ANALYSIS AND DESIGN OF SWITCHING AND FUZZY SYSTEMS

A DISSERTATION

SUBMITTED TO THE DEPARTMENT OF ELECTRICAL AND ELECTRONICS ENGINEERING

AND THE INSTITUTE OF ENGINEERING AND SCIENCES

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

By

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September 2002

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

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ABSTRACT

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In this thesis we consider the controller design problems for switching and fuzzy systems. In switching systems, the system dynamics and/or control input take different forms in different parts of the underlying state space. In fuzzy systems, the system dynamics and/or control input consist of certain logical expressions. From this point of view, it is reasonable to expect certain similarities between these systems. We show that under certain conditions, a switching system may be converted into an equivalent fuzzy system. While the changes in the system variables in a switching system may be abrupt, such changes are typically smooth in a fuzzy system. Therefore obtaining such an equivalent fuzzy system may inherit the stability properties of the original switching system while smoothing the system dynamics. Motivated from this idea we propose various switching strategies for certain classes of nonlinear systems and provide some stability results. Due to the difficulties in designing such switching rules for nonlinear systems, most of the results are developed for certain specific type of systems. Due to the logical structure, obtaining rigorous stability results are very difficult for fuzzy systems. We propose a fuzzy controller design method and prove a stability result under certain conditions. The proposed method may also be applied to function approximation. We also consider a different stabilization method, namely phase portrait matching, in which the main aim is to choose the control input appropriately so that the dynamics of the closed-loop system is close to a given desired dynamics. If this is achieved, then the phase portrait of the closed-loop system will also be close to a desired phase portrait. We propose various schemes to achieve this task.

Keywords : Fuzzy Systems, Switching Systems, Stabilization, Controller Design, Driftless Systems, Phase Portrait Matching, Lyapunov Functions, Periodic Switching.

ÖZET

ANAHTARLAMALI VE BULANIK SİSTEMLERİN ANALİZ VE TASARIMI

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Ağustos 2002

Bu tezde, anahtarlamalı ve bulanık sistemler için denetleyici tasarım problemlerini ele alıyoruz. Anahtarlamalı sistemlerde, sistem dinamiği ve/veya denetleyici girdisi, durum uzayının değişik bölgelerinde değişik biçimler almaktadır. Bulanık sistemlerde, sistem dinamiği ve/veya denetleyici girdisi belirli bazı mantıksal ifadelerden oluşmaktadır. Bu açıdan bakıldığında, sistemler arasında belli benzerlikler olmasını beklemek makul görülmektedir. Belli koşullar altında, anahtarlamalı bir sistemin eşdeğer bir bulanık sisteme dönüştürülebileceğini gösteriyoruz. Sistem parametrelerinin değişmesi anahtarlamalı sistemlerde ani olur iken, benzer değişimler bulanık bir sistem için genellikle düzgün/kesiksiz olmaktadır. Dolayısıyla, bu şekilde elde edilen bir bulanık sistem, orjinal anahtarlamalı sistemin kararlılık özelliklerini gösterirken aynı zamanda sistem dinamiğini düzgünleştirebilir. Bu düşünceden hareketle, doğrusal olmayan belli bir sınıf sistem için, değişik anahtarlama yöntemleri öneriyor ve bazı kararlılık sonuçları gösteriyoruz. Doğrusal olmayan sistemler için benzer anahtarlama kurallarının tasarımındaki zorluk dolayısıyla, geliştirilen sonuçların çoğu belirli tip sistemler için olmaktadır. Bulanık sistemler için genel bir kararlılık sonucu elde etmek, mantıksal yapısı nedeniyle oldukça zordur. Bulanık denetleyiciler için bir tasarım yöntemi öneriyor ve belirli koşullar altında kararlılığını gösteriyoruz. Önerilen yöntem fonksiyon yakınsama için de uygulanabilir. Ayrıca faz portresi eşleme ismiyle, amacı kapalı döngü sistem dinamiğinin verilen bir dinamiğe mümkün olduğunca yakınsamasını sağlayacak uygun kontrol girdisinin hesaplanması olan, farklı bir kararlaştırma yöntemini ele aldık. Eğer bu sağlanabilirse, kapalı döngü sistemin faz portresi istenilen bir faz portresine yakınsanabilir. Bunun için değişik tasarımlar öneriyoruz.

Anahtar Kelimler : Bulanık Sistemler, Anahtarlamalı Sistemler, Kararlılık, Denetleyici Tasarımı, Sürüklenmesiz Sistemler, Faz Portresi Eşleme, Lyapunov Fonksiyonları, Periyodik Anahtarlama.

ACKNOWLEDGEMENTS

First, I want to express my deep gratitude to my parents, grandmother and brothers, who have always been there behind me with their love, support, encouragement and prayers during the hard times of my PhD adventure. I dedicated this thesis to my dear parents.

I would like to thank my supervisor Prof. Dr. Ömer Morgül for his invaluable guidance, and patience. He has always been a kind and motivating advisor. I want to mention about his guidance especially on how to write for possible readers, rather than myself. I should acknowledge his encouragements and supports when I get stuck and unable to see the light.

I want to express my special thanks to Prof. Bülent Özgüler, Prof. Enis Çetin, Prof. H. Altay Güvenir and Prof Aydan M. Erkmen for reading and commenting on the thesis. I would also express my sincere thanks to Prof. Aydan M. Erkmen for the short but guiding discussions.

During these years, Hakan Köroğlu and Ercan Solak have been close friends. They were the two who contributed to me most in our thoughtful technical and nontechnical discussions. Nejat Kamacı, Mustafa Sungur, Engin Avcı, H. Bayram Erdem, Ömer Salihoğlu have been amiable housemates. I would like to thank to M.B.Y. for her constant love, patience, and support during these years. There has been many other friends, a partial list of them being Ayşegül Şahin*, Elif Öğüt, K. Güçlü Köprülü, Karim Saadaoui, Lütfiye Durak, Mustafa Akbaş, Necmi Bıyıklı, Tolga Kartaloğlu, and Z. Gürkan Figen with whom I enjoyed life in Bilkent. I want to take this opportunity to express my thanks to all of them.

Finally, I would like to thank my department and the whole Bilkent community for supplying us such a distinguished university environment.

Dedicated to
Naciye and **Hüseyin**
Akgül

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List of Acronyms

FBF : Fuzzy Basis Function.

FLC : Fuzzy Logic Controller.

FLS : Fuzzy Logic System.

GMP : Generalized Modus Ponens.

GMT : Generalized Modus Tollens.

LHP : Left half plane.

LTI : Linear Time Invariant.

MF : Membership Function.

MIMO : Multi-Input Multi-Output.

ODE : Ordinary Differential Equation.

PD : Proportional Derivative.

PID : Proportional Integral Derivative (Controller).

RPM : Revolution Per Minute.

SS : Switching System.

SISO : Single-Input Single-Output.

Chapter 1

Introduction

There is a continuous search for the models of the phenomena occurring around us. The understanding of the environment and its dynamics is one of the challenges to the human life to carry on. Therefore we try to associate a model with each phenomenon we see around us. The closer the model to the real world the better results we obtain in interacting with it. The rainfall pattern throughout a year in a country is an example of such a phenomenon. This local model is vital for the farmers in that country. Another example would be cattle dealing. The knowledge of the animal requirements like, feeding, breeding, and health care are components of the model that affects the output of the process, cattle dealing. Generally a human is in the position to control certain aspects of the phenomenon under consideration. The knowledge, gathered as the experience of years, is the model which can be expressed in general by subjective expressions.

Increase in the population resulted in the problem of increasing the production efficiently to meet the resulting demand. One of the solution is to use machines in the production lines to increase the efficiency. This tendency brings the need for engineers that can build machines which automatically perform certain tasks. This speeds up the advances in the control theory, concerning with the development of strategies which force a given system to exhibit desirable behavior. The systems considered might be physical, chemical, biological or even social in nature. The design attempts start with the modelling of the system, which is usually referred to as the plant [1]. In general, the model might be expressed with nonlinear differential equations.

Consider the problem of designing an automatic camera that can adjust its parameters depending on the distance of the target, the illumination of the environment, etc. One good starting point is to find some professional photographers and ask how s/he would decide on these adjustments. Or, consider to design an auto-pilot for an automobile. Again the knowledge of the drivers of that specific car model will be useful to start the design. Since these experts generally will not be able to express their experience in a differential equation form, all the information the interviewer will obtain is a collection of sentences. Generally these sentences will be conditionals like IF "certain conditions exist" THEN "perform certain actions". However, in general the conditional parts of the statements or the consequent parts might not be expressed so precisely, e.g. IF "the road is slippery due to the rain" THEN "don't exceed 30km/h". Such conditions are vague and require further effort to map them into the measurable domains.

If the given conditions can be measured precisely and the control action is also expressed precisely then one might come up with a controller structure that switches in between the specified control actions depending on the measured conditions. This type of controllers are known as switching controllers in the literature. The tools used in the design of such systems are the control theory developed so far plus the classical set theory and logic for the interpretation of the given rules. On the other hand if expressions are vague in nature and/or subjective then these expressions first must be mapped onto the measurable domains so that the interfaces between the sensors and actuators can be established. In the literature, the fuzzy set theory and the fuzzy logic provide the necessary tools to interpret these vague expressions and the conditional statements. Fuzzy control theory is an approach that utilizes fuzzy logic and the control theory tools to build controllers based on such vague specifications.

Many nonlinear systems can be approximated as a sum of piecewise linear systems. Hence a switching system that switches between these linear models will be a good approximation for the nonlinear model. The controller design for Linear Time Invariant (LTI) systems is one of the subject that has almost reached its mature state in the literature. The controllers designed for each of the linear systems, which constitute the nonlinear model, will be the switching controller to improve the performance of the system under consideration. Following the same reasoning fuzzy systems can also be used. Their use, may improve the performance of a designed switching controller such as smoothing some of the system variables.

In this thesis we consider two logic based system design approaches, namely the switching systems and the fuzzy systems. We investigate the modelling of both approaches and seek some methods to convert one model to the other so that the results obtained for one model can be applied to the other model. Fuzzy systems are known to be universal approximators. Hence any model can be approximated by a fuzzy system up to a given accuracy. One of the problems in control theory is the stabilization of a given plant. So we can conclude that if a system is stabilizable by a controller then there should be a fuzzy system which can approximate it for a given accuracy. Another problem we investigate in this thesis is to find a method to approximate a given input/output data by a fuzzy system. The determination of an input/output data is in fact another problem. We try to solve this problem by using phase portrait approach. The other topics we study studied in this thesis can be summarized under the topic of modelling and stabilization problems of switching and fuzzy systems.

In the following section we will give a brief literature survey on the switching and fuzzy systems and give some motivation on the subject. Then we will conclude this chapter with the organization of the thesis.

1.1 Switching Systems

The idea of switching is well known in control theory. In past decades, switching has been used in adaptive control to assure stability. In [2] switching between multiple adaptive models was used to improve the transient response of adaptive control systems in a stable fashion [3]. There are several other reasons for switching control to become popular. Any nonlinear system can be modelled with a linear system in a certain region by linearization. A switching system whose dynamics switches in between these linearized models will approximate the dynamics of the nonlinear system. The controllers which is designed for the linearized models will constitute the switching controller for the nonlinear system.

Motivated by the current interest in switching systems several authors have studied optimal control [4] and stability questions for switching systems [5]- [6]. More details about these studies are provided in Section 2.

In most of the works related with switching systems, the concept of a common Lyapunov unction is utilized to obtain some stability results. The necessary and sufficient conditions which guarantee the existence of a common Lyapunov function are not known, the problem may even be undecidable (see [7]). Hence many of the current work on this area focus on special cases, and such studies lead to special switching strategies to stabilize such systems.

In this thesis our main concern related with switching systems is the determination of a stabilizing control signal for a class of nonlinear systems. The design methodology and modelling aspects constitute a base for the models of fuzzy systems.

1.2 Fuzzy Systems

A fuzzy system is a real time expert system implementing a part of a human operator's or process engineer's expertise which does not lend itself to being easily expressed in a difference/differential equation but rather in situation/action rules like the IF-THEN statements given previously.

It is generally agreed that an important point in the evolution of the modern concept of uncertainty was the publication of a seminal paper by Lotfi A. Zadeh [8]. In this paper, Zadeh introduced a theory whose objects-*fuzzy sets*-are sets with boundaries that are not precise. The membership in a fuzzy set is not a matter of affirmation or denial, but rather a matter of *degree*.

The first successful industrial application of the Fuzzy Logic Controller (FLC) was a cement kiln control system developed by the Danish cement plant manufacturer F. L. Smidth in 1979.

A representation theorem, mainly due to Kosko [9], states that any continuous nonlinear function can be approximated as closely as needed with a finite set of fuzzy variables, values, and rules. Later X.-J. Zeng and M. G. Singh published their work on the approximation problem of SISO and MIMO fuzzy systems (see [10], [11]). These results describe the representational power of fuzzy control in principle, but it does not answer the questions of how many rules are needed and how they can be found.

FLCs can be used in various ways. One of them is to use the FLC directly as the controller for the system under consideration. The other choice is to use a standard controller like PID or lead-lag and tune the parameters of this controller with a Fuzzy Logic System (FLS). More details on these studies can be found in Section 3. Our main concern on this subject is the design methodologies that results in a fuzzy controller providing closed-loop stabilization.

1.3 Organization of the Thesis

In this thesis we consider the modelling and control problems of switching and fuzzy systems. The chapters and their contents can be summarized as follows.

Chapter 2: Switching Systems

In this chapter we have three main topics: Classical set theory, models for switching systems, and stability analysis. In classical set theory section, we give the necessary tools like, operations on sets, relations, and interpretation of conditional statements, to model a switching system. In models for switching systems section, we consider the various forms of models suggested in the literature. In stability analysis section, we give the basic stability results in the literature some of which will be utilized in Chapter 7.

Chapter 3: Fuzzy Systems

This chapter is composed of three main topics: Fuzzy set theory, models for fuzzy systems, and stability analysis. Fuzzy set theory section provides the necessary tools like operations on fuzzy sets, fuzzy relations, and interpretation of conditional statements with fuzzy propositions, to model a fuzzy system. The models for fuzzy systems section provides various forms of models suggested in the literature. In stability analysis section, we give the basic stability results available in the literature.

Chapter 4: Comparison of Switching and Fuzzy Systems

In this chapter we investigate the resemblance of switching and fuzzy system regarding the model structure and the stability results available in the literature. In discussing the modelling issues we propose a method to obtain an fuzzy system associated with a given switching system.

Chapter 5: Phase Portrait Matching

In this chapter we propose a method to calculate the control signal of a given system so that the closed loop dynamics is close to a desired dynamics. We devise two approaches to calculate the control signal. The first method aims to minimize the phase difference between the desired phase portrait and the closed-loop system. The second method aims to minimize the norm of the difference of the desired dynamics and the dynamics of the closed-loop system. The control signal calculated by these schemes might be out of the physical bounds of the given system. We propose two saturation schemes which might be used when the calculated control signal magnitude is out of some prescribed limits.

Chapter 6: Fuzzy Controller and System Design

In this chapter we have three main sections. In the first section we give the general design aspects for fuzzy controllers. In the remaining sections of this chapter we propose two methods for the design of fuzzy controllers. We show that the method we propose in the second section yields a stable closed-loop system under certain conditions. We also outline a design procedure related with the proposed method. In the third section we propose a method to approximate the calculated control signal by using fuzzy rules. If the input/output data is known for a stabilizing controller then this method can also be used to construct a fuzzy controller to be used in the stabilization problem.

Chapter 7: Switching Controller and System Design

This chapter is composed of 5 sections. In the first section we provide the tools available in the literature to design a switching system or controller. In the second section we consider a chaotic system which has piecewise linear dynamics. We model this system as a switching system, propose a synchronization scheme and give various stability results related with the proposed scheme. In the third section we propose a periodic switching strategy to stabilize a class of nonlinear systems. In the fourth section we propose another method for stabilization of a class of nonlinear system using a switching strategy so that the closed loop system becomes stable. In the fifth section we propose a switching strategy to stabilize a certain class of driftless systems.

Chapter 8: Applications

In this chapter we present some of the applications and simulation results related to the methods we proposed in the thesis. In the first section we present some simulation results related to the chaotic synchronization scheme given in Section 7.2. We also present some simulation results related with the fuzzy approximation technique we proposed in Section 6.3. In the second section, we propose two chaotic message transmission schemes for a class of switching chaotic systems, prove some convergence results. We also present some simulation results related to the proposed chaotic masking scheme. In the third section, we present some simulation results related to the methods we proposed for the stabilization of switching and fuzzy systems. We also present some stability results related with the switching schemes we proposed for a class of systems.

Chapter 2

Switching Systems

Many complicated systems, such as airplanes manufacturing and transportation systems etc, in general have many operating points, each of which should be handled carefully. We also see that today's products incorporate logical decision-making into even the simplest control loops (e.g., embedded systems) [12]. These systems are generally modelled almost at every operating point and a suitable control action is applied accordingly. Hence in such systems, both the system model used in designing controllers, and the control action changes, or switches, depending on operating points. Such systems are natural examples of switching systems. Formal definition of switching systems will be given later, see Section 2.2, but at this point we may state that by switching systems we mean a system for which the system dynamics and/or control action changes (or switches) according to a rule. In [13] there has been made distinction between the switching systems and the hybrid systems. A hybrid system is defined as being one that the switching rule considers the history of previous modes while the switching systems just checks some conditions at that time and discards the past modes of the system. There are studies where this distinction is omitted as well, see [7]. Although in our study the rule of the switching system is a function of only the current states of the switching system we accept to have a system with a switching rule which might be a function of the switching history, see e.g. (2.1).

The idea of switching is well known in control theory. In past decades, switching has been used in adaptive control to assure stability. In [2] switching between multiple adaptive models was used to improve the transient response of adaptive control systems in a stable fashion [3]. There are several other reasons for switching control to become popular. Any nonlinear system can be modelled with a linear system in a certain region by linearization. A switching system whose dynamics switches in between these linearized models will approximate the dynamics of the nonlinear system. The controllers which is designed for the linearized models will constitute the switching controller for the nonlinear system. A desired performance for the system might require too many number of controllers, which in return requires a high computational power. In fact the improvements which puts a high computational power to a small chip and the recent developments in sensor and actuator technology are the main fuels of the tendency to switching systems. By the sensors it is possible to measure the mode of the system and by the computers it is possible to decide in real-time how to feed the actuators to force the system to have an acceptable performance.

Motivated by the current interest in switching systems several authors have studied optimal control [4] and stability questions for switching systems [5], [14], [15], [16], and [6]. In [15], an exactly known linear systems and a given set of control laws are considered, and a method to determine whether the closed-loop system is stable under all possible switching sequences, is presented. In [5] the idea of multiple Lyapunov functions as a tool for studying the stability of switching systems is introduced, and some existence results for a set of controllers to be stabilizing in the sense of Lyapunov are presented. The stabilization of a linear plant by switching two linear control laws is considered in [14] and a method for determining a stabilizing switching sequence is described. In [16] a sufficient condition for robust output feedback stabilization with synchronous controller switching is presented. The necessary and sufficient conditions to test for quadratic stability and stabilizability and for stabilizability with a quadratic storage function for switching controller systems are presented in [6].

Most of these works utilize the concept of a common Lyapunov function to study stability of switching systems. The necessary and sufficient conditions which guarantee the existence of a common Lyapunov function are not known, the problem may even be undecidable (see [7]). Hence many of the current work on this area focus on special cases, and such studies lead to special switching strategies to stabilize such systems.

This chapter is organized as follows. In Section 2.1 we will review shortly the classical set theory. This would be constructive for the following sections and chapters. In Section 2.2 the modelling issues of switching systems will be given. In Section 2.3 the stability results in the literature will be reviewed.

2.1 Classical Set Theory

A classical set is a collection of objects of any kind. The concept of a set has become one of the most fundamental notions of mathematics. So-called set theory was founded by the German mathematician George Cantor (1845-1918). In set theory the notions 'set' and 'element' are primitive. They are not defined in terms of other concepts. Letting A be a set, "$x \in A$" means that x is an element of the set A and "$x \notin A$" means that x does not belong to the set A. The way in which these elements are specified is immaterial: for example, there is no difference between the set consisting of the elements $A = \{2, 3, 5, 7\}$ and $B = \{\text{the set of all prime numbers less then } 11\}$ [17]. Given a certain property, (e.g. having four legs), an interesting question is the following: Does there exist a set whose elements are exactly those having the given property (e.g. having four legs)? We are inclined to answer affirmatively to this question. However accepting an affirmative answer to this question might lead to different versions of Russell's Paradox. For this reason mathematicians have confined each of their discussions to some universal set or universe of discourse [18].

For any element x and a set A in the universe of discourse X it can unambiguously be determined whether $x \in A$ or $x \notin A$. A classical set may be finite, countable or uncountable. It can be described either by listing up the individual elements of the set or by stating a property for the membership. The set $C = \{red, orange, yellow, green, blue\}$ is an example of a finite set that is described by its elements. The set $T = \{x \in \mathbb{Z} | x \geq 0\}$, is an example of countable set that is described by a property. The real interval $[0, 1]$ is an example of an uncountable set. Two sets are very important, namely, the universe X, containing all elements of the universe of discourse, end the empty set $\emptyset$, containing no elements at all.

If $P(x)$ is a predicate stating that x has a property P, then a set can also be denoted by $\{x|P(x)\}$. This leads us to a third way of defining a set A, which utilizes the concept of characteristic function $\mu_A(\cdot)$ given below.

Definition 1 *Characteristic Function :*
Let A be defined in a universe of discourse X.

$$\mu_A : X \rightarrow \{0,1\}$$

is a characteristic function of the set A iff the following holds for all x:

$$\mu_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

Note that the output of the predicate $P(x)$ is Boolean, that is either TRUE or FALSE, whereas the output of a characteristic function μ_A is $\{0,1\}$. These two functions can be related with the following expression: $P(x) \Leftrightarrow (\mu_P(x) = 1)$ [17]. Also note that this definition of a classical set has a natural extension to the definition of a fuzzy set, see Section 3.1.

In the subsequent sections we will give some of the definitions for the set operations, define the relations and briefly define implications. These definitions will be modified in Section 3.1 using characteristic functions.

2.1.1 Operations on Sets

Classical set theory uses several operations like complement, intersection, union etc. The following definitions give the description of the operators which can be defined over set(s).

Definition 2 *Set Operations :*
Let A and B be two classical sets in a universe of discourse X. Then various set operations can be defined as follows:

$$\begin{array}{llcl} \textit{Complement of } A, & A' & = & \{x|x \notin A\} \\ \textit{Intersection of } A \textit{ and } B, & A \bigcap B & = & \{x|x \in A \textit{ and } x \in B\} \\ \textit{Union of } A \textit{ and } B, & A \bigcup B & = & \{x|x \in A \textit{ or } x \in B\} \end{array}$$

$$
\begin{array}{lrcl}
\textit{Difference of } A \textit{ from } B, & A - B & = & \{x | x \in A \textit{ or } x \notin B\} \\
\textit{Symetric difference of } A \textit{ and } B, & A + B & = & (A - B) \bigcup (B - A) \\
\textit{Power set of } A, & P(A) & = & \{U | U \subseteq A\} \\
\textit{Cartesian product of } A \textit{ and } B, & A \times B & = & \{(x, y) | x \in A \textit{ and } x \in B\} \\
\textit{Power } n \textit{ of } A, & A^n & = & \underbrace{A \times \cdots \times A}_{ntimes}
\end{array}
$$

There are some useful properties of these set operators which have been omitted. Interested readers can check any elementary set theory books like [17], [18].

2.1.2 Relations

A relation R can be considered as a set of tuples, where the tuples are ordered pairs. A binary (dyadic) tuple is denoted by (x, y), an example of ternary tuple is (x, y, z), and an example of n-ary tuple is $(x_1, \ldots, x_n)$. Let X be the domain of people, i.e. $X = \{$ Figen, Murat, Filiz, Timur, Selçuk$\}$ and let Y be the domain of height, e.g. $Y = \{1.65, 1.74, 1.78, 1.80, 1.82\}$. Then a relation which can be characterized as "has a height of" defined on $X \times Y$ is given as $R = \{$(Figen, 1.65), (Murat, 1.78), (Filiz, 1.74), (Timur, 1.80), (Selçuk, 1.82)$\}$. Consider the domain of natural numbers, $\mathbb{N}$. Then " $\leq$ " is a relation on $\mathbb{N} \times \mathbb{N}$ and defined as $\{(m, n) | m \leq n\}$. Clearly $(4, 1)$ is not in this relation. As should be noted this relation is a subset of $\mathbb{N} \times \mathbb{N}$. If we compare the definition of a set using predicates and those for a relation given above, we can say that a relation is a set in a product space. Hence just like classical sets, classical relations can be described by characteristic functions.

Definition 3 *Relation :*
Let a relation R be defined in a universe of discourse $X = X_1 \times \cdots \times X_n$.

$$\mu_R : X_1 \times \cdots \times X_n \rightarrow \{0, 1\}$$

is a characteristic function of R iff the following holds for all $x = (x_1, \ldots, x_n) \in X$.

$$\mu_R(x) = \begin{cases} 1, & (x_1, \ldots, x_n) \in R \\ 0, & (x_1, \ldots, x_n) \notin R \end{cases}$$

Relations can have several properties like *reflexive*, *symmetric*, or *transitive.* We exclude their definitions and leave it to the reader to check the references [17], [18] for more details.

2.1.3 Implications

Making statements is a basic human activity; it is part of that complex activities that collectively we call language. A fundamental property of statements is that they may be true or false; which is called the **truth value** of the given statement. Typically, a statement is understood as saying something about the world, and its truth value is assessed on that basis. If the world is in fact the way a statement says it is, then we call that statement **TRUE**; if not, we call it **FALSE**. There can be statements where the truth value can not be judged without supplementary information. Let us call these kind of statements as open propositions for the motivations of the subsequent discussions.

Definition 4 *Variable, Replacement Set :*
A **variable** *is a symbol which represents an unspecified or arbitrary element of a specified set. Such a set is also called the* **replacement set**.

Definition 5 *Open Proposition :*
An **open proposition** *is a declarative sentence which*

(a) contains a finite n number of variables, where $n \geq 1$,

(b) is neither true nor false, but

(c) becomes TRUE or FALSE when the variables are replaced by elements from their replacement set.

Let x represent an element of the set C, where C is the set of names of capital cities in the world and y represents an element of the set S, where S is the set of names of the states in the world. Then the following is an example for an open proposition;

$$x \text{ is the capital city of the state of } y.$$

p	q	p ⇒ q
0	0	1
0	1	1
1	0	0
1	1	1

Table 2.1: Truth table for p implies q

Any proposition can also be related to each other. Sometimes two proposition are so related that the first cannot be TRUE without the second being TRUE as well. For example *"A linear system* $\dot{\mathbf{x}} = A\mathbf{x}$ *is stable if and only if all eigenvalues of* A *has negative real parts."*. There are also conditional propositions like;

IF "e is less then 0.01" AND "$\dot{e}$ is positive" THEN "Set the control signal u to 1.2"

where the propositions "e is less then 0.01" and "$\dot{e}$ is positive" constitute the **antecedent** or the **premise** part of the IF-THEN statement while the proposition "Set the control signal u to 1.2" is the **consequent** part. In a statement like "IF p THEN q", where p and q are are propositions, it is said that p implies q which also has a representation $p \to q$. The truth value of this implication is given in Table 2.1.

There are many different ways to form and relate a set of propositions. The systematic study of how propositions can be related in ways that have repercussions for their respective truth value is called **logic** [19]. In [20] (p.29) the application of logic to the basic switching network is given. Also in [18] (p.183) there is an interesting quotation on the same subject which reveals how one can save a small fortune by application of Boolean Algebra, which is one of the tools in logic. We will skip these because our main concern will be the interpretation of the IF-THEN statements using the given sets defined in the antecedent and consequent part. This topic will also be elaborated in Section 3.1.3.

2.2 Models for Switching Systems

Depending on the problem under consideration there are different modelings of switching systems. Let us see some of the systems which are considered as switching systems.

Example 1 *Collisions:*

Consider the case of the vertical and horizontal motion of a ball of mass m in a room under gravity with constant g. Let x and y denote the horizontal and vertical position of the ball, v_x and v_y denote their respective velocities. In this case, the dynamics are given by,

$$\dot{x} = v_x, \quad \dot{v}_x = 0, \quad \dot{y} = v_y, \quad \dot{v}_y = -mg.$$

Further, upon hitting the boundaries $\{(x,y)|y = 0 \text{ or } y = d_{height}\}$, v_y is instantly set to $-\rho v_y$ where $\rho \in [0,1]$. Likewise, upon hitting $\{(x,y)|x = 0 \text{ or } x = d_{width}\}$ v_x is set to $-\rho v_x$. Here the continuous state $x(\cdot)$ changes impulsively on hitting prescribed regions of the state space [12].

Example 2 *Nonholonomic Integrator :*

Consider the nonholonomic integrator with the following dynamics:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x}) \\ \mathbf{g}(\mathbf{x}) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -x_2 & x_1 \end{bmatrix} \end{aligned}$$

Let $\mathcal{R}_i$, $i = 1, \ldots, 4$ be some regions on $\mathbb{R}^3$, see e.g. (4.10). In each region $\mathcal{R}_i$ the control signal, $\mathbf{u}(\mathbf{x})$, changes as: $\mathbf{u} = \mathbf{u}_i(\mathbf{x})$, when $\mathbf{x} \in \mathcal{R}_i$. In [21] a set of stabilizing $\mathbf{u}_i(\mathbf{x})$ is given.

Example 3 *Transmission System :*

Consider the dynamics of a transmission system whose simplified model is given below [12],

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= [-a(x_2/v) + u] \end{aligned}$$

where x_1 is the ground speed, x_2 is the engine RPM, $\{u \in [0,1]\}$ is the throttle position, and $v \in \{1,2,3,4\}$ is the gear shift position. The function a is positive for positive argument.

In Example 1 and 3, the change in dynamics can be modelled as the parameter change of the system. The change in the dynamics in Example 2 is due to the change in the control algorithms depending on the value of the states. There are other examples for different type of switching systems in [12]. As is the case with all modelling issues, a proper model of a switching system should be general enough to encompass a large variety of physical phenomena, yet be structured enough to present a workable model through which interesting problems can be mathematically framed [7]. Although we will restrict our study to the systems where the dynamic changes is due to the change in control algorithm as in Example 2, we will also include the models on which there is any stability result in the literature.

Most of the studies on the stability are performed on linear switching systems. The following dynamics is a common model for most of these studies;

$$\begin{aligned} \dot{\mathbf{x}} &= A_i \mathbf{x} \\ i^+ &= s(\mathbf{x}, i) \end{aligned} \tag{2.1}$$

where $A_i \in \mathbb{R}^{n \times n}$, $\mathbf{x} \in \mathbb{R}^n$ and $s : \mathbb{R}^n \times \mathcal{I} \to \mathcal{I}$ and $\mathcal{I} = \{1, \ldots, N\}$. Here $s(\cdot)$ is the function that determines when to change the value of i which is the index specifying the operation mode of the system. Example 1 can be modelled by Equation (2.1). In (2.1) there is no control signal $\mathbf{u}$. Consider the following dynamics;

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u} \\ \mathbf{u} &= K_i \mathbf{x} \\ i^+ &= s(\mathbf{x}, i) \end{aligned}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $\mathbf{x} \in \mathbb{R}^n$, and $\mathbf{u} \in \mathbb{R}^m$. Comparing these equations with (2.1) we see that $A_i = A + BK_i$ and $\mathbf{u} = \mathbf{u}_i(\mathbf{x})$ which is determined by the function $s(\cdot)$ and $K_i's$. In a general form the above dynamics can be stated as;

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u} \\ \mathbf{u} &= \mathbf{u}_i(\mathbf{x}) \\ i^+ &= s(\mathbf{x}, i) \end{aligned} \tag{2.2}$$

where $\mathbf{f}(\cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \mathbf{x} \in \mathbb{R}^n$, $\mathbf{g}(\cdot) \in \mathbb{R}^{n \times m}$, and $\mathbf{u} \in \mathbb{R}^m$. This model can be used to express the system given in Example 2. Suppose that the function $s(\cdot)$ in (2.2) is switching to the i^{th} controller when the state $\mathbf{x}$ is in a region specified by $\mathcal{R}_i$. Then the control signal can be expressed with the following equations;

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &= \textstyle\sum_{i=1}^{m} \mathbf{u}_i(\mathbf{x}) S_i(\mathbf{x}) \\ S_i(\mathbf{x}) &= \begin{cases} 1, & \mathbf{x} \in \mathcal{R}_i \\ 0, & \mathbf{x} \notin \mathcal{R}_i \end{cases} \end{aligned} \tag{2.3}$$

where $S_i(\mathbf{x})$ is the characteristic function for the set of points in region $\mathcal{R}_i$ as in Definition 3. The following dynamics can be used to express the system in Example 3.

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{p}) \\ \mathbf{p}^{+} &= s(\mathbf{x}, \mathbf{p}) \end{aligned} \tag{2.4}$$

where $\mathbf{p}$ is the parameter vector. In Example 3 this is equivalent to v, the variable for gear shift position.

2.3 Stability Analysis

Consider the system given by the dynamics (2.1). The stability of a switching system does not only depend on the vector fields $A_i\mathbf{x}$ but also depends on the order of the sequence. Since the case where there is no switching is also a valid switching sequence, every matrix A_i must be stable on their own. Note that, there exists stable matrices A_1 and A_2 such that under a certain switching strategy the overall system turns out to be unstable. The converse is also possible, that is there exists unstable matrices A_1 and A_2 such that under a certain switching strategy the overall system turns out to be stable [13]. Hence it is necessary to develop certain tools to study the stability of the switching systems when the switching sequence is arbitrary, and to incorporate these tools into the controller design so that the closed-loop system becomes stable. Let us consider the Lyapunov Stability Theorem for a Linear Time Invariant (LTI) system.

Theorem 1 *Consider the Lyapunov equation given below;*

$$A^T P + PA = -Q \tag{2.5}$$

A necessary and sufficient condition for a LTI system $\dot{\mathbf{x}} = A\mathbf{x}$ to be strictly stable is that, for any symmetric positive definite matrix Q, the unique matrix P solution of the Lyapunov Equation (2.5) be symmetric positive definite.

Proof 1 *See [22]*

The following Lemma gives a sufficient condition for a given switching system as in (2.1) be stable for arbitrary switching.

Lemma 1 *The switching system (2.1) is exponentially stable if there exists a positive definite matrix P such that $V(\mathbf{x}) = \mathbf{x}^T P\mathbf{x}$ is a common quadratic Lyapunov function for all A_i, i.e.,*

$$A_i^T P + PA_i = -Q_i \tag{2.6}$$

for some positive definite matrices Q_i [3].

So if one can find a common positive definite matrix for a given set of matrices $\{A_1, \ldots, A_N\}$ then the stability of the switching system can be proved. Finding a common positive definite matrix for the Lyapunov function is a hot topic of research see e.g. [3], [23], and [24]. The associated controller design problem is still an open problem and currently under investigation by many researchers. The following subsections define some constraints on the set $\{A_1, \ldots, A_N\}$ to prove the stability of the switching systems using Lemma 1.

2.3.1 Commuting Stable Matrices

In [3] the stability of a switching system defined as in (2.1) is considered. The set of matrices $\{A_1, \ldots, A_N\}$ defining the switching system is assumed to commute pairwise. The related stability result is given below,

Theorem 2 *Consider the switching system in (2.1) with $\mathcal{A} = \{A_1, \ldots, A_N\}$ where the matrices A_i are asymptotically stable and commute pairwise. Then*

i) The system is exponentially stable for any arbitrary switching sequence between the elements of $\mathcal{A}$.

ii) Given a symmetric positive definite matrix P_0, let $P_1, \ldots, P_N$ be the unique symmetric positive definite solution to the Lyapunov equations

$$A_i^T P_i + P_i A_i = P_{i-1}, \quad i = 1, \ldots, N. \tag{2.7}$$

Then the function $V(\mathbf{x}) = \mathbf{x}^T P_N \mathbf{x}$ is a common Lyapunov function for each of the individual systems $\dot{\mathbf{x}} = A_i \mathbf{x}$ $i = 1, \ldots, N$, and hence a Lyapunov function for the switching system (2.1).

iii) *For a given choice of the matrix P_0, the matrices $\{A_1, \ldots, A_N\}$ can be chosen in any order in (2.7) to yield the same solution P_N*

vi) *The matrix P_N can also be expressed in integral form as;*

$$P_N = \int_0^\infty e^{A_N^T t_N} \cdots \left[\int_0^\infty e^{A_2^T t_2} \left[\int_0^\infty e^{A_1^T t_1} P_0 e^{A_1 t_1} dt_1 \right] e^{A_2 t_2} dt_2 \right] \cdots e^{A_N t_N} dt_N \quad (2.8)$$

where, as in iii) above, the order in which the matrices $\{A_1, \ldots, A_N\}$ appear can be replaced by any permutation.

Proof 2 *See [3]* $\square$.

2.3.2 Multiple Lyapunov Functions

Consider the system given in (2.1) for $N = 2$. Let $V_1(\mathbf{x}) = \mathbf{x}^T P_1 \mathbf{x}$ and $V_2(\mathbf{x}) = \mathbf{x}^T P_2 \mathbf{x}$ be the two Lyapunov-like functions for the systems defined by $\dot{\mathbf{x}} = A_1 \mathbf{x}$ and $\dot{\mathbf{x}} = A_2 \mathbf{x}$, respectively, that is these functions decrease along the trajectories in a certain region. Let us assume that the at time $t = 0$ the active dynamics was A_1. Let the system switch to the dynamics A_2 after $\tau_{A_1,1}$ seconds. If the system then waits for a duration of $\tau_{A_2,1}$ seconds to switch back to A_1 we would have a sequence of time durations like $\tau = \{\tau_{A_1,1}, \tau_{A_2,1}, \tau_{A_1,2}, \tau_{A_2,2}, \ldots\}$. Let $t_j = t_{j-1} + \tau_{A_1,j} + \tau_{A_2,j}$ where $t_0 = 0$. If one can orchestrate the switching times so that $V_1(t_{j-1} + \tau_{A_1,j} + \tau_{A_2,j}) < V_1(t_{j-1})$ and $V_2(t_j + \tau_{A_1,j+1}) < V_2(t_j - \tau_{A_2,j})$ for all j then the system is asymptotically stable because of the decreasing tendency. This idea can be generalized to $N > 2$. There are many research on this subject among which [13] uses multiple Lyapunov functions as a tool for analyzing Lyapunov stability. In [25] the multiple Lyapunov function approach is utilized to stabilize a linear system using finite-state hybrid output feedback and a stabilizing switching sequence for a switched linear system with unstable individual matrices is obtained. This method is utilized in [26] and [27] to determine a switching strategy which is called *dwell time switching*. This approach will be considered in Section 7.1.1.

2.3.3 Other Stability Results

In this section we will include some other stability results on switching systems. What follows is some theorems stating the stability conditions for some specific structure for the system dynamics.

Triangular Matrices

Theorem 3 *If there exists a nonsingular matrix* $T \in \mathbb{R}^{n \times n}$ *such that* $\tilde{A}_i = TA_iT^{-1}$*,* $i = 1, \ldots, N$*, are upper-triangular then there exists a positive definite matrix* P *such that (2.6) holds.*

Proof 3 *See [23].*

Commonly symmetrizable Matrices

Theorem 4 *Assume the stable matrices* $A_1, \ldots, A_N \in \mathbb{R}^{n \times n}$ *are commonly symmetrizable, i.e., there exists a common similarity transformation under which every* A_i *is similar to a symmetric matrix. Then the system (2.1) is exponentially stable for arbitrary switching.*

Proof 4 *See [7].*

Theorem 5 *Assume the stable matrices* $A_1, \ldots, A_N \in \mathbb{R}^{n \times n}$ *are nearly commonly symmetrizable, i.e., there exists a exist a nonsingular matrix* T *such that*

$$TA_iT^{-1} = S_i + E_i, \quad S_i = S_i^T, \quad \|E_i\| < \epsilon, \quad \epsilon > 0. \tag{2.9}$$

If ϵ *is sufficiently small, then the switching system, (2.1), is exponentially stable.*

Proof 5 *See [7].*

N = 2 Case

Theorem 6 *Assume that* $N = 2$ *and* $A_1, A_2 \in \mathbb{R}^{n\times n}$. *Then* A_1, A_2 *have a common quadratic Lyapunov function if and only if the matrices* $\alpha A_1 + (1-\alpha)A_2$ *and* $\alpha A_1 + (1-\alpha)A_2^{-1}$ *are stable for all* $\alpha \in [0, 1]$.

Proof 6 *See [28]*

Chapter 3

Fuzzy Systems

In our daily life there are many phenomena which can be classified as being certain and uncertain. The class of uncertain phenomena can be classified once more as random and fuzzy. Therefore, we have three categories of phenomena and their associated mathematical models [29]:

1. **Deterministic Mathematical Models:** This is a class of models where the relationships between objects are fixed or known with certainty. According to our current theory the relation between the distance for a falling object with mass m on Earth and the elapsed time is given by the formula $h = 0.5gt^2$. This is an example for a deterministic mathematical model.

2. **Random (Stochastic) Mathematical Models:** This is a class of models where the relationships between objects are uncertain in nature. In other words it is hard to formulate the relationship between objects using an equation. Instead there is the observations given as a distribution. The relation between the outcome of rolling of dies might be an example for an event that can be modelled by random mathematical model.

3. **Fuzzy Mathematical Models:** This is a class of models where objects and relationships between objects are subjective and expressed with words that are vague to define. Think of someone who describes a person being tall. S/He might not be able to give a precise measure in meters where the boundary for a person

to be tall starts. Even s/he gives a measure it will be a subjective decision and might well be different for someone else.

A fuzzy system is a real time expert system implementing a part of a human operator's or process engineer's expertise which does not lend itself to being easily expressed in a difference/differential equation but rather in situation/action rules. However, fuzzy control differs from main-stream expert system technology in several aspects. One main feature of fuzzy control systems is that there are symbolic IF-THEN rules and qualitative, fuzzy variables and values such as:

if "pressure is high" *and* "slightly increasing" *then* "energy supply is negative medium".

Most of the researchers in the area of fuzzy control have a strong control engineering and systems theory background. From their perspective, fuzzy control can be seen as a heuristic and modular way of defining nonlinear, table based systems. Reconsider the rule above : it is nothing but an informal "nonlinear Proportional Derivative(PD)-element".

A representation theorem, mainly due to Kosko [9], states that any continuous nonlinear function can be approximated as closely as needed with a finite set of fuzzy variables, values, and rules. Later X.-J. Zeng and M. G. Singh published their work on the approximation problem of SISO and MIMO fuzzy systems (see [10], [11]). These results describe the representational power of fuzzy control in principle, but it does not answer the questions of how many rules are needed and how they can be found.

It is generally agreed that an important point in the evolution of the modern concept of uncertainty was the publication of a seminal paper by Lotfi A. Zadeh [8]. In this paper, Zadeh introduced a theory whose objects-*fuzzy sets*-are sets with boundaries that are not precise. The membership in a fuzzy set is not a matter of affirmation or denial, but rather a matter of *degree*.

The literature in fuzzy control has been growing rapidly in recent years, making it difficult to present a comprehensive survey of the wide variety of applications that have been made. Historically, the important milestones in the development of fuzzy control may be summarized as shown in Table 3.1, [30], [31]. The first successful

1972	Zadeh	A rationale for fuzzy control [32]
1973	Zadeh	Linguistic approach [33]
1974	Mamdani & Assilian	Steam engine control [34]
1976	Rutherford *et al.*	Analysis of control algorithms [35]
1977	Ostergaard	Heat exchanger and cement kiln control [36]
1977	Willaeys *et al.*	Optimal fuzzy control [37]
1979	Komolov *et al.*	Finite automation [38]
1980	Tong *et al.*	Wastewater treatment process [39]
1983	Takagi and Sugeno	Derivation of fuzzy control rules [40]
1984	Sugeno and Murakami	Parking control of a model car [41]
1985	Togai and Watanabe	Fuzzy chip [42]
1986	Yamakawa	Fuzzy controller hardware system [43]
1988	Dubois and Prade	Approximate reasoning [44]

Table 3.1: Some important studies in fuzzy control.

industrial application of the Fuzzy Logic Controller (FLC) was a cement kiln control system developed by the Danish cement plant manufacturer F. L. Smidth in 1979.

FLCs can be used in various ways. One of them is to use the FLC directly as the controller for the system under consideration. The other choice is to use a standard controller like PID or lead-lag and tune the parameters of this controller with a Fuzzy Logic System (FLS). A few of the studies on parameter tunings are given in [31], [45], [46] and [47]. In [46] the PID parameters are tuned according to some rules derived by considering a typical system output. In [47] some changes to the model of fuzzy PI controller is introduced to reduce the overshoot even more. In [31] a heuristic approach is presented to design the fuzzy controllers which tune the parameters of a PID and a Lead-Lag controller.

In the following sections we will first give some necessary definitions and background to analyze fuzzy systems. After that a short overview will be presented on how fuzzy set theory is used in implications to determine the relations between some variables of interest. Next the class of fuzzy systems and their mathematical representation will follow.

3.1 Fuzzy Set Theory

Fuzzy set theory can be seen as the extension of classical set theory in many aspects. In classical set theory the characteristic function which determine whether an object belongs to a set or not is determined depending on the value of that function being either 0 or 1. Let us repeat the definition of characteristic function given in the previous chapter. Let X be the universe of discourse, and let $A \subset X$. Then the characteristic function, $\mu_A : X \to \{0, 1\}$, of the set A is defined as:

$$\mu_A(x) = \begin{cases} 1, & x \in \mathrm{A} \\ 0, & x \notin \mathrm{A} \end{cases}$$

Fuzzy sets on the other hand fills the gap for sets with boundaries that can not be defined precisely. To motivate fuzzy reasoning, let us try to define the sets described by the phrases like "tall man", or "hot weather". An appropriate characteristic function for such sets described by such vague expression can be obtained by expanding its range to the closed interval $[0, 1]$. The characteristic function of a fuzzy set is called the membership function (MF). In Figure 3.1 the 3 mostly encountered membership function profiles are given. These membership functions are chosen to have different mean values for their support which will be defined later. This choice is for better visualization.

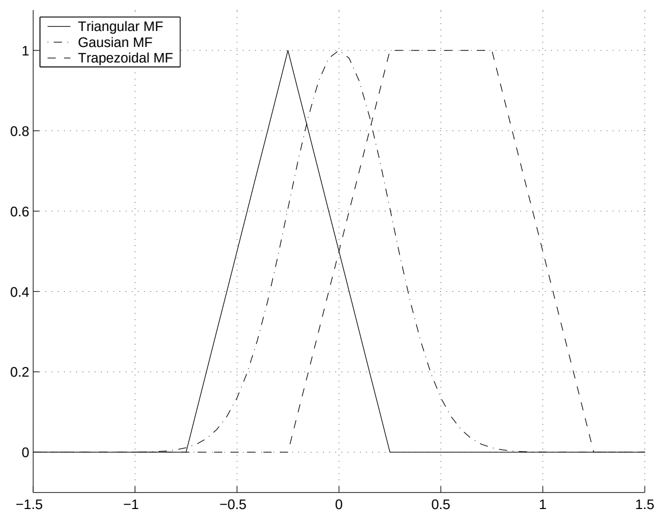


Figure 3.1: The general structure of a fuzzy systems.

In Figure 3.1, the horizontal axis represents the quantity of the concept being defined. For example the temperature, which can be measured, might be the horizontal axis in defining the membership function of the set "hot weather". We will represent the membership functions of the fuzzy set $\mathcal{S}$ by $\mu_{\mathcal{S}}(\cdot)$.

If the universe of discourse of the membership function is continuous as in the "temperature", then the function can be represented as a continuous or piece-wise continuous functions as in Figure 3.1. In most of the cases the control algorithms will be implemented by one or the other digital computing device which have a finite number of bit resolution. Hence even if we have measurements of temperature, its values after passing from an analog to digital converter will be quantized to a discrete universe of discourse. If the domain is discrete then we have tuples to represent fuzzy sets. Interested readers can resort to [30] or any related book such as [17], [48] for definitions fuzzy sets/relations whose universe of discourse is discrete and for more information on fuzzy systems. Here we will give the definition and examples of membership functions for continuous universe of discourse.

Definition 6 *Support :*
The support of a function, $\mu_{\mathcal{S}}(\cdot) : \mathcal{X} \longrightarrow [0,1]$ *is the set* $\{\mathbf{x}|\mu_{\mathcal{S}}(\mathbf{x}) > 0\}$ $\square$.

Definition 7 *Convex Function:*
Let $x_1, x_2 \in \mathcal{X} = \mathbb{R}$ *be any two elements within the support of the function,* $\mu_{\mathcal{S}}(\cdot) : \mathcal{X} \longrightarrow [0,1]$. *If* $\mu_{\mathcal{S}}(\alpha x_1 + (1-\alpha)x_2) \geq \alpha\mu_{\mathcal{S}}(x_1) + (1-\alpha)\mu_{\mathcal{S}}(x_2)$ *is satisfied* $\forall\ x_1, x_2$, $\forall\ \alpha \in [0,1]$ *then the function* $\mu_{\mathcal{S}}(\cdot)$ *is convex. This definition can be extended to* $\mathcal{X} = \mathbb{R}^n$. $\square$

Definition 8 *Membership Function :*
The membership function $\mu_{\mathcal{S}}(\cdot)$ *of a fuzzy set* $\mathcal{S}$ *is a function* $\mu_{\mathcal{S}}(\cdot) : \mathcal{X} \longrightarrow [0,1]$. *The form of a membership function can be arbitrary but as a general practice, it is required to have a convex membership function with finite support to ease certain manipulations* $\square$.

In Figure 3.1 some of the membership function profiles are given. These membership functions can be expressed as in (3.1) where m is the point the membership function is equal to 1 and w is a measure of the spread of the support. Tg stands for triangular,

Ga for Gaussian, and Tz for trapezoidal membership function. While a and d are the borders of the support, b and c determine the interval where the trapezoidal membership function is 1.

$$\mu_{Tg}(x) = \begin{cases} 1 - \frac{|x-m|}{w}, & |x-m| < w \\ 0, & |x-m| \geq w \end{cases}$$

$$\mu_{Ga}(x) = e^{\frac{-|x-m|^2}{2w^2}}$$

$$\mu_{Tz}(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a < x < b \\ 1, & b < x < c \\ \frac{x-d}{c-d}, & c < x < d \\ 0, & d < x \end{cases} \tag{3.1}$$

A fuzzy set $\mathcal{S}$ might be defined as those elements in $\mathcal{X}$ where the associated membership function $\mu_{\mathcal{S}}(\mathbf{x}) > 0$ i.e. $\mathcal{S} = \{\mathbf{x} | \mu_{\mathcal{S}}(\mathbf{x}) > 0, \ \mathbf{x} \in \mathcal{X}\}$. In fact those elements having a membership value of zero is not in the set. By defining the fuzzy set as $\mathcal{S} = \{\mathbf{x} | \mu_{\mathcal{S}}(\mathbf{x}) > 0, \ \mathbf{x} \in \mathcal{X}\}$ we make no difference with the definition of a classical set made by the use of characteristic function and have ignored the graded membership value of the element to the set. Therefore it is best to define a fuzzy set as a collection of tuples.

Definition 9 *Fuzzy Set [17]:*
Let $\mathcal{X}$ be the universe of discourse and $\mathcal{S} \subset \mathcal{X}$ be a set which has membership function is $\mu_{\mathcal{S}}(\cdot) : \mathcal{X} \longrightarrow [0,1]$. Then the fuzzy set $\mathcal{S}$ is defined as a collection of tuples given as:

$$\mathcal{S} = \{(x, \mu_{\mathcal{S}}(x)) | x \in \mathcal{X}\}$$

There are many different approaches to determine the membership function of a fuzzy set. These approaches depend mainly on the problem at hand. Depending on the problem one can use the methods which are based on frequency of the occurrence of the object as in [29] or use some clustering algorithms like fuzzy-c means as in [31]. Some other approaches can be found in [49]. All of these different approaches lead us to the following definitions which are used to classify fuzzy sets so that some of the operations are simplified.

Let us give some more definitions on membership function which will be used in the following sections. Let $\mathcal{I}_n$ denote the set $\mathcal{I}_n = \{1, \dots, n\}$.

Definition 10 *Normal Membership Function:*
Let $\mathcal{S}$ be a fuzzy set defined by the membership function, $\mu_{\mathcal{S}}(\cdot) : \mathcal{X} \longrightarrow [0, 1]$, $i \in \mathcal{I}_n$. Then the membership function, $\mu_{\mathcal{S}}(\cdot)$, is said to be normal if $\sup \mu_{\mathcal{S}}(\cdot) = 1.\Box$

Definition 11 *Completeness :*
Let $\mathcal{S}_1, \dots, \mathcal{S}_n$ be fuzzy sets defined by the membership functions, $\mu_{\mathcal{S}_i}(\cdot) : \mathcal{X} \longrightarrow [0, 1]$, $i \in \mathcal{I}_n$. Then these fuzzy sets are said to be complete on $\mathcal{X}$ if for any $x \in \mathcal{X}$, there exists $\mathcal{S}_i$ such that $\mu_{\mathcal{S}_i}(x) > 0.\Box$

Definition 12 *Consistency :*
Let $\mathcal{S}_1, \dots, \mathcal{S}_n$ be fuzzy sets defined by the membership functions, $\mu_{\mathcal{S}_i}(\cdot) : \mathcal{X} \longrightarrow [0, 1]$, $i \in \mathcal{I}_n$. The fuzzy sets are said to be consistent if $\mu_{\mathcal{S}_i}(x_0) = 1$ for some $x_0 \in \mathcal{X}$, then for all $j \neq i$ $\mu_{\mathcal{S}_j}(x_0) = 0.\Box$

Another problem in fuzzy system design is the optimization of the functional form. In these approaches the membership functions are assumed to have a predefined structure as in Figure 3.1. After the parametrization of these membership functions, the parameters are optimized to minimize a predefined cost function which might aim to reduce the number of partitions of the universe of discourse or aim to increase the performance of the system.

3.1.1 Operations on Fuzzy Sets

The expression, "$\mathcal{A}$ *AND* $\mathcal{B}$" in propositional logic is true if and only if both expressions $\mathcal{A}$ and $\mathcal{B}$ are true. In fuzzy set theory their interpretation is not so simple, because graded characteristic functions (membership function) are used. Zadeh proposed [8] the following definitions for the union, intersection and complement operations on fuzzy sets.

Definition 13 *Let $\mathcal{X}$ be the universe of discourse, $\mathcal{A}, \mathcal{B} \subset \mathcal{X}$ be two fuzzy sets and let $\mu_{\mathcal{A}}(\cdot)$ and $\mu_{\mathcal{B}}(\cdot)$ be their membership functions, respectively. Then the membership function of the sets $\mathcal{A} \cup \mathcal{B}$, $\mathcal{A} \cap \mathcal{B}, \mathcal{A}^c$ may be given as follows :*

$$\begin{array}{lll} \forall x \in X: \ \mu_{\mathcal{A}\cap\mathcal{B}}(x) & = & \min(\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)), \\ \forall x \in X: \ \mu_{\mathcal{A}\cup\mathcal{B}}(x) & = & \max(\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)), \\ \forall x \in X: \ \mu_{\mathcal{A}^c}(x) & = & 1 - \mu_{\mathcal{A}}(x). \qquad \Box \end{array}$$

If we update the list of set operations for classical sets given in the Chapter 2, with the definition of Zadeh we have the following list.

Fuzzy Set $\mathcal{A}$,	$\mathcal{A}$	$=$	$\{(x, \mu_{\mathcal{A}}(x)) \mid x \in \mathcal{X}\}$
Fuzzy Set $\mathcal{B}$,	$\mathcal{B}$	$=$	$\{(x, \mu_{\mathcal{B}}(x)) \mid x \in \mathcal{X}\}$
Fuzzy Set $\mathcal{U}$,	$\mathcal{U}$	$=$	$\{(x, \mu_{\mathcal{U}}(x)) \mid x \in \mathcal{X}\}$
Complement of $\mathcal{A}$,	$\mathcal{A}'$	$=$	$\{(x, 1 - \mu_{\mathcal{A}}(x)) \mid x \in \mathcal{X}\}$
Inersection of $\mathcal{A}$ and $\mathcal{B}$,	$\mathcal{A} \bigcap \mathcal{B}$	$=$	$\{(x, min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)\}) \mid x \in \mathcal{X}\}$
Union of $\mathcal{A}$ and $\mathcal{B}$,	$\mathcal{A} \bigcup \mathcal{B}$	$=$	$\{(x, max\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)\}) \mid x \in \mathcal{X}\}$
Difference of $\mathcal{A}$ from $\mathcal{B}$,	$\mathcal{A} - \mathcal{B}$	$=$	$\{(x, max\{\mu_{\mathcal{A}}(x), 1 - \mu_{\mathcal{B}}(x)\}) \mid x \in \mathcal{X}\}$
Symetric difference of $\mathcal{A}$ and $\mathcal{B}$,	$\mathcal{A} + \mathcal{B}$	$=$	$(\mathcal{A} - \mathcal{B}) \bigcup (\mathcal{B} - \mathcal{A})$
Power set of $\mathcal{A}$,	$P(\mathcal{A})$	$=$	$\{(x, \mu_{\mathcal{U}}(x)) \mid \mu_{\mathcal{A}}(x)) \geq \mu_{\mathcal{U}}(x), \ x \in \mathcal{X}\}$
Cartesian product of $\mathcal{A}$ and $\mathcal{B}$,	$\mathcal{A} \times \mathcal{B}$	$=$	$\{(x_1, x_2, min\{\mu_{\mathcal{A}}(x_1), \mu_{\mathcal{B}}(x_2)\}) \mid x \in \mathcal{X}\}$
Power n of $\mathcal{A}$,	$\mathcal{A}^n$	$=$	$\underbrace{\mathcal{A} \times \cdots \times \mathcal{A}}_{ntimes}$

Note that in the definition of the cartesian product of the set $\mathcal{A}$ and $\mathcal{B}$, a fuzzy set defined by $\mathcal{A} \times \mathcal{B}$ is not tuple but triple where the membership function defining the product has a domain of product space. This will be clear in Section 3.1.2 in which we give the definition of a relation. If the values of $\mu_{\mathcal{A}}(x)$ and $\mu_{\mathcal{B}}(x)$ are restricted to the set $\{0, 1\}$ then the results reduce to the classical set operations. Therefore this is a very simple extension of the classical set operations. There are other extensions, for example,

$$\begin{array}{lll} \forall x \in \mathrm{X}: \ \mu_{\mathcal{A}\cap\mathcal{B}}(x) & = & \mu_{\mathcal{A}}(x) \cdot \mu_{\mathcal{B}}(x), \\ \forall x \in \mathrm{X}: \ \mu_{\mathcal{A}\cup\mathcal{B}}(x) & = & \min(1, \mu_{\mathcal{A}}(x) + \mu_{\mathcal{B}}(x)). \end{array}$$

There is a class of operators called *t-norm* for intersection operations and another called *s-norm* for union operations. Readers can resort to [17], [30], and [48] for a more comprehensive literature survey on fuzzy sets and operations defined on them.

3.1.2 Fuzzy Relations

In Chapter 2 we have described the classical relation as being a classical set in a product space. This applies also for fuzzy relations. As we can define fuzzy sets with membership functions, so we can do with fuzzy relations.

Definition 14 *Fuzzy Relation :*
Let $\mathcal{U} \times \mathcal{V}$ be the universe of discourse and $\mu_{\mathcal{R}} : \mathcal{U} \times \mathcal{V} \longrightarrow [0,1]$ be the membership function of a relation $\mathcal{R}$. (As in the case of fuzzy sets the membership function of a relation can have any arbitrary form.) Then the fuzzy relation $\mathcal{R}$ is defined as a collection of triple as:

$$\mathcal{R} = \{(u, v, \mu_{\mathcal{R}}(u,v)) | u \in \mathcal{U} \text{ and } v \in \mathcal{V}\}$$

This definition can be extended to relations whose membership function is defined as $\mu_{\mathcal{R}} : \mathcal{U}_1 \times \cdots \times \mathcal{U}_n \longrightarrow [0,1]$. □

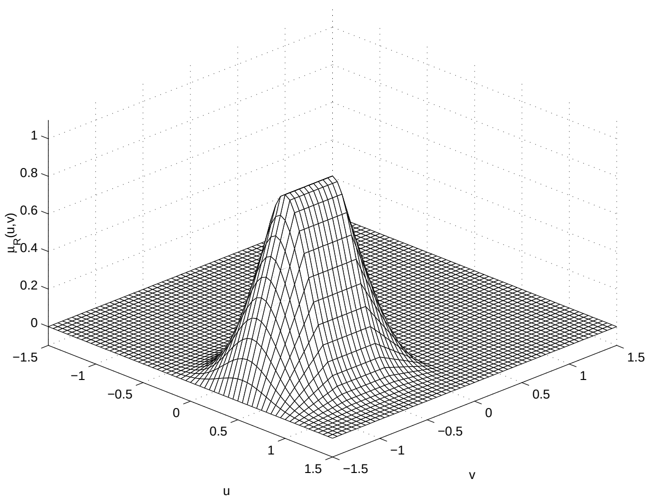


Figure 3.2: An example of a relation $\mathcal{R}$ between u and v.

In Figure 3.2 a relation between u and v is given. In fact this relation can be defined as the cartesian product of two fuzzy sets as;

$$\mathcal{R} = \{(u, v, min\{\mu_{Ga}(u), \mu_{Tz}(v)\})|u \in \mathcal{U} \text{ and } v \in \mathcal{V}\}$$

Here $\mu_{\mathcal{R}}(u, v) = min\{\mu_{Ga}(u), \mu_{Tz}(v)\}$ where $\mu_{Ga}(\cdot)$ is a Gausian while $\mu_{Tz}(\cdot)$ is a trapezoidal membership function whose parameterized forms are given in Section 3.1. There are two very important operations on fuzzy sets and fuzzy relation, namely projection and cylindrical extension. Let $\mathcal{R}$ be a fuzzy relation on $\mathcal{U} = \mathcal{U}_1 \times \cdots \times \mathcal{U}_n$. Let $(i_1, \ldots, i_k)$ be a subsequence of $(1, \ldots, n)$, and let $(j_1, \ldots, j_l)$ be the complementary subsequence of $(1, \ldots, n)$, i.e. $(j_1, \ldots, j_l) = (1, \ldots, n) - (i_1, \ldots, i_k)$. Let $\mathcal{V} = \mathcal{U}_{i_1} \times \cdots \times \mathcal{U}_{i_k}$.

Definition 15 *Projection :*
The fuzzy set/relation obtained by the projection of $\mathcal{R}$ on $\mathcal{V}$ is defined by

$$proj\ \mathcal{R}\ on\ V = \{(u_{i_1}, ..., u_{i_k}, \sup_{u_{j_1}, ..., u_{j_l}} \mu_{\mathcal{R}}(u_1, ..., u_n))|u \in \mathcal{U}\}$$

where $\mu_{\mathcal{R}}(\cdot)$ is the membership function defining the relation $\mathcal{R}$. □

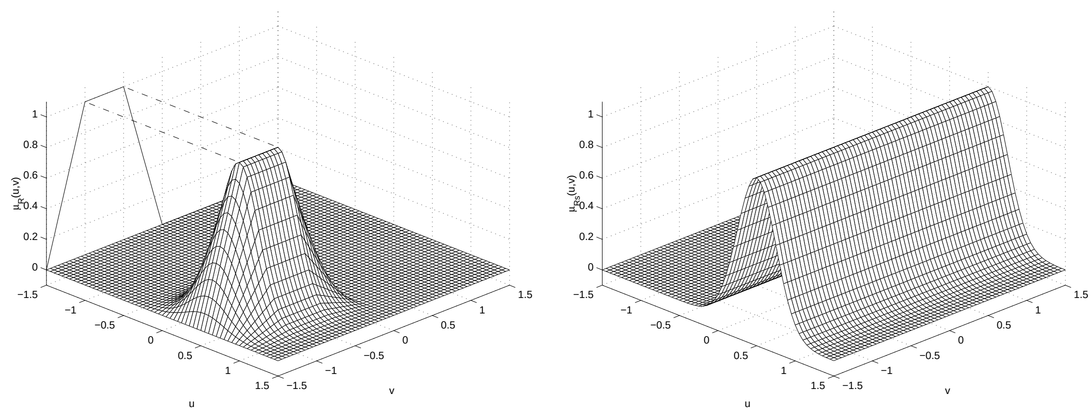
Definition 16 *Cylindrical Extension :*
Let $\mathcal{S}$ be a fuzzy set defined on $\mathcal{V}$. Then the cylindrical extension of $\mathcal{S}$ into $\mathcal{U}$ is a relation defined on $\mathcal{U}$ and it is given as:

$$ce(\mathcal{S}) = \{(u_1, ..., u_n, \mu_{\mathcal{S}}(u_{i_1}, ..., u_{i_k}))|u \in \mathcal{U}\}$$

where $\mu_{\mathcal{S}}(\cdot)$ is the membership function defining the fuzzy set $\mathcal{S}$. □

In Figure 3.3 the projection and the cylindrical extension is exemplified for $n = 2$. Note that the dimension of the set obtained by the projection operation is smaller than the dimension of the relation being projected. In Figure 3.3(a) the dimension of the relation is 2 while the dimension of the set obtained by projection is 1. In Figure 3.3(b) we observe that the dimension of the fuzzy set being extended is less then the dimension of the relation obtained by the cylindrical extension operation.

Suppose now that we have the membership function of the relation between the "temperature" and "quality of crop". Let us also have the temperature measurements



(a) (b)

Figure 3.3: (a)Projection of the $\mathcal{R}$ on $\mathcal{V}$, (b) Cylindrical extension of $\mathcal{S}$ into $\mathcal{U}$.

for the season expressed as a fuzzy set. Knowing these two facts- a related question is to deduce some information about "the quality of the crop" for that season. An answer to this question may be given by using the composition and the combination of a fuzzy set and a fuzzy relation with the aid of cylindrical extension and projection. This operation is denoted by the symbol "$\circ$". A fuzzy relation, as its name indicates, is a relation between its subsets. This operation, which is called composition operation, is necessary when one knows the relation of a certain subject and some related facts and wants to calculate the facts which s/he does not know [31].

Definition 17 *Composition operation :*
Let $\mathcal{S}$ be a fuzzy set defined on $\mathcal{V}$ and $\mathcal{R}$ be a fuzzy relation defined on $\mathcal{U}$. Then the composition of $\mathcal{S}$ and $\mathcal{R}$ resulting in a fuzzy set $\mathcal{T}$ defined on $\mathcal{U}-\mathcal{V}$ is given by

$$\mathcal{T} = \mathcal{S} \circ \mathcal{R} = \ \mathit{proj}\ (\ \mathit{ce}(\mathcal{S}) \cap \mathcal{R}\ )\ \mathit{on}\ \mathcal{V}\Box$$

An example is given in Figure 3.4. On the right wall of the figure the membership function $\mu_{\mathcal{S}}(u)$, which has a triangular shape, is given. The membership function of the relation $\mu_{ce(\mathcal{S})\cap\mathcal{R}}(u,v)$ is given on the product space. The membership function on the left wall of the figure is the projection of the relation defined by $\mu_{ce(\mathcal{S})\cap\mathcal{R}}(u,v)$, i.e. $\mu_{\mathcal{T}}(u,v)$.

Fuzzy systems are defined by IF-THEN statements and by the related operators. An IF-THEN statement defined with fuzzy sets constitute a relation among the variables

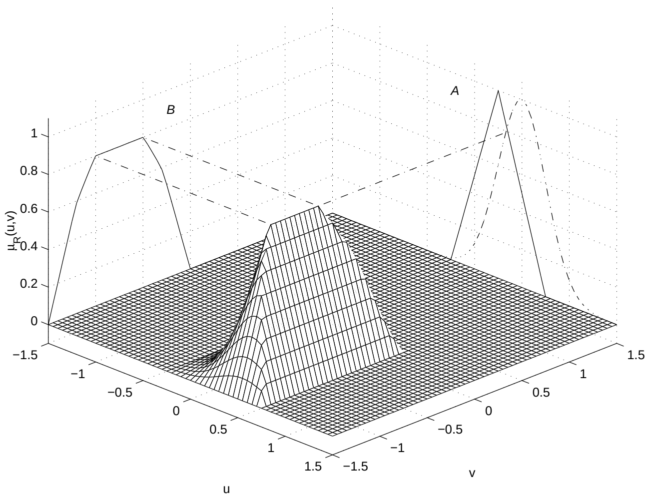


Figure 3.4: The fuzzy set $\mathcal{B}$, obtained by the projection of the composition of $\mathcal{A}$ with $\mathcal{R}$.

of interest. The composition operation is used to extract information about the THEN part of the statement.

3.1.3 Fuzzy Logic and Approximate Reasoning

In classical logic the truth value of $p \Rightarrow q$ is defined as shown in the Table.2.1. This implication or IF-THEN (if p then q) statement can be expressed with different combination of union, intersection and complement operations.

Some equivalent expressions of $p \Rightarrow q$ would be $\neg\ p\ \vee\ q$ and $(p\ \wedge\ q) \vee\ \neg\ p$ where "$\vee$", "$\wedge$" and "$\neg$" stand for union, intersection and complement operations, respectively. Here p and q can be considered as sets which are defined by a characteristic function whose domain is the set $\{0, 1\}$. To extend the implication operation into fuzzy domain again the codomain of the characteristic function of p and q must be extended to the closed unit interval, [0,1], and instead of the classical set operations, their fuzzy counterparts must be used. Having done these, one can calculate the truth value of an IF-THEN statement. A typical example which may be encountered in fuzzy control might be the following type of statement,

if "pressure is rather high" then "energy supply is negative big".

The classical set operators $p\ \Rightarrow\ q$, $\neg\ p\ \vee\ q$, and $(p\ \wedge\ q) \vee\ \neg\ p$, have all the same truth table. When these operators are replaced with fuzzy operators, then the equivalent of $p \Rightarrow q$ may differ according to the selected union, intersection and complement operators. There are a number of relations that can be used to represent the meaning of an IF-THEN statement. Some of these are equivalent to the logical implication $p \Rightarrow q$ and some are not.

Suppose that there is an IF-THEN statement as given below and it is desired to calculate the truth value of this implication.

$$\text{IF } \mathcal{A} \text{ THEN } \mathcal{B}.$$

Let $\mathcal{A}$ and $\mathcal{B}$ be two fuzzy propositions defined on $\mathcal{X}$ and $\mathcal{Y}$, respectively. Let $\mathcal{R}$ be the relation defined with the above IF-THEN statement. One way of expressing $\mathcal{R}$, by using the so-called Mamdani implication, is given as follows:

$$\begin{aligned} \mathcal{R} &= \mathrm{ce}(\mathcal{A}) \wedge \mathrm{ce}(\mathcal{B}), \\ \mu_{\mathcal{R}}(x, y) &= \min(\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(y)) \text{ for } x \in \mathcal{X}, y \in \mathcal{Y}. \end{aligned} \tag{3.2}$$

There are other implication operations. An extended list of these can be found in [17], [50].

3.2 Models for Fuzzy Systems

Fuzzy systems is a name for the systems which have a direct relationship with fuzzy concepts (like fuzzy sets, linguistic variables, and so on) and fuzzy logic. The most popular fuzzy systems in the literature may be classified into three types: pure (Mamdani type) fuzzy systems, Takagi-Sugeno type fuzzy system, and fuzzy systems with fuzzifier and defuzzifier.Figure 3.5 shows the general structure for fuzzy systems. The inner doted rectangle shows the Mamdani type fuzzy systems that is its input and output are just fuzzy sets. The whole system can be categorized as fuzzy system with fuzzifier and defuzzifier. A fuzzy system can be designed as a part of a control system, e.g. the controller in Figure 6.1 can be a fuzzy system.

Let us describe the duties of the sub-blocks. In real world we take measurements of

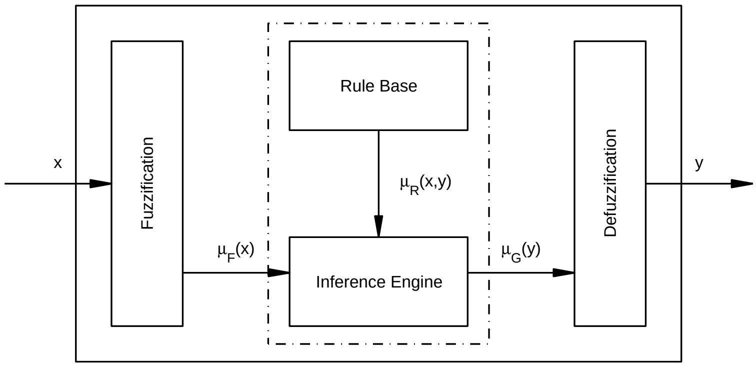


Figure 3.5: The general structure of a fuzzy systems.

the states or the output of a system. Fuzzification is the process of association of a given measurement, such as x, with a membership function $\mu_F(x)$. With the set defined by $\mu_F(x)$ it is straightforward to apply the operations defined on fuzzy sets so far. When we have obtained the set defined by $\mu_G(y)$ as the output of the Inference Engine it requires to be transferred back to the measurement domain of the real world. This process is called defuzzification. The block Rule Base stores all the rules expressed in IF-THEN statements. The Inference Engine has two input, one is the relations expressed by the rules the other is the fuzzy set obtained by fuzzification operation. The purpose of this block is to make approximate reasoning using its inputs. In fuzzy logic and approximate reasoning, there are two important fuzzy implication inference rules named the Generalized Modus Tollens (GMT) and the Generalized Modus Ponens (GMP):

$$\begin{aligned} Premise1 &: \mathcal{B} \\ Premise2 &: \text{IF } \mathcal{A} \text{ THEN } \mathcal{B} \\ Consequence &: \bar{\mathcal{A}} \qquad\qquad (GMT) \end{aligned}$$

$$\begin{aligned} Premise1 &: \mathcal{A} \\ Premise2 &: \text{IF } \mathcal{A} \text{ THEN } \mathcal{B} \\ Consequence &: \bar{\mathcal{B}} \qquad\qquad (GMP) \end{aligned}$$

The GMT is closely related to the backward goal-driven inference which is commonly used in expert systems, especially in the realm of medical diagnosis. The GMP is closely related to the forward data-driven inference which is particularly useful in the fuzzy logic controllers. So GMP is used in this thesis as the implication inference rule. The process of GMP or GMT as described above is called as firing the associated rule. As the consequence of this process, a fuzzy set is obtained.

In a fuzzy system generally there are more then one rule in the rule base. There are two ways to utilize all the rules. First is to take the union of the fuzzy sets obtained by firing each rule. This is called individual-rule-firing based inference. Second is to fire the rule obtained by taking the union of the relations defined by each rule which is called composition based inference. Although in general these two approaches will lead to different results, in [51] it is shown that under certain fuzzification operation the two is equivalent.

Let us make the following definition for the sake of completeness. A rule base where there is no IF-THEN rules with the same antecedent part and a different consequent part are said to be consistent. If for any input to the fuzzy system there is at least one rule to be fired then the rule base is said to be complete.

We will review the Mamdani type and the Takagi-Sugeno type of fuzzy systems in the subsequent sections.

3.2.1 Mamdani Type Fuzzy Systems

The basic configuration for a pure (Mamdani type) FLS is shown in Figure 3.5 within the inner part of the dotted rectangle. Here $\mathbf{x} \in \mathcal{F} \subset \mathbb{R}^n$ represents the input and $\mathbf{y} \in \mathcal{G} \subset \mathbb{R}^m$ represents the output. The fuzzy rule base consists of a collection of fuzzy IF-THEN rules. Each of these rules define a fuzzy relation $\mathcal{R}_l$, $l = 1, \ldots, r$ with an associated membership function $\mu_{\mathcal{R}_i}(\mathbf{x}, \mathbf{y}) : \mathcal{F} \times \mathcal{G} \longrightarrow [0, 1]$. The fuzzy IF-THEN rules are of the following form:

$$\begin{aligned} \mathcal{R}_l : \text{IF} \quad & x_1 \text{ is } F_1^l \text{ AND } \cdots \text{ AND } x_n \text{ is } F_n^l \\ & \text{THEN } y \text{ is } G^l. \end{aligned} \tag{3.3}$$

where F_i^l and G^l are fuzzy sets, $\mathbf{x} = (x_1, ..., x_n)^T \in \mathcal{F}$ and $y \in \mathcal{G}$ are input and output linguistic variables, respectively, and $l = 1, 2, ..., r$.

This model is rather abstract because we have only measurements and have to convert that into fuzzy sets to use the Mamdani type fuzzy system. This is the reason we have a fuzzification and a defuzzification blocks to establish the interface to the real world. In [31] the computational procedures for Mamdani type fuzzy systems and fuzzy systems with fuzzification and defuzzification block are explained in detail.

3.2.2 Takagi-Sugeno Type Fuzzy Systems

Instead of considering the fuzzy IF-THEN rules as in (3.3), Takagi and Sugeno proposed to use the following fuzzy IF-THEN rules:

$$\begin{aligned} \mathcal{R}_l : \text{IF} \quad & x_1 \text{ is } F_1^l \text{ AND } \cdots \text{ AND } x_n \text{ is } F_n^l \\ & \text{THEN } y^l = c_0^l + c_1^l \times x_1 + \cdots + c_n^l \times x_n. \end{aligned} \tag{3.4}$$

where F_i^l are fuzzy sets, c_i are real-valued parameters, y^l is the system output due to rule $\mathcal{R}_l$. The static input/output relation given above may also be extended to dynamic one as given below:

$$\begin{aligned} \mathcal{R}_l : \text{IF} \quad & x_1 \text{ is } F_1^l \text{ AND } \cdots \text{ AND } x_n \text{ is } F_n^l \\ & \text{THEN} \quad \begin{aligned} \dot{\mathbf{x}} &= A_l\mathbf{x} + B_l\mathbf{u} \\ \mathbf{y} &= C_l\mathbf{x} + D_l\mathbf{u} \end{aligned} \end{aligned} \tag{3.5}$$

where $A_l \in \mathbb{R}^{n\times n}$, $B_l \in \mathbb{R}^{n\times m}$,$C_l \in \mathbb{R}^{k\times n}$, $D_l \in \mathbb{R}^{k\times m}$, $u \in \mathbb{R}^m$, $\mathbf{x} \in \mathbb{R}^n$, $y \in \mathbb{R}^k$, and $l = 1, \ldots, r$.

In the literature there is a vast amount of results which consider the latter type as the representation of a nonlinear system obtained by local linearization. This interpretation of the Takagi-Sugeno model yields fruitful solution to many problems like stabilization or function approximation. Nevertheless using the locally linearized system might not be the optimum approach, see [52]. Suppose that each of the fuzzy sets F_i^l has a membership function $\mu_{F_i^l}(x_i)$. Then the membership function of the relation in the premise part of (3.5) will be calculated as:

$$\mu_{P_l}(\mathbf{x}) = \prod_{i=1}^{n} \mu_{F_i^l}(x_i) \tag{3.6}$$

in which the AND operation was performed by the arithmetic multiplication operation. Then considering all of the rules and combining the resulting membership function will lead us to the following output of the Takagi-Sugeno type fuzzy systems:

$$\begin{array}{rcl}\dot{\mathbf{x}} & = & \sum_{l=1}^{r} \alpha_l(\mathbf{x})(A_l\mathbf{x} + B_l\mathbf{u}) \\ \mathbf{y} & = & \sum_{l=1}^{r} \alpha_l(\mathbf{x})(C_l\mathbf{x} + D_l\mathbf{u})\end{array} \tag{3.7}$$

where the so-called Fuzzy Basis Function (FBS) $\alpha_l(\mathbf{x})$ are defined as follows,

$$\alpha_l(\mathbf{x}) = \frac{\mu_{P_l}(\mathbf{x})}{\sum_{i=1}^{r} \mu_{P_i}(\mathbf{x})} \tag{3.8}$$

If the fuzzy sets in the antecedent part of the IF-THEN statements are normal, complete and consistent then the above expressions simplifies to the following equations [10]:

$$\begin{array}{rcl}\dot{\mathbf{x}} & = & \sum_{l=1}^{r} \mu_{P_l}(\mathbf{x})(A_l\mathbf{x} + B_l\mathbf{u}) \\ \mathbf{y} & = & \sum_{l=1}^{r} \mu_{P_l}(\mathbf{x})(C_l\mathbf{x} + D_l\mathbf{u})\end{array} \tag{3.9}$$

Clearly, (3.7) now represents a nonlinear system, and (3.5) is an equivalent Takagi-Sugeno type fuzzy system representation of (3.7).

3.3 Stability Analysis

The stability analysis of a closed-loop system with a fuzzy controller was one of the big problems in the early stages of fuzzy control theory. Especially when the fuzzy controller is of Mamdani type it is a rather complex problem to assure stability. In fact this has been one of the subject of the debate between L.A. Zadeh and M. Athens in EUFIT'99, Aachen, Germany, September 1999. Later with the introduction of the Takagi-Sugeno type fuzzy controller the stability analysis and design of a fuzzy controller becomes one of the main concerns of fuzzy control theory. The superiority of Takagi-Sugeno type controllers is that it provides a model in the consequent part of the IF-THEN rules which eases the analysis and design. In this section we will first show the basic results on stability related with Mamdani, and Takagi-Sugeno type. Then we will give some of the stability results in the literature.

3.3.1 Mamdani Type Fuzzy System

The fuzzy system defined by (3.3) is a static mapping. Consider the fuzzy system composed of the following type of IF-THEN rules:

$$\mathcal{R}_l : \text{IF } \mathbf{x}(n) \text{ is } F^l \text{ THEN } \mathbf{x}(n+1) \text{ is } G^l. \tag{3.10}$$

The fuzzy system can be considered as a collection of discrete dynamical system interpolated by the corresponding membership functions. Let the relation for the rule $\mathcal{R}_l$ be represented by a relation defined by the membership function as $\mu_{\mathcal{R}_l}(\mathbf{x}(n)) : \mathcal{F} \times \mathcal{G} \longrightarrow [0\ 1]$. Let us assume that composition based inference is used for the fuzzy system. Denote the union of these fuzzy relations $\mathcal{R}$ by a membership function $\mu_{\mathcal{R}}(\mathbf{x}(n), \mathbf{x}(n+1)) = \cup_{l=1}^{r} \mu_{\mathcal{R}_l}(\mathbf{x}(n), \mathbf{x}(n+1))$. Then the dynamical equation in the fuzzy domain can be written as:

$$\mathcal{X}_{n+1} = \mathcal{X}_n \circ \mathcal{R}$$

where $\mathcal{X}_n$ is the fuzzy relation corresponding to the state at the n^{th} discrete time with a membership function $\mu_{\mathcal{X}_n}(\mathbf{x}) : \mathcal{F} \longrightarrow [0\ 1]$. Let the fuzzy relation $\mathcal{X}_0$ correspond to the initial state of the system. Then we have the following:

$$\begin{aligned}
\mathcal{X}_1 &= \mathcal{X}_0 \circ \mathcal{R} \\
\mathcal{X}_2 &= \mathcal{X}_1 \circ \mathcal{R} = \mathcal{X}_0 \circ \mathcal{R} \circ \mathcal{R} = \mathcal{X}_0 \circ \mathcal{R}^2 \\
\mathcal{X}_2 &= \mathcal{X}_0 \circ \mathcal{R}^3 \\
\vdots &= \vdots \\
\mathcal{X}_{n+1} &= \mathcal{X}_0 \circ \mathcal{R}^n
\end{aligned}$$

Hence a free motion fuzzy system is stable in the limit if a relation $\mathcal{R}^\infty$ exist such that [48]:

$$\lim_{k \longrightarrow \infty} \mathcal{R}^k = \lim_{k \longrightarrow \infty} \underbrace{\mathcal{R} \times \cdots \times \mathcal{R}}_{ktimes} = \mathcal{R}^\infty \tag{3.11}$$

otherwise we call the model oscillatory with period τ, i.e.:

$$\mathcal{R}^{k+\tau} = \mathcal{R}^k \quad \forall k = 1, 2, \ldots \tag{3.12}$$

3.3.2 Takagi-Sugeno Type Fuzzy System

Consider the discrete time fuzzy system with the following IF-THEN rules:

$$\begin{aligned} \mathcal{R}_l : \text{IF} \quad & x_1(k) \text{ is } F_1^l \text{ AND } \cdots \text{ AND } x_n(k) \text{ is } F_n^l \\ & \text{THEN} \quad \mathbf{x}(k+1) = A_l \mathbf{x}(k) \end{aligned} \tag{3.13}$$

where $A_l \in \mathbb{R}^{n\times n}$, $\mathbf{x}(k) \in \mathbb{R}^n$, and $l = 1, \ldots, r$. If the same assumptions and manipulations are carried out while obtaining the result given by the Equation (3.9) then the output of the fuzzy system whose rules are (3.13) would be:

$$\mathbf{x}(k+1) \;=\; \textstyle\sum_{l=1}^{r} \alpha_l(\mathbf{x}(k)) A_l \mathbf{x}(k) \tag{3.14}$$

where the basis function, $\alpha_l(\mathbf{x}(k))$, is given by the following equation:

$$\alpha_l(\mathbf{x}(k)) = \frac{\mu_{P_l}(\mathbf{x}(k))}{\sum_{i=1}^{r} \mu_{P_i}(\mathbf{x}(k))} \tag{3.15}$$

where

$$\mu_{P_l}(\mathbf{x}(k)) = \prod_{i=1}^{n} \mu_{F_i^l}(x_i(k)) \tag{3.16}$$

Tanaka and Sugeno [53] suggested an important criterion for the stability of the T-S type fuzzy model given by (3.14).

Theorem 7 *([53]): The equilibrium state of the fuzzy system (3.14) (namely* $\mathbf{x} = 0$*) is globally asymptotically stable if there exists a common positive definite matrix* P *such that*

$$A_i^T P A_i - P < 0 \quad \forall i = 1, 2, \ldots, r. \tag{3.17}$$

Proof 7 *See [53].* □

In [53] a similar results is given for the continuous fuzzy system given with the following dynamics:

$$\dot{\mathbf{x}} \;=\; \textstyle\sum_{l=1}^{r} \alpha_l(\mathbf{x}) A_l \mathbf{x} \tag{3.18}$$

where α_l is as in (3.8).

Theorem 8 *([53]): The system defined by (3.18) is asymptotically stable if there exists a common positive matrix P such that the following inequality holds:*

$$A_i^T P + P A_i < 0 \quad \forall i = 1, 2, \ldots, r. \tag{3.19}$$

Proof 8 *See [53].* □

The usefulness of these result of course depends on the systematic way of finding a positive definite matrix P which is still an area of research. Some of the related results may be found in [54] [55].

3.3.3 Other Stability Results

In this section we will include some other stability results on fuzzy systems.

Theorem 9 *([54]): Suppose that A_i in (3.14) is Schur for all $i = 1, 2, \ldots, r$ and A_i's are pairwise computative, i.e.*

$$A_j A_{j+1} = A_{j+1} A_j, \quad j = 1, 2, \ldots, r$$

Consider the following r Lyapunov equations:

$$\begin{aligned} A_1^T P_1 A_1 - P_1 &= -Q \\ A_2^T P_2 A_2 - P_2 &= -P_1 \\ \vdots &= \vdots \\ A_r^T P_r A_r - P_r &= -P_{r-1} \end{aligned}$$

where $Q > 0$ and $P_i, i = 1, 2, \ldots, r$ is the unique positive definite symmetric solution of each equation. Then we always have:

$$A_i^T P_r A_i - P_r < 0 \tag{3.20}$$

Proof 9 *See [54].* □

Theorem 10 *([55]): Consider a fuzzy system (3.14), suppose there exists an operator norm* $\|\cdot\|$ *and a constant* c_l*,* $l = 1, 2, \ldots, r$ *such that:*

$$\|A_l\| \leq c_l \tag{3.21}$$

then the system (3.14) is asymptotically stable if;

$$\sum_{l=1}^{r} \mu_{P_l}(\mathbf{x}(k))c_l < 1 \tag{3.22}$$

Proof 10 *See [55].* □

Chapter 4

Comparison of Switching and Fuzzy Systems

In our daily life we have many examples of systems which changes their dynamics from one operating condition to another one. For example the governing equation of a transmission system of a car is changed at certain speed values. An air-condition system changes its dynamics at a preset temperature from cooling to heating or vice versa. This type of systems are analyzed under the title of switching systems. In Chapter 2 we have investigated the properties of switching systems in more detail.

Sometimes it is not possible to define a system with ODEs. Instead we might have a vague description of the system behavior which is usually based on the experiences of an operator. A fuzzy system is a model obtained by utilizing the descriptions of this type. It consists of a set of IF-THEN rules defining the relation between the input and the output of the system. A detailed analysis on fuzzy systems has been given in Chapter 3.

Many advantages of using switching systems has been reported in the literature. In [21] it is shown that a switching system can be used to stabilize a nonholonomic integrator but with chattering in the control signal due to switching. We also have many studies on fuzzy systems which has its own advantages like smoothing the system variables [56]. In this chapter we examine the similarity of both systems under certain model assumptions and propose methods to convert a switching system into a fuzzy system. The resultant fuzzy system performs as good as the switching system with no chattering. Due to the similarity in the models there are many stability results in fuzzy system theory which is obtained by extending the results in switching systems into fuzzy domain. We will conclude this chapter with a brief discussion on the comparison of the stability results.

4.1 Comparison of the Models

The type of system which will be considered in this chapter can be defined by the dynamics

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u} \tag{4.1}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{f}(\mathbf{x}) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\mathbf{g}(\mathbf{x}) : \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times m}$, and $\mathbf{u} \in \mathbb{R}^m$. In this section we will compare only the models of the switching and the fuzzy controllers. The analysis can be extended to the case where the systems are expressed in closed form without a control signal. Let us first repeat the descriptions of the switching and the fuzzy systems for ease of reading. In Section 2.2 we have presented different models depending on the problem. It is easy to show the similarities with the fuzzy controller if we consider the model for switching controller given by (2.3):

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &= \textstyle\sum_{i=1}^{r} \mathbf{u}_{SS_i}(\mathbf{x}) S_i(\mathbf{x}) \\ S_i(\mathbf{x}) &= \begin{cases} 1, & \mathbf{x} \in \mathcal{R}_i \\ 0, & \mathbf{x} \notin \mathcal{R}_i \end{cases} \end{aligned} \tag{4.2}$$

where $S_i(\mathbf{x})$ is the characteristic function and $\mathbf{u}_{SS_i}(\mathbf{x})$ is the corresponding controller for the set of points in region $\mathcal{R}_i \subset \mathbb{R}^n$ and r is the total number of regions. Let us consider a Takagi-Sugeno type fuzzy system whose rule base have IF-THEN statements modified from (3.4):

$$\begin{aligned} \mathcal{L}_j : \text{IF} \quad & (x_1 \text{ is } F_{1,j}) \text{ AND } \cdots \text{ AND } (x_n \text{ is } F_{n,j}) \\ & \text{THEN } \mathbf{u} = \mathbf{u}_{FLS_j}(\mathbf{x}). \end{aligned} \tag{4.3}$$

where $F_{i,j}$ are the linguistic expression defined by the fuzzy set $\mu_{F_{i,j}}(\cdot) : \tilde{\mathcal{R}}_{i,j} \longrightarrow \mathbb{R}$, $\mathbf{u}_{FLS_i}(\mathbf{x})$ is the output of the fuzzy controller due to rule $\mathcal{L}_j$, and $\tilde{\mathcal{R}}_{i,j} \subset \mathbb{R}$. If the fuzzy sets $F_{i,j}$ are normal, consistent and complete then for a complete rule base, the sum of the product of the membership functions in the premise part of (4.3) is equal to 1. Under this condition, if multiplication is used for the "AND" operator and the inference operator, addition is used for the aggregation of the rules and center of gravity is used for the defuzzification of the inferred fuzzy sets then the output of the fuzzy controller would be expressed with the following expression;

$$\mathbf{u}(\mathbf{x}) = \sum_{j=1}^{r} \mathbf{u}_{FLS_j}(\mathbf{x})\mu_{\mathcal{L}_j}(\mathbf{x}), \quad \mu_{\mathcal{L}_j}(\mathbf{x}) = \prod_{k=1}^{n} \mu_{F_{k,j}}(x_j) \tag{4.4}$$

where r is the number of rules in the rule base and $\mu_{\mathcal{L}_j}(\mathbf{x})$ is the membership function defining the partitioned state-space for $\tilde{\mathcal{R}}_j = \prod_{i=1}^{n} \tilde{\mathcal{R}}_{i,j}$ that is the cartesian product of $\tilde{\mathcal{R}}_{i,j}$'s.

If the two models (4.4) and (4.2) are compared then it is observed that the difference is the characteristic/membership function defining the regions. In switching system the change from being a member to non-member is abrupt while in fuzzy system it is smooth. This is the reason why the output of a fuzzy system does not contain any abrupt changes which is in fact the case for switching system. Furthermore from [10] and [11] we can conclude that with proper choice of membership functions, fuzzy systems can be reduced to switching systems. Conversely, a switching system can be replaced by a fuzzy system which has fuzzified $\mathcal{R}$ to achieve smooth system variables. In this section we will try to analyze and propose a method to convert a switching system model to a fuzzy system model.

We want to have a fuzzy system whose performance is similar to a given switching system under consideration. The Takagi-Sugenuo model, (4.3) and (4.4), is rather suitable for this purpose because we can take the consequent part of the rules to be equal to the controller functions, $\mathbf{u}_{SS_j}(\mathbf{x})$, for each $\mathcal{R}_j$, that is $\mathbf{u}_{FLS_j}(\mathbf{x}) = \mathbf{u}_{SS_j}(\mathbf{x})$. The problem then reduces to fuzzifying $S_j(\mathbf{x})$ to $\mu_{\mathcal{L}_j}(\mathbf{x})$ so that (4.4) and (4.2) can be made arbitrary close to each other [10], [11]. If $\mathcal{R}_j$'s are convex sets then the partitioning of the subspaces with the membership functions, $\mu_{F_{i,j}}(x_i)$, can be easily performed so that every $\mathcal{R}_j$ can be expressed in the premise part of the rule as given in (4.3).

In some cases the set $\mathcal{R}_i$ may not be convex. Depending on the problem it is sometimes necessary to express the boundaries of the regions in transformed spaces

[21], [56]. In such situations new variables can be defined by some transformation of the states so that the premise part of the rules can be defined as in (4.3). In the following section an example simulation is given for this case.

Defining new variable(s) for the premise part of the rules can help to construct a suitable rule base for a fuzzy controller. Note that this is not sufficient, since the change of the variable does not provide any means to the fuzzification of the transition regions. Let $\tilde{S}_i(\mathbf{x}) : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function defining the region $\mathcal{R}_i$, that is,

$$\mathcal{R}_i = \{\mathbf{x} \in \mathbb{R}^n : \tilde{S}_i(\mathbf{x}) > 0\}, \quad \tilde{S}_i(\mathbf{x}) : \mathbb{R}^n \longrightarrow \mathbb{R} \tag{4.5}$$

The second example in [56] can be expressed in this form. The relation between the characteristic function in (4.2) and $\tilde{S}_i(\mathbf{x})$ can be defined as:

$$S_i(\mathbf{x}) = \begin{cases} 1, & \tilde{S}_i(\mathbf{x}) > 0 \\ 0, & \tilde{S}_i(\mathbf{x}) \leq 0 \end{cases} \tag{4.6}$$

Note that, the range of $\tilde{S}_i(\mathbf{x})$ is in general an open set in $\mathbb{R}$ while that of $S_i(\mathbf{x})$ is $\{0, 1\}$. Not also that $\tilde{S}_i(\mathbf{x})$ is not unique. Let $h(s) : \mathbb{R} \longrightarrow \mathbb{R}$ be a function defined as follows:

$$h(s) = \begin{cases} 1, & v < s \\ \frac{s+v}{2v}, & -v < s \leq v \\ 0, & s \leq -v \end{cases} \tag{4.7}$$

where v is a free parameter. To make smooth transition between models one might use (4.7) to obtain a membership function for the region defined by $S_i(\mathbf{x})$ as follows:

$$\mu_{\mathcal{L}_i}(\mathbf{x}) = h(\tilde{S}_i(\mathbf{x})) \tag{4.8}$$

The $h(s)$ given in (4.7) utilizes the absolute value of $\tilde{S}_i(\mathbf{x})$ to assign a membership value. Another approach might be to use the minimum distance between $\mathbf{x}$ and the surface defined by $\tilde{S}_i(\mathbf{x}) = 0$. There are no general rules for the fuzzification method and it requires optimization for better performance.

With the transformation in (4.7), $u(\mathbf{x})$ in (4.4) will be equal to that of (4.2) except in a band of width $2v$ around the points of $\mathbf{x}$ where $\tilde{S}_i(\mathbf{x}) = 0$. A weighted contribution of each model, $\mathbf{u}_i(\mathbf{x})$, will be summed up to calculate the desired $u(\mathbf{x})$ for the fuzzy system within the band. Note that as $v \to 0$, (4.8) approaches to (4.6). Hence by choosing v sufficiently small, we expect that the resulting difference between (4.4) and (4.2) will be small as well.

4.1.1 Simulation

In [21] the stabilization of a non-holonomic integrator is performed using four switching controllers, $g_i(\mathbf{x})$. The system is given in (4.9).

$$\begin{aligned} \dot{x_1} &= u_1, \\ \dot{x_2} &= u_2, \\ \dot{x_3} &= x_1 u_2 - x_2 u_1 \end{aligned} \tag{4.9}$$

The aim is to choose u_1 and u_2 such that $x_i(t) \to 0$ as $t \to \infty$. The partitioning of the state-space into $\mathcal{R}_i$s, given in [21], is as follows:

$$\begin{array}{lcllclcll} \mathcal{R}_1 & = & \{\mathbf{x} \in \mathbb{R}^3 : & 0 & \leq & x_1^2 + x_2^2 & < & \pi_2(x_3^2) & \} \\ \mathcal{R}_2 & = & \{\mathbf{x} \in \mathbb{R}^3 : & \pi_1(x_3^2) & < & x_1^2 + x_2^2 & < & \pi_4(x_3^2) & \} \\ \mathcal{R}_3 & = & \{\mathbf{x} \in \mathbb{R}^3 : & \pi_3(x_3^2) & < & x_1^2 + x_2^2 & & & \} \\ \mathcal{R}_4 & = & \{0\} & & & & & & \end{array} \tag{4.10}$$

where $\pi_i(x_3^2) = i(1 - e^{-\sqrt{x_3^2}})$, $i = 1, \ldots, 4$. In Figure 4.1(a) the functions $\pi_i(x_3^2)$ are given. The controllers, $\mathbf{u}_{SS_i}(\mathbf{x}) : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, are given in [21] as follows:

$$\begin{array}{lclclcl} \mathbf{u}_{SS_1} & = & \begin{bmatrix} 1 \\ 1 \end{bmatrix} & & \mathbf{u}_{SS_2} & = & \begin{bmatrix} x_1 + \frac{x_2 x_3}{x_1^2 + x_2^2} \\ x_2 - \frac{x_1 x_3}{x_1^2 + x_2^2} \end{bmatrix} \\ \mathbf{u}_{SS_3} & = & \begin{bmatrix} -x_1 + \frac{x_2 x_3}{x_1^2 + x_2^2} \\ -x_2 - \frac{x_1 x_3}{x_1^2 + x_2^2} \end{bmatrix} & & \mathbf{u}_{SS_4} & = & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array} \tag{4.11}$$

There is no unique function for $\tilde{\mathcal{S}}_j(\mathbf{x})$. One possible set of function might be:

$$\begin{aligned} \tilde{\mathcal{S}}_1(\mathbf{x}) &= -\ (x_1^2 + x_2^2 - \pi_2(x_3^2)) \\ \tilde{\mathcal{S}}_2(\mathbf{x}) &= -\ (x_1^2 + x_2^2 - \pi_1(x_3^2))(x_1^2 + x_2^2 - \pi_4(x_3^2)) \\ \tilde{\mathcal{S}}_3(\mathbf{x}) &= \ (x_1^2 + x_2^2 - \pi_3(x_3^2)) \\ \tilde{\mathcal{S}}_4(\mathbf{x}) &= \begin{cases} 1, & \mathbf{x} = 0 \\ 0, & \text{Otherwise} \end{cases} \end{aligned}$$

In Figure 4.1(b)(c)(d) the surface of the functions $\tilde{\mathcal{S}}_i(\mathbf{x})$ are given together with the corresponding boundary function $\pi_i(x_3^2)$ for $i = 1, 2, 3$. After defining the functions

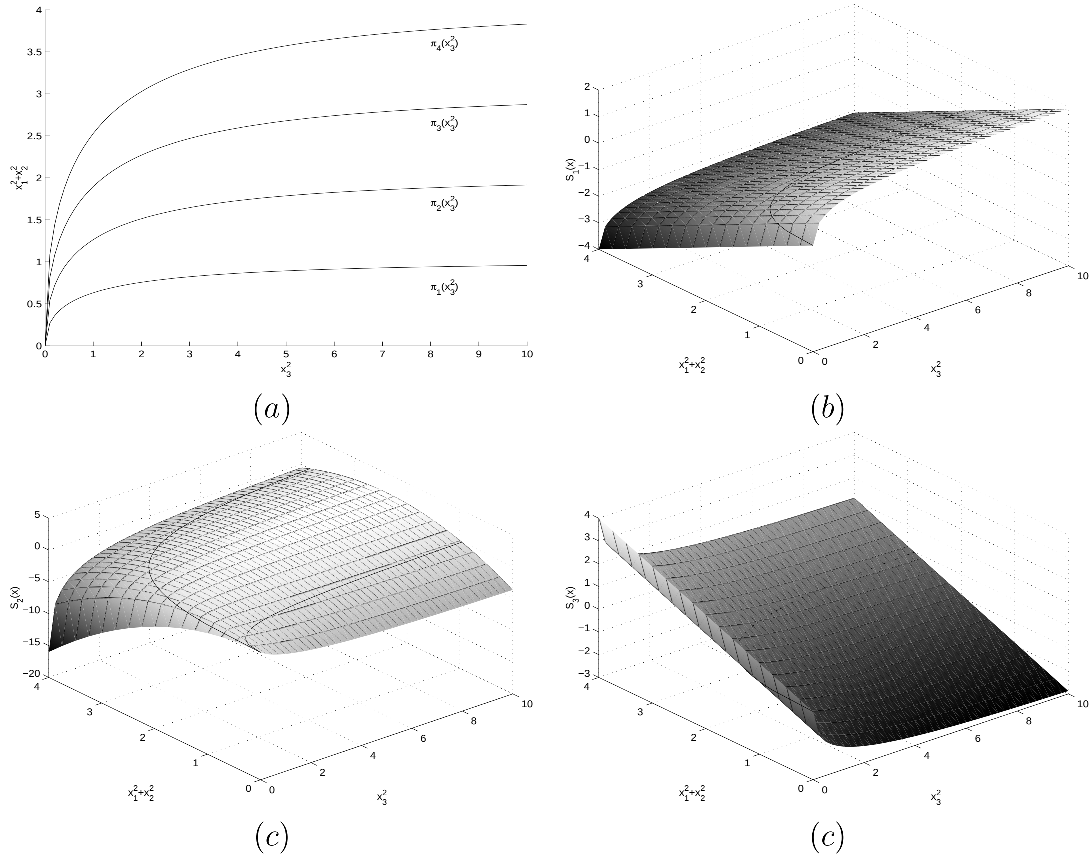


Figure 4.1: The functions defining the boundaries of the regions $\mathcal{R}_i$ (a), and the functions $\tilde{\mathcal{S}}_j(\mathbf{x})$, $i = 1, 2, 3$(b)(c)(d)
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$\tilde{\mathcal{S}}_i(\mathbf{x})$ we determine a suitable value for v in (4.7) to obtain the membership function $\mu_{\mathcal{R}_i}(\mathbf{x})$.

The partitioning given in (4.10) is not suitable to use directly in constructing the premise part of the rules, i.e. $\mu_{\mathcal{R}_i}(\mathbf{x})$, for the fuzzy controller. We can define a new variable ,$w(\mathbf{x})$, to overcome this difficulty as given in (4.12).

$$w(\mathbf{x}) = (x_1^2 + x_2^2)/(1 - \exp^{-\sqrt{x_3^2}}) \tag{4.12}$$

The resultant partition of the state-space can be expressed in terms of $w(\mathbf{x})$ as follows:

$$\begin{aligned}
\hat{\mathcal{R}}_1 &= \{\mathbf{x} \in \mathbb{R}^3 : 0 \leq w(\mathbf{x}) < 2\} \\
\hat{\mathcal{R}}_2 &= \{\mathbf{x} \in \mathbb{R}^3 : 1 < w(\mathbf{x}) < 4\} \\
\hat{\mathcal{R}}_3 &= \{\mathbf{x} \in \mathbb{R}^3 : 3 < w(\mathbf{x})\} \\
\hat{\mathcal{R}}_4 &= \{0\}
\end{aligned} \tag{4.13}$$

Defining the membership functions $\mu_{\mathcal{R}_i}(\mathbf{x})$ for (4.13) is much simpler then defining than for (4.10). Equation (4.7) serves as a fuzzification operation for $\tilde{\mathcal{S}}_j(\mathbf{x})$. The borders of the transformed regions $\hat{\mathcal{R}}_i$ are defined by integer numbers like 1,2,3,4 with respect to $w(\mathbf{x})$. These regions can be defined by characteristic function which takes the value 1 within the corresponding interval and 0 otherwise as given in (4.13). One can choose the corresponding membership functions such that a value of 0.5 is assigned the borders of these regions. Hence one possible choice of membership function $\mu_{\hat{\mathcal{R}}_2}(\mathbf{x})$ can be given as a trapezoidal membership function, e.g.

$$\mu_{\hat{\mathcal{R}}_2}(w(\mathbf{x})) = \begin{cases}
0, & w(\mathbf{x}) < -0.5 \\
w(\mathbf{x}) + 0.5, & -0.5 < w(\mathbf{x}) < 0.5 \\
1, & 0.5 < w(\mathbf{x}) < 1.5 \\
w(\mathbf{x}) - 2.5, & 1.5 < w(\mathbf{x}) < 2.5 \\
0, & 2.5 < w(\mathbf{x})
\end{cases}$$

The transition regions can be narrowed so that the membership functions approach to the corresponding characteristic function defining the regions in (4.6).

We simulated both the switching system given in (4.9) with $\mathcal{R}_i$ as given in [21], and the fuzzy system given by (4.4), (4.7), (4.8) and (4.13) with $v = 1$; in both cases the control signals for the i^{th} region, $\mathbf{u}_{SS_i}(\mathbf{x})$, is chosen as given in [21].

The simulation results are shown in Figure 4.2 and Figure 4.3. It is seen from the figures that the fuzzy system can perform competitive time response and free from chattering. This smoothing property of fuzzy control might be advantageous for certain systems. As we have mentioned in the previous section the fuzzification function $h(s)$ and the characteristic function $\tilde{S}_i(\mathbf{x})$ defining the region are not unique and the performance will vary for different choices.

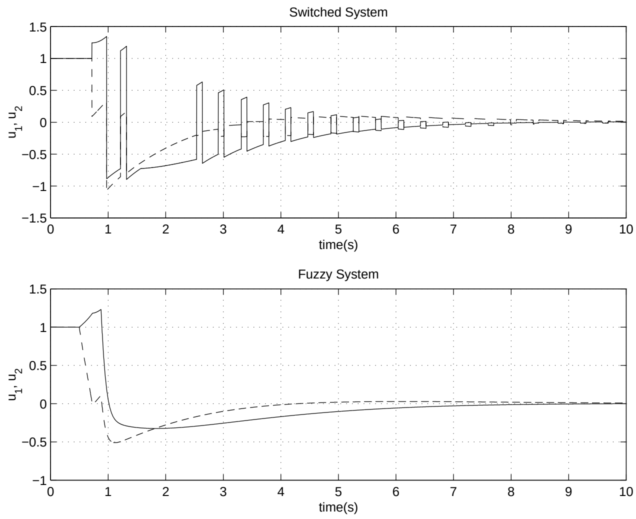


Figure 4.2: Control signals for switching and the corresponding fuzzy system

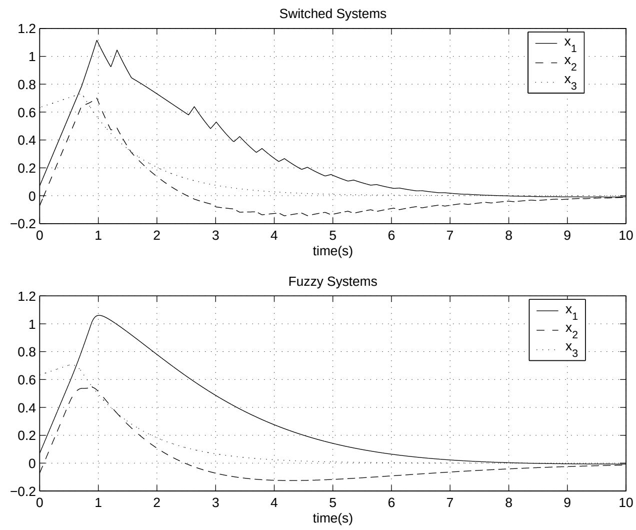


Figure 4.3: States of the switching and the corresponding fuzzy system.

4.2 Comparison of the Stability Results

If we reconsider the stability results for switching and fuzzy control we see that the results for fuzzy systems are the same for both, see e.g. Theorem 1 and Theorem 8. We can say that the achievements in switching control can directly be applied to fuzzy systems of Takagi-Sugeno type because of the resemblance of their models.

In the previous section we have considered the problem of designing a fuzzy controller from a given switching system. In this way a smothering effect on the state variables is expected. The reverse problem, which can be defined as the determination of a switching system for given a fuzzy system, is also interesting. In this way the stability results of the switching system might be used in deciding the stability of the fuzzy system. Considering the modelling of switching and fuzzy systems in Section 2.2 and Section 3.2 we see that a fuzzy system can be considered as a collection of systems interpolated with the membership functions. Also we know that a property defined by a membership function is fully possessed by the element of concern if its value is 1 and if the property is not possessed at all then its value is 0. If the membership function has a value of 0.5 for a specific measurement of the property then the ambiguity is in its peak state, i.e. this is a perfect point to make the decision boundary which is one of the main concern in the design of a switching system. The associated switching system is expected to have chattering in its states or control signals, that is the switching system will not be able to produce the exact output of the fuzzy system. Note that a fuzzy system can be designed to produce an output as close as to a given switching system with proper choice of membership functions.

There are some studies on the reverse problem [57]. Thathachar and Viswnath consider a discrete time fuzzy systems given by (3.14) [57]. They demonstrate the equivalence of stability properties of fuzzy systems and linear time invariant switching systems. They prove that the necessary and sufficient condition for a given fuzzy system (3.14) to be stable is that the corresponding switching system (2.1) be stable. The stability of the system is defined as the convergence of the state vector to zero for both systems. The problem we have worked in the previous section uses the membership functions as one of the main design parameters. Doing the same in the reverse problem constitute one of the direction for our future work.

Chapter 5

Phase Portrait Matching

In most of the controller design methods either time domain parameters or frequency domain parameters are given as design specifications. In this chapter we try to investigate another design scheme where the design specification is the given dynamics of the closed loop system. More specifically, we assume that there is a desired phase portrait behavior which is given, and the problem is to design an appropriate controller so that the closed loop dynamics generate a phase portrait which is close to the desired one in a certain sense. The problem has many unknowns but not enough number of equations to solve them. Some physical constraints and assumptions are utilized to overcome these difficulties. In this study we consider the systems given by the following dynamics:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x}). \tag{5.1}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{g} : \mathbb{R}^{n\times m} \to \mathbb{R}^n$, and $\mathbf{u}(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^m$ is the control input. We further assume that $\mathbf{x} = 0$ is an equilibrium point for the system given by (5.1) when $u(\mathbf{x}) = 0$. For a given vector function $\mathbf{d}(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^n$ the problem is to find the control signal $\mathbf{u}(\mathbf{x})$ such that the closed loop dynamics is close to the dynamics $\dot{\mathbf{x}} = \mathbf{d}(\mathbf{x})$ or more specifically, the phase portrait generated by (5.1) is close to the phase portrait generated by the desired dynamics.

Let us try to see some of the main problems of this approach. There might be some $\mathbf{x} \in \mathbb{R}^n$ where $\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x}) = \mathbf{d}(\mathbf{x})$ has no solution for $\mathbf{u}(\mathbf{x})$ because $\mathbf{d}(\mathbf{x})$ is arbitrary. Therefore in general any approach to this problem will try to approximate the given

$\mathbf{d}(\mathbf{x})$ in a sense if certain conditions are not met. One of the approaches might be to minimize the norm of the difference between $\mathbf{d}(\mathbf{x})$ and $\mathbf{f}(\mathbf{x})+\mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x})$. In this approach neither the speed of the system nor its phase are aimed to be matched exactly. Another approach will be to match the evolution direction of the system as far as the physical constraints permit. In this approach the speed of the system is the property which is being sacrificed to make the phase difference zero under certain conditions.

We will propose a method for the solution of the problems mentioned above. This method can be used to stabilize a given nonlinear system if certain conditions are met. It also can be used to approximate a given function, see Section 8.3.1. In the following chapters we will use these methods prior to the design procedure of a fuzzy controller and in saturating the control signal of switching controller. This chapter is organized as follows. In Section 5.1 we will analyze phase matching problem. We will first try to identify the potential problems and introduce a solution for a restricted set of systems. Then we will shortly review the projection operators which will be used to describe different solutions of the problem. This section will be completed by giving some methods to saturate the control signal while preserving the phase difference at minimum. The final section will be the minimization of the norm of the vector difference between $\mathbf{d}(\mathbf{x})$ and $\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x})$.

5.1 Minimization of the Phase Difference

To motivate the idea of phase matching let us consider the simplest case where the degree of the system is 2 and there is only one control signal; that is $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$, and $u(\mathbf{x}) : \mathbb{R}^2 \to \mathbb{R}$ for (5.1). In Figure 5.1 two possible configurations of the vectors for two different $\mathbf{x}$ is given. The dotted line corresponds to the values of $\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x})$ obtained by various values of control signal $u(\mathbf{x})$. Note that $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x})$, see (5.1). The thicker vector labelled as $\beta\mathbf{d}(\mathbf{x})$ and $\mathbf{d}(\mathbf{x})$ has no phase difference in between in Figure 5.1(a). In Figure 5.1(b) the phase difference is π that is the direction of the desired phase portrait at that point of $\mathbf{x}$ is just the opposite. Note that β might be a function of $\mathbf{x}$ as well.

5.1.1 Problems

Let us try to consider the possible problems. Looking at the Figure 5.1 (a) to have a zero phase difference between the vector fields $\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x})$ and $\mathbf{d}(\mathbf{x})$ we require the following to hold.

$$\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x}) = \beta\mathbf{d}(\mathbf{x}). \tag{5.2}$$

where $\beta(\cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $|\beta|$ determines the ratio of the magnitudes of $\mathbf{d}(\mathbf{x})$ and that of the possible $\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x})$ vector. Note that here $\beta(\cdot)$ is a function of $\mathbf{x}$. Keeping in mind this fact we will ignore the argument to simplify the notation.

Problem Type 1: $\beta < 0$

Consider Figure 5.1(b) together with (5.2). As it is seen the solution might contain the values of $\beta < 0$. This is not acceptable because the system, with the calculated control signal, evolves just the opposite to the desired direction. There is no control signal to exactly match the phase of $\mathbf{d}(\mathbf{x})$ and $\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x})$. Therefore the solution obtained from (5.2) should be checked for this condition and the control signal must be updated appropriately.

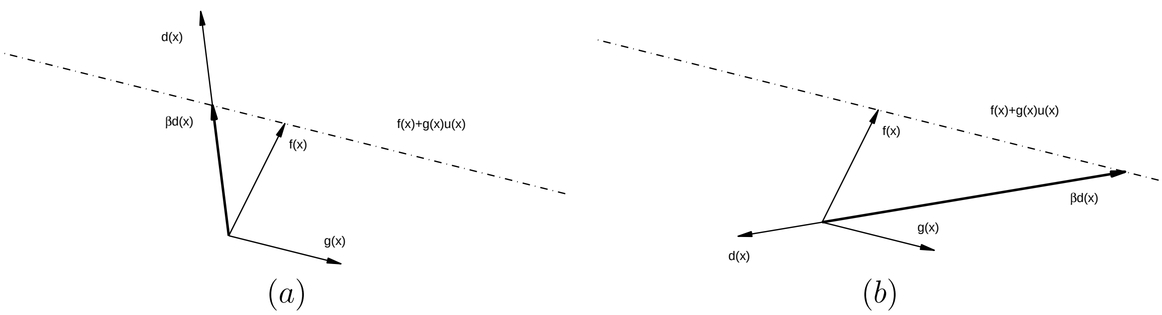


Figure 5.1: The characteristic of an affine system at some points x where (a)$\beta > 0$ (b)$\beta < 0$.

Problem Type 2: $\mathbf{d}(\mathbf{x})$ is in the column space of $\mathbf{g}(\mathbf{x})$

This problem occurs when the angle between $\mathbf{d}(\mathbf{x})$ and the plane spanned by the columns of $\mathbf{g}(\mathbf{x})$ becomes too small. In such circumstances the solution vector diverges to infinity.

This case, which can be visualized from Figure 5.1(b), is equivalent to $\mathbf{d}(\mathbf{x})$ being in the span of the column vectors of $\mathbf{g}(\mathbf{x})$. Since there is no solution to the control signal for exact phase matching, an appropriate approximation must be used as in problem of type 1.

Problem Type 3: $\mathbf{f}(\mathbf{x})$ is in the column space of $\mathbf{g}(\mathbf{x})$

The last problem that we will consider is the case where the phase difference between the vectors $\mathbf{f}(\mathbf{x})$ and the plane spanned by the columns of $\mathbf{g}(\mathbf{x})$ is close to zero. This situation is depicted in Figure 5.2 for positive and negative values of β. Consider the

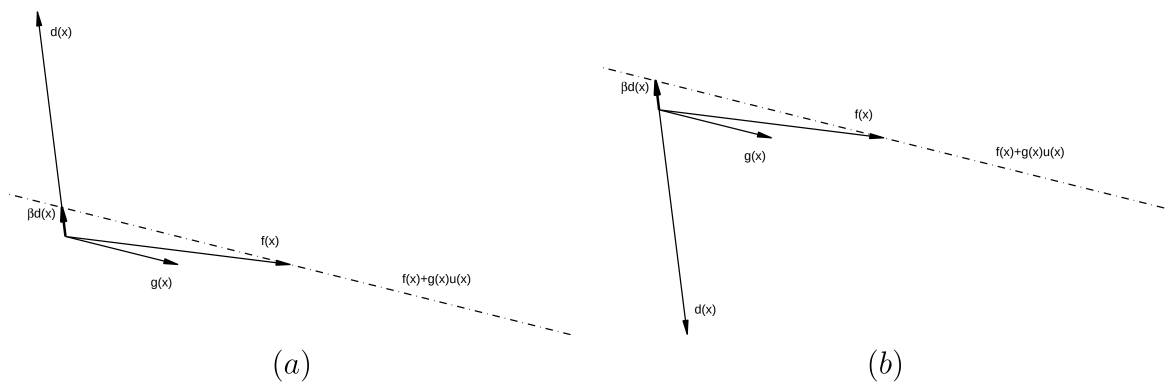


Figure 5.2: The situation with small phase difference between $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ when (a)$\beta > 0$ (b)$\beta < 0$.

degenerate case where $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ are parallel to each other. In this case independent of the value of $\mathbf{d}(\mathbf{x})$, the value of β would be zero while the control signal $u(\mathbf{x})$ would be some constant. The problem here becomes clear when we consider Figure 5.2. If $\beta = 0$ then the system is at equilibrium since $\dot{\mathbf{x}} = \beta\mathbf{d}(\mathbf{x})$. From this argument it is obvious that for small β the system will slow down and eventually stop if there is some $\mathbf{x} \in \mathbb{R}^n$ where $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ are parallel. So we can assume that there is a problem when $|\beta| < \epsilon$, where $\epsilon \in \mathbb{R}$ is a small positive number, and a different control signal is required within this region of the state space.

5.1.2 Solution when n = m+1

Let us first try to analyze the case where $\beta > \epsilon$. Reordering (5.2) lead us to the following matrix equation:

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} -\mathbf{g}(\mathbf{x}) & \mathbf{d}(\mathbf{x}) \end{bmatrix} \begin{bmatrix} u(\mathbf{x}) \\ \beta \end{bmatrix} = W(\mathbf{x}) \begin{bmatrix} u(\mathbf{x}) \\ \beta \end{bmatrix}. \tag{5.3}$$

which has a solution under certain conditions. If $\mathbf{d}(\mathbf{x})$ is not in the span of the columns of $\mathbf{g}(\mathbf{x})$ and the matrix $W(\mathbf{x})$ is a square matrix (i.e. $m + 1 = n$) then there is a unique solution for $u(\mathbf{x})$ and β.

Although theoretically we can apply any finite control signal to a given system, some of them might not be possible in practice. Therefore we can assume that there is a maximum and a minimum value for $u(\mathbf{x}) : \mathbb{R}^2 \rightarrow [U_{min}, U_{max}]$ in the simplest case. If we consider the case where $\beta < 0$ as depicted in Figure 5.3 we see that the best solution that can be provided is to apply the $u(\mathbf{x}) = U_{min}$ to minimize the phase difference.

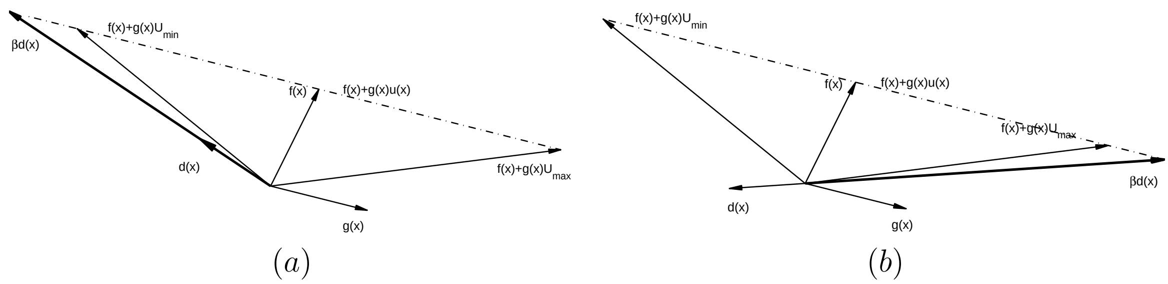


Figure 5.3: The range of $\dot{\mathbf{x}}$ vectors from $\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})U_{min}$ to $\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})U_{min}$ when (a) $\beta > 0$ (b) $\beta < 0$.

The same strategy can be applied to the case $det([\mathbf{g}(\mathbf{x})\ \mathbf{d}(\mathbf{x})]) \simeq 0$ which in return results in a value of β that is too big. So the problems of type 1 and type 2 can be determined by checking whether the calculated value of $u(\mathbf{x})$ is out of its boundaries or not. If so the applied control should be one of the boundary values which minimizes the phase difference, $\phi = \angle(\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u(\mathbf{x})) - \angle\mathbf{d}(\mathbf{x})$.

In the solution of (5.3) if $\beta = 0$ then for that calculated value of $u(\mathbf{x})$ the system will stop at that point hence a different value of $u(\mathbf{x})$ is needed to get rid of these regions. These regions can be defined as $\mathcal{R} = \{\mathbf{x} | 0 < \beta(\mathbf{x}) < \epsilon\}$ where $\epsilon > 0$ is a given small constant. One of the solution to this problem would be to check whether the desired system is converging towards the origin or diverging. Note that the norm of the desired state vectors will decrease if $\mathbf{x}^T\mathbf{d}(\mathbf{x}) < 0$ and increase if $\mathbf{x}^T\mathbf{d}(\mathbf{x}) > 0$. The following equation gives the rate of change in the norm of the state vectors.

$$\frac{d}{dt}\|x\|^2 = 2\mathbf{x}^T\dot{\mathbf{x}} = 2\mathbf{x}^T(\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u(\mathbf{x})). \tag{5.4}$$

If $\mathbf{x}^T\mathbf{d}(\mathbf{x}) < 0$ then a value of $u(\mathbf{x})$ should be selected such that $\frac{d}{dt}\|x\|^2 < 0$, and $\frac{d}{dt}\|x\|^2 > 0$ otherwise. Since the trajectories are everywhere tangent to the vector fields, with the proposed methodology the phase portraits generated by the closed loop dynamics will be close to the phase portrait generated by the desired dynamics. Note that since $\beta \neq 1$, the speeds of respective trajectories will be different, in general.

5.1.3 Projection Operators

Before proceeding further let us define the projection operator which will be a useful tool in the analysis of the proposed methods.

Definition 18 *Projection Matrix :*
Let $\mathbf{v}, \mathbf{g} \in \mathbb{R}^n$ be any vectors. The projection of $\mathbf{v}$ on to the span of the vector $\mathbf{g}$ can be calculated as:

$$proj_{\mathbf{g}}\mathbf{v} = \frac{\mathbf{g}^T\mathbf{v}}{\mathbf{g}^T\mathbf{g}}\mathbf{g} = \mathbf{g}\frac{\mathbf{g}^T\mathbf{v}}{\mathbf{g}^T\mathbf{g}} = \frac{\mathbf{g}\mathbf{g}^T}{\mathbf{g}^T\mathbf{g}}\mathbf{v} = G\mathbf{v}$$

The projection matrix $G \in \mathbb{R}^{n\times n}$ whose range space is the span of the vector $\mathbf{g}$ is defined as:

$$G = \frac{\mathbf{g}\mathbf{g}^T}{\mathbf{g}^T\mathbf{g}} \tag{5.5}$$

where $\mathbf{g} \neq 0$ $\square$.

In the subsequent section the vector $\mathbf{g}$ will be a function of $\mathbf{x} \in \mathbb{R}^n$. It well might be the case that $\mathbf{g}^T(\mathbf{x})\mathbf{g}(\mathbf{x}) = 0$ for some $\mathbf{x}$. Certainly, for those values of $\mathbf{x}$, the projection operator is not defined.

Fact 1 *Let G be a projection operator as given in the Definition 18. Then the following holds for any $n \in \mathbb{N}$*

$$G^n = G \tag{5.6}$$

.

Proof : By using (5.5), we obtain:

$$G^2 = \left(\frac{\mathbf{g}\mathbf{g}^T}{\mathbf{g}^T\mathbf{g}}\right)\left(\frac{\mathbf{g}\mathbf{g}^T}{\mathbf{g}^T\mathbf{g}}\right) = \frac{\mathbf{g}\mathbf{g}^T}{\mathbf{g}^T\mathbf{g}} = G$$

By repeating this argument we obtain (5.6). □

Fact 2 *Let G be a projection operator as given in the Definition 18. Then G is symmetric and positive semi-definite.*

Proof : It is obvious that the projection operator G is symmetric, see Definition 18. For any $\mathbf{y} \in \mathbb{R}^n$ we obtain:

$$\mathbf{y}^T G\mathbf{y} = \mathbf{y}^T G^2\mathbf{y} = \mathbf{y}^T G^T G\mathbf{y} = \|G\mathbf{y}\|^2 \geq 0.$$

by using (5.6). Hence G is positive semi-definite as well. □

Definition 19 *Orthogonal Complement of a Projection Matrix :*
The complement of the space spanned by $\mathbf{g}(\mathbf{x})$ *is defined as* $\{\mathbf{x}|\mathbf{x}^T G\mathbf{y} = 0, \mathbf{y} \in \mathbb{R}^n\}$. *The projection matrix on to the complement space,* $G^\perp$, *is defined as:*

$$G^\perp = I - G \tag{5.7}$$

where $I \in \mathbb{R}^{n\times n}$ *is the unit matrix.* □.

Fact 3 *Let* G *be a projection matrix as in Definition 18 and let* $G^\perp$ *be the projection matrix as in Definition 19. Then any vector* $\mathbf{x} \in \mathbb{R}^\mathbf{n}$ *can be written as the sum of the projection of* $\mathbf{x}$ *onto the space spanned by* $\mathbf{g}$ *and the projection of* $\mathbf{x}$ *onto the orthogonal complement of this space, that is :*

$$\mathbf{x} = G\mathbf{x} + G^\perp\mathbf{x} \tag{5.8}$$

Moreover, $G\mathbf{x} \perp G^\perp\mathbf{x}$, *i.e.*

$$(G\mathbf{x})^T\ G^\perp\mathbf{x} = 0 \tag{5.9}$$

□.

Proof : (5.8) follows from (5.7). Since for any $\mathbf{y} \in \mathbb{R}^n$,

$$\begin{aligned}(G\mathbf{x})^T\, G^{\perp}\mathbf{y} &= \mathbf{x}^T G^T G^{\perp}\mathbf{y} \\ &= \mathbf{x}^T G G^{\perp}\mathbf{y} \\ &= \mathbf{x}^T (G - G^2)\mathbf{y} \\ &= 0\end{aligned}$$

It follows that (5.9) also holds. □.

5.1.4 Other Solutions

The solutions given in Section 5.1.2 will work for $n = 2$ and $m = 1$. It is easy to saturate the control signal for $m = 1$. This saturation of control signal must be revised for higher dimensions. Consider Figure 5.4 which gives a possible situation for $n = 3$ and $m = 2$. This point of view will lead us to have some other way to calculate β and $\mathbf{u}(\mathbf{x})$ as well as provide some motivation to the subsequent section. There are two parallel and two

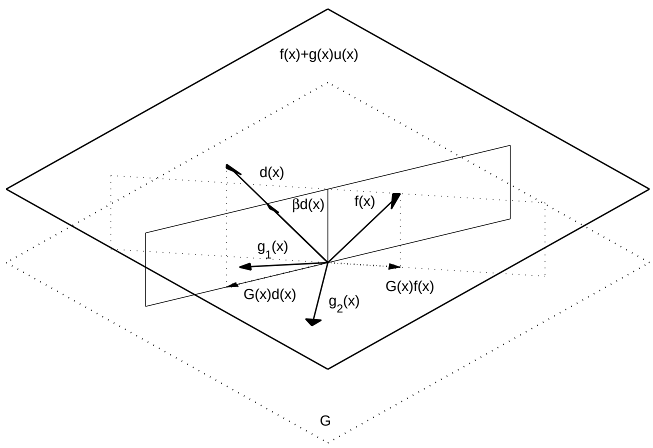


Figure 5.4: An affine system with $n = 3$ and $m = 2$.

vertical planes in Figure 5.4. The horizontal plane shown with dotted border is the span of the columns of $\mathbf{g}(\mathbf{x})$ in (5.1). The horizontal plane shown with solid borders represents the space of $\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x})$ for all possible values of $\mathbf{u}(\mathbf{x})$. The projection matrix G, which maps a given vector $\mathbf{x}$ onto the plane $\mathcal{G}$, can be defined as given in Definition 18. Let $\mathbf{g}(\mathbf{x}) = [\mathbf{g}_1(\mathbf{x}) \ldots \mathbf{g}_m(\mathbf{x})]$ then :

$$G(\mathbf{x}) = \sum_{i=1}^{m} \frac{\hat{\mathbf{g}}_i(\mathbf{x})\hat{\mathbf{g}}_i(\mathbf{x})^T}{\hat{\mathbf{g}}_i(\mathbf{x})^T\hat{\mathbf{g}}_i(\mathbf{x})} \tag{5.10}$$

where $\hat{\mathbf{g}}_i(\mathbf{x})$ are the vectors which are obtained by Gram-Schmidt orthogonalization of $\mathbf{g}_i(\mathbf{x})$ and $\hat{\mathbf{g}}_i(\mathbf{x})^T\hat{\mathbf{g}}_i(\mathbf{x}) \neq 0$. The projection matrix to the space $\mathcal{G}^\perp$, where $\perp$ stands for orthogonal complement, is defined as $G^\perp(\mathbf{x}) = I - G(\mathbf{x})$ and $I \in \mathbb{R}^{n\times n}$ unit matrix. The vertical plane shown with dotted borders is the span of $\mathbf{f}(\mathbf{x})$ and $G(\mathbf{x})\mathbf{f}(\mathbf{x})$ and the one shown with solid borders is the span of $\mathbf{d}(\mathbf{x})$ and $G(\mathbf{x})\mathbf{d}(\mathbf{x})$. It is obvious from Figure 5.4 that the projection of the vectors $\beta\mathbf{d}(\mathbf{x})$ and $\mathbf{f}(\mathbf{x})$ onto the space $\mathcal{G}^\perp$ are equal. This geometrical idea can be proven analytically as follows. Assume that the Equation (5.2) has a solution for $\mathbf{u}(\mathbf{x})$ and β. Then multiplying both sides with $G^\perp(\mathbf{x})$, we obtain:

$$(I - G(\mathbf{x}))\mathbf{f}(\mathbf{x}) = \beta(I - G(\mathbf{x}))\mathbf{d}(\mathbf{x}) \tag{5.11}$$

(Note that we have $G^\perp(\mathbf{x})\mathbf{g}(\mathbf{x}) = 0$). If we take the inner product of the two vectors, we obtain the following equation:

$$\begin{aligned} ((I - G(\mathbf{x}))\beta\mathbf{d}(\mathbf{x}))^T(I - G(\mathbf{x}))\mathbf{f}(\mathbf{x}) &= \beta\mathbf{d}(\mathbf{x})^T(I - G(\mathbf{x}))^2\mathbf{f}(\mathbf{x}) \\ &= \beta\mathbf{d}(\mathbf{x})^T(I - G(\mathbf{x}))\mathbf{f}(\mathbf{x}) \end{aligned} \tag{5.12}$$

where we used Fact 1. On the other hand, using (5.11), we obtain:

$$\begin{aligned} ((I - G(\mathbf{x}))\beta\mathbf{d}(\mathbf{x}))^T(I - G(\mathbf{x}))\mathbf{f}(\mathbf{x}) &= \|(I - G(\mathbf{x}))\beta\mathbf{d}(\mathbf{x})\|\|(I - G(\mathbf{x}))\mathbf{f}(\mathbf{x})\| \\ &= \|\beta(I - G(\mathbf{x}))\mathbf{d}(\mathbf{x})\|^2 \\ &= \beta^2\|(I - G(\mathbf{x}))\mathbf{d}(\mathbf{x})\|^2 \\ &= \beta^2\mathbf{d}(\mathbf{x})^T(I - G(\mathbf{x}))^2\mathbf{d}(\mathbf{x}) \\ &= \beta^2\mathbf{d}(\mathbf{x})^T(I - G(\mathbf{x}))\mathbf{d}(\mathbf{x}) \end{aligned} \tag{5.13}$$

From (5.12) and (5.13), we obtain:

$$\beta = \frac{\mathbf{d}(\mathbf{x})^T(I - G(\mathbf{x}))\mathbf{f}(\mathbf{x})}{\mathbf{d}(\mathbf{x})^T(I - G(\mathbf{x}))\mathbf{d}(\mathbf{x})} \tag{5.14}$$

Similarly repeating the same manipulations in (5.13), we obtain:

$$\begin{aligned} ((I - G(\mathbf{x}))\beta\mathbf{d}(\mathbf{x}))^T(I - G(\mathbf{x}))\mathbf{f}(\mathbf{x}) &= \|(I - G(\mathbf{x}))\beta\mathbf{d}(\mathbf{x})\|\|(I - G(\mathbf{x}))\mathbf{f}(\mathbf{x})\| \\ &= \|(I - G(\mathbf{x}))\mathbf{d}(\mathbf{x})\|^2 \\ &= \mathbf{f}(\mathbf{x})^T(I - G(\mathbf{x}))^2\mathbf{d}(\mathbf{x}) \\ &= \mathbf{f}(\mathbf{x})^T(I - G(\mathbf{x}))\mathbf{d}(\mathbf{x}) \end{aligned} \tag{5.15}$$

Hence from (5.12) and (5.15), we obtain :

$$\beta = \frac{\mathbf{f}(\mathbf{x})^T(I - G(\mathbf{x}))\mathbf{f}(\mathbf{x})}{\mathbf{f}(\mathbf{x})^T(I - G(\mathbf{x}))\mathbf{d}(\mathbf{x})} \tag{5.16}$$

From Fact 2 it follows that the denominator of (5.14) and numerator or (5.16) are nonnegative. So the sign of β is determined by the sign of the expression $\mathbf{d}(\mathbf{x})^T(I - G(\mathbf{x}))\mathbf{f}(\mathbf{x})$. Whenever this expression is less than zero, we have Problem of Type 1 stated previously in this section. If $\mathbf{d}(\mathbf{x})$ is in the span of the columns of $\mathbf{g}(\mathbf{x})$ then the inner product of $\mathbf{d}(\mathbf{x})$ with any vector in the complement of the space spanned by the columns of $\mathbf{g}(\mathbf{x})$ would be zero. Under this assumption on $\mathbf{d}(\mathbf{x})$ if we consider the denominators of (5.14) and (5.16) we see that β would be $\mp\infty$. This corresponds to the problem which was stated as type 2.

Let us consider (5.11), and let $\mathbf{z}(\mathbf{x}) : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \mathbf{z}(\mathbf{x}) \neq 0$ be an arbitrary vector function. From (5.11) we obtain:

$$\begin{aligned} (I - G(\mathbf{x}))\mathbf{f}(\mathbf{x}) &= \beta(I - G(\mathbf{x}))\mathbf{d}(\mathbf{x}) \\ \mathbf{z}(\mathbf{x})^T(I - G(\mathbf{x}))\mathbf{f}(\mathbf{x}) &= \beta\mathbf{z}(\mathbf{x})^T(I - G(\mathbf{x}))\mathbf{d}(\mathbf{x}) \end{aligned} \tag{5.17}$$

Hence β can be found as:

$$\beta = \frac{\mathbf{z}(\mathbf{x})^T(I - G(\mathbf{x}))\mathbf{f}(\mathbf{x})}{\mathbf{z}(\mathbf{x})^T(I - G(\mathbf{x}))\mathbf{d}(\mathbf{x})} \tag{5.18}$$

In (5.18) if we equate $\mathbf{z}(\mathbf{x}) = \mathbf{d}(\mathbf{x})$ we obtain (5.14) in the same way if set $\mathbf{z}(\mathbf{x}) = \mathbf{f}(\mathbf{x})$ we obtain (5.16). This result can be summarized as follows.

Corollary 1 *Let the Equation (5.2) have a solution for* $\mathbf{u}(\mathbf{x})$ *and* β*, and let* $\mathbf{z}(\mathbf{x}) \neq 0$ *be an arbitrary vector field. Then* β *can be given as in (5.18) and is independent of the choice of* $\mathbf{z}(\mathbf{x})$*.*

Proof : Obvious from (5.11), since $\beta(\cdot)$ is a scalar. □

The solvability condition in the corollary given above can be relaxed in certain cases.

Fact 4 *Let the range of* $G^\perp(\mathbf{x})$ *be a one dimensional vector space. Let* $G^\perp(\mathbf{x})\mathbf{d}(\mathbf{x}) \neq 0$*. Then (5.2)always have a solution* $\beta(\cdot)$*. This solution is given by (5.18), where* $\mathbf{z}(\mathbf{x}) \neq 0$ *is an arbitrary vector field.*

Proof : Since the range of $G^\perp(\mathbf{x}) = I - G(\mathbf{x})$ is one dimensional, then the vectors $G^\perp(\mathbf{x})\mathbf{f}(\mathbf{x})$ and $G^\perp(\mathbf{x})\mathbf{d}(\mathbf{x})$ lie in the same one-dimensional subspace. Hence if $G^{\perp}(\mathbf{x})\mathbf{d}(\mathbf{x}) \neq 0$, then there exists a $\beta(\mathbf{x})$ such that

$$G^{\perp}(\mathbf{x})\mathbf{f}(\mathbf{x}) = \beta(\mathbf{x})G^{\perp}(\mathbf{x})\mathbf{d}(\mathbf{x}) \tag{5.19}$$

Note that, since $G^{\perp}(\mathbf{x})\mathbf{f}(\mathbf{x})$ and $G^{\perp}(\mathbf{x})\mathbf{d}(\mathbf{x})$ lie in the same one dimensional subspace, the angle θ between them is either 0 or π, and β given by (5.19) is $\beta > 0$ if $\theta = 0$, and $\beta < 0$ if $\theta = \pi$.

To show that (5.2) always has a solution $\mathbf{u}(\mathbf{x})$ in this case ($\beta(\cdot)$ given by (5.19)), let us rewrite (5.2) as follows:

$$\begin{aligned} \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x}) &= G(\mathbf{x})[\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x})] + G^{\perp}(\mathbf{x})[\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x})] \\ &= G(\mathbf{x})\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x}) + G^{\perp}(\mathbf{x})\mathbf{f}(\mathbf{x}) \end{aligned} \tag{5.20}$$

where we used the fact $G^{\perp}(\mathbf{x})\mathbf{g}(\mathbf{x}) = 0$ and $G(\mathbf{x})\mathbf{g}(\mathbf{x}) = \mathbf{g}(\mathbf{x})$. Since $G(\mathbf{x})\mathbf{f}(\mathbf{x})$ is in the column space of $G(\mathbf{x})$, there exists a vector $\gamma_f(\mathbf{x}) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that:

$$G(\mathbf{x})\mathbf{f}(\mathbf{x}) \quad = \quad \mathbf{g}(\mathbf{x})\gamma_f(\mathbf{x}) \tag{5.21}$$

Similarly, since $G(\mathbf{x})\mathbf{d}(\mathbf{x})$ is in the column space of $G(\mathbf{x})$, there exists a vector $\gamma_d(\mathbf{x}) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that:

$$G(\mathbf{x})\mathbf{d}(\mathbf{x}) \quad = \quad \mathbf{g}(\mathbf{x})\gamma_d(\mathbf{x}) \tag{5.22}$$

Let $\beta(\cdot)$ be the solution of (5.19), $\gamma_f(\mathbf{x})$ and $\gamma_d(\mathbf{x})$ be as given in (5.21) and (5.22). Let us consider the following control law:

$$\mathbf{u}(\mathbf{x}) = -\gamma_f(\mathbf{x}) + \beta(\mathbf{x})\gamma_d(\mathbf{x}) \tag{5.23}$$

By using (5.19),(5.21),(5.23) in (5.20), we obtain:

$$\begin{aligned} \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x}) &= G(\mathbf{x})\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x}) + G^{\perp}(\mathbf{x})\mathbf{f}(\mathbf{x}) \\ &= \mathbf{g}(\mathbf{x})\gamma_f(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x}) + \beta(\mathbf{x})G^{\perp}(\mathbf{x})\mathbf{d}(\mathbf{x}) \\ &= \beta(\mathbf{x})\mathbf{g}(\mathbf{x})\gamma_d(\mathbf{x}) + \beta(\mathbf{x})G^{\perp}(\mathbf{x})\mathbf{d}(\mathbf{x}) \\ &= \beta(\mathbf{x})G(\mathbf{x})\gamma_d(\mathbf{x}) + \beta(\mathbf{x})G^{\perp}(\mathbf{x})\mathbf{d}(\mathbf{x}) \\ &= \beta(\mathbf{x})\mathbf{d}(\mathbf{x}) \end{aligned}$$

Hence $\beta(\mathbf{x})$ given by (5.19) and $\mathbf{u}(\mathbf{x})$ given by (5.23) is a solution of (5.2). Note that, $\beta(\mathbf{x})$ can be given by (5.18), where $\mathbf{z}(\mathbf{x}) \neq 0$ is arbitrary. □

Corollary 2 *Let* $n = m + 1$, *and assume that the columns of* $\mathbf{g}(\mathbf{x})$ *are linearly independent, i.e.* $rank(\mathbf{g}(\mathbf{x})) = m$. *Then the range of* $G^{\perp}(\mathbf{x})$ *is one dimensional.*

Proof : This result immediately follows from the fact that $rank(\mathbf{g}(\mathbf{x})) = m$ and $n = m + 1$. $\square$

Once β is known it is straightforward to calculate the required control signal as follows;

$$\begin{aligned}
\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x}) &= \beta\mathbf{d}(\mathbf{x}) \\
G(\mathbf{x})\mathbf{f}(\mathbf{x}) + (I - G(\mathbf{x}))\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x}) &= \beta G(\mathbf{x})\mathbf{d}(\mathbf{x}) + \beta(I - G(\mathbf{x}))\mathbf{d}(\mathbf{x}) \\
G(\mathbf{x})\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x}) &= \beta G(\mathbf{x})\mathbf{d}(\mathbf{x}) \\
\mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x}) &= G(\mathbf{x})(\beta\mathbf{d}(\mathbf{x}) - \mathbf{f}(\mathbf{x}))
\end{aligned}$$

In passing from the second line to the third one Equation (5.11) is used. In the last line since the right hand side is within the column space of $\mathbf{g}(\mathbf{x})$ the control signal can be calculated as;

$$\begin{aligned}
\mathbf{u}(\mathbf{x}) &= [\mathbf{g}(\mathbf{x})^T\mathbf{g}(\mathbf{x})]^{-1}\mathbf{g}(\mathbf{x})^T G(\mathbf{x})(\beta\mathbf{d}(\mathbf{x}) - \mathbf{f}(\mathbf{x})) \\
&= [\mathbf{g}(\mathbf{x})^T\mathbf{g}(\mathbf{x})]^{-1}\mathbf{g}(\mathbf{x})^T(\beta\mathbf{d}(\mathbf{x}) - \mathbf{f}(\mathbf{x}))
\end{aligned} \tag{5.24}$$

provided that $\mathbf{g}(\mathbf{x})\mathbf{g}(\mathbf{x})^T$ is invertible. Note that since $G(\mathbf{x}) = G(\mathbf{x})^T$ we have $\mathbf{g}(\mathbf{x})^T G(\mathbf{x}) = \mathbf{g}(\mathbf{x})^T G(\mathbf{x})^T = [G(\mathbf{x})\mathbf{g}(\mathbf{x})]^T = \mathbf{g}(\mathbf{x})^T$.

5.1.5 Saturation of the Control Signal

In Figure 5.3 the solution for $u(\mathbf{x})$ was discussed when $n = 2$ and $m = 1$. The control signal was saturated at its minimum or maximum value whichever minimizes the phase difference between $\mathbf{d}(\mathbf{x})$ and $\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x})$. Let us investigate the same problem for higher dimensions. In this section we will introduce two saturation schemes. One of these schemes is saturation with planar boundaries, and the other is saturation with circular boundaries. Let us investigate first the planar one. For $n = 3$ and $m = 2$, suppose that we have the constraints on $\mathbf{u}(\mathbf{x})$ as $\mathbf{u}(\mathbf{x}) : \mathbb{R}^3 \rightarrow [U_{1,min}, U_{1,max}] \times [U_{2,min}, U_{2,max}]$.

In Figure 5.5 the vectors $\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x})$, in which $\mathbf{u}(\mathbf{x})$ is within its boundaries, is shown by the polygon. As it is seen $\beta\mathbf{d}(\mathbf{x})$ requires a $\mathbf{u}(\mathbf{x})$ which lies out of its boundaries.

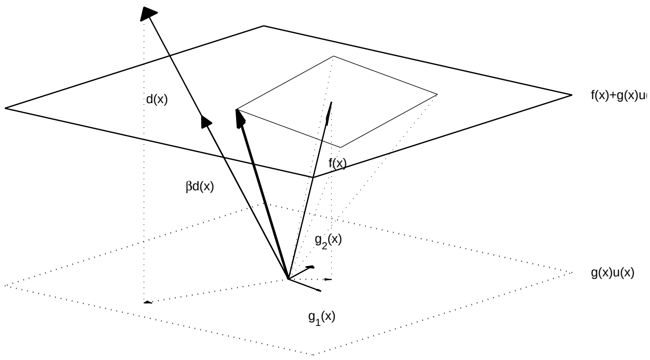


Figure 5.5: An affine system with constraints on $\mathbf{u}(\mathbf{x})$.

We have to find another value of $\mathbf{u}(\mathbf{x})$ such that $\mathbf{f}(\mathbf{x})+\mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x})$ lies on the boundary of the polygon and has a minimum angle between $\mathbf{f}(\mathbf{x})+\mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x})$ and $\mathbf{d}(\mathbf{x})$. The vector which requires an acceptable value of $\mathbf{u}(\mathbf{x})$ and has a minimum phase difference with $\mathbf{d}(\mathbf{x})$ is shown with the thicker vector. In the following discussion we will give an outline

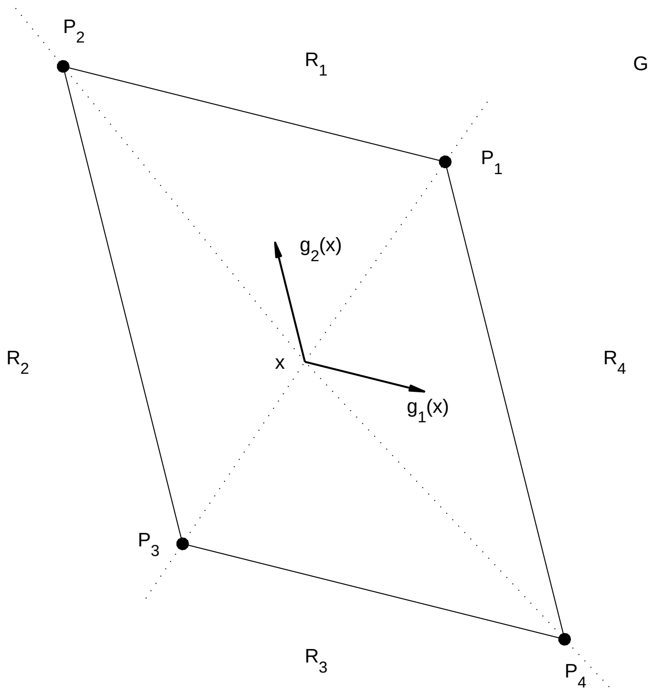


Figure 5.6: Phase matched saturation of $\mathbf{u}(\mathbf{x})$ with planar boundaries.

for saturation with planar boundaries. In Figure 5.6 the polygon is shown on the span of the columns of $\mathbf{g}(\mathbf{x})$. There are four vectors $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$, and $\mathbf{p}_4$, which determine the corners of the polygon. These vectors are calculated as;

$$\begin{aligned}
\mathbf{p}_1 &= \hat{\mathbf{p}}_1 + (I - G(\mathbf{x}))\mathbf{f}(\mathbf{x}) \\
\mathbf{p}_2 &= \hat{\mathbf{p}}_2 + (I - G(\mathbf{x}))\mathbf{f}(\mathbf{x}) \\
\mathbf{p}_3 &= \hat{\mathbf{p}}_3 + (I - G(\mathbf{x}))\mathbf{f}(\mathbf{x}) \\
\mathbf{p}_4 &= \hat{\mathbf{p}}_4 + (I - G(\mathbf{x}))\mathbf{f}(\mathbf{x})
\end{aligned}$$

where,

$$\begin{aligned}
\hat{\mathbf{p}}_1 &= \mathbf{g}_1(\mathbf{x})U_{1,max} + \mathbf{g}_2(\mathbf{x})U_{2,max} & \hat{\mathbf{p}}_3 &= \mathbf{g}_1(\mathbf{x})U_{1,min} + \mathbf{g}_2(\mathbf{x})U_{2,min} \\
\hat{\mathbf{p}}_2 &= \mathbf{g}_1(\mathbf{x})U_{1,min} + \mathbf{g}_2(\mathbf{x})U_{2,max} & \hat{\mathbf{p}}_4 &= \mathbf{g}_1(\mathbf{x})U_{1,max} + \mathbf{g}_2(\mathbf{x})U_{2,min}
\end{aligned}$$

The four related regions $R_i, i = 1, \ldots, 4$ as shown in Figure 5.7, can be defined as follows:

$$\begin{aligned}
R_1 &= \{\mathbf{x} | \mathbf{x} = \gamma_1 \hat{\mathbf{p}}_1 + \gamma_2 \hat{\mathbf{p}}_2 + (I - G(\mathbf{x}))\mathbf{f}(\mathbf{x}),\ \gamma_1, \gamma_2 > \mathbb{R}^+\} \\
R_2 &= \{\mathbf{x} | \mathbf{x} = \gamma_2 \hat{\mathbf{p}}_1 + \gamma_2 \hat{\mathbf{p}}_3 + (I - G(\mathbf{x}))\mathbf{f}(\mathbf{x}),\ \gamma_1, \gamma_2 > \mathbb{R}^+\} \\
R_3 &= \{\mathbf{x} | \mathbf{x} = \gamma_3 \hat{\mathbf{p}}_1 + \gamma_2 \hat{\mathbf{p}}_4 + (I - G(\mathbf{x}))\mathbf{f}(\mathbf{x}),\ \gamma_1, \gamma_2 > \mathbb{R}^+\} \\
R_4 &= \{\mathbf{x} | \mathbf{x} = \gamma_4 \hat{\mathbf{p}}_1 + \gamma_2 \hat{\mathbf{p}}_1 + (I - G(\mathbf{x}))\mathbf{f}(\mathbf{x}),\ \gamma_1, \gamma_2 > \mathbb{R}^+\}
\end{aligned}$$

The point P_i is the projection of the tip of the vector $\mathbf{p}_i$ onto the column space of $\mathbf{g}(\mathbf{x})$. Each pair of these vectors form a plane passing through the origin. If we consider Figure (5.5) the intersection of these planes will form an inverted pyramid whose base is the polygon. These planes will have normal vectors defined by;

$$\begin{aligned}
\mathbf{n}_{R_1} &= \tfrac{\mathbf{p}_2 \times \mathbf{p}_1}{\|\mathbf{p}_2 \times \mathbf{p}_1\|} & \mathbf{n}_{R_3} &= \tfrac{\mathbf{p}_4 \times \mathbf{p}_3}{\|\mathbf{p}_4 \times \mathbf{p}_3\|} \\
\mathbf{n}_{R_2} &= \tfrac{\mathbf{p}_3 \times \mathbf{p}_2}{\|\mathbf{p}_3 \times \mathbf{p}_2\|} & \mathbf{n}_{R_4} &= \tfrac{\mathbf{p}_1 \times \mathbf{p}_4}{\|\mathbf{p}_1 \times \mathbf{p}_4\|}
\end{aligned}$$

Let $\mathbf{n} \in \mathbb{R}^n$ be an arbitrary nonzero vector, and let $\mathcal{S}$ be a plane defined by $\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{n}^T\mathbf{x} = 0\}$, i.e. $\mathbf{n}$ is the normal of $\mathcal{S}$. If $\|\mathbf{n}\| = 1$, then the projection matrix on the plane $\mathcal{S}$ would be $S = I - \mathbf{n}\mathbf{n}^T$ where $I \in \mathbb{R}^{n \times n}$ is the unit matrix. It is known that the angle between a vector on the plane $\mathcal{S}$ and a vector $\mathbf{d} \in \mathbb{R}^n$ is minimum when $\mathbf{x} = \gamma\ proj_{\mathcal{S}}\ \mathbf{d} = \gamma S\mathbf{d} = \gamma(I - \mathbf{n}\mathbf{n}^T)\mathbf{d}$ where $\gamma \in \mathbb{R}^+$. So the candidate vectors, $\mathbf{c}_{R_i} \in \mathbb{R}^n$, which might have minimum phase with $\mathbf{d}(\mathbf{x})$ would be [58];

$$\mathbf{c}_{R_i} = \gamma_{R_i}(I - \mathbf{n}_{R_i}\mathbf{n}_{R_i}^T)\mathbf{d}(\mathbf{x})$$

where $i = 1, \ldots, 4$. After the calculation of $\mathbf{c}_{R_i}$ it is necessary to check whether they are on the borders of the polygon or not. It might be the case that none of them are on the borders, which are formed by connecting the points $\mathbf{p}_i$. In such cases the minimum angle occurs at one of these points. As it is seen, although we utilize the control signal to its full range, we have to do rather tedious calculations when m becomes higher.

Saturation with circular boundaries are calculationwise simpler but does not utilize the control signal to its full range. Consider Figure 5.7 where the planar boundaries are shown as the dotted lines. The circle has its center at the origin and has the largest

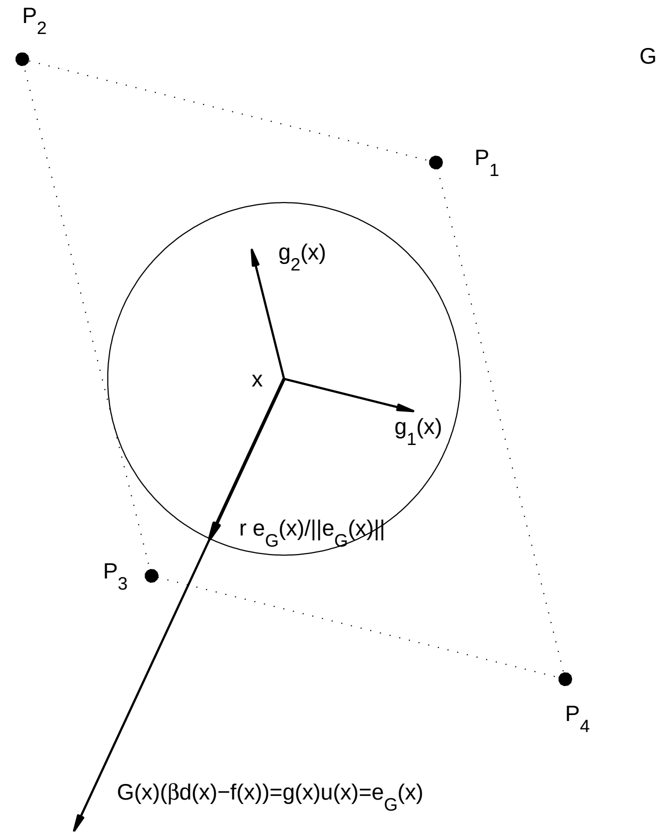


Figure 5.7: Phase matched saturation of $\mathbf{u}(\mathbf{x})$ with circular boundaries.

diameter that the lines formed by P_i's can confine. Let the distance of each line to the origin be d_i. Then the diameter of the circle will be $r = \min_i\{d_i\}$. Let us recall the equation used in the derivation of (5.24):

$$\mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x}) = G(\mathbf{x})(\beta\mathbf{d}(\mathbf{x}) - \mathbf{f}(\mathbf{x}))$$

Let us assume again that the calculated vector $\beta\mathbf{d}(\mathbf{x})$ require a control signal which is out of the boundary, i.e. $\|\mathbf{u}(\mathbf{x})\| > r$. Since the boundary is circular all we require is to scale the vector $G(\mathbf{x})(\beta\mathbf{d}(\mathbf{x})-\mathbf{f}(\mathbf{x}))$. The required control signal can be found as:

$$\begin{aligned}\mathbf{u}(\mathbf{x}) &= r[\mathbf{g}(\mathbf{x})^T\mathbf{g}(\mathbf{x})]^{-1}\tfrac{\mathbf{g}(\mathbf{x})^T G(\mathbf{x})(\beta\mathbf{d}(\mathbf{x})-\mathbf{f}(\mathbf{x}))}{\|G(\mathbf{x})(\beta\mathbf{d}(\mathbf{x})-\mathbf{f}(\mathbf{x}))\|} \\ &= r[\mathbf{g}(\mathbf{x})^T\mathbf{g}(\mathbf{x})]^{-1}\tfrac{\mathbf{g}(\mathbf{x})^T(\beta\mathbf{d}(\mathbf{x})-\mathbf{f}(\mathbf{x}))}{\|G(\mathbf{x})(\beta\mathbf{d}(\mathbf{x})-\mathbf{f}(\mathbf{x}))\|}\end{aligned} \tag{5.25}$$

provided that $\mathbf{g}(\mathbf{x})\mathbf{g}(\mathbf{x})^T$ is invertible. This method of saturation is computationwise simpler but is more strict on the control signal compared to the saturation with planar boundaries. These two saturation methods can be used to avoid problem of type 1 and type 2. Problem of type 3 has a solution which we give in the following section.

5.2 Minimization of the Norm

Consider the system given in (5.1). Let $\mathbf{h}(\mathbf{x}) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a vector function, and $G(\mathbf{x})$ be the projection matrix given in (5.10) such that $\mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x}) = G(\mathbf{x})\mathbf{h}(\mathbf{x})$. Let $\mathbf{d}(\mathbf{x}) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the vector function defining the desired phase portrait, and $\mathbf{e}(\mathbf{x}) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be defined as;

$$\begin{aligned}\mathbf{e}(\mathbf{x}) &= \mathbf{d}(\mathbf{x})-\mathbf{f}(\mathbf{x})-\mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x}) \\ &= G(\mathbf{x})\mathbf{d}(\mathbf{x})+(I-G(\mathbf{x}))\mathbf{d}(\mathbf{x})-G(\mathbf{x})\mathbf{f}(\mathbf{x})-(I-G(\mathbf{x}))\mathbf{f}(\mathbf{x})-G(\mathbf{x})\mathbf{h}(\mathbf{x}) \\ &= G(\mathbf{x})\left(\mathbf{d}(\mathbf{x})-\mathbf{f}(\mathbf{x})-\mathbf{h}(\mathbf{x})\right)+(I-G(\mathbf{x}))\left(\mathbf{d}(\mathbf{x})-\mathbf{f}(\mathbf{x})\right)\end{aligned} \tag{5.26}$$

Our aim in this section is to find the control signal that minimizes the norm of the vector $\mathbf{e}(\mathbf{x})$. We will try to find the vector function $\mathbf{h}(\mathbf{x})$ first and then using the relation given above we will calculate the control signal $\mathbf{u}(\mathbf{x})$. Let $\mathbf{e}_G(\mathbf{x}) = \mathbf{d}(\mathbf{x})-\mathbf{f}(\mathbf{x})-\mathbf{h}(\mathbf{x})$ and $\mathbf{e}_{G^\perp}(\mathbf{x}) = \mathbf{d}(\mathbf{x})-\mathbf{f}(\mathbf{x})$

$$\begin{aligned}\tfrac{1}{2}\tfrac{\partial}{\partial\mathbf{h}(\mathbf{x})}\|\mathbf{e}(\mathbf{x})\|^2 &= \tfrac{\partial}{\partial\mathbf{e}_G(\mathbf{x})}\|\mathbf{e}(\mathbf{x})\|^2\tfrac{\partial}{\partial\mathbf{h}(\mathbf{x})}\mathbf{e}_G(\mathbf{x}) \\ &= -\tfrac{\partial}{\partial\mathbf{e}_G(\mathbf{x})}(\mathbf{e}(\mathbf{x})^T\mathbf{e}(\mathbf{x})) \\ &= -\tfrac{\partial}{\partial\mathbf{e}_G(\mathbf{x})}(\mathbf{e}_G(\mathbf{x})^T G(\mathbf{x})\mathbf{e}_G(\mathbf{x})+\mathbf{e}_{G^\perp}(\mathbf{x})^T(I-G(\mathbf{x}))\mathbf{e}_{G^\perp}(\mathbf{x})) \\ &= -G(\mathbf{x})\mathbf{e}_G(\mathbf{x})\end{aligned}$$

Let $\mathbf{s}\in\mathbb{R}^n$ be an arbitrary vector. The derivative $\frac{\partial}{\partial\mathbf{h}(\mathbf{x})}\|\mathbf{e}(\mathbf{x})\|^2 = 0$ if $\mathbf{e}_G(\mathbf{x}) = (I-G(\mathbf{x}))\mathbf{s}$ or equivalently;

$$\begin{aligned}\mathbf{d}(\mathbf{x})-\mathbf{f}(\mathbf{x})-\mathbf{h}(\mathbf{x}) &= (I-G(\mathbf{x}))\mathbf{s} \\ \mathbf{h}(\mathbf{x}) &= \mathbf{d}(\mathbf{x})-\mathbf{f}(\mathbf{x})-(I-G(\mathbf{x}))\mathbf{s}\end{aligned}$$

Considering the assumption $\mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x}) = G(\mathbf{x})\mathbf{h}(\mathbf{x})$ we see that there is no contribution by the term $(I - G(\mathbf{x}))\mathbf{s}$ to the control signal due to the multiplication with the projection matrix $G(\mathbf{x})$. Therefore the vector $\mathbf{h}(\mathbf{x})$ which minimizes the norm square of the vector $\mathbf{e}(\mathbf{x})$ is given by;

$$\mathbf{h}(\mathbf{x}) = \mathbf{d}(\mathbf{x}) - \mathbf{f}(\mathbf{x}) \tag{5.27}$$

If we insert (5.27) into (5.26) the error vector would be $(I - G(\mathbf{x}))(\mathbf{d}(\mathbf{x}) - \mathbf{f}(\mathbf{x}))$. To test (5.27), set $\mathbf{h}(\mathbf{x}) = \mathbf{d}(\mathbf{x}) - \mathbf{f}(\mathbf{x}) - \mathbf{s}$. Then the error vector would be $G(\mathbf{x})\mathbf{s} + (I - G(\mathbf{x}))(\mathbf{d}(\mathbf{x}) - \mathbf{f}(\mathbf{x}))$. Since the additional component is in the orthogonal complement of the previous error vector the norm will increase for $\mathbf{s} \neq 0$.

The required control signal can be found as:

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &= [\mathbf{g}(\mathbf{x})^T\mathbf{g}(\mathbf{x})]^{-1}\mathbf{g}(\mathbf{x})^T G(\mathbf{x})\mathbf{h}(\mathbf{x}) \\ &= [\mathbf{g}(\mathbf{x})^T\mathbf{g}(\mathbf{x})]^{-1} G(\mathbf{x})\mathbf{h}(\mathbf{x}) \end{aligned} \tag{5.28}$$

provided that $\mathbf{g}(\mathbf{x})\mathbf{g}(\mathbf{x})^T$ is invertible.

In Section 5.1.1 we have defined problem of type 3 as the case when $\mathbf{f}(\mathbf{x})$ being in the span of the columns of $\mathbf{g}(\mathbf{x})$. In such cases the drifting effect of $\mathbf{f}(\mathbf{x})$ can be diminished by the application of suitable control signal, practically making it zero. In phase matching whenever this kind of problem occurs a control signal which is obtained by the minimization of the norm can be applied.

Chapter 6

Fuzzy Controller and System Design

Systems with fuzzy controllers and their stability have been investigated in many studies in the literature, see e.g. [17], [54], [57], [59], [60]. In most of these works, basically two types of fuzzy systems have been considered: Mamdani type or Takagi-Sugeno (T-S) type. The Mamdani type fuzzy system is well recognized and received by the society. The design process depends heavily on the knowledge of the system behavior. Since the T-S type fuzzy system can be interpreted as an interpolator of a collection of linear systems, the stability analysis and design is more analytical compared to the Mamdani type fuzzy system design. In T-S type fuzzy systems mainly the modelling issue is considered and the key issue is how to design the membership functions, or how to automatically partition the input space into the minimal set of local models which offer maximal approximation capability [61]. There are some adaptive algorithms developed to adjust the parameters of the membership functions which assume a certain shape like trapezoidal or gaussian (see [61], [62], [63]).

In this chapter we will first give some guidelines for the design of a fuzzy controller. In general, such guidelines are aimed at the design of a Mamdani type fuzzy controller [17], [59]. However, these guidelines can also be applied to the design of T-S type fuzzy controllers. Nevertheless the proposed design methods for T-S type fuzzy controllers in the literature might show deviations from these guidelines.

This chapter is organized as follows. In Section 6.1 we will give some coarse guidelines in the design of a fuzzy controller. In Section 6.2 we will propose a design methodology for the systems which have a completeness property. In Section 6.3 we will propose another design methodology which is based on data mining. (This method is also related to the phase portrait matching given in Chapter 5). Both of these proposed methods are for T-S type fuzzy controller.

6.1 General Design Aspects

There is no general systematic methodology for the construction of fuzzy controllers for challenging control applications that is guaranteed to result in a high-performance closed-loop control system. In Figure 6.1 a control system with negative unit feedback is given. In this section we will provide general guidelines and some methods to design a fuzzy controller for the controller block in Figure 6.1.

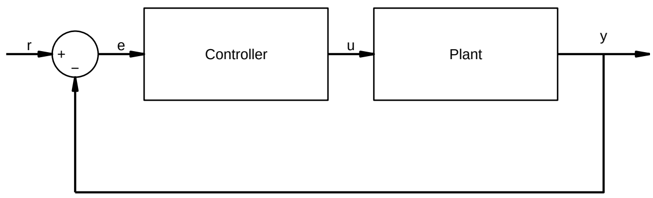


Figure 6.1: Control system with negative unit feedback.

Conventional control techniques provide many different approaches like feedback-linearization or sliding-mode control to the construction of nonlinear controllers. When there is a reasonably good model for the plant, which satisfies the necessary assumptions then conventional control can offer quite a viable solution to a control problem. It is said that more then 90% of all controllers in operation are PID controllers [59]. Therefore before attempting to design a fuzzy controller one should check the performance of the conventional designs. It is unwise to ignore past successes in control in the excitement over trying fuzzy control. Before listing the steps in the design of a fuzzy controller let us first identify the parameters of the design.

6.1.1 Fuzzy Controller Design Parameters

In Section 3.2 we have shown the basic building blocks of a fuzzy system. These blocks are fuzzification , rule base, inference engine and defuzzification blocks, see Figure 3.5. Apart from these processing blocks there might be a need to have some preprocessing block before the fuzzification block and a postprocessing block after the defuzzification block. Preprocessing block is used in general as a transformation from the original domain of measurement to another domain which makes the formation of the fuzzy sets and the rule base easier. The simplest preprocessing block might be a scaling which normalizes the domain of the membership function. Postprocessing is in general a scaling that is used when a normalization is performed in the preprocessing block.

Fuzzification Block

The design parameter of the fuzzification block is the choice of fuzzification strategy. There are two choices which depend on the inference used. If inference is composition based then the crisp input is associated with a fuzzy relation having the same domain as the relation obtained by the composition of the fuzzy relation of each IF-THEN rule. If inference is individual-rule-firing based then the crisp input is associated with a fuzzy set having the same domain as the universe of discourse of the variable representing the crisp input.

Rule Base Block

Rule base consists of IF-THEN rules. The antecedent part of the rule constitute a relation whose domain is the product space of the membership function domains. The consequent part is either an equation of control signal as a function of the inputs to the controller or a proposition on the control signal described by a fuzzy set. Therefore the design parameters of the rule base are:

- Choice of process state and control output variables,
- Choice of the contents of the rule antecedent and the rule consequent parts,
- Choice of membership function set for the plant state and control output variables,

- Derivation of the set of rules.

Inference Engine Block

There are two basic types of approaches employed in the design of the inference engine of a fuzzy controller: composition based inference and individual-rule-firing based inference. Both types of inference were given in Section 3.2. The latter type of inference is predominant because of its computational efficiency and less demand of memory. The design parameters of an inference engine are:

- Choice of inference engine, i.e. composition based inference or individual-rule-firing based inference,
- Choice of representing the meaning of a single rule, i.e. the implication operation on an IF-THEN statement (e.g min($\cdot$) operator),
- Choice of representing the meaning of a set of rules, i.e. the operation to combine the implied results from each IF-THEN statement (e.g max($\cdot$) operator).

Defuzzification Block

This block performs the so-called defuzzification which converts the fuzzy set of modified control output values into a single point-wise value, which is the inverse procedure of fuzzification. The design parameter of defuzzification block is the choice of defuzzification operators.

6.1.2 Fuzzy Controller Design Guidelines

In this section we will summarize the main features of the design process. The goal is to try to provide some basic design steps that are generic to all fuzzy controllers. The following list gives the design steps for nonadaptive fuzzy controller:

1) Begin by trying a conventional like PID or lead-lag controller. If the specifications are not met then try some more complicated methods like state-feedback, feedback

linearization, sliding-mode control. If this is successful then there is no need for a fuzzy controller.

2) Try to understand the behavior of the plant, how it react to inputs, what are the effects of disturbances, and what fundamental limitations it presents.A clear understanding comes from studying the physics of the process , developing mathematical models, using system identification methods, doing analysis, performing simulations, and using heuristic knowledge about the plant dynamics. The heuristic knowledge may come from, for example a human operator of the process or a control engineer. Sometimes, knowledge of the plant's behavior comes from actually trying out/simulating a controller on the system [31]. Depending on the knowledge gathered go over the following steps.

3) Decide on which fuzzy model is suitable for the control problem.

4) Choose a proper input to the fuzzy controller. This decision determines the antecedent part of IF-THEN rules. A good choice might be proportional, derivative, and integral of the error signal, or the states of the plant.

5) Choose the membership function for the antecedent (and consequent) part of the IF-THEN rules. If scaling is required then this should be considered with the choice of membership functions.

6) Decide on the operators like the "AND" operator used in the antecedent part, the inference engine, fuzzification and defuzzification operators. Decide on the operator to calculate the truth value of a single rule as in (3.2) and the operator to combine a set of rules (maximum operator or the addition are possible candidates).

7) Simulate the system to check for the specified objectives. In case of poor performance check out the following:

    a - Try tuning the fuzzy controller using the scaling of gains, if they exist.

    b - Revise the rule base and add more rules if possible. Analyze the rule base for inconsistency and incompleteness, see [64].

c - Try increasing the number of membership function used to represent an input, which increases the accuracy of the fuzzy controller on that variable.

It must be emphasized that the above guidelines do not constitute a systematic design procedure. As with conventional control design, a process of trial and error is generally needed. In [17], [59] and many other books there are general outlines for the choice of each parameter given above. Although the guideline given above is suitable also for fuzzy controller with T-S type, the design methods proposed in the literature for T-S type systems are in general more analytical. In most of the cases the proposed method depends mainly on the assumptions made on the system and the stability results provided. In the following section we will propose two design methods for T-S type controllers.

6.2 Fuzzy Controller Design for Systems with Completeness Property

In this section, we will analyze the stability of a T-S type fuzzy controller and give a sufficient condition on the membership functions to guarantee the stability. In Figure 3.1, some examples on the structure of the membership functions such as triangular, gaussian, and trapezoidal, were given. In general, a predefined structure on the membership function is assumed in the studies of current literature. In most of the studies the stability conditions of the plant which is independent of the membership functions are sought. This is similar to the stability condition of switching controller which is independent of the switching sequence.

In a T-S model the antecedent part of the IF-THEN rules are mostly composed of several propositions combined with AND operator. Each proposition is about one of the states or a measured feature of the system. The combinations of these propositions in the antecedent part of the rule constitutes a membership function for a fuzzy relation. There are some adaptive algorithms developed to adjust the parameters of the membership functions which assumes a certain shape like trapezoidal or gaussian [61], [62], [63]. A priori assumption of a structure on the membership functions will be a limiting factor on the performance of a FLC. In the sequel, we will propose a design procedure, which is based on local linearization, to obtain the membership functions of the fuzzy controller. This procedure does not require any assumption on the structure of the membership functions. The design procedure guarantees the stability of the closed-loop system under certain conditions.

6.2.1 Problem Statement and Definitions

In this subsection we consider the stability of the systems given by the following dynamics:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}) = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u} \tag{6.1}$$

where $\mathbf{x} \in \mathbb{R}^{\mathbf{n}}$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and $\mathbf{u} \in \mathbb{R}^m$ is the output of a fuzzy controller. We further assume that $\mathbf{x} = 0$ is an equilibrium point for the system given by (6.1) when $\mathbf{u} = 0$. We will call the system given by (6.1) as (asymptotically) stable if $\mathbf{x} = 0$ is (asymptotically) stable in the sense of Lyapunov. The problems investigated in this section are the stability of a T-S type fuzzy controller system and the design of a fuzzy controller such that (6.1) with feedback is stable. One of the common feature of the studies in the literature is the fact that the properties of the membership functions, like their value and functional structures, are not fully exploited [54], [57], [65]. The problem considered in this section is to design a fuzzy controller whose membership functions has no predefined structure for the system given by (6.1) such that the closed loop system is stable.

6.2.2 Stability Analysis

Let us first define the fuzzy controller and show the equivalent closed loop fuzzy system. Let, for $i = 1, 2, \ldots, r$, $S_i \in \mathbb{R}^n$ be regions in state space. The i^{th} rule ($\mathcal{L}_i$) of a fuzzy controller, which has a control law $\mathbf{u}_i(\mathbf{x})$ associated with the set S_i for $i = 1, 2, \ldots, r$ is given in the following clause;

$$\mathcal{L}_i : \text{IF “}x \text{ is in } S_i\text{” THEN } \mathbf{u}(\mathbf{x}) = \mathbf{u_i}(\mathbf{x}). \tag{6.2}$$

If the center of gravity is used as the defuzzification operation then the output of the fuzzy controller is given by the following equation:

$$\mathbf{u}(\mathbf{x}) = \frac{\sum_{\mathbf{i=1}}^{\mathbf{r}} \mu_{\mathbf{S_i}}(\mathbf{x})\mathbf{u_i}(\mathbf{x})}{\sum_{\mathbf{i=1}}^{\mathbf{r}} \mu_{\mathbf{S_i}}(\mathbf{x})} \tag{6.3}$$

where $\mu_{S_i}(\cdot)$ is the membership function associated with the set S_i of the i^{th} rule, $\mathcal{L}_i$. For the sake of simplicity, assume that the denominator in (6.3) equals 1, that is,

$$\sum_{i=1}^{r} \mu_{S_i}(\mathbf{x}) = 1. \tag{6.4}$$

Fact 5 *Consider the nonlinear system given by (6.1) and a fuzzy controller given by (6.2),(6.3). If (6.4) is satisfied then the closed-loop system is equivalent to a fuzzy system whose rules are as follows:*

$$\mathcal{L}_i : \mathit{IF} \text{ “}x \text{ is in } S_i\text{” } \mathit{THEN}\ \dot{\mathbf{x}} = \mathbf{F_i}(\mathbf{x}) \tag{6.5}$$

where $\mathbf{F_i}(\mathbf{x}) = \mathbf{F}(\mathbf{x}, \mathbf{u_i}(\mathbf{x}))$, $i = 1, 2, \ldots, r$.

Proof: The close-loop dynamics of (6.1) with the fuzzy controller, defined by (6.2), (6.3), and (6.4), is as follows;

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x}) \textstyle\sum_{i=1}^{r} \mu_{S_i}(\mathbf{x})\mathbf{u}_i(\mathbf{x}) \\ &= \textstyle\sum_{i=1}^{r} \mu_{S_i}(\mathbf{x})\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x}) \textstyle\sum_{i=1}^{r} \mu_{S_i}(\mathbf{x})\mathbf{u}_i(\mathbf{x}) \\ &= \textstyle\sum_{i=1}^{r} \mu_{S_i}(\mathbf{x})\mathbf{F}_i(\mathbf{x}) \end{aligned} \tag{6.6}$$

Since (6.4) is satisfied the last expression is equivalent to a fuzzy system of the form (6.5) with center of gravity defuzzification operation, see Chapter 3. □

The system defined by the rules as in (6.5) is the general form of a fuzzy system. In most of the studies in the literature the system dynamics, $\mathbf{F_i}(\mathbf{x})$, in the i^{th} rule, is taken to be linear as $\mathbf{F}_i(\mathbf{x}) = H_i\mathbf{x}$, where $H_i \in \mathbb{R}^{n\times n}$ is a constant matrix. The following definition will be used in the sequel.

Definition 20 *(See [6]) Completeness :*
Let $Z_1(\mathbf{x}), Z_2(\mathbf{x}), \ldots, Z_r(\mathbf{x}) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, *be given vector functions. The collection of vector functions* $Z_1(\mathbf{x}), Z_2(\mathbf{x}), \ldots, Z_r(\mathbf{x})$ *is said to be complete if for any* $\mathbf{x}_0 \in \mathbb{S} = \mathbb{R}^n$ *there exists* $i \in \{1, 2, \ldots, r\}$ *such that* $Z_i(\mathbf{x}_0) \le 0$. *Furthermore, the collection*

$Z_1(\mathbf{x}), Z_2(\mathbf{x}), \ldots, Z_r(\mathbf{x})$ is said to be strictly complete if for any $\mathbf{x}_0 \in \mathbb{S} \setminus 0$, there exists $i \in \{1, 2, \ldots, r\}$ such that $Z_i(\mathbf{x}_0) < 0$. If $\mathbb{S}$ is a subset of $\mathbb{R}^n$, $\mathbb{S} \subset \mathbb{R}^n$, then locally complete and locally strictly complete set of vector functions can be defined similarly.

The rules in switching control correspond to the antecedent part of the IF-THEN rules in fuzzy controllers. In [6] the stability of a switching controller is guarantied with the completeness of some matrices Z_i. The switching rule is to switch to the system which minimizes $\mathbf{x}^T Z_i \mathbf{x}$. The following theorem gives a sufficient condition on the membership functions for the stability of the fuzzy system (6.5).

Theorem 11 *Consider the fuzzy controller given by (6.2), let $P \in \mathbb{R}^{n \times n}$ be a positive definite matrix and define $Z_i(\mathbf{x}) = \mathbf{F}_i^T(\mathbf{x})P\mathbf{x} + \mathbf{x}^T P \mathbf{F}_i(\mathbf{x})$. Then the fuzzy control system defined by (6.1) and (6.2) is (asymptotically) stable if the vector functions, $Z_i(\mathbf{x})$'s, are (strictly) complete and the membership functions, $\mu_{S_i}(\mathbf{x})$, are defined as follows:*

$$\mu_{S_i}(\mathbf{x}) = \begin{cases} 0 & Z_i(\mathbf{x}) \geq 0 \\ c_i(\mathbf{x}) & Z_i(\mathbf{x}) < 0 \end{cases} \tag{6.7}$$

where $c_i(\mathbf{x})$ is an arbitrary function satisfying $0 < c_i(\mathbf{x}) \leq 1$, $\sum_{i=1}^{k} c_i(\mathbf{x}) = 1$ and k is the total number of indices i for which $Z_i(\mathbf{x}) < 0$.

Proof: Let us define a Lyapunov function $V(\mathbf{x}) = \mathbf{x}^T P \mathbf{x}$. Then we have the following:

$$\begin{aligned} \dot{V}(\mathbf{x}) &= \dot{\mathbf{x}}^T P \mathbf{x} + \mathbf{x}^T P \dot{\mathbf{x}} \\ &= \textstyle\sum_{i=1}^{r} \mu_{S_i}(\mathbf{x}) \mathbf{F}_i^T(\mathbf{x}) P \mathbf{x} + \mathbf{x}^T P \sum_{i=1}^{r} \mu_{S_i}(\mathbf{x}) \mathbf{F}_i(\mathbf{x}), \\ &= \textstyle\sum_{i=1}^{r} \mu_{S_i}(\mathbf{x}) [\mathbf{F}_i^T(\mathbf{x}) P \mathbf{x} + \mathbf{x}^T P \mathbf{F}_i(\mathbf{x})] \\ &= \textstyle\sum_{i=1}^{r} \mu_{S_i}(\mathbf{x}) Z_i(\mathbf{x}). \end{aligned} \tag{6.8}$$

where $Z_i(\mathbf{x}) = \mathbf{F}_i^T(\mathbf{x})P\mathbf{x} + \mathbf{x}^T P \mathbf{F}_i(\mathbf{x})$. The completeness property of $Z_i(\mathbf{x})$'s will guarantee that there is at least one vector function, $Z_i(\mathbf{x}) \leq 0$. Since $\mu_{S_i} \geq 0$ are chosen as given in (6.7), $\dot{V}(\mathbf{x}) \leq 0$ will be satisfied. In the same way if $Z_i(\mathbf{x})$'s, are strictly complete then $\dot{V}(\mathbf{x}) < 0$ and hence the FS is asymptotically stable. $\Box$.

Remark 1 *When the subsystems are linear, i.e., $F_i(\mathbf{x}) = H_i \mathbf{x}$, $H_i \in \mathbb{R}^{n \times n}$, the vector fucntions are given as $Z_i(\mathbf{x}) = \mathbf{x}^T Z_i \mathbf{x}$, where the matrix, $Z_i = (H_i^T P + P H_i)$. If for any*

$\mathbf{x} \in \mathbb{R}^n$ there exists a single $i \in 1, 2, \ldots, r$ such that $\mathbf{x}^T Z_i \mathbf{x} \leq 0$ ($\mathbf{x}^T Z_i \mathbf{x} < 0$)) then for $c_i(\mathbf{x}) = 1$, Theorem 11 reduces to the one given in [6] for switching systems.

Consider the membership function given by (6.7). By Remark 1 if for any $\mathbf{x}$, there is only one i that satisfies $Z_i(\mathbf{x}) \leq 0$ then setting $c_i(\mathbf{x}) = 1$ will be the solution and the fuzzy system will reduce to a switching system. However this might be a strict condition on the vector functions, $Z_i(\mathbf{x})$.

Suppose that the vector functions, $Z_i(\mathbf{x})$, are (strictly) complete and let k of them be negative for some $\mathbf{x} \in \mathbb{R}^n$, that is, $Z_{i_j}(\mathbf{x}) < 0$, for $j = 1 \ldots k$. If we define $c_{i_j}(\mathbf{x})$ for $j = 1 \ldots k$ as;

$$c_{i_j}(\mathbf{x}) = \frac{Z_{i_j}(\mathbf{x})}{\sum_{l=1}^{k} Z_{i_l}(\mathbf{x})} \tag{6.9}$$

then the controller that yields a more negative $Z_i(\mathbf{x})$ contributes more to the final control signal as a consequence of its membership function value being close to one.

Remark 2 *When the membership functions are chosen to satisfy (6.9) then we have;*

- $1 \geq c_i(\mathbf{x}) > 0$,
  $Z_{i_j}(\mathbf{x}) < 0 \Rightarrow c_i(\mathbf{x}) > 0$, $k = 1 \Rightarrow c_{i_1}(\mathbf{x}) = 1$,
- $\sum_{i=1}^{r} \mu_{S_i}(\mathbf{x}) = \sum_{j=1}^{k} c_{i_j} = \sum_{j=1}^{k} \frac{Z_{i_j}(\mathbf{x})}{\sum_{l=1}^{k} Z_{i_l}(\mathbf{x})} = 1.$

Note that the definition of the $c_i(\mathbf{x})$ in (6.9) does not impose any pre-assumed structure like gaussian or triangle.

So far we do not consider the controllers, $u_i(\mathbf{x})$, in the design procedure of stabilizing membership functions. It is assumed that $u_i(\mathbf{x})$ is chosen such that the vector functions $Z_i(\mathbf{x}) = F_i(\mathbf{x})^T P \mathbf{x} + \mathbf{x}^T P F_i(\mathbf{x})$ are (strictly) complete where P is a p.d. matrix.

6.2.3 Design Procedure

Next we will consider the design of controllers $\mathbf{u}_i(\mathbf{x})$. These controllers should be chosen in such a way that the vector functions $Z_i(\mathbf{x}) = \mathbf{F}_i(\mathbf{x})^T P \mathbf{x} + \mathbf{x}^T P \mathbf{F}_i(\mathbf{x})$ becomes (strictly) complete, where $P \in \mathbb{R}^{n \times n}$ is an appropriate positive-definite (p.d.) matrix.

It might be difficult to find such a set of control signals to satisfy the completeness assumption globally but one can find a set of $\mathbf{u}_i(\mathbf{x})$ such that the completeness property is satisfied in a nominal operating part of the state space. One particular way is to use pole placement for the local models obtained by linearization of (6.1).

To elaborate further on linearization, let $\mathbf{x}_i \in \mathbb{R}^n$ be a point and let $S_i \subset \mathbb{R}^n$ be a region around $\mathbf{x}_i$ such that the following linearization is valid;

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_i) + A_i(\mathbf{x} - \mathbf{x}_i) + \hat{\mathbf{f}}(\mathbf{x} - \mathbf{x}_i) \tag{6.10}$$

$$\mathbf{g}(\mathbf{x}) = B_i + \hat{\mathbf{g}}(\mathbf{x} - \mathbf{x}_i) \tag{6.11}$$

where A_i is the Jacobian matrix of $\mathbf{f}$ evaluated at $\mathbf{x}_i$ and $B_i = \mathbf{g}(\mathbf{x}_i)$. Due to the linearization, both $\|\hat{\mathbf{f}}(\mathbf{x} - \mathbf{x}_i)\| \to 0$ and $\|\hat{\mathbf{g}}(\mathbf{x} - \mathbf{x}_i)\| \to 0$ as $\mathbf{x} \to \mathbf{x}_i$. Now let us assume that a feedback law $u = -K_i\mathbf{x}$ where $K_i^T \in \mathbb{R}^n$ is applied. By using (6.10) and (6.11) in (6.1), we obtain;

$$\dot{\mathbf{x}} = \mathbf{F}_i(\mathbf{x}) = H_i\mathbf{x} + \mathbf{r}(\mathbf{x}, \mathbf{x}_i)$$

where $H_i = A_i - B_iK_i$ and

$$\mathbf{r}(\mathbf{x}, \mathbf{x}_i) = \mathbf{f}_i(\mathbf{x}_i) - A_i\mathbf{x}_i + \hat{\mathbf{f}}(\mathbf{x} - \mathbf{x}_i) - \hat{\mathbf{g}}(\mathbf{x} - \mathbf{x}_i)K_i\mathbf{x} \tag{6.12}$$

Now let $P \in \mathbb{R}^{n\times n}$ be an appropriate p.d. matrix, and define $\hat{Z}_i(\mathbf{x}) = \mathbf{x}^T(H_i^TP + PH_i)\mathbf{x}$. Then we have;

$$Z_i(\mathbf{x}) \quad = \quad \mathbf{F}_i^T(\mathbf{x})P\mathbf{x} + \mathbf{x}^TP\mathbf{F}_i(\mathbf{x}) = \hat{Z}_i(\mathbf{x}) + 2\mathbf{x}^TP\mathbf{r}(\mathbf{x}, \mathbf{x}_i) \tag{6.13}$$

Note that $\|\mathbf{f}(\mathbf{x}_i) - A_i\mathbf{x}_i\| = \|\hat{\mathbf{f}}(-\mathbf{x}_i)\|$, see (6.10) and $\|\hat{\mathbf{f}}(\mathbf{x}_i)\|/\|\mathbf{x}_i\| \to 0$ as $\mathbf{x}_i \to 0$; furthermore the last 2 terms in (6.12) contain higher order terms. Hence as $\mathbf{x} - \mathbf{x}_i \to 0$ and $\mathbf{x} \to 0$, the term $2\mathbf{x}^TP\mathbf{r}(\mathbf{x}, \mathbf{x}_i)$ in (6.13) can be made arbitrarily small. From these observations, we can conclude the following:

Let $S \subset \mathbb{R}^n$ be a region containing origin and let $\mathbf{x} \in S$ be such that $\hat{Z}_i(\mathbf{x}) < 0$. Then we have $Z_i(\mathbf{x}) < 0$ as well, provided that S is sufficiently small and $\mathbf{x}_i$ is appropriately chosen. This argument shows that if the linearization points are appropriately chosen, and if the state feedback, based on linearized systems, yields a strictly complete set of functions $\hat{Z}_i(\mathbf{x})$, then the original set of functions $Z_i(\mathbf{x})$ will be locally strictly complete as well. Based on this idea, we propose the following design procedure.

Design Procedure

Step 1) Choose r linearization points, x_i where $i = 1, \ldots, r$, such that the linearization yields controllable (A_i, B_i) pairs.

Step 2) Calculate the gain matrices K_i so that $(A_i - B_i K_i)$ has its poles at desired locations on the open Left Half Plane (LHP).

Step 3) Chose a p.d. matrix P to calculate $Z_i(\mathbf{x})$ and calculate the membership functions, $\mu_i(\mathbf{x})$, using (6.7), (6.9).

Step 4) If all of the control gains, K_i, are different go to Step 6.

Step 5) Combine the membership functions which corresponds to a control gains $K_{j_1} = \cdots = K_{j_m} = K_j$ as follows;

$$\mu_{S_j}(\mathbf{x}) = \mu_{j_1}(\mathbf{x}) + \cdots + \mu_{j_m}(\mathbf{x}) \tag{6.14}$$

Step 6) Construct the rules from the unique $\mu_{S_j}(\mathbf{x})$ and K_j where $j = 1, \ldots, l$, $l \leq r$ as;

$$\mathcal{L}_j : \text{IF "}x \text{ is } S_j\text{" THEN } \mathbf{u}(\mathbf{x}) = -K_j \mathbf{x}, \tag{6.15}$$

Step 7) Add the crisp set $\{0\}$ to the one which corresponds to the set obtained by linearization around the origin or set the gain $K = 0 \in \mathbb{R}^{1\times n}$ to the crisp set $\{0\}$, which makes the total number of rules $r + 1$.

Remark 3 *The region S_i in the state space given in (6.14) are defined as $S_i = \{x | \mu_{S_i}(\mathbf{x}) > 0\}$.*

Remark 4 *The problem of finding a common p.d. matrix P to yield a strictly complete set Z_i is not trivial and currently a well known research topic [54]. The choice of P in Step 3 will change the shape of the membership function hence the performance of the overall system.*

Remark 5 *Definition of an index for the size of the domain of attraction is an ongoing research and it is excluded from the discussions in this study. The determination of the domain of attraction can easily be done graphically for systems with dimension 2. It is illustrated in Section 8.3.3 with an example. The origin is one of the points around which the system is linearized and the corresponding pair (A_i, B_i) is required to be controllable as stated in Step 1. Preliminary analysis has shown that the size of the domain of attraction of a closed loop system with multiple controller is larger than that of a single controller around the origin.*

Remark 6 *In [64] the fuzzy controllers whose rule bases are obtained by a set of input/output data are investigated. The fuzzy model is chosen as Mamdani type. It is shown in [64] that the model for Takagi-Sugeno as in (6.15) has an equivalent Mamdani type fuzzy model. Hence the arguments related with the completeness, consistency, compactness and flexibility concepts discussed in [64] applies also to the proposed fuzzy controller. As it is noted in [64], these concepts for the general form of T-S type fuzzy model, where the consequent part is a nonlinear function of the inputs instead of linear combination as in (6.15), require further investigation. Nevertheless for the proposed fuzzy controller the indices defined in [64] can be used to check the flexibility, consistency, completeness, and compactness of the rule base. The performance of the fuzzy controller can be optimized by using an optimization tool like evolution strategies or genetic algorithms and the indices for these concepts.*

6.3 Fuzzy Controller Design Using Data Mining

Fuzzy systems are nonlinear systems and can approximate a given function with any prescribed accuracy [10], [11]. This in fact gives fuzzy controllers the ability to achieve the performance of any control algorithm in conventional control. The problem then turns out to be the method of tunning the parameters of the fuzzy controller. Suppose that we are given the input/output data of a desired controller, that is $\mathbf{u}(\mathbf{x})$ in (6.1) is known. The phase matching methods given in Chapter 5 can be used to approximate $\mathbf{u}(\mathbf{x})$. The problem in this section is to design a fuzzy controller if there is some information about the desired input-output mapping.

In this section we will use T-S type fuzzy controllers. For simplicity, we represent $\mathbf{u}(\mathbf{x})$ as $\mathbf{u}(\mathbf{x}) = \widetilde{\mathbf{K}}(\mathbf{x})\mathbf{x}$. To approximate $\widetilde{\mathbf{K}}(\mathbf{x})$ we use a rule base which consists of total r rules, and assume that i^{th} rule is given as;

$$\mathcal{L}_i : \text{IF "}\mathbf{x} \text{ is in } S_i\text{" THEN } \mathbf{u}(\mathbf{x}) = -\mathbf{K}_i\mathbf{x}. \tag{6.16}$$

where S_i is the i^{th} set associated with the membership function $\mu_{S_i}(\mathbf{x}) : \mathbb{R}^n \to [0,1]$ and $\mathbf{K}_i \in \mathbb{R}^{1\times n}$. The membership functions are assumed to be normal , i.e. $\sum_{i=1}^{r} \mu_{S_i}(\mathbf{x}) = 1$. If multiplication is used for logical AND operation and implication operation, and the center of gravity method is used as the defuzzification operation then the output of the fuzzy controller with r number of rules would be;

$$\mathbf{u}(\mathbf{x}) = \sum_{i=1}^{r} \mu_{S_i}(\mathbf{x})(-\mathbf{K}_i\mathbf{x}) = -\left(\sum_{i=1}^{r} \mu_{S_i}(\mathbf{x})\mathbf{K}_i\right)\mathbf{x} = -\widetilde{\mathbf{K}}(\mathbf{x})\mathbf{x}. \tag{6.17}$$

where $\widetilde{\mathbf{K}}(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^{1\times n}$ is as described above. In this scheme, designing a fuzzy controller is equivalent to find the membership function, $\mu_{S_i}(\mathbf{x})$, and the vectors, $\mathbf{K}_i$. This problem is decomposed into three sub parts. First we calculate $\widetilde{\mathbf{K}}(\mathbf{x})$ for the calculated $\mathbf{u}(\mathbf{x})$ and then for the calculated values of $\widetilde{\mathbf{K}}(\mathbf{x})$ we find a set of vectors $\mathbf{K}_i$. Finally we calculate the membership functions, $\mu_{S_i}(\mathbf{x})$, using $\widetilde{\mathbf{K}}(\mathbf{x})$ and $\mathbf{K}_i$.

6.3.1 Calculation of $\widetilde{\mathbf{K}}(\mathbf{x})$

If (6.17) is expanded we will have $\mathbf{u}(\mathbf{x}) = -\widetilde{\mathbf{K}}(\mathbf{x})\mathbf{x} = \widetilde{k}_1(\mathbf{x})x_1 + \widetilde{k}_2(\mathbf{x})x_2 + \ldots + \widetilde{k}_n(\mathbf{x})x_n$. Now assume that $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{u}(\mathbf{x})$ are known, and the problem is to find appropriate gains $\widetilde{k}_1(\mathbf{x}), \ldots, \widetilde{k}_n(\mathbf{x})$ There are more unknowns than the number of equations. To overcome this difficulty one might try to find the $\widetilde{K}(\mathbf{x})$ with the smallest norm.

Let us consider the case for $n = 2$, and $m = 1$, i.e. $\mathbf{u}(\mathbf{x}) = \widetilde{k}_1(\mathbf{x})x_1 + \widetilde{k}_2(\mathbf{x})x_2$. In Figure 6.2 we see the line given by $\mathbf{u}(\mathbf{x}) = -\widetilde{\mathbf{K}}(\mathbf{x})\mathbf{x}$. The $\tilde{\mathbf{K}}(\mathbf{x})$ with minimum norm is shown with the solid vector. The solution vector must have the same direction as the direction of the normal vector of the line(or hyperplane for higher values of n). Since the normal of this line is $\mathbf{x}$. the solution vector can be written as:

$$\tilde{\mathbf{K}}(\mathbf{x}) = \alpha\mathbf{x}^T \tag{6.18}$$

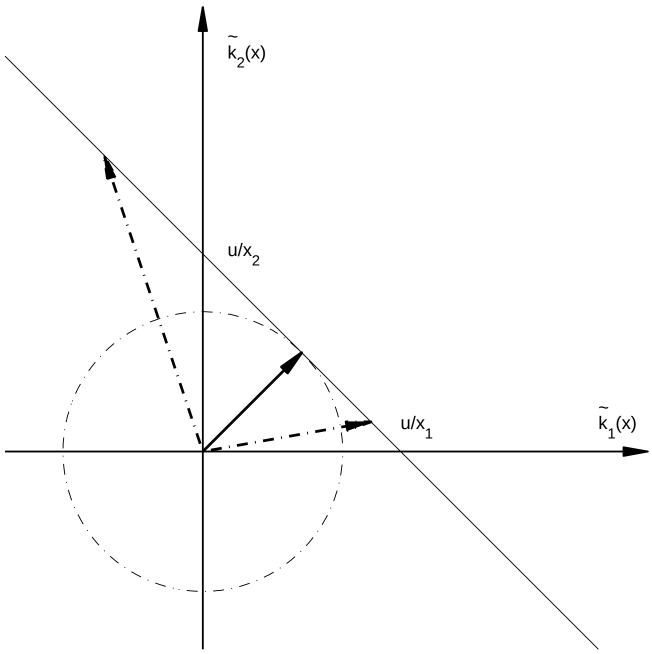


Figure 6.2: The control gain $\tilde{\mathbf{K}}(\mathbf{x})$ with minimum norm.

where $\alpha \in \mathbb{R}$ is a constant. Consider the following equations:

$$\begin{aligned} u(\mathbf{x}) &= -\widetilde{\mathbf{K}}(\mathbf{x})\mathbf{x} \\ &= -\alpha\mathbf{x}^T\mathbf{x} \\ &= -\alpha\|\mathbf{x}\|^2 \end{aligned} \tag{6.19}$$

Therefore we have;

$$\alpha = -\frac{u(\mathbf{x})}{\|\mathbf{x}\|^2}$$

Hence $\widetilde{K}(\mathbf{x})$ can be calculated as;

$$\widetilde{K}(\mathbf{x}) = -\frac{\mathbf{u}(\mathbf{x})\mathbf{x}^T}{\|\mathbf{x}\|^2}. \tag{6.20}$$

6.3.2 Calculation of $\mathbf{K}_i$

A ball, $\mathcal{B}_R$, in $\mathbb{R}^n$ with radius $R = max_x\{\|\widetilde{\mathbf{K}}(\mathbf{x})\|\}$ will include all the necessary control gain vectors, the fuzzy controller is required to generate. In Figure 6.3 this ball is shown for $n = 2$. Let us consider a set of points, $\widehat{\mathbf{K}}_i \in \mathbb{R}^n$, $i = 1, \cdots, r$ such that the convex hull $\mathcal{S}$ of $\widehat{\mathbf{K}}_i$ covers the considered ball, i.e. $\mathcal{B}_R \subset \mathcal{S} = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{x} = \sum_{i=1}^r \lambda_i\widehat{\mathbf{K}}_i,\ \lambda_i \geq 0,$ $\sum_{i=1}^r \lambda_i = 1\}$. If we can find such points $\widehat{\mathbf{K}}_i$, which are also called the vertices of the convex hull $\mathcal{S}$, then from (6.17) it is obvious that the proposed fuzzy controller will be

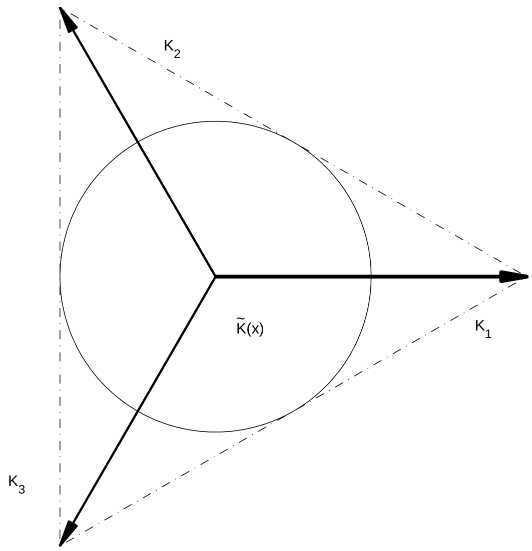


Figure 6.3: The ball $\mathcal{B}_R$ inside the convex hull of $\mathbf{K}_i(\mathbf{x})$ for $n = 2$.

able to produce any required value of $\widetilde{\mathbf{K}}(\mathbf{x})$. Given $\widetilde{\mathbf{K}}(\mathbf{x})$, and hence the ball $\mathcal{B}_R$, there are many ways to find such $\widehat{\mathbf{K}}_i$. Obviously, if r is sufficiently large then it is always possible to achieve $\mathcal{B}_R \subset \mathcal{S}$, since any ball can be covered with convex polygons with sufficiently many corners. However, this will increase the corner points, hence the number of rules in our fuzzy controller. Alternatively, we may try to reduce the number of vertices, $\widehat{\mathbf{K}}_i$, sufficient enough to have $\mathcal{B}_R \subset \mathcal{S}$. Note that in $\mathbb{R}^n$, $r \geq n+1$ should be satisfied. By using simple geometric ideas, one can easily find such points $\widehat{\mathbf{K}}_i$, $i = 1, \cdots, r$ in an algorithmic way, as given in Section 6.3.3, so that $\mathcal{S}$ covers the ball with a radius d, i.e. $\mathcal{B}_d \subset \mathcal{S}$. Here d is the minimum of the distances of the hyperplanes, formed by any n of the vertices, to the origin. For example for $n = 2$ (as used in Figure 6.3), we may chose the vertices as $\widehat{\mathbf{K}}_1 = [1\ \ 0]^T$, $\widehat{\mathbf{K}}_2 = [-1/2\ \sqrt{3}/2]^T$, $\widehat{\mathbf{K}}_3 = [-1/2\ \ -\sqrt{3}/2]^T$, and for $n = 3$ we may chose $\widehat{\mathbf{K}}_1 = [1\ \ 0\ \ 0]^T$, $\widehat{\mathbf{K}}_2 = [-0.3333\ \ 0.9428\ \ 0]^T$, $\widehat{\mathbf{K}}_3 = [-0.3333\ \ -0.4714\ \ 0.8165]^T$, and $\widehat{\mathbf{K}}_4 = [-0.3333\ \ -0.4714\ \ -0.8165]^T$. If the minimum of the distances of the hyperplanes, formed by any n of these points, to the origin is d, then we may chose

$$\mathbf{K}_i = \left(\frac{R}{d}\right) \widehat{\mathbf{K}}_i \ \ i = 1, \ldots, n+1 \tag{6.21}$$

Then the convex hull $\mathcal{S}$ of points $\mathbf{K}_i$ will cover $\mathcal{B}_R$.

6.3.3 Calculation of Vertices, $\widehat{\mathbf{K}}_i$

In this section we will propose a method to calculate the vertices of the convex hull $\mathcal{S}$ with minimum number, i.e r = n+1. Let us define $V_{\widehat{\mathbf{K}}}$ as the volume of the convex hull of the vertices $\widehat{\mathbf{K}}_i$ where $i = 1, \ldots, r$. Let also V_S be the volume of the biggest sphere that can be contained within the convex hull $\mathcal{S}$. One approach in calculation of $\widehat{\mathbf{K}}_i$ might be to minimize the difference $V_{\widehat{\mathbf{K}}} - V_S$. The method we propose provides a solution where the inner product between any two vertices is equal to some constant $-c$ where $c \in [0,\ 1]$. The inner product of the vertices would be negative because the angle in between them is greater than $\pi/2$, see Figure 6.3 for the case $n = 2$. For higher values of n this angle would decrease. To show that it should be higher than $\pi/2$ suppose that the angle is $\pi/2$ radians. Then since we want the angle be the same for all vertices, this means that we will not be able to have the vectors other than only one quadrant. Hence it will not be possible to map some part of the ball $\mathcal{B}_R$. Therefore we conclude that the angle $\pi/2$ is the limit as $n \longrightarrow \infty$.

Let us start with the case $n = 2$ where $r = n + 1$. Consider the matrix composed of the vertices, $\widehat{\mathbf{K}}_i$:

$$\begin{bmatrix} \widehat{\mathbf{K}}_1^T & \widehat{\mathbf{K}}_2^T & \widehat{\mathbf{K}}_3^T \end{bmatrix} = \begin{bmatrix} 1 & -c & -c \\ 0 & x & \bar{x} \end{bmatrix}$$

If we equate the inner products as $\widehat{\mathbf{K}}_1\widehat{\mathbf{K}}_2^T = \widehat{\mathbf{K}}_2\widehat{\mathbf{K}}_3^T = -c$ we will have the following:

$$\begin{aligned} c^2 + x\bar{x} &= -c \\ x\bar{x} &= -c(c+1) \end{aligned} \tag{6.22}$$

On the other hand if we equate the norms as $\|\widehat{\mathbf{K}}_2^T\| = \|\widehat{\mathbf{K}}_3^T\|$then we will have the following equality:

$$\begin{aligned} c^2 + x^2 &= c^2 + \bar{x}^2 \\ x^2 &= \bar{x}^2 \end{aligned} \tag{6.23}$$

If we compare Equation (6.22) and (6.23) we conclude that $\bar{x} = -x$ and hence $x^2 = c^2 + c$). Let us try to find the value of c by normalizing either the vertex $\widehat{\mathbf{K}}_2$

or $\widehat{\mathbf{K}}_3$ as follows:

$$\begin{aligned} \|\widehat{\mathbf{K}}_1^T\|^2 &= \|\widehat{\mathbf{K}}_2^T\|^2 \\ 1 &= c^2 + x^2 \\ &= c^2 + c^2 + c \\ &= 2c^2 + c \end{aligned}$$

This quadratic equation has two solutions $c_1 = -1$ and $c_2 = \frac{\sqrt{3}}{2}$. Since c should be positive from our assumption the solution for $n = 2$ is $c = \frac{\sqrt{3}}{2}$. The gains obtained by this calculation is given in Figure 6.3. Let us consider the calculation of the vertices for $n = 3$. In the same way suppose that we have constructed the following matrix using the vertices:

$$\begin{bmatrix} \widehat{\mathbf{K}}_1^T & \widehat{\mathbf{K}}_2^T & \widehat{\mathbf{K}}_3^T & \widehat{\mathbf{K}}_4^T \end{bmatrix} = \begin{bmatrix} 1 & -c & -c & -c \\ 0 & x & \bar{x} & \bar{x} \\ 0 & 0 & y & \bar{y} \end{bmatrix}$$

In this form of the vertices we have a problem when we equate the norm of $\widehat{\mathbf{K}}_2^T$ to $\widehat{\mathbf{K}}_3^T$ assuming $\bar{x} = -x$. Their norm will differ by an amount of y^2. To overcome this difficulty let us introduce a gain, G, for the adjustment of this difference as follows:

$$\begin{bmatrix} \widehat{\mathbf{K}}_1^T & \widehat{\mathbf{K}}_2^T & \widehat{\mathbf{K}}_3^T & \widehat{\mathbf{K}}_4^T \end{bmatrix} = \begin{bmatrix} 1 & -c & -c & -c \\ 0 & G_1 x & \bar{x}/G_1 & \bar{x}/G_1 \\ 0 & 0 & y & \bar{y} \end{bmatrix}$$

where we assume that $\bar{x} = -x$ and $\bar{y} = -y$. Note that the introduction of the gain G_1 does not effect the formula of x in terms of c, i.e. $x^2 = c^2 + c$ also for $n = 3$. Knowing the value of x, let us calculate the gain G_1 in terms of the variable c. By normalizing the vertex $\widehat{\mathbf{K}}_2^T$ we can calculate the gain G_1 as:

$$\begin{aligned} c^2 + G_1^2 x^2 &= 1 \\ G_1^2 &= (1 - c^2)/(x^2) \\ &= [(1 - c)(1 + c)]/[c(c + 1)] \\ &= (1 - c)/c \end{aligned} \tag{6.24}$$

Now we require to calculate the value of y by taking the inner product of $\widehat{\mathbf{K}}_3^T$ and $\widehat{\mathbf{K}}_4^T$ as follows:

$$\begin{gathered} c^2 + x^2/G_1^2 - y^2 = -c \\ c^2 + x^2/G_1^2 + c = y^2 \end{gathered} \tag{6.25}$$

Let us take the norm of either $\widehat{\mathbf{K}}_3^T$ or $\widehat{\mathbf{K}}_4^T$ to calculate the value of c that satisfies all these properties:

$$\begin{aligned} \|\widehat{\mathbf{K}}_1^T\|^2 &= \|\widehat{\mathbf{K}}_3^T\|^2 \\ 1 &= c^2 + x^2/G_1^2 + y^2 \\ &= 2c^2 + 2x^2/G^2 + c \\ &= 3c^2 + 2c \end{aligned} \tag{6.26}$$

Again we have two solutions, one of which is $c_1 = -1$ and the other is $c_2 = 1/3$. Since $c > 0$ the solution is $c = 1/3$. In Figure 6.4 the gain vectors are given together with the ball $\mathcal{B}_R$. For higher dimensions we can apply the same sequence of operations to obtain

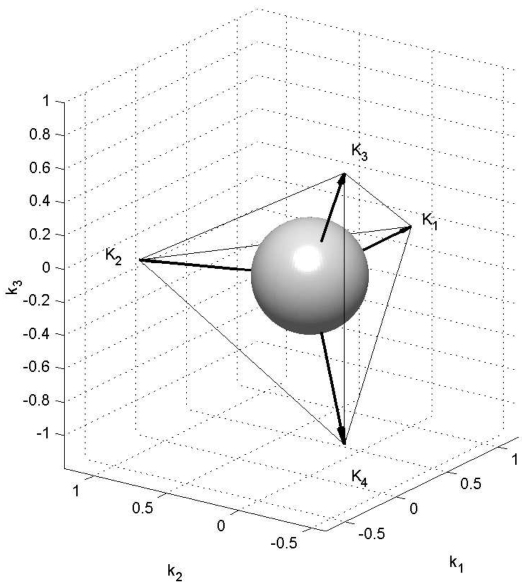


Figure 6.4: The ball $\mathcal{B}_R$ inside the convex hull of $\mathbf{K}(\mathbf{x})$ for $n = 3$.

the vertices. For example for $n = 4$ we would have the following matrix:

$$\begin{bmatrix} \widehat{\mathbf{K}}_1^T & \widehat{\mathbf{K}}_2^T & \widehat{\mathbf{K}}_3^T & \widehat{\mathbf{K}}_4^T & \widehat{\mathbf{K}}_5^T \end{bmatrix} = \begin{bmatrix} 1 & -c & -c & -c & -c \\ 0 & G_1 x & \bar{x}/G_1 & \bar{x}/G_1 & \bar{x}/G_1 \\ 0 & 0 & G_2 y & \bar{y}/G_2 & \bar{y}/G_2 \\ 0 & 0 & 0 & z & \bar{z} \end{bmatrix}$$

where $\bar{x} = -x$, $\bar{y} = -y$, and $\bar{z} = -z$. Note also that the calculated values for x, G_1, and y are the same function of c for $n = 4$. Therefore the process is incremental. First the calculation of G_2 in terms of the variable c must be performed by normalizing $\widehat{\mathbf{K}}_5^T$. Then the inner product of $\widehat{\mathbf{K}}_4^T$ and $\widehat{\mathbf{K}}_5^T$ must be equated to $-c$ to solve the new variable z in terms of c. Finally the value of c should be calculated by normalizing either $\widehat{\mathbf{K}}_4^T$ or $\widehat{\mathbf{K}}_5^T$.

The methodology can be adapted to higher values of n by introducing a new variable and the corresponding gain as shown in the previous examples.

6.3.4 Calculation of $\mu_{S_i}(\mathbf{x})$

Using (6.17) and the assumption that the MFs are normal we obtain the following equation;

$$\begin{bmatrix} \mathbf{K}_1^T & \mathbf{K}_2^T & \cdots & \mathbf{K}_{n+1}^T \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \mu_{S_1}(\mathbf{x}) \\ \mu_{S_2}(\mathbf{x}) \\ \cdots \\ \mu_{S_{n+1}}(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbf{K}}(\mathbf{x})^T \\ 1 \end{bmatrix} \tag{6.27}$$

Note that due to the construction and the fact that $\mathcal{B}_R \subset \mathcal{S}$, (6.27) always have a solution for $\mu_{S_i}(\mathbf{x})$ such that $\mu_{S_i}(\mathbf{x}) \geq 0$ and $\sum_{i=1}^{r} \mu_{S_i}(\mathbf{x}) = 1$. Note also that this design method can also be generalized as the function approximation using fuzzy systems.

Remark 7 *It is well known that fuzzy systems can approximate certain systems with arbitrary degree of accuracy, see [10], [11]. There are many fuzzy approximation schemes proposed in the literature, and the scheme given above may be considered as a new fuzzy approximation scheme. This method can also be used in fuzzy controller design once the control signal,* $\mathbf{u}(\mathbf{x})$ *is known. The purpose of implementing the control signal with a fuzzy controller might be some of the advantages of fuzzy control implementation like low cost implementation, shorter time to market, lower computation requirements, to achieve certain smoothness in system signals, etc.*

Chapter 7

Switching Controller and System Design

Design strategies of switching controllers and systems are versatile and generally specific to the system or problem under consideration. For example, in [21], Hespanha and Morse have considered some of the dynamical properties of the system to form their switching strategies and switching regions. In [26], [27] and [66] so-called supervisory based switching is utilized in solving the stabilization problem.

In some studies the stability results given in Section 2.3 are utilized. For example, in [67] Liberzon elaborates the multiple Lyapunov function approach together with dwell time switching to explore a link between the problem of stabilizing a linear system using finite-state hybrid output feedback and the problem of finding a stabilizing switching sequence for a switching linear system with unstable individual matrices. In [6] the authors provide some results on the stability of a switching system with an assumption of the existence of a common quadratic Lyapunov function and the completeness property of a certain set of matrices which are calculated using the dynamics of the switching system.

This chapter is organized as follows: In Section 7.1 we propose some of the most commonly encountered switching strategies. In the following sections we present the switching strategies to model or stabilize some class of systems. In Section 7.2 we consider a special class of chaotic systems, which are characterized by piecewise-linear dynamics that can be modelled as a switching system. For this class of systems, we propose two observer-based synchronization schemes. We prove various synchronization results and comment on the robustness of the proposed schemes, see [68]. In Section 7.3 we present a switching strategy which is composed of multiple phases such that in each phase a control signal is applied to stabilize some of the state variables. In Section 7.4 we propose a switching strategy again composed of multiple phases but this time the control signal is chosen only to stabilize a single state variable at a time while decreasing the norm of the state vector in each phase. In Section 7.5 we present a switching strategy to stabilize driftless systems which satisfy certain conditions.

7.1 Switching Strategies

In Section 2.3 we have introduced some stability results on the switching systems. In fact most of them directly specify a switching strategy within their proofs. For example the results related with the multiple Lyapunov function give rise to the switching strategy known as the dwell time switching. There are other switching strategies where the future values of the states are estimated and then the dynamics are switched in order to obtain the best performance. This strategy is known as supervisory switching. Finding a general switching strategy might be a hard problem but if one has enough information about the specific system under consideration generally there is a simple solution. Under the title of “Logic Based Switching” we will introduce some of the related examples.

7.1.1 Dwell Time Switching

Dwell time switching may be used to achieve stability for a switching system by using multiple Lyapunov functions. As mentioned in Section 2.3.2 the dwell time between the switchings is chosen such that the sequence of the Lyapunov function values constitute a decreasing sequence.

Consider the switching system given below:

$$\dot{\mathbf{x}} = \mathbf{f}_i(\mathbf{x}) + \mathbf{g}_i(\mathbf{x})\mathbf{u}_i(\mathbf{x}). \tag{7.1}$$

Let us define the vector fields $\mathbf{F}_1(\mathbf{x})$ and $\mathbf{F}_2(\mathbf{x})$ as $\mathbf{F}_i(\mathbf{x}) = \mathbf{f}_i(\mathbf{x}) + \mathbf{g}_i(\mathbf{x})\mathbf{u}_i(\mathbf{x})$, $i = 1, 2$. Let us further assume that with appropriate selection of the control laws, the systems $\dot{\mathbf{x}} = \mathbf{F}_1(\mathbf{x})$ and $\dot{\mathbf{x}} = \mathbf{F}_2(\mathbf{x})$ are both exponentially stable. Hence, there necessarily exist positive functions $V_1(\mathbf{x})$, $V_1(\mathbf{x}) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ which satisfy the following:

$$\tfrac{dV_i}{dt} = \tfrac{\partial V_i}{\partial \mathbf{x}}(\mathbf{x})\mathbf{f}_i(\mathbf{x}) \quad \leq \quad -\lambda_i V_i(\mathbf{x}), \tag{7.2}$$

$$a_i\|\mathbf{x}\|^2 \quad \leq \quad V_i(\mathbf{x}) \quad \leq \quad b_i\|\mathbf{x}\|^2 \tag{7.3}$$

where λ_i, a_i, b_i are positive constants for $i = 1, 2$, see [69] p.163. Note that although each system in (7.1) is exponentially stable, this does not imply that the overall system (7.1) is stable for arbitrary switching. As in Section 2.3.2, let us divide the time axis into intervals indexed by j, and let us assume that the length of the j^{th} interval τ_j is given as $\tau_j = \tau_{1,j} + \tau_{2,j}$, where $\tau_{i,j}$ is the duration in which the i^{th} dynamics is activated. As in Section 2.3.2, let us set $t_j = t_{j-1} + \tau_j$, where $t_0 = 0$. Obviously, for the stability the condition $V_i(t_j) < V_i(t_{j-1})$ must hold, $i = 1, 2$ $j = 1, 2, \ldots$. Now we will find a time length τ_D, called the dwell time, such that this condition holds when $\tau_{i,j} \geq \tau_D$. By using (7.2) and (7.3), we obtain:

$$\begin{aligned}
V_1(t_j) &\leq \tfrac{b_1}{a_2}V_2(t_j), && \text{using Equation (7.3)}\\
&\leq \tfrac{b_1}{a_2}e^{(-\lambda_2\tau_D)}V_2(t_{j-1} + \tau_{1,j}), && \text{using Equation (7.2)}\\
&\leq \tfrac{b_1}{a_2}\tfrac{b_2}{a_1}e^{(-\lambda_2\tau_D)}V_1(t_{j-1} + \tau_{1,j}), && \text{using Equation (7.3)}\\
&\leq \tfrac{b_1}{a_2}\tfrac{b_2}{a_1}e^{(-(\lambda_1+\lambda_2)\tau_D)}V_1(t_{j-1}), && \text{using Equation (7.2)}
\end{aligned}$$

So to satisfy $V_1(t_j) < V_1(t_{j-1})$ we need to calculate the dwell time τ_D from the following inequality:

$$\frac{b_1}{a_2}\frac{b_2}{a_1}e^{(-(\lambda_1+\lambda_2)\tau_D)} < 1 \tag{7.4}$$

Note that if τ_D is sufficiently large, i.e. $\tau_D \geq \frac{1}{\lambda_1+\lambda_2}\ln\frac{b_1 b_2}{a_1 a_2}$, this condition is always satisfied. With this choice of dwell time τ_D the sequence of Lyapunov function values at switching times will constitute a decreasing sequence which makes the switching system stable. For more information see [26] and [27].

7.1.2 Supervisory Switching

In this switching scheme there is a block which is composed of multiple estimators, each corresponding to a controller which can be switched to feed the system under consideration. These estimators provide output estimate of the system if the corresponding controller is to be switched. According to a chosen performance criteria and the provided outputs of the estimators the switching is performed by the so-called Estimator Based Supervisor [26], [27]. The function of this block can be expressed as the switching signal, $s(\mathbf{x}, \cdot)$ given in (2.1),(2.2), and (2.4). The output of the estimator is subtracted from the actual output and the corresponding error is calculated.

In [26], Morse proposed to switch to the controller which has a minimum output error. But this switching is not performed by just checking the current values of the outputs. The concept of Dwell Time Switching given in Section 7.1.1 is utilized in the switching process. The switching intervals is assured to be at least the duration of $\tau_D - \tau_C$ where τ_D is the dwell time and τ_C is the computation time required to make the calculation of the norm of the output errors. When a switching is just made the supervisor waits for a period of $\tau_D - \tau_C$ and samples the output and starts calculation of the output error. After finishing this calculation it is checked if the total time elapsed from the last switching is at least τ_D or not. If so then the norm of the output errors are checked. If there is a better candidate the switching is performed and the cycle begins from the top. If there is no better controller candidate then the system waits for a $\tau_D - \tau_C$ seconds and makes the re-sampling to calculate the norm of the output error.

A detailed analysis of this switching scheme can be found in [26] and [27] applied to set point control. Also in [66] the application of supervisory control to a unicycle can be found.

7.1.3 Logic Based Switching

Logic Based Switching may be considered as the more general name for a large class of studies in the switching system literature. Many systems can one or the other way be represented as a collection of subsystems. Depending on the specific properties they posses, different switching approaches are proposed [21], [66]. The supervisory switching given in the previous section is also considered to be a logic based switching strategy.

For example, in [21] the control of a nonholonomic integrator is stabilized by using a switching scheme in between regions in the state space. The nonholonomic integrator has the following dynamics, see (4.9):

$$\begin{aligned}\dot{x_1} &= u_1, \\ \dot{x_2} &= u_2, \\ \dot{x_3} &= x_1u_2 - x_2u_1\end{aligned}$$

In [21], Hespanha and Morse points the fact that no matter what control law is used, whenever x_1 and x_2 are both zero, $\dot{x}_3$ will be zero and x_3 will remain constant. Furthermore whenever x_1 and x_2 are "small", only "large" control signals will be able to produce significant changes in x_3. A plausible strategy to make the origin an attractor of the closed loop system is to keep the state away from the axes $x_1 = x_2 = 0$ while x_3 is large and, as x_3 decreases, to let x_1 and x_2 became small. The regions are constructed under this logical inference which was given in (4.10). The analysis of invariant sets, stability and convergence of the system with more detail can be found in [21].

In [66] the specific problem considered is that of parking a shelled mobile robot of unicycle type with unknown parameters.The techniques of supervisory control are employed to design a hybrid feedback control law that solves this problem. In this chapter we will propose some switching strategies which might be included in logic based switching class.

7.2 A Switching Synchronization Scheme for a Class of Chaotic Systems

In this section, we will consider a special class of chaotic systems, which are characterized by piecewise-linear dynamics that can be modelled as a switching system. Although the resulting dynamics seems to be rather restricted, nevertheless this class of systems contains many chaotic systems, including most of the chaotic electronic oscillators, already proposed in the literature. For this class of systems, we propose two observer-based synchronization schemes. We prove various synchronization results and comment on the robustness of the proposed schemes.

Although the concept of synchronization of chaotic systems may seem somewhat paradoxical, it has been known since the seminal work [70] that it is possible, and even more surprisingly this property is robust in certain cases, see e.g. [71]. In recent years, many aspects of chaotic dynamics including synchronization and control of chaotic systems have received considerable attention among scientists in many different fields. The literature is quite rich on this subject, and interested reader may consult to e.g. [72], [73].

Most of the synchronized chaotic systems consist of two parts : a generator of chaotic signals (drive system), and a receiver (response system). The response system is usually a duplicate of a part (or the whole) of the drive system. A chaotic signal generated by the drive system, which is called the synchronization signal, is usually transmitted to the response system to achieve the synchronization. One of the motivations for the synchronization is the possibility of sending messages through chaotic systems for secure communication, see [74].

Various synchronization schemes are proposed in the literature, see e.g. [73], and in most of these works a systematic procedure to determine the response system and the synchronization signal is not given. A particular synchronization scheme which utilizes such a systematic procedure is the observer-based synchronization scheme, see e.g. [71], [75]. In this approach, typically the response system is a duplicate of the drive system, and a synchronization error term, which is the difference between the synchronization signal and a similar signal generated in the response system, is injected into the response system through a gain vector, which is called the observer gain. General procedures and conditions to determine this gain vector to guarantee synchronization for a given arbitrary drive system can be obtained , see e.g. [71]. However, if the properties of a given particular drive system are not taken into consideration, the required gain may be quite high. High gain is not desired especially when the synchronization signal is corrupted with noise, as the noise will also be amplified by the gain. Also high gain values may cause large transients, and this might cause saturation in certain cases. One way to eliminate the high gain values is to incorporate the system properties into the observer-based design. In this approach, a particular class of systems may be considered and the observer-based approach could be modified for this particular class of systems.

This section is organized as follows. In the next subsection, we introduce the class of systems under consideration. In Section 7.2.3, we propose our synchronization schemes. We also prove various synchronization and stability properties, and comment on the robustness of the proposed schemes. In Section 8.1.1, we give various simulation results which indicate the effectiveness of the proposed schemes.

7.2.1 A Brief Review of Chaos

In this thesis we are not interested in chaotic systems but we use them in our simulations to demonstrate our methods. Nevertheless let us describe the systems defined as a chaotic systems. The word "chaos" is used to describe a state of complete disorder and confusion in our daily life. We note that there is no universally agreed definition of chaos in current literature, [73]. For a critique as well as various examples and counter examples resulting from different definitions of chaos, see [76]. We will give one of such definitions, which is well-accepted in the literature, [7]. Let us consider the following system:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{7.5}$$

where $\mathbf{x} \in \Omega \subset \mathbb{R}^n$, and $\mathbf{f}(\cdot) : \Omega \longrightarrow \mathbb{R}^n$. Let us denote by $\Phi(t, \mathbf{x}_0)$ the solution of (7.5) at time t from the initial condition $\mathbf{x}(\mathbf{x}_0)$. similarly denote by $\Phi(t, \Omega_1)$ the set of points that is reached at time t starting from the initial conditions in the set $\Omega_1 \subset \Omega$, i.e. $\Phi(t, \Omega_1) = \{x_t | x_t = \Phi(t, \mathbf{x}_0), \mathbf{x}_0 \in \Omega_1\}$.

Definition 21 *Topological Transitivity :*
The dynamical system (7.5) is said to be topologically transitive on Ω if for any pair of open sets $\Omega_1, \Omega_2 \subset \Omega$ there exists a time $t > 0$ such that $\Phi(t, \Omega_1) \bigcap \Omega_2 \neq 0$.

Definition 22 *Sensitive Dependence on Initial Conditions :*
The dynamical system (7.5) has sensitive dependence on initial conditions if there exists

$\delta > 0$ *such that, for any* $\mathbf{x} \in \Omega$ *and any neighborhood* $\mathcal{N}$ *of* $\mathbf{x}$*, there exists* $\mathbf{z} \in \mathcal{N}$ *and* $t > 0$ *such that* $\|\Phi(t, \mathbf{x}) - \Phi(t, \mathbf{z})\| > \delta$

Definition 23 *Dense Set :*
A set Ω_1 *is dense in another set* $\Omega \supset \Omega_1$ *if for any point in* $\mathbf{x} \in \Omega$ *there is a point* $\mathbf{z} \in \Omega_1$ *that is arbitrarily close to* $\mathbf{x}$*.*

Definition 24 *Chaotic System :*
The dynamical system (7.5) is said to be chaotic if

i *it has sensitive dependence on initial conditions,*

ii *it is topologically transitive on* Ω*,*

iii *its periodic solutions are dense in* Ω

Put in other words, chaotic system is unpredictable by its sensitive dependence on initial conditions, its state space cannot be decomposed into disjoint invariant sets by transitivity and it has an underlying regular structure by the density of periodic solutions, [7].

7.2.2 Problem Statement

In this subsection we will consider a special class of chaotic systems given by piecewise-linear dynamics. To be specific, let $-\infty < k_1 < k_2 < \ldots < k_{m-1} < \infty$ be given constants, and set $k_0 = -\infty,\ k_m = \infty$, where $m \geq 2$. We define the regions $\mathcal{R}_i \in \mathbf{R}$ as :

$$\mathcal{R}_i = \{x \in R \mid k_{i-1} \leq x \leq k_i \ \} \quad , \ i = 1, 2, \ldots, m \ . \tag{7.6}$$

We consider the systems given as :

$$\dot{z} = f(z) \quad , \tag{7.7}$$

where $z = (z_1 \ldots z_n)^T \in \mathbb{R}^n$, here the superscript T denotes the transpose, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a piecewise-linear map defined as :

$$f(z) = A_i z + b_i \quad , \quad z_1 \in \mathcal{R}_i \quad , \quad i = 1, \ldots, m \ . \tag{7.8}$$

Here, for $i = 1, \ldots, m$, $A_i \in \mathbb{R}^{n \times n}$ constant matrices and $b_i \in \mathbb{R}^n$ are constant vectors. To ensure the existence and continuity of the solutions of (7.7), we will assume that f is a continuous function. This requirement puts some conditions on A_i and b_i, some of which will be exploited in designing the synchronization schemes. These continuity requirements can easily be obtained by using $A_i z + b_i = A_{i+1} z + b_{i+1}$ for $i = 1, \ldots, m-1$, $z_1 = k_i$, and $z \in \mathbb{R}^n$ is arbitrary otherwise. After some straightforward algebra, we see that the following holds

$$A_{i+1} = A_i + \hat{h}_i e_1^T \quad , \quad i = 1, \ldots, m-1 \quad , \tag{7.9}$$

for some vectors $\hat{h}_i \in \mathbb{R}^n$, here e_1 is the first unit vector, i.e. $e_1 = (1\ 0 \ldots 0)^T \in \mathbb{R}^n$. In other words, A_{i+1} and A_i only differs in their first columns. Hence, there exist a constant matrix $A \in \mathbb{R}^{n \times n}$ and constant vectors $h_i \in \mathbb{R}^n$ such that the following holds :

$$A_i = A + h_i e_1^T \quad , \quad i = 1, \ldots, m \quad . \tag{7.10}$$

We note that this representation is not unique, since $h_i e_1^T$ only affects the first column of A.

For synchronization, we need a synchronization signal to be sent to the receiver. A natural choice for the synchronization signal is $y = z_1$. Note that in the context of observer-based synchronization schemes, we have the following output function for the system (7.7)

$$y = Cz \quad , \quad C = e_1^T \quad . \tag{7.11}$$

As is customary in observer-based schemes, we assume that the pair (C, A) is observable, i.e. the following matrix Q is nonsingular :

$$Q = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} \quad . \tag{7.12}$$

see e.g. [71], [75].

The class of systems which could be described by the equations given above might be limited. However, there are meaningful classes of chaotic systems which could be represented in this framework. Such a class is the Lur'e type systems which are frequently encountered and investigated in the literature, see e.g. [71], [77], and the references therein. These systems can be represented as :

$$\dot{z} = A_L z - b_L f(y) \quad , \quad y = C_L z \quad , \tag{7.13}$$

where $A_L \in \mathbb{R}^{n \times n}$, $b_L, C_L^T \in \mathbb{R}^n$ and $f : \mathbb{R} \to \mathbb{R}$ is an arbitrary function. Let us assume that $f(\cdot)$ is given by a piecewise-linear characteristics as

$$f(y) = c_i y + d_i \quad , \quad y \in \mathcal{R}_i \quad , \quad i = 1, \ldots, m \quad , \tag{7.14}$$

where c_i, d_i are scalar constants. By using (7.14) in (7.13) we obtain the structure given by (7.7)- (7.8) with

$$A_i = A_L - c_i b_L C_L \quad , \quad b_i = -d_i b_L \quad . \tag{7.15}$$

If $y = z_1$, i.e. $C_L = e_1^T$, then (7.11) is also satisfied. If $y \neq z_1$, then by using the coordinate change $\tilde{z} = Qz$, where Q is given by (7.12), with $C = C_L, A = A_L$, the system (7.13)-(7.14) could be transformed into the form (7.7), (7.8) and (7.11).

Remark 8 *If the nonlinearity $f(\cdot)$ is not piecewise-linear, then it may be approximated by a piecewise-linear one with any desired accuracy. If the original system exhibits a chaotic behavior, and if the approximation error is sufficiently small, then it is reasonable to expect similar behavior when the piecewise-linear approximation is used instead of the original $f(\cdot)$. However, the effect of this approximation error on the synchronization requires further investigation.* □

Remark 9 *Most of the chaotic electronic oscillators proposed in the literature can be represented by the dynamics given above. In particular, the well-known Chua's chaotic oscillator, which is studied extensively in the literature, also belongs to such class of oscillators. This is particularly important, since it is known that Chua's oscillator is equivalent to a large class of chaotic systems already proposed in the literature.* □

7.2.3 An Observer-Based Synchronization Scheme

Consider the chaotic system given by (7.7), (7.8), (7.11). We assume that (7.10)-(7.12) also hold. For this system we first propose the following observer :

$$\dot{\hat{z}} = A\hat{z} + h_i z_1 + b_i + K(z_1 - \hat{z}_1) \quad , \quad z_1 \in \mathcal{R}_i \quad , \tag{7.16}$$

where $\hat{z} \in \mathbb{R}^n$ is the receiver state, and $K \in \mathbb{R}^n$ is a gain vector yet to be determined. Note that since $y = z_1$ is the synchronization signal, which is available, the observer structure given by (7.16) is realizable at the receiver.

Remark 10 *To relate (7.16) with the observer-based synchronization scheme proposed in e.g. [71], let us consider the following :*

$$\dot{\hat{z}} = A_j\hat{z} + b_j + u \quad , \quad \hat{z}_1 \in \mathcal{R}_j \quad , \tag{7.17}$$

where $j = 1, \ldots, m$, and u is an appropriate input to the observer. It is easy to show that if we choose the following u for (7.17)

$$u = h_i z_1 + b_1 - h_j \hat{z}_1 - b_j + K(z_1 - \hat{z}_1) \quad , \quad z_1 \in \mathcal{R}_i \quad , \quad \hat{z}_1 \in \mathcal{R}_j \quad , \tag{7.18}$$

then (7.17) reduces to (7.16). Hence, the synchronization scheme proposed in (7.16) is a special observer-based synchronization scheme. Note that in this formalism, the control action (7.18) can be interpreted as a set of "IF-THEN" rules given by : "if $z_1 \in \mathcal{R}_i$ and $\hat{z}_1 \in \mathcal{R}_j$, then u is given by (7.18)". Hence, we have a set of rules, and the control action switches between them. This formalism could also be used in designing the fuzzy logic controllers for such chaotic systems. Also note that there are different switching synchronization systems already proposed for some chaotic systems. □

To prove the synchronization property, let us define the synchronization error as $e = z - \hat{z}$. Upon differentiation and using (7.7), (7.16) we obtain :

$$\dot{e} = (A - KC)e \quad . \tag{7.19}$$

Since the pair (C, A) is assumed to be observable, the gain K can be appropriately chosen so that $A - KC$ is stable (i.e. all eigenvalues are in the left half of the complex plane), hence the error e decays exponentially to zero for any initial condition $e(0) = z(0) - \hat{z}(0)$. Moreover, the decay rate could be adjusted arbitrarily by proper choice of K, see [71], [75].

Remark 11 *Note that by using the techniques given in [71], one can also design other observer-based synchronization schemes for the systems given by (7.7). However, such schemes are for general nonlinear systems and particular structures of nonlinearities, e.g. the piecewise-linearity in present case, is not fully utilized. As a result, the required gain vectors in [71] usually depend on the Lipschitz constant γ of the nonlinearity, and the largest gain would be proportional to $(n\gamma)^n$, see [7]. Usually $\gamma \gg 1$, and for large dimensions usually rather large gains are required to guarantee the synchronization. In the present case however, the required gain K is independent of any such Lipschitz constant, and the only requirement is the stability of $A - KC$. As a result, the synchronization is achieved with smaller gains. This is particularly important if the synchronization signal is corrupted by noise, since in this case the gain vector also multiplies the noise as well.* $\square$

In (7.16), the synchronization signal enters into the observer dynamics through two ways. The first one is due to the switching rule $h_i z_1$, which may be considered as a nonlinear processing of z_1, and the second one is due to the linear injection term $K(z_1 - \hat{z}_1)$. While the linear term is desirable and is present in many synchronization schemes, the nonlinear processing of synchronization signal is less desirable, especially when noise is present in the transmission of synchronization signal. One possible remedy is to use $\hat{z}_1$ in the nonlinear processing stated above. This observation leads to our second observer structure given as

$$\dot{\hat{z}} = A_i \hat{z} + b_i + K(z_1 - \hat{z}_1) \quad , \quad z_1 \in \mathcal{R}_i \quad , \tag{7.20}$$

However, in this case the error dynamics will not be as simple as (7.19). In fact, by using (7.7), (7.8), (7.11) and (7.20) we obtain

$$\dot{e} = (A_i - KC)e = F_i e \quad , \quad z_1 \in \mathcal{R}_i \quad , \tag{7.21}$$

where for simplicity we set $F_i = A_i - KC = A + h_i C - KC$. To have a stable error, it is necessary that all F_i be stable. Our next result states that this is possible by choosing

K appropriately.

Theorem 12 *Let (C, A) be observable. Then, there exists $K \in \mathbb{R}^n$ such that all F_i, $i = 1, \ldots, m$ are stable.*

Proof : Set $\det(\lambda I - A) = \lambda^n + \alpha_1 \lambda^{n-1} + \ldots + \alpha_n$ as the characteristic polynomial of A. Define the vectors $u_1 = (1\ \alpha_1 \ldots \alpha_{n-1})^T$, $u_2 = (0\ 1\ \alpha_1 \ldots \alpha_{n-2})^T$, ..,$u_n = (0\ 0 \ldots 1)^T$, and define the matrices $U = (u_1 u_2 \ldots u_n)$, $R = UQ$, where Q is given by (7.12). By using simple algebra and Cayley-Hamilton Theorem (i.e. $A^n + \alpha_1 A^{n-1} + \ldots + \alpha_n I = 0$) and (7.11) we obtain :

$$RAR^{-1} = \begin{pmatrix} -\alpha_1 & 1 & 0 & \ldots & 0 \\ -\alpha_2 & 0 & 1 & \ldots & 0 \\ & & \vdots & & \\ -\alpha_{n-1} & 0 & 0 & \ldots & 1 \\ -\alpha_n & 0 & 0 & \ldots & 0 \end{pmatrix} \quad , \quad CR = C \tag{7.22}$$

Let us define $\hat{h}_i = Rh_i = (\hat{h}_{i1} \ldots \hat{h}_{in})^T$, $\hat{K} = RK = (\hat{k}_1 \ldots \hat{k}_n)^T$, and set $\hat{F}_i = RF_iR^{-1}$, $i = 1, \ldots, m$. By using (7.22) we obtain the following characteristic polynomial for $\hat{F}_i$

$$\det(\lambda I - \hat{F}_i) = \lambda^n + (\hat{k}_1 - \gamma_{i1})\lambda^{n-1} + (\hat{k}_2 - \gamma_{i2})\lambda^{n-2} + \ldots + (\hat{k}_n - \gamma_{in}) \quad , \tag{7.23}$$

where $\gamma_{ij} = \hat{h}_{ij} - \alpha_j$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. By using Routh criterion, after some straightforward algebra it could be shown that given γ_{ij}, one can find the weights $\hat{k}_j$ such that (7.23) yields stable polynomials for $i = 1, \ldots, m$. For example, for $n = 2$, it suffices to choose

$$\hat{k}_1 > \max_i\{\gamma_{i1}\} \quad , \quad \hat{k}_2 > \max_i\{\gamma_{i2}\} \quad , \tag{7.24}$$

and for $n = 3$, it suffices to choose

$$\hat{k}_1 > \max_i\{\gamma_{i1}\} \quad , \quad \hat{k}_3 > \max_i\{\gamma_{i3}\} \quad , \quad \hat{k}_2 > \max_i\{\gamma_{i2} + (\hat{k}_3 - \gamma_{i3})/(\hat{k}_1 - \gamma_{i1}) \ \} \quad . \tag{7.25}$$

Since F_i and $\hat{F}_i$ are similar, they have the same eigenvalues, hence all F_i are stable as well. The required gain is $K = R^{-1}\hat{K}$. □

Note that although each F_i in (7.21) is stable with a proper choice of K, this does not necessarily imply that the resulting error is also stable. This is due to switching in (7.21), and there are some examples which indicate that switching between stable systems may cause instability in certain cases, see [7]. In such examples, the instability is often due to a particular (often periodic) switching. When applied to chaotic systems, as in this work, the switching is also chaotic. In such cases it is reasonable to expect that chaotic switching between stable systems as in (7.21) might yield stable error dynamics. However, this conjecture requires further investigation. The necessary and sufficient conditions on F_i to guarantee the stability of (7.21) independent of the switching pattern are not known. Some preliminary results indicate that the problem might even be undecidable, see [7]. Some sufficient conditions to guarantee the stability exist in the literature. The simplest one is the existence of a common quadratic Lyapunov function $V = e^T Pe$ for (7.21), where P is a symmetric and positive definite matrix, i.e. there exist symmetric and positive definite matrices P, Q_i such that $F_i^T P + PF_i = -Q_i$ holds for $i = 1, \ldots, m$. Indeed in this case, by differentiating $V = e^T Pe$, and using (7.21) we obtain $\dot{V} = -e^T Q_i e < 0$, and by using standard Lyapunov stability arguments, we conclude the asymptotic stability of error. In fact, it can easily be concluded that the decay is exponential in this case. Unfortunately, the necessary and sufficient conditions for the existence of such a common Lyapunov function is also not known. A sufficient condition is the existence of a common set of eigenvectors $\{v_1 \ldots v_n\}$ for F_i, i.e. $F_i v_j = \lambda_{ij} v_j$ holds for $i = 1, \ldots, m,\ j = 1, \ldots, n$. Indeed, in this case if we set $V = (v_1 \ldots v_n)$, after some straightforward calculations it is easy to show that $P = (VV^T)^{-1}$ yields a common Lyapunov function. The required condition may seem to be restrictive, but it holds in certain cases, including our first simulation example given in the next section.

Remark 12 *We also note that the proposed observer has a switching behavior, resulting from the piecewise-linear dynamics of the drive system. Due to this structure, the control action in the observer can be interpreted as some "IF-THEN" rules. This interpretation may be used in designing, or in incorporation of fuzzy control techniques into the synchronization schemes. However, this point requires further investigation.*

Finally we note on the robustness of the synchronization scheme proposed in this paper. If the error dynamics given by (7.19) or (7.21) is exponentially stable, then the corresponding scheme is robust with respect to noise and parameter mismatch. This claim can be justified by using the robustness of exponentially stable systems, see e.g. [71], [75]. As an example, consider the synchronization scheme given by (7.20) and assume that the synchronization signal is corrupted by noise. In this case, z_1 in (7.20) should be replaced by $z_1 + n$, where n represents the noise. The corresponding error dynamics now will be as follows

$$\dot{e} = F_i e - Kn \quad . \tag{7.26}$$

Hence, the noise n enters as an input into an exponentially stable system, and if n is bounded it produces a bounded error as well. To elaborate further, let us assume that $V = e^T P e$ is a common Lyapunov function for (7.21). By differentiating V, using $F_i^T P + P F_i = -Q_i$ and (7.26), we obtain :

$$\dot{V} = -e^T Q_i e + 2e^T P K n \quad , \tag{7.27}$$

Now assume that the noise is bounded and set $n_{max} = \max_t \mid n(t) \mid$. Let $\lambda_{min}(\cdot)$ and $\lambda_{max}(\cdot)$ denote the minimum and maximum eigenvalues of symmetric matrices, respectively, and set $\alpha = \lambda_{max}(P)$, $\beta = \min_i\{\lambda_{min}(Q_i)\}$. From (7.27) we obtain

$$\dot{V} \leq -\beta \|e\|^2 + 2\alpha n_{max} \|K\| \|e\| \quad , \tag{7.28}$$

where $\| \cdot \|$ denotes the standard Euclidean norm. By using standard invariance arguments, see [78], it is easy to deduce that the error will be bounded by $e_\infty = 2\alpha \|K\| n_{max}/\beta$ as $t \to \infty$. Note that the noise affects the error linearly. The same also holds for the gains. Hence, larger gains are undesirable since they amplify the effect of noise as well, as is expected.

7.3 Periodic Switching

In this section we will consider the stabilization of the systems given with the dynamics $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x})$ where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{f}(\cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\mathbf{g}(\cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}^{n \times m}$, and $\mathbf{u}(\cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$. As it is obvious there are m control signals and n states to be stabilized. If $m = n$ the system can be stabilized easily. The proposed periodic switching strategy is composed of multiple phases as given below;

Procedure

Phase 1 **:** Choose a suitable control signal, $\mathbf{u}(\mathbf{x})$ to stabilize a set of m states, (i.e. $x_1, x_2, \ldots, x_m$). Apply this control signal for T_1 seconds.

...

Phase i **:** Choose a suitable control signal, $\mathbf{u}(\mathbf{x})$ to stabilize a set of m states (distinct from previously selected sets). Apply this control signal for T_i seconds.

...

After all possible set selections are made, return to the Phase 1 again.

A suitable control signal must not only stabilize the associated state variables but also guarantee that the state vector is bounded. Let $\mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}) \cdots f_n(\mathbf{x})]^T$ and $\mathbf{g}(\mathbf{x}) = [\mathbf{g}_1(\mathbf{x}) \cdots \mathbf{g}_m(\mathbf{x})]$ where $f_i(\cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}$, $\mathbf{g}_j(\cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, for $i = 1, \ldots, n$, and $j = 1, \ldots, m$. Let us try to find the control signal to stabilize the m state variables, that is, $x_{i_1}, \ldots, x_{i_m}$. If we expand the equation defining the dynamics of the system we would have the following:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) + \textstyle\sum_{i=1}^m \mathbf{g}_i(\mathbf{x}) u_i(\mathbf{x}) \\ \dot{x}_k &= f_k(\mathbf{x}) + \textstyle\sum_{i=1}^m g_{i,k}(\mathbf{x}) u_i(\mathbf{x}) \end{aligned}$$

where $\mathbf{g}_i(\mathbf{x}) = [g_{i,1}(\mathbf{x}) \cdots g_{i,m}(\mathbf{x})]^T$, $g_{i,j}(\cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}$, $\mathbf{u}(\mathbf{x}) = [u_1(\mathbf{x}) \cdots u_m(\mathbf{x})]^T$, $u_i(\cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}$, and $k = 1, 2, \ldots, n$. An easy way to stabilize the m state variables $x_{i_1}, \ldots, x_{i_m}$ is to calculate the control signal using the following matrix equation:

$$\begin{array}{ccccccc} \begin{bmatrix} -\lambda_{i_1} x_{i_1} \\ -\lambda_{i_2} x_{i_2} \\ \vdots \\ -\lambda_{i_m} x_{i_m} \end{bmatrix} & = & \begin{bmatrix} f_{i_1}(\mathbf{x}) \\ f_{i_2}(\mathbf{x}) \\ \vdots \\ f_{i_m}(\mathbf{x}) \end{bmatrix} & + & \begin{bmatrix} g_{1,i_1}(\mathbf{x}) & \cdots & g_{m,i_1}(\mathbf{x}) \\ g_{1,i_2}(\mathbf{x}) & \cdots & g_{m,i_2}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ g_{1,i_m}(\mathbf{x}) & \cdots & g_{m,i_m}(\mathbf{x}) \end{bmatrix} & & \begin{bmatrix} u_1(\mathbf{x}) \\ u_2(\mathbf{x}) \\ \vdots \\ u_m(\mathbf{x}) \end{bmatrix} \\ \dot{\mathbf{x}}_{\mathcal{I}_m} & = & \mathbf{f}_{\mathcal{I}_m}(\mathbf{x}) & + & \tilde{G}(\mathbf{x}) & & \mathbf{u}(\mathbf{x}) \end{array}$$

where $\lambda_{i_1}, \ldots, \lambda_{i_m}$ are arbitrary positive constants. If $\tilde{G}(\mathbf{x})$ has an inverse then the control signal can be calculated as;

$$\mathbf{u}(\mathbf{x}) = \tilde{G}(\mathbf{x})^{-1}(\dot{\mathbf{x}}_{\mathcal{I}_m} - \mathbf{f}_{\mathcal{I}_m}(\mathbf{x})) \tag{7.29}$$

If $\tilde{G}(\mathbf{x})$ has no inverse then $\mathbf{u}(\mathbf{x})$ can not be calculated with (7.29). There are two possible reasons for $\tilde{G}(\mathbf{x})$ to be a singular matrix.

1. The chosen index set $\mathcal{I}_m$,
2. The state of the system being in a certain region of the state space, that is, $\{\mathbf{x} | \det(\tilde{G}(\mathbf{x})) = 0\}$

The first case is a problem related with the dynamics of the system, i.e. the structure of $\mathbf{g}(\mathbf{x})$. Since the chosen state variables, x_{i_k}, cannot be controlled separately, the corresponding phase can be dropped. If the singularity occurs for a specific values of $\mathbf{x}$ then another phase is required with a control signal just to take the system away from that region of the state space. This argument shows that switching may be useful in avoiding such singularities. If $\tilde{G}(\mathbf{x})$ is not singular but close to it then the solution vector $\mathbf{u}(\mathbf{x})$ will have a large norm which is most probably out of physical capability of the actual system. In such cases we propose to use the saturation method with circular boundary proposed in Section 5.1.5.

7.4 A Switching Strategy Emulating Common Lyapunov Function

The Theorem 1 and Lemma 1 presented in Section 2.3 give a sufficient condition for the stability of a switching system of the form $\dot{\mathbf{x}} = A_i\mathbf{x}$, $i = 1, 2, \ldots, N$. The existence of a common positive definite matrix P which satisfies $A_i^T P + PA_i < 0$, $i = 1, 2, \ldots, N$ guarantees that, independent of the switching sequence, the switching system is stable. With the aid of P, we can define a common Lyapunov function $V = \mathbf{x}^T P\mathbf{x}$. It is obvious from the assumptions given above that $V(\mathbf{x})$ decreases along the solution of $\dot{\mathbf{x}} = A_i\mathbf{x}$, independent of the switching sequence.

In the sequel, we will propose a switching strategy which may be related to the ideas given above. Let us consider the system given by:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x}) \tag{7.30}$$

We propose to chose $\mathbf{u}(\mathbf{x})$ based on the following two goals. Choose the control law $\mathbf{u}(\mathbf{x})$ so that:

1. The overall state norm of $\mathbf{x} \in \mathbb{R}^n$ decreases,
2. Meanwhile the norm of some state variables, say $x_{i_1}, \ldots, x_{i_k}$, also decreases.

Note that, if condition 1 is always satisfied by using a single (i.e. unswitched) control law, then condition 2 is redundant, and the system is always stable with this single control law. However, in some cases $\mathbf{u}(\mathbf{x})$ may become singular for certain $\mathbf{x} \in \mathbb{R}^n$, and switching different state components x_{i_j} for second condition given above, we may generate a reasonable switching strategy. If an appropriate switching strategy is found, then it is reasonable to expect that the overall system remains stable. To determine an appropriate control law, which satisfies the conditions given above, may be found with the aid of the following equations:

$$\frac{d}{dt}\|\mathbf{x}\|^2 + \lambda\|\mathbf{x}\|^2 = 0 \tag{7.31}$$

$$\frac{d}{dt}|x_{i_j}|^2 + \lambda_{i_j}|x_j|^2 = 0 \tag{7.32}$$

where λ and λ_{i_j} are arbitrary positive numbers. By using (7.30) and (7.31), and noting that $\|\mathbf{x}\|^2 = \mathbf{x}^T\mathbf{x}$, we obtain:

$$\begin{aligned} \tfrac{d}{dt}\|\mathbf{x}\|^2 + \lambda\|\mathbf{x}\|^2 &= 2\mathbf{x}^T\dot{\mathbf{x}} + \lambda\mathbf{x}^T\mathbf{x} \\ &= 2\mathbf{x}^T[\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x})] + \lambda\mathbf{x}^T\mathbf{x} \end{aligned}$$

Hence the required $\mathbf{u}(\mathbf{x})$ satisfying (7.31) may be found as follows:

$$\mathbf{x}^T\mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x}) = -\mathbf{x}^T\mathbf{f}(\mathbf{x}) - \frac{\lambda}{2}\mathbf{x}^T\mathbf{x} \tag{7.33}$$

From (7.32) we obtain:

$$\begin{aligned} \tfrac{d}{dt}\|x_{i_j}\|^2 + \lambda\|x_{i_j}\|^2 &= 2x_{i_j}\dot{x}_{i_j} + \lambda x_{i_j}^2 \\ &= 2x_{i_j}[f_{i_j}(\mathbf{x}) + \bar{g}_{i_j}(\mathbf{x})\mathbf{u}(\mathbf{x})] + \lambda x_{i_j}^2 \end{aligned}$$

where $i_j \in \mathcal{I}_n$, $\mathbf{g}(\mathbf{x}) = [\bar{g}_1^T(\mathbf{x}) \cdots \bar{g}_n^T(\mathbf{x})]^T$, $\bar{g}_i(\cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}^{1\times m}$. Assuming that $x_{i_j} \neq 0$, to satisfy (7.32), we may choose $\mathbf{u}(\mathbf{x})$ as follows:

$$\bar{g}_{i_j}(\mathbf{x})\mathbf{u}(\mathbf{x}) = -f_{i_j}(\mathbf{x}) - \frac{\lambda_{i_j}}{2}\mathbf{x}_{i_j} \tag{7.34}$$

Let $\mathcal{I}_{m-1} = \{i_1, \ldots, i_{m-1}\}$ where $i_j \in \mathcal{I}_n$. The following matrix equation can now be written using the above derivations:

$$\begin{array}{cccc} \begin{bmatrix} \mathbf{x}^T\mathbf{g}(\mathbf{x}) \\ \bar{g}_{i_1}(\mathbf{x}) \\ \vdots \\ \bar{g}_{i_{m-1}}(\mathbf{x}) \end{bmatrix} & \begin{bmatrix} u_1(\mathbf{x}) \\ u_2(\mathbf{x}) \\ \vdots \\ u_m(\mathbf{x}) \end{bmatrix} & = & \begin{bmatrix} -\mathbf{x}^T\mathbf{f}(\mathbf{x}) - \frac{\lambda}{2}\mathbf{x}^T\mathbf{x} \\ -f_{i_1}(\mathbf{x}) - \frac{\lambda_{i_1}}{2}x_{i_1} \\ \vdots \\ -f_{i_{m-1}}(\mathbf{x}) - \frac{\lambda_{i_{m-1}}}{2}x_{i_{m-1}} \end{bmatrix} \\ \bar{G}(\mathbf{x}) & \mathbf{u}(\mathbf{x}) & = & \mathbf{b}_{\mathcal{I}_{m-1}}(\mathbf{x}) \end{array}$$

where $i_k \in \mathcal{I}_{m-1}$. If $\bar{G}(\mathbf{x})$ is invertible then the control signal can be calculated as:

$$\mathbf{u}(\mathbf{x}) = \bar{G}(\mathbf{x})^{-1}\mathbf{b}_{\mathcal{I}_{m-1}}(\mathbf{x}) \tag{7.35}$$

Similar to $\tilde{G}(\mathbf{x})$ in Section 7.3 there are two possible reasons for $\bar{G}(\mathbf{x})$ to be a singular matrix.

1. The chosen index set $\mathcal{I}_{m-1}$,
2. The state of the system being in a certain region of the state space, that is, $\{\mathbf{x} | \det(\bar{G}(\mathbf{x})) = 0\}$

The solutions we propose for these two cases are the same as for $\tilde{G}(\mathbf{x})$. If $\bar{G}(\mathbf{x})$ is not singular but close to it then we propose to use the saturation method with circular boundary proposed in Section 5.1.5.

7.4.1 Event Driven Switching

We have stated that if the overall state norm $\|\mathbf{x}\|$ decreases for all values of $\mathbf{u}(\mathbf{x})$ applied to the system then the second condition is redundant. But it is rather difficult to find such a control signal for any given system. The switching introduced above is a remedy to overcome this difficulty. By the control signals in each phase, the overall state norm and one of the states are forced to decrease. These phases are applied circularly, i.e. in phase $1:$ ($\|\mathbf{x}\|\downarrow$ & $|\mathbf{x}_1|\downarrow$), in phase $2:$ ($\|\mathbf{x}\|\downarrow$ & $|\mathbf{x}_2|\downarrow$), ..., in phase $n:$ ($\|\mathbf{x}\|\downarrow$ & $|\mathbf{x}_n|\downarrow$), in phase $n+1:$ ($\|\mathbf{x}\|\downarrow$ & $|\mathbf{x}_1|\downarrow$), where $\downarrow$ symbol denotes a decrease in the corresponding norm. But there might be other approaches to stabilize a given system, not necessarily by forcing all the individual states to decrease, together with the norm of the overall state vector.

The solution of the Equation (7.31) used in the above switching strategy guaranties the norm of the state to decrease but does not guaranty the state vector to reach to the origin. As an example consider the so-called driftless systems, i.e. $\mathbf{f}(\mathbf{x}) = 0$ in (7.30). Consider some region in the state space such that $\mathcal{S} = \{\mathbf{x}|\bar{g}_i(\mathbf{x}) = 0\}$. As long as the the solution of the dynamic equation of the closed loop system stays in $\mathcal{S}$, the norm of the overall state vector will decrease, but the state x_i would not change its value because $\dot{x}_i = \bar{g}_i(\mathbf{x})$. Hence the norm will never be zero. In such cases the increase of a state variable, opposite to the previous approach, might serve to by pass the region, $\mathcal{S}$, if there exist a path.

The equations (7.31) and (7.32) used in constructing (7.35) can be utilized in many different ways. The constants λ_{i_j} in (7.32) are required to be positive. But just for the reasons given above they may be chosen to be negative if necessary. Suppose that we require the state x_2 be less then x_3 in absolute value to avoid some singularity. Let us define the ratio of states $z(\mathbf{x}) = x_2^2/x_3^2$. The solution of the following differential equation can be used instead of (7.32) to force the solution so that x_2 is less than x_3.

$$\tfrac{d}{dt}z(\mathbf{x}) + \lambda_z z \quad = \quad 0 \tag{7.36}$$

where λ_z is an arbitrary positive constant. Note that the Equation (7.36) is affine in $\mathbf{u}(\mathbf{x})$, i.e:

$$\begin{aligned}
\tfrac{d}{dt}z(\mathbf{x}) + \lambda_z z(\mathbf{x}) &= \tfrac{2x_2\dot{x}_2x_3^2 - 2x_3\dot{x}_3x_2^2}{x_3^4} \\
&= \tfrac{2x_2x_3^2(f_2(\mathbf{x})+\bar{g}_2(\mathbf{x})\mathbf{u}(\mathbf{x})) - 2x_3x_2^2(f_2(\mathbf{x})+\bar{g}_3(\mathbf{x})\mathbf{u}(\mathbf{x}))}{x_3^4} \\
&= \tfrac{2x_2x_3^2f_2(\mathbf{x}) - 2x_3x_2^2f_3(\mathbf{x})}{x_3^4} + \tfrac{2x_2x_3^2\bar{g}_2(\mathbf{x})\mathbf{u}(\mathbf{x}) - 2x_3x_2^2\bar{g}_3(\mathbf{x})\mathbf{u}(\mathbf{x})}{x_3^4} \\
&= \tfrac{2x_2x_3^2f_2(\mathbf{x}) - 2x_3x_2^2f_3(\mathbf{x})}{x_3^4} + \tfrac{2x_2x_3^2\bar{g}_2(\mathbf{x}) - 2x_3x_2^2\bar{g}_3(\mathbf{x})}{x_3^4}\mathbf{u}(\mathbf{x})
\end{aligned} \tag{7.37}$$

where we assumed that $x_3 \neq 0$. Equation (7.37) can be used as one of the rows of $\bar{G}(\mathbf{x})$ in (7.35). This way we may expect the value of x_2^2 be less then x_3^2. (7.37) should be handled carefully because we are trying to stabilize the ratio. This might well force x_2 to go zero before the other states. It should be checked that such situation does not cause any instability problems or does not cause any function $\bar{g}_i(\mathbf{x}) = 0$ which will freeze the state x_i.

Another open problem in the earlier approach is the order of the states to be decreased. We have not made any suggestions in Section 7.4. One approach to this problem might be to select the state with the highest absolute value. This approach has been demonstrated on different systems in Section 8.3.2. In this approach at the end of each phase an event is produced if there is another state whose absolute value is higher then the one used in the current phase, and switching to another controller is performed accordingly.

7.5 A Switching Strategy to Stabilize some of the Driftless Systems

In this section we consider a stabilizing switching strategy which can be applied to some driftless systems which have the following dynamics:

$$\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x}) \tag{7.38}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{g}(\cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}^{n \times m}$, and $\mathbf{u}(\cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$. As it is seen from (7.38) the system does not drift when $\mathbf{u}(\mathbf{x}) = \mathbf{0}$. We propose a switching strategy which depends on the utilization of this fact. Let $\mathbf{g}(\mathbf{x}) = [\bar{g}_1^T(\mathbf{x}) \cdots \bar{g}_n^T(\mathbf{x})]^T$ where $\bar{g}_i(\cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}^{1 \times m}$. Then the i^{th} state can be expressed as:

$$\dot{x}_i = \bar{g}_i(\mathbf{x})\mathbf{u}(\mathbf{x}) \tag{7.39}$$

Let us provide some definition to be utilized in defining the switching strategy. Let $\mathbf{u}(\mathbf{x}) = [u_1(\mathbf{x}) \cdots u_m(\mathbf{x})]^T$, and $\mathbf{u}_j(\mathbf{x}) = [0 \cdots 0 \; u_j(\mathbf{x}) \; 0 \cdots 0]^T$, for $j = 1, \ldots, n$.

Definition 25 *Distinctly Stabilizable States :*
If there exists an i and a j such that

$$\bar{g}_i(\mathbf{x})\mathbf{u}_j(\mathbf{x}) \neq 0, \quad \forall \mathbf{x} \in \mathbb{R}^n \tag{7.40}$$

Then the state x_i is distinctly stabilizable by the control signal $\mathbf{u}_j(\mathbf{x}) \neq 0$. A typical choice is $u_j(\mathbf{x}) = -\lambda_{i,j} x_i$. $\square$

Note that since in general $m < n$ in a control system we have some of the states coupled with each other, i.e. application of $\mathbf{u}_j(\mathbf{x})$ changes more then one state variable.

Definition 26 *State Dependent Distinctly Stabilizable States :*
Consider a state, x_i such that $\bar{g}_i(\mathbf{x})$ is only a function of distinctly stabilizable states, x_j, where $i \in \mathcal{I}_n$ and the indices of these states constitute the index set $\mathcal{I}_j$. If there exist a control signal $\mathbf{u}_k(\mathbf{x})$ such that $\bar{g}_i(\mathbf{x})\mathbf{u}_k(\mathbf{x}) \neq 0$ where $k \in \{\mathcal{I}_n - \mathcal{I}_j\}$ then the state x_i is state dependent distinctly stabilizable state. A typical choice is $u_k(\mathbf{x}) = -\bar{g}_{i,k}(\mathbf{x})\lambda_{i,k} x_i$ where $\bar{g}_i(\mathbf{x}) = [\bar{g}_{i,1}(\mathbf{x}) \ \bar{g}_{i,2}(\mathbf{x}) \ \cdots \ \bar{g}_{i,m}(\mathbf{x})]$. $\square$

Definition 27 *Distinctly Non-Stabilizable States :*
If a state x_i is not distinctly stabilizable or stated dependent distinctly stabilizable the it is called distinctly non-stabilizable state. $\square$

Consider the dynamics of nonholonomic integrator which is given as:

$$\begin{aligned}
\dot{x_1} &= u_1, \\
\dot{x_2} &= u_2, \\
\dot{x_3} &= x_1 u_2 - x_2 u_1
\end{aligned}$$

For this system x_1 and x_2 are distinctly stabilizable state whereas x_3 is distinctly non-stabilizable state. Also consider the chained system of degree 3 given with the following dynamics:

$$\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_1 x_3 \\
\dot{x}_3 &= u_2
\end{aligned}$$

In this system x_1 and x_3 are distinctly stabilizable state whereas x_2 is state dependent distinctly stabilizable state.

Let the system given in (7.38) has states either distinctly stabilizable or state dependent distinctly stabilizable states. Let the state dependent distinctly stabilizable states be $x_{i_{\mathcal{S}}}$ where $i_{\mathcal{S}} \in \mathcal{I}_n$. Let the distinctly stabilizable states be x_j where $j \in \mathcal{I}_n$. The switching strategy is as follows:

1) Stabilize the state dependent distinctly stabilizable states, $x_{i_{\mathcal{S}}}$, then,

2) Stabilize those state(s) that makes $\bar{g}_{i_{\mathcal{S}}}(\mathbf{x}) = 0$. In this way the state dependent distinctly stabilizable states will be freezed at zero independent of the control signal applied thereafter,

3) Stabilize the distinctly stabilizable state(s) x_j.

This switching strategy is rather specific but there are some plants, like chained systems, where each of the states can be stabilized distinctly one by one using the switching strategy given above. A simulation example is provided in Section 8.3.2.

Chapter 8

Applications

In this chapter we will consider the application of several methods proposed in the thesis.

In Section 8.1 we will consider two modelling approaches. The first one is the modelling of a chaotic system by a switching system. This will be utilized to provide a switching synchronization scheme for a class of chaotic system given in Section 7.2. Next we will use the fuzzy approximation proposed in Section 6.3 to approximate a time varying nonlinear function.

In Section 8.2 we will provide an application of the synchronization scheme proposed in Section 7.2 to observer based chaotic message transmission.

In Section 8.3 we will demonstrate the method we have proposed for the stabilization of different systems. These methods can be categorized as : phase portrait matching, switching, and fuzzy system approaches. In Section 8.3.2 we will utilize various switching approaches (periodic switching, the switching emulating common Lyapunov function, the event driven switching).

Throughout the simulation examples, the initial state values are given to some certain digits. This does not mean that they are required to be precise. It is due to the random number generator which is used to generate the initial conditions.

8.1 System Modelling

8.1.1 Using Switching System

In this section, we present two simulation examples. The first one is called the generalized Chua's circuit, and is given by the following equations :

$$\dot{z}_1 = \alpha[z_2 - h(z_1)] \quad , \tag{8.1}$$

$$\dot{z}_2 = z_1 - z_2 + z_3 \quad , \tag{8.2}$$

$$\dot{z}_3 = -\beta z_2 \quad , \tag{8.3}$$

where the nonlinearity $h(\cdot)$ is given by the following piecewise-linear characteristic

$$h(z_1) = m_{2q-1} z_1 + \sum_{j=1}^{2q-1} (m_{j-1} - m_j)(\mid z_1 + c_j \mid - \mid z_1 - c_j \mid) \quad . \tag{8.4}$$

Here α, β, q, m_k and c_l denote various coefficients. For various coefficients of these values, this system exhibits various chaotic behaviors, see e.g. [79]. In our simulations, we chose $\alpha = 9$, $\beta = 14.286$, $q = 2$, $m_0 = 0.9/7$, $m_1 = -3/7$, $m_2 = 3.5/7$, $m_3 = -2.4/7$, $c_1 = 1$, $c_2 = 2.15$, $c_3 = 4$, and for these parameters this system is known to exhibit a three-scroll chaotic behavior, as shown in Figure 8.1 (a); for details and also for an electronic implementation of the relevant circuit, see [79]. This system can easily be transformed into the structure given by (7.7), (7.8), together with (7.10) and (7.11). Note that in our formalism, we have $m = 7$, with $k_1 = -k_6 = -c_3$, $k_2 = -k_5 = -c_2$, $k_3 = -k_4 = -c_1$. By using

$$A = \begin{pmatrix} 0 & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{pmatrix} \quad , \quad b = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \quad , \tag{8.5}$$

we can transform (8.1)-(8.4) to the structure given by (7.6)-(7.10) with $h_1 = h_7 = -m_3 b$, $h_2 = h_6 = -m_2 b$, $h_3 = h_5 = -m_1 b$, $h_4 = -m_0 b$, $b_1 = -b_7 = -d_1 b$, $b_2 = -b_6 = -d_2 b$, $b_3 = -b_5 = -d_3 b$, $b_4 = 0$, where $d_1 = [-m_0 c_1 + m_1(c_1 - c_2) + m_2(c_2 - c_3) + m_3 c_3]$, $d_2 = [-m_0 c_1 + m_1(c_1 - c_2) + m_2 c_2]$, $d_3 = [c_1(m_1 - m_0)]$. In the formalism of Theorem 12, we have $\alpha_1 = 1$, $\alpha_2 = b - a$, $\alpha_3 = 0$, and the required transformation in (7.22) is given as $R = (r_1 r_2 r_3)$ with $r_1 = (1\ 1\ b)^T$, $r_2 = (0\ a\ 0)^T$, $r_3 = (0\ 0\ a)^T$. By using the coefficients given above and (7.25), we obtain $\hat{k}_1 > 2.86$, $\hat{k}_3 > 55.103$, and remaining gain $\hat{k}_2$ can be found from (7.25). If we choose $\hat{k}_1 = 4$, $\hat{k}_3 = 57.144$, we obtain $\hat{k}_2 > 5.842$; and with the selection of $\hat{k}_2 = 13$, we obtain the gain vector $K = R^{-1}\hat{K} = (4\ 1\ 0)^T$. In this case all matrices $F_i = A_i - KC$ are stable; furthermore direct calculation shows that all F_i have a common set of eigenvectors, hence a common Lyapunov function exists. Therefore, we conclude that the error dynamics given by (7.21) is exponentially stable. We simulated the observer structure given by (7.20) together with the given system (8.1)-(8.4), and the simulation results are shown in Figure 8.1. The initial conditions

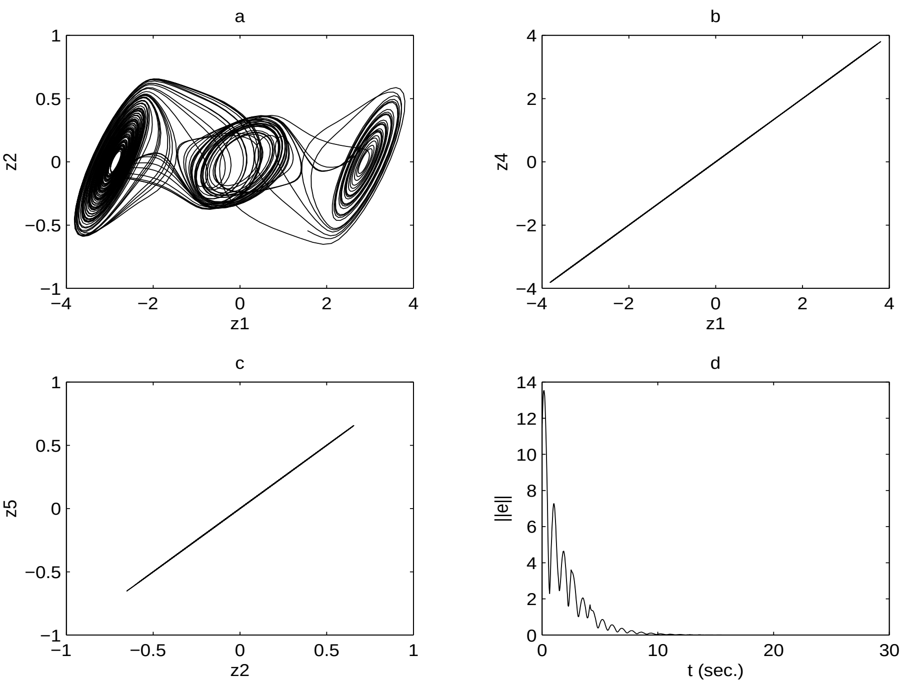


Figure 8.1: Simulation result for Chua's circuit (a) z_1 vs z_2, (b) z_1 vs. $z_4 = \hat{z}_1$, (c) z_2 vs. $z_5 = \hat{z}_2$, (d) $\|e\|$ vs time.

are chosen as $z(0) = (4\ 1\ -4)^T$, $\hat{z}(0) = -z_0$, hence $\|e(0)\| = \|z(0) - \hat{z}(0)\| = 11.48$, which is quite large. In Figure 8.1-b and 1-c, the plots of z_1 versus $\hat{z}_1$ and z_2 versus $\hat{z}_2$ are shown, respectively (note that in the figures, we used $z_4 = \hat{z}_1,\ z_5 = \hat{z}_2$). Also note that these figures are plotted after the transients. Finally, in Figure 8.1-d, we plotted the synchronization error magnitude $\|e\|$.

In the second simulation, we used the Brockett system, see [75] for details. This system is in Lur'e form and could be transformed into the structure given by (7.6)-(7.10) as described in the Section 7.2.2. Here we have $m = 3$, $k_1 = -k_2 = -1$. By using

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1.25 & -1 \end{pmatrix} \quad , \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad , \tag{8.6}$$

we can transform the Brockett system into the structure given by (7.6)-(7.10) with $h_1 = h_3 = -3.6b$, $h_2 = 1.8b$, $b_1 = -b_3 = -5.4b$, $b_2 = 0$. This system exhibits chaotic behaviour as shown in Figure 8.2-a. To calculate the required gain vector in (7.20), we use the procedure given in Theorem 12. In the formalism of Theorem 12, we have $\alpha_1 = 1$, $\alpha_2 = 1.25$, $\alpha_3 = 0$, and the required transformation in (7.22) is given as $R = (r_1 r_2 r_3)$ with $r_1 = (1\ 1\ 1.25)^T$, $r_2 = (0\ 1\ 1)^T$, $r_3 = (0\ 0\ 1)^T$. By using the coefficients given above and (7.25), we obtain $\hat{k}_1 > -1$, $\hat{k}_3 > 1.8$, and remaining gain $\hat{k}_2$ can be found from (7.25). If we choose $\hat{k}_1 = 0$, $\hat{k}_3 = 2$, we obtain $\hat{k}_2 > 4.35$; and with the selection of $\hat{k}_2 = 4.5$, we obtain the gain vector $K = R^{-1}\hat{K} = (0\ 4.5\ -2.5)^T$. In this case all matrices $F_i = A_i - KC$ are stable. However, unlike the previous case the matrices F_i do not have a common set of eigenvectors, hence existence of a common Lyapunov function is not guaranteed in this case. Since the necessary and sufficient conditions for the existence of a common Lyapunov function are not known yet, we do not claim the nonexistence of a common Lyapunov function as well. We simulated the system given by (7.7)-(7.10) and the observer given by (7.20). The resulting error in synchronization is given in Figure 8.2-b. As can be seen from the figure, the error system is exponentially stable. Hence, as mentioned in Section 7.2.2, this system should be robust with respect to noise. To demonstrate this point, we also assumed that the syncronization signal $z_1(t)$ is corrupted with a noise term $n(t)$, i.e. in (7.20) we used $z_1 + n$. The noise is assumed to be random and uniformly distributed in $[0\ a]$, where $a > 0$ is a constant. We considered the case $a = 0.2$ and $a = 0.02$, and the resulting synchronization errors are shown in Figure 8.2-c and 8.2-d, respectively. Both figures are plotted after the transients. Note that since the peak amplitude of z_1 is around 2, the case $a = 0.2$ corresponds to a 10% perturbation, and as can be seen in Figure 8.2-c, the synchronization error magnitude is of the same order of a. The case $a = 0.02$ corresponds to 1% perturbation, and as can be seen in Figure 8.2-d, the synchronization error magnitude is still of the same order of a. For the Brockett system, the initial conditions are chosen as $z(0) = (2\ 1\ -1)^T$,

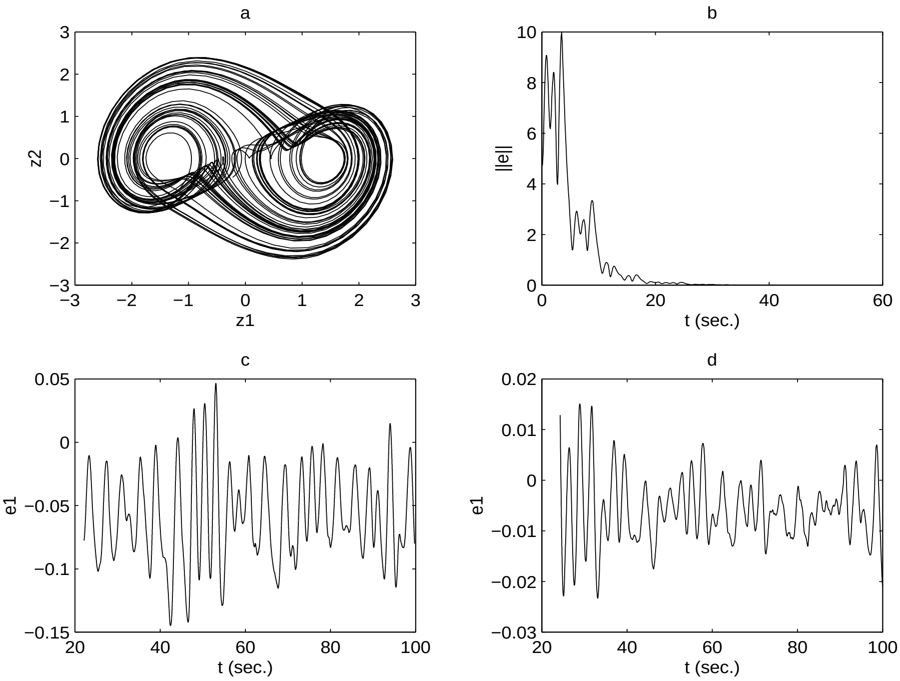


Figure 8.2: Simulation result for Brockett system (a) z_1 vs z_2, (b) $\|e\|$ vs time, (c) $e_1 = z_1 - \hat{z}_1$ vs time for $a = 0.2$, (d) $e_1 = z_1 - \hat{z}_1$ vs time for $a = 0.02$.

$\hat{z}(0) = -z_0$, hence $\|e(0)\| = \|z(0) - \hat{z}(0)\| = 4.89$, which is quite large. These simulations show the effectiveness of the proposed technique.

8.1.2 Using Fuzzy Controller Obtained by Data Mining

In this section we will consider to build a fuzzy controller which approximates a given function. Consider the Duffing system given with the following differential equation:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -0.25z_1 - z_1^3 + 11cos(t) \end{aligned}$$

Let us denote the right hand side of second equation as $f(z,t)$, i.e.

$$f(z,t) = -0.25z - z^3 + 11cos(t)$$

We will build a fuzzy system whose output $u(z,t)$ approximates $f(z,t)$ in a certain range. The form of the rules of the fuzzy system would be as follows:

$$\mathcal{L}_i : \text{IF "}\mathbf{x} \text{ is in } S_i\text{" THEN } \mathbf{u}(\mathbf{x}) = -\mathbf{K}_i\mathbf{x}.$$

where $\mathbf{x} = [z\ t]^T$, $z \in [-1\ 1]$, $t \in [0\ 2\pi]$, $K_i \in \mathbb{R}^{1\times 2}$, and $i = 1,2,3$. The domain of interest is determined as $\mathcal{D} = \{\mathbf{x}|\mathbf{x} \in [-1\ 1]\times[0\ 2\pi]\}$ hence the output of the fuzzy system would be valid in this region. $\mathcal{S}_i$ is a fuzzy relation denoted by the membership function $\mu_i(\mathbf{x}) : \mathcal{D} \longrightarrow [0\ 1]$ corresponding to the gain K_i. Using the procedure which we have proposed in Section 6.3 we first calculate $\widetilde{\mathbf{K}}(\mathbf{x})$ using (6.20). Then $R = \max_{\mathbf{x}}\{\|\widetilde{\mathbf{K}}(\mathbf{x})\|\}$ is calculated as 110.2600. We chose the gain vectors $\mathbf{K}_i$ as:

$$\begin{aligned} \mathbf{K}_1 &= [\ 220.5200 \quad 0\ ] \\ \mathbf{K}_2 &= [\ -110.2600 \quad 190.9759\ ] \\ \mathbf{K}_3 &= [\ -110.2600 \quad -190.9759\ ] \end{aligned}$$

We then calculate the membership functions $\mu_i(z,t)$, $i = 1,2,3$ using (6.27). Figure 8.3 shows the membership function for the calculated gains above.

These membership functions together with the gains $\mathbf{K}_i$, $i = 1,2,3$ specify the fuzzy system. The output of the fuzzy system and the error between the reconstructed function is given in Figure 8.4. As can be seen the error made in approximating the given function is on the order of 10^{-14}.

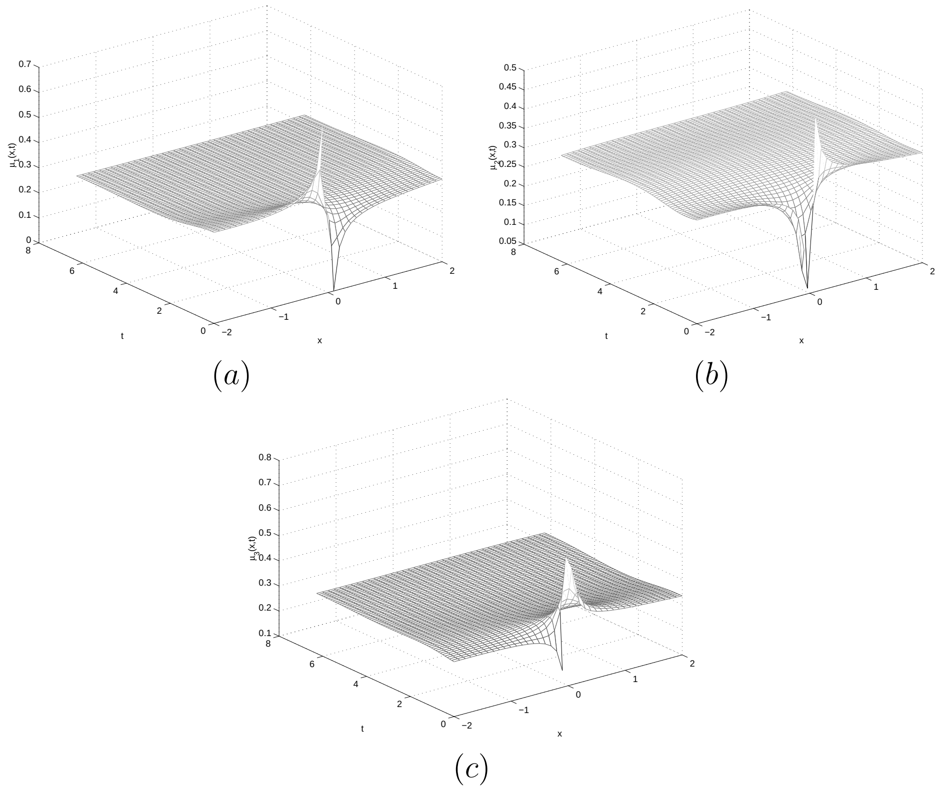


Figure 8.3: The membership function of the fuzzy system approximating $f(z,t)$.

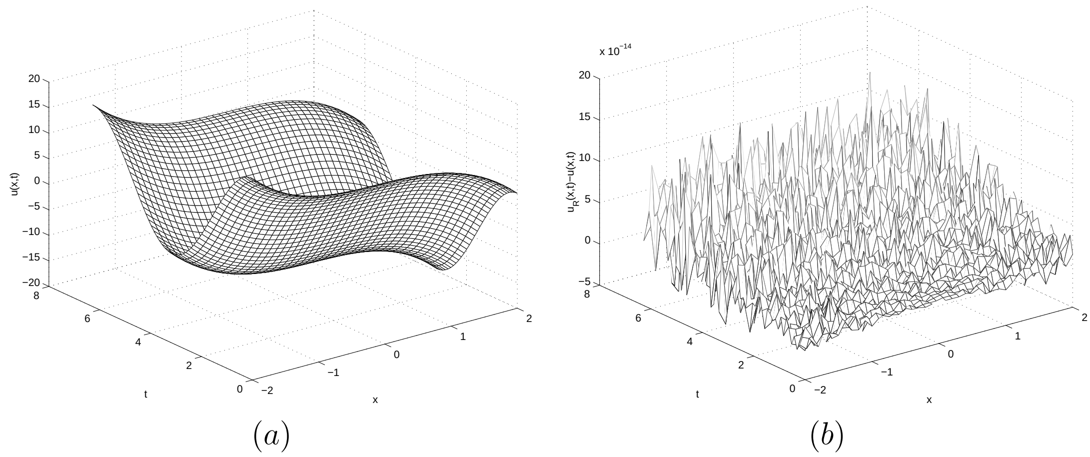


Figure 8.4: (a) The output of the fuzzy system, and (b) the error made in the modelling.

8.2 Observer Based Chaotic Message Transmission

After the seminal works of Pecora and Carrol [70], [80], the idea of synchronization and control of chaotic systems has received a great deal of interest among researchers from various fields. Many scientific journals in fact devoted special issues on these and related subjects, see e.g. IEEE Transactions on Circuits and Systems, part 1, October 1997, and December 2000 issues, International Journal of Bifurcation and Chaos, March and April 2000 issues. For more information on this subject and for more references, see [73], [81]. While the synchronization of two chaotic systems is an interesting subject on its own, one of the main motivations for the research in this area is the possibility of using chaotic signals for secure communication, see [82], [83], [84]. Use of chaos may also increase the performance of communication systems, see e.g. [85]. Most of the synchronization schemes consist of two parts : a generator of chaotic signals, which is called the drive system, and a receiver, which is also called a response system. A signal generated by the drive system may be used as an input to the receiver to achieve synchronization. An extensive list of references for various aspects of chaotic systems may be found in [72].

Chaotic systems may be used in message transmission in various ways, see e.g [84], [85]. One of the widely used technique is called the chaotic masking, and in this scheme the message is added to the chaotic signal used for synchronization, and this signal is sent to the receiver. Under certain conditions the message may be recovered at the receiver, see [82], [83], [86], [87]. Another possibility is to use the message as an appropriate input to the drive system, and send the synchronization signal to the receiver. This scheme may be called as chaotic modulation, and as in chaotic masking scheme, the message may be recovered in the receiver under certain conditions.

8.2.1 Observer Based Chaotic Masking Scheme

Consider the switching system given by (7.6)-(7.8) and the observer given by (7.16). Assume that for an appropriate gain vector K, the error system given by (7.19) is exponentially stable. As noted in Section 7.2.3 this is always possible if the pair (C, A) is observable. Let $m(t)$ be the message to be transmitted. We modify the system and the observer as follows:

$$\begin{aligned}\dot{\mathbf{z}} &= A_i\mathbf{z} + \mathbf{b}_i + Km \\ y &= z_1 + m\end{aligned} \tag{8.7}$$

$$\begin{aligned}\dot{\hat{\mathbf{z}}} &= A\hat{\mathbf{z}} + +\mathbf{h}_i z_1 + K(y - \hat{y}) \\ \hat{y} &= \hat{z}_1\end{aligned} \tag{8.8}$$

where $z_1 \in R_i$. Let us define the error $\mathbf{e}$ as $\mathbf{e} = \mathbf{z} - \hat{\mathbf{z}}$. By using (8.7) and (8.8), we obtain the following error dynamics:

$$\dot{\mathbf{e}} = (A - KC)\mathbf{e} \tag{8.9}$$

i.e. the error dynamics is the same as (7.19). This is due to the fact that the same signal Km is used in both (8.7) and (8.8), hence cancel each other in the error dynamics. Let us define the received message $m_r(t)$ as.

$$\begin{aligned}m_r(t) &= y(t) - \hat{y}(t) \\ &= C\mathbf{e}(t) + m(t)\end{aligned} \tag{8.10}$$

where $C = [1\ 0\ 0]$. Since $\|\mathbf{e}(t)\| \leq Me^{-\alpha t}\|\mathbf{e}(0)\|$ for some $m > 0$,$\alpha > 0$, it follows that the following holds:

$$\|m_r(t) - m(t)\| \leq \|C\|Me^{-\alpha t}\|\mathbf{e}(0)\| \tag{8.11}$$

that is $m_r(t) \longrightarrow m(t)$ as $t \longrightarrow \infty$. Hence asymptotic recovery of the message is possible. Note that, due to the observability of (C, A), the $\alpha > 0$ in (8.11) can be assigned arbitrarily. However, large $\alpha > 0$ may require large gain vector K, which in turn may force the system (8.7) to leave its chaotic domain.

This scheme is called chaotic masking, since the message to be transmitted $m(t)$ is added to the chaotic signal $z_1(t)$. Since chaotic signals exhibits random-like behavior, it is very difficult for an intruder to extract the message $m(t)$ from the masked message $y(t)$.

Several remarks are now in order:

(i) To increase the performance of this scheme, first a pure synchronization period, in which no message is sent, may be used for a finite period $T > 0$. At the end of this period, $\|\mathbf{e}(t)\| \leq Me^{-\alpha t}\|\mathbf{e}(0)\|$ would be negligibly small. After this period, the message may be sent. If for some reasons, e.g. noise, the error in message recovery is not acceptable, the same synchronization period may be applied again. For an application of this idea, see [74].

(ii) The proposed scheme is robust with respect to noise and parameter mismatch. To elaborate further, let us assume that the parameters in (8.7) and (8.8) vary by an amount of $\|\Delta\mu\|$, and assume that the transmitted message is corrupted by a noise $n(t)$. Let us assume that $|n(t)| \leq n_M$ for some $n_M > 0$. Then we have the following:

$$\|\mathbf{e}(t)\| \leq c_1\|\Delta\mu\| + c_2 n_M \tag{8.12}$$

where c_1 and c_2 are some positive constants. This result is due to the fact that the error dynamics is exponentially stable. For more details, see [88]

(iii) This scheme is particularly effective when the message, $m(t)$, is discrete (i.e. piecewise constant). In this case, by comparing the recovered message with a threshold value, it is possible to reconstruct the message exactly. For application of this idea, see [88].

Simulation Results:

As an example of the ideas presented above, we considered the switching system given by (8.1)-(8.4) in Section 8.1.1. The gain of the observer, as in Section 8.1.1, is chosen as $K = [4\ 1\ 0]^T$. It can easily be shown that the matrix $A - KC$ in (7.19) is stable with this selection. for the message to be transmitted, we chose $m(t) = 0.1sin(t)$, and simulated the system given by (8.7) and (8.8). The result of this simulation is shown in Figure 8.5. Here, the transmitted signal $y = z_1 + m$ is shown in Figure 8.5(a), z_1 vs z_2 graph is shown in Figure 8.5(b), the message $m(t)$ and the recovered message $m_r(t)$ are shown in Figure 8.5(c), and (d), respectively. As can be seen from the figure, the solution of (8.7) are chaotic and the message is recovered at the observer side with reasonable accuracy.

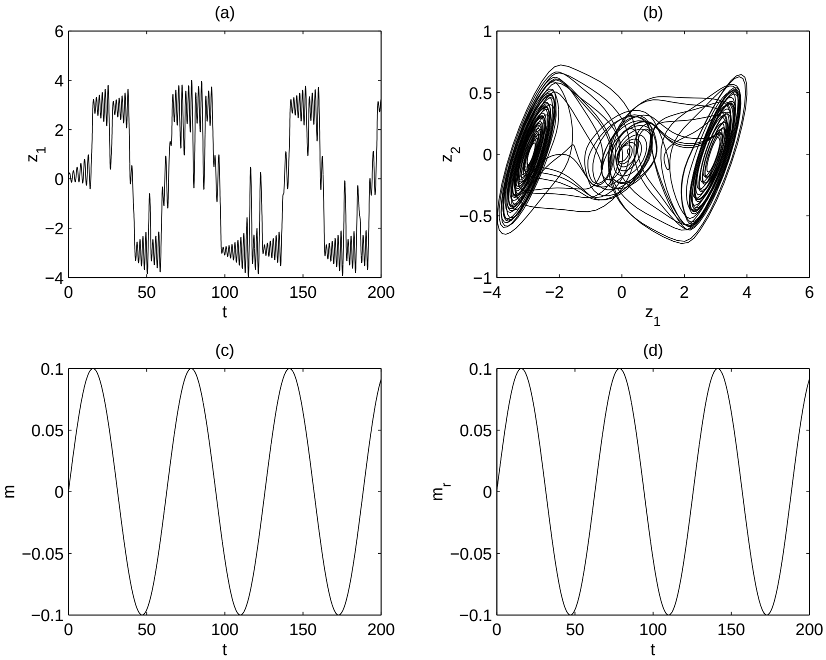


Figure 8.5: (a) The transmitted signal $y = z_1 + m$ (b) the graph of z_1 vs z_2 (c) the message $m(t)$ (d) the recovered message $m_r(t)$

8.2.2 An Observer Based Chaotic Modulation Scheme

As in Section 8.2.1, consider the switching system given by (7.6)-(7.8) and the observer given by (7.16). Assume that for an appropriate gain vector K, the error system given by (7.19) is exponentially stable. Let $m(t)$ be the message to be transmitted. We modify the system and the observer as follows:

$$\begin{aligned} \dot{\mathbf{z}} &= A_i\mathbf{z} + \mathbf{b}_i + Bm \\ y &= z_1 \end{aligned} \tag{8.13}$$

$$\begin{aligned} \dot{\hat{\mathbf{z}}} &= A\hat{\mathbf{z}} + +\mathbf{h}_i z_1 + b_i + K(y - \hat{y}) \\ \hat{y} &= \hat{z}_1 \end{aligned} \tag{8.14}$$

where $z_1 \in \mathcal{R}_i$ and $B \in \mathcal{R}^n$ is a constant vector. Let us define the error $\mathbf{e}$ as $\mathbf{e} = \mathbf{z} - \hat{\mathbf{z}}$. By using (8.13) and (8.13), we obtain the following error dynamics:

$$\dot{\mathbf{e}} = (A - KC)\mathbf{e} + Bm \tag{8.15}$$

Since y is available, we can measure $e_m = y - \hat{y} = C\mathbf{e}$, where $C = [1\ 0\ 0]^T$.

By using the Laplace transform in (8.15), we may relate the signals e_m and m. Let s denote the Laplace variable, and let the variables with capital letters such as $E_m(s)$,

$M(s)$, etc., denote the Laplace transform of the corresponding variables. From (8.15), we obtain :

$$E_m(s) = G(s)M(s) \quad , \quad G(s) = C(sI - A + KC)^{-1}b \quad . \tag{8.16}$$

Hence $e_m(t)$ is a *filtered* version of the message $m(t)$. Consequently, in some cases we may recover the message from e_m. These cases include the following.

Case 1 : Let the message m be a *band-limited* signal, whose frequency spectrum is in the range $[0, \Omega]$. If $G(s)$ given by (8.16) is a low-pass filter whose cut-off frequency ω_c satisfies $\omega_c \gg \Omega$, then we can recover m. In most of the cases, by selecting the gain K appropriately, we can design such a $G(s)$. Moreover, since e_m is available, we may use another filter $G_1(s)$ to obtain another signal e_f as follows :

$$E_f(s) = G_1(s)E_m(s) = G_1(s)G(s)M(s) \quad . \tag{8.17}$$

If we can choose G_1 and G such that $\mid 1 - G_1(j\omega)G(j\omega) \mid$ is sufficiently small for $\omega \in [0, \Omega]$, then we may use $e_f(t)$ as the recovered message. We note that while in the masking scheme presented in the previous section we have asymptotic recovery of the message, in the present case there will always be error in the recovery, however this error may be made arbitrarily small by carefully selecting the gain K.

Case 2 : Let the message m be a discrete signal, e.g. $m(t) \in \{0, 1\}$. In this case, it is possible to recover the message without error. To see this, we set $G(s) = n(s)/d(s)$ where $n(s)$ and $d(s)$ are polynomials as given below

$$n(s) = a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \quad , \quad d(s) = s^n + b_{n-1}s^{m-1} + \ldots + b_1 s + b_0 \quad , \tag{8.18}$$

where a_i and b_i are various constants, $i = 0, 1, \ldots, n-1$. We note that we may choose K appropriately so that $d(s)$ becomes a stable polynomial, i.e. all of its roots have negative real parts, moreover these roots may be selected arbitrarily, see [71], [75]. Let $\tilde{m}$ denote 0 or 1. After the transients, $e_m(t)$ will converge to $a_0\tilde{m}/b_0$. As for the transients, let T denote the interval length in which $m = 0$ or $m = 1$, and let s_i, $i = 1, 2, \ldots, n$ denote the roots of $d(s) = 0$. Since the transients contain the terms $e^{s_i t}$, these terms will become negligible within the interval T if $-\mathbb{R}\{s_i\} \gg 1/T$, $i = 1, 2, \ldots, n$. Hence, the recovered message m_r may be given as

$$m_r(t) = \frac{b_0}{a_0} e_m(t) \quad , \tag{8.19}$$

provided that $a_0 \neq 0$. We note that the roots s_i of $d(s)$ may be selected arbitrarily when A, C is an observable pair, see [71], [75], hence arbitrary fast decay of transient is possible. Furthermore, since m is discrete, we may even reconstruct the signal by comparing m_r with a threshold. Such a reconstructed signal m_c may be given as

$$m_c(t) = \begin{cases} 1 & \text{if} \quad m_r(t) > 0.5 \\ 0 & \text{if} \quad m_r(t) < 0.5 \end{cases} \quad . \tag{8.20}$$

Obviously, this idea could be applied to any discrete message, i.e. when $m(t) \in \{a_1, \ldots, a_l\}$.

Case 3 : In (8.13), instead of using the message directly, we may use a signal related to the message. Let $r(t)$ denote the message to be sent and let us choose the signal m used in (8.13) as follows :

$$M(s) = \frac{k}{G_p(s)} R(s) \quad , \tag{8.21}$$

where k is a scaling constant, $G_p(s)$ is a new transfer function, M and R denote the Laplace transform of m and r, respectively. Let $G_p(s)$ be given as $G_p(s) = n_p(s)/d_p(s)$, where $n_p(s)$ and $d_p(s)$ are polynomials in s, see (8.18). Obviously, the polynomial $n_p(s)$ given by (8.18) should be stable to have bounded $m(t)$. Let, for any polynomial $q(s)$, $\deg(q)$ denote the degree of $q(s)$. In general, $\deg(n_p) \leq \deg(d_p)$, see (8.18), hence the derivatives of r should be available to generate m. Under these conditions, from (8.16) and (8.21) it follows that

$$R(s) = \frac{G_p(s)}{kG(s)} E_m(s) \quad . \tag{8.22}$$

Note that $e_m(t)$ is available, hence to recover $r(t)$, the overall transfer function in (8.22) should be realizable. For this, *i :* $n(s)$ in (8.18) should be a stable polynomial, and *ii :* $\deg(d) + \deg(n_p) \leq \deg(n) + \deg(d_p)$ should hold. This is always satsified if, e.g., we have $n_p(s) = 1$, and $\deg(d_p) = \deg(d)$. Under these conditions, the transfer overall function in (8.22) is realizable, hence we can recover $r(t)$. A special case is to choose $G_p(s) = G(s)$. In this case (8.22) becomes $R(s) = E_m/k$, hence we have $r(t) \to e_m(t)/k$ as $t \to \infty$. However, note that in this case to generate $m(t)$ in the driver, we need to know the gain K of the receiver, cf. (8.16), (8.21).

Simulation Results

As an example of the ideas presented above, we again consider the switching system given in (8.1)-(8.4) in Section 8.1.1. the gain of the observer, as in Section 8.1.1, is chosen as $K = [4\ 1\ 0]^T$. For the message to be transmitted, we chose a periodic pulse train with amplitude 1 and period $T = 10$ sec., see Figure 8.6 In (8.13) we choose $B = [0\ 0\ 1]^T$. With this choice, and noting that $C = [1\ 0\ 0]$, $G(s)$ given by (8.16) can be found as:

$$G(s) = \frac{\alpha}{(s+4)(s^2+s+\beta)}$$

Hence, as explained above in case 2, if T is sufficiently large, $e_m(t) \longrightarrow \frac{\alpha}{4\beta}m(t)$, hence m(t) can be recovered as $m(t) = \frac{4\beta}{\alpha}e_m(t)$. Moreover, message can be reconstructed by using (8.20). We simulated the system given (8.13) and (8.14) with the parameters as

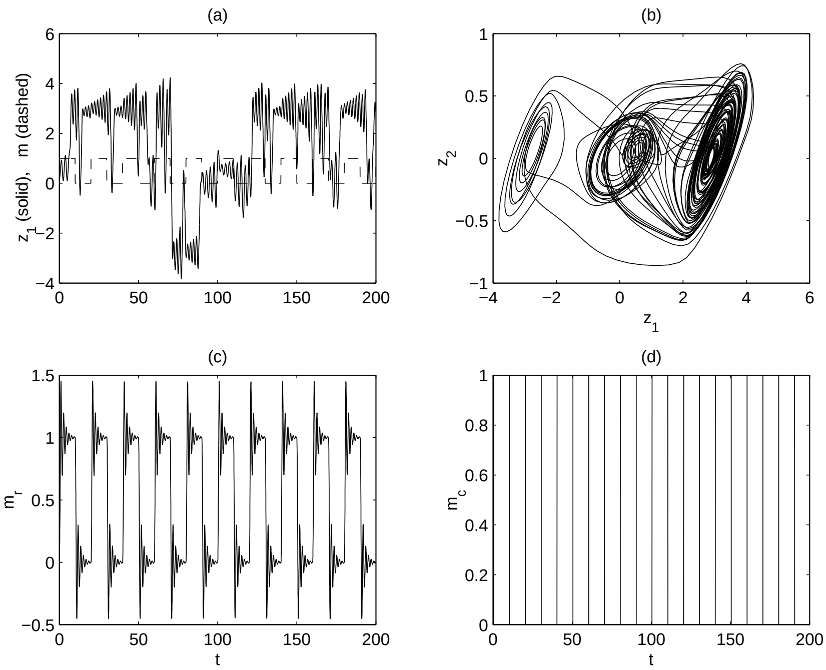


Figure 8.6: (a) The transmitted signal $z_1(t)$ (solid) and the message $m(t)$, (b) z_1 vs z_2 , (c) the recovered message, and (d) the reconstructed message.

given above. Results of that simulation is shown in Figure 8.6. The transmitted signal $z_1(t)$ (solid) and the message $m(t)$ are shown in Figure 8.6 (a), z_1 vs z_2 graph is given in Figure 8.6 (b), the recovered and reconstructed messages are shown in Figure 8.6 (c) and (d) respectively. As can be seen the messages can be recovered at the observer site without any error.

8.3 Stabilization

8.3.1 Using PPM

In this section we will investigate the stabilization problem of a certain class of systems using phase portrait matching. We will try to minimize either the phase difference or the norm of the error, see Chapter 5.

Let us reconsider the system given by the following differential equation:

$$\dot{\mathbf{x}} = f(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$$

In Chapter 5 we have investigated in detail the problem of finding a suitable $\mathbf{u}$ such that the closed loop dynamics is close to a desired phase portrait, $\mathbf{d}(\mathbf{x})$. It is reasonable to choose $\mathbf{d}(\mathbf{x})$ to stabilize the system under consideration. A typical phase portrait would be $\mathbf{d}(\mathbf{x}) = -P\mathbf{x}$ where P is a positive definite matrix. To elaborate further let us consider the change in the norm square of the state vector with the assumption that $f(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u} = \mathbf{d}(\mathbf{x})$ has a solution.

$$\begin{aligned}
\tfrac{1}{2}\tfrac{d\|\mathbf{x}\|^2}{dt} = \mathbf{x}^T\dot{\mathbf{x}} &= \mathbf{x}^T\mathbf{f}(\mathbf{x}) + \mathbf{x}^T\mathbf{g}(\mathbf{x})\mathbf{u}(\mathbf{x}) \\
&= \mathbf{x}^T\mathbf{f}(\mathbf{x}) + \mathbf{x}^T G(\mathbf{x})\mathbf{h}(\mathbf{x}) \\
&= \mathbf{x}^T G(\mathbf{x})\mathbf{f}(\mathbf{x}) + \mathbf{x}^T G^{\perp}(\mathbf{x})\mathbf{f}(\mathbf{x}) + \mathbf{x}^T G(\mathbf{x})\mathbf{h}(\mathbf{x}) \qquad (8.23)\\
&= \mathbf{x}^T G^{\perp}(\mathbf{x})\mathbf{f}(\mathbf{x}) + \mathbf{x}^T G(\mathbf{x})[\mathbf{h}(\mathbf{x}) + \mathbf{f}(\mathbf{x})] \\
&= \mathbf{x}^T G^{\perp}(\mathbf{x})\mathbf{f}(\mathbf{x}) + \mathbf{x}^T G(\mathbf{x})\mathbf{d}(\mathbf{x})
\end{aligned}$$

Let us consider setting $\mathbf{d}(\mathbf{x}) = -\gamma(\mathbf{x})G(\mathbf{x})\mathbf{x}$ where $\gamma(\mathbf{x}) : \mathbb{R}^n \longrightarrow \mathbb{R}^+$. The system is stable if the right hand side in (8.23) is negative for all values of $\mathbf{x}$. If we insert $\mathbf{d}(\mathbf{x})$ into (8.23) we obtain the following:

$$\tfrac{d}{dt}(\tfrac{1}{2}\|\mathbf{x}\|^2) = \mathbf{x}^T G^{\perp}(\mathbf{x})\mathbf{f}(\mathbf{x}) - \gamma(\mathbf{x})\mathbf{x}^T G(\mathbf{x})\mathbf{x} < 0 \qquad (8.24)$$

Hence we may chose $\gamma(\mathbf{x})$ as:

$$\gamma(\mathbf{x}) > \tfrac{\mathbf{x}^T G^{\perp}(\mathbf{x})\mathbf{f}(\mathbf{x})}{\mathbf{x}^T G(\mathbf{x})\mathbf{x}} \qquad (8.25)$$

Let us consider the phase portrait matching using minimization of the phase difference or the stabilization problem. Although this topic needs further investigation, the following discussions provide some remedy to stabilize a given system under certain conditions. The change in the norm square of the state would be:

$$\frac{1}{2}\frac{d\|\mathbf{x}\|^2}{dt} = \mathbf{x}^T\dot{\mathbf{x}} \;=\; \mathbf{x}^T[\beta\mathbf{d}(\mathbf{x})] \tag{8.26}$$

where β can be given as in (5.18). Let us investigate the case where $\mathbf{d}(\mathbf{x})$ is set to $\mathbf{d}(\mathbf{x}) = -G(\mathbf{x})\mathbf{x}$.

$$\begin{aligned} \frac{d\frac{1}{2}\|\mathbf{x}\|^2}{dt} &= -\beta\mathbf{x}^T G(\mathbf{x})\mathbf{x} \\ &= -[\frac{\mathbf{z}(\mathbf{x})^T(I-G(\mathbf{x}))\mathbf{f}(\mathbf{x})}{\mathbf{z}(\mathbf{x})^T(I-G(\mathbf{x}))\mathbf{d}(\mathbf{x})}]\mathbf{x}^T G(\mathbf{x})\mathbf{x} \end{aligned} \tag{8.27}$$

Note that in Section 5.1 we have defined the case of $\beta < 0$ as the problem of type 1. If $\beta > 0$, then (8.27) shows that the system with the choice of $\mathbf{d}(\mathbf{x})$ is stable. If we have problem of type 1 then we have proposed to saturate the control signal. Since our problem is stabilization, let us change our proposition with the application of the control signal which is calculated by minimization of the norm method. That is we suggest a switching controller in minimization of the phase difference. With appropriate choice of $\gamma(\mathbf{x})$, (8.24) can be satisfied in such cases. Hence the switching control scheme guaranties the stability of the closed loop system.

Simulations

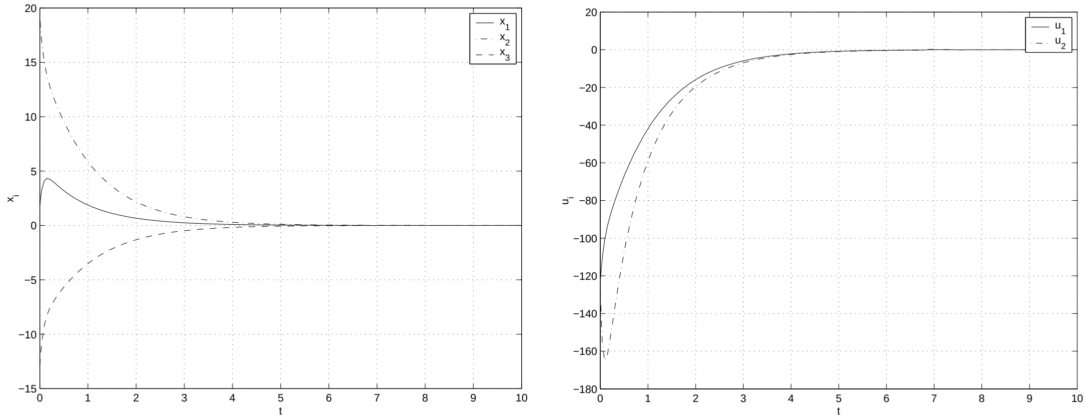
Let us consider the Lorenz chaotic system given with the following dynamics:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} 10(x_2 - x_1) \\ 28x_1 - x_2 - x_1x_3 \\ -\frac{8}{3}x_3 + x_1x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

We will first apply the norm minimization method given above with the following strategy. If $\mathbf{x}^T G^{\perp}(\mathbf{x})\mathbf{f}(\mathbf{x}) < 0$, since the free system is already stable we set $\gamma(\mathbf{x}) = 1$. If $\mathbf{x}^T G^{\perp}(\mathbf{x})\mathbf{f}(\mathbf{x}) > 0$ then we set

$$\gamma(\mathbf{x}) \;=\; 10\frac{\mathbf{x}^T G^{\perp}(\mathbf{x})\mathbf{f}(\mathbf{x})}{\mathbf{x}^T G(\mathbf{x})\mathbf{x}} \tag{8.28}$$

The bounds on the control signals is kept high as $-1000 < u_i < 1000$, $i = 1, 2$. The initial condition was chosen as $x_0 = [1.7413\ 19.2599\ -12.8563]^T$. The simulation results are shown in Figure 8.7. In Figure 8.7(a) the states of the system is given. In Figure

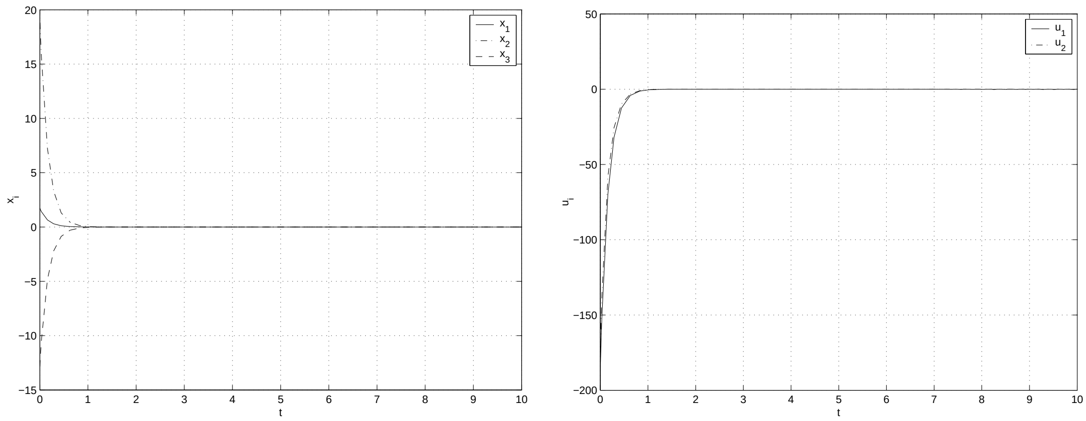


(a) (b)

Figure 8.7: (a) The stabilization of Lorenz chaotic system using norm minimization and (b) the corresponding control signals.

8.7(b) the control signals are shown.

Next we will consider the phase minimization approach with the same parameters given above. The simulation results is given in Figure 8.8.



(a) (b)

Figure 8.8: (a) The stabilization of Lorenz chaotic system using phase difference minimization and (b) the corresponding control signals.

In Figure 8.8(a) the states of the system is given. In Figure 8.8(b) the control signals are shown. If compared with the previous simulations the faster settling time is the first thing that takes attention. Although further investigation is needed to arrive a more complete conclusion we relate this fact to the high gain applied in phase minimization periods, mainly due to the value of β.

8.3.2 Using Switching

We will first consider the stabilization of the so-called chained system given below by using periodic switching.

$$\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_1 x_3 \\
\vdots &= \vdots \\
\dot{x}_{n-1} &= u_1 x_n \\
\dot{x}_n &= u_2
\end{aligned} \tag{8.29}$$

Next we will apply the switching methods introduced in Chapter 7 to nonholonomic integrator system, Lorenz chaotic system, and the unicycle.

Periodic Switching

This switching strategy, which is given in Section 7.3, is composed of multiple phases. Let n be the system order and m be the number of control signals. Then there might be at most $C(n, m)$ number of phases to choose from where $C(n, m)$ is the m tuple combinations of n element. In each phase we seek to find the control signal which forces to decrease m state variables while those state variables, not considered in the current phase, should be bounded.

We will consider a chained system with $n = 3$ and let there be two phases of the periodic switching. The dynamics of the system then can be written as follows:

$$\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_1 x_3 \\
\dot{x}_3 &= u_2
\end{aligned} \tag{8.30}$$

In phase 1 let us consider the states x_2 and x_3 with the control signal:

$$\mathbf{u}(\mathbf{x}) = \begin{bmatrix} -\lambda_2 \frac{x_2}{x_3} \\ -\lambda_3 x_3 \end{bmatrix} \tag{8.31}$$

where we assume that $x_3 \neq 0$, and λ_2, λ_3 are positive constants. The closed loop dynamics of (8.30) will be;

$$\begin{aligned}
\dot{\mathbf{x}}_1 &= -\lambda_2 \tfrac{x_2}{x_3} \\
\dot{\mathbf{x}}_2 &= -\lambda_2 x_2 \\
\dot{\mathbf{x}}_3 &= -\lambda_3 x_3
\end{aligned} \tag{8.32}$$

for a period of T_1 seconds. Let x_1 and x_3 be the chosen states in the second phase with the control signal:

$$\mathbf{u}(\mathbf{x}) = \begin{bmatrix} -\lambda_1 x_1 \\ -\lambda_3 x_3 \end{bmatrix} \tag{8.33}$$

The closed loop dynamics of (8.30) will be;

$$\begin{aligned}
\dot{\mathbf{x}}_1 &= -\lambda_1 x_1 \\
\dot{\mathbf{x}}_2 &= -\lambda_1 x_1 x_3 \\
\dot{\mathbf{x}}_3 &= -\lambda_3 x_3
\end{aligned} \tag{8.34}$$

for a period of T_2 seconds. In phase 1 and phase 2 we have not considered the state x_1, x_2, respectively. These state variables should be bounded for the stabilization of the closed loop system. Let $T = T_1 + T_2$ be the period of the switching. Assume that the states are $\mathbf{x}(kT)$ at time $t = kT$. The solution of Equation (8.32) at time $t = kT + T_1$ (i.e. phase 1) would be:

$$\begin{aligned}
\mathbf{x}_1(kT+T_1) &= x_1(kT) - \tfrac{\lambda_2}{\lambda_2-\lambda_3}(1 - e^{-(\lambda_2-\lambda_3)T_1})\tfrac{x_2(kT)}{x_3(kT)} \\
\mathbf{x}_2(kT+T_1) &= e^{-\lambda_2 T_1} x_2(kT) \\
\mathbf{x}_3(kT+T_1) &= e^{-\lambda_3 T_1} x_3(kT)
\end{aligned} \tag{8.35}$$

The states x_2 and x_3 are stabilized with (8.31). The value of the state x_1 depends on the evolution of the ratio $\frac{x_2(kT)}{x_3(kT)}$. Although the states x_2 and x_3 are converging to zero their ratio might not, and therefore may cause x_1 to diverge. The solution of Equation (8.34) at time $t = kT + T_1 + T2 = (k+1)T$ (i.e. phase 2) would be:

$$\begin{aligned}
\mathbf{x}_1((k+1)T) &= e^{-\lambda_1 T_2} x_1(kT) - \tfrac{\lambda_2}{\lambda_2-\lambda_3}(1 - e^{-(\lambda_2-\lambda_3)T_1})e^{-\lambda_1 T_2}\tfrac{x_2(kT)}{x_3(kT)} \\
\mathbf{x}_2((k+1)T) &= e^{-\lambda_2 T_1} x_2(kT) \\
&\quad -\tfrac{\lambda_1}{\lambda_1-\lambda_3}(x_1(kT) - \tfrac{\lambda_2}{\lambda_2-\lambda_3}(1 - e^{-(\lambda_2-\lambda_3)T_1})\tfrac{x_2(kT)}{x_3(kT)}) \\
&\quad \times(1 - e^{-(\lambda_1-\lambda_3)T_2})e^{-\lambda_3 T_1} x_3(kT) \\
\mathbf{x}_3((k+1)T) &= e^{-\lambda_3(T_1+T_2)} x_3(kT)
\end{aligned} \tag{8.36}$$

In (8.36) the convergence of x_3 is obvious. The convergence of x_1 and x_2 depends again on the ratio $\frac{x_2(kT)}{x_3(kT)}$. Simulation with the control signal given by (8.31) and (8.33) shows this fact that the convergence of x_1 heavily depends on the convergence of the ratio $\frac{x_2(kT)}{x_3(kT)}$.

A remedy for x_1 to be bounded might be to increase x_3 in the second phase not more then the decrease in the first phase which in return cause the ratio $\frac{x_2(kT)}{x_3(kT)}$ to converge to zero as well. The control signal in phase 2 will then be:

$$\mathbf{u}(\mathbf{x}) = \begin{bmatrix} -\lambda_2 \frac{x_2}{x_3} \\ \lambda_4 x_3 \end{bmatrix} \tag{8.37}$$

where λ_4 is a positive constant and we assume that $x_3 \neq 0$. Let us define a new variable $z(kT) = \frac{x_2(kT)}{x_3(kT)}$. Using (8.35) for phase 1, (8.30) and (8.37) for phase 2 we can write a matrix equation as follows:

$$\begin{bmatrix} x_1((k+1)T) \\ z((k+1)T) \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} \begin{bmatrix} x_1(kT) \\ z(kT) \end{bmatrix} \tag{8.38}$$

where a_{ij}, $i, j = 1, 2$ are constants which are functions of the constants $\lambda_i, i = 1, 2, 3, 4$ and $T_j, j = 1, 2$. Considering Equation (8.38), (8.35) and solution for the closed loop system given by (8.30) and (8.37) the following conditions should be satisfied for convergence of the chained system:

1. A should have eigenvalues in unit disc, so that $x_1(kT) \longrightarrow 0$ and $z(kT) \longrightarrow 0$
2. $\lambda_4 T_2 - \lambda_3 T_1 < 0$, so that $x_3(kT) \longrightarrow 0$
3. $\lambda_2 > \lambda_3$, so that $x_2(kT) \longrightarrow 0$

With these conditions convergence is not possible for any choice of parameters. But it is possible for a quite wide range of parameters, e.g. $\lambda_1 = 4, \lambda_2 = 2, \lambda_3 = 1$, and $\lambda_4 = 0.5$ with $T_1 = T_2 = 2$ seconds. In Figure 8.9 the stabilized system states are given. The eigenvalues of A for this choice of λ_i and T_j are $0.4\ 10^{-6}$ and 0.7276. For different systems the dependence of the entries of A to the parameters will vary. For example for chained system increasing the parameter λ_4 decreases the eigenvalues of the matrix

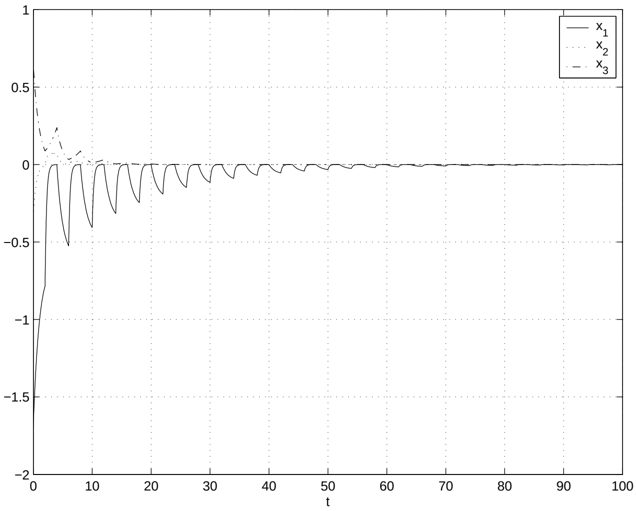


Figure 8.9: States of the chained system stabilized using periodic switching.

A. So if we change λ_4 to $\lambda_4 = 0.3$, where the other parameters being the same, the eigenvalues of the matrix A become $0.4\ 10^{-6}$ and 1.0271. Hence even in this scheme, instability is possible.

The drawback of this method is that the states not considered in a certain phase should be bounded. Therefore an analysis which guarantees the boundedness of these states should be performed. We have given such an analysis for the system given by (8.30).

Switching Strategy Emulating Common Lyapunov Function

This switching strategy, composed of multiple phases, is given in Section 7.4. This switching strategy depends on the use of differential equations of the variables under consideration, e.g. $\|\mathbf{x}\|^2$, $|x_i|^2$, or $z(\mathbf{x})$, see e.g. (7.31), (7.32), (7.37). In each phase the differential Equation (7.31) related with $\|\mathbf{x}\|^2$ is used, see the derivation of (7.35). The other equations can be obtained from the differential equations of $|x_i|^2$ or a dummy variable $z(\mathbf{x})$ depending on the problem under consideration, see e.g. (7.32), (7.37).

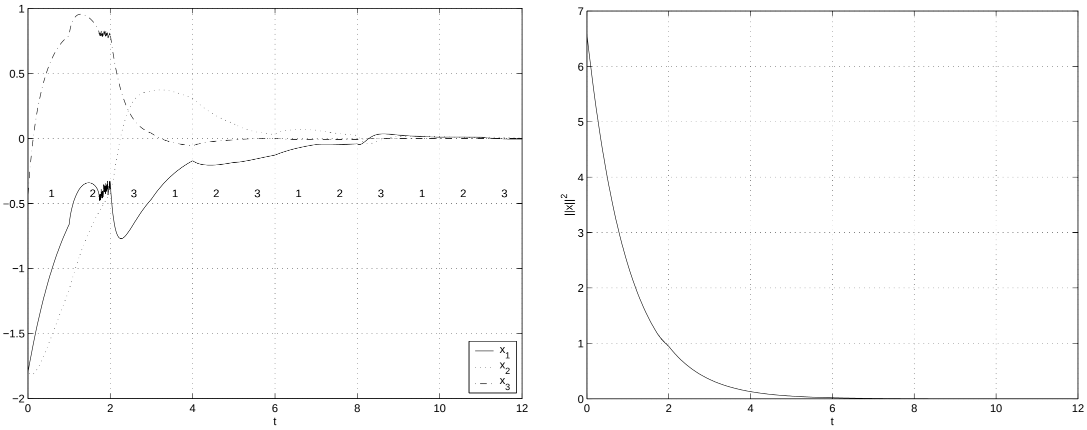
In this section we will consider different aspects of this switching scheme on different systems. The first system we will consider is the nonholonomic integrator with the following dynamics, which was given previously as (4.9):

$$\begin{aligned}\dot{x_1} &= u_1,\\ \dot{x_2} &= u_2,\\ \dot{x_3} &= x_1u_2 - x_2u_1\end{aligned}$$

As was mentioned before, if the values of x_1 and x_2 are zero then no control signal would be able to effect the state x_3. If x_1 and x_2 have small values then a rather big control signal is required to change the state x_3. Hence, if the problem is stabilization then the control scheme must first guaranty that the convergence of x_3 to the origin is faster than that of x_1 and x_2. The convergence of a state is determined by λ_{i_j} in (7.32) and the chosen λ in (7.31). Hence for nonholonomic integrator if we choose $\lambda_3 > \lambda_1$ and $\lambda_3 > \lambda_2$ then the state x_3 would be expected to converge faster then x_1 and x_2. It might be possible that the initial state is given so that x_3 is rather big compared to the other states. To overcome this difficulty we may choose λ_3 a rather large positive value.

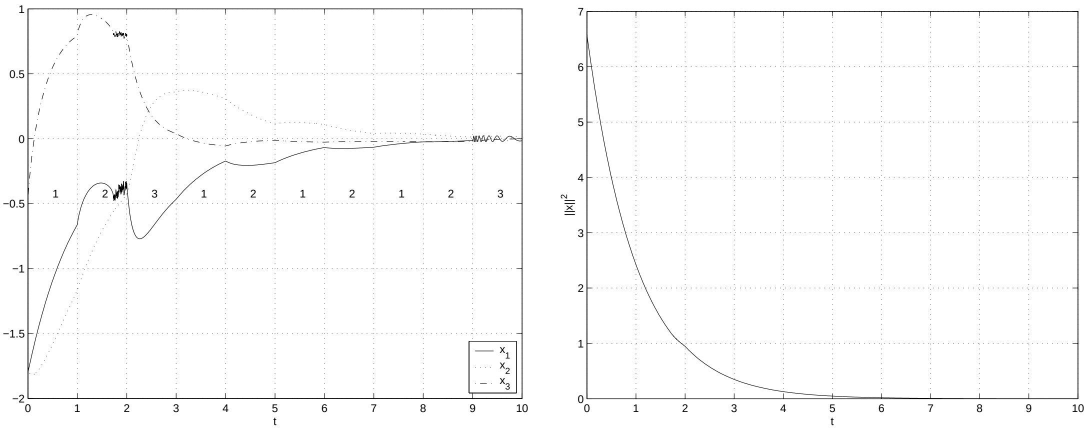
The first simulation is performed by using the switching scheme that stabilizes the states sequentially using Equation (7.35). In Figure 8.10 the states of the system and the norm of the state vector with respect to time are given. The numbers $1, 2, 3$ on the figure indicates the switching phase sequences, e.g. a number such as 3 indicates that during that interval the phase 3 is active and x_3 is forced to decrease. In this simulation the initial condition was chosen as, $\mathbf{x}_0 = [-1.800058 - 1.768276 - 0.467815]^T$. The switching periods of each phase last $T = 1$ second. The constants in Equation (7.32) is chosen to be; $\lambda_1 = 2, \lambda_2 = 2, \lambda_3 = 6$, and $\lambda = 1$;. In Figure 8.10 (a) the three states are given. The numbers in each time slot is the state index i of the state x_i being stabilized. The stabilization of the states start with x_1 and cycles as shown on the figure. In Figure 8.10 (b) the norm of the overall state vector is shown. As it is seen it converges exponentially to the origin as the solution to the Equation (7.32).

In the second simulation we consider the same system, with different order in the sequence of the state stabilization which has been proposed in Section 7.4.1. Namely, in each phase we choose to stabilize the state with the highest norm. The simulation given in Figure 8.11 has the same parameters as before. In general, it is not possible to say the superiority of one scheme to the other. But it can be argued that the latter strategy



(a) (b)

Figure 8.10: Stabilization of nonholonomic integrator using sequentially stabilizing the states. (a) States vs. time (b) $\|\mathbf{x}\|$ vs. time.



(a) (b)

Figure 8.11: Stabilization of nonholonomic integrator starting from the state with the highest absolute value. (a) States vs. time (b) $\|\mathbf{x}\|$ vs. time.

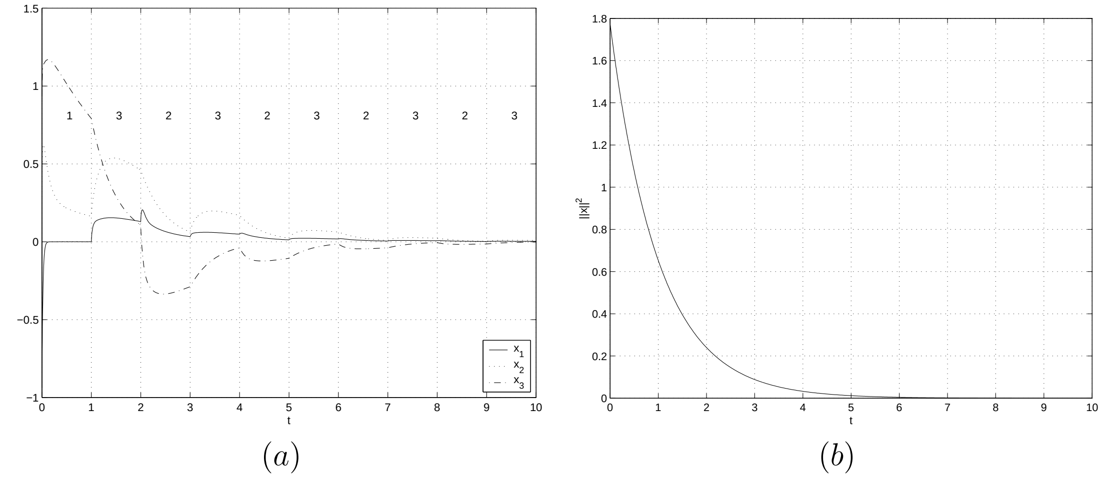


Figure 8.12: Stabilization of Lorenz Chaotic system, starting from the state with the highest absolute value($\lambda_1 = 100$). (a) States vs. time (b) $\|\mathbf{x}\|$ vs. time.

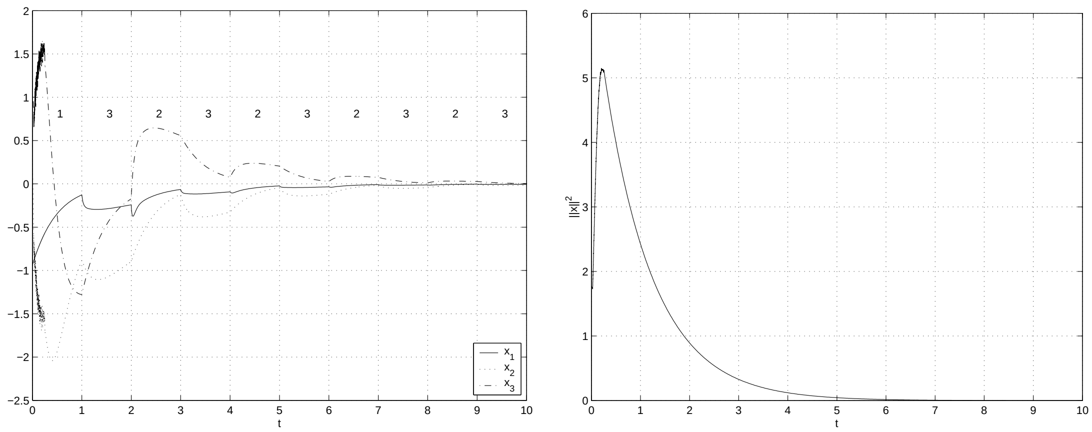
of switching, which tries to stabilize the state with maximum deviation, is somewhat using more feedback about the state of the system. If the differential equations of the closed loop system in each phase can be calculated explicitly then one may also design a stabilization sequence of the states as well.

Next we consider the Lorenz system given below, which is known to exhibit chaotic behavior for $u_1 = u_2 = 0$.

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} 10(x_2 - x_1) \\ 28x_1 - x_2 - x_1x_3 \\ -\frac{8}{3}x_3 + x_1x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

In the simulation given in Figure 8.12 the initial condition was chosen as $\mathbf{x}_0 = [-0.937640\ 0.288568\ 0.901906]^T$. The duration of each phase is chosen as $T = 1$ seconds. The constants of Equation (7.32) are chosen to be $\lambda_1 = 100, \lambda_2 = 4, \lambda_3 = 4$ and that of (7.31) is chosen as $\lambda_4 = 1$. In this simulation the state with maximum deviation is stabilized in each phase.

Note that although we expect that $\|\mathbf{x}\|$ should decrease monotonically to zero, in some simulations we observed that $\|\mathbf{x}\|$ may increase initially, then decreases monotonically to zero. Such a simulation result is given in Figure 8.13, where we used $\lambda_1 = \lambda_2 = \lambda_3 = 4$, and $\lambda_4 = 1$. One possible reason for this might be the numerical error in calculating $[\bar{G}(\mathbf{x})]^{-1}$ in (7.35).



(a) (b)

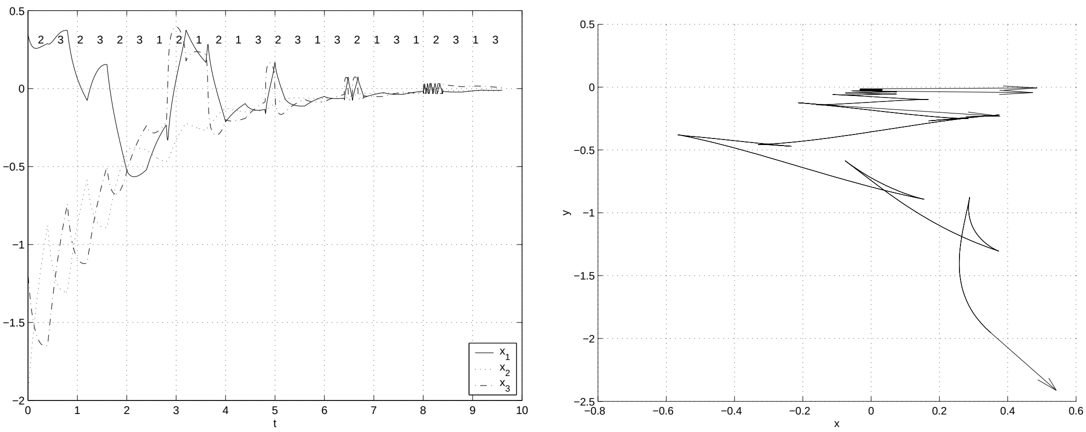
Figure 8.13: Stabilization of Lorenz Chaotic system, starting from the state with the highest absolute value ($\lambda_1 = 4$). (a) States vs. time (b) $\|\mathbf{x}\|$ vs. time.

The next simulation is performed on the unicycle system given with the following differential equations [89]:

$$\begin{aligned} \dot{x_1} &= u_1 cos(x_3), \\ \dot{x_2} &= u_1 sin(x_3), \\ \dot{x_3} &= u_2 \end{aligned}$$

where x_1 and x_2 are the x and y coordinate of a car which does not drift, x_3 is the angle in radians between the x axis and the front direction vector of the car, u_1 is the speed of and u_2 is the rate of change of the angle of the car with respect to x axis. Stabilization of a unicycle is equivalent to parking a car to the origin, so that the front direction vector is aligned with the x axis. In the simulation given in Figure 8.14, the initial state is $\mathbf{x}_0 = [0.349170 \ -1.951609 \ -1.174541]^T$. Each phase has a duration of $T = 0.4$ seconds. The related constants are chosen to be $\lambda_1 = 4, \lambda_2 = 4, \lambda_3 = 4$, and $\lambda = 1$ in (7.32) and (7.31), respectively.

In Figure 8.14(a) the evolution of the states are given. As it is seen, although they all converge to the origin it does not make too much sense about the motion of the car. In Figure 8.14(b) the movement of the car is plotted on the $x-y$ plane where the arrows show the front direction of the car. As it is seen the motion of the car is full of back and forth movements. After a time, the state x_3 became too small compared to the other states and the switching strategy tries to stabilize those with big deviation. Although at the end, the car is parked at the desired location with a suitable orientation, the parking



(a) (b)

Figure 8.14: Stabilization of unicycle starting from the state with the highest absolute value: (a) States vs. time (b) The trajectory of the unicycle.

strategy is not so effective. From this example we can restate the fact that, knowing the system and possible solution candidates allow us to come up with more reliable and applicable solutions.

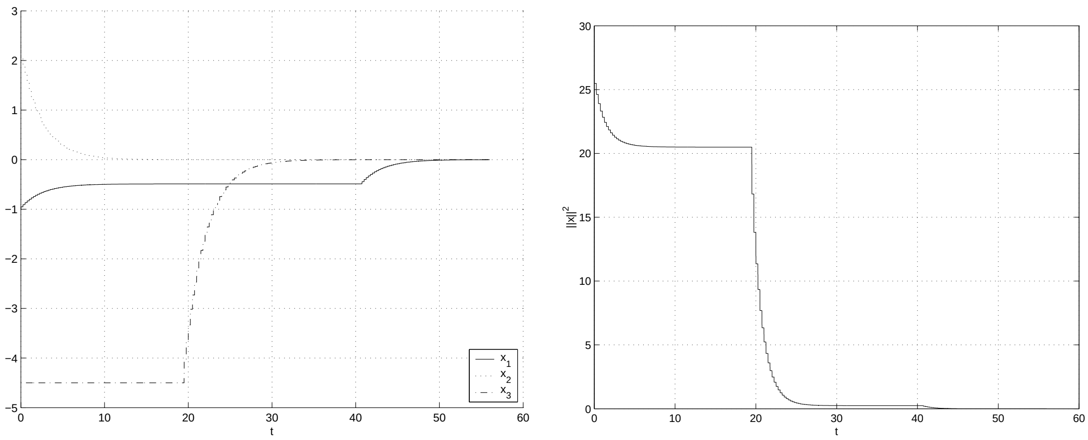
Switching Strategy for a Class of Driftless System

The main difference of this strategy to the previously introduced ones is that it stops after a finite number of phases. We will consider the chained system defined by the differential equation given as (8.30):

$$\begin{aligned}\dot{x}_1 &= u_1\\ \dot{x}_2 &= u_1 x_3\\ \dot{x}_3 &= u_2\end{aligned}$$

The states x_1 and x_3 are distinctly stabilizable states while x_2 is state dependent stabilizable state (see the definition in Section 7.5). The system will be stabilized in three phases. By a stabilized state we mean that its absolute value is less then a given positive number, $\epsilon > 0$.

The switching strategy applies as follows; stabilize first x_2 using the control signal u_1 while setting $u_2 = 0$. Since $u_2 = 0$ the state x_3 is constant throughout the first phase.



(a) (b)

Figure 8.15: Stabilization of chained systems using the properties of the states being distinctly stabilizable. (a) States vs. time (b) $\|\mathbf{x}\|$ vs. time.

A candidate for the control signal would be:

$$\mathbf{u}(\mathbf{x}) = \begin{bmatrix} -\lambda_2 \frac{x_2}{x_3} \\ 0 \end{bmatrix}$$

where we assume that $x_3(t) \neq 0$. This phase last till we have $|x_2| < \epsilon$. During this phase the state x_1 evolves freely. In the second phase we stabilize the state x_3, so that the coupling between the states x_1 and x_2 via u_1 is broken. We apply the following control signal till $|x_3| < \epsilon$.

$$\mathbf{u}(\mathbf{x}) = \begin{bmatrix} 0 \\ -\lambda_3 x_3 \end{bmatrix}$$

In the beginning of the third phase both $|x_2|$ and $|x_3|$ are less then ϵ. What remains is the stabilization of x_1. We set $u_2 = 0$ so that x_3 remains in its small value while we use the control signal u_1 to bring the state x_1 below the specified value. In the third phase the following control signal can be applied:

$$\mathbf{u}(\mathbf{x}) = \begin{bmatrix} -\lambda_1 x_1 \\ 0 \end{bmatrix}$$

8.15. During the third phase the value of x_2 will change but since the value of x_3 is made less then ϵ the change will be small. For better results ϵ can be chosen a smaller value. The simulation which has been performed with the constants $\lambda_i = 10$, $i = 1, 2, 3$ is shown in Figure 8.15.

8.3.3 Using Fuzzy Controller for Systems with Completeness Property

Consider the system, also used in [90], given by the following equations;

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_1^3 - x_2 \\ -x_1 + x_2^3 + u \end{bmatrix} \tag{8.39}$$

This system is linearized around the nine points, $(1.5, 1.5)$, $(0, 1.5)$, $(-1.5, 1.5)$, $(1.5, 0)$, $(0, 0)$, $(-1.5, 0)$, $(1.5, -1.5)$, $(0, -1.5)$, $(-1.5, -1.5)$ and we obtained the local linear models as $\dot{\mathbf{x}} = A_i\mathbf{x} + B_i\mathbf{u}$ in each linearization point, where A_i is the jacobian of the vector field with respect to $\mathbf{x}$ and $B_i = [0\ 1]^T$. The control gains K_i, are obtained by pole placement method as follows;

$$K_1 = [\ -7.0625\ \ 18.00\ ],\quad K_2 = [\ -83.00\ \ 24.75\ ],\quad K_3 = [\ -7.0625\ \ 18.00\ ],$$
$$K_4 = [\ -7.0625\ \ 11.25\ ],\quad K_5 = [\ -83.00\ \ 18.00\ ],\quad K_6 = [\ -7.0625\ \ 11.25\ ],$$
$$K_7 = [\ -7.0625\ \ 18.00\ ],\quad K_8 = [\ -83.00\ \ 24.75\ ],\quad K_9 = [\ -7.0625\ \ 18.00\ ].$$

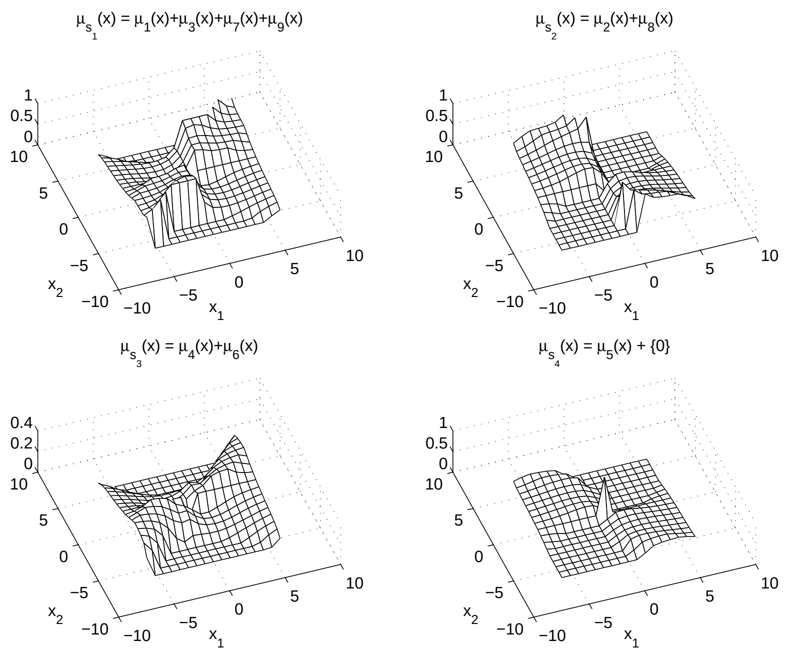
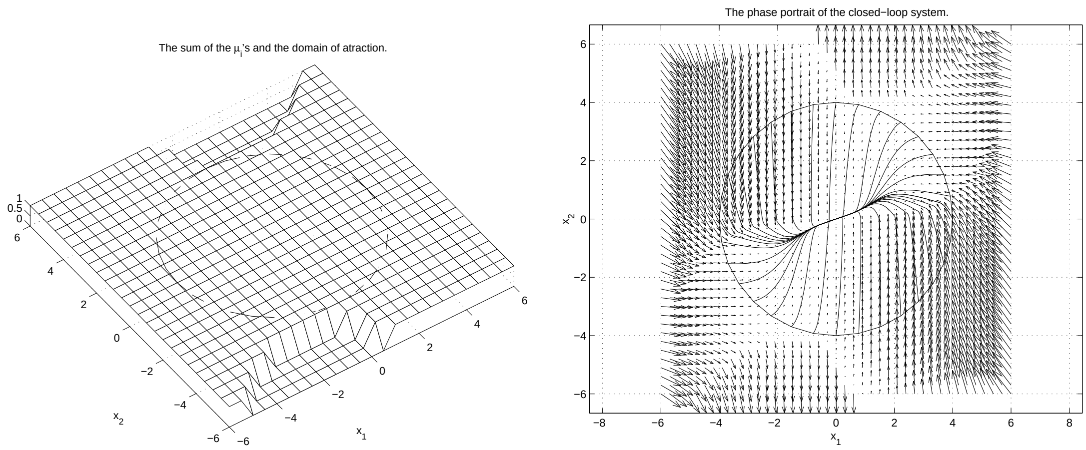


Figure 8.16: Membership functions of the Fuzzy Logic Controller.

The eigenvalues of the matrices, $(A_i - B_iK_i)$, are assigned at $-9 \mp j$ on the open complex LHP. The i^{th} control signal is calculated as $u_i(\mathbf{x}) = -K_i\mathbf{x}$. Due to the symmetry of the nonlinearity, some of the gains are the same. This



(a) (b)

Figure 8.17: (a) Sum of the membership functions and the domain of attraction, (b) The phase portrait of the closed-loop system and the simulations.

results in rules with different antecedent parts but with the same consequent part. These rules can be combined together by taking the union of the membership functions in the antecedent part and leave the consequent part unchanged. Note that in this example $K_1 = K_3 = K_7 = K_9 = [-7.0625\ 18.00]$, $K_2 = K_3 = [-83.00\ 24.75]$, $K_4 = K_6 = [-7.0625\ 11.25]$, $K_5 = [-83.00\ 18.00]$. By combining these as given in Section 6.2, we obtain 4 such rules. In the calculation of $Z_i(\mathbf{x})$ we also need a common positive definite matrix P. In this example P is taken as the identity matrix of appropriate dimension, which makes the set boundary an n dimensional circle instead of ellipsoidal. Figure 8.16 shows the appropriately unified membership functions which are determined using Equation (6.7), (6.9).

Because of the definition of locally strictly complete vector functions and the procedure devised to calculate the membership functions, none of the regions S_i includes the origin. For the computation of S_i see Remark 3 in Section 6.2.3. In this simulation the origin is added to $\mu_{S_4}(\mathbf{x})$. In Figure 8.17(a) the sum of the membership functions is shown. As it is seen at the upper right and lower left corner there are regions where all of the membership functions are zero. This means that the set of vector functions $Z_i(\mathbf{x})$ is locally complete with $\mathbb{S} = \{\|x\| < 4\} \subset \mathbb{R}^2$. The phase portrait of the closed-loop system is shown in Figure 8.17(b). Obviously, the trajectories inside the domain of attraction converge to the origin.

Consider the system, given by the following equations;

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ -x_1/(1+x_2^2) + u \end{bmatrix} \tag{8.40}$$

Discrete version of this system is used in [91]. This system is linearized around the nine points, $(0,0)$, $(rcos(\pi/4n), rsin(\pi/4n))$,where $n = 1, \cdots, 9$, and $r = 1.5$ and the local linear models in the form $\dot{\mathbf{x}} = A_i\mathbf{x} + B_iu$, are obtained. The control gains K_i's, after Step 5 are obtained by pole placement method as follows;

$$\begin{aligned} K_1 &= [\ -0.4706 \quad 4.5017\ ], \quad K_2 = [\ -0.3077 \quad 5.00\ ] \\ K_3 &= [\ -0.4706 \quad 5.4983\ ], \quad K_4 = [\ -1.0000 \quad 5.00\ ] \end{aligned}$$

The eigenvalues of the matrices, $(A_i - B_iK_i)$, are assigned at -1 and -5 on the open complex LHP. The i^{th} control signal is calculated as $u_i(\mathbf{x}) = -K_i\mathbf{x}$. In this example P is taken as the identity matrix of appropriate dimension, which makes the set boundary an n dimensional circle instead of ellipsoidal.

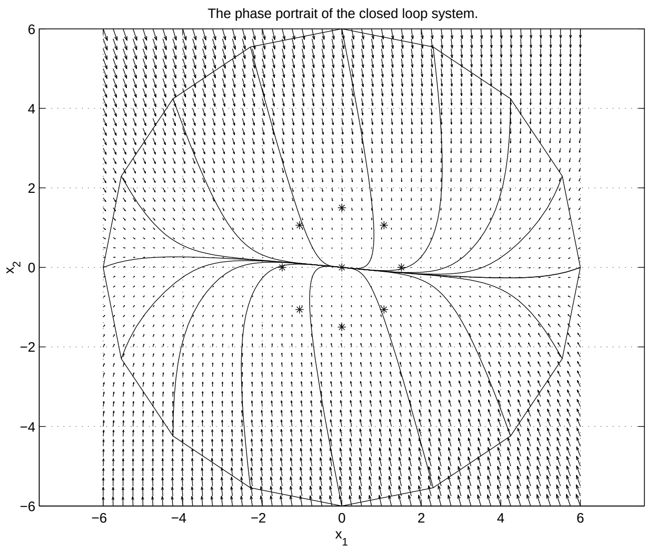


Figure 8.18: The phase portrait of the closed-loop system and the simulations.

The phase portrait of the closed-loop system is shown in Figure 8.18. The dots are the points of linearization. Obviously, the trajectories inside the domain of attraction converge to the origin.

Chapter 9

Conclusions

We considered the modelling and stabilization problems for the switching and fuzzy systems. In the following section we will summarize or results and main contributions. In Section 9.2 we will give the open problems that can be the extensions of the studies summarized in Section 9.1.

9.1 Results and Contributions

9.1.1 Modelling

Switching Systems

In Section 4 we investigate the differences and similarities between the models of switching and fuzzy systems. We show that the switching rules in switching systems are associated with the membership functions in fuzzy systems. We provided a method to define the membership functions of the associated fuzzy system for a given switching system. We demonstrated two approaches. In one of the approaches we utilized the functions which bound the switching regions. In the other approach we utilized a transformation which simplifies the analysis to obtain the required membership functions. We also showed that, if the membership function of the associated fuzzy system is degenerated then the output of the fuzzy system is exactly the same as the output of the switching system.

In Section 7.2 we modelled a chaotic system which has piecewise linear dynamics as a switching system. Utilizing some properties of this model, we proposed an observer based synchronization scheme and proved some convergence results. In Section 8.2, we proposed a chaotic masking and a chaotic modulation scheme by using the synchronization scheme we proposed in Section 7.2. We also presented some convergence results related with the proposed schemes. Both approaches are demonstrated with simulations in Section 8.2.1, and 8.2.2 respectively.

Fuzzy systems

In Section 6.3 we proposed a method for the approximation of a calculated function by using fuzzy rules. In this method the input/output data or the function to be approximated must be known. In this method, the control input is obtained by using state feedback with constant gain and the fuzzy control rules, have the forms as given in (6.16). This form contains some gain vectors, and we provided a method to obtain these gains as well as related membership functions which are used in the fuzzy approximation. This method can also be used to construct fuzzy controllers for stabilization if the input/output data is known or calculated by some means like phase portrait matching proposed in Chapter 5. In Section 8.1.2 we have demonstrated the effectiveness of the method with an example.

9.1.2 Phase Portrait Matching

In Chapter 5 we proposed a method to calculate the control signal of a given system so that the phase portrait of the closed loop system is as close as possible to the phase portrait of a given desired phase portrait. We proposed two methods to calculate the required control signal. In one of these methods, the aim is to minimize the phase between the vectors giving the closed-loop dynamics and the desired dynamics. In the second method the aim is to minimize the norm of the error between these vectors. We proposed two saturation methods in case the calculated control signal is out of some prescribed bounds. We used these methods in various stabilization approaches that we proposed.

9.1.3 Stabilization

Control Signal Calculated by Phase Portrait Matching

In Section 8.3.1 we proposed a method to use the control signal which is calculated by phase portrait matching to stabilize a class of given nonlinear systems. We also provided some examples to demonstrate the effectiveness of the method.

Switching Controllers

The stabilization methods that we proposed have a common characteristics. They are all composed of multiple phases. The control signals calculated in each phase guarantee the decrease of the norms of certain set of states.

In Section 7.3 we proposed a switching strategy to stabilize a class of nonlinear systems. The control signal is calculated so that in each phase a set of states is stabilized while the other states are kept bounded. This method is demonstrated in Section 8.3.2 with a simulation.

In Section 7.4 we proposed another approach to calculate the control signal for each phase. In this approach, the norm of the overall state vector as well as the norms of some selected states are decreased by an appropriate selection of the control input. We proposed two methods to determine these selected states. In the first method, these states are selected in a sequential way, while in the second method, the state which has the highest norm is selected in each phase. For both approaches several simulations on this method is provided in Section 8.3.2.

In Section 7.5 we proposed a method to stabilize a class of driftless systems. To apply this method the system must have certain strict properties. These strict properties on the other hand allow us to stabilize the system states one by one. One such system is the so-called chained system of degree 3. The stabilization of the chained system is given in Section 8.3.2.

Fuzzy Controllers

In Section 6.2 we proposed a fuzzy controller design method assuming that the system under consideration possesses a completeness property. In the proposed method the completeness property is obtained by the choice of the linearization points of the given nonlinear system. We provided a method to calculate the normal membership functions of the fuzzy controller. The devised calculation is so that the control action that has the most contribution to the stabilization is associated with a higher membership value. The method is demonstrated with simulations in Section 8.3.3

9.2 Research Directions

9.2.1 Obtaining an Equivalent Switching System for a Given Fuzzy System

In Chapter 4 we proposed a method to obtain an equivalent fuzzy system associated with a given switching system. One interesting problem is to find an equivalent switching system for a given fuzzy system. One possible approach in determining the boundary of the regions for switching system might be obtained by equating the related membership function to 0.5 since this is the degree where being a member or not is equal to each other. Once the switching model is obtained the projection of the stability results on the given fuzzy system is another related open problem.

9.2.2 Design of the Desired Phase Portrait

In Chapter 5 we proposed the method of phase portrait matching in calculating the control signal for a class of given nonlinear systems. The proposed method was devised so that the minimization of the phase difference or the norm indices are minimized. In

Section 8.3.1 we modified the proposed method for stabilization where the desired phase portrait was the vector function to be designed. Consider the problem of a mobile robot that has to reach a location where there are several obstacles on its path. If there were no obstacles then we can apply the method given in 8.3.1 directly. In case of the obstacles the design of the phase portrait must be revised. Consider a two dimensional Gaussian function which has its peak at the location of the obstacles. The gradient of this function can be adjusted so that the phase portrait vectors point away from the obstacle. In this way a superposition of the Gaussian functions might help us to build the desired phase portrait which will guide the mobile robot to find its way.

The phase portrait matching can also be related to switching system and stabilization as follows. In some cases, matching of a single desired phase portrait might not be possible due to singularities. In such cases, a set of desired portraits (or equivalently a set of desired dynamics) may be chosen, and each of such desired dynamics may be matched in different regions of the underlying state space. The resulting feedback system will be a switching system, and various stability results given and/or proposed in this thesis may be used to prove the stability of the overall system.

9.2.3 Fuzzy Controller Design

In Section 6.2 we proposed a method to design a fuzzy controller for systems having completeness property which is obtained by the choice of the linearization points. One open problem related to this approach is the location of the linearization points. The effect of choosing different form of consequent part for the rules of the fuzzy system is another open problem which will effect the performance of the system. The determination of the size of the domain of attraction is another research direction that needs attention. Our preliminary results indicate that the size of the domain of attraction of the closed-loop system might be increased with appropriate selection of such points and rules. But this point requires further investigation.

In Section 6.3 we proposed a method for the design of fuzzy systems in which constant gain vectors were utilized. It might result in simpler membership functions if we choose a nonlinear feedback rule which can be related with the dynamics of the system under consideration.

9.2.4 Switching Controller Design

The switching strategies proposed in Section 7.4 provide various ways to calculate the stabilizing control signal. One of the problems with these approaches is that although the norm of the overall state vector decreases it is not guarantied to converge to zero. The main reason for this situation is the dynamical structure of the system under consideration. One remedy to overcome this situation might be to let some of the states increase up to some bounds so that such problematic regions can be avoided. The determination of state sequences which should increase/decrease, and the determination of the durations of each phase are some of the open problems.

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