AN INVENTORY MODEL FOR RECYCLABLE GOODS WITH A DISPOSAL OPTION

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MASTER OF SCIENCE

By

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September 2002

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Master of Science.

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Abstract

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WITH A DISPOSAL OPTION

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In this study, we develop and analyze a control policy for a continuous review inventory system of recyclable goods with a disposal option. We assume that the return and demand flows are independent and the net demand pattern is governed by a Brownian motion process. Under the fixed procurement lead-time and backordering assumptions, we derive the analytical expressions of the cost rate function for the cases where the net demand rate is zero and positive. A numerical analysis is conducted to see the effects of the net demand rate over the policy parameters and to illustrate the advantages of using the model.

Keywords: Inventory Control, Reusable products, Recycling, Disposal, Brownian Motion, Cash flow management

Özet

GERİ DÖNÜSÜMLÜ MALLAR İÇİN ELDEN ÇIKARMA SEÇENEKLİ BİR ENVANTER POLİTİKASI

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Bu çalışmada, sürekli gözden geçirilen envanter sistemlerinde, geri dönüşümlü mallar için, elden çıkarma seçenekli bir kontrol politikası geliştirilmiş ve analiz edilmiştir. Geri dönüş ve talep akışlarının birbirinden bağımsız olduğu ve net talebin Brownian rassal sürecine göre hareket ettiği varsayılmıştır. Sabit tedarik süresi ve geri ısmarlama varsayımı altında, net talep oranının sıfır ve pozitif olduğu değerler için ortalama maliyet fonksiyonun analitik ifadesi bulunmuştur. Net talep oranının politika parametreleri üzerindeki etkisini ve modelin kullanımının yaralarını görebilmek için sayısal analiz verilmiştir.

Anahtar sözcükler: Envanter kontrolü, Geri kullanılabilir mallar, Geri dönüşüm, Elden çıkarma, Brownian rassal süreci, Nakit akışı yönetimi

To my mother and Gül . . .

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Chapter 1

Introduction and Literature Review

Reuse of products and materials becomes more and more important as a result of growing environmental and economical concerns due to increasing industrialization. In many countries governments set recycling quotas and take-back obligations for producers. Waste reduction techniques are used more intensively to prevent landfilling or incineration of reusable goods. Beside the governmental restrictions and environmental legislation, customer expectations for a 'green' image have become an important marketing strategy. Not only these environmental motivations but also the economical concerns play an important role in reuse activities. Companies want to retrieve the useful parts or materials integrated in used products. These separated parts and materials can be used in production of new products or sold in other markets.

The efforts for the reuse of products and materials present many complications that affect production systems. Unlike the conventional production environment, the reuse of materials requires the management of the flow of returned items. The field that deals with the issues on distribution planning, inventory management and production planning of systems with return flows is called reverse logistics.

Another new area called Product Recovery Management (PRM) has emerged to arrange reuse efforts in a more systematic manner. The objective of PRM is given by Thierry et al. [29] as "recovering the economical and ecological value as reasonable as possible, thereby reducing the ultimate quantities of waste". First step in achieving this objective is to classify the types of returned items. A review paper on reverse logistics by Fleischmann [4] defines three main categories for returned items as packages, spare parts and consumer goods. This categorization determines the forms of reuse.

According to Thierry et al. [29] a returned item can be directly reused/resold, recovered or disposed. It can be reused or resold directly after simple cleaning operations. Packages, bottles and containers are the most common examples of direct reuse. Alternatively, they can be recovered according to the product recovery options: repair, refurbishing, remanufacturing, cannibalization and recycling.

In the repair option, a used product is brought to working order by mending or changing some parts; but the repaired product has lower quality than a new product. Refurbishing option is similar to repair with higher quality standards but less strict than those for new products. Also, in refurbishing, the product is disassembled into modules, inspected and replaced, if necessary.

Contrary to repair and refurbishing options, the goal of remanufacturing is to bring used products to the same quality levels as for new products. Therefore, a used product is disassembled into modules and parts; and, each part or module is rigorously tested to replace the critical components with new ones.

Different from the first three options, the purpose of cannibalization is to recover reusable parts in a used product rather than the product itself. Hence, in cannibalization, limited disassembly of a used product occurs to find the reusable parts.

Finally, in the recycling option, the aim is to recover materials from used products. With this property, it is the most different recovery option from the others since other options try to save all features of the used product or its parts. However, in recycling the goal is to separate product into materials to use them in production of new parts or sell in secondary markets.

Another important issue for a company applying the concept of reuse is the control of its inventories. Producers have to integrate their existing inventory system with the return flow of used items. In a typical inventory system of reusable products, the materials can be supplied from an outside supplier or recovered from returned items. As a result of this, the inventory level does not always decrease but may also increase due to returns. Hence, this makes the analysis of the inventory process more complicated. Also, this fluctuating structure is the main difference between the traditional inventory systems and the inventory systems of reusable items. However, similar to conventional inventory systems, the decision maker wants to know when to order and how much to order. The objective of inventory management of reusable products can be stated as minimizing the associated costs with respect to required service level while controlling the outside orders together with inside recovery process.

The earlier models proposed in the literature of reverse logistics are the adaptations of the well known EOQ formula to the systems with returns. The studies on this framework begin with Schrady [26]. He considers a system with deterministic demand and return rates with fixed procurement and recover lead-times. He suggests a control policy and provides the EOQ type expressions for the optimal order and recovery lotsizes . A similar model is proposed by Richter [23], [24] with a disposal option and variable setup numbers per cycle. He derives the optimal policy parameters and discusses the relationship between the production and repair setup numbers and waste disposal rates.

Although the EOQ type models provides intuition about the relationship between the disposal rate and the cost function, the main drawback of these models is that the demands and returns are assumed to be known with certainty. For many real life cases, this assumption does not hold.

For stochastic demand and return patterns we can divide the literature into two main categories according to the forms of reuse. These are the well-known repair/maintenance systems and general product recovery systems. Although we mentioned repair as a recovery option, the repair/maintenance systems in the inventory literature differ by two characteristics from the general product recovery systems. The main distinction of these systems is that the demand

and return patterns are perfectly correlated such that every return is followed by a demand or vice versa. Therefore, the fluctuations caused by returns is not observed in inventory levels. Secondly, the repair/maintenance systems are closed-loop systems which means that the number of items circulating in the system is constant. The objective of these models is to determine the optimal number of spare parts while minimizing the total cost with respect to required service level. Good surveys on repair/maintenance models are [21], [18], [1] and [15].

From the point of product recovery management, proposed models in the inventory literature of the reusable items can be classified according to their complexities in many ways. However, the major distinction is between the studies considering the recovery and the storage facilities separately and the studies considering only the storage facility where the returned items enter the serviceable inventory upon arrival. We base our literature review on this distinction from the perspective of the periodic review and continuous review models.

One of the first models considering the recycling option is given by Cohen et al. [2]. They deal with a periodic review inventory system with lost sales in which after a fixed number of periods a fixed portion of items return to the inventory. The recycling facility is not considered separately so that the returned items enter the serviceable inventory directly. It is also assumed that a fixed fraction of the on-hand inventory decays at the end of each review period. Demands are assumed to be independent and identically distributed random variables with known density functions. The outside procurement of items is possible with zero lead-time. The optimal policy is given as a one parameter order-up-to policy. Due to the limitations of dynamic programming, a myopic approximation is derived and shown that it provides good approximations for different recycle periods.

Kelle and Silver [13] proposes a similar model given by Cohen et al. [2] without the decay. They offer a periodic review model for reusable containers to obtain the purchasing policy for a finite time horizon. Different from Cohen et al. [2], they work with the net demand which is the demand minus the number of returned containers. The cumulative net demand is approximated by a normal distribution and a stochastic model is derived and transformed into the deterministic, dynamic lot-sizing problem to determine the optimal order quantities.

The first model that separates the serviceable and recoverable inventories is proposed by Simpson [27]. He considers a periodic review repairable inventory system with back-ordering in which the joint probability density function of demand and return patterns is known. It is assumed that only the repairable items are returned and enter the repairable inventory upon arrival. The testing and repair operations are assumed instantaneous. The serviceable inventory is decreased by demands and increased by purchasing new items from an outside supplier and/or repairing items available in the repairable inventory. The repairing and purchasing lead-times are assumed to be zero. The repairable inventory is decreased by repairs and/or disposed units and increased by returns. At the beginning of each period purchase, repair and/or disposal orders are given with the fixed purchase and repair costs per unit and zero salvage value. For an n-period repairable inventory system a three parameter $(\theta_n, \delta_n, \xi_n)$ policy is proposed. According to this policy whenever the serviceable inventory level is less than θ_n repairs take place to bring the inventory level up-to θ_n. After the repair decision if the serviceable inventory is less than δ_n a purchase order is given to increase the serviceables up-to δ_n. Finally, if the total of repairable and serviceable inventories is greater than $\theta_n + \xi_n$ the unrepaired units are disposed to decrease the repairable inventory level down-to $\theta_n + \xi_n$. He proved the optimality of this policy by using the Kuhn-Tucker saddle point theorems.

Inderfurth [8] extends the work of Simpson [27] for remanufacturable items when there exist fixed lead-times for procurement and recovery. He develops different policies according to the relationship between remanufacturing and procurement lead-times. For simplicity he considers a special case in which each of the returned items which is not remanufactured in the preceding period is disposed of. When the lead-times are identical he offers an optimal (L, U) policy. According to this policy when the inventory position is smaller than L, all returned items are remanufactured and a procurement order is given to increase the inventory position up to L. When the inventory position is greater than U, then a disposal order is given to decrease the inventory position down to U and the remaining recoverable products are remanufactured. Between this limits all returned items are remanufactured and neither disposal nor procurement order is given. Then, he derives the functional equations of dynamic programming and suggests an extension of the (L, U) policy when stockkeeping of returned items is allowed. For identical lead-times this policy is same as the one offered by Simpson [27].

Recently, Inderfurt et al. [11] considers a product recovery system where returned items are either disposed of or recovered according to different remanufacturing options. The outside procurement of serviceables is not allowed; and, hence, only the lead-times of different remanufacturing options are taken into account. The objective is to determine the number of items remanufactured for each option and amount of disposal in each period while minimizing the total cost function including remanufacturing, disposal, backordering and holding costs. They show that the structure of the optimal policy is quite complicated and not suitable for practical use. For that reason a near-optimal remanufacture-up-to, dispose-down-to, (nM, U), control policy is offered and its properties are discussed for different period and cost structures.

Another recent study by Toktay et al. [30] presents a periodic review, one-parameter policy to control the supply chain for Kodak's single-use flash camera in which only a portion of used cameras return. The system is modelled as a closed queueing network including vendor, shipping, production, distribution, retailer and customer nodes. It is assumed that unsatisfied demands are lost and fixed ordering cost is ignored. For various cases return flow parameters and unobservable inventory are estimated with different methods and procurement quantities are determined. The simulation results are provided to investigate the effects of informational structure and the accuracies of approximation methods.

The literature that we have reviewed so far comprises the periodically reviewed models. However, most of the literature on the inventory management of reusable products is devoted to the continuously reviewed models.

The earliest work in this area belongs to Heyman [7]. He proposes a one-parameter disposal policy for a single-item, continuous review inventory system with purchase, repair and disposal options. It is assumed that repair and purchase times are negligible. Demands and returns occur according to independent processes and only the serviceable inventory is considered. If the inventory level is smaller than the keep level - maximum number of items allowed in the inventory - returned items enter the inventory. Otherwise they are disposed. For Poisson demand and return processes the system is modelled as an $M/M/1/N$ queue and the optimal keep level is derived analytically. Approximate results are obtained for general demand and return processes.

Yuan and Cheung [40] suggests an (s, S) control policy for a continuous review system with dependent Poisson demand and return streams in which each item returns to inventory with positive probability. The replenishment lead-time is assumed to be zero and backorders are allowed. The system is modelled as a two state Markov process and an algorithm is developed to search the optimal policy parameters. For positive lead-times and with disposal probability, Kiesmüller and Van der Laan [14], study the same model for periodic review case. They propose a two parameter order-up-to policy and discuss the effects of regarding the dependency between the return and demand processes.

Muckstadt and Isaac [17] analyze an inventory system for repairable items with distinct modelling of repair and storage facilities. All returned units enter the repair facility which works as a first-come, first-served queuing system with Poisson arrivals. No assumption about service time distributions or the number of repair servers is made. Serviceable inventory is increased by repaired units and/or outside procurement and decreased by Poisson demands. The procurement lead-time is fixed and backorders are allowed. To control the inventory system, continuous review (Q, r) policy is offered. Inventory position is modelled as a Markov chain and the steady-state distribution is stated. Since it is analytically intractable to obtain the joint distribution of inventory position and number of units in the repair system, the net inventory is approximated by a normal distribution to derive the optimal policy parameters. In the second part of the paper the results are extended for a two echelon system and an algorithm is introduced to find the optimal policy parameters.

Van der Laan et al. [33] study the same model stated by Muckstadt and Isaac [17]. They propose two alternative approximation methods to find the expected number of backorders and extend them with the disposal option. In the first approximation the net demand during lead-time is assumed to have a normal distribution. Whereas in the second one the difference between demand and the output process of the repair facility is assumed to have Brownian motion with drift. Both approximations are compared with the results of Muckstadt and Isaac [17] and it is found that the second method gives near optimal results for all cases. Similar to the one proposed by Heyman [7], the disposal option is imposed on the model by restricting the number of items allowed in the repair shop. A returned item is disposed if there is no waiting room in the repair shop. Repair facility is modelled as an $M/M/c/c+N$ queueing system and an heuristic optimization procedure is given to find the optimal parameters of (s, Q, N) policy.

Van der Laan et al. [34] discuss a numerical comparison of three different control policies for the system studied by Van der Laan et al. [33] by including non-zero holding costs for remanufacturables. These policies are the (s_p, Q_p, s_d, N), (s_p, Q_p, N) and (s_p, Q_p, s_d) policies in which s_p is the inventory position where the procurement order of size Q_p is given, s_d is the inventory position where the returned products are disposed and N is the capacity of the remanufacturing facility. The exact cost rate expression for the (s_p, Q_p, s_d, N) strategy is derived and compared with the costs of other two strategies numerically with respect to varying return and remanufacturing rate scenarios. It is found that the (s_p, Q_p, s_d) policy operates more efficiently than the (s_p, Q_p, s_d, N) policy in most of the situations. Moreover, the (s_p, Q_p, s_d, N) policy gives a lower bound for the other two strategies and provides a reasonable cost reduction.

Two continuous review PUSH and PULL strategies are suggested by Van der Laan et al. [35] for a single-product, hybrid production/inventory system with different stocking points for remanufacturables and serviceables. Under the

(s_m, Q_m, Q_r) PUSH-strategy, whenever remanufacturable inventory contains Q_r modules remanufacturing starts and all Q_r modules enter the remanufacturing process. Manufacturing takes places in batches of size Q_m and starts whenever the serviceable inventory drops to the level s_m. Under the (s_m, Q_m, s_r, S_r) PULL-strategy, remanufacturing starts whenever the serviceable inventory position is at or below s_r and remanufacturable inventory contains sufficient products to increase the serviceable inventory position to S_r. Manufacturing starts whenever the serviceable inventory position drops to the level s_m and the manufacturing batch size equals Q_m. They extend the model of Muckstadt and Isaac [17] by considering correlated Coxian-2 distributed demand and return flows, deterministic remanufacturing and manufacturing lead-times with nonzero fixed remanufacturing costs. Also, an exact procedure to calculate the expected total cost is derived. Traditional systems are compared with the systems with remanufacturing and the cases are discussed when the PUSH or PULL strategies may be economically favorable.

Under the same PUSH and PULL control strategies and model assumptions, the effects of lead-time duration and lead-time variability on total costs are investigated by Van der Laan et al. [36] with correlated Poisson demands and returns. In the numerical studies the costs for both strategies are observed when the lead-time durations are varied and the distributions are assumed to be Bernoulli. They show that the manufacturing lead-times have a larger effect on costs then remanufacturing lead-times.

Van der Laan and Salomon [37] include the disposal option to the model offered by Van der Laan et al.[35]. They propose (s_m, Q_m, Q_r, s_d) PUSH-disposal and $(s_m, Q_m, s_r, S_r, s_d)$ PULL-disposal policies. For the PUSH-disposal policy s_d is the serviceable inventory position level where the returned products are disposed upon arrival. Under the PULL-disposal policy whenever the remanufacturable inventory level hits s_d, disposal occurs. They identify the cases when one of the policies outperforms the other and conclude that these disposal strategies are not very robust to the changes in demand and return patterns.

Recently, Inderfurth and Van der Laan [10] apply the (s_m, Q_m, Q_r, s_d)

PUSH-disposal policy to a periodic review production/inventory system with independent Poisson demands and returns. The remanufacturing lead-time is considered as a planning variable and they offer a policy improvement due to lead-time variation.

Further extensions can be found in the related articles (see Van der Laan [38], Van der Laan and Teunter [39], Teunter et al. [31])which consider average cost and net present value analysis of hybrid systems.

There is some similarity between the so called cash-balancing models and reusable product inventories in terms of the model dynamics and the tools used for the analysis. In cash-balancing models the objective is to control the cash level of a bank which is affected by customer withdrawals and deposits. The money deposits and withdrawals correspond to the return of used products and the demand for new products in an inventory system with returns, respectively. Also in cash management models, there is the possibility of transferring the money from or to the central bank due to the intensity of money inflows and outflows. These operations correspond to the outside procurement and disposal options in the inventory management context. However, there is an important distinction between these models, regarding the procurement lead-time and holding costs of reusable products. Most of the time, considering lead-time for transactions in cash-balancing models may be meaningless due to very fast Electronic Data Interchange opportunities or small transportation times of money from the central bank to a local branch. Besides, a recovery process does not exist for these models since returned money is directly added to the cash inventory of a bank. There is a vast literature on cash-management models. Although they propose similar policies, most of the time they have different emphases on solution procedures, cost structures and control mechanisms. Since the detailed review of these models is beyond the scope of our study, below we cover the most related papers with our problem in terms of the structure and the control policy.

Constantinides [3] studies a continuous-time cash management system with stochastic demands and returns. The analysis is based on the net demand, (demand minus return) which is assumed to be generated by a Wiener process.

He considers the optimal (d, D, U, u) policy in which d is the lower bound to transact up-to D and u is the upper bound to transact down-to U. He discusses properties of the optimal policy parameters for the cases when the expected net demand is zero and non-zero.

Penttinen [20] presents closed form myopic and stationary solutions for a periodic-review stochastic cash-balance model. The optimal policy addressed by Constantinides [3] is used to control the transactions. The exact and approximate solutions are given for myopic policy parameters when demand has the logistic distribution. For comparison, stationary results are presented for exponentially distributed inflows and outflows. As a result of numerical studies, it is found that, for high shortage costs, myopic solutions perform almost same and in case of no fixed costs, stationary model gives better results for disposal level.

Hinderer and Waldmann [9] extend the same policy when there exist randomly varying factors influencing cash flow environment. It is assumed that the cash flow in each period is a random variable with a discrete density which depends on the environmental process. Structural results are given and the optimality of this transfer rule is discussed for different cash flow and environmental processes.

There is also an independent class of models in the inventory theory literature in which the demand is modelled by the Brownian motion process. Although these studies are not in the context of the inventory management of reusable items, we mention the most significant ones for completeness of our literature review. For instance, an interesting study by Sulem [28] offers an (s, S) control policy for a system in which the demand is modelled by the Wiener process. The lead-time is assumed to be zero and the backordering is allowed. The cost structure includes the fixed and variable ordering costs, the shortage cost and the holding cost which are discounted with a constant interest rate. Under the impulse control assumption, the solution of the optimal (s, S) is given by two algebraic equations. Finally, the expressions for the deterministic demand and the zero discount factor cases are provided.

Nieobber and Dekker [19] proposes a model for refinery tankage assessment and stock control problem. They offer a periodic-review target-stock control policy where the demand for small parcels are modelled by a Brownian motion process.

Moinzadeh and Nahmias [16] consider a continuous review system where there is an agreement between buyer and seller for fixed deliveries. They offer a two parameter (s, S) control policy for adjustments. Since the exact analysis is analytically intractable, demand process is approximated by a Brownian motion process. By using the deterministic version of the problem and the results from the Brownian motion theory, simple expressions for the optimal policy parameters are obtained.

Finally, Girlich [5] applies the Brownian motion approximation for demand process to a simple example and discusses the advantages of using this approximation for complex inventory systems.

In this study, we consider a model similar to the one proposed by Constantinides [3] with non-zero fixed procurement lead-time. Returns and demands are assumed to be stochastic where the net demand constitutes a Brownian motion process. Only the serviceable inventory is considered and an (S, s, r, Q) policy is proposed to control the order quantity and timing of disposal and procurement decisions. Unlike the models with disposal option discussed so far, the disposal of items occur in the serviceable inventory. The objective is to determine the policy parameters which minimize the total expected cost while satisfying the required service level. The required expressions of Brownian motion moving in a strip and the general closed-form structural expressions are derived explicitly. By using these results, the long-run expected cost per unit time is derived for both zero net demand and positive net demand cases which correspond to Brownian motion process without drift and with drift respectively. Then, the numerical optimization techniques are employed to find the optimal policy parameters and the optimal cost rate function.

The rest of this thesis is organized as follows:

In Chapter 2, we introduce the inventory model in detail, explain the control policy and present some structural expressions. In Chapter 3, we provide preliminary results for Brownian motion which are used in the following chapters.

In Chapter 4, operating characteristics of the inventory system are derived for the zero net demand case. In Chapter 5, we extend the model for the positive net demand case and derive the explicit expressions for the total cost function. In Chapter 6, a detailed numerical study is provided and the behavior of the total cost function is discussed for both zero and positive net demand cases. Finally, in Chapter 7 concluding remarks are presented.

Chapter 2

Model Description and Control Policy

We study a single-item, single-location continuous review inventory system of reusable items with independent, continuous stochastic demand and return flows.

In the inventory literature of reusable items the vast amount of attention is given to the systems including remanufacturing and manufacturing operations together. The continuous review models in this context, separate the recovery and storage facilities and offer approximate solutions by using the results from queuing theory (see, e.g. Muckstadt and Isaac [17], Van der Laan et al. [33], Van der Laan et al. [35]). Although these models elaborate the remanufacturing concept in detail, there exist simpler systems for reusable products in which the separate modelling of recovery is not needed. These are the systems allowing direct reuse and recycling operations. Such systems are first modelled by Cohen et al. [2] and Kelle and Silver [13] from the periodic review perspective. The first and the only study which considers one stocking point for continuous review inventory system with returns is proposed by Heyman [7]. He offers a one-parameter optimal disposal policy for Poisson demands and returns. However, the proposed model does not include the procurement lead-time and fixed costs for ordering, recovery and disposal.

As outlined above, it is clear that the existing reverse logistics literature includes very limited number of studies on continuous review inventory systems with a single stocking location. Most of the studies consider sophisticated hybrid manufacturing/remanufacturing systems. Hence, our major motivation in this study is to offer a control policy for the systems which apply the reuse concept with less complicated processes.

With this motivation we address the systems including direct reuse or recycling options. The most common examples of direct reuse are bottles and containers which can be reused after simple cleaning and maintenance operations. For the recycling option typical examples may be plastic packages, waste papers and metals in discarded cars.

In the system that we consider a returned item enters recovery process, upon return from the customer. This process includes cleaning, maintenance or recycling operations. After the recovery process, each recovered item has the same quality as a new product or material, and enters the inventory. However, the parameters such as holding cost of returned items, recovery cost and recovery lead-time are neglected. Therefore, it is assumed that a returned item is instantaneously recovered and enters the serviceable inventory with a holding cost per unit per time, h.

We only consider the serviceable inventory which is decreased by the customer demands and disposal of items and increased by recovered items and outside procurement of new products. However, the output of the recovery process, in general, is not enough to satisfy all the customer demands. Hence, the outside procurement of new items is possible with a fixed positive procurement lead-time, L. The procurement cost structure includes fixed and variable ordering costs denoted by K_o and C_o respectively . It is also assumed that the unsatisfied demands during lead-time are backordered.

The disposal of the items is possible when the inventory level hits the predetermined threshold disposal level after the lead-time. Moreover, the threshold disposal level can be interpreted as a physical capacity constraint on a warehouse or stocking point. Each time the disposal order is given a fixed number of items are disposed with the unit disposal cost, C_d and the ordering cost, K_d.

Most of the models that we have discussed in the previous chapter apply the disposal upon arrival of returned items. Unlike these models, we consider disposal on the serviceable inventory. This kind of disposal policy seems more reasonable when there exists significant legal restrictions on producers. Due to the new arrangements in the environmental legislation of many developed countries, the companies producing goods sold in glass, metal or plastic packages are responsible for collecting and recycling those packages in specified portions according to their types. For instance, in Turkey, the quota proportions of year 2001 are given as 30% for plastic and glass, 25% for metal and 20% for paper packages (see [32]).

As mentioned above, we consider continuous demand and return flows. Consequently, the net demand, the difference between the returned items and the customer demands at any time, is used to define the inventory level. This netting approach is very common in most of the cash-balancing models. Furthermore, we assume that the increments in the inventory level during small time units are normally distributed. Hence, the inventory level constitutes a Brownian Motion process where the drift parameter is denoted by, μ. Unlike the classical inventory processes, there exist decreases and increases in the inventory level due to the fluctuating structure of the Brownian motion. It is also assumed that the return rate is smaller than the demand rate, since the loss or breakage of issued items is possible. Hence, the μ can take zero or negative values. Clearly, the net demand rate should be interpreted as the absolute value of the drift parameter. In Chapters 4 and 5 we will present the operating characteristics for both cases.

After stating the motivation of the study and some basic characteristics of the model, we can summarize the main assumptions of the model as follows:

Assumptions:

1. Inventory system is reviewed continuously.

2. The procurement lead-time is positive and fixed.

3. Inventory level process is governed by a Brownian Motion.

4. Demands that are not satisfied during the lead-time are back ordered.

5. Disposal is instantaneous.

From the cash management perspective, our model can be seen as an extension of the model proposed by Constantinides [3]. Below we propose the extension of the four parameter optimal control policy of this model when there exists fixed positive lead-time. Under the assumptions given above, the inventory policy we consider is stated as follows:

Policy: A replenishment order of size Q is placed when the inventory position drops to the reorder point r and, whenever the inventory position hits the disposal trigger level S, the excess inventory is disposed to bring the inventory position down-to s immediately, except during the lead-time period L.

We will refer to this policy as the (S, s, r, Q) policy. The decision variables are the order size Q, the reorder point r, the disposal trigger level S and the dispose down-to level s. All policy parameters are assumed to be non-negative and there is at most one order outstanding at any time.

Furthermore, we assume that no units are disposed of when there is an outstanding order. The main reason for this assumption is that the inventory level may hit the reorder point multiple times during lead-time due to its fluctuating structure. If we allow for disposals during lead-time, this may cause order crossing which conflicts with the one order outstanding assumption. Moreover, the disposal of items does not seem realistic when there is an order outstanding since it would further decrease inventories.

Since the policy is not imposed during lead-time, there exist two different paths for the process before and after the lead-time. After the lead-time period the process can be seen as a Brownian motion moving within different strips. These strips are determined by the position of the inventory at the end of the lead-time period and the policy parameters S, s and r. Whereas during the lead-time period it can be seen as a released Brownian motion moving between minus and plus infinity. Also, it is assumed that the inventory level after a batch of Q units arrives ,namely x, is between the reorder point and the threshold disposal level; that is $r < x \leq S$. If we let $x < r$, according to the structure of the Brownian motion, it is possible that the process stays under the reorder point forever. On the other hand when $x > S$, the disposal policy may be applied at the end of the lead-time by disposing the excess inventory down-to s, and the model can be solved by relaxing this part of the assumption. However, regarding the negative and the zero μs, the probability of this event becomes negligible. Hence, we hold this additional assumption for the simplicity of derivations of the operating characteristics.

Under the control policy stated above, the inventory process repeats itself at specific epochs where the inventory process hits the level r to initiate the order of Q units. We define these epochs as regenerative points. This repetitive structure enables us to derive the long-run characteristic of the system. Therefore, a regenerative cycle can be defined as follows.

Cycle: A regenerative cycle is the time between two consecutive reorder instances.

To derive the long-run characteristics of the system and formulate the optimization problem, we need to define the expected total cost per unit time, $TC(S, s, r, Q)$ as a function of the decision variables S, s, r, Q. We denote the expected cycle cost, expected cycle length, expected on-hand inventory carried in a cycle, expected on-hand inventory carried during lead-time, expected number of disposals occuring after the lead-time period and time-weighted expected back-order carried during lead-time with $E(CC)$, $E(CL)$, $E(OH)$, $E(OH_L)$, $E(N)$ and $E(BO)$, respectively. A detailed notation is given at the end of this chapter.

According to renewal reward theorem (see Ross [25]) we can represent our expected total cost function as follows:

$$TC(S, s, r, Q) = \lim_{t \to \infty} \frac{CC(t)}{t} = \frac{E(CC)}{E(CL)}$$

Where $CC(t)$ is the total cycle cost incurred by time t. Hence, we can state the optimization problem as follows:

$$\min\ TC(S,s,r,Q) = \frac{K_0 + hE(OH) + [K_d + C_d(S-s)]E(N) + C_oQ}{E(CL)} \tag{2.1}$$

s.t.

$$\frac{E(BO)}{E(BO)+E(OH_L)} \leq (1-\alpha) \tag{2.2}$$

Note that, the conventional service level measures can not be used in our model since we are not able to derive the expected unit backorder during lead-time due to the complex structure of the Brownian Motion. Therefore, we define a new service level measure α, which is defined according to the expected time weighted backorder and the expected on-hand carried during the lead-time. The ratio given by (2.2) can be seen analogous to the cost tradeoff of the newsboy problem where $E(BO)$ and $E(OH_L)$ correspond to the understocking and the overstocking costs of the newsboy problem, respectively.

Before the derivation of the structural expressions, we need to define some random variables which are crucial for our analysis. Therefore, let the position of the inventory level at time t be denoted by $X(t)$. At any time t, let T_u be the time needed to complete the regenerative cycle, if the inventory level $X(t) = u$. We will call this the remaining cycle time at inventory level u. Recall that the inventory level right after the replenishment order has arrived is denoted by x. Then, T_x is the time until the cycle is completed immediately after the replenishment lead time.

Also, let $T_{u,v}$ be the time the inventory process, starting at u, hits the level v for the first time. Moreover, let $T^z_{u,v}$ be the time it takes the process, starting at u, hits the level v the first time, without hitting z. Finally, let $T_{u,vz}$ be the time the process, starting at u, escapes from the strip $[v,z]$ for the first time.

For the derivation of the expected on-hand inventory expressions, we adopt the trapezoidal area approximations used commonly in the classical inventory literature. We observe that a better approximation is obtained, if two possible locations of x as $r < x \leq s$ and $s \leq x \leq S$ are considered separately. These cases are considered below. It will become clear in the sequel that both cases yield the same expected cycle length expression unlike the expected on-hand inventory. This point will be further discussed in the course of the derivations.

Case 1: $r < x \leq s$

According to the proposed control policy, whenever the inventory level hits S, the excess inventory is disposed and the inventory level is reduced to s. This allows a repetitive behavior for the process within a regenerative cycle. In particular, the inventory level may hit S and is reduced to s several times before the regenerative cycle is completed. Hence, we analyze the process after the lead time in two stages.

The first stage corresponds to the first escape time from the strip $[r, s]$. If the escape is at the level r; that is, if the process, starting at x, hits r before s, then the cycle is completed. Note that for this scenario T_x equals to $T^s_{x,r}$. A typical realization of this scenario can be seen in Figure 2.1.

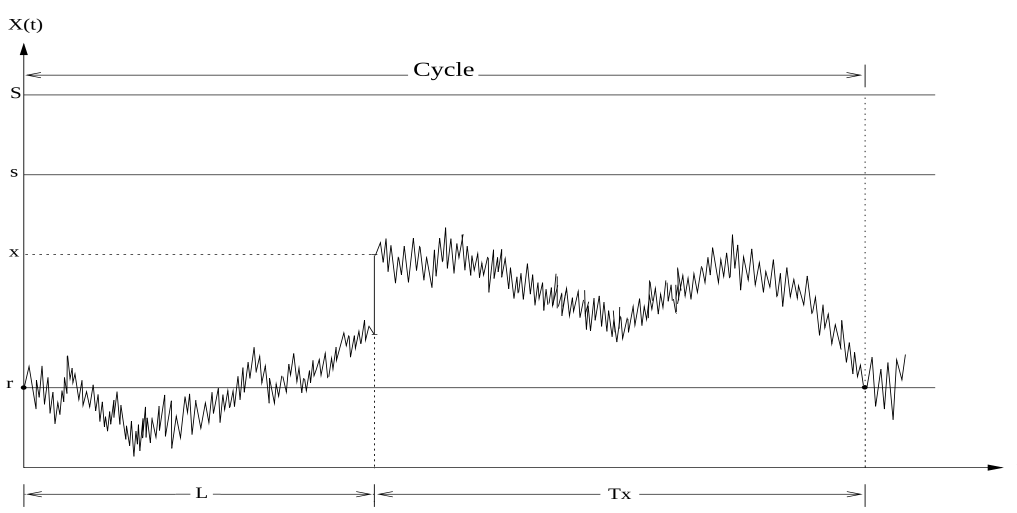

Figure 2.1: Realization of the inventory level process for Case 1

Otherwise the second stage starts, which is the time until the cycle is completed if the process is at level s. We refer this stage as T_s. Similarly, at the second stage, if the process starting at s, hits r before S then the cycle is completed (see Figure 2.2).

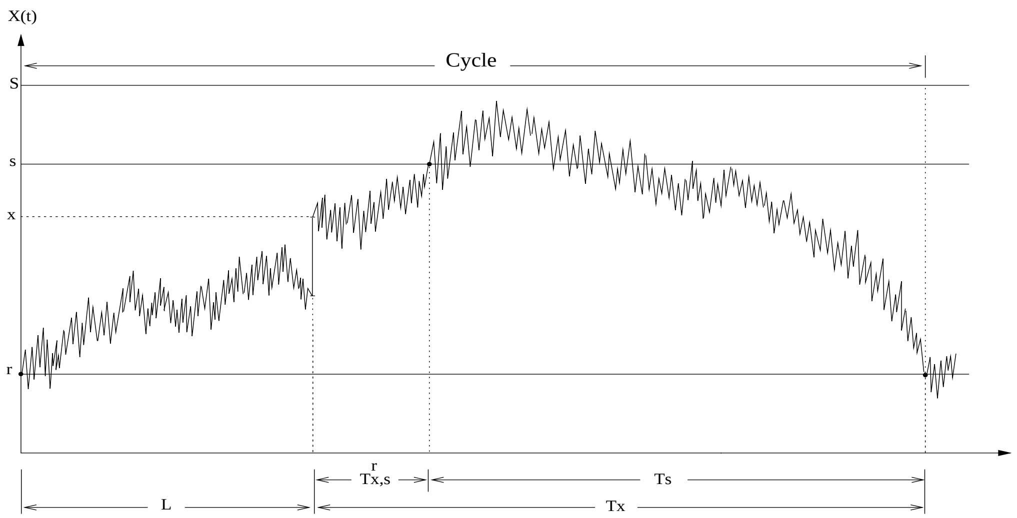

Figure 2.2: Realization of the inventory level process for Case 1

Otherwise, the level S is reached before r and it is immediately reduced to s. Clearly, since the process repeats itself; whenever it reaches S, a new T_s starts. Figure 2.3 illustrates an example of this realization.

It is clear that for both realizations given by Figure 2.2 and Figure 2.3, T_x is equal to $T^r_{x,s} + T_s$.

According to the realizations given above, there exist two possible scenarios for T_x which can be seen more clearly with the following relation.

$$T_x = \begin{cases} T^s_{x,r} & \text{if} \quad T_{x,r} < T_{x,s} \\ T^r_{x,s} + T_s & \text{if} \quad T_{x,s} < T_{x,r} \end{cases} \tag{2.3}$$

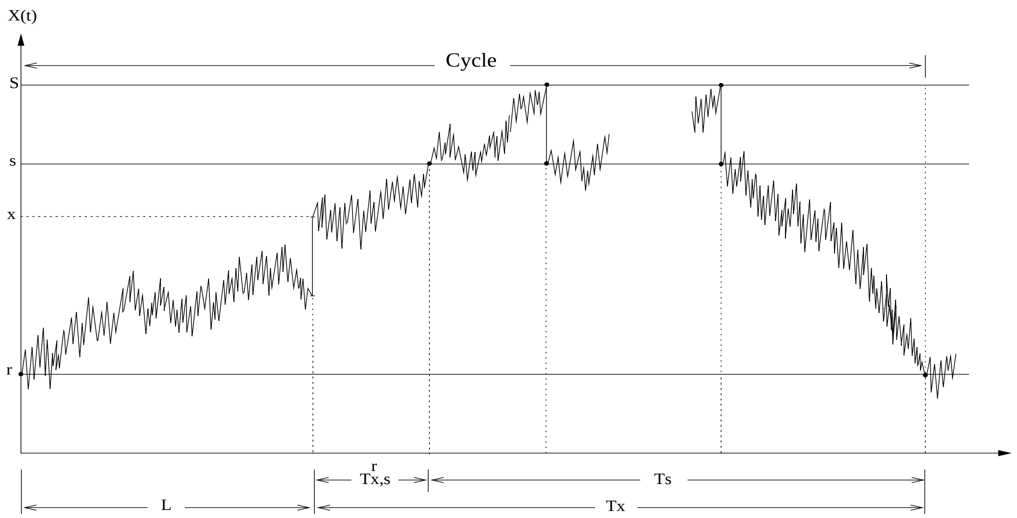

Figure 2.3: Realization of the inventory level process for Case 1

Case 2: $s \leq x \leq S$

In this case we again consider the behavior of the process in two stages. The first stage corresponds to the first escape time of the process, from the strip $[s, S]$, starting at x. We denote this time by $T_{x,sS}$. Due to the proposed disposal policy, regardless of the inventory level from which the escape is realized, the second stage, T_s, starts at s and continues until the cycle is completed.

Hence, at the end of the first stage, a second stage as defined in Case 1 starts for all realizations, unlike the previous case where the second stage starts only if the process hits s before r. Therefore, in Case 2, the first stage is always terminated at level s. Possible realizations of this case can be seen in Figure 2.4 and Figure 2.5.

From Figure 2.4 and Figure 2.5 we observe that for all possible realizations T_x can be given as follows:

$$T_x = T_{x,sS} + T_s \tag{2.4}$$

Note that, due to the disposal policy, there is not any difference between hitting S before s or vice versa. Therefore, T_x includes first escaping time from the strip $[s, S]$, $T_{x,sS}$.

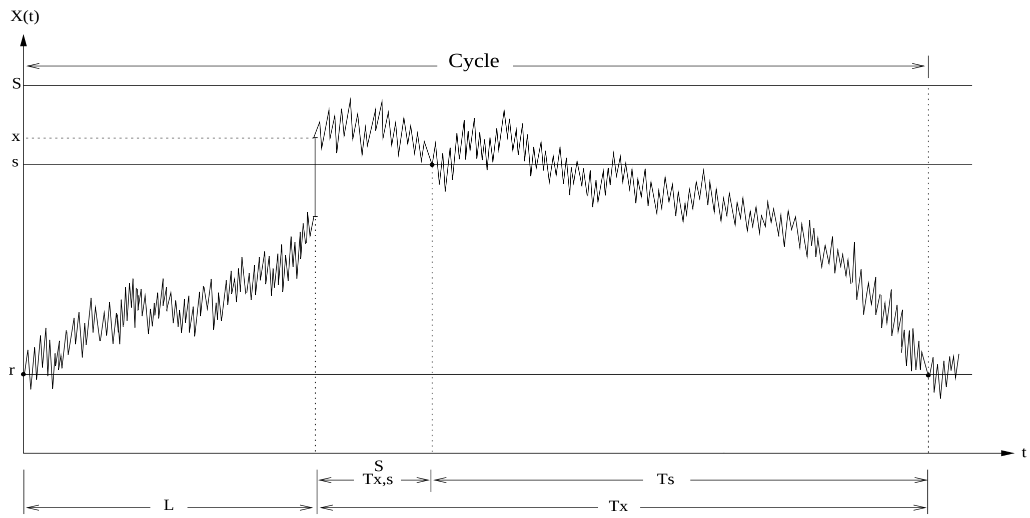

Figure 2.4: Realization of the inventory level process for Case 2

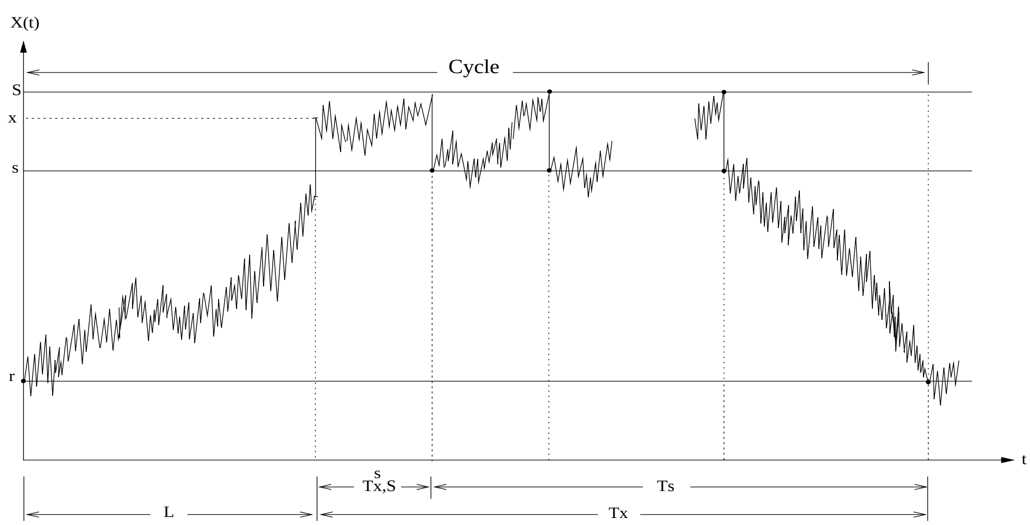

Figure 2.5: Realization of the inventory level process for Case 2

2.1 Structural Expressions

In this section we state the structural expressions for both cases which will be used in the derivation of the operating characteristics when the net demand takes zero or positive values. We start with the expressions for the expected on-hand after the lead-time for Case 1 and Case 2. Then, we derive the common expression of the expected cycle-length after the lead-time.

2.1.1 Expected On-Hand Inventory After The Lead-Time

Case 1: $r < x \leq s$

To derive the expected on-hand inventory after the lead-time, the trapezoidal areas are used for the realizations given above. This kind of linear approximation is common in the inventory literature and convenient for analytical derivations (see Hadley and Within [6]).

Let OH_u be the inventory carried until the end of the cycle if the process is at point u. Then, the inventory carried until the end of the cycle after the lead-time can be written as:

$$OH_x = \begin{cases} \frac{(x+r)}{2} T^s_{x,r} & \text{if} \quad T_{x,r} < T_{x,s} \\ \frac{(x+s)}{2} T^r_{x,s} + OH_s & \text{if} \quad T_{x,s} < T_{x,r} \end{cases} \tag{2.5}$$

In the first line of (2.5) the product of $T^s_{x,r}$ with the average of x and r gives the trapezoidal area $A1$ in Figure 2.6, which corresponds to the case where the process hits r before s.

The second line corresponds to the case where s is hit before r (see Figure 2.7 and Figure 2.8). The first term of the second line is the trapezoidal area, $A2$, found by multiplying the average of x and s by the time until the process,

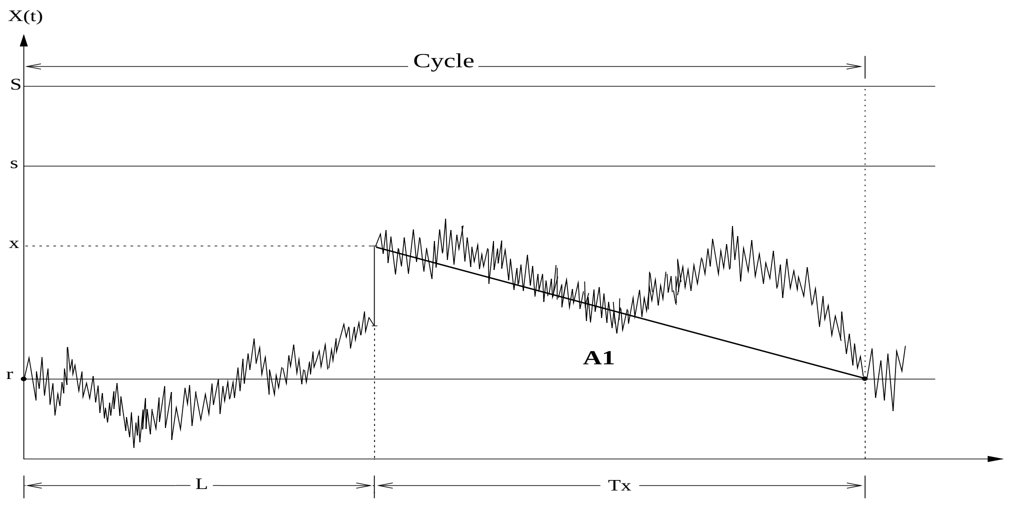

Figure 2.6: Trapezoidal areas of Case 1

starting at x, hits s before r for the first time, namely $T^r_{x,s}$, whereas the second one is the inventory carried until the end of the cycle from the point s, denoted by OH_s.

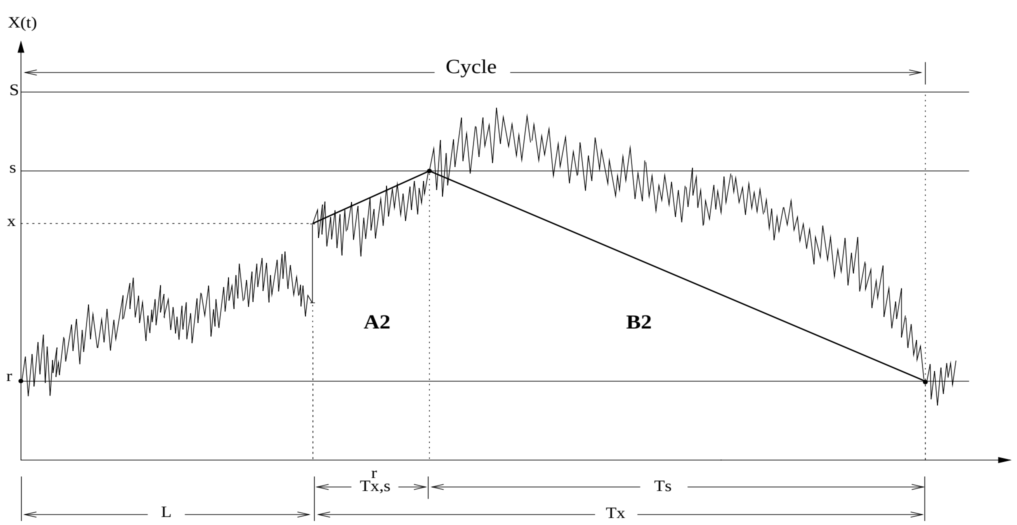

Figure 2.7: Trapezoidal areas of Case 1

The derivation of OH_s requires the computation of the trapezoidal areas $B1$ and $B2$ given in Figure 2.7 and Figure 2.8. $B1$ is found by multiplying the average of s and S by $T^r_{s,S}$, whereas $B2$ is found by multiplying the average of s and r by

$T^S_{s,r}$. Also note that, in each cycle, there is always only one $B2$, while the number of $B1$ may be zero in some realizations.

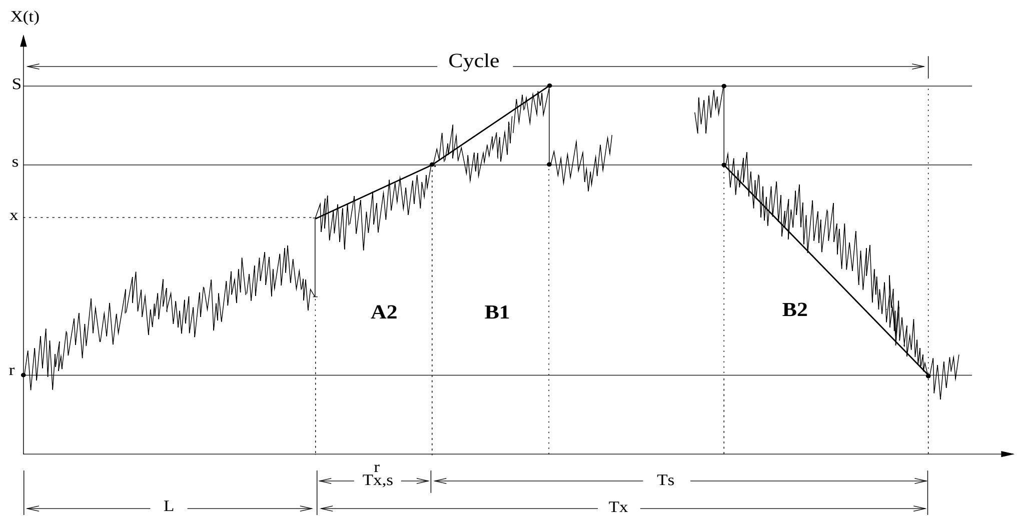

Figure 2.8: Trapezoidal areas of Case 1

Let $P^z_{u,v}$ be the probability that the process, starting at u, hits v before z. Also, let $I(\cdot)$ denote the indicator function. Then, we can give the expectations of on-hand inventories by the following results.

Lemma 2.1

$$i)\ E(OH_s) = \frac{1}{P^S_{s,r}}\left[\frac{(s+r)}{2}E(T^S_{s,r}\ I(T_{x,r} < T_{x,S})) + \frac{(s+S)}{2}E(T^r_{s,S}\ I(T_{x,S} < T_{x,r}))\right]$$

$ii)$ *For* $r < x \leq s$:

$$\begin{aligned} E(OH_x) &= \frac{(x+r)}{2}E(T^s_{x,r}\ I(T_{x,r} < T_{x,s})) + \frac{(x+s)}{2}E(T^r_{x,s}\ I(T_{x,s} < T_{x,r})) \\ &+ E(OH_s)P^r_{x,s} \end{aligned}$$

Proof : See Appendix A.

Case 2: $s \leq x \leq S$

The on-hand inventory carried during T_x can be defined as a random variable as follows:

$$OH_x = \begin{cases} \frac{(x+s)}{2} T_{x,s}^S + OH_s & \text{if} \quad T_{x,s} < T_{x,S} \\ \frac{(x+S)}{2} T_{x,S}^s + OH_s & \text{if} \quad T_{x,S} < T_{x,s} \end{cases} \tag{2.6}$$

The first line of (2.6) corresponds to the case where s is hit before S. Therefore, the product of $T_{x,s}^S$ with the average of x and s gives the trapezoidal area $A3$ shown in Figure 2.9.

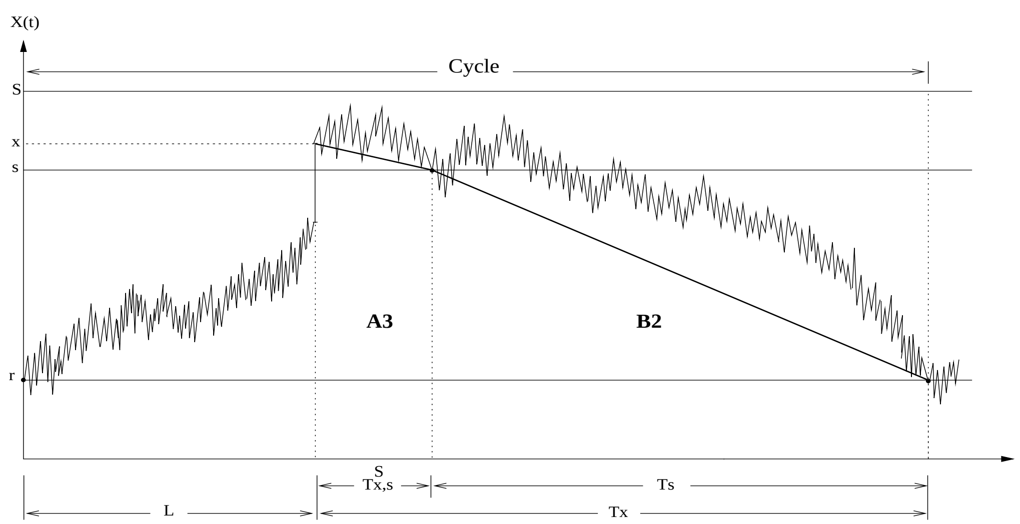

Figure 2.9: Trapezoidal areas of Case 2

Similarly, in the second line of (2.6), the trapezoidal area $A4$ depicted in Figure 2.10 is found by multiplying $T_{x,S}^s$ with the average of x and s for the case where the process hits S before s.

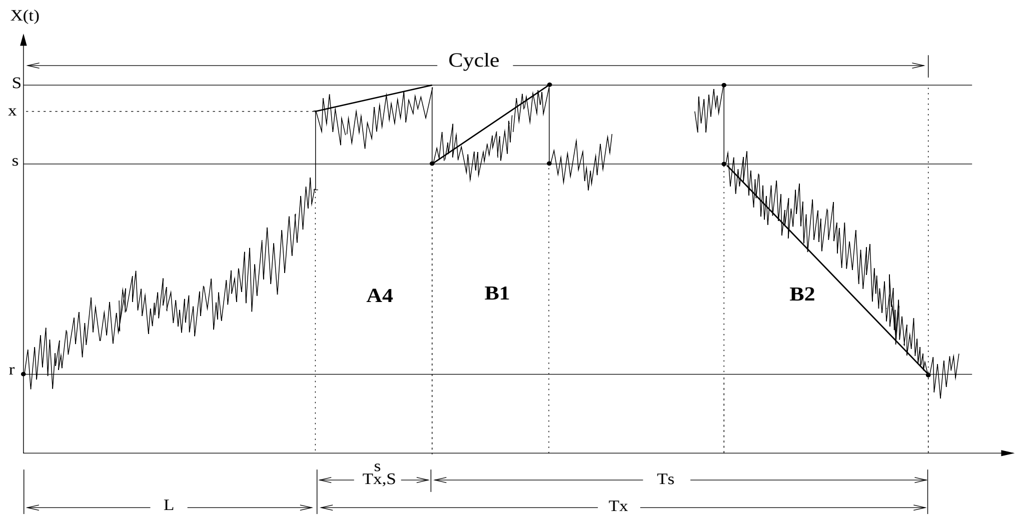

Figure 2.10: Trapezoidal areas of Case 2

Since the second stage, T_s, is common in Case1 and Case2, the corresponding expected on-hand inventories will be the same. Therefore, using the part (i) of Lemma 2.1 and the equation (2.6), we provide the following result without proof.

Lemma 2.2 *For $s \leq x \leq S$:*

$$\begin{aligned} E(OH_x) &= \tfrac{(x+s)}{2} E(T^S_{x,s}\, I(T_{x,s} < T_{x,S})) + \tfrac{(x+S)}{2} E(T^s_{x,S}\, I(T_{x,S} < T_{x,s})) \\ &+ E(OH_s) \end{aligned}$$

As we mentioned earlier, separate analysis according to the position of the process at the end of the lead-time provides a better approximation of the expected on-hand inventory. In particular, if we had considered the two cases jointly by assuming that, $r < x \leq S$, then for the realizations where the process hits the level S first, the corresponding trapezoidal area between x and s would have been disappeared. As a result of this, expected on-hand would have been approximated roughly. So, in order to increase the precision, this partition is imposed to the problem. However, we can not make the same claim for the cycle length. It is an intuitive result that the cycle length is independent of any kind of partition. One can show that the expected cycle length expressions will be the same not only for both cases but also when there exists no partition. For this reason, in the next section we will introduce the derivation of the expected cycle length after the lead-time just for Case 1, when $r < x \leq s$.

2.1.2 Expected Cycle-Length After The Lead-Time

According to (2.3), we can give the expectations of T_x and T_s with the following result.

Lemma 2.3

$$i)\ E(T_s) = \frac{E(T_{s,rS})}{P^S_{s,r}}$$

ii) For $r < x \leq s$:

$$E(T_x) = E(T_{x,rs}) + E(T_s)P^r_{x,s}$$

Proof : See Appendix A.

Notation

S	:	Disposal trigger level
s	:	Dispose down-to level
r	:	Reorder point
Q	:	Order quantity
L	:	Fixed lead-time
α	:	Service Level
y	:	The position of the inventory level at the end of the fixed lead-time
x	:	The position of the inventory level immediately after the order arrived ($x = y + Q$)
CL	:	Cycle Length (Time between two consecutive reorder instances)
OH	:	Total inventory carried in a cycle
OH_L	:	Inventory carried during lead-time
BO	:	Time weighted back order during lead-time
N	:	Number of disposals in a cycle
K_o	:	Fixed ordering cost per order
C_o	:	Variable ordering cost per unit
K_d	:	Fixed disposal cost per disposal batch
C_d	:	Variable disposal cost per unit
h	:	Holding cost per unit per time
TC	:	Cost rate
CC	:	Cycle cost
$CC(t)$	:	Total cycle cost within a cycle incurred by time t since beginning of a cycle
$X(t)$	:	Position of the Brownian Motion (B.M.) at time t since beginning of a cycle
μ	:	Drift coefficient of the B.M.
T_u	:	Remaining cycle length for a B.M. starting at u
$T_{u,v}$	:	The time until the B.M. hits v for the first time starting at u

$T^z_{u,v}$	:	The time until the B.M. hits v before z for the first time starting at u
$T_{u,vz}$	:	The time until the B.M. escapes from strip $[v, z]$ for the first time starting at u, $min(T_{u,v}, T_{u,z})$
$P^z_{u,v}$	:	Probability that the B.M. hits v before z for the first time starting from u
OH_u	:	Inventory carried until the end of cycle for a B.M. starting at u
$I(\cdot)$	:	Indicator Function

Chapter 3

Preliminary Results for Brownian Motion

In this chapter we introduce the basic characteristics of the Brownian motion and present some preliminary results which will be used in the derivation of the operating characteristics.

As we mentioned above, for the systems of recyclable or reusable items, if return and demand flows are assumed to be continuous then the Brownian Motion is one of the most suitable process for this kind of systems since it has normally distributed, independent increments.

Brownian motion is a symmetric random walk in which the step size and the time interval needed to take this step go to zero in the limit. The formal definition of Brownian motion is given by Ross [25] as follows:

Definition 3.1 *A stochastic process* $\{X(t), t \geq 0\}$ *is said to be a Brownian Motion process if:*

i) $X(0) = 0$*;*

ii) $\{X(t), t \geq 0\}$ *has stationary and independent increments;*

iii) For every $t > 0$*,* $X(t)$ *is normally distributed with mean* 0 *and variance* $\sigma^2 t$

The independent increments property assumes that the increment of one interval is not affected by the increment realized in the previous interval. Moreover, the stationary increments property states that the distribution of those increments depends only the length of the time interval in which the increment occurs.

The positive constant σ, the variance parameter of $X(t)$, can be interpreted as a time scaling factor. The process is usually called as *standard Brownian motion* when $\sigma = 1$. In the sequel of this study we will assume $\sigma = 1$.

As can be seen from the figures given in the previous chapter, the general structure of the inventory level at time t, $X(t)$, is governed by Brownian motion. However, there are some jumps in order arrival and disposal instances which prevent the direct analysis of a cycle. Hence, we analyze the lead-time period and after the lead-time period separately.

Since we do not impose the (S, s, r, Q) policy during the lead-time period no jumps occur and the inventory level process behaves as a Brownian motion without any boundaries. Also, recall that in our model the net demand rate, μ, corresponds to the drift coefficient of Brownian motion and can take zero or negative values. When μ is equal to zero, by using the Definition 3.1 the density of $X(t)$ during lead-time can be given as

$$f_{X(t)}(x) = \frac{1}{\sqrt{2\pi t}} e^{-(x-r)^2/2t} \quad , -\infty < x < \infty \tag{3.1}$$

Note that the mean parameter of the normal distribution is equal to r, since the starting point of the cycle is r, $X(0) = r$.

If we consider a negative drift coefficient, $\mu < 0$, then the mean is equal to $r + \mu t$ and the density of $X(t)$ is given as

$$f_{X(t)}(x) = \frac{1}{\sqrt{2\pi t}} e^{-(x-r-\mu t)^2/2t} \quad , -\infty < x < \infty \tag{3.2}$$

In the previous chapter we discussed the analysis of the period after the lead-time and presented the structural expressions of $E(OH_x)$ and $E(T_x)$. The possible paths of this period are nothing but the same Brownian motion starting from different points and moving in different strips. In the sequel, using the results from Resnick [22], Karatzas and Shreve [12] and Ross [25], we give the expressions of the expected hitting times and the hitting probabilities which will be used in the derivation of the expected cycle-length and the expected on-hand inventory for zero and negative drift cases.

Proposition 3.1 *Let $b < x < a$. Then for a Brownian motion $\{X(t), t \geq 0\}$ with $X(0) = x$, $\mu = 0$ and moving in strip $[b, a]$, we have:*

$$i)\ P^b_{x,a} = \frac{x-b}{a-b} \qquad and \qquad P^a_{x,b} = \frac{a-x}{a-b}$$

$$ii)\ E(T^b_{x,a}\, I(T_{x,a} < T_{x,b})) = \frac{(x-b)(a+x-2b)(a-x)}{3(a-b)}$$

$$E(T^a_{x,b}\, I(T_{x,b} < T_{x,a})) = \frac{(x-b)(2a-x-b)(a-x)}{3(a-b)}$$

$$iii)\ E(T_{x,ba}) = (x-b)(a-x)$$

Proof : See Appendix B.

Proposition 3.2 *Let $b < x < a$, then for a Brownian motion $\{X(t), t \geq 0\}$ with $X(0) = x$, nonzero drift μ and moving in strip $[b, a]$, we have:*

$$i)\ P^b_{x,a} = \frac{e^{-2\mu x} - e^{-2\mu b}}{e^{-2\mu a} - e^{-2\mu b}} \qquad and \qquad P^a_{x,b} = \frac{e^{-2\mu a} - e^{-2\mu x}}{e^{-2\mu a} - e^{-2\mu b}}$$

$$\begin{aligned} ii)\ E(T^b_{x,a}\, I(T_{x,a} < T_{x,b})) &= \frac{x\,(e^{-2\mu x} + e^{-2\mu b})}{\mu\,(e^{-2\mu a} - e^{-2\mu b})} \\ &+ \frac{2b\,e^{-2\mu b} - a\,(e^{-2\mu a} + e^{-2\mu b})}{\mu\,(e^{-2\mu a} - e^{-2\mu b})^2}\, e^{-2\mu x} \\ &+ \frac{a\,e^{-4\,\mu b} + (a-2b)\,e^{-2\mu(a+b)}}{\mu\,(e^{-2\mu a} - e^{-2\mu b})^2} \end{aligned}$$

$$
\begin{aligned}
E(T^a_{x,b}\, I(T_{x,b} < T_{x,a})) &= \frac{-x\,\left(e^{-2\mu x} + e^{-2\mu a}\right)}{\mu\,\left(e^{-2\mu a} - e^{-2\mu b}\right)} \\
&+ \frac{2a\ e^{-2\mu a} - b\,\left(e^{-2\mu a} + e^{-2\mu b}\right)}{\mu\,\left(e^{-2\mu a} - e^{-2\mu b}\right)^2}\ e^{-2\mu x} \\
&+ \frac{b\ e^{-4\,\mu a} + (b-2a)\ e^{-2\mu(a+b)}}{\mu\,\left(e^{-2\mu a} - e^{-2\mu b}\right)^2}
\end{aligned}
$$

$$
iii)\ \ E(\,T_{x,ba}) = -\frac{x}{\mu} + \frac{(a-b)}{\mu\,\left(e^{-2\mu a} - e^{-2\mu b}\right)}\ e^{-2\mu x} + \frac{b\ e^{-2\mu a} - a\ e^{-2\mu b}}{\mu\,\left(e^{-2\mu a} - e^{-2\mu b}\right)}
$$

Proof : See Appendix B.

Chapter 4

Operating Characteristics for Zero Net Demand Rate

In this chapter we derive the operation characteristics of an inventory system for recyclable/reusable items in which the expected net demand, μ, is equal to zero. In this particular case it is assumed that the return and the demand rates are equal. Actually, this is a special case of a more general problem in which will be presented and discussed in the following chapter where the demand rate is greater than the return rate.

4.1 Expected Cycle Length

If at the end of the lead-time, the inventory process is as x, then the expected remaining cycle time $E(T_x)$ is given by the following lemma:

Lemma 4.1

$$E(T_x) \quad = \quad (x-r)(S+s-x-r)$$

Proof :

Recall from Lemma 2.3 that the structural expression of the expected length of stage two is given as

$$E(T_s) = \frac{E(T_{s,rS})}{P^S_{s,r}}$$

and, from Proposition 3.1 (i) and (iii), we have

$$\begin{aligned} E(T_{s,rS}) &= (s-r)(S-s) \\ P^S_{s,r} &= \frac{S-s}{S-r} \end{aligned}$$

Thus

$$E(T_s) = (s-r)(S-r) \tag{4.1}$$

Similarly, from Proposition 3.1

$$E(T_{x,rs}) = (x-r)(s-x) \tag{4.2}$$

$$P^r_{x,s} = \frac{x-r}{s-r} \tag{4.3}$$

Also, recall from Lemma 2.3 that for $r < x \leq s$

$$E(T_x) = E(T_{x,rs}) + E(T_s)P^r_{x,s} \tag{4.4}$$

Therefore, by substituting (4.1),(4.2),(4.3) in (4.4), we find

$$E(T_x) = (x-r)(S+s-x-r)$$

Q.E.D.

For both cases defined in Chapter 2, the expected cycle length is the sum of the fixed lead-time, L, and the expected remaining cycle time after the lead-time, $E(T_x)$, given that $X(L) = y$, where y is the position of the inventory level at the end of lead-time before the arrival of an order. If we rewrite $X(L)$ by conditioning on x and use the results of Lemma 4.1, we can provide the following result without proof.

Corollary 4.1

i) $E(CL|X(L) = x - Q) \quad = \quad L + (x - r)(S + s - x - r)$

ii) For $x = r + Q$,

$$E(CL) \quad \cong \quad L + 2Q(S + s - Q - 2r)$$

To hedge against stockouts during the lead-time and satisfy the required service level, usually more than the expected lead-time demand is hold in the inventory. This excess amount is called as "safety stock", denoted by SS, and is equal to the reorder point minus the expected lead-time demand.

$$SS = r - |\mu|L$$

Moreover, we can state that on the average $y = SS$. Hence, for the tractability of the analytical derivations, we can make the substitution $x = r - |\mu|L + Q$. Since we are considering the case when $\mu = 0$ in this chapter, we substitute x by $r + Q$ in Corollary 4.1.

4.2 Expected On-Hand

The following lemma gives the expected on-hand inventories after the lead-time period for Case 1 and Case 2. Recall that the structural expressions of these expectations are provided by Lemmas 2.1 and 2.2.

Lemma 4.2
i) For $r < x \leq s$

$$E(OH_x) \quad = \quad \frac{(x - r)\left[(s - x)(s + r + 4x) + (S - r)(S + r + 4s)\right]}{6} \tag{4.5}$$

ii) For $s \leq x \leq S$

$$E(OH_x) \quad = \quad \frac{(s - r)(S - r)(S + r + 4s) - (s - x)(S - x)(S + s + 4x)}{6} \tag{4.6}$$

Proof : See Appendix C.

The on-hand inventory during a cycle is defined as the on-hand inventory during lead-time, $E(OH_L)$, plus the on-hand inventory carried after the lead-time, $E(OH_x)$, according to Case 1 and Case 2. Using the results of Lemma 4.2, the expected on-hand can be given by the following corollary without proof:

Corollary 4.2

i) $E(OH|X(L) = x - Q)$

$$= E(OH_L) + \begin{cases} \frac{(x-r)[(s-x)(s+r+4x)+(S-r)(S+r+4s)]}{6} & \textit{if} \quad r < x \leq s \\ \\ \frac{(s-r)(S-r)(S+r+4s)-(s-x)(S-x)(S+s+4x)}{6} & \textit{if} \quad s \leq x \leq S \end{cases}$$

ii) *For* $x = r + Q$,

$E(OH)$

$$= \quad E(OH_L) + \begin{cases} \frac{Qs^2+3sQ^2-6r^2Q-9rQ^2-4Q^3+S^2Q+4SsQ}{6} & \textit{if} \quad r < (r+Q) \leq s \\ \\ \frac{3Ss^2-6Ssr+6sr^2-3s^2r-3r^3-2SsQ+6SQr+s^2Q+3sQ^2}{6} & \\ +\frac{6srQ+3Sr^2-12Qr^2-12rQ^2+S^2Q+3SQ^2-4Q^3}{6} & \textit{if} \quad s \leq (r+Q) \leq S \end{cases}$$

In order to complete the derivation of $E(OH)$, we need to derive the expected on-hand inventory during the lead-time, $E(OH_L)$. Recall that we employed the trapezoidal approximation in the derivation of $E(OH_x)$ for tractability. However, the analysis of the lead-time period is much simpler than after the lead-time period since there are no disposal during this period. Hence, $E(OH_L)$ is derived exactly. The following theorem provides the result explicitly.

Theorem 4.1

$$E(OH_L) \;=\; \int_0^L \sqrt{\frac{t}{2\pi}}\; e^{-r^2/2t} dt + r \int_0^L \Phi\left(\frac{r}{\sqrt{t}}\right)\; dt \tag{4.7}$$

Proof :

According to the assumption that the control policy is not imposed during the lead-time, the inventory process behaves as a free Brownian Motion fluctuating around r. This assumption enables the exact derivation of the $E(OH_L)$. Therefore, to calculate the $E(OH_L)$, we should find the expectation of the positive inventory level at time t, $E\left([X(t)]^+\right)$, and then take the average of this value over the lead-time period. Thus we have

$$E(OH_L) \;=\; \int_0^L E\left([X(t)]^+\right) dt \tag{4.8}$$

Recall from Chapter 3 that $X(t)$ is normally distributed with mean r and variance t and its density is given by (3.1). Therefore,

$$\begin{aligned} E\left([X(t)]^+\right) &= \int_0^\infty x f_{X(t)}(x) dx \\ &= \sqrt{\frac{t}{2\pi}}\; e^{-r^2/2t} + r\; \Phi\left(\frac{r}{\sqrt{t}}\right) \end{aligned} \tag{4.9}$$

where $\Phi\left(\frac{r}{\sqrt{t}}\right)$ is the cumulative density function of Standard Normal distribution. Substituting (4.9) in (4.8) yields

$$E(OH_L) \;=\; \int_0^L \sqrt{\frac{t}{2\pi}}\; e^{-r^2/2t} dt + r \int_0^L \Phi\left(\frac{r}{\sqrt{t}}\right)\; dt$$

Q.E.D.

Clearly, $E(OH_L)$ is a function of the reorder point, r, and it needs to be computed numerically since it can not be represented as an implicit function.

4.3 Expected Backorder

The expected backorder, $E(BO)$ is derived exactly and presented in Theorem 4.10.

Theorem 4.2

$$E(BO) \quad = \quad E(OH_L) - rL \tag{4.10}$$

Proof :

The method for the derivation of $E(BO)$ is same as the method used to derive $E(OH_L)$. However, this time we need to derive the expected negative inventory level at time t, $E\left([X(t)]^-\right)$. Therefore,

$$E(BO) \quad = \quad -\int_0^L E\left([X(t)]^-\right) dt \tag{4.11}$$

Using the density function (3.1), we obtain

$$\begin{aligned} E\left([X(t)]^-\right) &= \int_{-\infty}^0 x f_{X(t)}(x) dx \\ &= -\sqrt{\frac{t}{2\pi}}\, e^{-r^2/2t} + r\left[1 - \Phi\left(\frac{r}{\sqrt{t}}\right)\right] \end{aligned} \tag{4.12}$$

Substituting (4.12) in (4.11) yields

$$\begin{aligned} E(BO) &= \int_0^L \sqrt{\frac{t}{2\pi}}\, e^{-r^2/2t} dt + r\int_0^L \Phi\left(\frac{r}{\sqrt{t}}\right) dt - rL \\ &= E(OH_L) - rL \end{aligned}$$

Q.E.D.

4.4 Expected Number of Disposals

If the inventory level increases too much due to returns then the disposal of items occurs. In order to follow-up the average disposal cost, we need to derive the average number of disposals $E(N)$ in a cycle. Therefore, given that the inventory level at the end of lead-time is $y = x - Q$, the probability mass function of the number of disposals N is:

$$P(N = n|X(L) = x - Q) = \begin{cases} P^S_{x,r} & , n = 0 \\ \\ P^r_{x,S} P^S_{s,r} (P^r_{s,S})^{(n-1)} & , n > 0 \end{cases} \tag{4.13}$$

If there is no disposal, $(n = 0)$, then it means that the process goes directly down to r from x without hitting S which occurs with probability , $P^S_{x,r}$. If there is only one disposal, then it means that the process goes up to S from x without hitting r which occurs with probability $P^r_{x,S}$ and after disposal it directly goes down to r from s without hitting S, which occurs with probability $P^S_{s,r}$. If there are n disposals before the cycle ends, then the probability of them are given by the term $(P^r_{s,S})^{(n-1)}$. The following corollary states the expected number of disposals, $E(N)$ explicitly. The derivation of the $E(N)$ is exact, however for simplicity we use the $x = r + Q$ approximation.

Corollary 4.3

i) $E(N|X(L) = x - Q) \;=\; \dfrac{x - r}{S - s}$

ii) For $x = r + Q$:

$$E(N) = \frac{Q}{S - s}$$

Proof :

i) From Proposition 3.1 (*i*) we know

$$P^r_{x,S} = \frac{x - r}{S - r}, \qquad P^r_{s,S} = \frac{s - r}{S - r}, \qquad P^S_{s,r} = \frac{S - s}{S - r}$$

and by using (4.13) we can write the conditional expectation as

$$
\begin{aligned}
E(N|X(L)=x-Q) &= \sum_{n=0}^{\infty} n\ P(N|X(L)=x-Q) \\
&= \sum_{n=1}^{\infty} n\ P^r_{x,S} P^S_{s,r} (P^r_{s,S})^{(n-1)} \\
&= P^r_{x,S} P^S_{s,r} \left[1 + 2P^r_{s,S} + 3(P^r_{s,S})^2 + \cdots\right] \\
&= \frac{P^r_{x,S}}{P^S_{s,r}} \\
&= \frac{x-r}{S-s} \qquad (4.14)
\end{aligned}
$$

ii) By simply substituting $x = r + Q$ in (4.14), we obtain

$$E(N) = \frac{Q}{S-s}$$

Q.E.D.

4.5 Expected Cycle Cost

The expected cycle cost under the (S, s, r, Q) policy can be obtained by using the (ii) of Corollary 4.2 and 4.3 and the cost parameters. Hence, the general form of the expected cycle cost when $\mu = 0$ with the approximation $x = r + Q$ is given as

$$E(CC) = K_o + hE(OH) + K_d E(N) + (C_o + C_d)Q \qquad (4.15)$$

Using Corollary (4.1) (ii) and (4.15), we can construct the objective function given by (2.1). Moreover, the constraint (2.2) can be constructed by using (4.7) and (4.10).

Chapter 5

Operating Characteristics for Positive Net Demand Rate

In this chapter we extend the results of Chapter 4 for more general case in which the inventory level process is governed by a Brownian motion with negative drift coefficient, μ. In this case, the demand rate is greater than the return rate resulting in positive net demand. Compared to the zero net demand case, in the long-run when $\mu < 0$ the effects of some characteristics such as outside procurement, backorders and number of disposals may increase due to high demand flows. To analyze these effects in detail we need to derive the operating characteristics of the model. In our derivations we apply the methods used in the previous chapter with minor modifications.

5.1 Expected Cycle Length

The expected remaining cycle time after the replenishment lead-time can be given by the following lemma.

Lemma 5.1

$$E(T_x) = -\frac{(x-r)}{\mu} + \frac{(S-s)}{\mu} \cdot \frac{(e^{-2\mu x} - e^{-2\mu r})}{(e^{-2\mu S} - e^{-2\mu s})}$$

Proof : See Appendix D.

Employing the result of Lemma 5.1, we give the conditional and the approximate expected cycle length when $\mu \leq 0$ with the following corollary without proof.

Corollary 5.1

$$i)\ E(CL|X(L) = x - Q) \quad = \quad L - \frac{(x-r)}{\mu} + \frac{(S-s)}{\mu} \cdot \frac{(e^{-2\mu x} - e^{-2\mu r})}{(e^{-2\mu S} - e^{-2\mu s})}$$

ii) For $x = r - |\mu|L + Q$,

$$E(CL) \quad \cong \quad -\frac{Q}{\mu} + \frac{(S-s)}{\mu} \cdot \frac{e^{-2\mu r}(e^{-2\mu(Q-|\mu|L)} - 1)}{(e^{-2\mu S} - e^{-2\mu s})}$$

5.2 Expected On-Hand

The expected on-hand inventories after the lead-time period for Case 1 and Case 2 are given by the following lemma.

Lemma 5.2

i) For $r < x \leq s$

$$E(OH_x) \quad = \quad -\frac{x^2}{2\mu} + \frac{(s-r)x}{\mu} \cdot \frac{e^{-2\mu x}}{(e^{-2\mu s} - e^{-2\mu r})} + e^{-2\mu x} F_1(S, s, r; \mu) + G_1(S, s, r; \mu) \tag{5.1}$$

where

$$F_1(S, s, r; \mu) \quad = \quad \frac{sr}{\mu} \left[\frac{e^{-2\mu S}(e^{-2\mu s} + e^{-2\mu r}) - 2e^{-4\mu s}}{(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu s} - e^{-2\mu r})^2} \right] - \frac{r^2}{\mu} \left[\frac{e^{-2\mu r}(e^{-2\mu S} - e^{-2\mu s})}{(e^{-2\mu S} - e^{-2\mu r})(e^{-2\mu s} - e^{-2\mu r})^2} \right]$$

$$
\begin{aligned}
&- \frac{s^2}{2\mu}\left[\frac{e^{-2\mu s}\left[\left(e^{-2\mu S}-e^{-2\mu s}\right)+\left(e^{-2\mu S}-e^{-2\mu r}\right)\right]}{\left(e^{-2\mu S}-e^{-2\mu s}\right)\left(e^{-2\mu s}-e^{-2\mu r}\right)^2}\right. \\
&- \left.\frac{e^{-2\mu r}\left(e^{-2\mu s}-e^{-2\mu r}\right)}{\left(e^{-2\mu S}-e^{-2\mu s}\right)\left(e^{-2\mu s}-e^{-2\mu r}\right)^2}\right] \\
&+ \frac{Ss}{\mu}\cdot\frac{e^{-2\mu s}}{\left(e^{-2\mu S}-e^{-2\mu s}\right)\left(e^{-2\mu s}-e^{-2\mu r}\right)} \\
&- \frac{S^2}{2\mu}\cdot\frac{\left(e^{-2\mu S}+e^{-2\mu r}\right)}{\left(e^{-2\mu S}-e^{-2\mu s}\right)\left(e^{-2\mu S}-e^{-2\mu r}\right)} \\
&+ \frac{Sr}{\mu}\left[\frac{e^{-2\mu S}}{\left(e^{-2\mu S}-e^{-2\mu s}\right)\left(e^{-2\mu S}-e^{-2\mu r}\right)}\right. \\
&- \left.\frac{e^{-2\mu r}}{\left(e^{-2\mu S}-e^{-2\mu r}\right)\left(e^{-2\mu s}-e^{-2\mu r}\right)}\right]
\end{aligned}
$$

$$
\begin{aligned}
G_1(S,s,r;\mu) &= \frac{s^2}{2\mu}\left[\frac{e^{-2\mu(s+r)}\left[\left(e^{-2\mu S}-e^{-2\mu s}\right)+\left(e^{-2\mu S}-e^{-2\mu r}\right)\right]}{\left(e^{-2\mu S}-e^{-2\mu s}\right)\left(e^{-2\mu s}-e^{-2\mu r}\right)^2}\right. \\
&- \left.\frac{e^{-4\mu r}\left(e^{-2\mu s}-e^{-2\mu r}\right)}{\left(e^{-2\mu S}-e^{-2\mu s}\right)\left(e^{-2\mu s}-e^{-2\mu r}\right)^2}\right] \\
&+ \frac{r^2}{2\mu}\left[\frac{e^{-4\mu s}\left(e^{-2\mu S}-e^{-2\mu r}\right)}{\left(e^{-2\mu S}-e^{-2\mu r}\right)\left(e^{-2\mu s}-e^{-2\mu r}\right)^2}\right. \\
&+ \left.\frac{e^{-4\mu r}\left[\left(e^{-2\mu S}-e^{-2\mu s}\right)-\left(e^{-2\mu s}-e^{-2\mu r}\right)\right]}{\left(e^{-2\mu S}-e^{-2\mu r}\right)\left(e^{-2\mu s}-e^{-2\mu r}\right)^2}\right] \\
&+ \frac{sr}{\mu}\left[\frac{e^{-2\mu(s+r)}\left[\left(e^{-2\mu s}-e^{-2\mu r}\right)-2\left(e^{-2\mu S}-e^{-2\mu s}\right)\right]}{\left(e^{-2\mu S}-e^{-2\mu s}\right)\left(e^{-2\mu s}-e^{-2\mu r}\right)^2}\right] \\
&- \frac{Ss}{\mu}\cdot\frac{e^{-2\mu(s+r)}}{\left(e^{-2\mu S}-e^{-2\mu s}\right)\left(e^{-2\mu s}-e^{-2\mu r}\right)} \\
&+ \frac{S^2}{2\mu}\cdot\frac{e^{-2\mu r}\left(e^{-2\mu S}+e^{-2\mu r}\right)}{\left(e^{-2\mu S}-e^{-2\mu s}\right)\left(e^{-2\mu S}-e^{-2\mu r}\right)}
\end{aligned}
$$

$$- \quad \frac{Sr}{\mu}\left[\frac{e^{-2\mu(S+r)}}{(e^{-2\mu S}-e^{-2\mu s})(e^{-2\mu S}-e^{-2\mu r})}\right.$$

$$\left. - \quad \frac{e^{-4\mu r}}{(e^{-2\mu S}-e^{-2\mu r})(e^{-2\mu s}-e^{-2\mu r})}\right]$$

ii) For $s \leq x \leq S$

$$\begin{aligned} E(OH_x) &= -\frac{x^2}{2\mu} + \frac{(S-s)x}{\mu} \cdot \frac{e^{-2\mu x}}{(e^{-2\mu S}-e^{-2\mu s})} - e^{-2\mu x} F_2(S,s;\mu) \\ &+ G_2(S,s,r;\mu) \end{aligned} \tag{5.2}$$

where

$$\begin{aligned} F_2(S,s;\mu) &= \frac{(S-s)^2}{2\mu}\left[\frac{(e^{-2\mu S}+e^{-2\mu s})}{(e^{-2\mu S}-e^{-2\mu s})^2}\right] \\ G_2(S,s,r;\mu) &= \frac{s^2}{2\mu}\left[\frac{e^{-2\mu S}(e^{-2\mu s}+e^{-2\mu r}) + e^{-2\mu s}(e^{-2\mu S}-e^{-2\mu r})}{(e^{-2\mu S}-e^{-2\mu s})^2}\right] \\ &+ \frac{r^2}{2\mu} \cdot \frac{(e^{-2\mu S}+e^{-2\mu r})}{(e^{-2\mu S}-e^{-2\mu r})} - \frac{sr}{\mu} \cdot \frac{e^{-2\mu s}}{(e^{-2\mu S}-e^{-2\mu s})} \\ &- \frac{Ss}{\mu}\left[\frac{e^{-2\mu s}(e^{-2\mu S}+e^{-2\mu s})}{(e^{-2\mu S}-e^{-2\mu s})^2}\right] \\ &+ \frac{S^2}{2\mu}\left[\frac{2e^{-2\mu(S+s)}(e^{-2\mu s}-e^{-2\mu r})}{(e^{-2\mu S}-e^{-2\mu s})^2(e^{-2\mu S}-e^{-2\mu r})}\right. \\ &\left. + \frac{e^{-2\mu r}(e^{-2\mu S}+e^{-2\mu r})}{(e^{-2\mu S}-e^{-2\mu s})(e^{-2\mu S}-e^{-2\mu r})}\right] \\ &+ \frac{Sr}{\mu}\left[\frac{e^{-2\mu S}(e^{-2\mu s}-e^{-2\mu r})}{(e^{-2\mu S}-e^{-2\mu s})(e^{-2\mu S}-e^{-2\mu r})} - \frac{e^{-2\mu r}}{(e^{-2\mu S}-e^{-2\mu r})}\right] \end{aligned}$$

Proof : See Appendix D.

Using Lemma 5.2, we give the expected on-hand inventory carried during a cycle with the following corollary without proof.

Corollary 5.2

i) $E(OH|X(L) = x - Q)$

$$= E(OH_L) + \begin{cases} -\frac{x^2}{2\mu} + \frac{(s-r)x}{\mu} \cdot \frac{e^{-2\mu x}}{(e^{-2\mu s} - e^{-2\mu r})} \\ +e^{-2\mu x} F_1(S,s,r;\mu) + G_1(S,s,r;\mu) & \text{if} \quad r < x \leq s \\ \\ -\frac{x^2}{2\mu} + \frac{(S-s)x}{\mu} \cdot \frac{e^{-2\mu x}}{(e^{-2\mu S} - e^{-2\mu s})} \\ -e^{-2\mu x} F_2(S,s;\mu) + G_2(S,s,r;\mu) & \text{if} \quad s \leq x \leq S \end{cases}$$

ii) For $x = r - |\mu|L + Q$,

$$E(OH) = E(OH_L)$$

$$+ \begin{cases} -\frac{(r-|\mu|L+Q)^2}{2\mu} + \frac{(s-r)(r-|\mu|L+Q)}{\mu} \cdot \frac{e^{-2\mu(r-|\mu|L+Q)}}{(e^{-2\mu s} - e^{-2\mu r})} \\ +e^{-2\mu(r-|\mu|L+Q)} F_1(S,s,r;\mu) + G_1(S,s,r;\mu) & \text{if} \quad r < (r - |\mu|L + Q) \leq s \\ -\frac{(r-|\mu|L+Q)^2}{2\mu} + \frac{(S-s)(r-|\mu|L+Q)}{\mu} \cdot \frac{e^{-2\mu(r-|\mu|L+Q)}}{(e^{-2\mu S} - e^{-2\mu s})} \\ -e^{-2\mu(r-|\mu|L+Q)} F_2(S,s;\mu)) + G_2(S,s,r;\mu) & \text{if} \quad s \leq (r - |\mu|L + Q) \leq S \end{cases}$$

Finally, to complete the derivation of $E(OH)$, the expected on-hand during lead-time is derived exactly and given as follows:

Theorem 5.1

$$E(OH_L) \;=\; \int_0^L \sqrt{\frac{t}{2\pi}}\, e^{-(r+\mu t)^2/2t} dt + r \int_0^L \Phi\left(\frac{r+\mu t}{\sqrt{t}}\right) dt + \mu \int_0^L t\, \Phi\left(\frac{r+\mu t}{\sqrt{t}}\right) dt$$

Proof :

To calculate the expected on-hand during lead-time we use the same procedure employed when $\mu = 0$. Therefore, we can give the structural expression as

$$E(OH_L) = \int_0^L E\left([X(t)]^+\right) dt \tag{5.3}$$

From Chapter 3 we know that $X(t)$ is normally distributed with mean $r + \mu t$ and variance t and its density is given by (3.2). Thus,

$$E\left([X(t)]^+\right) = \int_0^\infty x f_{X(t)}(x) dx$$

$$= \sqrt{\frac{t}{2\pi}}\, e^{-(r+\mu t)^2/2t} + (r + \mu t)\, \Phi\left(\frac{r + \mu t}{\sqrt{t}}\right) \tag{5.4}$$

Substituting (5.4) in (5.3) yields

$$E(OH_L) = \int_0^L \sqrt{\frac{t}{2\pi}}\, e^{-(r+\mu t)^2/2t} dt + r \int_0^L \Phi\left(\frac{r + \mu t}{\sqrt{t}}\right) dt + \mu \int_0^L t\, \Phi\left(\frac{r + \mu t}{\sqrt{t}}\right) dt$$

Q.E.D.

$E(OH_L)$ is a function of the reorder point, r and the drift coefficient, μ. Note that when $\mu = 0$ the above expression converges to (4.7).

5.3 Expected Backorder

The expected backorder is given by the following theorem.

Theorem 5.2

$$E(BO) = E(OH_L) - rL - \frac{\mu L^2}{2} \tag{5.5}$$

Proof :

Recall that the structural expression of $E(BO)$ is given as follows

$$E(BO) \quad = \quad -\int_0^L E\left([X(t)]^-\right) dt \tag{5.6}$$

Using the density function (3.2), we obtain

$$\begin{aligned} E\left([X(t)]^-\right) &= \int_{-\infty}^{0} x f_{X(t)}(x) dx \\ &= -\sqrt{\frac{t}{2\pi}}\, e^{-(r+\mu t)^2/2t} + (r+\mu t)\left[1 - \Phi\left(\frac{r+\mu t}{\sqrt{t}}\right)\right] \end{aligned} \tag{5.7}$$

Substituting (5.7) in (5.6) yields

$$\begin{aligned} E(BO) &= \int_0^L \sqrt{\frac{t}{2\pi}}\, e^{-(r+\mu t)^2/2t} dt + r\int_0^L \Phi\left(\frac{r+\mu t}{\sqrt{t}}\right) dt + \mu \int_0^L t\,\Phi\left(\frac{r+\mu t}{\sqrt{t}}\right) dt \\ &- \; rL - \frac{\mu L^2}{2} \\ &= E(OH_L) - rL - \frac{\mu L^2}{2} \end{aligned}$$

Q.E.D.

If $\mu < 0$, the second part of (5.5) can be interpreted as the difference between the area of a triangle which has the base length of L and the height $|\mu|L$ and the area of a rectangle having sides r and L. In other words if we consider merely the lead-time period, $(rL - \frac{|\mu|L^2}{2})$ is the area under the drift line. Therefore, as much as the $E(OH_L)$ approaches to this area, the $E(BO)$ converges to zero.

5.4 Expected Number of Disposals

Using the probability mass function given by (4.13), the expected number of disposals is given by the following result.

Corollary 5.3

i) $E(N|X(L) = x - Q) \;=\; \dfrac{e^{-2\mu x} - e^{-2\mu r}}{e^{-2\mu S} - e^{-2\mu s}}$

ii) For $x = r - |\mu|L + Q$,

$$E(N) \;=\; \frac{e^{-2\mu r}\big(e^{-2\mu(Q-|\mu|L)} - 1\big)}{e^{-2\mu S} - e^{-2\mu s}}$$

Proof :

i) From Proposition 3.2 (*i*) we know

$$P^r_{x,S} \;=\; \frac{e^{-2\mu x} - e^{-2\mu r}}{e^{-2\mu S} - e^{-2\mu r}}, \qquad P^r_{s,S} = \frac{e^{-2\mu s} - e^{-2\mu r}}{e^{-2\mu S} - e^{-2\mu r}}, \qquad P^S_{s,r} = \frac{e^{-2\mu S} - e^{-2\mu s}}{e^{-2\mu S} - e^{-2\mu r}}$$

and by using (4.13) we can write the conditional expectation as

$$\begin{aligned} E(N|X(L) = x - Q) &= \sum_{n=0}^{\infty} n \; P(N|X(L) = x - Q) \\ &= \sum_{n=1}^{\infty} n \; P^r_{x,S}(P^r_{s,S})^{(n-1)} P^S_{s,r} \\ &= P^r_{x,S} P^S_{s,r} \left[1 + 2P^r_{s,S} + 3(P^r_{s,S})^2 + \ldots\right] \\ &= \frac{P^r_{x,S}}{P^S_{s,r}} \\ &= \frac{e^{-2\mu x} - e^{-2\mu r}}{e^{-2\mu S} - e^{-2\mu s}} \end{aligned} \tag{5.8}$$

ii) Substituting $x = r - |\mu|L + Q$ in (5.8) yields

$$E(N) \;=\; \frac{e^{-2\mu r}\big(e^{-2\mu(Q-|\mu|L)} - 1\big)}{e^{-2\mu S} - e^{-2\mu s}}$$

Q.E.D.

5.5 Expected Cycle Cost

The expected cycle cost under the (S, s, r, Q) policy can be obtained by using the (ii) of Corollary 5.2 and 5.3 and the cost parameters. Hence, the general form of the expected cycle cost when $\mu < 0$ with the approximation $x = r - |\mu|L + Q$ is given as

$$E(CC) = K_o + hE(OH) + [K_d + C_d(S - s)]\, E(N) + C_oQ \quad (5.9)$$

Using Corollary (5.1) (ii) and (5.9), we can construct the objective function given by (2.1). Moreover, the constraint (2.2) can be constructed by using Theorem 5.1 and 5.2.

Chapter 6

Numerical Analysis

In this chapter we present the results of the numerical study and discuss the sensitivity and the performance of the (S, s, r, Q) policy. The chapter consists of three sections.

In the first section, we aim to examine the essential features of the policy parameters for the zero expected net demand case. During the analysis of this case, we treat the disposal trigger level as the physical warehouse capacity to identify the interesting properties of the policy. Hence, we construct our numerical analysis to study the sensitivity of the policy parameters under various values of warehouse capacity, lead-time and service level with respect to a wide range of cost parameters.

In the second section, we try to isolate the important characteristics of the model for the positive expected net demand case. We conduct a similar analysis presented in the first section by relaxing the warehouse capacity assumption. The numerical results provided in this section reflects the sensitivity of the policy parameters with respect to various values of lead-time, service level and expected net demand rate under different cost structures.

In the final section, we focus on the performance of the (S, s, r, Q) policy vis a vis other simpler control policies when the expected net demand rate is positive. To highlight the advantages of our model, we introduce two new policies based on the (S, s, r, Q) policy. We compare the performances of the policies with the original policy and discuss over the numerical results.

6.1 Sensitivity Analysis for Zero Expected Net Demand Case

For our numerical study, we developed a computer program in MATLAB programming language. We used the nonlinear optimization routines to optimize the objective function (2.1) subject to (2.2) given in Chapter 2. Additional constraints are added to the code regarding the policy parameters and the position of the inventory at the end of the lead-time. As mentioned in Chapter 2, the policy parameters must satisfy the condition $r \leq s < S$. Also, recall that the position of the inventory at the end of the lead-time must satisfy the condition $r < x \leq S$ and is approximated by $r + Q$. Hence, we have three additional constraints given as $s - r \geq 0$, $S - s > 0$ and $S - (r + Q) \geq 0$ which have to be satisfied during the optimization process. However, to prevent the computational errors in case of equalities, the numerical accuracy is set to 10^{-4} . This substitution avoids the optimal policy parameters being too close to each other. The MATLAB Optimization Toolbox function "fmincon', which implements an iterative optimization algorithm, is used by setting the termination tolerance on the function value to 10^{-6}, the maximum number of iterations and the maximum number of function evaluations to 400.

To analyze the sensitivity of the (S, s, r, Q) policy when $\mu = 0$, we design an experiment held in two stages. For both stages, we fix the holding cost, $h = 1$ and the variable purchasing cost per unit $C_o = 0$. Moreover, the dispose trigger level S is assumed to be the warehouse capacity and considered as a given parameter during the optimization.

The objective of the first stage is to observe the effects of different cost arrangements, various service levels and warehouse capacities on the decision variables and the cost rate. Therefore, the lead-time is fixed to an arbitrary value, $L = 5$. The rest of the parameters of this stage is given in Table 6.1.

Parameter	Value
Ordering Cost (K_o)	500, 1000
Fixed Disposal Cost (K_d)	250, 500, 1000, 2000
Variable Disposal Cost (C_d)	5, 10, 25
Service Level (α)	0.95, 0.99, 0.999
Warehouse Capacity (S)	60, 40, 15, 5

Table 6.1: Parameter Set 1

To observe the effects more clearly, we use relatively high ordering and fixed disposal costs; where, the values of K_d are taken as $0.5K_o, K_o$ and $2K_o$.

The results for the first stage are given by Table A.1 and Table A.2.

From the results we observe that as C_d increases, s^* increases to decrease the number of units to be disposed, $(S - s^*)$. Moreover, an increase in K_d causes a decrease in s^* to reduce the expected number of disposals. Also, we observe an increase in s^* as K_o increases. However, s^* is quite insensitive to the changes in K_o,K_d and C_d for small S values (see Table A.2) since the effect of S on the expected number disposal and the number of units to be disposed decreases and Q^* emerges as the dominating factor. Therefore, when $S = 15$, s^* equals to the half of the S and does not change neither with the service level α, nor with the cost parameters. On the other hand, for all other S values, the s^* increases as α increases.

When $\mu = 0$ the left-hand side of the constraint (2.2) is the function of the lead-time and the reorder point, r. Hence, r^* is determined by L and service level α. Since L is fixed in the first stage, r^* increases as α increases to reduce the backorders and does not change as S and/or the cost parameters increase.

Similar intuitive results can be seen for the replenishment quantity, Q. We observe that Q^* increases as K_o increases to obtain more units in each order and decrease the frequency of orders. Moreover, an increase in K_d and/or C_d causes a decrease in Q^* to balance the expected number of disposals together with s^*. As we mentioned above, the reorder point increases as the service level increases. If the capacity of the warehouse is small then the probability of hitting S increases as r^* increases. Therefore, for small S and high α, Q^* approaches zero (see Table A.2). Especially, when $S = 5$ and $\alpha = 0.999$, Q^* equals zero for all cost values since r^* is very close to S. Also, we observe that as α increases, Q^* increases for $S = 60, 40$ and decreases for $S = 5, 15$.

The cost rate, TC, is unimodal in S; it first decreases and then increases as S decreases. The smallest TC^*s are found for $S = 40$. As we expected, we observe an increase in TC^* as the cost parameters and service level increase.

In the second stage, we set the values of the ordering cost and the warehouse capacity as $K_o = 500$ and $S = 40$, in order to analyze the effects of the lead-time over the optimization parameters and the cost rate function. The other parameters are taken according to Table 6.2.

Parameter	Value
Fixed Disposal Cost (K_d)	250, 500, 1000, 2000
Variable Disposal Cost (C_d)	5, 10, 25
Service Level (α)	0.95, 0.99, 0.999
Lead Time (L)	1, 5, 15

Table 6.2: Parameter Set 2

From Table A.3 we observe that when L increases r^* increases to reduce the backorders. Also, there is a slight decrease in Q^* as L increases to decrease the number of disposals caused by high reorder point. Similarly, s^* increases with L in order to adopt itself to the increase in r^*. The results for the cost rate function are also very intuitive. TC^* increases as L increases. Note that, for high service levels, the increase in TC^* is greater.

Below, we give Figure 6.1 to illustrate a typical realization of the cost rate function for the zero net demand case when $S = 40$ and $r^* = 1.85$. Note that, it is a unimodal smooth shaped function for the cases that we encountered.

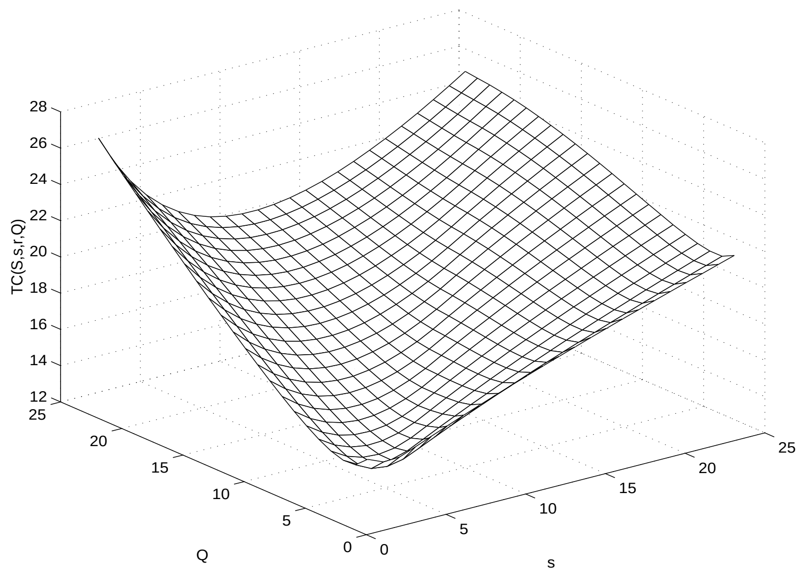

Figure 6.1: Realization of the cost rate function when $\mu = 0$

Finally, in Table A.4, we provide the numerical results for the case when the decision maker does not give any order. Hence, we set Q equal to zero and optimize the policy parameters S, s and r subject to the constraint (2.2).

From the analytical expressions it can be seen that when $Q \to 0$, the cycle length equals L, and the expected on-hand inventory after the lead-time period and the expected number of disposals become zero. There remain only K_o and $hE(OH_L)$ in the expected cycle cost expression. Therefore, from Table A.4, we see that TC^* increases as α and/or K_o increases.

However, our mathematical model is designed for the cases where $Q > 0$. Also we observe that, there is a discontinuity in the cost rate function when $Q = 0$. The limit from the right according to Q does not exist. Moreover, as Q approaches zero, the cost rate function goes to infinity. Therefore, we can state that for our cost rate function do nothing policy is not optimal.

6.2 Sensitivity Analysis for Positive Expected Net Demand Case

The numerical analysis of the negative expected net demand case is similar to the zero expected net demand case. We use the same MATLAB optimization function “fmincon” with the same tolerance, iteration and evaluation settings. Recall that for nonzero μ we substitute x with $(r-|\mu|L+Q)$. Therefore, the two of the additional constraints are found as $S-(r-|\mu|L+Q)\geq 0$ and $Q-|\mu|L\geq 0$. The other additional constraints are the same with the zero expected net demand case and equal to $s-r\geq 0$ and $S-s>0$.

We investigate the sensitivity of (S,s,r,Q) policy when $\mu<0$ according to the cost parameters, the service level, the lead-time and the net demand rate. Throughout the analysis, the fixed parameters are $h=1$ and $C_o=0$. Unlike the previous case, the disposal trigger S is considered as an optimization parameter.

First, we analyze the sensitivity of the optimization parameters and the cost rate function to the changes in the cost parameters and the service level. For this stage, the lead-time and the net demand rate are fixed as $L=15$ and $\mu=-0.01$, respectively. The parameter set of the first stage is given by Table 6.3 and the results are presented in Table A.5.

Parameter	Value
Ordering Cost (K_o)	500, 1000
Fixed Disposal Cost (K_d)	250, 500, 1000, 2000
Variable Disposal Cost (C_d)	5, 10, 25
Service Level (α)	0.95, 0.99, 0.999

Table 6.3: Parameter Set 3

From Table A.5, we observe that S^* generally increases as K_d and C_d increases. Also note that, s^* decreases as K_d increases to decrease the expected number of disposals together with S^*. However, when $\alpha=0.999$, S^* and s^* remain constant as K_d increases . Since a decrease in s^* creates an exponential increase in the denominator of the expected number of disposals, we expect a decrease in s^* as C_d increases. Nevertheless, this intuitive result can be seen only when $K_o = 1000$.

Note that, when the drift is nonzero, the left-side of the constraint (2.2) is the function of the lead-time, the net demand rate and the reorder point. Therefore, r^* is effected only by L, μ and α. Since L and μ are fixed, r^* increases as α increases to reduce the backorders.

As we expected, Q^* increases as K_o increases. For small K_o, Q^* is very insensitive to the changes in the service level and the disposal costs. However, when $K_o = 1000$, Q^* decreases as K_d and C_d increase to reduce the expected number of disposals. Moreover, we observe that as α increases, Q^* first decreases and then increases.

Secondly, we examine the effects of the lead-time over the policy parameters. We studied the case in which $K_o = 500$ and $\mu = -0.01$. We use the parameter set given in Table 6.2. The results of the second stage are presented in Table A.6.

From Table A.6, it can be seen that as L increases, r^* increases to reduce the backorders whereas Q^* decreases to balance the disposals. We do not observe a specific pattern for S^* and s^* as the lead-time increases. Also note that, for small lead-times, S^* and s^* are more sensitive to the changes in K_d and C_d. For all lead-time settings, when $\alpha = 0.999$, we observe that the policy parameters are very insensitive to the disposal costs.

Finally, we study the effects of the net demand rate over the policy parameters. The fixed parameters are $L = 5$ and $K_o = 500$. We analyze the system under the following parameter set.

Parameter	Value
Fixed Disposal Cost (K_d)	250, 500, 1000, 2000
Variable Disposal Cost (C_d)	5, 10, 25
Service Level (α)	0.95, 0.99, 0.999
Net Demand Rate (μ)	-0.01, -0.1, -1

Table 6.4: Parameter Set 4

The results are given in Table A.7. Note that, as net demand rate increases, (μ decreases), the effect of K_d and C_d disappears since the probability of hitting S during a cycle approaches zero. Therefore, the optimization algorithm converges to an arbitrary value according to the initial guesses of S and s. Throughout the numerical analysis, the initial guesses of the policy parameters are taken as $So = 75$, $so = 40$, $ro = 15$ and $Qo = 10$. From Table A.7, it is clear that when $\mu = -1$, S^* and s^* are equal to their initial guesses for any cost and service level setting. Also note that the results for the reorder point and the order quantity are quite intuitive. As μ decreases, r^* increases to decrease the expected number of backorders and Q^* increases to satisfy the demand.

Further results related with the negative net demand case can be found in Tables A.8-A.13.

Figure 6.2 illustrates a typical realization of the cost rate function for the positive net demand case when $S^* = 33.9$ and $r^* = 1.88$. We observe a unimodal smooth shaped cost rate function.

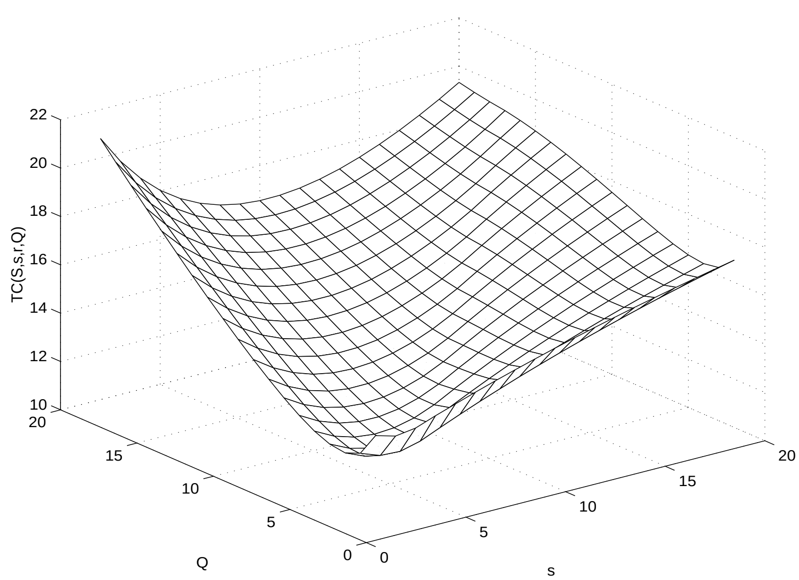

Figure 6.2: Realization of the cost rate function when $\mu = -0.01$

6.3 Performance Analysis for Positive Expected Net Demand Case

In this section, we compare the (S, s, r, Q) policy when $\mu < 0$ with two different policies: 1) The EOQ based heuristic policy , 2) The no-disposal policy. Then, we discuss over the results to highlight the advantages of using the (S, s, r, Q) policy. In the first comparison, we asses the performance of the (S, s, r, Q) policy when it is compared with a very rough approximation based on the well-known EOQ formula. To differentiate our policy parameters from the heuristic policy parameters, we denote the heuristic policy with (S_2, s_2, r_2, Q_2) and rewrite the (S, s, r, Q) policy as (S_1, s_1, r_1, Q_1).

According to the heuristic policy, the order quantity, Q_2 and the disposal quantity, $(S_2 - s_2)$, are approximated by the EOQ formula with the appropriate costs. Then, S_2, s_2 and r_2 are optimized subject to the constraints defined in the previous section.

The heuristic order quantity and the heuristic disposal quantity can be given as follows:

$$Q_2 = \sqrt{\frac{2K_o|\mu|}{h}} \quad \text{and} \quad S_2 - s_2 = \sqrt{\frac{2K_d|\mu|}{h}} \tag{6.1}$$

Note that, the heuristic disposal quantity $(S_2 - s_2)$ is analogous to the order quantity since a disposal order is nothing but a symmetric replenishment order. Therefore, the only difference in $(S_2 - s_2)$ is the fixed disposal cost K_d.

The policies are compared with respect to the parameter set given by Table 6.5.

We define our performance measure as the percentage decrease in the cost rate function obtained by the (S_1, s_1, r_1, Q_1) policy which can be computed as follows:

$$\triangle\,\% = \frac{TC_2(S_2^*, s_2^*, r_2^*, Q_2) - TC_1(S_1^*, s_1^*, r_1^*, Q_1^*)}{TC_2(S_2^*, s_2^*, r_2^*, Q_2)} \times 100 \tag{6.2}$$

Parameter	Value
Ordering Cost (K_o)	500
Fixed Disposal Cost (K_d)	250, 500, 1000
Variable Disposal Cost (C_d)	5, 10, 25
Service Level (α)	0.95, 0.99, 0.999
Lead Time (L)	5
The Net Demand Rate (μ)	-0.01

Table 6.5: Parameter Set 5

The results are presented in Table A.14.

We first observe that $\triangle\%$ values are quite significant for all cost parameters and service levels. The main reason of this observation is that the heuristic policy underestimates the optimal order quantity and the optimal disposal level.

The underestimation of the optimal order quantity follows from the fact that there are no backorders in the *EOQ* formula. Since it is worthwhile to increase the order quantity moderately to reduce the backorders, Q_1^* is always greater than Q_2^*.

Similarly, for all cost and service level settings, S_2^* is smaller than S_1^* and s_2^* is greater than s_1^* resulting in the underestimation of the optimal disposal quantity. This result arises because our policy tends to decrease the frequency of disposals as much as possible. Moreover, *EOQ* based policy is not sensitive enough to the increases in disposal costs.

Also, note that r^* is same for both policies since it is determined by the binding constraint (2.2).

From Table A.14, we note that as K_d increases, $(S_1^* - s_1^*)$ increases and Q_1^* decreases to reduce the number of disposals. On the other hand, for the heuristic policy, since Q_2^* is fixed and the increase in $(S_2^* - s_2^*)$ is not enough to decrease the number of disposals, TC_2^* increases more than TC_1^*. Therefore, $\triangle\%$ increases with K_d.

The (S_1, s_1, r_1, Q_1) policy responds similarly to the changes in C_d as it does for the changes in K_d. However, when C_d increases, the increase in TC_1^* is greater than the increase in TC_2^* causing a slight decrease in $\triangle\%$. Also, we observe that $\triangle\%$ decreases as α increases.

Finally, we find that for higher net demand rates, $\triangle\%$ first decreases and than increases. Also, we observe that the affect of the disposal option disappears for high net demand rates. For this case, $\triangle\%$ values are very insensitive to the changes in the disposal costs (see Table A.15 and Table A.16).

In the second comparison, we aim to highlight the importance of the disposal option. Therefore, we propose the no disposal policy in which S and s are fixed. When we set S and s to very high values, the probability of giving a disposal order in a cycle approaches to zero. Consequently, the affect of the policy parameters S and s over the cost rate function becomes negligible and the (S, s, r, Q) policy behaves as the conventional (Q, r) policy. We denote the no disposal policy by (S_3, s_3, r_3, Q_3) and compare it with the (S_1, s_1, r_1, Q_1) policy. Throughout the analysis, we fix $S_3 = 200$ and $s_3 = 190$, then optimize the r_3 and Q_3 respectively.

The rest of the parameters used in the analysis are given by Table 6.5.

Similar to the previous comparison, we can give the performance measure as follows:

$$\triangle\,\% = \frac{TC_3(S_3, s_3, r_3^*, Q_3^*) - TC_1(S_1^*, s_1^*, r_1^*, Q_1^*)}{TC_3(S_3, s_3, r_3^*, Q_3^*)} \times 100 \qquad (6.3)$$

The results of the numerical comparison are presented in Table A.17.

From Table A.17, we observe that the improvement in the cost rate function obtained by the (S_1, s_1, r_1, Q_1) policy is quite significant for all cost and service level settings. As we expected, the (S_3, s_3, r_3, Q_3) policy is very insensitive to the changes in the disposal costs since the affect of the disposal parameters becomes negligible due to fixed S_3 and s_3. Note that, the behavior of the (S_3, s_3, r_3, Q_3) policy is same with the the typical (Q, r) policy in which r_3^* increases as α increases to reduce the backorders whereas Q_3^* decreases as backorders decrease.

Moreover, it can be seen from Table A.17 that the (S_3, s_3, r_3, Q_3) policy underestimates the optimal order quantity causing a greater cost rate function.

Also, it is clear that r^* is same for both policies due to the binding constraint of the objective function.

From Table A.17, we can state that as K_d and/or C_d increases, generally, $\triangle\%$ decreases since TC_3^* remains constant while TC_1^* increases. However, for $\alpha = 0.999$, TC_3^* increases slightly while TC_1^* remains constant as K_d increases. Hence, for this service level, $\triangle\%$ increases somewhat as K_d increases. Finally, we observe a small decrease in $\triangle\%$ as α increases.

Chapter 7

Conclusion

In this thesis, we develop and analyze a control policy, the (S, s, r, Q) policy, for a continuous review inventory system of recyclable goods with a disposal option. We assume that the return and demand flows are independent and the inventory process is governed by the Brownian motion process. Under the fixed procurement lead-time and backordering assumptions, we derive the expressions of the operating characteristics for the cases where the net demand rate is zero and positive. Then, we optimize the cost rate function with respect to the service level criterion.

The existing reverse logistics literature includes mostly the models considering distinct inventories for serviceables and recoverables. These models are mathematically quite complex and consider the systems including sophisticated recovery options. However, the (S, s, r, Q) policy proposed in this study is designed for simpler systems applying recycling or direct reuse. Therefore, we are able to provide the analytical cost rate expressions using very general simplifying assumptions employed commonly in the inventory theory literature. On the other hand, it is the first model in the literature of the inventory management of recyclable items using the Brownian motion to define the inventory level process. From this point of view, it can be seen as the extension of the cash management models to the positive lead-time case.

An interesting extension of the (S, s, r, Q) policy is found when the average number of returns is equal to the average number of demands. This is the case where the expected net demand is equal to zero. Although the zero net demand case seems unrealistic, we give the analysis of this case to provide an insight to the special case of a more general problem. From the numerical study, we conclude that, when $\mu = 0$, no disposal option is not optimal. However, it would be interesting to compare this result against simulation.

For more realistic systems where the demand rate is greater than the return rate, we conclude that using the (S, s, r, Q) policy is meaningful when the $|\mu|/\sigma$ is small ($\sigma = 1$). This result is quite intuitive since for high demand flows the significance of the disposal option disappears and the system behaves as the typical (Q, r) model. However, for the systems where the demand rate is somewhat greater than the return rate (i.e. $\mu = -0.01$), the performance comparison with a similar model to (Q, r) shows that the (S, s, r, Q) policy outperforms that policy in all experimental points with the mean improvement of 29.7% in the cost rate function. The maximum improvements (i.e. 32.38%) in this experiment are observed when the service level is equal to 95%. Moreover, we also see that when the order and disposal quantities are approximated by the *EOQ* formula, the system incurs higher number of disposals and backorders. The numerical results show that, of all the experimental points tested for different net demand rates, the performance of the (S, s, r, Q) policy outperforms the *EOQ* based policy with the mean improvement of 12.64%. The maximum improvements (i.e. 37.94%) in this experiment are observed when $\mu = -0.01$.

Although the (S, s, r, Q) policy provides some nice results for the positive net demand case, the analytical expression of the cost rate function seems quite tedious. To eliminate this mathematical complexity, one extension may be to apply the Taylor approximation to the exponential terms in the cost rate function. Finally, we can state that an interesting suggestion for future research may be the incorporation of the nonlinear cost structure into the model.

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Appendix

A.1 Appendix A

Proof of Lemma 2.1

$i)$ In the following derivations, for any random variable T, we use the notation $T_{:1}$, $T_{:2}$, ... to denote a sequence of independent random variables identical to T,

So that, $OH_{s:1}$, $OH_{s:2}$, ... will denote independent random variables identical to OH_s. Similarly, $T_{u,v:1}$, $T_{u,v:2}$, ... are independent and identical to $T_{u,v}$ and $T^z_{u,v:1}$, $T^z_{u,v:2}$, ... are independent and identical to $T^z_{u,v}$. Then, the on-hand inventory carried during T_s can be written as:

$$OH_{s:1} = \begin{cases} \frac{(s+r)}{2}\, T^S_{s,r:1} & \text{if} \quad T_{s,r:1} < T_{s,S:1} \\ \frac{(s+S)}{2}\, T^r_{s,S:1} + OH_{s:2} & \text{if} \quad T_{s,S:1} < T_{s,r:1} \end{cases}$$

$$OH_{s:2} = \begin{cases} \frac{(s+r)}{2}\, T^S_{s,r:2} & \text{if} \quad T_{s,r:2} < T_{s,S:2} \\ \frac{(s+S)}{2}\, T^r_{s,S:2} + OH_{s:3} & \text{if} \quad T_{s,S:2} < T_{s,r:2} \end{cases}$$

$$\vdots$$

According to the stochastic equations given above we can write

$$\begin{aligned} E(OH_{s:1}) &= E\left[\frac{(s+r)}{2}\, T^S_{s,r:1}\, I(T_{s,r:1} < T_{s,S:1})\right] \\ &+ E\left[\left(\frac{(s+S)}{2}\, T^r_{s,S:1} + OH_{s:2}\right) I(T_{s,S:1} < T_{s,r:1})\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{(s+r)}{2}E(T^S_{s,r:1}\ I(T_{s,r:1} < T_{s,S:1})) + \frac{(s+S)}{2}E(T^r_{s,S:1}\ I(T_{s,S:1} < T_{s,r:1})) \\
&+ E(OH_{s:2})E(I(T_{s,S:1} < T_{s,r:1}))
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(s+r)}{2}E(T^S_{s,r:1}I(T_{s,r:1} < T_{s,S:1})) + \frac{(s+S)}{2}E(T^r_{s,S:1}I(T_{s,S:1} < T_{s,r:1})) \\
&+ E(OH_{s:2})P(T_{s,S:1} < T_{s,r:1})
\end{aligned}
$$

and in general

$$
\begin{aligned}
E(OH_{s:i}) &= \frac{(s+r)}{2}E(T^S_{s,r:i}I(T_{s,r:i} < T_{s,S:i})) + \frac{(s+S)}{2}E(T^r_{s,S:i}I(T_{s,S:i} < T_{s,r:i})) \\
&+ E(OH_{s:(i+1)})P(T_{s,S:i} < T_{s,r:i})
\end{aligned}
$$

Since for $i = 1, 2, \ldots$

$$
\begin{aligned}
P(T_{s,S:i} < T_{s,r:i}) &= P^r_{s,S} \\
E(T^S_{s,r:i}I(T_{s,r:i} < T_{s,S:i})) &= E(T^S_{s,r}I(T_{s,r} < T_{s,S})) \\
E(T^r_{s,S:i}I(T_{s,S:i} < T_{s,r:i})) &= E(T^r_{s,S}I(T_{s,S} < T_{s,r}))
\end{aligned}
$$

We let

$$
W = \frac{(s+r)}{2}E(T^S_{s,r}\ I(T_{s,r} < T_{s,S})) + \frac{(s+S)}{2}E(T^r_{s,S}\ I(T_{s,r} < T_{s,S}))
$$

If we substitute each equation in the previous one, we obtain

$$
\begin{aligned}
E(OH_{s:1}) &= W + \left[W + \left[W + [\ldots].P^r_{s,S}\right].P^r_{s,S}\right].P^r_{s,S} \\
&= W.\left[1 + P^r_{s,S} + (P^r_{s,S})^2 + (P^r_{s,S})^3 + \ldots\right] \\
&= \frac{W}{1 - P^r_{s,S}} \\
&= \frac{1}{P^S_{s,r}}\left[\frac{(s+r)}{2}E(T^S_{s,r}\ I(T_{s,r} < T_{s,S})) + \frac{(s+S)}{2}E(T^r_{s,S}\ I(T_{s,S} < T_{s,r}))\right]
\end{aligned}
$$

ii) From the relation (2.5) the expected on-hand inventory carried after the lead-time can be written as:

$$
\begin{aligned}
E(OH_x) &= \frac{(x+r)}{2}E(T^s_{x,r}\ I(T_{x,r} < T_{x,s})) + \frac{(x+s)}{2}E(T^r_{x,s}\ I(T_{x,s} < T_{x,r})) \\
&+ E(OH_s).E(I(T_{x,s} < T_{x,r}))
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(x+r)}{2} E(T^{s}_{x,r}\; I(T_{x,r} < T_{x,s})) + \frac{(x+s)}{2} E(T^{r}_{x,s}\; I(T_{x,s} < T_{x,r})) \\
&+ E(OH_s).P^{r}_{x,s}
\end{aligned}
$$

Q.E.D.

Proof of Lemma 2.3

i) As before, let $T_{u,vz:1}$, $T_{u,vz:2}$, ... be independent and identical to $T_{u,vz}$; and $T_{u:1}$, $T_{u:2}$, ... be independent and identical to T_u. Then, T_s can be written as follows:

$$
T_{s:1} = \begin{cases} T^{S}_{s,r:1} & \text{if} \quad T_{s,r:1} < T_{s,S:1} \\ T^{r}_{s,S:1} + T_{s:2} & \text{if} \quad T_{s,S:1} < T_{s,r:1} \end{cases}
$$

$$
T_{s:2} = \begin{cases} T^{S}_{s,r:2} & \text{if} \quad T_{s,r:2} < T_{s,S:2} \\ T^{r}_{s,S:2} + T_{s:3} & \text{if} \quad T_{s,S:2} < T_{s,r:2} \end{cases}
$$

$$
\vdots
$$

Thus, according to the relations given above we can write:

$$
\begin{aligned}
E(T_{s:1}) &= E(T^{S}_{s,r:1}\; I(T_{s,r:1} < T_{s,S:1})) + E\left[(T^{r}_{s,S:1} + T_{s,r:2})\; I(T_{s,S:1} < T_{s,r:1})\right] \\
&= E(T^{S}_{s,r:1}\; I(T_{s,r:1} < T_{s,S:1})) + E(T^{r}_{s,S:1}\; I(T_{s,S:1} < T_{s,r:1})) \\
&+ E(T_{s:2}).E(I(T_{s,S:1} < T_{s,r:1})) \\
&= E(T_{s,rS:1}) + E(T_{s:2}).P(T_{s,S:1} < T_{s,r:1})
\end{aligned}
$$

in general for $i = 1, 2, \ldots$

$$
E(T_{s:i}) = E(T_{s,rS:i}) + E(T_{s:(i+1)}).P(T_{s,S:i} < T_{s,r:i})
$$

Similarly,

$$E(T_{s,rS:i}) = E(T_{s,rS})$$

Then, we have

$$\begin{aligned} E(T_{s:1}) &= E(T_{s,rS}) + \left[E(T_{s,rS}) + \left[E(T_{s,rS}) + [\ldots] . P^r_{s,S}\right] . P^r_{s,S}\right] . P^r_{s,S} \\ &= E(T_{s,rS}) . \left[1 + P^r_{s,S} + (P^r_{s,S})^2 + (P^r_{s,S})^3 + \ldots\right] \\ &= \frac{E(T_{s,rS})}{1 - P^r_{s,S}} \\ &= \frac{E(T_{s,rS})}{P^S_{s,r}} \end{aligned}$$

ii) From (2.3) the expected length of T_x can be written as:

$$\begin{aligned} E(T_x) &= E(T^s_{x,r}\, I(T_{x,r} < T_{x,s})) + E(T^r_{x,s}\, I(T_{x,s} < T_{x,r})) \\ &+ E(T_s).E(I(T_{x,s} < T_{x,r})) \\ &= E(T_{x,rs}) + E(T_s).P^r_{x,s} \end{aligned}$$

Q.E.D.

A.2 Appendix B

Proof of Proposition 3.1

i) From Karatzas and Shreve [12], we have the following Moment Generating Functions (M.G.F.) for a Brownian motion moving in strip $[0, a]$.

$$\phi_{-T^0_{x,a}}(\alpha) = E(e^{-\alpha T^0_{x,a}} I(T_{x,a} < T_{x,0})) = \frac{e^{x\sqrt{2\alpha}} - e^{-x\sqrt{2\alpha}}}{e^{a\sqrt{2\alpha}} - e^{-a\sqrt{2\alpha}}} \quad \text{(A.2.1)}$$

$$\phi_{-T^a_{x,0}}(\alpha) = E(e^{-\alpha T^a_{x,0}} I(T_{x,0} < T_{x,a})) = \frac{e^{(a-x)\sqrt{2\alpha}} - e^{-(a-x)\sqrt{2\alpha}}}{e^{a\sqrt{2\alpha}} - e^{-a\sqrt{2\alpha}}} \quad \text{(A.2.2)}$$

Observe that

$$\lim_{\alpha\to 0} E(e^{-\alpha T^0_{x,a}} I(T_{x,a} < T_{x,0})) = P(T_{x,a} < T_{x,0}) = P^0_{x,a}$$
$$\lim_{\alpha\to 0} E(e^{-\alpha T^a_{x,0}} I(T_{x,0} < T_{x,a})) = P(T_{x,0} < T_{x,a}) = P^a_{x,0}$$

Hence,

$$P^0_{x,a} = \lim_{\alpha\to 0} \frac{e^{x\sqrt{2\alpha}} - e^{-x\sqrt{2\alpha}}}{e^{a\sqrt{2\alpha}} - e^{-a\sqrt{2\alpha}}} = \frac{0}{0}$$

If we apply L'Hôpital's rule by taking derivatives, we find

$$P^0_{x,a} = \lim_{\alpha\to 0} \frac{\frac{x}{\sqrt{2\alpha}}(e^{x\sqrt{2\alpha}} + e^{-x\sqrt{2\alpha}})}{\frac{a}{\sqrt{2\alpha}}(e^{a\sqrt{2\alpha}} + e^{-a\sqrt{2\alpha}})} = \frac{x}{a}$$

Similarly,

$$P^a_{x,0} = \lim_{\alpha\to 0} \frac{e^{(a-x)\sqrt{2\alpha}} - e^{-(a-x)\sqrt{2\alpha}}}{e^{a\sqrt{2\alpha}} - e^{-a\sqrt{2\alpha}}} = \frac{0}{0}$$
$$= \lim_{\alpha\to 0} \frac{\frac{(a-x)}{\sqrt{2\alpha}}(e^{(a-x)\sqrt{2\alpha}} + e^{-(a-x)\sqrt{2\alpha}})}{\frac{a}{\sqrt{2\alpha}}(e^{a\sqrt{2\alpha}} + e^{-a\sqrt{2\alpha}})} = \frac{a-x}{a}$$

If we convert the strip from $[0, a]$ to $[b, a]$ we obtain

$$P^b_{x,a} = \frac{x-b}{a-b} \qquad \text{and} \qquad P^a_{x,b} = \frac{a-x}{a-b}$$

ii) To derive the $E(T^0_{x,a}\, I(T_{x,a} < T_{x,0}))$ and $E(T^a_{x,0}\, I(T_{x,0} < T_{x,a}))$, we employ the M.G.F property

$$\begin{aligned}
\phi'_{-T^0_{x,a}}(0) &= E(-T^0_{x,a}\, I(T_{x,a} < T_{x,0})) \\
\phi'_{-T^a_{x,0}}(0) &= E(-T^a_{x,0}\, I(T_{x,0} < T_{x,a}))
\end{aligned}$$

We can write the first derivative of (A.2.1) with respect to α as follows:

$$\begin{aligned}
\phi'_{-T^0_{x,a}}(\alpha) &= \frac{\frac{x}{\sqrt{2\alpha}}(e^{x\sqrt{2\alpha}} + e^{-x\sqrt{2\alpha}})(e^{a\sqrt{2\alpha}} - e^{-a\sqrt{2\alpha}})}{\left(e^{a\sqrt{2\alpha}} - e^{-a\sqrt{2\alpha}}\right)^2} \\
&- \frac{\frac{a}{\sqrt{2\alpha}}(e^{x\sqrt{2\alpha}} - e^{-x\sqrt{2\alpha}})(e^{a\sqrt{2\alpha}} + e^{-a\sqrt{2\alpha}})}{\left(e^{a\sqrt{2\alpha}} - e^{-a\sqrt{2\alpha}}\right)^2} \\
&= \frac{(x-a)\left[e^{(x+a)\sqrt{2\alpha}} - e^{-(x+a)\sqrt{2\alpha}}\right] - (x+a)\left[e^{(x-a)\sqrt{2\alpha}} - e^{-(x-a)\sqrt{2\alpha}}\right]}{\left(e^{a\sqrt{2\alpha}} - e^{-a\sqrt{2\alpha}}\right)^2 \sqrt{2\alpha}} \\
\phi'_{-T^0_{x,a}}(0) &= \frac{0}{0}
\end{aligned}$$

Using the L'Hôpital's rule, we find

$$\begin{aligned}
\phi'_{-T^0_{x,a}}(0) &= \lim_{\alpha\to 0}\left[\frac{(x^2-a^2)\left[e^{(x+a)\sqrt{2\alpha}} + e^{-(x+a)\sqrt{2\alpha}}\right]}{2a\left(e^{2a\sqrt{2\alpha}} - e^{-2a\sqrt{2\alpha}}\right)\sqrt{2\alpha} + \left(e^{a\sqrt{2\alpha}} - e^{-a\sqrt{2\alpha}}\right)^2}\right. \\
&\left. - \frac{(x^2-a^2)\left[e^{(x-a)\sqrt{2\alpha}} + e^{-(x-a)\sqrt{2\alpha}}\right]}{2a\left(e^{2a\sqrt{2\alpha}} - e^{-2a\sqrt{2\alpha}}\right)\sqrt{2\alpha} + \left(e^{a\sqrt{2\alpha}} - e^{-a\sqrt{2\alpha}}\right)^2}\right] \\
&= \frac{0}{0} \\
&= \lim_{\alpha\to 0}\left[\frac{(x^2-a^2)(x+a)\left[e^{(x+a)\sqrt{2\alpha}} - e^{-(x+a)\sqrt{2\alpha}}\right]}{4a^2\left(e^{2a\sqrt{2\alpha}} + e^{-2a\sqrt{2\alpha}}\right)\sqrt{2\alpha} + 4a\left(e^{2a\sqrt{2\alpha}} - e^{-2a\sqrt{2\alpha}}\right)}\right. \\
&\left. - \frac{(x^2-a^2)(x-a)\left[e^{(x-a)\sqrt{2\alpha}} - e^{-(x-a)\sqrt{2\alpha}}\right]}{4a^2\left(e^{2a\sqrt{2\alpha}} + e^{-2a\sqrt{2\alpha}}\right)\sqrt{2\alpha} + 4a\left(e^{2a\sqrt{2\alpha}} - e^{-2a\sqrt{2\alpha}}\right)}\right] \\
&= \frac{0}{0}
\end{aligned}$$

$$= \lim_{\alpha\to 0}\left[\frac{(x^2-a^2)(x+a)^2\left[e^{(x+a)\sqrt{2\alpha}}+e^{-(x+a)\sqrt{2\alpha}}\right]}{8a^3\left(e^{2a\sqrt{2\alpha}}-e^{-2a\sqrt{2\alpha}}\right)\sqrt{2\alpha}+12a^2\left(e^{2a\sqrt{2\alpha}}+e^{-2a\sqrt{2\alpha}}\right)} - \frac{(x^2-a^2)(x-a)^2\left[e^{(x-a)\sqrt{2\alpha}}+e^{-(x-a)\sqrt{2\alpha}}\right]}{8a^3\left(e^{2a\sqrt{2\alpha}}-e^{-2a\sqrt{2\alpha}}\right)\sqrt{2\alpha}+12a^2\left(e^{2a\sqrt{2\alpha}}+e^{-2a\sqrt{2\alpha}}\right)}\right]$$

$$= \frac{(x^2-a^2)\left[(x+a)^2-(x-a)^2\right]}{12a^2}$$

$$= \frac{x(x^2-a^2)}{3a} = E(-T^0_{x,a}\ I(T_{x,a} < T_{x,0}))$$

Hence,

$$E(T^0_{x,a}\ I(T_{x,a} < T_{x,0})) = \frac{x(a^2-x^2)}{3a}$$

When we write this equation for the $[b,a]$ strip, we get

$$E(T^b_{x,a}\ I(T_{x,a} < T_{x,b})) = \frac{(x-b)(a+x-2b)(a-x))}{3(a-b)}$$

Similarly, the first derivative of (A.2.2) with respect to α can be written as follows:

$$\phi'_{-T^a_{x,0}}(\alpha) = \frac{\frac{(a-x)}{\sqrt{2\alpha}}(e^{(a-x)\sqrt{2\alpha}}+e^{-(a-x)\sqrt{2\alpha}})(e^{a\sqrt{2\alpha}}-e^{-a\sqrt{2\alpha}})}{\left(e^{a\sqrt{2\alpha}}-e^{-a\sqrt{2\alpha}}\right)^2} - \frac{\frac{a}{\sqrt{2\alpha}}(e^{(a-x)\sqrt{2\alpha}}-e^{-(a-x)\sqrt{2\alpha}})(e^{a\sqrt{2\alpha}}+e^{-a\sqrt{2\alpha}})}{\left(e^{a\sqrt{2\alpha}}-e^{-a\sqrt{2\alpha}}\right)^2}$$

$$= \frac{(2a-x)\left[e^{x\sqrt{2\alpha}}-e^{-x\sqrt{2\alpha}}\right]-x\left[e^{(2a-x)\sqrt{2\alpha}}-e^{-(2a-x)\sqrt{2\alpha}}\right]}{\left(e^{a\sqrt{2\alpha}}-e^{-a\sqrt{2\alpha}}\right)^2\sqrt{2\alpha}}$$

$$\phi'_{-T^a_{x,0}}(0) = \frac{0}{0}$$

From the L'Hôpital's rule, we find

$$\phi'_{-T^a_{x,0}}(0) = \lim_{\alpha\to 0}\left[\frac{x(2a-x)\left[e^{x\sqrt{2\alpha}}+e^{-x\sqrt{2\alpha}}\right]}{2a\left(e^{2a\sqrt{2\alpha}}-e^{-2a\sqrt{2\alpha}}\right)\sqrt{2\alpha}+\left(e^{a\sqrt{2\alpha}}-e^{-a\sqrt{2\alpha}}\right)^2} - \frac{x(2a-x)\left[e^{(2a-x)\sqrt{2\alpha}}+e^{-(2a-x)\sqrt{2\alpha}}\right]}{2a\left(e^{2a\sqrt{2\alpha}}-e^{-2a\sqrt{2\alpha}}\right)\sqrt{2\alpha}+\left(e^{a\sqrt{2\alpha}}-e^{-a\sqrt{2\alpha}}\right)^2}\right]$$

$$
\begin{aligned}
&= \frac{0}{0} \\
&= \lim_{\alpha\to 0}\left[\frac{x^2(2a-x)\left[e^{x\sqrt{2\alpha}}-e^{-x\sqrt{2\alpha}}\right]}{4a^2\left(e^{2a\sqrt{2\alpha}}+e^{-2a\sqrt{2\alpha}}\right)\sqrt{2\alpha}+4a\left(e^{2a\sqrt{2\alpha}}-e^{-2a\sqrt{2\alpha}}\right)}\right. \\
&\quad \left.-\frac{x(2a-x)^2\left[e^{(2a-x)\sqrt{2\alpha}}-e^{-(2a-x)\sqrt{2\alpha}}\right]}{4a^2\left(e^{2a\sqrt{2\alpha}}+e^{-2a\sqrt{2\alpha}}\right)\sqrt{2\alpha}+4a\left(e^{2a\sqrt{2\alpha}}-e^{-2a\sqrt{2\alpha}}\right)}\right] \\
&= \frac{0}{0} \\
&= \lim_{\alpha\to 0}\left[\frac{x^3(2a-x)\left[e^{x\sqrt{2\alpha}}+e^{-x\sqrt{2\alpha}}\right]}{8a^3\left(e^{2a\sqrt{2\alpha}}-e^{-2a\sqrt{2\alpha}}\right)\sqrt{2\alpha}+12a^2\left(e^{2a\sqrt{2\alpha}}+e^{-2a\sqrt{2\alpha}}\right)}\right. \\
&\quad \left.-\frac{x(2a-x)^3\left[e^{(2a-x)\sqrt{2\alpha}}+e^{-(2a-x)\sqrt{2\alpha}}\right]}{8a^3\left(e^{2a\sqrt{2\alpha}}-e^{-2a\sqrt{2\alpha}}\right)\sqrt{2\alpha}+12a^2\left(e^{2a\sqrt{2\alpha}}+e^{-2a\sqrt{2\alpha}}\right)}\right] \\
&= \frac{x(2a-x)\left[x^2-(2a-x)^2\right]}{12a^2} \\
&= -\frac{x(2a-x)(a-x)}{3a} = E(-T^a_{x,0}\, I(T_{x,0} < T_{x,a}))
\end{aligned}
$$

Therefore,

$$
E(T^a_{x,0}\, I(T_{x,0} < T_{x,a})) = \frac{x(2a-x)(a-x)}{3a}
$$

For the $[b, a]$ strip, we have

$$
E(T^a_{x,b}\, I(T_{x,b} < T_{x,a})) = \frac{(x-b)(2a-x-b)(a-x)}{3(a-b)}
$$

First escaping time from strip $[0, a]$, $E(T_{x,0a})$ can be found by

$$
\begin{aligned}
E(T_{x,0a}) &= E(T^0_{x,a}\, I(T_{x,a} < T_{x,0})) + E(T^a_{x,0}\, I(T_{x,0} < T_{x,a})) \\
&= \frac{x(a^2-x^2)+x(2a-x)(a-x)}{3a} \\
&= x(a-x)
\end{aligned}
$$

iii) Using the same conversion we found

$$
E(T_{x,ba}) = (x-b)(a-x) \qquad Q.E.D.
$$

Proof of Proposition 3.2

Let $X_t = \mu t + B_t$, $t \geq 0$, be a Brownian motion with constant drift $\mu \in$ R , on some probability space (Ω, F, P). Fix some interval $[b, a]$, $-\infty < b < a < +\infty$, and calculate

$$f(x) \triangleq E_x[T_a I_{\{T_a < T_b\}}], \quad x \in [b, a]$$

where $T_r \triangleq inf t \geq 0 : X_t = r$.

We shall heuristically derive a differential equation which is satisfied by $f(.)$. Then we shall solve it for $f(.)$. Can argue that

$$\left\{ \begin{array}{rcl} E[X_{t+h} - X_t | X_t = x] & = & \mu h + o(h) \\ E[(X_{t+h} - X_t)^2 | X_t = x] & = & h + o(h) \\ E[(X_{t+h} - X_t)^3 | X_t = x] & = & o(h), \quad k \geq 3 \end{array} \right\} \tag{A.2.3}$$

We apply Taylor's expension to $f(.)$ (assuming $f(.)$ is smooth enough):

$$f(X_h) = f(X_0) + f^{'}(X_0)(X_h - X_0) + \frac{1}{2} f^{''}(X_0)(X_h - X_0)^2 + \cdots$$

and take conditional expectation, given $X_0 = x \in (b, a)$, of both sides. Using (A.2.3), we find

$$E_x[f(X_h)] = f(x) + \left[\mu f^{'}(x) + \frac{1}{2} f^{''}(x) \right] h + o(h) \tag{A.2.4}$$

Next calculate $E_x[f(X_h)]$. Suppose $h > 0$ is so small that $P_x(T_a \wedge T_b < h)$ is negligible, i.e. the process does not leave the interval $[b, a]$ by time h with high probability.

Let $\theta_h : \Omega \rightarrow \Omega$ be the shift-operator, defined by $X(t)(\theta_h(\omega)) \triangleq X(t + h)(\omega)$, $t \geq 0$. Geometrically, the shift-operator "shifts" the time origin h time units into the future, and cuts off the history of path before h. The Markov property is often expressed by using shift-operators:

$$\begin{aligned} & E[(g_1(X_{t_1}) \ldots g_k(X_{t_k})) \circ \theta_h | X_s, 0 \leq s \leq h] \\ & \qquad = \quad E[(g_1(X_{t_1}) \ldots g_k(X_{t_k}))], P_{X_h} - \text{a.s.} \end{aligned} \tag{A.2.5}$$

for every $k \geq 1$, any bounded Borel functions $g_1, \ldots, g_k$, and $0 \leq t_1 < \cdots < t_k$. By the Markov property of X, we have

$$f(X_h) = E_{X_h}[T_a I_{\{T_a<T_b\}}] = E[(T_a I_{\{T_a<T_b\}}) \circ \theta_h | X(s), 0 \leq s \leq h] \qquad \text{(A.2.6)}$$

(It can be shown that $T_a I_{\{T_a<T_b\}}$ is $\sigma(X_t : t \geq 0)$-measurable. Therefore it is the limit of random variables in the form of $g_1(X_{t_1}) \cdots g_k(X_{t_k})$).
Take any $\omega \in \Omega$ such that $X_s(\omega) \in [a, b]$ for all $0 \leq s \leq h$. Then $T_a(\omega) = h + T_a(\theta_h(\omega))$, and $T_b(\omega) = h + T_b(\theta_h(\omega))$. Therefore,

$$T_a I_{\{T_a<T_b\}} \circ \theta_h(\omega) = (T_a - h) I_{\{T_a-h<T_b-h\}}(\omega) = (T_a - h) I_{\{T_a<T_b\}}(\omega).$$

By plugging this into (A.2.6), and taking conditional expectation of both sides given $X_0 = x$, we find

$$E_x[f(X_h)] = E_x[(T_a - h) I_{\{T_a<T_b\}}] + o(h) = f(x) - h P_x(T_a < T_b) + o(h).$$

Recall that h is chosen so that $P_x(T_a \wedge T_b < h)$ is negligible. Finally, together with (A.2.4), we obtain

$$-h P_x(T_a < T_b) \;\; = \;\; \left[\mu f_1^{'}(x) + \frac{1}{2} f_1^{''}(x)\right] h + o(h)$$

When we divide both sides by h, and let h go to zero, we see that $f(.)$ must satisfy the second order differential equation

$$\frac{1}{2} f^{''}(x) + \mu f^{'}(x) \;\; = \;\; -P_x(T_a < T_b), \qquad x \in (a, b) \qquad \text{(A.2.7)}$$

together with boundary conditions $f(a) = f(b) = 0$.
By a similar heuristic argument, one can show that $g(x) \triangleq P_x(T_a < T_b)$, $x \in [b, a]$, is the solution of the differential equation

$$\frac{1}{2} g^{''}(x) + \mu g^{'}(x) \;\; = \;\; 0, \qquad x \in (a, b) \qquad \text{(A.2.8)}$$

with the boundary conditions $g(a) = 1$ and $g(b) = 0$.

i) Let $g_1(x) = P_x(T_a < T_b) = P^b_{x,a}$ and $g_2(x) = P_x(T_b < T_a) = P^a_{x,b}$.

Clearly, $g_1(x)$ can be found by solving the differential equation (A.2.8). Therefore, we find

$$P^b_{x,a} \quad = \quad \frac{e^{-2\mu x} - e^{-2\mu b}}{e^{-2\mu a} - e^{-2\mu b}} \tag{A.2.9}$$

Since $g_2(x) = 1 - g_1(x)$

$$P^a_{x,b} = \frac{e^{-2\mu a} - e^{-2\mu x}}{e^{-2\mu a} - e^{-2\mu b}} \tag{A.2.10}$$

ii) Let $f_1(x) = E_x[T_a I_{\{T_a < T_b\}}] = E(T^b_{x,a}\ I(T_{x,a} < T_{x,b}))$.

Therefore, by substituting (A.2.9) in (A.2.7), we obtain

$$\frac{1}{2}f_1''(x) + \mu f_1'(x) \quad = \quad -\frac{e^{-2\mu x} - e^{-2\mu b}}{e^{-2\mu a} - e^{-2\mu b}} \qquad x \in (a,b) \tag{A.2.11}$$

Let

$$p_1 \quad = \quad -\frac{1}{e^{-2\mu a} - e^{-2\mu b}}$$

$$q_1 \quad = \quad \frac{e^{-2\mu b}}{e^{-2\mu a} - e^{-2\mu b}}$$

Rewrite (A.2.11)

$$f_1''(x) + 2\mu f_1'(x) \quad = \quad 2p_1 e^{-2\mu x} + 2q_1 \tag{A.2.12}$$

Denote $r = f_1'(x)$, $r^2 = f_1''(x)$ and $F(x) = 2p_1 e^{-2\mu x} + 2q_1$. Hence, we can write the reduced equation when $F(x) = 0$ as follows

$$r^2 + 2\mu r \quad = \quad 0$$

The roots of this equation are $r_1 = -2\mu$ and $r_2 = 0$. Therefore, the general solution of the homogeneous equation can be given as

$$y_h \quad = \quad C_1 e^{-2x} + C_2$$

Using the variation of parameters method, we obtain the functions

$$u_1 = e^{-2x}, \qquad u_2 = 1$$

$$v_1 = -\frac{p_1}{\mu}\, x - \frac{q_1}{2\,\mu^2}\, e^{2\mu x} + C_1$$

$$v_2 = \frac{q_1}{\mu}\, x - \frac{p_1}{2\,\mu^2}\, e^{-2\mu x} + C_2$$

If we substitute these functions in the general solution of the form $y = v_1 u_1 + v_2 u_2$, we get

$$f_1(x) = -p_1 \frac{x}{\mu}\, e^{-2\mu x} + q_1 \frac{x}{\mu} + C_1\, e^{-2\mu x} + C_2 \qquad \text{(A.2.13)}$$

Since $f_1(a) = f_1(b) = 0$

$$f_1(a) = -p_1 \frac{a}{\mu}\, e^{-2\mu a} + q_1 \frac{a}{\mu} + C_1\, e^{-2\mu a} + C_2 = 0 \qquad \text{(A.2.14)}$$

$$f_1(b) = -p_1 \frac{b}{\mu}\, e^{-2\mu b} + q_1 \frac{b}{\mu} + C_1\, e^{-2\mu b} + C_2 = 0 \qquad \text{(A.2.15)}$$

Side by side summation of (A.2.14) and (A.2.15) yields

$$C_1 = \frac{p_1[a\, e^{-2\mu a} - b\, e^{-2\mu b}] - q_1(a-b)}{\mu\,(e^{-2\mu a} - e^{-2\mu b})}$$

$$C_2 = \frac{b\, e^{-2\mu a}[p_1\, e^{-2\mu b} - q_1] - a\, e^{-2\mu b}[p_1\, e^{-2\mu a} - q_1]}{\mu\,(e^{-2\mu a} - e^{-2\mu b})}$$

Substituting p_1 and q_1 yields

$$C_1 = \frac{2b\, e^{-2\mu b} - a\,(e^{-2\mu a} + e^{-2\mu b})}{\mu\,(e^{-2\mu a} - e^{-2\mu b})^2}$$

$$C_2 = \frac{a\, e^{-4\,\mu b} + (a - 2b)\, e^{-2\mu(a+b)}}{\mu\,(e^{-2\mu a} - e^{-2\mu b})^2}$$

Finally, by plugging p_1, q_1, C_1 and C_2 into (A.2.13), we find

$$f_1(x) = \frac{x\,(e^{-2\mu x} + e^{-2\mu b})}{\mu\,(e^{-2\mu a} - e^{-2\mu b})} + \frac{2b\, e^{-2\mu b} - a\,(e^{-2\mu a} + e^{-2\mu b})}{\mu\,(e^{-2\mu a} - e^{-2\mu b})^2}\, e^{-2\mu x}$$

$$+ \quad \frac{a\ e^{-4\,\mu b} + (a-2b)\ e^{-2\mu(a+b)}}{\mu\ (e^{-2\mu a} - e^{-2\mu b})^2}$$

Similarly, let $f_2(x) = E_x[T_b I_{\{T_b<T_a\}}] = E(T^a_{x,b}\ I(T_{x,b} < T_{x,a}))$.
Then, $f_2(x)$ is the solution of the following differential equation with the boundary conditions $f_2(a) = f_2(b) = 0$

$$\frac{1}{2} f_2''(x) + \mu f_2'(x) \quad = \quad -P^a_{x,b} \tag{A.2.16}$$

Using (A.2.10), we get

$$\frac{1}{2} f_2''(x) + \mu f_2'(x) \quad = \quad -\frac{e^{-2\mu a} - e^{-2\mu x}}{e^{-2\mu a} - e^{-2\mu b}} \tag{A.2.17}$$

Let

$$p_2 \quad = \quad \frac{1}{e^{-2\mu a} - e^{-2\mu b}}$$

$$q_2 \quad = \quad -\frac{e^{-2\mu a}}{e^{-2\mu a} - e^{-2\mu b}}$$

Rewrite (A.2.17)

$$f_2''(x) + 2\mu f_2'(x) \quad = \quad 2p_2 e^{-2\mu x} + 2q_2 \tag{A.2.18}$$

The roots of the reduced equation are equal to $r_1 = -2\mu$ and $r_2 = 0$. Hence, the functions required to apply the variation of parameters method are found as

$$u_1 \quad = \quad e^{-2x}, \qquad\qquad u_2 = 1$$

$$v_1 \quad = \quad -\frac{p_2}{\mu}\ x - \frac{q_2}{2\ \mu^2}\ e^{2\mu x} + C_1$$

$$v_2 \quad = \quad \frac{q_2}{\mu}\ x - \frac{p_2}{2\ \mu^2}\ e^{-2\mu x} + C_2$$

Substituting these functions in $y = v_1 u_1 + v_2 u_2$ yields

$$f_2(x) \quad = \quad -p_2 \, \frac{x}{\mu} \, e^{-2\mu x} + 2_1 \, \frac{x}{\mu} + C_1 \, e^{-2\mu x} + C_2 \qquad \text{(A.2.19)}$$

Since $f_2(a) = f_2(b) = 0$

$$f_2(a) \quad = \quad -p_2 \, \frac{a}{\mu} \, e^{-2\mu a} + 2_1 \, \frac{a}{\mu} + C_1 \, e^{-2\mu a} + C_2 = 0 \qquad \text{(A.2.20)}$$

$$f_2(b) \quad = \quad -p_2 \, \frac{b}{\mu} \, e^{-2\mu b} + 2_1 \, \frac{b}{\mu} + C_1 \, e^{-2\mu b} + C_2 = 0 \qquad \text{(A.2.21)}$$

Side by side summation of (A.2.20) and (A.2.21) yields

$$C_1 \quad = \quad \frac{p_2[a \, e^{-2\mu a} - b \, e^{-2\mu b}] - q_2(a-b)}{\mu \, (e^{-2\mu a} - e^{-2\mu b})}$$

$$C_2 \quad = \quad \frac{b \, e^{-2\mu a}[p_2 \, e^{-2\mu b} - q_2] - a \, e^{-2\mu b}[p_2 \, e^{-2\mu a} - q_2]}{\mu \, (e^{-2\mu a} - e^{-2\mu b})}$$

If we substitute p_2 and q_2

$$C_1 \quad = \quad \frac{2a \, e^{-2\mu a} - b \, (e^{-2\mu a} + e^{-2\mu b})}{\mu \, (e^{-2\mu a} - e^{-2\mu b})^2}$$

$$C_2 \quad = \quad \frac{b \, e^{-4 \, \mu a} + (b - 2a) \, e^{-2\mu(a+b)}}{\mu \, (e^{-2\mu a} - e^{-2\mu b})^2}$$

Finally, putting the pieces together we obtain

$$\begin{aligned} f_2(x) \quad &= \quad \frac{-x \, (e^{-2\mu x} + e^{-2\mu a})}{\mu \, (e^{-2\mu a} - e^{-2\mu b})} + \frac{2a \, e^{-2\mu a} - b \, (e^{-2\mu a} + e^{-2\mu b})}{\mu \, (e^{-2\mu a} - e^{-2\mu b})^2} \, e^{-2\mu x} \\ &+ \quad \frac{b \, e^{-4 \, \mu a} + (b - 2a) \, e^{-2\mu(a+b)}}{\mu \, (e^{-2\mu a} - e^{-2\mu b})^2} \end{aligned}$$

iii) First escaping time from strip $[b, a]$, $E(T_{x,ba})$ can be found by

$$E(T_{x,ba}) \quad = \quad E(T^b_{x,a} \, I(T_{x,a} < T_{x,b})) + E(T^a_{x,b} \, I(T_{x,b} < T_{x,a}))$$

$$
\begin{aligned}
&= -\frac{x}{\mu} + \frac{2b\ e^{-2\mu b} - a\ \left(e^{-2\mu a} + e^{-2\mu b}\right) + 2a\ e^{-2\mu a} - b\left(e^{-2\mu a} + e^{-2\mu b}\right)}{\mu\ \left(e^{-2\mu a} - e^{-2\mu b}\right)^2}\ e^{-2\mu x} \\
&+ \frac{a\ e^{-4\ \mu b} + (a-2b)\ e^{-2\mu(a+b)} + b\ e^{-4\ \mu a} + (b-2a)\ e^{-2\mu(a+b)}}{\mu\ \left(e^{-2\mu a} - e^{-2\mu b}\right)^2} \\
&= -\frac{x}{\mu} + \frac{(a-b)}{\mu\ \left(e^{-2\mu a} - e^{-2\mu b}\right)}\ e^{-2\mu x} + \frac{b\ e^{-2\mu a} - a\ e^{-2\mu b}}{\mu\ \left(e^{-2\mu a} - e^{-2\mu b}\right)}
\end{aligned}
$$

Q.E.D.

A.3 Appendix C

Proof of Lemma 4.2

From Lemma 2.1 (i) the structural expression for the expected on-hand carried during stage two is given as

$$E(OH_s) = \frac{1}{P^S_{s,r}}\left[\frac{(s+r)}{2}E(T^S_{s,r}\, I(T_{s,r} < T_{s,S})) + \frac{(s+S)}{2}E(T^r_{s,S}\, I(T_{s,S} < T_{s,r}))\right] \tag{A.3.1}$$

Employing the results from Proposition 3.1 we have

$$\begin{aligned} E(T^S_{s,r}\, I(T_{s,r} < T_{s,S})) &= \frac{(s-r)(2S-s-r)(S-s)}{3(S-r)} \\ E(T^r_{s,S}\, I(T_{s,S} < T_{s,r})) &= \frac{(s-r)(S+s-2r)(S-s)}{3(S-r)} \\ P^S_{s,r} &= \frac{S-s}{S-r} \end{aligned}$$

Substituting these in (A.3.1) gives

$$E(OH_s) = \frac{(S-r)(s-r)(S+r+4s)}{6} \tag{A.3.2}$$

i) Recall that for $r < x \leq s$, from Lemma 2.1

$$\begin{aligned} E(OH_x) &= \frac{(x+r)}{2}E(T^s_{x,r}\, I(T_{x,r} < T_{x,s})) + \frac{(x+s)}{2}E(T^r_{x,s}\, I(T_{x,s} < T_{x,r})) \\ &+ E(OH_s)P^r_{x,s} \end{aligned} \tag{A.3.3}$$

Using the results of Proposition 3.1 we obtain

$$\begin{aligned} E(T^s_{x,r}\, I(T_{x,r} < T_{x,s})) &= \frac{(x-r)(2s-x-r)(s-x)}{3(s-r)} \\ E(T^r_{x,s}\, I(T_{x,s} < T_{x,r})) &= \frac{(x-r)(s+x-2r)(s-x)}{3(s-r)} \\ P^r_{x,s} &= \frac{x-r}{s-r} \end{aligned}$$

Putting them together with (A.3.3) yields

$$E(OH_x) = \frac{(x-r)\left[(s-x)(s+r+4x) + (S-r)(S+r+4s)\right]}{6}$$

ii) Also, recall that for $s \leq x \leq S$, from Lemma 2.2

$$\begin{aligned} E(OH_x) &= \frac{(x+s)}{2} E(T^S_{x,s}\, I(T_{x,s} < T_{x,S})) + \frac{(x+S)}{2} E(T^s_{x,S}\, I(T_{x,S} < T_{x,s})) \\ &+ E(OH_s) \end{aligned} \tag{A.3.4}$$

From Proposition 3.1

$$E(T^S_{x,s}\, I(T_{x,s} < T_{x,S})) = \frac{(x-s)(2S-x-s)(S-x)}{3(S-s)}$$

$$E(T^s_{x,S}\, I(T_{x,S} < T_{x,s})) = \frac{(x-s)(S+x-2s)(S-x)}{3(S-s)}$$

Putting the pieces together with (A.3.4), we get

$$E(OH_x) = \frac{(s-r)(S-r)(S+r+4s) - (s-x)(S-x)(S+s+4x)}{6}$$

Q.E.D.

A.4 Appendix D

Proof of Lemma 5.1

From Lemma 2.3 the structural expression of the expected length of stage two is given as

$$E(T_s) = \frac{E(T_{s,rS})}{P^S_{s,r}}$$

Clearly, from Proposition 3.2, for a B.M. with drift μ it can be written that

$$\begin{aligned} E(T_{s,rS}) &= -\frac{s}{\mu} + \frac{(S-r)}{\mu\,(e^{-2\mu S} - e^{-2\mu r})}\, e^{-2\mu s} + \frac{r\, e^{-2\mu S} - S\, e^{-2\mu r}}{\mu\,(e^{-2\mu S} - e^{-2\mu r})} \\ &= \frac{(S-r)e^{-2\mu s} - (s-r)e^{-2\mu S} - (S-s)e^{-2\mu r}}{\mu\,(e^{-2\mu S} - e^{-2\mu r})} \end{aligned}$$

$$P^S_{s,r} = \frac{e^{-2\mu S} - e^{-2\mu s}}{e^{-2\mu S} - e^{-2\mu r}}$$

Thus

$$E(T_s) = \frac{(S-r)e^{-2\mu s} - (s-r)e^{-2\mu S} - (S-s)e^{-2\mu r}}{\mu\,(e^{-2\mu S} - e^{-2\mu s})} \qquad \text{(A.4.1)}$$

Similarly, from Proposition 3.2

$$\begin{aligned} E(T_{x,rs}) &= -\frac{x}{\mu} + \frac{(s-r)}{\mu\,(e^{-2\mu s} - e^{-2\mu r})}\, e^{-2\mu x} + \frac{r\, e^{-2\mu s} - s\, e^{-2\mu r}}{\mu\,(e^{-2\mu s} - e^{-2\mu r})} \\ &= \frac{(s-r)e^{-2\mu x} - (x-r)e^{-2\mu s} - (s-x)e^{-2\mu r}}{\mu\,(e^{-2\mu s} - e^{-2\mu r})} \qquad \text{(A.4.2)} \end{aligned}$$

$$P^r_{x,s} = \frac{e^{-2\mu x} - e^{-2\mu r}}{e^{-2\mu s} - e^{-2\mu r}} \qquad \text{(A.4.3)}$$

Also, recall from Lemma 2.3 that for $r < x \leq s$

$$E(T_x) = E(T_{x,rs}) + E(T_s).P^r_{x,s} \qquad \text{(A.4.4)}$$

Therefore, by substituting (A.4.1),(A.4.2),(A.4.3) in (A.4.4), we obtain

$$E(T_x) = \frac{[(s-r)e^{-2\mu x} - (x-r)e^{-2\mu s} - (s-x)e^{-2\mu r}](e^{-2\mu S} - e^{-2\mu s})}{\mu\,(e^{-2\mu s} - e^{-2\mu r})(e^{-2\mu S} - e^{-2\mu s})}$$

$$
\begin{aligned}
&+ \frac{[(S-r)e^{-2\mu s}-(s-r)e^{-2\mu S}-(S-s)e^{-2\mu r}](e^{-2\mu x}-e^{-2\mu r})}{\mu\,(e^{-2\mu s}-e^{-2\mu r})(e^{-2\mu S}-e^{-2\mu s})} \\
&= \frac{(S-s)[e^{-2\mu(s+x)}-e^{-2\mu(s+r)}+e^{-2\mu(r+x)}+e^{-4\mu r}]}{\mu\,(e^{-2\mu s}-e^{-2\mu r})(e^{-2\mu S}-e^{-2\mu s})} \\
&+ \frac{(x-r)[e^{-4\mu s}-e^{-2\mu(S+s)}+e^{-2\mu(S+r)}-e^{-2\mu(s+r)}]}{\mu\,(e^{-2\mu s}-e^{-2\mu r})(e^{-2\mu S}-e^{-2\mu s})} \\
&= \frac{(S-s)[e^{-2\mu x}(e^{-2\mu s}-e^{-2\mu r})-e^{-2\mu r}(e^{-2\mu s}-e^{-2\mu r})]}{\mu\,(e^{-2\mu s}-e^{-2\mu r})(e^{-2\mu S}-e^{-2\mu s})} \\
&+ \frac{(x-r)[e^{-2\mu r}(e^{-2\mu S}-e^{-2\mu s})-e^{-2\mu s}(e^{-2\mu S}-e^{-2\mu s})]}{\mu\,(e^{-2\mu s}-e^{-2\mu r})(e^{-2\mu S}-e^{-2\mu s})} \\
&= -\frac{(x-r)}{\mu}+\frac{(S-s)}{\mu}\cdot\frac{(e^{-2\mu x}-e^{-2\mu r})}{(e^{-2\mu S}-e^{-2\mu s})}
\end{aligned}
$$

Q.E.D.

Proof of Lemma 5.2

Recall that from Lemma 2.1

$$
E(OH_s) = \frac{1}{P_{s,r}^S}\left[\frac{(s+r)}{2}E(T_{s,r}^S\, I(T_{s,r}<T_{s,S}))+\frac{(s+S)}{2}E(T_{s,S}^r\, I(T_{s,S}<T_{s,r}))\right]
$$

Employing the results from Proposition 3.2 we have

$$
\begin{aligned}
E(T_{s,r}^S\, I(T_{s,r}<T_{s,S})) &= -\frac{s(e^{-2\mu s}+e^{-2\mu S})}{\mu(e^{-2\mu S}-e^{-2\mu r})}+\frac{2Se^{-2\mu S}-r(e^{-2\mu S}+e^{-2\mu r})}{\mu(e^{-2\mu S}-e^{-2\mu r})^2}\,e^{-2\mu s} \\
&+ \frac{re^{-4\mu S}+(r-2S)e^{-2\mu(S+r)}}{\mu(e^{-2\mu S}-e^{-2\mu r})^2} \\
E(T_{s,S}^r\, I(T_{s,S}<T_{s,r})) &= \frac{s(e^{-2\mu s}+e^{-2\mu r})}{\mu(e^{-2\mu S}-e^{-2\mu r})}+\frac{2re^{-2\mu r}-S(e^{-2\mu S}+e^{-2\mu r})}{\mu(e^{-2\mu S}-e^{-2\mu r})^2}\,e^{-2\mu s} \\
&+ \frac{Se^{-4\mu r}+(S-2r)e^{-2\mu(S+r)}}{\mu(e^{-2\mu S}-e^{-2\mu r})^2} \\
P_{s,r}^S &= \frac{e^{-2\mu S}-e^{-2\mu s}}{e^{-2\mu S}-e^{-2\mu r}}
\end{aligned}
$$

For simplicity we will perform the derivation of $E(OH_s)$ by dividing it into two parts as follows:

$$
\begin{aligned}
E(OH_s) &= \frac{\frac{(s+r)}{P^S_{s,r}} E(T^S_{s,r}\ I(T_{s,r} < T_{s,S})) + \frac{(s+S)}{P^S_{s,r}} E(T^r_{s,S}\ I(T_{s,S} < T_{s,r}))}{2} \\
E(OH_s) &= \frac{I + II}{2}
\end{aligned}
$$

First we concentrate on the following equation

$$
\begin{aligned}
&\frac{1}{P^S_{s,r}} E(T^S_{s,r}\ I(T_{s,r} < T_{s,S})) \\
&= \frac{(e^{-2\mu S} - e^{-2\mu r})}{(e^{-2\mu S} - e^{-2\mu s})} \left[-\frac{s(e^{-2\mu s} + e^{-2\mu S})}{\mu(e^{-2\mu S} - e^{-2\mu r})} \right. \\
&+ \frac{2Se^{-2\mu S} - r(e^{-2\mu S} + e^{-2\mu r})}{\mu(e^{-2\mu S} - e^{-2\mu r})^2}\ e^{-2\mu s} \\
&+ \left. \frac{re^{-4\mu S} + (r - 2S)e^{-2\mu(S+r)}}{\mu(e^{-2\mu S} - e^{-2\mu r})^2} \right] \\
&= -\frac{s(e^{-2\mu S} + e^{-2\mu s})}{\mu(e^{-2\mu S} - e^{-2\mu s})} + \frac{2Se^{-2\mu(S+s)} - r\left[e^{-2\mu(S+s)} + e^{-2\mu(s+r)}\right]}{\mu(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu S} - e^{-2\mu r})} \\
&+ \frac{re^{-4\mu S} + (r - 2S)e^{-2\mu(S+r)}}{\mu(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu S} - e^{-2\mu r})} \\
&= -\frac{s(e^{-2\mu S} + e^{-2\mu s})}{\mu(e^{-2\mu S} - e^{-2\mu s})} \\
&+ \frac{r\left[e^{-2\mu(S+r)} - e^{-2\mu(S+s)} + e^{-4\mu S} - e^{-2\mu(s+r)}\right]}{\mu(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu S} - e^{-2\mu r})} \\
&+ \frac{2S\left[e^{-2\mu(S+s)} - e^{-2\mu(S+r)}\right]}{\mu(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu S} - e^{-2\mu r})} \\
&= -\frac{s(e^{-2\mu S} + e^{-2\mu s})}{\mu(e^{-2\mu S} - e^{-2\mu s})} + \frac{r(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu S} + e^{-2\mu r})}{\mu(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu S} - e^{-2\mu r})}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{2S\ e^{-2\mu S}(e^{-2\mu s} - e^{-2\mu r})}{\mu(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu S} - e^{-2\mu r})} \\
&= \quad -\frac{s(e^{-2\mu S} + e^{-2\mu s})}{\mu(e^{-2\mu S} - e^{-2\mu s})} + \frac{r(e^{-2\mu S} + e^{-2\mu r})}{\mu(e^{-2\mu S} - e^{-2\mu r})} \\
&+ \quad \frac{2S\ e^{-2\mu S}(e^{-2\mu s} - e^{-2\mu r})}{\mu(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu S} - e^{-2\mu r})}
\end{aligned} \tag{A.4.5}
$$

To obtain I we multiply (A.4.5) with $(s+r)$. Hence,

$$
\begin{aligned}
I \quad &= \quad -\frac{s(s+r)(e^{-2\mu S} + e^{-2\mu s})}{\mu(e^{-2\mu S} - e^{-2\mu s})} + \frac{r(s+r)(e^{-2\mu S} + e^{-2\mu r})}{\mu(e^{-2\mu S} - e^{-2\mu r})} \\
&+ \quad \frac{2S(s+r)\ e^{-2\mu S}(e^{-2\mu s} - e^{-2\mu r})}{\mu(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu S} - e^{-2\mu r})} \\
&= \quad -\frac{s^2(e^{-2\mu S} + e^{-2\mu s})}{\mu(e^{-2\mu S} - e^{-2\mu s})} - \frac{2sr\ e^{-2\mu S}(e^{-2\mu s} - e^{-2\mu r})}{\mu(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu S} - e^{-2\mu r})} \\
&+ \quad \frac{r^2(e^{-2\mu S} + e^{-2\mu r})}{\mu(e^{-2\mu S} - e^{-2\mu r})} + \frac{2Ss\ e^{-2\mu S}(e^{-2\mu s} - e^{-2\mu r})}{\mu(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu S} - e^{-2\mu r})} \\
&+ \quad \frac{2Sr\ e^{-2\mu S}(e^{-2\mu s} - e^{-2\mu r})}{\mu(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu S} - e^{-2\mu r})}
\end{aligned} \tag{A.4.6}
$$

Similarly for II, we begin by computing

$$
\frac{1}{P_{s,r}^S} E(T_{s,S}^r\ I(T_{s,S} < T_{s,r}))
$$

$$
\begin{aligned}
&= \quad \frac{(e^{-2\mu S} - e^{-2\mu r})}{(e^{-2\mu S} - e^{-2\mu s})} \Bigg[\frac{s(e^{-2\mu s} + e^{-2\mu r})}{\mu(e^{-2\mu S} - e^{-2\mu r})} \\
&+ \quad \frac{2re^{-2\mu r} - S(e^{-2\mu S} + e^{-2\mu r})}{\mu(e^{-2\mu S} - e^{-2\mu r})^2}\ e^{-2\mu s} \\
&+ \quad \frac{Se^{-4\mu r} + (S - 2r)e^{-2\mu(S+r)}}{\mu(e^{-2\mu S} - e^{-2\mu r})^2} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{s(e^{-2\mu s}+e^{-2\mu r})}{\mu(e^{-2\mu S}-e^{-2\mu s})} + \frac{2re^{-2\mu(s+r)} - S\left[e^{-2\mu(S+s)}+e^{-2\mu(s+r)}\right]}{\mu(e^{-2\mu S}-e^{-2\mu s})(e^{-2\mu S}-e^{-2\mu r})} \\
&+ \frac{Se^{-4\mu r}+(S-2r)e^{-2\mu(S+r)}}{\mu(e^{-2\mu S}-e^{-2\mu s})(e^{-2\mu S}-e^{-2\mu r})} \\
&= \frac{s(e^{-2\mu s}+e^{-2\mu r})}{\mu(e^{-2\mu S}-e^{-2\mu s})} + \frac{S\left[e^{-2\mu(S+r)}-e^{-2\mu(S+s)}+e^{-4\mu r}-e^{-2\mu(s+r)}\right]}{\mu(e^{-2\mu S}-e^{-2\mu s})(e^{-2\mu S}-e^{-2\mu r})} \\
&+ \frac{2r\left[e^{-2\mu(s+r)}-e^{-2\mu(S+r)}\right]}{\mu(e^{-2\mu S}-e^{-2\mu s})(e^{-2\mu S}-e^{-2\mu r})} \\
&= \frac{s(e^{-2\mu s}+e^{-2\mu r})}{\mu(e^{-2\mu S}-e^{-2\mu s})} - \frac{S(e^{-2\mu s}-e^{-2\mu r})(e^{-2\mu S}+e^{-2\mu r})}{\mu(e^{-2\mu S}-e^{-2\mu s})(e^{-2\mu S}-e^{-2\mu r})} \\
&- \frac{2r\ e^{-2\mu r}}{\mu(e^{-2\mu S}-e^{-2\mu r})} \qquad \text{(A.4.7)}
\end{aligned}
$$

Multiplying (A.4.7) with $(s+S)$ yields

$$
\begin{aligned}
II &= \frac{s(s+S)(e^{-2\mu s}+e^{-2\mu r})}{\mu(e^{-2\mu S}-e^{-2\mu s})} - \frac{S(s+S)(e^{-2\mu s}-e^{-2\mu r})(e^{-2\mu S}+e^{-2\mu r})}{\mu(e^{-2\mu S}-e^{-2\mu s})(e^{-2\mu S}-e^{-2\mu r})} \\
&- \frac{2r(s+S)\ e^{-2\mu r}}{\mu(e^{-2\mu S}-e^{-2\mu r})} \\
&= \frac{s^2(e^{-2\mu s}+e^{-2\mu r})}{\mu(e^{-2\mu S}-e^{-2\mu s})} + \frac{2Ss\ e^{-2\mu r}}{\mu(e^{-2\mu S}-e^{-2\mu r})} - \frac{S^2(e^{-2\mu s}-e^{-2\mu r})(e^{-2\mu S}+e^{-2\mu r})}{\mu(e^{-2\mu S}-e^{-2\mu s})(e^{-2\mu S}-e^{-2\mu r})} \\
&- \frac{2sr\ e^{-2\mu r}}{\mu(e^{-2\mu S}-e^{-2\mu r})} - \frac{2Sr\ e^{-2\mu r}}{\mu(e^{-2\mu S}-e^{-2\mu r})} \qquad \text{(A.4.8)}
\end{aligned}
$$

Summation of (A.4.6) and (A.4.8) gives

$$
\begin{aligned}
I+II &= \frac{s^2}{\mu}\left[\frac{(e^{-2\mu s}+e^{-2\mu r})}{(e^{-2\mu S}-e^{-2\mu s})} - \frac{(e^{-2\mu S}+e^{-2\mu s})}{(e^{-2\mu S}-e^{-2\mu s})}\right] \\
&+ \frac{2Ss}{\mu}\left[\frac{e^{-2\mu S}(e^{-2\mu s}-e^{-2\mu r})}{(e^{-2\mu S}-e^{-2\mu s})(e^{-2\mu S}-e^{-2\mu r})} + \frac{e^{-2\mu r}}{(e^{-2\mu S}-e^{-2\mu r})}\right]
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{2sr}{\mu}\left[\frac{e^{-2\mu S}(e^{-2\mu s}-e^{-2\mu r})}{(e^{-2\mu S}-e^{-2\mu s})(e^{-2\mu S}-e^{-2\mu r})}+\frac{e^{-2\mu r}}{(e^{-2\mu S}-e^{-2\mu r})}\right] \\
&+ \frac{r^2(e^{-2\mu S}+e^{-2\mu r})}{\mu(e^{-2\mu S}-e^{-2\mu r})}-\frac{S^2(e^{-2\mu s}-e^{-2\mu r})(e^{-2\mu S}+e^{-2\mu r})}{\mu(e^{-2\mu S}-e^{-2\mu s})(e^{-2\mu S}-e^{-2\mu r})} \\
&+ \frac{2Sr}{\mu}\left[\frac{e^{-2\mu S}(e^{-2\mu s}-e^{-2\mu r})}{(e^{-2\mu S}-e^{-2\mu s})(e^{-2\mu S}-e^{-2\mu r})}-\frac{e^{-2\mu r}}{(e^{-2\mu S}-e^{-2\mu r})}\right] \\
&= -\frac{s^2(e^{-2\mu S}-e^{-2\mu r})}{\mu(e^{-2\mu S}-e^{-2\mu s})}+\frac{2Ss}{\mu}\left[\frac{e^{-2\mu(S+s)}-e^{-2\mu(s+r)}}{(e^{-2\mu S}-e^{-2\mu s})(e^{-2\mu S}-e^{-2\mu r})}\right] \\
&- \frac{2sr}{\mu}\left[\frac{e^{-2\mu(S+s)}-e^{-2\mu(s+r)}}{(e^{-2\mu S}-e^{-2\mu s})(e^{-2\mu S}-e^{-2\mu r})}\right]+\frac{r^2(e^{-2\mu S}+e^{-2\mu r})}{\mu(e^{-2\mu S}-e^{-2\mu r})} \\
&- \frac{S^2(e^{-2\mu s}-e^{-2\mu r})(e^{-2\mu S}+e^{-2\mu r})}{\mu(e^{-2\mu S}-e^{-2\mu s})(e^{-2\mu S}-e^{-2\mu r})} \\
&+ \frac{2Sr}{\mu}\left[\frac{e^{-2\mu S}(e^{-2\mu s}-e^{-2\mu r})}{(e^{-2\mu S}-e^{-2\mu s})(e^{-2\mu S}-e^{-2\mu r})}-\frac{e^{-2\mu r}}{(e^{-2\mu S}-e^{-2\mu r})}\right] \\
&= -\frac{s^2(e^{-2\mu S}-e^{-2\mu r})}{\mu(e^{-2\mu S}-e^{-2\mu s})}+\frac{2Ss\ e^{-2\mu s}}{\mu(e^{-2\mu S}-e^{-2\mu s})}-\frac{2sr\ e^{-2\mu s}}{\mu(e^{-2\mu S}-e^{-2\mu s})} \\
&+ \frac{r^2(e^{-2\mu S}+e^{-2\mu r})}{\mu(e^{-2\mu S}-e^{-2\mu r})}-\frac{S^2(e^{-2\mu s}-e^{-2\mu r})(e^{-2\mu S}+e^{-2\mu r})}{\mu(e^{-2\mu S}-e^{-2\mu s})(e^{-2\mu S}-e^{-2\mu r})} \\
&+ \frac{2Sr}{\mu}\left[\frac{e^{-2\mu S}(e^{-2\mu s}-e^{-2\mu r})}{(e^{-2\mu S}-e^{-2\mu s})(e^{-2\mu S}-e^{-2\mu r})}-\frac{e^{-2\mu r}}{(e^{-2\mu S}-e^{-2\mu r})}\right] \qquad \text{(A.4.9)}
\end{aligned}
$$

Dividing (A.4.9) by two yields $E(OH_s)$. Hence,

$$
\begin{aligned}
E(OH_s) &= -\frac{s^2}{2\mu}\cdot\frac{(e^{-2\mu S}-e^{-2\mu r})}{(e^{-2\mu S}-e^{-2\mu s})}+\frac{Ss}{\mu}\cdot\frac{e^{-2\mu s}}{(e^{-2\mu S}-e^{-2\mu s})} \\
&- \frac{sr}{\mu}\cdot\frac{e^{-2\mu s}}{(e^{-2\mu S}-e^{-2\mu s})}+\frac{r^2}{2\mu}\cdot\frac{(e^{-2\mu S}+e^{-2\mu r})}{(e^{-2\mu S}-e^{-2\mu r})} \\
&- \frac{S^2}{2\mu}\left[\frac{(e^{-2\mu s}-e^{-2\mu r})(e^{-2\mu S}+e^{-2\mu r})}{(e^{-2\mu S}-e^{-2\mu s})(e^{-2\mu S}-e^{-2\mu r})}\right]
\end{aligned}
$$

$$+ \quad \frac{Sr}{\mu}\left[\frac{e^{-2\mu S}(e^{-2\mu s}-e^{-2\mu r})}{(e^{-2\mu S}-e^{-2\mu s})(e^{-2\mu S}-e^{-2\mu r})}-\frac{e^{-2\mu r}}{(e^{-2\mu S}-e^{-2\mu r})}\right] \qquad \text{(A.4.10)}$$

$i)$ Recall that for $r < x \leq s$, from Lemma 2.1

$$\begin{aligned} E(OH_x) &= \frac{(x+r)}{2}E(T^s_{x,r}\; I(T_{x,r} < T_{x,s})) + \frac{(x+s)}{2}E(T^r_{x,s}\; I(T_{x,s} < T_{x,r})) \\ &+ E(OH_s).P^r_{x,s} \\ &= I_x + II_x + III_x \end{aligned}$$

Using the results of Proposition 3.2 we obtain

$$\begin{aligned} E(T^s_{x,r}\; I(T_{x,r} < T_{x,s})) &= -\frac{x(e^{-2\mu x}+e^{-2\mu s})}{\mu(e^{-2\mu s}-e^{-2\mu r})} + \frac{2se^{-2\mu s}-r(e^{-2\mu s}+e^{-2\mu r})}{\mu(e^{-2\mu s}-e^{-2\mu r})^2}\, e^{-2\mu x} \\ &+ \frac{re^{-4\mu s}+(r-2s)e^{-2\mu(s+r)}}{\mu(e^{-2\mu s}-e^{-2\mu r})^2} \\ E(T^r_{x,s}\; I(T_{x,s} < T_{x,r})) &= \frac{x(e^{-2\mu x}+e^{-2\mu r})}{\mu(e^{-2\mu s}-e^{-2\mu r})} + \frac{2re^{-2\mu r}-s(e^{-2\mu s}+e^{-2\mu r})}{\mu(e^{-2\mu s}-e^{-2\mu r})^2}\, e^{-2\mu x} \\ &+ \frac{se^{-4\mu r}+(s-2r)e^{-2\mu(s+r)}}{\mu(e^{-2\mu s}-e^{-2\mu r})^2} \\ P^r_{x,s} &= \frac{e^{-2\mu x}-e^{-2\mu r}}{e^{-2\mu s}-e^{-2\mu r}} \qquad \text{(A.4.11)} \end{aligned}$$

Concentrate on the first summand I_x, we have

$$\begin{aligned} I_x &= \frac{(x+r)}{2}\left[-\frac{x(e^{-2\mu x}+e^{-2\mu s})}{\mu(e^{-2\mu s}-e^{-2\mu r})} + \frac{2se^{-2\mu s}-r(e^{-2\mu s}+e^{-2\mu r})}{\mu(e^{-2\mu s}-e^{-2\mu r})^2}\, e^{-2\mu x}\right. \\ &+ \left.\frac{re^{-4\mu s}+(r-2s)e^{-2\mu(s+r)}}{\mu(e^{-2\mu s}-e^{-2\mu r})^2}\right] \\ &= -\frac{x(x+r)(e^{-2\mu x}+e^{-2\mu s})}{2\mu(e^{-2\mu s}-e^{-2\mu r})} + \frac{s(x+r)\left[e^{-2\mu(s+x)}-e^{-2\mu(s+r)}\right]}{\mu(e^{-2\mu s}-e^{-2\mu r})^2} \\ &- \frac{r(x+r)\left[e^{-2\mu(s+x)}+e^{-2\mu(r+x)}-e^{-4\mu s}-e^{-2\mu(s+r)}\right]}{2\mu(e^{-2\mu s}-e^{-2\mu r})^2} \end{aligned}$$

$$
\begin{aligned}
&= -\frac{x^2(e^{-2\mu x}+e^{-2\mu s})}{2\mu(e^{-2\mu s}-e^{-2\mu r})} \\
&- \frac{rx}{2\mu}\left[\frac{(e^{-2\mu x}+e^{-2\mu s})}{(e^{-2\mu s}-e^{-2\mu r})}+\frac{(e^{-2\mu(s+x)}+e^{-2\mu(r+x)}-e^{-4\mu s}-e^{-2\mu(s+r)})}{(e^{-2\mu s}-e^{-2\mu r})^2}\right] \\
&+ \frac{sx\left[e^{-2\mu(s+x)}-e^{-2\mu(s+r)}\right]}{\mu(e^{-2\mu s}-e^{-2\mu r})^2}+\frac{sr\left[e^{-2\mu(s+x)}-e^{-2\mu(s+r)}\right]}{\mu(e^{-2\mu s}-e^{-2\mu r})^2} \\
&- \frac{r^2\left[e^{-2\mu(s+x)}+e^{-2\mu(r+x)}-e^{-4\mu s}-e^{-2\mu(s+r)}\right]}{2\mu(e^{-2\mu s}-e^{-2\mu r})^2} \\
&= -\frac{x^2(e^{-2\mu x}+e^{-2\mu s})}{2\mu(e^{-2\mu s}-e^{-2\mu r})}+\frac{(s-r)x\ e^{-2\mu s}(e^{-2\mu x}-e^{-2\mu r})}{\mu(e^{-2\mu s}-e^{-2\mu r})^2} \\
&+ \frac{sr\ e^{-2\mu s}(e^{-2\mu x}-e^{-2\mu r})}{\mu(e^{-2\mu s}-e^{-2\mu r})^2}+\frac{r^2(e^{-2\mu s}-e^{-2\mu x})(e^{-2\mu s}+e^{-2\mu r})}{2\mu(e^{-2\mu s}-e^{-2\mu r})^2} \qquad \text{(A.4.12)}
\end{aligned}
$$

Similarly II_x can be derived as follows

$$
\begin{aligned}
II_x &= \frac{(x+s)}{2}\left[\frac{x(e^{-2\mu x}+e^{-2\mu r})}{\mu(e^{-2\mu s}-e^{-2\mu r})}+\frac{2re^{-2\mu r}-s(e^{-2\mu s}+e^{-2\mu r})}{\mu(e^{-2\mu s}-e^{-2\mu r})^2}\ e^{-2\mu x}\right. \\
&+ \left.\frac{se^{-4\mu r}+(s-2r)e^{-2\mu(s+r)}}{\mu(e^{-2\mu s}-e^{-2\mu r})^2}\right] \\
&= \frac{x(x+s)(e^{-2\mu x}+e^{-2\mu r})}{2\mu(e^{-2\mu s}-e^{-2\mu r})}+\frac{r(x+s)\left[e^{-2\mu(r+x)}-e^{-2\mu(s+r)}\right]}{\mu(e^{-2\mu s}-e^{-2\mu r})^2} \\
&- \frac{s(x+s)\left[e^{-2\mu(s+x)}+e^{-2\mu(r+x)}-e^{-4\mu r}-e^{-2\mu(s+r)}\right]}{2\mu(e^{-2\mu s}-e^{-2\mu r})^2} \\
&= \frac{x^2(e^{-2\mu x}+e^{-2\mu r})}{2\mu(e^{-2\mu s}-e^{-2\mu r})} \\
&+ \frac{sx}{2\mu}\left[\frac{(e^{-2\mu x}+e^{-2\mu r})}{(e^{-2\mu s}-e^{-2\mu r})}-\frac{(e^{-2\mu(s+x)}+e^{-2\mu(r+x)}-e^{-4\mu r}-e^{-2\mu(s+r)})}{(e^{-2\mu s}-e^{-2\mu r})^2}\right]
\end{aligned}
$$

$$+ \quad \frac{rx\left[e^{-2\mu(r+x)} - e^{-2\mu(s+r)}\right]}{\mu(e^{-2\mu s} - e^{-2\mu r})^2} + \frac{sr\left[e^{-2\mu(r+x)} - e^{-2\mu(s+r)}\right]}{\mu(e^{-2\mu s} - e^{-2\mu r})^2}$$

$$- \quad \frac{s^2\left[e^{-2\mu(s+x)} + e^{-2\mu(r+x)} - e^{-4\mu r} - e^{-2\mu(s+r)}\right]}{2\mu(e^{-2\mu s} - e^{-2\mu r})^2}$$

$$= \quad \frac{x^2(e^{-2\mu x} + e^{-2\mu r})}{2\mu(e^{-2\mu s} - e^{-2\mu r})} + \frac{(s-r)x\ e^{-2\mu r}(e^{-2\mu s} - e^{-2\mu x})}{\mu(e^{-2\mu s} - e^{-2\mu r})^2}$$

$$- \quad \frac{sr\ e^{-2\mu r}(e^{-2\mu s} - e^{-2\mu x})}{\mu(e^{-2\mu s} - e^{-2\mu r})^2} - \frac{s^2(e^{-2\mu x} - e^{-2\mu r})(e^{-2\mu s} + e^{-2\mu r})}{2\mu(e^{-2\mu s} - e^{-2\mu r})^2} \quad \text{(A.4.13)}$$

Summation of (A.4.12) and (A.4.13) yields

$$I_x + II_x \quad = \quad -\frac{x^2}{2\mu}\left[\frac{(e^{-2\mu x} + e^{-2\mu s}) - (e^{-2\mu x} + e^{-2\mu r})}{(e^{-2\mu s} - e^{-2\mu r})}\right]$$

$$+ \quad \frac{(s-r)x}{\mu}\left[\frac{e^{-2\mu s}(e^{-2\mu x} - e^{-2\mu r}) + e^{-2\mu r}(e^{-2\mu s} - e^{-2\mu x})}{(e^{-2\mu s} - e^{-2\mu r})^2}\right]$$

$$+ \quad \frac{sr}{\mu}\left[\frac{e^{-2\mu s}(e^{-2\mu x} - e^{-2\mu r}) - e^{-2\mu r}(e^{-2\mu s} - e^{-2\mu x})}{(e^{-2\mu s} - e^{-2\mu r})^2}\right]$$

$$+ \quad \frac{r^2(e^{-2\mu s} - e^{-2\mu x})(e^{-2\mu s} + e^{-2\mu r})}{2\mu(e^{-2\mu s} - e^{-2\mu r})^2}$$

$$- \quad \frac{s^2(e^{-2\mu x} - e^{-2\mu r})(e^{-2\mu s} + e^{-2\mu r})}{2\mu(e^{-2\mu s} - e^{-2\mu r})^2}$$

$$= \quad -\frac{x^2}{2\mu} + \frac{(s-r)x}{\mu} \cdot \frac{e^{-2\mu x}}{(e^{-2\mu s} - e^{-2\mu r})}$$

$$+ \quad \frac{sr}{\mu}\left[\frac{e^{-2\mu s}(e^{-2\mu x} - e^{-2\mu r}) - e^{-2\mu r}(e^{-2\mu s} - e^{-2\mu x})}{(e^{-2\mu s} - e^{-2\mu r})^2}\right]$$

$$+ \quad \frac{r^2}{2\mu}\left[\frac{(e^{-2\mu s} - e^{-2\mu x})(e^{-2\mu s} + e^{-2\mu r})}{(e^{-2\mu s} - e^{-2\mu r})^2}\right]$$

$$- \quad \frac{s^2}{2\mu}\left[\frac{(e^{-2\mu x} - e^{-2\mu r})(e^{-2\mu s} + e^{-2\mu r})}{(e^{-2\mu s} - e^{-2\mu r})^2}\right] \quad \text{(A.4.14)}$$

To obtain III_x, we multiply (A.4.10) with (A.4.11).Therefore,

$$
\begin{aligned}
III_x &= \frac{(e^{-2\mu x} - e^{-2\mu r})}{(e^{-2\mu s} - e^{-2\mu r})}\left\{-\frac{s^2}{2\mu}\cdot\frac{(e^{-2\mu S} - e^{-2\mu r})}{(e^{-2\mu S} - e^{-2\mu s})} + \frac{Ss}{\mu}\cdot\frac{e^{-2\mu s}}{(e^{-2\mu S} - e^{-2\mu s})}\right. \\
&- \frac{sr}{\mu}\cdot\frac{e^{-2\mu s}}{(e^{-2\mu S} - e^{-2\mu s})} + \frac{r^2}{2\mu}\cdot\frac{(e^{-2\mu S} + e^{-2\mu r})}{(e^{-2\mu S} - e^{-2\mu r})} \\
&- \frac{S^2}{2\mu}\left[\frac{(e^{-2\mu s} - e^{-2\mu r})(e^{-2\mu S} + e^{-2\mu r})}{(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu S} - e^{-2\mu r})}\right] \\
&+ \left.\frac{Sr}{\mu}\left[\frac{e^{-2\mu S}(e^{-2\mu s} - e^{-2\mu r})}{(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu S} - e^{-2\mu r})} - \frac{e^{-2\mu r}}{(e^{-2\mu S} - e^{-2\mu r})}\right]\right\} \\
&= -\frac{s^2}{2\mu}\left[\frac{(e^{-2\mu S} - e^{-2\mu r})(e^{-2\mu x} - e^{-2\mu r})}{(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu s} - e^{-2\mu r})}\right] \\
&+ \frac{Ss}{\mu}\left[\frac{e^{-2\mu s}(e^{-2\mu x} - e^{-2\mu r})}{(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu s} - e^{-2\mu r})}\right] \\
&- \frac{sr}{\mu}\left[\frac{e^{-2\mu s}(e^{-2\mu x} - e^{-2\mu r})}{(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu s} - e^{-2\mu r})}\right] \\
&+ \frac{r^2}{2\mu}\left[\frac{(e^{-2\mu S} + e^{-2\mu r})(e^{-2\mu x} - e^{-2\mu r})}{(e^{-2\mu S} - e^{-2\mu r})(e^{-2\mu s} - e^{-2\mu r})}\right] \\
&- \frac{S^2}{2\mu}\left[\frac{(e^{-2\mu S} + e^{-2\mu r})(e^{-2\mu x} - e^{-2\mu r})}{(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu S} - e^{-2\mu r})}\right] \\
&+ \frac{Sr}{\mu}\left[\frac{e^{-2\mu S}(e^{-2\mu x} - e^{-2\mu r})}{(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu S} - e^{-2\mu r})}\right. \\
&- \left.\frac{e^{-2\mu r}(e^{-2\mu x} - e^{-2\mu r})}{(e^{-2\mu S} - e^{-2\mu r})(e^{-2\mu s} - e^{-2\mu r})}\right] \qquad \text{(A.4.15)}
\end{aligned}
$$

Finally by summing (A.4.14) and (A.4.15), we get $E(OH_x)$ when $r < x \leq$ as follows:

$$
E(OH_x) = -\frac{x^2}{2\mu} + \frac{(s-r)x}{\mu}\cdot\frac{e^{-2\mu x}}{(e^{-2\mu s} - e^{-2\mu r})}
$$

$$
\begin{aligned}
+ \quad & \frac{sr}{\mu}\left[\frac{e^{-2\mu s}\left(e^{-2\mu x}-e^{-2\mu r}\right)-e^{-2\mu r}\left(e^{-2\mu s}-e^{-2\mu x}\right)}{\left(e^{-2\mu s}-e^{-2\mu r}\right)^2}\right.\\
- \quad & \left.\frac{e^{-2\mu s}\left(e^{-2\mu x}-e^{-2\mu r}\right)}{\left(e^{-2\mu S}-e^{-2\mu s}\right)\left(e^{-2\mu s}-e^{-2\mu r}\right)}\right]\\
+ \quad & \frac{r^2}{2\mu}\left[\frac{\left(e^{-2\mu s}-e^{-2\mu x}\right)\left(e^{-2\mu s}+e^{-2\mu r}\right)}{\left(e^{-2\mu s}-e^{-2\mu r}\right)^2}+\frac{\left(e^{-2\mu S}+e^{-2\mu r}\right)\left(e^{-2\mu x}-e^{-2\mu r}\right)}{\left(e^{-2\mu S}-e^{-2\mu r}\right)\left(e^{-2\mu s}-e^{-2\mu r}\right)}\right]\\
- \quad & \frac{s^2}{2\mu}\left[\frac{\left(e^{-2\mu x}-e^{-2\mu r}\right)\left(e^{-2\mu s}+e^{-2\mu r}\right)}{\left(e^{-2\mu s}-e^{-2\mu r}\right)^2}+\frac{\left(e^{-2\mu S}-e^{-2\mu r}\right)\left(e^{-2\mu x}-e^{-2\mu r}\right)}{\left(e^{-2\mu S}-e^{-2\mu s}\right)\left(e^{-2\mu s}-e^{-2\mu r}\right)}\right]\\
+ \quad & \frac{Ss}{\mu}\left[\frac{e^{-2\mu s}\left(e^{-2\mu x}-e^{-2\mu r}\right)}{\left(e^{-2\mu S}-e^{-2\mu s}\right)\left(e^{-2\mu s}-e^{-2\mu r}\right)}\right]\\
- \quad & \frac{S^2}{2\mu}\left[\frac{\left(e^{-2\mu S}+e^{-2\mu r}\right)\left(e^{-2\mu x}-e^{-2\mu r}\right)}{\left(e^{-2\mu S}-e^{-2\mu s}\right)\left(e^{-2\mu S}-e^{-2\mu r}\right)}\right]\\
+ \quad & \frac{Sr}{\mu}\left[\frac{e^{-2\mu S}\left(e^{-2\mu x}-e^{-2\mu r}\right)}{\left(e^{-2\mu S}-e^{-2\mu s}\right)\left(e^{-2\mu S}-e^{-2\mu r}\right)}-\frac{e^{-2\mu r}\left(e^{-2\mu x}-e^{-2\mu r}\right)}{\left(e^{-2\mu S}-e^{-2\mu r}\right)\left(e^{-2\mu s}-e^{-2\mu r}\right)}\right]\\
= \quad & -\frac{x^2}{2\mu}+\frac{(s-r)x}{\mu}\cdot\frac{e^{-2\mu x}}{\left(e^{-2\mu s}-e^{-2\mu r}\right)}\\
+ \quad & \frac{sr}{\mu}\left[\frac{e^{-2\mu(x+S+s)}+e^{-2\mu(x+S+r)}-2e^{-2\mu(x+2s)}}{\left(e^{-2\mu S}-e^{-2\mu s}\right)\left(e^{-2\mu s}-e^{-2\mu r}\right)^2}\right.\\
+ \quad & \left.\frac{3e^{-2\mu(r+2s)}-2e^{-2\mu(S+s+r)}-e^{-2\mu(s+2r)}}{\left(e^{-2\mu S}-e^{-2\mu s}\right)\left(e^{-2\mu s}-e^{-2\mu r}\right)^2}\right]\\
+ \quad & \frac{r^2}{2\mu}\left[\frac{2e^{-2\mu(x+s+r)}-2e^{-2\mu(x+S+r)}+e^{-2\mu(S+2s)}-e^{-2\mu(r+2s)}}{\left(e^{-2\mu S}-e^{-2\mu r}\right)\left(e^{-2\mu s}-e^{-2\mu r}\right)^2}\right.\\
+ \quad & \left.\frac{e^{-2\mu(S+2r)}-2e^{-2\mu(s+2r)}+e^{-6\mu r}}{\left(e^{-2\mu S}-e^{-2\mu r}\right)\left(e^{-2\mu s}-e^{-2\mu r}\right)^2}\right]\\
- \quad & \frac{s^2}{2\mu}\left[\frac{2e^{-2\mu(x+S+s)}-e^{-2\mu(x+2s)}-2e^{-2\mu(x+s+r)}+e^{-2\mu(x+2r)}}{\left(e^{-2\mu S}-e^{-2\mu s}\right)\left(e^{-2\mu s}-e^{-2\mu r}\right)^2}\right.
\end{aligned}
$$

$$
\begin{aligned}
& - \quad \frac{2e^{-2\mu(S+s+r)} - e^{-2\mu(r+2s)} - 2e^{-2\mu(s+2r)} + e^{-6\mu r}}{\left(e^{-2\mu S} - e^{-2\mu s}\right)\left(e^{-2\mu s} - e^{-2\mu r}\right)^2}\Bigg] \\
& + \quad \frac{Ss}{\mu}\left[\frac{e^{-2\mu s}\left(e^{-2\mu x} - e^{-2\mu r}\right)}{\left(e^{-2\mu S} - e^{-2\mu s}\right)\left(e^{-2\mu s} - e^{-2\mu r}\right)}\right] \\
& - \quad \frac{S^2}{2\mu}\left[\frac{\left(e^{-2\mu S} + e^{-2\mu r}\right)\left(e^{-2\mu x} - e^{-2\mu r}\right)}{\left(e^{-2\mu S} - e^{-2\mu s}\right)\left(e^{-2\mu S} - e^{-2\mu r}\right)}\right] \\
& + \quad \frac{Sr}{\mu}\left[\frac{e^{-2\mu S}\left(e^{-2\mu x} - e^{-2\mu r}\right)}{\left(e^{-2\mu S} - e^{-2\mu s}\right)\left(e^{-2\mu S} - e^{-2\mu r}\right)} - \frac{e^{-2\mu r}\left(e^{-2\mu x} - e^{-2\mu r}\right)}{\left(e^{-2\mu S} - e^{-2\mu r}\right)\left(e^{-2\mu s} - e^{-2\mu r}\right)}\right] \\
& = \quad -\frac{x^2}{2\mu} + \frac{(s-r)x}{\mu} \cdot \frac{e^{-2\mu x}}{\left(e^{-2\mu s} - e^{-2\mu r}\right)} \\
& + \quad \frac{sr}{\mu}\left[\frac{e^{-2\mu x}\left[e^{-2\mu S}\left(e^{-2\mu s} + e^{-2\mu r}\right) - 2e^{-4\mu s}\right]}{\left(e^{-2\mu S} - e^{-2\mu s}\right)\left(e^{-2\mu s} - e^{-2\mu r}\right)^2}\right. \\
& + \quad \left.\frac{e^{-2\mu(s+r)}\left[3e^{-2\mu s} - 2e^{-2\mu S} - e^{-2\mu r}\right]}{\left(e^{-2\mu S} - e^{-2\mu s}\right)\left(e^{-2\mu s} - e^{-2\mu r}\right)^2}\right] \\
& + \quad \frac{r^2}{2\mu}\left[\frac{-2e^{-2\mu x}e^{-2\mu r}\left(e^{-2\mu S} - e^{-2\mu s}\right) + e^{-4\mu s}\left(e^{-2\mu S} - e^{-2\mu r}\right)}{\left(e^{-2\mu S} - e^{-2\mu r}\right)\left(e^{-2\mu s} - e^{-2\mu r}\right)^2}\right. \\
& + \quad \left.\frac{e^{-4\mu r}\left(e^{-2\mu S} - 2e^{-2\mu s} - e^{-2\mu r}\right)}{\left(e^{-2\mu S} - e^{-2\mu r}\right)\left(e^{-2\mu s} - e^{-2\mu r}\right)^2}\right] \\
& - \quad \frac{s^2}{2\mu}\left[\frac{e^{-2\mu x}\left[e^{-2\mu s}\left(2e^{-2\mu S} - e^{-2\mu r} - e^{-2\mu s}\right) - e^{-2\mu r}\left(e^{-2\mu s} - e^{-2\mu r}\right)\right]}{\left(e^{-2\mu S} - e^{-2\mu s}\right)\left(e^{-2\mu s} - e^{-2\mu r}\right)^2}\right. \\
& + \quad \left.\frac{e^{-4\mu r}\left(e^{-2\mu s} - e^{-2\mu r}\right) - e^{-2\mu(s+r)}\left(2e^{-2\mu S} - e^{-2\mu r} - e^{-2\mu s}\right)}{\left(e^{-2\mu S} - e^{-2\mu s}\right)\left(e^{-2\mu s} - e^{-2\mu r}\right)^2}\right] \\
& + \quad \frac{Ss}{\mu}\left[\frac{e^{-2\mu x}e^{-2\mu s} - e^{-2\mu(s+r)}}{\left(e^{-2\mu S} - e^{-2\mu s}\right)\left(e^{-2\mu s} - e^{-2\mu r}\right)}\right] \\
& - \quad \frac{S^2}{2\mu}\left[\frac{e^{-2\mu x}\left(e^{-2\mu S} + e^{-2\mu r}\right) - e^{-2\mu r}\left(e^{-2\mu S} + e^{-2\mu r}\right)}{\left(e^{-2\mu S} - e^{-2\mu s}\right)\left(e^{-2\mu S} - e^{-2\mu r}\right)}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{Sr}{\mu}\left[\frac{e^{-2\mu x}e^{-2\mu S} - e^{-2\mu(S+r)}}{(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu S} - e^{-2\mu r})} - \frac{e^{-2\mu x}e^{-2\mu r} - e^{-4\mu r}}{(e^{-2\mu S} - e^{-2\mu r})(e^{-2\mu s} - e^{-2\mu r})}\right] \\
&= \quad -\frac{x^2}{2\mu} + \frac{(s-r)x}{\mu}\cdot\frac{e^{-2\mu x}}{(e^{-2\mu s} - e^{-2\mu r})} \\
&+ \quad \frac{sr}{\mu}\left[\frac{e^{-2\mu x}\left[e^{-2\mu S}(e^{-2\mu s} + e^{-2\mu r}) - 2e^{-4\mu s}\right]}{(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu s} - e^{-2\mu r})^2}\right] \\
&+ \quad \frac{sr}{\mu}\left[\frac{e^{-2\mu(s+r)}\left[(e^{-2\mu s} - e^{-2\mu r}) - 2(e^{-2\mu S} - e^{-2\mu s})\right]}{(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu s} - e^{-2\mu r})^2}\right] \\
&- \quad \frac{r^2}{\mu}\left[\frac{e^{-2\mu x}e^{-2\mu r}(e^{-2\mu S} - e^{-2\mu s})}{(e^{-2\mu S} - e^{-2\mu r})(e^{-2\mu s} - e^{-2\mu r})^2}\right] \\
&+ \quad \frac{r^2}{2\mu}\left[\frac{e^{-4\mu s}(e^{-2\mu S} - e^{-2\mu r}) + e^{-4\mu r}\left[(e^{-2\mu S} - e^{-2\mu s}) - (e^{-2\mu s} - e^{-2\mu r})\right]}{(e^{-2\mu S} - e^{-2\mu r})(e^{-2\mu s} - e^{-2\mu r})^2}\right] \\
&- \quad \frac{s^2}{2\mu}\left[\frac{e^{-2\mu x}\left[e^{-2\mu s}\left[(e^{-2\mu S} - e^{-2\mu s}) + (e^{-2\mu S} - e^{-2\mu r})\right]\right.}{(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu s} - e^{-2\mu r})^2}\right. \\
&- \quad \left.\frac{\left.e^{-2\mu r}(e^{-2\mu s} - e^{-2\mu r})\right]}{(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu s} - e^{-2\mu r})^2}\right] \\
&+ \quad \frac{s^2}{2\mu}\left[\frac{e^{-2\mu(s+r)}\left[(e^{-2\mu S} - e^{-2\mu s}) + (e^{-2\mu S} - e^{-2\mu r})\right]}{(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu s} - e^{-2\mu r})^2}\right. \\
&- \quad \left.\frac{e^{-4\mu r}(e^{-2\mu s} - e^{-2\mu r})}{(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu s} - e^{-2\mu r})^2}\right] \\
&+ \quad \frac{Ss}{\mu}\cdot\frac{e^{-2\mu x}e^{-2\mu s}}{(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu s} - e^{-2\mu r})} \\
&- \quad \frac{Ss}{\mu}\cdot\frac{e^{-2\mu(s+r)}}{(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu s} - e^{-2\mu r})} \\
&- \quad \frac{S^2}{2\mu}\cdot\frac{e^{-2\mu x}(e^{-2\mu S} + e^{-2\mu r})}{(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu S} - e^{-2\mu r})}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{S^2}{2\mu} \cdot \frac{e^{-2\mu r}\left(e^{-2\mu S}+e^{-2\mu r}\right)}{\left(e^{-2\mu S}-e^{-2\mu s}\right)\left(e^{-2\mu S}-e^{-2\mu r}\right)} \\
&+ \quad \frac{Sr}{\mu}\left[\frac{e^{-2\mu x}e^{-2\mu S}}{\left(e^{-2\mu S}-e^{-2\mu s}\right)\left(e^{-2\mu S}-e^{-2\mu r}\right)} - \frac{e^{-2\mu x}e^{-2\mu r}}{\left(e^{-2\mu S}-e^{-2\mu r}\right)\left(e^{-2\mu s}-e^{-2\mu r}\right)}\right] \\
&- \quad \frac{Sr}{\mu}\left[\frac{e^{-2\mu(S+r)}}{\left(e^{-2\mu S}-e^{-2\mu s}\right)\left(e^{-2\mu S}-e^{-2\mu r}\right)} - \frac{e^{-4\mu r}}{\left(e^{-2\mu S}-e^{-2\mu r}\right)\left(e^{-2\mu s}-e^{-2\mu r}\right)}\right] \\
&= \quad -\frac{x^2}{2\mu} + \frac{(s-r)x}{\mu} \cdot \frac{e^{-2\mu x}}{\left(e^{-2\mu s}-e^{-2\mu r}\right)} \\
&+ \quad e^{-2\mu x}\left\{\frac{sr}{\mu}\left[\frac{e^{-2\mu S}\left(e^{-2\mu s}+e^{-2\mu r}\right)-2e^{-4\mu s}}{\left(e^{-2\mu S}-e^{-2\mu s}\right)\left(e^{-2\mu s}-e^{-2\mu r}\right)^2}\right]\right. \\
&- \quad \frac{r^2}{\mu}\left[\frac{e^{-2\mu r}\left(e^{-2\mu S}-e^{-2\mu s}\right)}{\left(e^{-2\mu S}-e^{-2\mu r}\right)\left(e^{-2\mu s}-e^{-2\mu r}\right)^2}\right] \\
&- \quad \frac{s^2}{2\mu}\left[\frac{e^{-2\mu s}\left[\left(e^{-2\mu S}-e^{-2\mu s}\right)+\left(e^{-2\mu S}-e^{-2\mu r}\right)\right]-e^{-2\mu r}\left(e^{-2\mu s}-e^{-2\mu r}\right)}{\left(e^{-2\mu S}-e^{-2\mu s}\right)\left(e^{-2\mu s}-e^{-2\mu r}\right)^2}\right] \\
&+ \quad \frac{Ss}{\mu} \cdot \frac{e^{-2\mu s}}{\left(e^{-2\mu S}-e^{-2\mu s}\right)\left(e^{-2\mu s}-e^{-2\mu r}\right)} \\
&- \quad \frac{S^2}{2\mu} \cdot \frac{\left(e^{-2\mu S}+e^{-2\mu r}\right)}{\left(e^{-2\mu S}-e^{-2\mu s}\right)\left(e^{-2\mu S}-e^{-2\mu r}\right)} \\
&+ \quad \left.\frac{Sr}{\mu}\left[\frac{e^{-2\mu S}}{\left(e^{-2\mu S}-e^{-2\mu s}\right)\left(e^{-2\mu S}-e^{-2\mu r}\right)} - \frac{e^{-2\mu r}}{\left(e^{-2\mu S}-e^{-2\mu r}\right)\left(e^{-2\mu s}-e^{-2\mu r}\right)}\right]\right\} \\
&+ \quad \left\{\frac{sr}{\mu}\left[\frac{e^{-2\mu(s+r)}\left[\left(e^{-2\mu s}-e^{-2\mu r}\right)-2\left(e^{-2\mu S}-e^{-2\mu s}\right)\right]}{\left(e^{-2\mu S}-e^{-2\mu s}\right)\left(e^{-2\mu s}-e^{-2\mu r}\right)^2}\right]\right. \\
&+ \quad \frac{r^2}{2\mu}\left[\frac{e^{-4\mu s}\left(e^{-2\mu S}-e^{-2\mu r}\right)+e^{-4\mu r}\left[\left(e^{-2\mu S}-e^{-2\mu s}\right)-\left(e^{-2\mu s}-e^{-2\mu r}\right)\right]}{\left(e^{-2\mu S}-e^{-2\mu r}\right)\left(e^{-2\mu s}-e^{-2\mu r}\right)^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{s^2}{2\mu}\left[\frac{e^{-2\mu(s+r)}\left[(e^{-2\mu S}-e^{-2\mu s})+(e^{-2\mu S}-e^{-2\mu r})\right]}{(e^{-2\mu S}-e^{-2\mu s})(e^{-2\mu s}-e^{-2\mu r})^2}\right.\\
&- \quad \left.\frac{e^{-4\mu r}(e^{-2\mu s}-e^{-2\mu r})}{(e^{-2\mu S}-e^{-2\mu s})(e^{-2\mu s}-e^{-2\mu r})^2}\right]\\
&- \quad \frac{Ss}{\mu}\cdot\frac{e^{-2\mu(s+r)}}{(e^{-2\mu S}-e^{-2\mu s})(e^{-2\mu s}-e^{-2\mu r})}\\
&+ \quad \frac{S^2}{2\mu}\cdot\frac{e^{-2\mu r}(e^{-2\mu S}+e^{-2\mu r})}{(e^{-2\mu S}-e^{-2\mu s})(e^{-2\mu S}-e^{-2\mu r})}\\
&- \quad \frac{Sr}{\mu}\left[\frac{e^{-2\mu(S+r)}}{(e^{-2\mu S}-e^{-2\mu s})(e^{-2\mu S}-e^{-2\mu r})}-\frac{e^{-4\mu r}}{(e^{-2\mu S}-e^{-2\mu r})(e^{-2\mu s}-e^{-2\mu r})}\right]\Bigg\}\\
&= \quad -\frac{x^2}{2\mu}+\frac{(s-r)x}{\mu}\cdot\frac{e^{-2\mu x}}{(e^{-2\mu s}-e^{-2\mu r})}+e^{-2\mu x}F_1(S,s,r;\mu)+G_1(S,s,r;\mu)
\end{aligned}
$$

ii) We have for $s \le x \le S$ from Lemma 2.2

$$
\begin{aligned}
E(OH_x) &= \frac{(x+s)}{2}E(T^S_{x,s}\,I(T_{x,s}<T_{x,S}))+\frac{(x+S)}{2}E(T^s_{x,S}\,I(T_{x,S}<T_{x,s}))\\
&+ E(OH_s)\\
&= I_x+II_x+III_x
\end{aligned}
$$

Similarly from Proposition 3.2

$$
\begin{aligned}
E(T^S_{x,s}\,I(T_{x,s}<T_{x,S})) &= -\frac{x(e^{-2\mu x}+e^{-2\mu S})}{\mu(e^{-2\mu S}-e^{-2\mu s})}+\frac{2Se^{-2\mu S}-s(e^{-2\mu S}+e^{-2\mu s})}{\mu(e^{-2\mu S}-e^{-2\mu s})^2}\,e^{-2\mu x}\\
&+ \frac{se^{-4\mu S}+(s-2S)e^{-2\mu(S+s)}}{\mu(e^{-2\mu S}-e^{-2\mu s})^2}\\
E(T^s_{x,S}\,I(T_{x,S}<T_{x,s})) &= \frac{x(e^{-2\mu x}+e^{-2\mu s})}{\mu(e^{-2\mu S}-e^{-2\mu s})}+\frac{2se^{-2\mu s}-S(e^{-2\mu S}+e^{-2\mu s})}{\mu(e^{-2\mu S}-e^{-2\mu s})^2}\,e^{-2\mu x}\\
&+ \frac{Se^{-4\mu s}+(S-2s)e^{-2\mu(S+s)}}{\mu(e^{-2\mu S}-e^{-2\mu s})^2}
\end{aligned}
$$

Thus, the first summand I_x can be obtained as follows

$$\begin{aligned}
I_x &= \frac{(x+s)}{2}\left[-\frac{x(e^{-2\mu x}+e^{-2\mu S})}{\mu(e^{-2\mu S}-e^{-2\mu s})}+\frac{2Se^{-2\mu S}-s(e^{-2\mu S}+e^{-2\mu s})}{\mu(e^{-2\mu S}-e^{-2\mu s})^2}\,e^{-2\mu x}\right.\\
&+\left.\frac{se^{-4\mu S}+(s-2S)e^{-2\mu(S+s)}}{\mu(e^{-2\mu S}-e^{-2\mu s})^2}\right]\\
&= -\frac{x(x+s)(e^{-2\mu x}+e^{-2\mu S})}{2\mu(e^{-2\mu S}-e^{-2\mu s})}+\frac{S(x+s)\left[e^{-2\mu(S+x)}-e^{-2\mu(S+s)}\right]}{\mu(e^{-2\mu S}-e^{-2\mu s})^2}\\
&- \frac{s(x+s)\left[e^{-2\mu(S+x)}+e^{-2\mu(s+x)}-e^{-4\mu S}-e^{-2\mu(S+s)}\right]}{2\mu(e^{-2\mu S}-e^{-2\mu s})^2}\\
&= -\frac{x^2(e^{-2\mu x}+e^{-2\mu S})}{2\mu(e^{-2\mu S}-e^{-2\mu s})}\\
&- \frac{sx}{2\mu}\left[\frac{(e^{-2\mu x}+e^{-2\mu S})}{(e^{-2\mu S}-e^{-2\mu s})}+\frac{(e^{-2\mu(S+x)}+e^{-2\mu(s+x)}-e^{-4\mu S}-e^{-2\mu(S+s)})}{(e^{-2\mu S}-e^{-2\mu s})^2}\right]\\
&+ \frac{Sx\left[e^{-2\mu(S+x)}-e^{-2\mu(S+s)}\right]}{\mu(e^{-2\mu S}-e^{-2\mu s})^2}+\frac{Ss\left[e^{-2\mu(S+x)}-e^{-2\mu(S+s)}\right]}{\mu(e^{-2\mu S}-e^{-2\mu s})^2}\\
&- \frac{s^2\left[e^{-2\mu(S+x)}+e^{-2\mu(s+x)}-e^{-4\mu S}-e^{-2\mu(S+s)}\right]}{2\mu(e^{-2\mu S}-e^{-2\mu s})^2}\\
&= -\frac{x^2(e^{-2\mu x}+e^{-2\mu S})}{2\mu(e^{-2\mu S}-e^{-2\mu s})}+\frac{(S-s)x\,e^{-2\mu S}(e^{-2\mu x}-e^{-2\mu s})}{\mu(e^{-2\mu S}-e^{-2\mu s})^2}\\
&+ \frac{Ss\,e^{-2\mu S}(e^{-2\mu x}-e^{-2\mu s})}{\mu(e^{-2\mu S}-e^{-2\mu s})^2}+\frac{s^2(e^{-2\mu S}-e^{-2\mu x})(e^{-2\mu S}+e^{-2\mu s})}{2\mu(e^{-2\mu S}-e^{-2\mu s})^2} \qquad \text{(A.4.16)}
\end{aligned}$$

Likewise, for II_x we have

$$\begin{aligned}
II_x &= \frac{(x+S)}{2}\left[\frac{x(e^{-2\mu x}+e^{-2\mu s})}{\mu(e^{-2\mu S}-e^{-2\mu s})}+\frac{2se^{-2\mu s}-S(e^{-2\mu S}+e^{-2\mu s})}{\mu(e^{-2\mu S}-e^{-2\mu s})^2}\,e^{-2\mu x}\right.\\
&+\left.\frac{Se^{-4\mu s}+(S-2s)e^{-2\mu(S+s)}}{\mu(e^{-2\mu S}-e^{-2\mu s})^2}\right]
\end{aligned}$$

$$= \frac{x(x+S)(e^{-2\mu x}+e^{-2\mu s})}{2\mu(e^{-2\mu S}-e^{-2\mu s})} + \frac{s(x+S)\left[e^{-2\mu(s+x)}-e^{-2\mu(S+s)}\right]}{\mu(e^{-2\mu S}-e^{-2\mu s})^2}$$

$$- \frac{S(x+S)\left[e^{-2\mu(S+x)}+e^{-2\mu(s+x)}-e^{-4\mu s}-e^{-2\mu(S+s)}\right]}{2\mu(e^{-2\mu S}-e^{-2\mu s})^2}$$

$$= \frac{x^2(e^{-2\mu x}+e^{-2\mu s})}{2\mu(e^{-2\mu S}-e^{-2\mu s})}$$

$$+ \frac{Sx}{2\mu}\left[\frac{(e^{-2\mu x}+e^{-2\mu s})}{(e^{-2\mu S}-e^{-2\mu s})} - \frac{(e^{-2\mu(S+x)}+e^{-2\mu(s+x)}-e^{-4\mu s}-e^{-2\mu(S+s)})}{(e^{-2\mu S}-e^{-2\mu s})^2}\right]$$

$$+ \frac{sx\left[e^{-2\mu(s+x)}-e^{-2\mu(S+s)}\right]}{\mu(e^{-2\mu S}-e^{-2\mu s})^2} + \frac{Ss\left[e^{-2\mu(s+x)}-e^{-2\mu(S+s)}\right]}{\mu(e^{-2\mu S}-e^{-2\mu s})^2}$$

$$- \frac{S^2\left[e^{-2\mu(S+x)}+e^{-2\mu(s+x)}-e^{-4\mu s}-e^{-2\mu(S+s)}\right]}{2\mu(e^{-2\mu S}-e^{-2\mu s})^2}$$

$$= \frac{x^2(e^{-2\mu x}+e^{-2\mu s})}{2\mu(e^{-2\mu S}-e^{-2\mu s})} + \frac{(S-s)x\, e^{-2\mu s}(e^{-2\mu S}-e^{-2\mu x})}{\mu(e^{-2\mu S}-e^{-2\mu s})^2}$$

$$- \frac{Ss\, e^{-2\mu s}(e^{-2\mu S}-e^{-2\mu x})}{\mu(e^{-2\mu S}-e^{-2\mu s})^2} - \frac{S^2(e^{-2\mu x}-e^{-2\mu s})(e^{-2\mu S}+e^{-2\mu s})}{2\mu(e^{-2\mu S}-e^{-2\mu s})^2} \quad \text{(A.4.17)}$$

Summation of (A.4.16) and (A.4.17) yields

$$I_x + II_x = -\frac{x^2}{2\mu}\left[\frac{(e^{-2\mu x}+e^{-2\mu S})-(e^{-2\mu x}+e^{-2\mu s})}{(e^{-2\mu S}-e^{-2\mu s})}\right]$$

$$+ \frac{(S-s)x}{\mu}\left[\frac{e^{-2\mu S}(e^{-2\mu x}-e^{-2\mu s})+e^{-2\mu s}(e^{-2\mu S}-e^{-2\mu x})}{(e^{-2\mu S}-e^{-2\mu s})^2}\right]$$

$$+ \frac{Ss}{\mu}\left[\frac{e^{-2\mu S}(e^{-2\mu x}-e^{-2\mu s})-e^{-2\mu s}(e^{-2\mu S}-e^{-2\mu x})}{(e^{-2\mu S}-e^{-2\mu s})^2}\right]$$

$$+ \frac{s^2(e^{-2\mu S}-e^{-2\mu x})(e^{-2\mu S}+e^{-2\mu s})}{2\mu(e^{-2\mu S}-e^{-2\mu s})^2}$$

$$- \frac{S^2(e^{-2\mu x}-e^{-2\mu s})(e^{-2\mu S}+e^{-2\mu s})}{2\mu(e^{-2\mu S}-e^{-2\mu s})^2}$$

$$
\begin{aligned}
&= -\frac{x^2}{2\mu} + \frac{(S-s)x}{\mu} \cdot \frac{e^{-2\mu x}}{(e^{-2\mu S} - e^{-2\mu s})} \\
&+ \frac{Ss}{\mu}\left[\frac{e^{-2\mu S}(e^{-2\mu x} - e^{-2\mu s}) - e^{-2\mu s}(e^{-2\mu S} - e^{-2\mu x})}{(e^{-2\mu S} - e^{-2\mu s})^2}\right] \\
&+ \frac{s^2}{2\mu}\left[\frac{(e^{-2\mu S} - e^{-2\mu x})(e^{-2\mu S} + e^{-2\mu s})}{(e^{-2\mu S} - e^{-2\mu s})^2}\right] \\
&- \frac{S^2}{2\mu}\left[\frac{(e^{-2\mu x} - e^{-2\mu s})(e^{-2\mu S} + e^{-2\mu s})}{(e^{-2\mu S} - e^{-2\mu s})^2}\right]
\end{aligned}
\tag{A.4.18}
$$

Note that $III_x = E(OH_s)$ is given by (A.4.10). Thus, by adding (A.4.10) to (A.4.18) we get

$$
\begin{aligned}
E(OH_x) &= -\frac{x^2}{2\mu} + \frac{(S-s)x}{\mu} \cdot \frac{e^{-2\mu x}}{(e^{-2\mu S} - e^{-2\mu s})} \\
&+ \frac{Ss}{\mu}\left[\frac{e^{-2\mu S}(e^{-2\mu x} - e^{-2\mu s}) - e^{-2\mu s}(e^{-2\mu S} - e^{-2\mu x})}{(e^{-2\mu S} - e^{-2\mu s})^2} + \frac{e^{-2\mu s}}{(e^{-2\mu S} - e^{-2\mu s})}\right] \\
&+ \frac{s^2}{2\mu}\left[\frac{(e^{-2\mu S} - e^{-2\mu x})(e^{-2\mu S} + e^{-2\mu s})}{(e^{-2\mu S} - e^{-2\mu s})^2} - \frac{(e^{-2\mu S} - e^{-2\mu r})}{(e^{-2\mu S} - e^{-2\mu s})}\right] \\
&- \frac{S^2}{2\mu}\left[\frac{(e^{-2\mu x} - e^{-2\mu s})(e^{-2\mu S} + e^{-2\mu s})}{(e^{-2\mu S} - e^{-2\mu s})^2} + \frac{(e^{-2\mu s} - e^{-2\mu r})(e^{-2\mu S} + e^{-2\mu r})}{(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu S} - e^{-2\mu r})}\right] \\
&- \frac{sr}{\mu} \cdot \frac{e^{-2\mu s}}{(e^{-2\mu S} - e^{-2\mu s})} + \frac{r^2}{2\mu} \cdot \frac{(e^{-2\mu S} + e^{-2\mu r})}{(e^{-2\mu S} - e^{-2\mu r})} \\
&+ \frac{Sr}{\mu}\left[\frac{e^{-2\mu S}(e^{-2\mu s} - e^{-2\mu r})}{(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu S} - e^{-2\mu r})} - \frac{e^{-2\mu r}}{(e^{-2\mu S} - e^{-2\mu r})}\right] \\
&= -\frac{x^2}{2\mu} + \frac{(S-s)x}{\mu} \cdot \frac{e^{-2\mu x}}{(e^{-2\mu S} - e^{-2\mu s})} \\
&+ \frac{Ss}{\mu}\left[\frac{e^{-2\mu(S+x)} - e^{-2\mu(s+x)} - e^{-2\mu(S+s)} - e^{-4\mu s}}{(e^{-2\mu S} - e^{-2\mu s})^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{s^2}{2\mu}\left[\frac{2e^{-2\mu(S+s)} + e^{-2\mu(S+r)} - e^{-2\mu(s+r)} - e^{-2\mu x}\left(e^{-2\mu S} + e^{-2\mu s}\right)}{\left(e^{-2\mu S} - e^{-2\mu s}\right)^2}\right] \\
&- \frac{S^2}{2\mu}\left[\frac{e^{-2\mu x}\left(e^{-2\mu S} + e^{-2\mu s}\right)\left(e^{-2\mu S} - e^{-2\mu r}\right)}{\left(e^{-2\mu S} - e^{-2\mu s}\right)^2\left(e^{-2\mu S} - e^{-2\mu r}\right)}\right. \\
&+ \left.\frac{3e^{-2\mu(S+s+r)} - 2e^{-2\mu(S+2s)} - e^{-2\mu(2S+r)} - e^{-2\mu(S+2r)} + e^{-2\mu(s+2r)}}{\left(e^{-2\mu S} - e^{-2\mu s}\right)^2\left(e^{-2\mu S} - e^{-2\mu r}\right)}\right] \\
&- \frac{sr}{\mu}\cdot\frac{e^{-2\mu s}}{\left(e^{-2\mu S} - e^{-2\mu s}\right)} + \frac{r^2}{2\mu}\cdot\frac{\left(e^{-2\mu S} + e^{-2\mu r}\right)}{\left(e^{-2\mu S} - e^{-2\mu r}\right)} \\
&+ \frac{Sr}{\mu}\left[\frac{e^{-2\mu S}\left(e^{-2\mu s} - e^{-2\mu r}\right)}{\left(e^{-2\mu S} - e^{-2\mu s}\right)\left(e^{-2\mu S} - e^{-2\mu r}\right)} - \frac{e^{-2\mu r}}{\left(e^{-2\mu S} - e^{-2\mu r}\right)}\right] \\
&= -\frac{x^2}{2\mu} + \frac{(S-s)x}{\mu}\cdot\frac{e^{-2\mu x}}{\left(e^{-2\mu S} - e^{-2\mu s}\right)} \\
&+ \frac{Ss}{\mu}\left[\frac{e^{-2\mu x}\left(e^{-2\mu S} + e^{-2\mu s}\right)}{\left(e^{-2\mu S} - e^{-2\mu s}\right)^2}\right] - \frac{Ss}{\mu}\left[\frac{e^{-2\mu s}\left(e^{-2\mu S} + e^{-2\mu s}\right)}{\left(e^{-2\mu S} - e^{-2\mu s}\right)^2}\right] \\
&- \frac{s^2}{2\mu}\left[\frac{e^{-2\mu x}\left(e^{-2\mu S} + e^{-2\mu s}\right)}{\left(e^{-2\mu S} - e^{-2\mu s}\right)^2}\right] \\
&+ \frac{s^2}{2\mu}\left[\frac{e^{-2\mu S}\left(e^{-2\mu s} + e^{-2\mu r}\right) + e^{-2\mu s}\left(e^{-2\mu S} - e^{-2\mu r}\right)}{\left(e^{-2\mu S} - e^{-2\mu s}\right)^2}\right] \\
&- \frac{S^2}{2\mu}\left[\frac{e^{-2\mu x}\left(e^{-2\mu S} + e^{-2\mu s}\right)}{\left(e^{-2\mu S} - e^{-2\mu s}\right)^2}\right] \\
&+ \frac{S^2}{2\mu}\left[\frac{2e^{-2\mu(S+s)}\left(e^{-2\mu s} - e^{-2\mu r}\right)}{\left(e^{-2\mu S} - e^{-2\mu s}\right)^2\left(e^{-2\mu S} - e^{-2\mu r}\right)} + \frac{e^{-2\mu r}\left(e^{-2\mu S} + e^{-2\mu r}\right)}{\left(e^{-2\mu S} - e^{-2\mu s}\right)\left(e^{-2\mu S} - e^{-2\mu r}\right)}\right] \\
&- \frac{sr}{\mu}\cdot\frac{e^{-2\mu s}}{\left(e^{-2\mu S} - e^{-2\mu s}\right)} + \frac{r^2}{2\mu}\cdot\frac{\left(e^{-2\mu S} + e^{-2\mu r}\right)}{\left(e^{-2\mu S} - e^{-2\mu r}\right)} \\
&+ \frac{Sr}{\mu}\left[\frac{e^{-2\mu S}\left(e^{-2\mu s} - e^{-2\mu r}\right)}{\left(e^{-2\mu S} - e^{-2\mu s}\right)\left(e^{-2\mu S} - e^{-2\mu r}\right)} - \frac{e^{-2\mu r}}{\left(e^{-2\mu S} - e^{-2\mu r}\right)}\right]
\end{aligned}
$$

$$
\begin{aligned}
= \quad & -\frac{x^2}{2\mu} + \frac{(S-s)x}{\mu} \cdot \frac{e^{-2\mu x}}{(e^{-2\mu S} - e^{-2\mu s})} \\
- \quad & e^{-2\mu x} \left\{ \frac{(S^2 - 2Ss + s^2)}{2\mu} \left[\frac{(e^{-2\mu S} + e^{-2\mu s})}{(e^{-2\mu S} - e^{-2\mu s})^2} \right] \right\} \\
+ \quad & \frac{s^2}{2\mu} \left[\frac{e^{-2\mu S}(e^{-2\mu s} + e^{-2\mu r}) + e^{-2\mu s}(e^{-2\mu S} - e^{-2\mu r})}{(e^{-2\mu S} - e^{-2\mu s})^2} \right] \\
- \quad & \frac{Ss}{\mu} \left[\frac{e^{-2\mu s}(e^{-2\mu S} + e^{-2\mu s})}{(e^{-2\mu S} - e^{-2\mu s})^2} \right] \\
+ \quad & \frac{S^2}{2\mu} \left[\frac{2e^{-2\mu(S+s)}(e^{-2\mu s} - e^{-2\mu r})}{(e^{-2\mu S} - e^{-2\mu s})^2 (e^{-2\mu S} - e^{-2\mu r})} + \frac{e^{-2\mu r}(e^{-2\mu S} + e^{-2\mu r})}{(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu S} - e^{-2\mu r})} \right] \\
- \quad & \frac{sr}{\mu} \cdot \frac{e^{-2\mu s}}{(e^{-2\mu S} - e^{-2\mu s})} + \frac{r^2}{2\mu} \cdot \frac{(e^{-2\mu S} + e^{-2\mu r})}{(e^{-2\mu S} - e^{-2\mu r})} \\
+ \quad & \frac{Sr}{\mu} \left[\frac{e^{-2\mu S}(e^{-2\mu s} - e^{-2\mu r})}{(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu S} - e^{-2\mu r})} - \frac{e^{-2\mu r}}{(e^{-2\mu S} - e^{-2\mu r})} \right] \\
= \quad & -\frac{x^2}{2\mu} + \frac{(S-s)x}{\mu} \cdot \frac{e^{-2\mu x}}{(e^{-2\mu S} - e^{-2\mu s})} \\
- \quad & e^{-2\mu x} \left\{ \frac{(S-s)^2}{2\mu} \left[\frac{(e^{-2\mu S} + e^{-2\mu s})}{(e^{-2\mu S} - e^{-2\mu s})^2} \right] \right\} \\
+ \quad & \left\{ \frac{s^2}{2\mu} \left[\frac{e^{-2\mu S}(e^{-2\mu s} + e^{-2\mu r}) + e^{-2\mu s}(e^{-2\mu S} - e^{-2\mu r})}{(e^{-2\mu S} - e^{-2\mu s})^2} \right] \right. \\
- \quad & \frac{Ss}{\mu} \left[\frac{e^{-2\mu s}(e^{-2\mu S} + e^{-2\mu s})}{(e^{-2\mu S} - e^{-2\mu s})^2} \right] \\
+ \quad & \frac{S^2}{2\mu} \left[\frac{2e^{-2\mu(S+s)}(e^{-2\mu s} - e^{-2\mu r})}{(e^{-2\mu S} - e^{-2\mu s})^2 (e^{-2\mu S} - e^{-2\mu r})} + \frac{e^{-2\mu r}(e^{-2\mu S} + e^{-2\mu r})}{(e^{-2\mu S} - e^{-2\mu s})(e^{-2\mu S} - e^{-2\mu r})} \right] \\
- \quad & \frac{sr}{\mu} \cdot \frac{e^{-2\mu s}}{(e^{-2\mu S} - e^{-2\mu s})} + \frac{r^2}{2\mu} \cdot \frac{(e^{-2\mu S} + e^{-2\mu r})}{(e^{-2\mu S} - e^{-2\mu r})}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{Sr}{\mu}\left[\frac{e^{-2\mu S}(e^{-2\mu s}-e^{-2\mu r})}{(e^{-2\mu S}-e^{-2\mu s})(e^{-2\mu S}-e^{-2\mu r})}-\frac{e^{-2\mu r}}{(e^{-2\mu S}-e^{-2\mu r})}\right]\Bigg\} \\
&= \quad -\frac{x^2}{2\mu}+\frac{(S-s)x}{\mu}\cdot\frac{e^{-2\mu x}}{(e^{-2\mu S}-e^{-2\mu s})}-e^{-2\mu x}F_2(S,s;\mu) \\
&+ \quad G_2(S,s,r;\mu)
\end{aligned}
$$

Q.E.D.

A.5 Tables of the Sensitivity Analysis for Zero Expected Net Demand Case

$L=5, \mu=0$			$\alpha=0.95$					$\alpha=0.99$					$\alpha=0.999$			
Ko	Kd	Cd	S	s^*	r^*	Q^*	TC^*	S	s^*	r^*	Q^*	TC^*	S	s^*	r^*	Q^*
		5	60	3.82	1.85	3.74	15.76	60	4.94	2.94	3.77	16.71	60	6.44	4.41	3.8
	250	10	60	3.83	1.85	3.73	15.85	60	4.94	2.94	3.77	16.80	60	6.44	4.41	3.8
		25	60	3.84	1.85	3.72	16.10	60	4.95	2.94	3.75	17.07	60	6.45	4.41	3.8
		5	60	3.82	1.85	3.73	15.84	60	4.93	2.94	3.77	16.79	60	6.43	4.41	3.8
500	500	10	60	3.82	1.85	3.72	15.92	60	4.94	2.94	3.76	16.88	60	6.44	4.41	3.8
		25	60	3.83	1.85	3.71	16.18	60	4.95	2.94	3.75	17.15	60	6.45	4.41	3.8
		5	60	3.81	1.85	3.72	15.99	60	4.93	2.94	3.75	16.95	60	6.43	4.41	3.8
	1000	10	60	3.82	1.85	3.71	16.08	60	4.93	2.94	3.75	17.04	60	6.43	4.41	3.8
		25	60	3.83	1.85	3.70	16.34	60	4.94	2.94	3.73	17.31	60	6.44	4.41	3.7
		5	60	4.81	1.85	5.48	17.74	60	5.94	2.94	5.53	18.72	60	7.46	4.41	5.6
	500	10	60	4.82	1.85	5.47	17.83	60	5.94	2.94	5.53	18.81	60	7.46	4.41	5.6
		25	60	4.83	1.85	5.46	18.09	60	5.96	2.94	5.51	19.08	60	7.48	4.41	5.5
		5	60	4.80	1.85	5.46	17.90	60	5.93	2.94	5.52	18.88	60	7.44	4.41	5.5
1000	1000	10	60	4.81	1.85	5.46	17.99	60	5.93	2.94	5.51	18.97	60	7.45	4.41	5.5
		25	60	4.82	1.85	5.44	18.25	60	5.95	2.94	5.49	19.24	60	7.47	4.41	5.5
		5	60	4.78	1.85	5.43	18.22	60	5.91	2.94	5.48	19.22	60	7.42	4.41	5.5
	2000	10	60	4.79	1.85	5.42	18.31	60	5.92	2.94	5.47	19.31	60	7.43	4.41	5.5
		25	60	4.81	1.85	5.40	18.57	60	5.93	2.94	5.45	19.58	60	7.45	4.41	5.5
		5	40	4.45	1.85	4.69	13.64	40	5.59	2.94	4.76	14.65	40	7.13	4.41	4.8
	250	10	40	4.46	1.85	4.68	13.78	40	5.60	2.94	4.75	14.79	40	7.14	4.41	4.8
		25	40	4.49	1.85	4.64	14.18	40	5.63	2.94	4.71	15.20	40	7.18	4.41	4.8
		5	40	4.43	1.85	4.66	13.83	40	5.57	2.94	4.73	14.85	40	7.11	4.41	4.8
500	500	10	40	4.44	1.85	4.65	13.97	40	5.58	2.94	4.71	14.99	40	7.12	4.41	4.8
		25	40	4.47	1.85	4.61	14.37	40	5.61	2.94	4.67	15.41	40	7.16	4.41	4.7
		5	40	4.40	1.85	4.60	14.21	40	5.54	2.94	4.66	15.25	40	7.07	4.41	4.7
	1000	10	40	4.41	1.85	4.59	14.35	40	5.55	2.94	4.65	15.39	40	7.08	4.41	4.7
		25	40	4.44	1.85	4.55	14.75	40	5.58	2.94	4.61	15.81	40	7.12	4.41	4.6
		5	40	5.77	1.85	6.81	16.22	40	6.94	2.94	6.91	17.28	40	8.52	4.41	7.0
	500	10	40	5.79	1.85	6.80	16.36	40	6.96	2.94	6.89	17.42	40	8.54	4.41	7.0
		25	40	5.83	1.85	6.75	16.78	40	7.01	2.94	6.84	17.85	40	8.60	4.41	6.9
		5	40	5.73	1.85	6.73	16.63	40	6.89	2.94	6.81	17.71	40	8.47	4.41	6.9
1000	1000	10	40	5.74	1.85	6.71	16.77	40	6.91	2.94	6.79	17.86	40	8.49	4.41	6.9
		25	40	5.79	1.85	6.66	17.18	40	6.96	2.94	6.74	18.29	40	8.54	4.41	6.8
		5	40	5.65	1.85	6.56	17.43	40	6.80	2.94	6.63	18.58	40	8.36	4.41	6.7
	2000	10	40	5.66	1.85	6.55	17.57	40	6.82	2.94	6.61	18.72	40	8.38	4.41	6.7
		25	40	5.71	1.85	6.50	17.99	40	6.87	2.94	6.57	19.15	40	8.43	4.41	6.6

Table A.1: Sensitivity Analysis when $S=40,60$ and $\mu=0$

$L=5, \mu=0$			$\alpha = 0.95$					$\alpha = 0.99$					$\alpha = 0.999$			
Ko	Kd	Cd	S	s^*	r^*	Q^*	TC^*	S	s^*	r^*	Q^*	TC^*	S	s^*	r^*	Q^*
		5	15	7.5	1.85	6.45	16.39	15	7.5	2.94	5.84	18.47	15	7.5	4.41	5.0
	250	10	15	7.5	1.85	6.33	16.77	15	7.5	2.94	5.73	18.90	15	7.5	4.41	4.9
		25	15	7.5	1.85	6.02	17.88	15	7.5	2.94	5.43	20.16	15	7.5	4.41	4.6
		5	15	7.5	1.85	5.78	18.85	15	7.5	2.94	5.21	21.25	15	7.5	4.41	4.4
500	500	10	15	7.5	1.85	5.71	19.21	15	7.5	2.94	5.14	21.65	15	7.5	4.41	4.3
		25	15	7.5	1.85	5.39	20.27	15	7.5	2.94	4.93	22.84	15	7.5	4.41	4.2
		5	15	7.5	1.85	4.64	23.43	15	7.5	2.94	4.38	26.43	15	7.5	4.41	3.7
	1000	10	15	7.5	1.85	4.58	23.75	15	7.5	2.94	4.32	26.80	15	7.5	4.41	3.7
		25	15	7.5	1.85	4.42	24.73	15	7.5	2.94	4.16	27.91	15	7.5	4.41	3.5
		5	15	7.5	1.85	6.88	24.80	15	7.5	2.94	6.23	28.65	15	7.5	4.41	5.3
	500	10	15	7.5	1.85	6.80	25.19	15	7.5	2.94	6.15	29.10	15	7.5	4.41	5.2
		25	15	7.5	1.85	6.58	26.36	15	7.5	2.94	5.96	30.42	15	7.5	4.41	5.0
		5	15	7.5	1.85	6.04	29.87	15	7.5	2.94	5.46	34.37	15	7.5	4.41	4.6
1000	1000	10	15	7.5	1.85	5.99	30.24	15	7.5	2.94	5.42	34.79	15	7.5	4.41	4.6
		25	15	7.5	1.85	5.85	31.33	15	7.5	2.94	5.29	36.02	15	7.5	4.41	4.5
		5	15	7.5	1.85	4.92	39.22	15	7.5	2.94	4.55	44.91	15	7.5	4.41	3.8
	2000	10	15	7.5	1.85	4.88	39.56	15	7.5	2.94	4.52	45.29	15	7.5	4.41	3.8
		25	15	7.5	1.85	4.79	40.56	15	7.5	2.94	4.43	46.43	15	7.5	4.41	3.7
		5	5	2.5	1.85	1.20	79.52	5	3.75	2.94	0.37	100.33	5	4.62	4.41	0.0
	250	10	5	2.5	1.85	1.18	80.26	5	3.75	2.94	0.35	100.64	5	4.62	4.41	0.0
		25	5	2.5	1.85	1.10	82.39	5	3.75	2.94	0.28	101.45	5	4.62	4.41	0.0
		5	5	2.5	1.85	0.74	92.00	5	3.75	2.94	0.00	102.97	5	4.62	4.41	0.0
500	500	10	5	2.5	1.85	0.72	92.51	5	3.75	2.94	0.00	102.97	5	4.62	4.41	0.0
		25	5	2.5	1.85	0.66	93.96	5	3.75	2.94	0.00	102.97	5	4.62	4.41	0.0
		5	5	2.5	1.85	0.00	101.96	5	3.75	2.94	0.00	102.97	5	4.62	4.41	0.0
	1000	10	5	2.5	1.85	0.00	101.96	5	3.75	2.94	0.00	102.97	5	4.62	4.41	0.0
		25	5	2.5	1.85	0.00	101.96	5	3.75	2.94	0.00	102.97	5	4.63	4.41	0.0
		5	5	2.5	1.85	1.22	155.94	5	3.75	2.94	0.38	197.26	5	4.64	4.41	0.0
	500	10	5	2.5	1.85	1.21	156.69	5	3.75	2.94	0.37	197.58	5	4.64	4.41	0.0
		25	5	2.5	1.85	1.17	158.90	5	3.75	2.94	0.34	198.48	5	4.64	4.41	0.0
		5	5	2.5	1.85	0.76	181.25	5	3.75	2.94	0.00	202.97	5	4.64	4.41	0.0
1000	1000	10	5	2.5	1.85	0.75	181.77	5	3.75	2.94	0.00	202.97	5	4.64	4.41	0.0
		25	5	2.5	1.85	0.72	183.29	5	3.75	2.94	0.00	202.97	5	4.64	4.41	0.0
		5	5	2.5	1.85	0.00	201.96	5	3.75	2.94	0.00	202.97	5	4.64	4.41	0.0
	2000	10	5	2.5	1.85	0.00	201.96	5	3.75	2.94	0.00	202.97	5	4.64	4.41	0.0
		25	5	2.5	1.85	0.00	201.96	5	3.75	2.94	0.00	202.97	5	4.64	4.41	0.0

Table A.2: Sensitivity Analysis when $S = 5, 15$ and $\mu = 0$

$Ko=50, \mu=0$			$\alpha=0.95$					$\alpha=0.99$					$\alpha=0.999$				
L	Kd	Cd	S	s^*	r^*	Q^*	TC^*	S	s^*	r^*	Q^*	TC^*	S	s^*	r^*	Q^*	TC^*
1	250	5	40	3.52	0.83	4.87	12.97	40	4.03	1.32	4.90	13.42	40	4.72	1.97	4.94	14.02
		10	40	3.53	0.83	4.86	13.10	40	4.05	1.32	4.89	13.55	40	4.73	1.97	4.93	14.16
		25	40	3.57	0.83	4.83	13.51	40	4.08	1.32	4.86	13.96	40	4.77	1.97	4.90	14.58
	500	5	40	3.51	0.83	4.84	13.15	40	4.02	1.32	4.87	13.61	40	4.71	1.97	4.92	14.22
		10	40	3.52	0.83	4.83	13.29	40	4.03	1.32	4.86	13.74	40	4.72	1.97	4.90	14.36
		25	40	3.56	0.83	4.81	13.69	40	4.07	1.32	4.83	14.15	40	4.76	1.97	4.87	14.77
	1000	5	40	3.49	0.83	4.80	13.52	40	3.97	1.32	4.82	13.99	40	4.68	1.97	4.86	14.61
		10	40	3.50	0.83	4.79	13.66	40	4.01	1.32	4.81	14.12	40	4.69	1.97	4.85	14.75
		25	40	3.53	0.83	4.76	14.06	40	4.04	1.32	4.78	14.53	40	4.73	1.97	4.82	15.17
5	250	5	40	4.45	1.85	4.69	13.64	40	5.59	2.94	4.76	14.65	40	7.13	4.41	4.86	16.01
		10	40	4.46	1.85	4.68	13.78	40	5.60	2.94	4.75	14.79	40	7.14	4.41	4.84	16.16
		25	40	4.49	1.85	4.64	14.18	40	5.63	2.94	4.71	15.20	40	7.18	4.41	4.80	16.59
	500	5	40	4.43	1.85	4.66	13.83	40	5.57	2.94	4.73	14.85	40	7.11	4.41	4.82	16.23
		10	40	4.44	1.85	4.65	13.97	40	5.58	2.94	4.71	14.99	40	7.12	4.41	4.81	16.38
		25	40	4.47	1.85	4.61	14.37	40	5.61	2.94	4.67	15.41	40	7.16	4.41	4.76	16.81
	1000	5	40	4.40	1.85	4.60	14.21	40	5.54	2.94	4.66	15.25	40	7.07	4.41	4.75	16.67
		10	40	4.41	1.85	4.59	14.35	40	5.55	2.94	4.65	15.39	40	7.08	4.41	4.73	16.82
		25	40	4.44	1.85	4.55	14.75	40	5.58	2.94	4.61	15.81	40	7.12	4.41	4.69	17.25
15	250	5	40	5.50	3.21	4.17	14.23	40	7.47	5.10	4.28	15.96	40	10.14	7.64	4.46	18.34
		10	40	5.50	3.21	4.15	14.36	40	7.48	5.10	4.26	16.10	40	10.15	7.64	4.43	18.49
		25	40	5.52	3.21	4.08	14.74	40	7.49	5.10	4.19	16.51	40	10.17	7.64	4.35	18.93
	500	5	40	5.48	3.21	4.12	14.41	40	7.44	5.10	4.23	16.17	40	10.11	7.64	4.39	18.59
		10	40	5.48	3.21	4.10	14.54	40	7.45	5.10	4.21	16.31	40	10.11	7.64	4.37	18.74
		25	40	5.49	3.21	4.04	14.93	40	7.46	5.10	4.14	16.72	40	10.13	7.64	4.29	19.18
	1000	5	40	5.43	3.21	4.04	14.79	40	9.21	5.10	7.77	17.43	40	10.04	7.64	4.27	19.08
		10	40	5.44	3.21	4.02	14.92	40	7.40	5.10	4.11	16.73	40	10.05	7.64	4.24	19.23
		25	40	5.45	3.21	3.96	15.30	40	7.41	5.10	4.05	17.14	40	10.07	7.64	4.17	19.67

Table A.3: Sensitivity Analysis when $S=40$, $Ko=500$ and $\mu=0$

$L=5, \mu=0$			$\alpha=0.95$					$\alpha=0.99$					$\alpha=0.999$				
Ko	Kd	Cd	S^*	s^*	r^*	Q	TC^*	S	s^*	r^*	Q	TC^*	S	s^*	r^*	Q	TC^*
500	250	5	74.83	39.59	1.85	0	101.96	74.83	39.59	2.94	0	102.97	74.83	39.59	4.41	0	104.42
		10	74.83	39.59	1.85	0	101.96	74.83	39.59	2.94	0	102.97	74.83	39.59	4.41	0	104.42
		25	74.83	39.59	1.85	0	101.96	74.83	39.59	2.94	0	102.97	74.83	39.60	4.41	0	104.42
	500	5	74.83	39.59	1.85	0	101.96	74.83	39.59	2.94	0	102.97	74.83	39.59	4.41	0	104.42
		10	74.83	39.59	1.85	0	101.96	74.83	39.59	2.94	0	102.97	74.83	39.59	4.41	0	104.42
		25	74.84	39.59	1.85	0	101.96	74.84	39.59	2.94	0	102.97	74.84	39.60	4.41	0	104.42
	1000	5	74.84	39.58	1.85	0	101.96	74.84	39.59	2.94	0	102.97	74.84	39.59	4.41	0	104.42
		10	74.84	39.58	1.85	0	101.96	74.84	39.59	2.94	0	102.97	74.84	39.59	4.41	0	104.42
		25	74.84	39.59	1.85	0	101.96	74.84	39.59	2.94	0	102.97	74.85	39.59	4.41	0	104.42
1000	500	5	74.84	39.60	1.85	0	201.96	74.84	39.60	2.94	0	202.97	74.84	39.60	4.41	0	204.42
		10	74.84	39.60	1.85	0	201.96	74.84	39.60	2.94	0	202.97	74.84	39.60	4.41	0	204.42
		25	74.84	39.60	1.85	0	201.96	74.85	39.60	2.94	0	202.97	74.85	39.60	4.41	0	204.42
	1000	5	74.85	39.59	1.85	0	201.96	74.85	39.59	2.94	0	202.97	74.85	39.60	4.41	0	204.42
		10	74.85	39.59	1.85	0	201.96	74.85	39.60	2.94	0	202.97	74.85	39.60	4.41	0	204.42
		25	74.85	39.60	1.85	0	201.96	74.85	39.60	2.94	0	202.97	74.85	39.60	4.41	0	204.42
	2000	5	74.87	39.59	1.85	0	201.96	74.87	39.59	2.94	0	202.97	74.87	39.59	4.41	0	204.42
		10	74.87	39.59	1.85	0	201.96	74.87	39.59	2.94	0	202.97	74.87	39.59	4.41	0	204.42
		25	74.87	39.59	1.85	0	201.96	74.87	39.59	2.94	0	202.97	74.87	39.59	4.41	0	204.42

Table A.4: Sensitivity Analysis when $Q = 0$, $L = 5$ and $\mu = 0$

A.6 Tables of the Sensitivity Analysis for Positive Expected Net Demand Case

$L = 15, \mu = -0.01$			$\alpha = 0.95$					$\alpha = 0.99$					$\alpha = 0.999$				
Ko	Kd	Cd	S^*	s^*	r^*	Q^*	TC^*	S^*	s^*	r^*	Q^*	TC^*	S^*	s^*	r^*	Q^*	TC^*
500	250	5	73.15	3.61	3.30	0.15	12.25	35.81	29.76	5.20	0.15	12.84	73.47	34.52	7.76	0.15	13.67
		10	73.45	3.38	3.30	0.15	12.25	73.39	3.67	5.20	0.15	12.84	73.46	34.55	7.76	0.15	13.67
		25	73.47	3.53	3.30	0.15	12.25	73.34	3.76	5.20	0.15	12.84	73.30	34.60	7.76	0.15	13.67
	500	5	73.20	3.24	3.30	0.15	12.25	69.72	30.42	5.20	0.15	12.84	73.47	34.52	7.76	0.15	13.67
		10	72.87	3.29	3.30	0.15	12.25	63.73	29.31	5.20	0.15	12.84	73.46	34.55	7.76	0.15	13.67
		25	73.53	3.53	3.30	0.15	12.25	73.38	3.75	5.20	0.15	12.84	73.30	34.60	7.76	0.15	13.67
	1000	5	73.50	3.23	3.30	0.15	12.25	72.31	31.70	5.20	0.15	12.84	73.47	34.52	7.76	0.15	13.67
		10	73.34	3.13	3.30	0.15	12.25	72.17	32.02	5.20	0.15	12.84	73.46	34.55	7.76	0.15	13.67
		25	73.50	3.34	3.30	0.15	12.25	69.30	28.66	5.20	0.15	12.84	73.30	34.60	7.76	0.15	13.67
1000	500	5	41.56	5.99	3.30	4.99	13.91	42.14	7.78	5.20	4.79	15.51	36.04	10.62	7.76	5.13	17.32
		10	42.78	5.94	3.30	4.90	13.99	34.29	7.73	5.20	4.70	15.59	37.37	10.53	7.76	4.97	17.44
		25	46.72	5.79	3.30	4.64	14.22	46.96	7.59	5.20	4.45	15.82	41.29	10.30	7.76	4.58	17.74
	1000	5	47.34	5.74	3.30	4.64	14.11	47.77	7.54	5.20	4.44	15.71	36.04	10.62	7.76	5.13	17.32
		10	48.60	5.71	3.30	4.56	14.17	48.93	7.50	5.20	4.37	15.78	37.37	10.53	7.76	4.97	17.44
		25	52.99	5.59	3.30	4.35	14.36	52.84	7.39	5.20	4.16	15.98	41.29	10.30	7.76	4.58	17.74
	2000	5	58.53	5.44	3.30	4.17	14.36	57.99	7.25	5.20	3.99	15.99	36.04	10.62	7.76	5.13	17.32
		10	60.80	5.40	3.30	4.09	14.41	59.68	7.22	5.20	3.93	16.04	37.37	10.53	7.76	4.97	17.44
		25	67.31	6.30	3.30	4.20	14.60	67.11	7.10	5.20	3.72	16.18	41.29	10.30	7.76	4.58	17.74

Table A.5: Sensitivity Analysis when $L = 15$ and $\mu = -0.01$

$Ko = 500, \mu = -0.01$			$\alpha = 0.95$					$\alpha = 0.99$					$\alpha = 0.999$				
L	Kd	Cd	S^*	s^*	r^*	Q^*	TC^*	S^*	s^*	r^*	Q^*	TC^*	S^*	s^*	r^*	Q^*	TC^*
1	250	5	36.68	3.78	0.84	5.26	11.54	37.14	4.27	1.32	5.25	12.02	3.70	5.18	1.98	5.59	12.49
		10	38.10	3.73	0.84	5.16	11.66	38.56	4.21	1.32	5.15	12.14	35.39	5.10	1.98	5.47	12.62
		25	42.67	3.56	0.84	4.87	11.96	34.12	4.05	1.32	4.87	12.44	40.15	4.87	1.98	5.08	12.97
	500	5	40.29	3.61	0.84	5.02	11.70	40.75	4.09	1.32	5.01	12.18	3.70	5.18	1.98	5.59	12.49
		10	41.67	3.56	0.84	4.93	11.80	42.13	4.05	1.32	4.93	12.28	35.39	5.10	1.98	5.47	12.62
		25	46.28	3.43	0.84	4.69	12.07	46.73	3.92	1.32	4.69	12.55	40.15	4.87	1.98	5.08	12.97
	1000	5	46.48	3.38	0.84	4.69	11.93	46.94	3.87	1.32	4.68	12.41	3.70	5.18	1.98	5.59	12.49
		10	47.98	3.34	0.84	4.62	12.01	48.43	3.83	1.32	4.62	12.49	35.39	5.10	1.98	5.47	12.62
		25	53.58	3.22	0.84	4.40	12.23	54.01	3.71	1.32	4.40	12.71	40.15	4.87	1.98	5.08	12.97
5	250	5	3.90	4.51	1.88	4.71	11.66	34.69	5.57	2.98	4.66	12.67	31.47	7.30	4.45	4.99	13.80
		10	35.13	4.46	1.88	4.61	11.78	35.91	5.52	2.98	4.56	12.79	32.93	7.21	4.45	4.83	13.95
		25	38.95	4.31	1.88	4.34	12.10	39.69	5.38	2.98	4.29	13.12	37.18	6.99	4.45	4.45	14.33
	500	5	37.42	4.33	1.88	4.46	11.84	38.21	5.39	2.98	4.41	12.85	31.47	7.30	4.45	4.99	13.80
		10	38.57	4.29	1.88	4.38	11.94	39.35	5.35	2.98	4.33	12.96	32.93	7.21	4.45	4.83	13.95
		25	42.23	4.18	1.88	4.16	12.23	42.97	5.24	2.98	4.11	13.25	37.18	6.99	4.45	4.45	14.33
	1000	5	34.11	4.11	1.88	4.14	12.10	34.88	5.17	2.98	4.08	13.12	31.47	7.30	4.45	4.99	13.80
		10	44.25	4.08	1.88	4.08	12.19	45.00	5.14	2.98	4.03	13.21	32.93	7.21	4.45	4.83	13.95
		25	48.05	3.99	1.88	3.91	12.43	48.73	5.06	2.98	3.86	13.45	37.18	6.99	4.45	4.45	14.33
15	250	5	73.15	3.61	3.30	0.15	12.25	35.81	29.76	5.20	0.15	12.84	73.47	34.52	7.76	0.15	13.67
		10	73.45	3.38	3.30	0.15	12.25	73.39	3.67	5.20	0.15	12.84	73.46	34.55	7.76	0.15	13.67
		25	73.47	3.53	3.30	0.15	12.25	73.34	3.76	5.20	0.15	12.84	73.30	34.60	7.76	0.15	13.67
	500	5	73.20	3.24	3.30	0.15	12.25	69.72	30.42	5.20	0.15	12.84	73.47	34.52	7.76	0.15	13.67
		10	72.87	3.29	3.30	0.15	12.25	63.73	29.31	5.20	0.15	12.84	73.46	34.55	7.76	0.15	13.67
		25	73.53	3.53	3.30	0.15	12.25	73.38	3.75	5.20	0.15	12.84	73.30	34.60	7.76	0.15	13.67
	1000	5	73.50	3.23	3.30	0.15	12.25	72.31	31.70	5.20	0.15	12.84	73.47	34.52	7.76	0.15	13.67
		10	73.34	3.13	3.30	0.15	12.25	72.17	32.02	5.20	0.15	12.84	73.46	34.55	7.76	0.15	13.67
		25	73.50	3.34	3.30	0.15	12.25	69.30	28.66	5.20	0.15	12.84	73.30	34.60	7.76	0.15	13.67

Table A.6: Sensitivity Analysis when $Ko = 500$ and $\mu = -0.01$

$L=5, Ko=500$			$\alpha=0.95$					$\alpha=0.99$					$\alpha=0.999$				
μ	Kd	Cd	S^*	s^*	r^*	Q^*	TC^*	S^*	s^*	r^*	Q^*	TC^*	S^*	s^*	r^*	Q^*	TC^*
		5	3.90	4.51	1.88	4.71	11.66	34.69	5.57	2.98	4.66	12.67	31.47	7.30	4.45	4.99	13.80
	250	10	35.13	4.46	1.88	4.61	11.78	35.91	5.52	2.98	4.56	12.79	32.93	7.21	4.45	4.83	13.95
		25	38.95	4.31	1.88	4.34	12.10	39.69	5.38	2.98	4.29	13.12	37.18	6.99	4.45	4.45	14.33
		5	37.42	4.33	1.88	4.46	11.84	38.21	5.39	2.98	4.41	12.85	31.47	7.30	4.45	4.99	13.80
-0.01	500	10	38.57	4.29	1.88	4.38	11.94	39.35	5.35	2.98	4.33	12.96	32.93	7.21	4.45	4.83	13.95
		25	42.23	4.18	1.88	4.16	12.23	42.97	5.24	2.98	4.11	13.25	37.18	6.99	4.45	4.45	14.33
		5	34.11	4.11	1.88	4.14	12.10	34.88	5.17	2.98	4.08	13.12	31.47	7.30	4.45	4.99	13.80
	1000	10	44.25	4.08	1.88	4.08	12.19	45.00	5.14	2.98	4.03	13.21	32.93	7.21	4.45	4.83	13.95
		25	48.05	3.99	1.88	3.91	12.43	48.73	5.06	2.98	3.86	13.45	37.18	6.99	4.45	4.45	14.33
		5	75.04	74.04	2.17	8.89	10.55	75.04	74.04	3.29	8.77	11.56	75.05	74.05	4.80	8.61	12.91
	250	10	75.04	74.04	2.17	8.89	10.55	75.04	74.04	3.29	8.77	11.56	75.05	74.05	4.80	8.61	12.91
		25	75.04	74.04	2.17	8.89	10.55	75.05	74.05	3.29	8.77	11.56	75.05	74.05	4.80	8.61	12.91
		5	75.04	74.04	2.17	8.89	10.56	75.04	74.04	3.29	8.77	11.56	75.05	74.05	4.80	8.61	12.91
-0.1	500	10	75.04	74.04	2.17	8.89	10.56	75.05	74.05	3.29	8.77	11.56	75.05	74.05	4.80	8.61	12.91
		25	75.05	74.05	2.17	8.89	10.56	75.05	74.05	3.29	8.77	11.56	75.05	74.05	4.80	8.61	12.91
		5	75.05	74.05	2.17	8.89	10.56	75.05	74.05	3.29	8.77	11.56	75.05	74.05	4.80	8.61	12.91
	1000	10	75.05	74.05	2.17	8.89	10.56	75.05	74.05	3.29	8.77	11.56	75.05	74.05	4.80	8.61	12.91
		25	75.05	74.05	2.17	8.89	10.56	75.05	74.05	3.29	8.77	11.56	75.05	74.05	4.80	8.61	12.91
		5	75.00	40.00	5.23	23.12	23.35	75.00	40.00	6.69	22.66	24.35	75.00	40.00	8.50	22.09	25.59
	250	10	75.00	40.00	5.23	23.12	23.35	75.00	40.00	6.69	22.66	24.35	75.00	40.00	8.50	22.09	25.59
		25	75.00	40.00	5.23	23.12	23.35	75.00	40.00	6.69	22.66	24.35	75.00	40.00	8.50	22.09	25.59
		5	75.00	40.00	5.23	23.12	23.35	75.00	40.00	6.69	22.66	24.35	75.00	40.00	8.50	22.09	25.59
-1	500	10	75.00	40.00	5.23	23.12	23.35	75.00	40.00	6.69	22.66	24.35	75.00	40.00	8.50	22.09	25.59
		25	75.00	40.00	5.23	23.12	23.35	75.00	40.00	6.69	22.66	24.35	75.00	40.00	8.50	22.09	25.59
		5	75.00	40.00	5.23	23.12	23.35	75.00	40.00	6.69	22.66	24.35	75.00	40.00	8.50	22.09	25.59
	1000	10	75.00	40.00	5.23	23.12	23.35	75.00	40.00	6.69	22.66	24.35	75.00	40.00	8.50	22.09	25.59
		25	75.00	40.00	5.23	23.12	23.35	75.00	40.00	6.69	22.66	24.35	75.00	40.00	8.50	22.09	25.59

Table A.7: Sensitivity Analysis when $L=5$, $Ko=500$ and $\mu=-0.01, -0.1, -1$

$L=1, \mu=-0.1$			$\alpha=0.95$					$\alpha=0.99$					$\alpha=0.999$				
Ko	Kd	Cd	S^*	s^*	r^*	Q^*	TC^*	S^*	s^*	r^*	Q^*	TC^*	S^*	s^*	r^*	Q^*	TC^*
500	250	5	75.04	74.04	0.89	9.79	10.58	75.04	74.04	1.39	9.77	11.06	75.04	74.04	2.05	9.76	11.71
		10	75.04	74.04	0.89	9.79	10.58	75.04	74.04	1.39	9.77	11.06	75.04	74.04	2.05	9.76	11.71
		25	75.05	74.05	0.89	9.79	10.58	75.05	74.05	1.39	9.77	11.06	75.05	74.05	2.05	9.76	11.71
	500	5	75.04	74.04	0.89	9.79	10.58	75.05	74.05	1.39	9.77	11.06	75.04	74.04	2.05	9.76	11.71
		10	75.04	74.04	0.89	9.79	10.58	75.05	74.05	1.39	9.77	11.06	75.04	74.04	2.05	9.76	11.71
		25	75.05	74.05	0.89	9.79	10.58	75.05	74.05	1.39	9.77	11.06	75.05	74.05	2.05	9.76	11.71
	1000	5	75.05	74.05	0.89	9.79	10.58	75.05	74.05	1.39	9.77	11.06	75.04	74.04	2.05	9.76	11.71
		10	75.05	74.05	0.89	9.79	10.58	75.05	74.05	1.39	9.77	11.06	75.04	74.04	2.05	9.76	11.71
		25	75.05	74.05	0.89	9.79	10.58	75.05	74.05	1.39	9.77	11.06	75.05	74.05	2.05	9.76	11.71
1000	500	5	75.06	74.06	0.89	13.93	14.73	75.07	74.07	1.39	13.92	15.21	75.06	74.06	2.05	13.91	15.87
		10	75.06	74.06	0.89	13.93	14.73	75.07	74.07	1.39	13.92	15.21	75.06	74.06	2.05	13.91	15.87
		25	75.06	74.06	0.89	13.93	14.73	75.07	74.07	1.39	13.92	15.21	75.07	74.07	2.05	13.91	15.87
	1000	5	75.07	74.07	0.89	13.93	14.73	75.07	74.07	1.39	13.92	15.21	75.06	74.06	2.05	13.91	15.87
		10	75.07	74.07	0.89	13.93	14.73	75.07	74.07	1.39	13.92	15.21	75.06	74.06	2.05	13.91	15.87
		25	75.07	74.07	0.89	13.93	14.73	75.07	74.07	1.39	13.92	15.21	75.07	74.07	2.05	13.91	15.87
	2000	5	75.07	74.03	0.89	13.93	14.73	75.08	74.03	1.39	13.92	15.21	75.06	74.06	2.05	13.91	15.87
		10	75.07	74.03	0.89	13.93	14.73	75.08	74.03	1.39	13.92	15.21	75.06	74.06	2.05	13.91	15.87
		25	75.08	74.03	0.89	13.93	14.73	75.08	74.04	1.39	13.92	15.21	75.07	74.07	2.05	13.91	15.87

Table A.8: Sensitivity Analysis when $L=1$ and $\mu=-0.1$

$L = 5, \mu = -0.1$			$\alpha = 0.95$					$\alpha = 0.99$					$\alpha = 0.999$				
Ko	Kd	Cd	S^*	s^*	r^*	Q^*	TC^*	S^*	s^*	r^*	Q^*	TC^*	S^*	s^*	r^*	Q^*	TC^*
500	250	5	75.04	74.04	2.17	8.89	10.55	75.04	74.04	3.29	8.77	11.56	75.05	74.05	4.80	8.61	12.91
		10	75.04	74.04	2.17	8.89	10.55	75.04	74.04	3.29	8.77	11.56	75.05	74.05	4.80	8.61	12.91
		25	75.04	74.04	2.17	8.89	10.55	75.05	74.05	3.29	8.77	11.56	75.05	74.05	4.80	8.61	12.91
	500	5	75.04	74.04	2.17	8.89	10.56	75.04	74.04	3.29	8.77	11.56	75.05	74.05	4.80	8.61	12.91
		10	75.04	74.04	2.17	8.89	10.56	75.05	74.05	3.29	8.77	11.56	75.05	74.05	4.80	8.61	12.91
		25	75.05	74.05	2.17	8.89	10.56	75.05	74.05	3.29	8.77	11.56	75.05	74.05	4.80	8.61	12.91
	1000	5	75.05	74.05	2.17	8.89	10.56	75.05	74.05	3.29	8.77	11.56	75.05	74.05	4.80	8.61	12.91
		10	75.05	74.05	2.17	8.89	10.56	75.05	74.05	3.29	8.77	11.56	75.05	74.05	4.80	8.61	12.91
		25	75.05	74.05	2.17	8.89	10.56	75.05	74.05	3.29	8.77	11.56	75.05	74.05	4.80	8.61	12.91
1000	500	5	75.06	74.06	2.17	13.06	14.73	75.06	74.06	3.29	12.98	15.78	75.06	74.06	4.80	12.87	17.17
		10	75.06	74.06	2.17	13.06	14.73	75.06	74.06	3.29	12.98	15.78	75.06	74.06	4.80	12.87	17.17
		25	75.06	74.06	2.17	13.06	14.73	75.06	74.06	3.29	12.98	15.78	75.07	74.07	4.80	12.87	17.17
	1000	5	75.06	74.06	2.17	13.06	14.73	75.06	74.06	3.29	12.98	15.78	75.06	74.06	4.80	12.87	17.17
		10	75.06	74.06	2.17	13.06	14.73	75.07	74.07	3.29	12.98	15.78	75.06	74.06	4.80	12.87	17.17
		25	75.06	74.06	2.17	13.06	14.73	75.07	74.07	3.29	12.98	15.78	75.07	74.07	4.80	12.87	17.17
	2000	5	75.07	74.00	2.17	13.06	14.73	75.07	74.00	3.29	12.97	15.78	75.06	74.06	4.80	12.87	17.17
		10	75.07	74.01	2.17	13.06	14.73	75.07	74.00	3.29	12.97	15.78	75.06	74.06	4.80	12.87	17.17
		25	75.07	74.01	2.17	13.06	14.73	75.08	74.00	3.29	12.97	15.78	75.07	74.07	4.80	12.87	17.17

Table A.9: Sensitivity Analysis when $L = 5$ and $\mu = -0.1$

$L = 15, \mu = -0.1$			$\alpha = 0.95$					$\alpha = 0.99$					$\alpha = 0.999$				
Ko	Kd	Cd	S^*	s^*	r^*	Q^*	TC^*	S^*	s^*	r^*	Q^*	TC^*	S^*	s^*	r^*	Q^*	TC^*
500	250	5	75.03	74.03	4.16	6.68	9.33	75.04	74.04	6.16	6.01	10.67	75.04	74.04	8.81	5.07	12.38
		10	75.03	74.03	4.16	6.68	9.33	75.04	74.04	6.16	6.01	10.67	75.04	74.04	8.81	5.07	12.38
		25	75.04	74.04	4.16	6.68	9.33	75.04	74.04	6.16	6.01	10.67	75.04	74.04	8.81	5.07	12.38
	500	5	75.03	74.03	4.16	6.68	9.33	75.04	74.04	6.16	6.01	10.67	75.04	74.04	8.81	5.07	12.38
		10	75.04	74.04	4.16	6.68	9.33	75.04	74.04	6.16	6.01	10.67	75.04	74.04	8.81	5.07	12.38
		25	75.04	74.04	4.16	6.68	9.33	75.04	74.04	6.16	6.01	10.67	75.04	74.04	8.81	5.07	12.38
	1000	5	75.04	74.04	4.16	6.68	9.33	75.04	74.04	6.16	6.01	10.67	75.04	74.04	8.81	5.07	12.38
		10	75.04	74.04	4.16	6.68	9.33	75.04	74.04	6.16	6.01	10.67	75.04	74.04	8.81	5.07	12.38
		25	75.04	74.04	4.16	6.68	9.33	75.04	74.04	6.16	6.01	10.67	75.04	74.04	8.81	5.07	12.38
1000	500	5	75.05	74.05	4.16	10.91	13.57	75.06	74.06	6.16	10.46	15.12	75.06	74.06	8.81	9.85	17.17
		10	75.05	74.05	4.16	10.91	13.57	75.06	74.06	6.16	10.46	15.12	75.06	74.06	8.81	9.85	17.17
		25	75.05	74.05	4.16	10.91	13.57	75.06	74.06	6.16	10.46	15.12	75.07	74.07	8.81	9.85	17.17
	1000	5	75.05	74.05	4.16	10.91	13.57	75.06	74.06	6.16	10.46	15.12	75.06	74.06	8.81	9.85	17.17
		10	75.05	74.05	4.16	10.91	13.57	75.06	74.06	6.16	10.46	15.12	75.06	74.06	8.81	9.85	17.17
		25	75.06	74.06	4.16	10.91	13.57	75.06	74.06	6.16	10.46	15.12	75.07	74.07	8.81	9.85	17.17
	2000	5	75.06	73.99	4.16	10.91	13.57	75.07	73.97	6.16	10.46	15.12	75.06	74.06	8.81	9.85	17.17
		10	75.06	73.99	4.16	10.91	13.57	75.07	73.97	6.16	10.46	15.12	75.06	74.06	8.81	9.85	17.17
		25	75.06	73.99	4.16	10.91	13.57	75.07	73.97	6.16	10.46	15.12	75.07	74.07	8.81	9.85	17.17

Table A.10: Sensitivity Analysis when $L = 15$ and $\mu = -0.1$

$L=1, \mu=-1$			$\alpha=0.95$					$\alpha=0.99$					$\alpha=0.999$				
Ko	Kd	Cd	S^*	s^*	r^*	Q^*	TC^*	S^*	s^*	r^*	Q^*	TC^*	S^*	s^*	r^*	Q^*	TC^*
500	250	5	75	40.00	1.48	29.66	30.14	75	40.00	2.04	29.62	30.66	75	40.00	2.77	29.58	31.34
		10	75	40.00	1.48	29.66	30.14	75	40.00	2.04	29.62	30.66	75	40.00	2.77	29.58	31.34
		25	75	40.00	1.48	29.66	30.14	75	40.00	2.04	29.62	30.66	75	40.00	2.77	29.58	31.34
	500	5	75	40.00	1.48	29.66	30.14	75	40.00	2.04	29.62	30.66	75	40.00	2.77	29.58	31.34
		10	75	40.00	1.48	29.66	30.14	75	40.00	2.04	29.62	30.66	75	40.00	2.77	29.58	31.34
		25	75	40.00	1.48	29.66	30.14	75	40.00	2.04	29.62	30.66	75	40.00	2.77	29.58	31.34
	1000	5	75	40.00	1.48	29.66	30.14	75	40.00	2.04	29.62	30.66	75	40.00	2.77	29.58	31.34
		10	75	40.00	1.48	29.66	30.14	75	40.00	2.04	29.62	30.66	75	40.00	2.77	29.58	31.34
		25	75	40.00	1.48	29.66	30.14	75	40.00	2.04	29.62	30.66	75	40.00	2.77	29.58	31.34
1000	500	5	75	39.99	1.48	42.75	34.22	75	40.36	2.04	42.72	43.76	75	39.75	2.77	42.69	44.46
		10	75	39.99	1.48	42.75	34.22	75	40.36	2.04	42.72	43.76	75	39.75	2.77	42.69	44.46
		25	75	39.99	1.48	42.75	34.22	75	40.36	2.04	42.72	43.76	75	39.75	2.77	42.69	44.46
	1000	5	75	39.99	1.48	42.75	34.22	75	40.36	2.04	42.72	43.76	75	39.75	2.77	42.69	44.46
		10	75	39.99	1.48	42.75	34.22	75	40.36	2.04	42.72	43.76	75	39.75	2.77	42.69	44.46
		25	75	39.99	1.48	42.75	34.22	75	40.36	2.04	42.72	43.76	75	39.75	2.77	42.69	44.46
	2000	5	75	39.99	1.48	42.75	34.22	75	40.36	2.04	42.72	43.76	75	39.75	2.77	42.69	44.46
		10	75	39.99	1.48	42.75	34.22	75	40.36	2.04	42.72	43.76	75	39.75	2.77	42.69	44.46
		25	75	39.99	1.48	42.75	34.22	75	40.36	2.04	42.72	43.76	75	39.75	2.77	42.69	44.46

Table A.11: Sensitivity Analysis when $L = 1$ and $\mu = -1$

$L=5, \mu=-1$			$\alpha=0.95$					$\alpha=0.99$					$\alpha=0.999$				
Ko	Kd	Cd	S^*	s^*	r^*	Q^*	TC^*	S^*	s^*	r^*	Q^*	TC^*	S^*	s^*	r^*	Q^*	TC^*
500	250	5	75	40.00	5.23	23.12	23.35	75	40.00	6.69	22.66	24.35	75	40.00	8.50	22.09	25.59
		10	75	40.00	5.23	23.12	23.35	75	40.00	6.69	22.66	24.35	75	40.00	8.50	22.09	25.59
		25	75	40.00	5.23	23.12	23.35	75	40.00	6.69	22.66	24.35	75	40.00	8.50	22.09	25.59
	500	5	75	40.00	5.23	23.12	23.35	75	40.00	6.69	22.66	24.35	75	40.00	8.50	22.09	25.59
		10	75	40.00	5.23	23.12	23.35	75	40.00	6.69	22.66	24.35	75	40.00	8.50	22.09	25.59
		25	75	40.00	5.23	23.12	23.35	75	40.00	6.69	22.66	24.35	75	40.00	8.50	22.09	25.59
	1000	5	75	40.00	5.23	23.12	23.35	75	40.00	6.69	22.66	24.35	75	40.00	8.50	22.09	25.59
		10	75	40.00	5.23	23.12	23.35	75	40.00	6.69	22.66	24.35	75	40.00	8.50	22.09	25.59
		25	75	40.00	5.23	23.12	23.35	75	40.00	6.69	22.66	24.35	75	40.00	8.50	22.09	25.59
1000	500	5	75	44.33	5.23	35.79	36.02	75	45.02	6.69	35.46	37.15	75	46.43	8.50	35.05	38.55
		10	75	44.33	5.23	35.79	36.02	75	45.02	6.69	35.46	37.15	75	46.43	8.50	35.05	38.55
		25	75	44.33	5.23	35.79	36.02	75	45.02	6.69	35.46	37.15	75	46.43	8.50	35.05	38.55
	1000	5	75	44.33	5.23	35.79	36.02	75	45.02	6.69	35.46	37.15	75	46.43	8.50	35.05	38.55
		10	75	44.33	5.23	35.79	36.02	75	45.02	6.69	35.46	37.15	75	46.43	8.50	35.05	38.55
		25	75	44.33	5.23	35.79	36.02	75	45.02	6.69	35.46	37.15	75	46.43	8.50	35.05	38.55
	2000	5	75	44.33	5.23	35.79	36.02	75	45.02	6.69	35.46	37.15	75	46.43	8.50	35.05	38.55
		10	75	44.33	5.23	35.79	36.02	75	45.02	6.69	35.46	37.15	75	46.43	8.50	35.05	38.55
		25	75	44.33	5.23	35.79	36.02	75	45.02	6.69	35.46	37.15	75	46.43	8.50	35.05	38.55

Table A.12: Sensitivity Analysis when $L=5$ and $\mu=-1$

$L = 15, \mu = -1$			$\alpha = 0.95$					$\alpha = 0.99$					$\alpha = 0.999$				
Ko	Kd	Cd	S^*	s^*	r^*	Q^*	TC^*	S^*	s^*	r^*	Q^*	TC^*	S^*	s^*	r^*	Q^*	TC^*
500	250	5	75	40.00	13.72	15.00	13.30	75	40.00	16.71	15.00	14.21	75	40.00	20.21	15.00	15.35
		10	75	40.00	13.72	15.00	13.30	75	40.00	16.71	15.00	14.21	75	40.00	20.21	15.00	15.35
		25	75	40.00	13.72	15.00	13.30	75	40.00	16.71	15.00	14.21	75	40.00	20.21	15.00	15.35
	500	5	75	40.00	13.72	15.00	13.30	75	40.00	16.71	15.00	14.21	75	40.00	20.21	15.00	15.35
		10	75	40.00	13.72	15.00	13.30	75	40.00	16.71	15.00	14.21	75	40.00	20.21	15.00	15.35
		25	75	40.00	13.72	15.00	13.30	75	40.00	16.71	15.00	14.21	75	40.00	20.21	15.00	15.35
	1000	5	75	40.00	13.72	15.00	13.30	75	40.00	16.71	15.00	14.21	75	40.00	20.21	15.00	15.35
		10	75	40.00	13.72	15.00	13.30	75	40.00	16.71	15.00	14.21	75	40.00	20.21	15.00	15.35
		25	75	40.00	13.72	15.00	13.30	75	40.00	16.71	15.00	14.21	75	40.00	20.21	15.00	15.35
1000	500	5	75	40.00	13.72	24.65	23.38	75	40.00	16.71	22.92	24.63	75	40.00	20.21	20.87	26.08
		10	75	40.00	13.72	24.65	23.38	75	40.00	16.71	22.92	24.63	75	40.00	20.21	20.87	26.08
		25	75	40.00	13.72	24.65	23.38	75	40.00	16.71	22.92	24.63	75	40.00	20.21	20.87	26.08
	1000	5	75	40.00	13.72	24.65	23.38	75	40.00	16.71	22.92	24.63	75	40.00	20.21	20.87	26.08
		10	75	40.00	13.72	24.65	23.38	75	40.00	16.71	22.92	24.63	75	40.00	20.21	20.87	26.08
		25	75	40.00	13.72	24.65	23.38	75	40.00	16.71	22.92	24.63	75	40.00	20.21	20.87	26.08
	2000	5	75	40.00	13.72	24.65	23.38	75	40.00	16.71	22.92	24.63	75	40.00	20.21	20.87	26.08
		10	75	40.00	13.72	24.65	23.38	75	40.00	16.71	22.92	24.63	75	40.00	20.21	20.87	26.08
		25	75	40.00	13.72	24.65	23.38	75	40.00	16.71	22.92	24.63	75	40.00	20.21	20.87	26.08

Table A.3: Sensitivity Analysis when $L = 15$ and $\mu = -1$

A.7 Tables of the Performance Analysis for Positive Expected Net Demand Case

$L=5,\ Ko=500,\ \mu=-0.01$													
α	Kd	Cd	S_1^*	s_1^*	r_1^*	Q_1^*	TC_1^*	S_2^*	s_2^*	r_2^*	Q_2	TC_2^*	$\triangle\%$
0.95	250	5	33.90	4.51	1.88	4.71	11.66	20.14	17.90	1.88	3.16	17.28	32.53
		10	35.13	4.46	1.88	4.61	11.78	20.34	18.10	1.88	3.16	17.39	32.29
		25	38.95	4.31	1.88	4.34	12.10	20.94	18.70	1.88	3.16	17.73	31.76
	500	5	37.42	4.33	1.88	4.46	11.84	22.36	19.20	1.88	3.16	18.27	35.21
		10	38.57	4.29	1.88	4.38	11.94	22.55	19.39	1.88	3.16	18.37	35.01
		25	42.23	4.18	1.88	4.16	12.23	23.13	19.96	1.88	3.16	18.67	34.53
	1000	5	43.11	4.11	1.88	4.14	12.10	25.36	20.88	1.88	3.16	19.50	37.94
		10	44.25	4.08	1.88	4.08	12.19	25.54	21.07	1.88	3.16	19.59	37.79
		25	48.05	3.99	1.88	3.91	12.43	26.09	21.61	1.88	3.16	19.86	37.43
0.99	250	5	34.69	5.57	2.98	4.66	12.67	21.09	18.85	2.98	3.16	18.26	30.60
		10	35.91	5.52	2.98	4.56	12.79	21.29	19.06	2.98	3.16	18.37	30.38
		25	39.69	5.38	2.98	4.29	13.12	21.90	19.66	2.98	3.16	18.71	29.91
	500	5	38.21	5.39	2.98	4.41	12.85	23.32	20.16	2.98	3.16	19.26	33.25
		10	39.35	5.35	2.98	4.33	12.96	23.51	20.35	2.98	3.16	19.36	33.07
		25	42.97	5.24	2.98	4.11	13.25	24.09	20.93	2.98	3.16	19.67	32.64
	1000	5	43.88	5.17	2.98	4.08	13.12	26.32	21.85	2.98	3.16	20.50	35.98
		10	45.00	5.14	2.98	4.03	13.21	26.51	22.03	2.98	3.16	20.59	35.85
		25	48.73	5.06	2.98	3.86	13.45	27.05	22.58	2.98	3.16	20.86	35.52
0.999	250	5	31.47	7.30	4.45	4.99	13.80	22.37	20.14	4.45	3.16	19.57	29.48
		10	32.93	7.21	4.45	4.83	13.95	22.58	20.34	4.45	3.16	19.69	29.17
		25	37.18	6.99	4.45	4.45	14.33	23.19	20.95	4.45	3.16	20.04	28.48
	500	5	31.47	7.30	4.45	4.99	13.80	24.61	21.45	4.45	3.16	20.59	32.95
		10	32.93	7.21	4.45	4.83	13.95	24.81	21.65	4.45	3.16	20.69	32.60
		25	37.18	6.99	4.45	4.45	14.33	25.39	22.22	4.45	3.16	21.00	31.76
	1000	5	31.47	7.30	4.45	4.99	13.80	27.63	23.16	4.45	3.16	21.84	36.81
		10	32.93	7.21	4.45	4.83	13.95	27.81	23.34	4.45	3.16	21.94	36.41
		25	37.18	6.99	4.45	4.45	14.33	28.36	23.89	4.45	3.16	22.20	35.46

Table A.14: Comparison of the (S, s, r, Q) policy with the EOQ based heuristic policy when $L = 5$, $Ko = 500$ and $\mu = -0.01$

$L = 5,\ Ko = 500,\ \mu = -0.1$													
α	Kd	Cd	S_1^*	s_1^*	r_1^*	Q_1^*	TC_1^*	S_2^*	s_2^*	r_2^*	Q_2	TC_2^*	△%
0.95	250	5	75.04	74.04	2.17	8.89	10.55	121.07	114.00	2.17	10.00	10.61	0.52
		10	75.04	74.04	2.17	8.89	10.55	121.71	114.63	2.17	10.00	10.61	0.52
		25	75.04	74.04	2.17	8.89	10.55	122.67	115.60	2.17	10.00	10.61	0.52
	500	5	75.04	74.04	2.17	8.89	10.56	126.30	116.30	2.17	10.00	10.61	0.52
		10	75.04	74.04	2.17	8.89	10.56	126.30	116.30	2.17	10.00	10.61	0.52
		25	75.05	74.05	2.17	8.89	10.56	126.30	116.30	2.17	10.00	10.61	0.52
	1000	5	75.05	74.05	2.17	8.89	10.56	127.82	113.68	2.17	10.00	10.61	0.52
		10	75.05	74.05	2.17	8.89	10.56	127.82	113.68	2.17	10.00	10.61	0.52
		25	75.05	74.05	2.17	8.89	10.56	128.46	114.32	2.17	10.00	10.61	0.52
0.99	250	5	75.04	74.04	3.29	8.77	11.56	122.47	115.39	3.29	10.00	11.63	0.58
		10	75.04	74.04	3.29	8.77	11.56	123.14	116.07	3.29	10.00	11.63	0.58
		25	75.05	74.05	3.29	8.77	11.56	122.46	115.38	3.29	10.00	11.63	0.58
	500	5	75.04	74.04	3.29	8.77	11.56	125.27	115.27	3.29	10.00	11.63	0.58
		10	75.05	74.05	3.29	8.77	11.56	127.54	117.54	3.29	10.00	11.63	0.58
		25	75.05	74.05	3.29	8.77	11.56	124.95	114.95	3.29	10.00	11.63	0.58
	1000	5	75.05	74.05	3.29	8.77	11.56	129.08	114.94	3.29	10.00	11.63	0.58
		10	75.05	74.05	3.29	8.77	11.56	129.08	114.94	3.29	10.00	11.63	0.58
		25	75.05	74.05	3.29	8.77	11.56	128.70	114.55	3.29	10.00	11.63	0.58
0.999	250	5	75.05	74.05	4.80	8.61	12.91	125.57	118.50	4.80	10.00	13.00	0.66
		10	75.05	74.05	4.80	8.61	12.91	123.96	116.89	4.80	10.00	13.00	0.66
		25	75.05	74.05	4.80	8.61	12.91	126.19	119.12	4.80	10.00	13.00	0.66
	500	5	75.05	74.05	4.80	8.61	12.91	127.80	117.80	4.80	10.00	13.00	0.66
		10	75.05	74.05	4.80	8.61	12.91	127.81	117.81	4.80	10.00	13.00	0.66
		25	75.05	74.05	4.80	8.61	12.91	125.55	115.55	4.80	10.00	13.00	0.66
	1000	5	75.05	74.05	4.80	8.61	12.91	130.55	116.41	4.80	10.00	13.00	0.66
		10	75.05	74.05	4.80	8.61	12.91	131.51	117.37	4.80	10.00	13.00	0.66
		25	75.05	74.05	4.80	8.61	12.91	129.95	115.81	4.80	10.00	13.00	0.66

Table A.15: Comparison of the (S, s, r, Q) policy with the EOQ based heuristic policy when $L = 5$, $Ko = 500$ and $\mu = -0.1$

$L = 5,\ Ko = 500,\ \mu = -1$													
α	Kd	Cd	S_1^*	s_1^*	r_1^*	Q_1^*	TC_1^*	S_2^*	s_2^*	r_2^*	Q_2	TC_2^*	$\triangle\%$
0.95	250	5	75.00	40.00	5.23	23.12	23.35	68.68	46.32	5.23	31.62	24.22	3.59
		10	75.00	40.00	5.23	23.12	23.35	68.68	46.32	5.23	31.62	24.22	3.59
		25	75.00	40.00	5.23	23.12	23.35	68.68	46.32	5.23	31.62	24.22	3.59
	500	5	75.00	40.00	5.23	23.12	23.35	73.42	41.80	5.23	31.62	24.22	3.59
		10	75.00	40.00	5.23	23.12	23.35	73.42	41.80	5.23	31.62	24.22	3.59
		25	75.00	40.00	5.23	23.12	23.35	73.42	41.80	5.23	31.62	24.22	3.59
	1000	5	75.00	40.00	5.23	23.12	23.35	84.28	39.56	5.23	31.62	24.22	3.59
		10	75.00	40.00	5.23	23.12	23.35	84.28	39.56	5.23	31.62	24.22	3.59
		25	75.00	40.00	5.23	23.12	23.35	84.28	39.56	5.23	31.62	24.22	3.59
0.99	250	5	75.00	40.00	6.69	22.66	24.35	68.68	46.32	6.69	31.62	25.32	3.81
		10	75.00	40.00	6.69	22.66	24.35	68.68	46.32	6.69	31.62	25.32	3.81
		25	75.00	40.00	6.69	22.66	24.35	68.68	46.32	6.69	31.62	25.32	3.81
	500	5	75.00	40.00	6.69	22.66	24.35	73.41	41.79	6.69	31.62	25.32	3.81
		10	75.00	40.00	6.69	22.66	24.35	73.41	41.79	6.69	31.62	25.32	3.81
		25	75.00	40.00	6.69	22.66	24.35	73.41	41.79	6.69	31.62	25.32	3.81
	1000	5	75.00	40.00	6.69	22.66	24.35	86.05	41.33	6.69	31.62	25.32	3.81
		10	75.00	40.00	6.69	22.66	24.35	86.05	41.33	6.69	31.62	25.32	3.81
		25	75.00	40.00	6.69	22.66	24.35	86.05	41.33	6.69	31.62	25.32	3.81
0.999	250	5	75.00	40.00	8.50	22.09	25.59	68.68	46.32	8.50	31.62	26.68	4.09
		10	75.00	40.00	8.50	22.09	25.59	68.68	46.32	8.50	31.62	26.68	4.09
		25	75.00	40.00	8.50	22.09	25.59	68.68	46.32	8.50	31.62	26.68	4.09
	500	5	75.00	40.00	8.50	22.09	25.59	73.37	41.75	8.50	31.62	26.68	4.09
		10	75.00	40.00	8.50	22.09	25.59	73.37	41.75	8.50	31.62	26.68	4.09
		25	75.00	40.00	8.50	22.09	25.59	73.37	41.75	8.50	31.62	26.68	4.09
	1000	5	75.00	40.00	8.50	22.09	25.59	79.84	35.12	8.50	31.62	26.68	4.09
		10	75.00	40.00	8.50	22.09	25.59	79.84	35.12	8.50	31.62	26.68	4.09
		25	75.00	40.00	8.50	22.09	25.59	79.84	35.12	8.50	31.62	26.68	4.09

Table A.16: Comparison of the (S, s, r, Q) policy with the EOQ based heuristic policy when $L = 5$, $Ko = 500$ and $\mu = -1$

$L=5,\ Ko=500,\ \mu=-0.01$													
α	Kd	Cd	S_1^*	s_1^*	r_1^*	Q_1^*	TC_1^*	S_3	s_3	r_3^*	Q_3^*	TC_3^*	$\triangle\%$
		5	33.90	4.51	1.88	4.71	11.66	200	190	1.88	2.12	17.24	32.38
	250	10	35.13	4.46	1.88	4.61	11.78	200	190	1.88	2.12	17.24	31.69
		25	38.95	4.31	1.88	4.34	12.10	200	190	1.88	2.12	17.24	29.84
		5	37.42	4.33	1.88	4.46	11.84	200	190	1.88	2.12	17.24	31.35
0.95	500	10	38.57	4.29	1.88	4.38	11.94	200	190	1.88	2.12	17.24	30.75
		25	42.23	4.18	1.88	4.16	12.23	200	190	1.88	2.12	17.25	29.11
		5	43.11	4.11	1.88	4.14	12.10	200	190	1.88	2.12	17.25	29.85
	1000	10	44.25	4.08	1.88	4.08	12.19	200	190	1.88	2.12	17.25	29.36
		25	48.05	3.99	1.88	3.91	12.43	200	190	1.88	2.12	17.26	28.00
		5	34.69	5.57	2.98	4.66	12.67	200	190	2.98	2.07	18.40	31.13
	250	10	35.91	5.52	2.98	4.56	12.79	200	190	2.98	2.07	18.40	30.48
		25	39.69	5.38	2.98	4.29	13.12	200	190	2.98	2.07	18.40	28.73
		5	38.21	5.39	2.98	4.41	12.85	200	190	2.98	2.07	18.40	30.15
0.99	500	10	39.35	5.35	2.98	4.33	12.96	200	190	2.98	2.07	18.40	29.58
		25	42.97	5.24	2.98	4.11	13.25	200	190	2.98	2.07	18.41	28.03
		5	43.88	5.17	2.98	4.08	13.12	200	190	2.98	2.07	18.41	28.72
	1000	10	45.00	5.14	2.98	4.03	13.21	200	190	2.98	2.07	18.41	28.26
		25	48.73	5.06	2.98	3.86	13.45	200	190	2.98	2.07	18.42	26.97
		5	31.47	7.30	4.45	4.99	13.80	200	190	4.45	2.00	19.95	30.82
	250	10	32.93	7.21	4.45	4.83	13.95	200	190	4.45	2.00	19.95	30.10
		25	37.18	6.99	4.45	4.45	14.33	200	190	4.45	2.00	19.96	28.20
		5	31.47	7.30	4.45	4.99	13.80	200	190	4.45	2.00	19.96	30.84
0.999	500	10	32.93	7.21	4.45	4.83	13.95	200	190	4.45	2.00	19.96	30.12
		25	37.18	6.99	4.45	4.45	14.33	200	190	4.45	2.00	19.96	28.22
		5	31.47	7.30	4.45	4.99	13.80	200	190	4.45	2.00	19.97	30.88
	1000	10	32.93	7.21	4.45	4.83	13.95	200	190	4.45	2.00	19.97	30.16
		25	37.18	6.99	4.45	4.45	14.33	200	190	4.45	2.00	19.97	28.26

Table A.17: Comparison of the (S, s, r, Q) policy with no-disposal option