1 \le j \le n,\quad k_{2j}^2 \sum_{i=1}^{n} x_{ij}^2 k_{1i}^2 = 1, \tag{8.28}$$

$$\forall j_1, j_2,\quad j_1 \ne j_2,\quad 1 \le j_1, j_2 \le n,\quad \sum_{i=1}^{n} x_{ij_1}x_{ij_2}k_{1i}^2 = 0. \tag{8.29}$$

Note that (8.29) is decoupled from (8.28), meaning that if one can solve (8.29) for k_{1i}, finding k_{2j} to satisfy (8.28) is trivial.

In fact, (8.29) involves $(n^2-n)/2$ equations with n unknowns k_{1i}^2. Since we are looking for a positive solution, this naturally becomes a linear programming problem. Given the matrix X, determining if (8.29) has a solution can be done using widely available linear programming packages. But it seems that there is no easily checkable necessary and sufficient condition to determine whether a linear set of homogeneous equations has a positive solution, or put another way, whether the kernel of a matrix has a positive vector. However in the case $n = 2$, the analysis is considerably simpler and the result is quite visual.

Planar Systems

The condition given by Fact 8.3 lends itself to a nice interpretation in terms of eigenvector interlacing when the switching system is planar.

When $n = 2$ the number of equations in (8.29) is just one and it is

$$x_{11}x_{12}k_{11}^2 + x_{21}x_{22}k_{12}^2 = 0,$$

and has a solution if and only if the quantity

$$x_{11}x_{12}x_{21}x_{22} < 0.$$

Namely the multiplication of the all elements in the matrix X is negative.

Let the eigenvector of A_i be denoted as v_{i1}, v_{i2}, thus we have

$$V_1 = [v_{11}\ v_{12}], \quad V_2 = [v_{21}\ v_{22}].$$

A routine calculation of the quantity $x_{11}x_{12}x_{21}x_{22}$ and a rearranging of terms show that this quantity is negative if and only if the following product p defined by

$$p = \det[v_{11}\ v_{21}]\det[v_{21}\ v_{12}]\det[v_{12}\ v_{22}]\det[v_{22}\ v_{11}]$$

is negative.

Before proceeding further we give the following lemma whose proof is very easy and therefore omitted.

Lemma 8.5 *Given two dimensional vectors x and y, y lies in the counter-clockwise half-plane above the line of vector x (i.e. y leads x in counter-clockwise direction) if and only if the determinant of the matrix $[x\ y]$ is positive.*

Now we use this lemma to obtain a geometric condition on the orientations of the eigenvectors. Note that the quantity p is negative if and only if the signs of the determinants are as given in Table 8.1.

For the sake of illustration here we investigate only the first row of the table and using the Lemma 8.5 show that p is negative if and only if the eigenvectors of A_1 and A_2 interlace, that is, when one traverses the unit circle in the counter-clockwise direction,

$\det[v_{11}\ v_{21}]$	$\det[v_{21}\ v_{12}]$	$\det[v_{12}\ v_{22}]$	$\det[v_{22}\ v_{11}]$
+	+	+	-
+	+	-	+
+	-	+	+
-	+	+	+
+	-	-	-
-	+	-	-
-	-	+	-
-	-	-	+

Table 8.1: Possible sign combinations to make p negative.

the eigenvectors are encountered in the order $v_{11}, v_{21}, v_{12}, v_{22}$. Indeed, first column says that v_{21} leads v_{11}, by the second column v_{12} leads v_{21}, Thus v_{21} is *between* the eigenvectors of the first matrix. Similarly one can easily show that an eigenvector of the first matrix is *between* those of the second matrix. The negative determinant ensures that being *between* is not trivial, namely by reversing the direction of an eigenvalue interlacing property is not destroyed. A typical situation is illustrated in Figure 8.2.

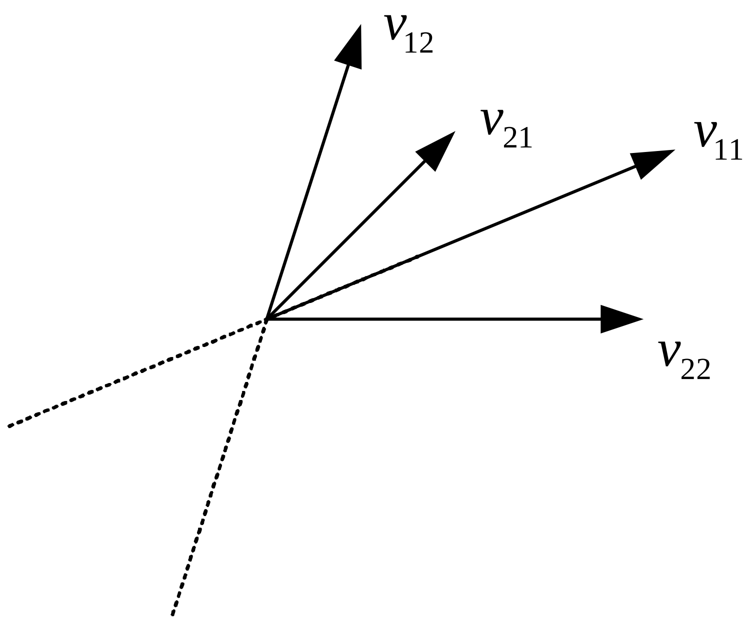

Figure 8.2: Interlacing eigenvectors

8.3.4 Stability of Discrete-Time Switching Systems

For the discrete-time case we have the system description very similar to 8.17 as

$$x(k+1) = A_{\sigma(k)}x(k), \quad x(0) = x_0, \tag{8.30}$$

where we have the system switching between the matrices in the set of stable matrices $\Sigma = \{A_1, \ldots, A_N\}$. We are interested in the stability of the system (8.30) for arbitrary switching path $\sigma : \mathbf{N} \to \{1, \ldots, N\}$.

The stability problem for these types of problems are also known in the literature as determining whether a given set of matrices has a **left converging product**. If the set Σ has this property then it is said to be a **LCP set**, [112]. The stability of (8.30) for arbitrary switching is equivalent to set Σ being a LCP set with limit as the zero matrix. LCP property is closely related to the existence of a matrix norm with certain properties, [113].

Theorem 8.7 *The system (8.30) is exponentially stable if and only if there exists a matrix norm $\|\cdot\|$ such that*

$$\|A_i\| < 1, \quad i = 1, \ldots, N.$$

Proof: See [114]. $\square$

While this result provides the existence, actual construction of the norm is very difficult in general. Here we give a procedure to construct such a norm. We first give a simple interpretation of an induced norm in terms of convex sets.

Let us consider a compact, convex, symmetric set $\Omega \in \mathbf{R}^n$ with non-empty interior which includes the origin, i.e., $0 \in \Omega$. By constraining the interior to be non-empty we exclude degenerate cases, like a line segment in the plane; a line segment is compact and convex but its interior is empty. Since the set is convex we have

$$\forall x_1, x_2 \in \Omega,\ \forall \lambda \in [0,1],\ \lambda x_1 + (1-\lambda)x_2 \in \Omega.$$

Also symmetry implies

$$\forall x \in \Omega \ \Rightarrow \ -x \in \Omega.$$

Now let us define the following quantity for a vector $x \in \mathbf{R}^n$;

$$n_\Omega(x) \triangleq \frac{1}{\max\{t \mid tx \in \Omega\}}$$

and $n_\Omega(0) = 0$. In other words we scale x until it is just on the boundary of Ω and take the inverse of scaling factor. Hence if x is on the boundary, we have $n_\Omega(x) = 1$.

Lemma 8.6 *$n_\Omega(x)$ is a norm on $\mathbf{R}^n$.*

Proof: See [115, p. 73] $\square$

Let us denote this norm as $\|x\|_\Omega$. Since every vector norm induces a matrix norm in $\mathbf{R}^n$, for a matrix A we have the induced norm $\|A\|_\Omega$ as

$$\|A\|_\Omega = \sup_{\|x\|_\Omega = 1} \|Ax\|_\Omega.$$

Denote the image of the boundary of Ω under A as $A\partial(\Omega)$. Then a simple manipulation shows that

$$\|A\|_\Omega = \min\{t \geq 0 \mid A(\partial\Omega) \subset t\Omega\}.$$

Therefore if $A(\partial\Omega) \subset \Omega$ and $A(\partial\Omega) \cap \partial\Omega = \emptyset$ then $\|A\|_\Omega < 1$.

Here we see another interpretation of Lyapunov stability in terms of convex sets. Namely, given a matrix A, if there exists a convex, compact, symmetric set Ω with non-empty interior including the origin such that A maps its boundary strictly inside Ω then the system $x(k+1) = Ax(k)$ is asymptotically stable.

Note that a norm so defined becomes 2-norm if Ω is a circle, 1-norm for a diamond and ∞-norm for a square.

In the light of Theorem 8.7, given a set of matrices $A_1, \ldots, A_N$, if there exists a convex, compact, symmetric set Ω with non-empty interior such that each A_i maps its boundary strictly inside Ω then the switching system (8.30) is exponentially stable for arbitrary switching. Now we prove the converse.

Theorem 8.8 *If the switching system (8.30) is exponentially stable for arbitrary switching among the set of matrices, $\Sigma = \{A_1, \ldots, A_N\}$, then there exists a convex, compact, symmetric set Ω with non-empty interior such that each A_i maps its boundary strictly inside Ω.*

Proof: Start with an arbitrary symmetric, convex polygon Ω_0 with vertices $\{\omega_1, \omega_2, \ldots, \omega_p\}$, i.e.,

$$\Omega_0 = \text{co}\{\omega_1, \omega_2, \ldots, \omega_p\},$$

where co$\{\ldots\}$ denotes the convex hull of a set. Also denote the iteration by Σ as $\Sigma^k \Omega_0$. Namely the set $\Sigma^k \Omega_0$ consists of all the elements of the form $q = A_{i(k-1)} \ldots A_{i(0)} \omega_j, \ j = 1 \ldots p$. Let us consider the union of all such sets and define it as

$$\Omega^\infty = \bigcup_{k=1}^{\infty} \Sigma^k \Omega_0.$$

Obviously we have

$$\Sigma \Omega^\infty = \Omega^\infty.$$

Let us now consider $\Omega = \text{co } \Omega^\infty$. Using the linearity of matrix multiplication and the convexity of Ω we have

$$\Sigma \Omega \subset \Omega.$$

Hence $\|A_i\|_\Omega \leq 1, \ i = 1 \ldots N$. The inclusion is not strict, because a vertex q_j of the polygon Ω may be mapped to the boundary $\partial \Omega$ after one iteration. There are finitely many vertices of Ω which violate the strict inclusion. Now we will show that Ω can be modified appropriately to get a strict inclusion of the iterates of all of its corners.

By assumed stability of the set Σ, every infinite iteration of ω_j enters Ω_0 after finitely many steps and does not leave it then on. Therefore the sequence starting from a problem generating vertex of Ω will be strictly inside Ω after some iteration. Let a sequence starting from ω_j hit after finite number of steps the boundary $\partial\Omega$. Hence we have $q_j = A_{i(k-1)} \ldots A_{i(0)}\omega_j \in \partial\Omega$ and $A_i q_j$ is strictly inside Ω for every $i = 1, \ldots, N$. Define δ_i as a positive number satisfying $\delta_i A_i q_j \in \partial\Omega$ and let $\delta = \min\{\delta_i\}$. Choose numbers ϵ_t such that $0 = \epsilon_1 < \epsilon_2 < \ldots < \epsilon_{k-1} < \delta$ and define the new vertices, $\{\omega_j, (1+\epsilon_2)A_{i(0)}\omega_j, (1+\epsilon_3)A_{i(1)}A_{i(0)}\omega_j, \ldots, (1+\epsilon_{k-1})q_j\}$. Now these new vertices, when considered in the convex hull will be mapped strictly inside new Ω. If the same procedure is applied to all the problem generating sequences, then we will have the modified hull Ω_ϵ mapped strictly inside itself. Hence $\|A_i\|_{\Omega_\epsilon} < 1,\ i = 1, \ldots, N.$ $\square$

Chapter 9

Conclusions

In this work we studied some problems revolving around the observability of nonlinear and switching systems and observer design. While a substantial amount of literature exists on the observability of nonlinear systems, the rather recent field of switching systems lacks a commensurate interest in their observability. The motivations for this thesis were twofold. On the one hand we wanted to modify and adapt existing nonlinear design methodologies to chaos synchronization and control, and on the other hand, to satisfy, to some extent, the need for a proper definition of observability and related conditions for switching systems.

For the adaptation of nonlinear observers, we first showed that chaos synchronization can indeed be put in nonlinear observer framework. By transforming one of the first chaos synchronization settings, coupled Lorentz system, to a form which can be formulated as a system-observer pair. This naturally generalized to include many other ad hoc synchronization schemes.

While some observer design methods can directly be applied to chaos synchronization in a rather straightforward way, still many others have very stringent requirements on the system structure. In particular the observers for systems in Brunowsky form require the nonlinearity to be globally Lipschitz. Using the fact that chaotic trajectories remain in a bounded region, we were able to relax this condition to nonlinearity being locally Lipschitzian. Then we used an extension result to have globally convergent nonlinear observers. Once we are equipped with an exponential observer, we studied two message encoding schemes that modulate messages with chaotic signals and proved the guaranteed message reconstruction.

One disadvantage of using the observer for Brunowsky form is the prohibitively high magnitudes of the gains. We then sought ways to overcome this drawback by reducing the gain magnitudes while still having globally convergent state estimates. We basically exploited the structure of the Vandermonde matrix. As we proved, there are still theoretical lower bounds on the gain magnitudes. The solution we arrived at, namely, assigning eigenvalues on the same circle, is not optimal in the strict sense, yet it considerably simplifies gain selection and reduces gain magnitudes.

Chaotic systems, as opposed to the system descriptions in the classical state space formulation, in general, do not specify the output function. Hence at least in theory it is possible to tailor an output function to facilitate observer design. One obvious criteria is to choose the output function to make the system observable. Although this problem is highly nonlinear even for simple nonlinear systems, we were able to parameterize the set of linear output functions that make a linear system observable. This result can be used to make the linear part of a nonlinear system observable. Yet, it is difficult to relate the result to the parameterization of nonlinear output functions.

We used observers to change the behaviour of forced chaotic oscillators by effectively modifying its bifurcation parameters. This result draws on the fact that forced chaotic oscillators can easily be described in Brunowsky form, enabling us to use the special observer structure for this class of systems. We indicated how an observer in the closed loop can be used to asymptotically change the structure of one system with certain behavior to another one with a desired behavior. Also, observer-based tracking is nicely solved in the same framework.

Scarcity of constructive methods in the analysis of switching systems led us to work on the observability of switching systems and alongside with it, on some stability issues. We started from a simple form of switching, namely, periodic switching. It turned out that observability of a periodically switching discrete-time system has an equivalent characterization in terms of the rank condition of a matrix formed using the observability matrices among which the system switches. We then used this result to construct an exponential observer.

For a periodically time-varying continuous-time system, we obtained a hybrid observer that combines discrete and continuous dynamics.

When the switching is not periodic, many control problems turn out to be intractable at best. In most cases, negative undecidability results put a limit on which generalization can be achieved in observability and stability characterizations. We have shown that even for simple systems switching between only two vector fields, the observability problem is quite complicated. For bimodal planar systems we were able to completely characterize the observability. For higher-order systems, we gave a recursive procedure which can be implemented using standard LMI packages. However, it only provides an algorithm with no guarantee on halting. Although no such results are available as of writing of the thesis, judging from the dual negative results for the stability problem, one should not be too surprised to discover that observability of switching systems is undecidable.

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