RELATIVE IDENTIFICATION OF SEQUENTIALLY RATIONALIZABLE CHOICE PROCEDURES

A Master's Thesis

by
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To my family

RELATIVE IDENTIFICATION OF SEQUENTIALLY RATIONALIZABLE CHOICE PROCEDURES

The Graduate School of Economics and Social Sciences
of
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by

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ABSTRACT

RELATIVE IDENTIFICATION OF SEQUENTIALLY RATIONALIZABLE CHOICE PROCEDURES

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Sequentially Rationalizable Choice functions are a famous family of boundedly rational choice procedures. Given a set A of alternatives, the Decision Maker applies sequentially to each choice problem a pair of asymmetric binary relations to eliminate dominated alternatives. By playing with the conditions of the used binary relations, we consider four types of Sequentially Rationalizable Choice theories. Although these choice theories seem to behave similarly, there exist simple behavioral differences among these theories. The method of relative identification helps us to find specifically the simple behavioral differences that disentangle each of the choice theories considered.

Keywords: Bounded Rationality, Choice Theory, Relative Identification.

ÖZET

SIRAYLA RASYONALİZE EDİLEN SEÇME PROSEDÜRLERİN GÖRECELİ TANIMLAMASI

Dula, Erion
Yüksek Lisans, İktisat Bölümü
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Sırayla rasyonalize edilen seçme fonksiyonları, sınırlı rasyonel seçme prosedürlerin önemli ailelerindendir. Verilmiş bir A kümesinin alternatifleri olarak, karar verici domine edilmiş alternatifleri elimine etmek için her bir seçme problemine sırayla bir çift asimetrik ikili ilişkiler uygular. İkili ilişkilerin durumlarıyla oynayarak, biz dört tip sırayla rasyonalize edilebilir seçme teorisiyle ilgileniyoruz. Bu seçim teorileri birbirlerine çok benzer davransa da, bu teoriler içerisinde basit davranışsal fark vardır. Göreceli tanımlama metodu spesifik olarak ilgilendiğimiz her bir seçim teorisini ayırt etmemizi sağlayan basit davranışsal farkları bulmamıza yardımcı oluyor.

Anahtar kelimeler: Sınırlı Rasyonellik, Seçme Teorisi, Göreceli Tanıma.

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CHAPTER I

INTRODUCTION

Recently there has been a growing literature on bounded rationality. There are many recent papers on this literature that propose choice theories to accommodate choice behavior that rational choice theory can not explain. One famous family of boundedly rational choice procedures is the Sequentially Rationalizable Choice (Manzini and Mariotti [1]). In the simplest case the decision maker uses only two binary relations to discriminate among the available alternatives. The decision maker (DM) uses the first rationale (binary relation) to eliminate inferiour alternatives and then uses the second rationale to select an alternative from what is left from the choice set. Put differently, the first rationale creates a shortlist of candidate alternatives and the second rationale chooses among the alternatives in shortlist. Thus, it can be thought as a two-stage choice procedure.

When the DM is acquiped with two rationales which are applied sequentially, the set of survived alternatives might not be a singleton. In the case when this survival set is always a singleton this choice procedure will be called Sequentially Rationalizable Choice function (SRC). Manzini and Mariotti [1] characterizes SRC for the case when the two rationales are asymmetric binary relations. It turns out that this kind of SRC, which they call Rational Shortlist Method (RSM), is characterized by two simple axioms called Expansion and Weak WARP, which later will be considered in my work. Now a natural thing to do would be to look at the cases when the conditions on the two rationales are more restricted. It would be interesting to see what will happen when the two rationales used in sequentially rational choice procedure need not be only asymmetric. Since the assumption of the binary relations being only asymmetric is too relaxed, I consider some choice procedures which are SRCs and nest the RSM considered in Manzini and Mariotti [1]. Thus by relaxing the conditions on two rationales, I consider SRCs from the simplest case to the most general.

I start with the SRC whose first rationale is incomplete and transitive and second rationale is complete and transitive. I will call it "Shortlisting", as it is treated by this name in Yildiz [2]. Then I continue with SRC whose both rationales are incomplete and transitive (SRCIT). I relax the condition on the first rationale to study SRCAT, whose first rationale is asymmetric and second is incomplete and transitive. Finally I consider the case of SRC studied in Mariotti and Manzini [1] (RSM) whose rationales are both asymmetric.

The next natural question to ask would be: what are the simple behavioral differences between these choice procedures? Actually this is the most important question that I ask in my thesis. To find the behavioral differences between these theories I will use the method of relative identification via second order regularities first introduced by Yildiz [3]. A second order regularity is a requirement of the form: If $a = c(S_1)$ and $b = c(S_2)$, then $c = c(S_3)$ for some alternatives a, b, c and choice sets S_1, S_2, S_3.

If there is set of axioms (second order regularities) that is satisfied by a choice theory but is not satisfied from another choice theory, then an outsider can observe that these theories are different just by checking that set of axioms. Thus, one can relatively identify these two choice theories. This method is helpful because one does not need to go through axiomatic characterization of each theory in order to differentiate them. Even in the case when one knows the axiomatic characterizations of two choice theories, it might be difficult to distinguish them behaviorally. This is indeed the case for the choice theories that I study. Consider both Shortlisting and RSM. They are both sequentially rationalizable choice procedures. From the point of view of an outsider they look very similar, since the only difference between them is the condition on the rationales. Thus one can expect these choice theories to act similarly. However, when we look at the axiomatic characterization of both theories it not that easy to discover the behavioral differences between these theories.. As follows from Yildiz [2], a choice function is shortlisting if and only if a given binary relation (defined in Yildiz [2]) is acyclic. It follows from Mariotti and Manzini [1], that a choice function is RSM if and only if it satisfies two second order regularities which are Expansion and Weak WARP. Since the characterizations of these two theories are not similar at all, it is rather difficult to identify the behavioral differences between these two similar choice procedures. That is why relative identification seems a natural method to tell the simple behavioral differences between these theories.

In my analysis, first I find find the set Q of second order regularities satisfied by Shortlisting. It can be easily seen that Shortlisting is nested by SRCIT, SRCAT and RSM. Thus, the set of second order regularities that SRCIT, SRCAT and RSM satisfy will be a subset of Q. Then I find which of the regularities that Shortlisting satisfies are satisfied by SRCIT, SRCAT and RSM. It turns out that we can relatively identify Shortlisting and SRCIT from SRCAT and RSM but it is impossible to relatively identify Shortlisting from SRCIT and SRCAT from RSM via second order regularities.

CHAPTER II

PRELIMINARIES

2.1 Definitions

Let A be a set of alternatives with $|A| \geq 2$. Let $P(A)$ denote the collection of all subsets of A with at least two elements. A choice function chooses one alternative from each member of $P(A)$. Thus, it is a mapping $c : P(A) \mapsto A$ such that for each S in $P(A)$, $c(S) \in S$. A choice theory τ is a collection of choice functions.

Definition 1: For a choice function c, a k^{th}-order regularity $(k - reg)$ is a statement of the form: If $a_1 = c(S_1)$ and $a_2 = c(S_2)$ .... and $a_k = c(S_k)$, then $a_{k+1} = c(S_{k+1})$ for some alternatives a_1,..., $a_{k+1} \in A$ and choice sets S_1,..., $S_{k+1} \in P(A)$.

Second order regularity is a specific case of k^{th}-order regularity .

Definition 2: For a choice function c, a second order regularity $(2 - reg)$ is a statement of the form: If $a = c(S_1)$ and $b = c(S_2)$, then $c = c(S_3)$ for some alternatives a, b, $c \in A$ and choice sets S_1, S_2, $S_3 \in P(A)$.

As an illustration consider **Expansion** (used in the axiomatic characterization of RSM) and **Path Existence**. It is easy to see that Expansion implies Path Existence.

Expansion (EXP): For each $a \in A$ and for each $S_1, S_2 \in P(A)$ such that $a \in S_1 \cap S_2$, if $a = c(S_1)$ and $a = c(S_2)$, then $a = c(S_1 \cup S_2)$.
It says that, if the same alternative is chosen from two sets of alternatives then it must be chosen also from the union of two sets.

Path Existence (PE): For each $a, b, c \in A$ and for each $S_1 \in P(A)$ such that $a, b, c \in S_1$, if $a = c(S_1)$ and $b = c(S_1 \setminus \{c\})$, then $c = c(b, c)$.

2.2 Relative Identification

A choice theory is k^{th}-order regular if it can be characterized by k^{th}-order regularities.

In our work we will second order regularities for relative identification, because 1-reg choice theories are characterized and it is known that SRCs are not 1-regular. RSM on the other hand is second order regular.

A choice theory τ satisfies a set of regularities Q if each $c \in \tau$ satisfies each regularity $q \in Q$.

Definition 3: Let τ_1 and τ_2 be two choice theories, a set of 2-regularities Q identifies τ_1 relative to τ_2 if τ_1 satisfies Q, but τ_2 fails to satisfy Q. Let $F = \{\tau_1, \tau_2, ..., \tau_k\}$ be a family of choice theories, a collection of 2-regularities Q relatively identifies F if for each distinct τ_i,$\tau_j \in F$, there exists $Q_{ij} \subset Q$ that identifies τ_i relative to τ_j.

2.3 Sequentially rationalizable procedures

Given $S \subset A$ and a binary relation $\succ$ in $A \times A$ we denote the set of $\succ$ -maximal elements of S by: $max(S; \succ) = \{x \in S \mid \nexists y \in S \text{ such that } y \succ x\}$

Definition 4: A choice function c is an **SRC** if there exists an order pair $(\succ_1, \succ_2)$ of asymmetric relations with $\succ_1 \subset A \times A$ and $\succ_2 \subset A \times A$ such that:

For each S in $P(A)$, $c(S) = max(max(S; \succ_1); \succ_2)$.

Thus, this choice can be interpreted as if the DM goes through two elimination processes. In the first round he eliminates the alternatives that are dominated according to rationale $\succ_1$ and chooses only the ones that are maximal. In the second round he retains only one alternative (from what is left) that is maximal according to rationale $\succ_2$.

For an illustration of how an **SRC** works consider the following example:

Suppose that a DM has to choose between a Chinese restaurant, an Italian restaurant and a fast food. Assume that in terms of taste the Italian restaurant (IR) is better than the Chinese Restaurant (CR) and the Chinese Restaurant is better than fast food (FF). Lets assume also that in terms of the speed of food service DM thinks that fast food dominates the fancy Italian Restaurant but has no idea about the speed of service in Chinese Restaurant. Thus in terms of the speed of service no comparisons are available between FF and CR or between CR and IR. Lets assume that DM is in hurry and he first decides on the basis of speed of service and then he considers the taste. When all the alternatives are available DM will choose CR since IR is eliminated by the speed criterion and later FF is eliminated by the taste criterion. Also when the alternative FF is not available DM will choose IR among IR and CR since the speed criterion can not be used. One can also notice that this choice procedure obviously violates **WARP** and deserves to be fitted in the family of rationally bounded choice procedures.

Now let the choice theory **Shortlisting** be denoted by τ^{SL}. For a choice function c, $c \in \tau^{SL}$ iff for each S in $P(A)$, $c(S) = max(max(S; \succ_1); \succ_2)$ where $\succ_1$ is an incomplete and transitive and $\succ_2$ is a complete and transitive binary relation.

Let the choice theory **SRCIT** be represented by τ^{SRCIT}. For a choice function c, $c \in \tau^{SRCIT}$ iff for each S in $P(A)$, $c(S) = max(max(S; \succ_1); \succ_2)$ where $\succ_1$ and $\succ_2$ are both incomplete and transitive binary relations.

Let the choice theory **SRCAT** be represented by τ^{SRCAT}. For a choice function c, $c \in \tau^{SRCAT}$ iff for each S in $P(A)$, $c(S) = max(max(S; \succ_1); \succ_2)$ where $\succ_1$ is an asymmetric and $\succ_2$ is a incomplete and transitive binary relation.

Finally, the choice theory **RSM** will be represented by τ^{RSM}. For a choice function c, $c \in \tau^{RSM}$ iff for each S in $P(A)$, $c(S) = max(max(S; \succ_1); \succ_2)$ where $\succ_1$ and $\succ_2$ are both asymmetric binary relations.

CHAPTER III

AXIOMS AND RESULTS

Here we consider four rationally bounded choice theories: τ^{SL}, τ^{SRCIT},τ^{SRCAT} and τ^{RSM}. Below we find the set of all second order regularities that these choice theories satisfy. It is easy to see that Shortlisting is nested from all the other three choice theories. So, if Q is the set of second order regularities that Shortlisting satisfies, then the set of 2-regularities that SRCIT, SRCAT and RSM satisfy will be a subset of Q. In appendix we find the set Q of second order regularities satisfied by Shortlisting. Then we find which second order regularity in Q is and which one is not satisfied by SRCIT, SRCAT or RSM. Relative identification of these choice theories will follow from **Definition 2** and the table below. Axioms and results are summarized below.

Now, consider the following axioms which are represented as second order regularities.

Expansion (EXP): For each $a \in A$ and for each $S_1, S_2 \in P(A)$ such that $a \in S_1 \cap S_2$, if $a = c(S_1)$ and $a = c(S_2)$, then $a = c(S_1 \cup S_2)$.

Path Existence (PE): For each $a, b, c \in A$ and for each $S_1 \in P(A)$ such that $a, b, c \in S_1$,

if $a = c(S_1)$ and $b = c(S_1 \setminus \{c\})$, then $c = c(b, c)$.

Weak WARP (WW): For each $a, b \in A$ and for each $S_1, S_2 \in P(A)$ such that $a, b \in S_1 \subseteq S_2$,

if $a = c(S_1)$ and $b = c(S_2)$, then $a = c(a, b)$.

Domination of Removed Alternative (DRA): For each $a, b, c \in A$ and for each $S_1 \in P(A)$ such that $a, b, c \in S_1$,

if $a = c(S_1)$ and $b = c(S_1 \setminus \{c\})$, then $a = c(a, c)$.

Domination of Rival Alternatives (DRIA): For each $a, b, c \in A$ and for each $S_1 \in P(A)$ such that $a, b, c \in S_1$,

if $a = c(S_1)$ and $b = c(S_1 \setminus \{c\})$, then $a = c(a, b, c)$.

Independence of Removed Alternatives (IRA): For each $a, b \in A$ and for each $S_1, S_2 \in P(A)$ such that $a, b \in S_1 \cap S_2$, $S_2 \subseteq S_1$ and $|S_1 \setminus S_2| \geq 1$,

if $a = c(S_1)$ and $b = c(S_2)$, then $a = c(S_1 \setminus \{b\})$.

Binary WARP (BW): For each $a, b \in A$ and for each $S_1, S_2 \in P(A)$ such that $a, b \in S_1$, $S_1 \setminus \{a\} \subseteq S_2$ and $a \notin S_2$,

if $a = c(S_1)$ and $b = c(S_2)$, then $a = c(a, b)$.

Proposition 1: Let F consist of the following boundedly rational choice theories: τ^{SL}, τ^{SRCIT}, τ^{SRCAT} and τ^{RSM}. Let Q consist of all the second order regularities in

	SL	SRCIT	SRCAT	RSM
EXP	+	+	+	+
PE	+	+	+	+
WW	+	+	+	+
BW	+	+	+	+
DRIA	+	+	—	—
IRA	+	+	—	—
DRA	+	+	—	—

the form of EXP, PE, WW, DRA, DRIA, IRA or BW .

i) For each $\tau \in F$, if q is a second order regularity that τ satisfies, then $q \in Q$.

ii) The second order regularities that each theory satisfies can be seen in the table above; it follows that Q can not relatively identify F. However, Q identifies τ^{SL} relative to τ^{SRCAT} and τ^{RSM}. Also, Q identifies τ^{SRCIT} relative to τ^{SRCAT} and τ^{RSM}.

Proof: From **Propostion 2** in Appendix we find the set Q of the second order regularities that Shortlisting satisfies. Since Shortlisting is nested from the other three choice theories part **i)** follows. Explanations over the table are available in appendix.

As it can be seen from the table above, according to **Definition 3**, Q can not relatively identify F. Q can not identify τ^{SL} relatively to τ^{SRCIT} and τ^{SRCAT} relatively to τ^{RSM}. However Q identifies τ^{SL} relatively to τ^{SRCAT} and τ^{RSM}. Furthermore, Q identifies τ^{SRCIT} relatively to τ^{SRCAT} and τ^{RSM}.

Given any choice function from some choice theory, by looking just at the table, we can see if this choice function is consistent or not with our four SRC theories. For example, if some choice function does not satisfy one of the axioms in Q than it can not be consistent with τ^{SL} or τ^{SRCIT}. If some choice function c satisfies IRA but does not satisfy DRIA, than c is consistent with none of the SRCs. So given any choice function c, one can check if c is consistent with one of the SRCs by checking the axioms in Q that c will satisfy. If a choice function c satisfies all the second order regularities in Q, then it will be consistent with τ^{SL} or τ^{SRCIT}. However this does not guarantee us that c will belong to one of these two choice theories, since this relatively identification is restricted only to second order regularities. If one can find all the third or higher order regularities that SRC theories satisfy, then we identify the entire behavioral content of the choice theory in hand. .

CHAPTER IV

CONCLUSION

The main purpose of this thesis is to point out the simple behavioral differences that exist among four distinct Sequentially Rationalizable Choice theories. Although these theories are all two-stage choice procedures, it is impossible to reveal simple behavioral differences if one looks just at their axiomatic characterization. However, with the method of relative identification it is possible to disentangle these theories. We find the set Q of all second order regularities satisfied by Shortlisting, SRCIT, SRCAT and RSM. We observe that one can not identify Shortlisting relatively to SRCIT and SRCAT relatively to RSM via second order regularities. Relative identification among all other combinations is possible. A choice function c is consistent with a choice theory τ if it satisfies the second order regularities that τ satisfies and fails to satisfy the ones that τ does not satisfy. Thus, given a choice function c, an outsider can see if c is consistent with one of SRCs just by checking the axioms in Q that c does satisfy. As said before, we restricted ourselves only to behavioral differences that are pointed out by second order regularities. It would be interesting to look in the future if third or higher order regularities would point out more behavioral differences among SRCs.

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APPENDIX

Proposition 2: Consider any second order regularity q satisfied by Shortlisting choice theory, denoted by τ^{SL}: If $a = (S_1)$ and $b = (S_2)$ then $c = (S_3)$ for some $a, b, c \in A$ and pairwise distinct S_1, S_2, $S_3 \in P(A)$. Then q is in the form of EXP, PE, WW, DRA, DRIA, IRA or BW.

Proof:

Lemma 1: If $a = b$, then q is in the form BE, i.e. given any statement in the form: $a = c(S_1)$ and $b = c(S_2) \Rightarrow c = c(S_3)$, if $a = b$ then $c = a$ and $S_1 \cup S_2 = S_3$.

Proof. Assume that $a = c(S_1) \wedge a = c(S_2) \Rightarrow c = c(S_3)$.

Step 1: $a \in S_3$.

Proof: Assume $a \notin S_3$. Now let $a \succ_1 t$,$\forall t \in S_1 \cup S_2 \setminus \{a\}$ and $d \succ_1 c$ for some $d \in S_3$. Then we have that $a = c(S_1)$, $b = c(S_2)$ but $c \neq d = (S_3)$. Thus must have that $a \in S_3$.

Step 2: $a = c$.

Proof: Assume $a \neq c$. Take $\succ_1$ such that $a \succ_1 t$,$\forall t \in S_1 \cup S_2 \cup S_3 \setminus \{a\}$. Since

$a \in S_3$ we have that $a = c(S_1) \wedge a = c(S_2) \wedge c \neq a = c(S_3)$. Thus must have that $a = c$.

Now we have the sentence $a = c(S_1) \wedge a = c(S_2) \Rightarrow a = c(S_3)$.

We want to show that $S_1 \cup S_2 = S_3$.

Step 3: $S_3 \subseteq S_1 \cup S_2$.

Proof: Assume that $\exists x \in S_3 \setminus (S_1 \cup S_2)$. Than obviously $x \neq a$. Take $\succ_1$ such that $x \succ_1 a \succ_1 t$,$\forall t \in S_1 \cup S_2 \cup S_3 \setminus \{x, a\}$. Then we have that $a = c(S_1) \wedge a = c(S_2) \wedge a \neq x = c(S_3)$. Thus $\nexists x$ such that $x \in S_3 \setminus (S_1 \cup S_2)$, i.e $S_3 \subseteq S_1 \cup S_2$.

Now we want to show that $S_1 \cup S_2 \subseteq S_3$.

Step 4: If $S_2 \subseteq S_3$ and $S_3 \subseteq S_1 \cup S_2$ then $(S_1 \cap S_3) \setminus S_2 \neq 0$.

Proof: Assume the contrary, i.e $(S_1 \cap S_3) \subseteq S_2$. Now we have that $S_3 \subseteq (S_1 \cup S_2) \cap S_3 = (S_1 \cap S_3) \cup (S_2 \cap S_3) = (S_1 \cap S_3) \cup S_2 \subseteq S_2$. Thus $S_3 \subseteq S_2$, which is a contradiction since we cant have $S_2 = S_3$.

Step 5: $S_1 \subseteq S_3$.

Proof: Assume that $S_1 \nsubseteq S_3$.

Case 1: $S_1 \cap S_3 = \{a\}$.

Now $S_3 \subseteq S_1 \cup S_2 \Rightarrow S_3 \setminus \{a\} \subseteq (S_1 \setminus \{a\}) \cup (S_2 \setminus \{a\}) \Rightarrow S_3 \setminus \{a\} \subseteq S_2 \setminus \{a\}$ $\Rightarrow S_3 \subseteq S_2$. Since $S_3 \neq S_2$ then $\exists y \in S_2$ such that $y \notin S_3$. Now take $x_1 \in S_3 \subseteq S_2$ such that $x_1 \neq a$. Obviously since $x_1 \neq a$ and $S_1 \cap S_3 = \{a\}$ we have that $x_1 \notin S_1$.

Thus take $\succ_1$ and $\succ_2$ such that $x_1 \succ_2 a \succ_2 z$, $\forall z \in S_1 \cup S_2 \setminus \{a, x_1\}$ and $y \succ_1 x_1$. Then we will have $a = c(S_1) \wedge a = c(S_2)$ but $a \neq x_1 = c(S_3)$. Again we have reached a contradiction.

Case 2: $|S_1 \cap S_3| \geq 2$.

So there exists b such that $b \neq a$ and $a, b \in S_1 \cap S_3$.

Subcase 2a: $S_2 \nsubseteq S_3$.

Since $S_1 \nsubseteq S_3$. there exist $x \in S_1$ such that $x \notin S_3$. Now if $b \notin S_2$ then let $b \succ_2 a \succ_2 z$, $\forall z \in S_1 \cup S_2 \setminus \{b, a\}$ and $x \succ_1 b$. Then we will have that $b = c(S_3)$. If $b \in S_2$, then we can choose some $t \in S_2$ such that $t \notin S_3$ (since $S_2 \nsubseteq S_3$) and we let $t \succ_1 b$. Again we will have $b = c(S_3)$, leading us to a contradiction.

Subcase 2b: $S_2 \subseteq S_3$.

Now by **Step 4** we have that $(S_1 \cap S_3) \setminus S_2 \neq 0$. So that $\exists z \in S_1 \cap S_3$ but $z \notin S$. Now let $z \succ_2 a \succ_2 y$, $\forall y \in S_1 \cup S_2 \cup S_3 \setminus \{a, y\}$ and $x \succ_1 z$ for some $x \in S_1 \setminus S_3$. Then again we will have that $a = c(S_1)$, $a = c(S_2)$ but $z = c(S_3)$. Again, a contradiction.

Thus we have proved by contradiction that $S_1 \subseteq S_3$.

Now simliarly one can prove that $S_2 \subseteq S_3$. Then, $S_1 \cup S_2 \subseteq S_3$ and $S_3 \subseteq S_1 \cup S_2$ will imply that $S_1 \cup S_2 = S_3$

□

Lemma 2: If $a \neq b$, $c \neq a$ and $b \neq c$ then q is in the form of PE, i.e given any statement in the form: $a = c(S_1)$ and $b = c(S_2) \Rightarrow c = c(S_3)$, if a, b and c are pairwise distinct, then $S_2 = S_1 \setminus \{c\}$ and $S_3 = b \cup c$.

Proof. Take any statement of the form $a = c(S_1)$ and $b = c(S_2) \Rightarrow c = c(S_3)$. Assume that a, b and c are pairwise distinct.

Step 1: $c \in S_1 \cup S_2$.

Proof: Assume that $c \notin S_1 \cup S_2$. Then must have that $c \notin S_1$ and $c \notin S_2$.

Case 1: $a \notin S_2$ and $b \notin S_1$.
Choose $\succ_1$ such that $a \succ_1 b \succ_1 x$,$\forall x \in S_1 \cup S_2 \setminus \{a, b\}$ and for some $z \in S_3$ such that $z \neq c$ let $z \succ_1 c \succ_1 t$,$\forall t \in S_3 \setminus \{z, c\}$. Now we have $c(S_3) = z$.

Case 2: $a \notin S_2$ and $b \in S_1$. Then choose the same $\succ_1$ as above.

Case 3: Is symmetric to Case 2.

Case 4: $a \in S_2$ and $b \in S_1$.
Since $S_1 \neq S_2$, then either $S_1 \backslash S_2 \neq \varnothing$ or $S_2 \backslash S_1 \neq \varnothing$. WLOG assume that $S_1 \backslash S_2 \neq \varnothing$. Thus there exists $k \in S_1$ such that $k \notin S_2$. Now let $b \succ_2 a \succ_2 x$,$\forall x \in S_1 \cup S_2 \setminus \{a, b\}$ and $k \succ_1 b$. Take any $z \in S_3$ and let $z \succ_1 c$ and we will have that $a = c(S_1)$, $b = c(S_2)$ but $c \neq (S_3)$.

In all of the cases above we have reached a contradiction. Thus must have that $c \in S_1 \cup S_2$.

Now WLOG let $c \in S_1$.

Step 2: $b \in S_1$.

Proof:Assume $b \notin S_1$. Now let $a \succ_1 t$,$\forall t \in S_1 \setminus \{a\}$ and $b \succ_1 x$,$\forall x \in S_2 \setminus \{b\}$. Take any $z \in S_3$ and let $z \succ_1 c$ and we will have that $a = c(S_1)$, $b = c(S_2)$ but $c \neq (S_3)$. Thus must have that $b \in S_1$.

Step 3: $a \in S_2$.

Proof: Assume $a \notin S_2$. Now let $a \succ_1 b \succ_1 t \succ_1 c$,$\forall t \in S_1 \cup S_2 \cup S_3 \setminus \{a, b, c\}$. Then we have that $a = c(S_1)$, $b = c(S_2)$ but $c \neq (S_3)$. Thus must have that $a \in S_2$.

Since $\succ_1$ is an incomplete relation we will use $a \sim_1 b$ whenever $a \not\succ_1 b$ and $b \not\succ_1 a$.

Step 4: $c \notin S_2$.

Proof: Assume $c \in S_2$. Now we have a sentence of the form $a = c(a, b, c, K_1)$ and $b = c(a, b, c, K_2) \Rightarrow c = c(S_3)$.

Since $K_1 \neq K_2$, then either $K_1 \setminus K_2 \neq \varnothing$ or $K_2 \setminus K_1 \neq \varnothing$.WLOG assume that $K_1 \setminus K_2 \neq \varnothing$. Thus there exists $d \in S_1$ such that $d \notin S_2$. Now let $a \sim_1 b$, $d \succ_1 b$ and $b \succ_2 a \succ_2 t$,$\forall t \in S_1 \cup S_2 \setminus \{a, b\}$. If we take any $z \in S_3$ and let $z \succ_1 c$ and we will have that $a = c(S_1)$, $b = c(S_2)$ but $c \neq (S_3)$. Thus reaching a contradiction. We must have that $c \notin S_2$.

Now we have a sentence of the form: $a = c(a, b, c, K_1)$ and $b = c(a, b, K_2) \Rightarrow c = c(S_3)$, where $c \notin K_2$. By letting $a \sim_1 b$, $c \succ_1 b$. and $b \succ_2 a \succ_2 t$,$\forall t \in S_1 \cup S_2 \setminus \{a, b\}$ we can have $a = c(S_1)$, $b = c(S_2)$ and we can force $c(S_3)$ to be different from c whenever $z \in S_3$ and $z \succ_1 c$. Actually the later can happen when $(S_3 \setminus \{c\}) \setminus (K_2 \cup b \cup a) \neq \varnothing$. In this case take $z \in S_3 \setminus (K_2 \cup b \cup a)$ and let $z \succ_1 c \succ_1 b$. We have that $c \neq c(S_3)$

and b is not effected from the choice in S_2. Thus we proved that we must have $S_3 \setminus \{c\} \subseteq (a \cup b \cup K_2)$.

Step 5: $b \in S_3$.

Proof: Assume $b \notin S_3$. Since $|S_3| \geq 2$, $\exists d \in S_3$ such that $d \neq c$. Ovbiously $d \neq b$. Now we must have that $d \in a \cup K_2$. Now let $d \sim_1 c$, $d \succ_2 c$ and $d \succ_1 t$, $\forall t \in S_3 \setminus \{d, c\}$. Also let $a \sim_1 b$, $c \succ_1 b$ and $b \succ_2 a \succ_2 t$, $\forall t \in S_1 \cup S_2 \setminus \{a, b\}$. Now whether or not $d = a$, we will have that $a = c(S_1)$, $b = c(S_2)$ but $d = c(S_3)$. Thus $b \in S_3$.

Step 6: $a \notin S_3$.

Proof: Assume $a \in S_3$. Now let $a \sim_1 b$, $c \succ_1 b$ and $b \succ_2 a \succ_2 t$, $\forall t \in S_1 \cup S_2 \cup S_3 \setminus \{a, b\}$. Then we will have that $a = c(S_1)$, $b = c(S_2)$ but $a = c(S_3)$. Thus $a \notin S_3$.

Now we have a sentence of the form $a = c(a, b, c, K_1)$ and $b = c(a, b, K_2) \Rightarrow c = c(b, c, K_3)$, where $c \notin K_2$, $a \notin K_3$ and $K_3 \subseteq K_2$.

Step 7: $|K_3| = 0$.

Proof: Assume the contrary, $|K_3| \geq 1$. Now take $d \in K_3 \subseteq K_2$.

Let $a \sim_1 b$, $c \succ_1 b$ and $b \succ_2 a \succ_2 d \succ_2 c \succ_2 t$, $\forall t \in S_1 \cup S_2 \cup S_3 \setminus \{a, b, c, d\}$. Then we will have that $a = c(S_1)$, $b = c(S_2)$ but $d = c(S_3)$, whenever $d \in K_1$ or not. Thus we must have that $|K_3| = 0$.

Now we are left with a sentence of the form $a = c(a, b, c, K_1)$ and $b = c(a, b, K_2) \Rightarrow c = c(b, c)$, where $c \notin K_2$.

Step 8: $K_1 \subseteq K_2$.

Proof: Assume the contrary, $K_1 \setminus K_2 \neq \varnothing$. Then $\exists d \in K_1$ such that $d \notin K_2$. Now take $\succ_1$ and $\succ_2$ such that $a \sim_1 b$, $d \succ_1 b \succ_1 c$ and $b \succ_2 a \succ_2 t$, $\forall t \in S_1 \cup S_2 \setminus \{a, b\}$. We have that $a = c(S_1)$, $b = c(S_2)$ but $b = c(S_3)$. Therefore $K_1 \subseteq K_2$.

Step 9: $K_2 \subseteq K_1$.

Proof: Assume the contrary, $K_2 \setminus K_1 \neq \varnothing$. Then $\exists e \in K_2$ such that $e \notin K_1$. Now take $\succ_1$ and $\succ_2$ such that $a \sim_1 b$, $e \succ_1 a$ and $a \succ_2 b \succ_2 t$, $\forall t \in S_1 \cup S_2 \setminus \{a, b\}$. We have that $a = c(S_1)$, $b = c(S_2)$ but $b = c(S_3)$. Therefore $K_2 \subseteq K_1$.

Since $K_2 \subseteq K_1$ and $K_1 \subseteq K_2$ we can conclude that $K_2 = K_1$.

Thus we have proved that $S_2 = S_1 \setminus \{c\}$ and $S_3 = b \cup c$ given that $a \neq b$, $a \neq c$ and $c \neq b$.

□

Lemma 3: If $a \neq b$, then q is in the form of WW, DRA, DRIA, IRA or BW.

Proof. We will first consider the case when $a \in S_2$ and then the case when $a \notin S_2$.

Case 1: $a \in S_2$.

Step 1: $b \in S_1$.

Proof: Assume $b \notin S_1$. Now let $b \succ_1 t$,$\forall t \in S_2 \setminus \{b\}$. If $\exists c \in S_3 \setminus S_1$ then let $c \succ_1 a$ and $a \succ_1 x$,$\forall x \in S_1/a$. We will have that $a = c(S_1)$, $b = c(S_2)$ but $a \neq (S_3)$. If $S_3 \subseteq S_1$, since $S_1 \neq S_3$, $\exists c \in S_1$ such that $c \notin S_3$. Now take $z \in S_3 \subseteq S_1$ and let $c \succ_1 z$, $b \succ_1 t$ $\forall t \in S_2 \setminus \{b\}$ and $z \succ_2 a \succ_2 t$,$\forall t \in S_1 \cup S_3 \setminus \{z, a\}$. Again we have that $a = c(S_1)$, $b = c(S_2)$ but $z = (S_3)$. Thus $b \in S_1$.

Now we have a sentence of the form $a = c(a, b, K_1)$ and $b = c(a, b, K_2) \Rightarrow a = c(S_3)$.

From now on we divide the problem in two cases, when $K_2 \setminus K_1 \neq 0$ and $K_2 \subseteq K_1$.

Case 1a: $\exists x \in K_2 \setminus K_1$.

Step 2: $b \in S_3$.

Proof: Assume $b \notin S_3$. If $\exists c \in K_3 \setminus K_1$, let $a \sim_1 b$, $c \succ_1 a$ and $a \succ_2 b \succ_2 t$,$\forall t \in S_1 \cup S_2 \setminus \{b, a\}$. We will have that $a = c(S_1)$, $b = c(S_2)$ but $a \neq (S_3)$.
If $K_3 \subseteq K_1$, then take $c \in K_3 \subseteq K_1$ and choose $\succ_1$ and $\succ_2$ such that $a \sim_1 b$, $b \succ_1 c$, $x \succ_1 a$ and $c \succ_2 a \succ_2 b \succ_2 t$,$\forall t \in S_1 \cup S_2 \cup S_3 \setminus \{a, b, c\}$. We will have that $a = c(S_1)$, $b = c(S_2)$ but $c = (S_3)$. Thus must have that $b \in S_3$.

Now we have a sentence of the form
$a = c(a, b, K_1)$ and $b = c(a, b, x, K_2) \Rightarrow a = c(a, b, K_3)$, where $x \notin K_1$.

Step 3: $K_3 \subseteq K_1$.

Proof: Assume the contrary, $K_3 \setminus K_1 \neq \varnothing$. Then $\exists c \in K_3$ such that $c \notin K_1$. Now take $\succ_1$ and $\succ_2$ such that $a \sim_1 b$, $c \succ_1 a$, $x \succ_1 b$ and $a \succ_2 b \succ_2 c \succ_2 t$, $\forall t \in S_1 \cup S_2 \cup S_3 \setminus \{a, b, c\}$. We have that $a = c(S_1)$, $b = c(S_2)$ but $c(S_3) = b$ if $x \notin S_3$ and $c(S_3) = c$ if $x \in S_3$. Therefore $K_3 \subseteq K_1$.

Step 4: $|K_3| = 0$.

Proof: Assume the contrary, $|K_3| \geq 1$. Now take $d \in K_3 \subseteq K_1$.
Also since $K_1 \neq K_3$, $\exists e \in K_1 \setminus K_3$ so that $e \in K_1$ and $e \notin K_3$. Now since $x \notin K_1$, must have that $x \notin K_3$ too. Now let $a \sim_1 b$, $e \succ_1 d$, $x \succ_1 a$ and $d \succ_2 a \succ_2 b \succ_2 t$,

$\forall t \in S_1 \cup S_2 \cup S_3 \setminus \{a, b, d\}$. If $d \in K_2$ then take $x \succ_1 d$. At the end we have that $a = c(S_1)$, $b = c(S_2)$ but $c(S_3) = d$. Thus $|K_3| = 0$.

Now we have a sentence of the form $a = c(a, b, K_1)$ and $b = c(a, b, x, K_2) \Rightarrow a = c(a, b)$, where $x \notin K_1$.

Step 5: $K_1 \subseteq K_2$.

Proof: Assume the contrary, $K_1 \setminus K_2 \neq \varnothing$. Then $\exists c \in K_1$ such that $c \notin K_2$. Now take $\succ_1$ and $\succ_2$ such that $a \sim_1 b$, $c \succ_1 b$ and $b \succ_2 a \succ_2 t$, $\forall t \in S_1 \cup S_2 \setminus \{a, b\}$. We have that $a = c(S_1)$, $b = c(S_2)$ but $c(a, b) = b$. Therefore $K_1 \subseteq K_2$.

At this point we have reached the axiom WW: If $a, b \in S_1 \subseteq S_2$ then $a = c(S_1)$ and $b = c(S_2) \Rightarrow a = c(a, b)$.

Now what happens if $\nexists x$ such that $x \in K_2/K_1$?

Case 1b: $K_2 \subseteq K_1$.

Start first by considering the case when $|K_1/K_2| = 1$. This means that $\exists! c$ such that $c \in K_1$ and $c \notin K_2$.

We have now a sentence of the form: $a = c(a, b, c, K_1)$ and $b = c(a, b, K_1) \Rightarrow a = c(a, K_3)$.

Step 6: $S_3 \subseteq S_1$.

Proof: Assume the contrary, $\exists x \in S_3/S_1$. Let $c \succ_1 b$, $x \succ_1 a$ and $b \succ_2 a \succ_2 c \succ_2 t$

$,\forall t \in K_1$. Then we have that $a = c(S_1)$, $b = c(S_2)$ but $a \neq (S_3)$. Thus $S_3 \subseteq S_1$.

Step 7: $c \in S_3$.

Proof: Assume $c \notin S_3$. Since $|S_3| \geq 2$, $\exists d \in S_3$ such that $d \neq a$. From **Step 6** we have that $d \in b \cup K_1$

Case 1: $d = b$.

Let $c \succ_1 b$ and $b \succ_2 a \succ_2 c \succ_2 t$ $,\forall t \in K_1$. Then we will have $a = c(S_1)$, $b = c(S_2)$ but $b = (S_3)$.

Case 2: $d \neq b$.

Then we have that $d \in K_1$. Now let $c \succ_1 b$, $c \succ_1 d$ and $b \succ_2 d \succ_2 a \succ_2 c \succ_2 t$ $,\forall t \in K_1$. Then we will have $a = c(S_1)$, $b = c(S_2)$ but $a \neq (S_3)$. Thus $c \in S_3$.

Now we have a sentence of the form $a = c(a, b, c, K_1)$ and $b = c(a, b, K_1) \Rightarrow a = c(a, c, K_3)$, where $K_3 \subseteq K_1 \cup b$.

Step 8: If $b \in K_3$ then q is in the form of DRIA.

Proof: Assume $b \in K_3$. If $|K_3| \geq 2$, then $\exists d \in K_3$ such that $d \neq b$. Since $d \neq b$ we have that $d \in K_1$ too. Now since $K_3 \setminus \{b\} \subseteq K_1$, $\exists e \in K_1$ such that $e \notin K_3 \setminus \{b\}$. Now let $a \sim_1 b$, $c \succ_1 b$, $e \succ_1 d$ and $b \succ_2 d \succ_2 a \succ_2 c \succ_2 e \succ_2 t$, $\forall t \in S_1 \setminus \{a, b, c, d, e\}$. Then we will have $a = c(S_1)$, $b = c(S_2)$ but $d = c(S_3)$. Thus $|K_3| \geq 2$ is not possible and since $b \in K_3$, must have that $K_3 = \{b\}$.

Thus we have reached DRIA: $a = c(S_1)$ and $b = c(S_1 \setminus \{c\}) \Rightarrow a = c(a, b, c)$.

Step 9: If $b \notin K_3$ then q is in the form of IRA or DRA.

(1) $a = c(S_1)$ and $b = c(S_1 \setminus \{c\}) \Rightarrow a = c(a, c)$ (DRA)

(2) $a = c(S_1)$ and $b = c(S_1 \setminus \{c\}) \Rightarrow a = c(S_1 \setminus \{b\})$ (IRA)

Proof: Assume $b \notin K_3$ and $|K_3| \neq 0$. Then we have that $K_3 \subseteq K_1$. Then $\exists d \in K_3 \subseteq K_1$ and also $\exists e \in K_1 \setminus K_3$. Now take $a \sim_1 b$, $c \succ_1 b$, $e \succ_1 d$ and $d \succ_2 b \succ_2 a \succ_2 c \succ_2 e \succ_2 t$, $\forall t \in S_1 \setminus \{a, b, c, d, e\}$. We will have $a = c(S_1)$, $b = c(S_2)$ but $d = c(S_3)$. Thus must have that $|K_3| = 0$ or $K_3 = K_1$. Indeed both of them will work and thus we yield the above axioms.

Now consider the case when $|K_1/K_2| \geq 2$.

Now our problem can be written in this form: $a = c(a, b, K_1, K_2)$ and $b = c(a, b, K_1) \Rightarrow a = c(a, K_3)$, where $|K_2| \geq 2$ and $K_1 \cup K_2 = \varnothing$.

As shown before, obviously we must have that $K_3 \subseteq b \cup K_1 \cup K_2$.

Step 10: If $b \in K_3$ then $K_3 = b \cup K_1 \cup K_2$, the we have only the trivial axiom below:
$a = c(S_1)$ and $b = c(S_2) \Rightarrow a = c(S_1)$.

Proof: Assume $b \in K_3$. Then we are left with a sentence of the form: $a = c(a, b, K_1, K_2)$ and $b = c(a, b, K_1) \Rightarrow a = c(a, b, K_3)$, where $|K_2| \geq 2$.

As shown before we must have that $K_3 \subseteq K_1 \cup K_2$. Now lets prove that $K_1 \cup K_2 \subseteq K_3$ too. We start by showing that $K_2 \subseteq K_3$. Assume that $K_2 \subseteq K_3 \neq \varnothing$, i.e $\exists x \in K_2$ such that $x \notin K_3$. Now take $a \sim_1 b$, $x \succ_1 b$ and $b \succ_2 a \succ_2 t$, $\forall t \in K_1 \cup K_2$. Then we have that $a = c(S_1)$, $b = c(S_2)$ but $b = c(S_3)$. Thus, must have that

$K_2 \subseteq K_3$.

Now we want to show that $K_1 \subseteq K_3$ too. Assume that $K_1 \subseteq K_3 \neq \varnothing$, i.e $\exists x \in K_1$ such that $x \notin K_3$. Since $|K_2| \geq 2$, we can take $c, d \in K_2 \subseteq K_3$. Now let $a \sim_1 b$, $c \succ_1 b$, $x \succ_1 d$ and $d \succ_2 b \succ_2 a \succ_2 t$, $\forall t \in K_1 \cup K_2 \subseteq \{d\}$. Then we will have $a = c(S_1)$, $b = c(S_2)$ but $d = c(S_3)$. Thus must have that $K_1 \subseteq K_3$ too.

Now $K_1 \subseteq K_3$ and $K_2 \subseteq K_3 \Rightarrow K_1 \cup K_2 \subseteq K_3$. Thus we have that $K_1 \cup K_2 = K_3$, which yield us nothing but the trivial axiom.

Step 11: Given a statement of the form: $a = c(a, b, K_1, K_2)$ and $b = c(a, b, K_1) \Rightarrow a = c(a, K_3)$. If $b \notin K_3$, $K_1 \cap K_2 = \varnothing$ and $|K_2| \geq 2$ then $K_3 = K_1 \cup K_2$.

Proof: Assume $b \notin K_3$. From the proofs done before it is trivial the fact that $K_3 \subseteq K_1 \cup K_2$. Let us first show that $K_2 \subseteq K_3$.

Assume that $\exists x \in K_2 \setminus K_3$, i.e $x \in K_2$ and $x \notin K_3$. Since $|K_3| \geq 1$ we can take $c \in K_3 \subseteq K_1 \cup K_2$. Obviously $c \neq x$. Now let $a \sim_1 b$, $x \succ_1 b$, $x \succ_1 c$ and $b \succ_2 c \succ_2 a \succ_2 t$, $\forall t \in K_1 \cup K_2$. We will have that $a = c(S_1)$, $b = c(S_2)$ but $c = c(S_3)$, whenever $c \in K_1$ or $c \in K_2$. Thus there exists no x such that $\exists x \in K_2 / K_3$, i.e $K_2 \subseteq K_3$.

Let us know show that $K_1 \subseteq K_3$.

Assume $\exists x \in K_1 \setminus K_3$, i.e $x \in K_1$ and $x \notin K_3$. Since $|K_2| \geq 2$, we can take $c, d \in K_2 \subseteq K_3$. Now let $a \sim_1 b$, $c \succ_1 b$, $x \succ_1 d$ and $d \succ_2 b \succ_2 a \succ_2 t$, $\forall t \in K_1 \cup K_2 \setminus \{d\}$. Then we will have $a = c(S_1)$, $b = c(S_2)$ but $d = c(S_3)$. Thus must have that $K_1 \subseteq K_3$.

So we proved that $K_1 \cup K_2 = K_3$, which implies IRA.

Above we found all the axioms of the form: $a = c(S_1)$ and $b = c(S_2) \Rightarrow a = c(S_3)$, only for the case when $a \in S_2$.

Now what happens if $a \notin S_2$?

Case 2: $a \notin S_2$.

Step 12: $b \in S_1$.

Proof: Assume $b \notin S_1$. Then obviously $S_3 \subseteq S_1$. Since $S_3 \neq S_1$, $\exists c \in S_1$ such that $c \notin S_3$. Also since $|S_3| \geq 2$ we can take $x \in S_3 \subseteq S_1$. Now choose $\succ_1$ and $\succ_2$ such that $x \succ_1 c$, $x \succ_1 a$ and $b \succ_2 x \succ_2 a \succ_2 t$,$\forall t \in S_1 \cup S_2 \cup S_3 \setminus \{a, b, x\}$. We will have that $a = c(S_1)$, $b = c(S_2)$ but $x = (S_3)$. Thus must have that $b \in S_1$.

Now we have a sentence of the form:

$a = c(a, b, K_1)$ and $b = c(b, K_2) \Rightarrow a = c(a, K_3)$, where $a \notin K_2$.

Step 13: $b \in K_3$.

Proof: Assume $b \notin K_3$. If $\exists c \in K_3 \setminus K_1$, let $c \succ_1 a$, $a \succ_1 t$ $\forall t \in K_1 \cup \{b\}$ and $b \succ_2 t$,$\forall t \in K_2$. We will have that $a = c(S_1)$, $b = c(S_2)$ but $a \neq (S_3)$.

If $K_3 \subseteq K_1$, then take $c \in K_3 \subseteq K_1$ and choose $\succ_1$ and $\succ_2$ such that $b \succ_1 c$ and $c \succ_2 a \succ_2 b \succ_2 t$,$\forall t \in K_1 \cup K_2 \cup K_3$. We will have that $a = c(S_1)$, $b = c(S_2)$ but

$c = (S_3)$. Thus must have that $b \in S_3$.

Now we have a sentence of the form: $a = c(a, b, K_1)$ and $b = c(b, K_2) \Rightarrow a = c(a, b, K_3)$, where $a \notin K_2$.

Step 14: $K_3 \subseteq K_1$.

Proof: Assume the contrary, $K_3 \setminus K_1 \neq \varnothing$. Then $\exists c \in K_3$ such that $c \notin K_1$. If $c \notin K_2$, take $\succ_2$ such that $c \succ_2 a \succ_2 b \succ_2 t$, $\forall t \in K_1 \cup K_2 \cup K_3$. We have that $a = c(S_1)$, $b = c(S_2)$ but $c(S_3) = c$. If $c \in K_2$, take $c \succ_1 a$ and $a \succ_2 b \succ_2 c \succ_2 t$, $\forall t \in K_1 \cup K_2 \cup K_3$. We have that $a = c(S_1)$, $b = c(S_2)$ but $c(S_3) = b$. Thus must have that $K_1 \subseteq K_3$.

Step 15: $|K_3| = 0$.

Proof: Assume the contrary, $|K_3| \geq 1$. Now take $d \in K_3 \subseteq K_1$.
Also since $K_1 \neq K_3$, $\exists c \in K_1 \setminus K_3$ so that $c \in K_1$ and $c \notin K_3$. If $d \notin K_2$, let $c \succ_1 d$ and $d \succ_2 a \succ_2 b \succ_2 t$, $\forall t \in K_1 \cup K_2 \cup K_3 \setminus \{d\}$. If $d \in K_2$ let $c \succ_1 d$, $a \succ_1 b$ and $b \succ_2 d \succ_2 a \succ_2 c \succ_2 t$, $\forall t \in K_1 \cup K_2 \cup K_3 \setminus \{c, d\}$. In both cases we will have that $a = c(S_1)$, $b = c(S_2)$ but $c(S_3) = d$. Thus $|K_3| = 0$.

Now we have a sentence of the form: $a = c(a, b, K_1)$ and $b = c(b, K_2) \Rightarrow a = c(a, b)$, where $a \notin K_2$.

Step 16: $K_1 \subseteq K_2$.

Proof: Assume the contrary, $K_1 \setminus K_2 \neq \varnothing$. Then $\exists c \in K_1$ such that $c \notin K_2$. Now take $\succ_1$ and $\succ_2$ such that $c \succ_1 b$ and $b \succ_2 a \succ_2 t$, $\forall t \in K_1 \cup K_2$. We have that $a = c(S_1)$, $b = c(S_2)$ but $c(a, b) = b$. Therefore $K_1 \subseteq K_2$.

Indeed we have reached the axiom BW: If $a, b \in S_1$, $S_1/a \subseteq S_2$ and $a \notin S_2$ then $a = c(S_1)$ and $b = c(S_2) \Rightarrow a = (a, b)$

$\square$

Explanations over the table:

It is very easy to verify that all the axioms in Q are satisfied by SRCIT too. Since SRCAT is nested from RSM, it is enough to prove that RSM satisfies EXP, PE,WW and BW. For the same reason it is enough to prove that SRCAT does not satisfy DRIA, IRA and DRA.

Now remember that SRCAT has its first rationale asymmetric and its second rationale transitive. Take the set of alternatives $A = \{a, b, c, d\}$.

Take $\succ_1$ and $\succ_2$ such that $a \succ_1 c$, $b \succ_1 d$, $c \succ_1 b$, $b \succ_2 a$, $d \succ_2 a$ and $d \succ_2 c$. Obviously $\succ_1$ is asymmetric and $\succ_2$ is transitive. Then IRA will not be satisfied from SRCAT.

If we take $\succ_1$ and $\succ_2$ such that $c \succ_1 b$, $d \succ_1 c$, $a \succ_1 d$, $c \succ_2 a$, $b \succ_2 a$ and $b \succ_2 d$, $\succ_1$ will be asymmetric and $\succ_2$ will be transitive. In this case DRIA and DRA will not be satisfied from SRCAT.

From the axiomatic characterization of RSM (Mariotti and Manzini [1]) we know that RSM must satisfy EXP and WW. PE is also satisfied by RSM, since EXP implies PE. Now, it remains only to prove that RSM satisfy BW.

Binary WARP (BW): For each $a, b \in A$ and for each $S_1, S_2 \in P(A)$ such that $a, b \in S_1$, $S_1 \setminus \{a\} \subseteq S_2$ and $a \notin S_2$, if $a = c(S_1)$ and $b = c(S_2)$, then $a = c(a, b)$.

Now assume the contrary, that under the above conditions $a = c(S_1)$, $b = c(S_2)$ but $b = c(a, b)$. Obviously, $a = c(S_1)$ implies $b \not\succ_1 a$. Then we have that $b \sim_1 a$ and $b \succ_2 a$. Since $b \succ_2 a$, there must exist some $d \in S_1$ such that $d \succ_1 b$. But then d will not let b be chosen from S_2. So we have reached a contradiction.