EXPLORATIONS ON $\mathcal{X}$-SELF SELECTIVITY

A Master's Thesis

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To my family

EXPLORATIONS ON $\mathcal{X}$-SELF SELECTIVITY

Graduate School of Economics and Social Sciences
of
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by

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I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

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ABSTRACT

EXPLORATIONS ON $\mathcal{X}$-SELF SELECTIVITY

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$\mathcal{X}$-self selectivity of a social choice function is defined as being self selective relative to the set of test functions $\mathcal{X}$ and all of its subsets. We explore the self-selectivity of social choice functions which satisfy independence of irrelevant alternatives, against different kinds of sets of test functions. We observe that testing against a smaller set can be sufficient to deduce that a given social choice function is also self-selective relative to a larger set, under certain conditions. Moreover, we show that $\mathcal{X}$-self selectivity is closed under set intersection and union. This leads to the notion of degree of self-selectivity, which allows us to compare the self-selectivities of two social choice functions under certain conditions.

Keywords: Social Choice, Self-Selectivity, Dictatorship, Independence of Irrelevant Alternatives, Tops-into.

ÖZET

$\mathcal{X}$-KENDINI SEÇERLIK ÜZERINE İNCELEMELER

Kerman, Toygar

Yüksek Lisans, İktisat Bölümü

Tez Yöneticisi: Prof. Dr. Semih Koray

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Bir sosyal seçim fonksiyonunun $\mathcal{X}$-kendini seçerliğini, o fonksiyonun $\mathcal{X}$ kümesi ve bütün alt kümelerine göreceli olarak kendini seçer olması olarak tanımlıyoruz. İlgisiz seçeneklerin bağımsızlığı özelliğini sağlayan sosyal seçim fonksiyonlarının değişik türde sınama fonksiyonları kümelerine karşı kendini seçerliğini inceliyoruz. Gözlemliyoruz ki, belirli koşullar altında, verilmiş sosyal seçim fonksiyonunu, daha küçük bir sınama fonksiyonları kümesine karşı sınamak, daha büyük bir sınama fonksiyonları kümesine göre de kendini seçer olduğunu anlamamız için yeterli oluyor. Buna ek olarak, $\mathcal{X}$-kendini seçerliğin, küme kesişimi ve birleşimi altında kapalı olduğunu gösteriyoruz. Bu özellik kendini seçerlik derecesi kavramına bizi götürüyor, bu da belirli koşullar altında verilmiş iki sosyal seçim fonksiyonunun kendini seçerliklerini karşılaştırmamıza olanak sağlıyor.

Anahtar Kelimeler: Sosyal Seçim, Kendini Seçerlik, Diktatörlük, İlgisiz Seçeneklerden Bağımsızlık, Üstten Seçerlik.

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CHAPTER 1

INTRODUCTION

A society facing a choice problem uses a certain social choice rule to resolve the problem. It is usually assumed that the social choice rule the society will use is predetermined. It is, however, not uncommon that the society is also to choose the rule that it will employ in making its decision. In case the chosen rule does not choose itself when it is also used in choosing the choice rule, it will create an inconsistency.

Let A stand for a finite nonempty set of alternatives, from which the choice is to be made. The society is assumed to be endowed with a preference profile on A, and to have a finite nonempty set $\mathcal{A}$ of social choice functions (SCFs) available to make its choice of an SCF. We assume that each agent's preferences on A is represented by a linear order, and that each agent ranks the available SCFs according to the outcomes they lead to. Thus, two SCFs leading to the same alternative are regarded as equivalent. So, the linear order profile over the alternatives leads to a complete preorder profile over the available SCFs[1]. We can test the self-selectivity of an SCF in a set $\mathcal{A}$ of available SCFs by applying

[1]A complete preorder ρ on a nonempty set B is a binary relation on B that is complete and transitive on B

it to the induced complete preorder profile of SCFs. We will say that an SCF F is self-selective relative to $\mathcal{A}$ at a preference profile R on A if, for any finite subset $\mathscr{B}$ of $\mathcal{A}$, it chooses itself from $\mathscr{B} \cup \{F\}$ at some linear order profile on $\mathscr{B} \cup \{F\}$ compatible with the complete preorder profile on $\mathscr{B} \cup \{F\}$ induced by R. An SCF is universally self-selective if it chooses itself relative to any possible $\mathcal{A}$ at each R.

Self-selectivity of SCFs was first introduced by Koray (2000). In that paper, the main result is that a unanimous SCF is universally self-selective if and only if it is dictatorial. Universal self-selectivity seems to be too strong an assumption for an SCF, as it is required that the SCF is self-selective at any preference profile, relative to any set of test functions. Thus, in our study, we restrict the set of SCFs against which the self-selectivity of a given SCF is to be tested to some set $\mathcal{X}$ of admissible SCFs.

The resulting notion of $\mathcal{X}$-self selectivity not only allows us to escape dictatoriality, but $\mathcal{X}$ may also capture certain properties concerning efficiency or fairness that the society in question may regard desirable. The technical difficulties associated with testing self-selectivity can be simplified by showing that $\mathcal{X}$-self selectivity is equivalent to $\mathcal{Y}$-self selectivity, where $\mathcal{Y}$ is a "small" subset of $\mathcal{X}$.

The notion of $\mathcal{X}$-self selectivity was introduced by Koray and Slinko (2008), where $\mathcal{X}$ is taken as the set of all selections of certain social choice correspondences π. They show that, under certain conditions imposed upon π, a selection of π is $\mathcal{X}$-self selective if and only if it is dictatorial or π-antidictatorial.[2]

[2] A π-antidictatorial SCF chooses at each preference profile R the worst alternative in $\pi(R)$ for the "anti-dictator".

In this study, the SCFs tested for self-selectivity are assumed to be neutral and to satisfy Independence of Irrelevant Alternatives (IIA). IIA is assumed to have some natural smoothness in the behavior of the SCF tested for self-selectivity as the size of the alternative set changes. We first show that testing our SCF against the set $\mathcal{T}$ of all tops-into SCFs is equivalent to testing it against the set $\mathcal{D}$ of all dictatorships, which is a subset of the former. We then consider the set $\mathcal{P}$ of all Pareto optimal SCFs, which is not a subset of $\mathcal{T}$, since a Pareto optimal alternative clearly need not be any agent's top choice. It turns out that there is no equivalency between testing against $\mathcal{D}$ and against $\mathcal{P}$. But instead, we see that taking a set $\mathcal{X}$ of SCFs, which contains $\mathcal{D}$ and satisfies certain other conditions, our test function makes $\mathcal{X}$-self selectivity a sufficient condition for $\mathcal{P}$-self selectivity.

We also show that $\mathcal{X}$-self selectivity is closed under set union. That is, if a neutral SCF which satisfies IIA is both $\mathcal{Y}$-self selective and $\mathcal{Z}$-self selective for two collections $\mathcal{Y}$ and $\mathcal{Z}$ of SCFs at some preference profile R, then it is also $(\mathcal{Y} \cup \mathcal{Z})$-self selective at R. This also inspires the notion of self-selectivity degree of SCFs, since the maximal set of SCFs, relative to which a neutral SCF satisfying IIA is self-selective at some preferene profile is unique. This is different than the self-selectivity degree indroduced by Altuntaş (2011). In that study, the self-selectivity degree is confined to a certain family of SCFs, namely the q-Condorcet rules. She shows that when we test a q-Condorcet rule against a q'-Condorcet rule with $q < q'$, then the q-Condorcet rule is self-selective. So, she measures the self-selectivity of the q-Condorcet rules within the Condorcet family. What we want to find here is the maximal set, relative to which a neutral SCF satisfying IIA is self-selective, where that maximal set can include SCFs from any family.

CHAPTER 2

PRELIMINARIES

Let N be a finite nonempty society that will be kept fixed. Let $\mathbb{N}$ be the set of natural numbers, set $I_m = \{1, \ldots, m\}$ and denote the set of all linear orders on I_m by $\mathscr{L}(I_m)$ for each $m \in \mathbb{N}$.

Definition 1. We say that a map $F : \bigcup\limits_{m \in \mathbb{N}} \mathscr{L}(I_m)^N \to \mathbb{N}$ is a social choice function (SCF) if and only if, for each $m \in \mathbb{N}$, $R \in \mathscr{L}(I_m)^N$, one has $F(R) \in I_m$.

The set of all SCFs will be denoted by $\mathscr{F}$. For each $m \in \mathbb{N}$, $R \in \mathscr{L}(I_m)^N$ and every permutation σ_m on I_m, we define the permuted linear order profile R_{σ_m} on I_m through the following biconditional: For all $i \in N$, $k, l \in I_m$, $kR^i_{\sigma_m}l$ if and only if $\sigma_m(k)R^i\sigma_m(l)$.

Definition 2. We say that $F \in \mathscr{F}$ is neutral if and only if for each $m \in \mathbb{N}$ and every permutation σ_m on I_m we have $\sigma_m(F(R_{\sigma_m})) = F(R)$.

The set of all neutral SCFs will be denoted by $\mathscr{N}$. Neutrality of an SCF F allows us to extend the domain of F to linear order profiles on any finite nonempty set in a natural manner. Take any finite set A with $|A| = m \in \mathbb{N}$, where $|A|$ is the cardinality of A. Let $\mu : I_m \to A$ be a bijection. Any linear order profile L on A induces a linear order profile L_μ on I_m, where $\forall i \in N$ and $\forall k, l \in I_m$, we have $kL^i_\mu l$ if and only if $\mu(k)L^i\mu(l)$. Now, since F is neutral, for any two bijections $\mu, \nu : I_m \to A$, we have $\mu(F(L_\mu)) = \nu(F(L_\nu))$. So, we define $F(L) = \mu(F(L_\mu))$, where $\mu : I_m \to A$ is any bijection.

Take any $m \in \mathbb{N}$, $R \in \mathscr{L}(I_m)^N$ and any nonempty finite subset $\mathcal{A}$ of $\mathscr{N}$. For each $i \in N$, we define $R^i_{\mathcal{A}}$ on $\mathcal{A}$ in the following manner: For all $F, G \in \mathcal{A}$ and $i \in N$, $FR^i_{\mathcal{A}}G$ if and only if $F(R)R^iG(R)$. That is, each agent evaluates the available SCFs according to the outcomes they lead to. Thus, $R^i_{\mathcal{A}}$ is a complete preorder on $\mathcal{A}$.

Given a complete preorder ρ on a finite nonempty set A, a linear order λ on A is said to be compatible with ρ if for all $x, y \in A$, $x\lambda y$ implies $x\rho y$. Now, for each $m \in \mathbb{N}$, $R \in \mathscr{L}(I_m)^N$ and every nonempty finite subset $\mathcal{A}$ of $\mathscr{N}$, set $\mathscr{L}(\mathcal{A}, R) = \left\{L \in \mathscr{L}(\mathcal{A})^N | L^i \text{ is a linear order on } \mathcal{A} \text{ compatible with } R^i_{\mathcal{A}}\right\}$, which is referred to as the set of all linear order profiles on $\mathcal{A}$ induced by R.

Definition 3. Given $F \in \mathscr{N}$, $m \in \mathbb{N}$, $R \in \mathscr{L}(I_m)^N$ and a finite subset $\mathcal{A}$ of $\mathscr{N}$, we say that F is self-selective at R relative to $\mathcal{A}$ if there exists some $L \in \mathscr{L}(\mathcal{A} \cup \{F\}, R)$ such that $F = F(L)$.

We say that F is self-selective at R if F is self-selective at R relative to any finite subset $\mathcal{A}$ of $\mathscr{N}$.

F is said to be universally self-selective if F is self-selective at each $R \in \bigcup_{m \in \mathbb{N}} \mathscr{L}(I_m)^N$.

We now give the definition of $\mathcal{X}$-self selectivity, which will be our main focus in this study.

Definition 4. Let $\mathcal{X}$ be a collection of SCFs and $R \in \mathscr{L}(I_m)^N$. We say that $F \in \mathscr{N}$ is $\mathcal{X}$-self selective at R if and only if F is self-selective at R relative to

$\mathcal{A}$, for any finite subset $\mathcal{A}$ of $\mathscr{X}$.

We say that F is $\mathcal{X}$-self selective if and only if F is $\mathcal{X}$-self selective at each $R \in \mathscr{L}(I_m)^N$.

Definition 5. Let F be an SCF and set
$\bar{I}_m = \{a \in I_m | a \text{ is top-ranked at } R_i \in \mathscr{L}(I_m), \text{for some } i \in N\}$ for each $m \in \mathbb{N}$.
We say that F is tops-into if and only if, for all $m \in \mathbb{N}$ and $R \in \mathscr{L}(I_m)^N$, one has $F(R) \in \bar{I}_m$. That is, at each $R \in \mathscr{L}(I_m)^N$, F chooses an alternative that is top-ranked by at least one agent.

Note that if an SCF F is not tops-into, it means that there exists at least one profile $R \in \mathscr{L}(I_m)^N$ such that F chooses an alternative which is not top-ranked by any agent at R.

Definition 6. We say that an SCF F is unanimous if, for all $m \in \mathbb{N}$, $R \in \mathscr{L}(I_m)^N$ and $a \in I_m$, we have

$$\left[\forall i \in N, \forall b \in I_m : aR^i b\right] \Rightarrow F(R) = a.$$

Definition 7. An SCF $F \in \mathscr{N}$ is called Paretian if and only if for all $R \in \bigcup\limits_{m \in \mathbb{N}} \mathscr{L}(I_m)^N$, $F(R)$ is Pareto optimal with respect to R.

We say that an SCF $F \in \mathscr{N}$ satisfies independence of irrelevant alternatives (IIA) if and only if for all $m \in \mathbb{N}$ and $R \in \mathscr{L}(I_m)^N$, we have

$$[\emptyset \neq B \subset I_m, F(R) \notin B] \Rightarrow F(R) = F(R|_{I_m \setminus B}),$$

where $R|_{I_m \setminus B}$ denotes the restriction of R to $I_m \setminus B$.

Definition 8 (Koray and Slinko (2008))**.** Let $\mathcal{X}$ be a collection of SCFs and let π be a social choice correspondence (SCC). We say that $\mathcal{X}$ is π-complete if and only if, for each $m \in \mathbb{N}$ and $R \in \mathscr{L}(I_m)^N$, we have $\bigcup_{F \in \mathcal{X}} F(R) = \pi(R)$.

Definition 9. For any $m \in \mathbb{N}$ and $R \in \mathscr{L}(I_m)^N$, define $P(R)$ to be the set of all Pareto optimal alternatives at R, and define $T(R)$ to be the set of all alternatives that are top-ranked at R by at least one agent. We refer to P and T as the Pareto SCC and top SCC, respectively.

Example 1 (Koray and Slinko (2008)). Let $\mathcal{D}$ be the set of all dictatorships. Then $\mathcal{D}$ is T-complete. However, $\mathcal{D}$ is not P-complete, since a Pareto optimal alternative may not be top-ranked by any agent.

Definition 10 (Koray and Slinko (2008))**.** Let π be any neutral SCC. Given $i \in N$, define the SCF F by $F(R) = \min R_i|_{\pi(R)}$ for any $m \in \mathbb{N}$ and any $R \in \mathscr{L}(I_m)^N$. We refer to F as the π-antidictatorship of agent i. Note that, F chooses the alternative in $\pi(R)$ which is least preferred by agent i within $\pi(R)$.

In particular, if π is the Pareto correspondence and agent i is the Pareto-antidictator, then F chooses the worst Pareto optimal alternative for agent i.

CHAPTER 3

INFERENCES OF $\mathcal{X}$-SELF SELECTIVITY

3.1 $\mathcal{X}$-Self Selectivity with Dictatorships, Tops-Into and Pareto optimal SCFs

Let $\mathcal{T}$, $\mathcal{D}$ and $\mathcal{P}$ denote the sets of all tops-into, dictatorial and Paretian SCFs, respectively.

Proposition 1. *Let $F \in \mathscr{N}$ satisfy IIA and $R \in \mathscr{L}(I_m)^N$. Now, F is $\mathcal{D}$-self selective at R if and only if F is $\mathcal{T}$-self selective at R.*

Proof. Let $F \in \mathscr{N}$ satisfy IIA and take $R \in \mathscr{L}(I_m)^N$. Assume that F is $\mathcal{T}$-self selective at R. Since all dictatorships are tops-into SCFs, we have, $\mathcal{D} \subset \mathcal{T}$. Hence, F is $\mathcal{D}$-self selective at R.

For the converse, suppose that F is $\mathcal{D}$-self selective, but not $\mathcal{T}$-self selective at R. Since F is not $\mathcal{T}$-self selective at R, there is a finite subset $\mathcal{A}$ of $\mathcal{T}$ such that F is not self-selective at R relative to $\mathcal{A}$, i.e., $\forall L \in \mathscr{L}(\mathcal{A} \cup \{F\}, R) : F(L) \neq F$. Pick any $\bar{L} \in \mathscr{L}(\mathcal{A} \cup \{F\}, R)$. Then $F(\bar{L}) = G$ for some $G \in \mathcal{A}$, and assume that $G(R) = x$ for some $x \in I_m$. Recall that G is a tops-into SCF, which implies that there exists a dictatorship D_G choosing the same alternative as G

at R. That is, $D_G(R) = x$. Moreover, we know that $F(R) \neq G(R)$, since otherwise we would have $F(\bar{L}') = F$, where $\bar{L}' \in \mathscr{L}(\mathcal{A} \cup \{F\}, R)$ is obtained from $\bar{L}$ by interchanging the positions of G and F, due to neutrality of F.

But then, $\mathscr{L}(\{G\} \cup \{F\}, R) = \{L'\}$, where $L' = \bar{L}|_{\{G,F\}}$. Since F satisfies IIA, $F(L') = G$. Moreover, by neutrality of F, we have $F(L'') = D_G$, where L'' is obtained from L' by substituting D_G for G. Now, however, $\{D_G\} \subset \mathcal{D}$, $\mathscr{L}(\{D_G\} \cup \{F\}, R) = \{L''\}$ with $F(L'') = D_G \neq F$, contradicting that F is $\mathcal{D}$-self selective at R. Thus, F is $\mathcal{T}$-self selective at R.

□

In the next proposition, we will make use of the correspondence $P \setminus T$, which chooses at each $R \in \mathscr{L}(I_m)^N$, all Pareto optimal alternatives that are not top-ranked by any agent. However, we will not assume that the set $\mathcal{X}$ is $(P \setminus T)$-complete, since then together with the fact that $\mathcal{D} \subset \mathcal{X}$, it would make $\mathcal{X}$ P-complete, which is a case we will consider later. Hence, we only require that for the alternatives which are Pareto optimal but not top-ranked by any agent, there exists a function in $\mathcal{X}$ that chooses each such alternative at R.

Proposition 2. *Let $F \in \mathscr{N}$ satisfy IIA, $\mathcal{X}$ be a collection of SCFs and let $R \in \mathscr{L}(I_m)^N$ be given. Define $\mathcal{S} = \{H \in \mathscr{F} | H(R) \in (P \setminus T)(R)\}$. Assume that $\mathcal{D} \subset \mathcal{X}$ and $\mathcal{S} \subset \mathcal{X}$. If F is $\mathcal{X}$-self selective at R, then F is $\mathcal{P}$-self selective at R.*

Proof. Assume that F is $\mathcal{X}$-self selective at R. Since $\mathcal{D} \subset \mathcal{X}$ by hypothesis, we have that F is $\mathcal{T}$-self selective at R by Proposition 1. Now take any $H \in P \setminus T$. We will now show that F is $(\mathcal{T} \cup \{H\})$-self selective at R. Note that $H(R)$ may or may not belong to $T(R)$. So, we consider two cases.

Case 1. $H(R) \in (P \setminus T)(R)$.

In this case, $H \in \mathcal{S} \subset \mathcal{X}$. Since F is $\mathcal{X}$-self selective at R, F is also $\{H\}$-self selective at R. By Proposition 4, it follows that F is $(\mathcal{T} \cup \{H\})$-self selective at R.

Case 2. $H(R) \notin (P \setminus T)(R)$.

Let $\mathcal{A}$ be any finite subset of $\mathcal{T}$ and consider $\mathcal{A} \cup \{H\} \cup \{F\}$. As $H(R) \in T(R)$ now, there is some $G \in \mathcal{T}$ with $H(R) = G(R)$. But then $\mathcal{A} \cup \{G\}$ is a finite subset of $\mathcal{T}$. Since F is $\mathcal{T}$-self selective, there is some $L \in \mathscr{L}(\mathcal{A} \cup \{G\} \cup \{F\}, R)$ such that $F(L) = F$. Let L' be the linear order profile obtained from L by substituting H for G in L. Now $L' \in \mathscr{L}(\mathcal{A} \cup \{H\} \cup \{F\}, R)$ and $F(L') = F$ by neutrality. So, again by Proposition 4, F is $(\mathcal{T} \cup \{H\})$-self selective at R.

Since this result holds for any $H \in P \setminus T$, by resorting to Proposition 4 once more, we conclude that F is $\mathcal{P}$-self selective at R. □

In showing that such an SCF with the properties as in the above proposition is $\mathcal{P}$-self selective, we made use of the fact that $\mathcal{S} \subset \mathcal{X}$. In the next proposition, we assume instead that $\mathcal{X}$ is P-complete, to reach a similar result.

Proposition 3. *Let $F \in \mathscr{N}$ satisfy IIA, $\mathcal{X}$ be any nonempty collection of SCFs, which is P-complete and $R \in \mathscr{L}(I_m)^N$ be given. If F is $\mathcal{X}$-self selective at R, then F is $\mathcal{P}$-self selective at R.*

Proof. Suppose that F is $\mathcal{X}$-self selective, but not $\mathcal{P}$-self selective at R. Then, there is a finite set $\mathcal{A} \subset \mathcal{P}$ such that $\forall L \in \mathscr{L}(\mathcal{A} \cup \{F\}, R)$: $F(L) \neq F$. Take any $L \in \mathscr{L}(\mathcal{A} \cup \{F\}, R)$. Then $F(L) = H \neq F$ for some $H \in \mathcal{A}$. Note that one also has $F(R) \neq H(R)$.

Now since F is $\mathcal{X}$-self selective, we have $\mathcal{A} \not\subset \mathcal{X}$. So, either H is in $\mathcal{X}$ and is chosen because of the existence of SCFs in $\mathcal{A} \setminus \mathcal{X}$, or H is not in $\mathcal{X}$ to begin with. Thus, we consider two cases.

Case 1. $H \in \mathcal{X}$.

Now since F satisfies IIA, we have $F(L|_{\{H,F\}}) = H \neq F$ in contradiction with the $\mathcal{X}$-self selectivity of F at R as $\{H\} \subset \mathcal{X}$.

Case 2. $H \notin \mathcal{X}$.

We again have $F(L|_{\{H,F\}}) = H$. Now since $\mathcal{X}$ is P-complete, $\exists G \in \mathcal{X}$: $G(R) = H(R)$. Moreover, $G \neq F$ as $G(R) \neq F(R)$. Denote L' for the linear order profile obtained from $L|_{\{H,F\}}$ by substituting G for H. By neutrality of F, we have $F(L') = G$ contradicting that F is $\mathcal{X}$-self selective at R. Since this holds for any $L \in \mathscr{L}(\mathcal{A} \cup \{F\}, R)$, we conclude that F is $\mathcal{P}$-self selective at R. $\square$

Since $\mathcal{T} \subset \mathcal{P}$, we can state a corollary to Proposition 1.

Corollary 1. *Let $F \in \mathscr{N}$ satisfy IIA, $\mathcal{X}$ be any nonempty collection of SCFs which is T-complete and $R \in \mathscr{L}(I_m)^N$ be given. If F is $\mathcal{X}$-self selective at R, then F is $\mathcal{T}$-self selective at R.*

3.2 $\mathcal{X}$-Self Selectivity Under Union and Intersection

Suppose that $F \in \mathscr{N}$ satisfies IIA. Let $\mathcal{X}$ and $\mathcal{Y}$ be two nonempty collections of SCFs. Moreover, let F be both $\mathcal{X}$-self selective and $\mathcal{Y}$-self selective at a given profile $R \in \mathscr{L}(I_m)^N$. Now, clearly F is self selective relative to any subset of $\mathcal{X} \cap \mathcal{Y}$, since any such subset is also a subset of both $\mathcal{X}$ and $\mathcal{Y}$. Hence, $\mathcal{X}$-self selectivity is indeed closed under set intersection.

In showing that $\mathcal{X}$-self selectivity is closed under set union as well, the fact that

F satisfies IIA will play a crucial role. IIA is needed for our SCF to be consistent among different layers of the given profile. For example, functions that choose an alternative at a given profile contingent upon the number of alternatives in the profile, will, in general, not satisfy IIA and therefore, we could not employ the method in our proof.

Proposition 4. *Let $F \in \mathscr{N}$ satisfy IIA and $R \in \mathscr{L}(I_m)^N$. Let $\mathcal{X}$ and $\mathcal{Y}$ be two collections of SCFs. Now, if F is both $\mathcal{X}$-self selective and $\mathcal{Y}$-self selective at R, then F is both $(\mathcal{X} \cup \mathcal{Y})$-self selective and $(\mathcal{X} \cap \mathcal{Y})$-self selective at R.*

Proof. Assume that $F \in \mathscr{N}$ satisfies IIA and F is $\mathcal{X}$-self selective and $\mathcal{Y}$-self selective at some $R \in \mathscr{L}(I_m)^N$. Since $(\mathcal{X} \cap \mathcal{Y}) \subset \mathcal{X}$ (and $\mathcal{Y}$), F is $(\mathcal{X} \cap \mathcal{Y})$-self selective at R.

Now suppose that F is not $(\mathcal{X} \cup \mathcal{Y})$-self selective at R. Then there is a finite subset $\mathcal{A}$ of $\mathcal{X} \cup \mathcal{Y}$ such that F is not self-selective at R relative to $\mathcal{A}$, i.e., $\forall L \in \mathscr{L}(\mathcal{A} \cup \{F\}, R) : F(L) \neq F$. Pick any $L \in \mathscr{L}(\mathcal{A} \cup \{F\}, R)$. Now $F(L) = H$ for some $H \in \mathcal{A}$ with $H \neq F$. Without loss of generality, assume that $H \in \mathcal{X}$.

Suppose that $F(R) = H(R)$. Now the linear order profile $\tilde{L}$ on $\mathcal{A} \cup \{F\}$ obtained from L by interchanging F and H also belongs to $\mathscr{L}(\mathcal{A} \cup \{F\}, R)$. Moreover, by neutrality of F, it follows that $F(\tilde{L}) = F$ in contradiction with our supposition. So, $F(R) \neq H(R)$. Thus, $\mathscr{L}(\{F\} \cup \{H\}, R) = \{L'\}$, for some L'. As F satisfies IIA and $L' = L|_{\{F,H\}}$, we have $F(L') = H$. On the other hand, since $\{H\} \subset \mathcal{X}$ and F is $\mathcal{X}$-self selective at R, we have $F(L') = F$, a contradiction. Hence, F is $(\mathcal{X} \cup \mathcal{Y})$-self selective at R. □

The above proposition states that whenever a neutral SCF which satisfies IIA is self-selective relative to two different sets of test functions and their subsets, then it is also self-selective relative to their union and all subsets of that union.

Another question emerges from this proposition, which we can define as the degree of self-selectivity. Since $\mathcal{X}$-self selectivity is closed under set union, there should exist a unique set for which our SCF is self-selective relative to it and all its subsets. Hence, each SCF has such a unique maximal set. However, to compare the degrees of self-selectivity of two SCFs, the method of comparison of self-selectivities of SCFs should be well defined, which we will mention in the next chapter.

We can state an immediate corollary to the above proposition, combining it with Proposition 1. We know that $\mathcal{D}$-self selectivity implies $\mathcal{T}$-self selectivity for a neutral SCF which satisfies IIA. But also, if this SCF is $\mathcal{D}$-self selective, this means that it is self-selective relative to each set that contains only one dictatorship. Hence, testing our function against dictatorships one by one, suffices to imply $\mathcal{T}$-self selectivity, without the need of testing against other subsets of $\mathcal{D}$.

Corollary 2. *Let $F \in \mathscr{N}$ satisfy IIA and $R \in \mathscr{L}(I_m)^N$ be given. Let $\mathcal{D} = \{D_1, D_2, \ldots, D_n\}$ denote the set of all dictatorships. Now, if F is self-selective at R relative to $\{D_i\} \cup \{F\}$ $\forall i \in N$, then F is $\mathcal{T}$-self selective at R.*

Proof. Suppose $F \in \mathscr{N}$ satisfies IIA and $R \in \mathscr{L}(I_m)^N$. Assume that $\forall i \in N$, F is self-selective at R relative to $\{D_i\} \cup \{F\}$. Then, by a previous proposition, F is self-selective at R relative to $\mathcal{A} \cup \{F\}$, for any $\mathcal{A} \subset \mathcal{D}$. So, F is $\mathcal{D}$-self selective at R and therefore $\mathcal{T}$-self selective at R. □

CHAPTER 4

DEGREE OF SELF-SELECTIVITY

As we mentioned in the previous chapter, $\mathcal{X}$-self selectivity gives rise to another concept, the degree of self-selectivity. To compare self selectivities of two SCFs, we give the following definition.

Definition 11. Let $F, G \in \mathscr{N}$ satisfy IIA and $R \in \mathscr{L}(I_m)^N$. Let $\mathcal{X}$ and $\mathcal{Y}$ be two nonempty collections of SCFs, which are the maximal sets such that F is $\mathcal{X}$-self selective and G is $\mathcal{Y}$-self selective at R. We say that F is more [less] self-selective than G at R if and only if $\mathcal{Y} \subset \mathcal{X}$ [$\mathcal{X} \subset \mathcal{Y}$].

We say that F is more [less] self-selective than G if and only if F is more [less] self-selective than G for all $R \in \mathscr{L}(I_m)^N$.

We will say that a set $\mathcal{X}$ is maximal for an SCF F at R, if $\mathcal{X}$ is the maximal set such that F is $\mathcal{X}$-self selective at a profile R. Note that to compare the self-selectivities of two SCFs, we need the condition that one set is included in the other. Thus, we may not be able to compare the self-selectivities of two functions, if one maximal set is not a subset of the other.

Note that this notion of the degree of self-selectivity is different than the degree of self-selectivity that has been introduced by Altuntaş (2011). In that study, the degree of self-selectivity is measured within the same family of SCFs, namely the Condorcet family. It is shown that as q decreases, the q-Condorcet rule becomes more self-selective against other rules of the same family.

By the above definition, we can measure the degree of self-selectivity of SCFs not within a family, but for all SCFs that are available. The only condition we need is that the set of SCFs of our functions which we are testing against, should be comparable as described above.

Making use of the fact that $\mathcal{X}$-self selectivity is closed under set union, we can state the following corollary.

Corollary 3. *Let $F, G \in \mathscr{N}$ satisfy IIA and $R \in \mathscr{L}(I_m)^N$. Suppose that both F and G are $\mathcal{X}$-self selective at R for some $\mathcal{X} \subset \mathscr{F}$, where $\mathcal{X}$ is maximal for G. If F is $\mathcal{Y}$-self selective at R for any $\mathcal{Y} \not\subset \mathcal{X}$, then F is more self-selective than G at R.*

Proof. Suppose that $F, G \in \mathscr{N}$ satisfy IIA and let $R \in \mathscr{L}(I_m)^N$ be given. Assume that both SCFs are $\mathcal{X}$-self selective at R, where $\mathcal{X}$ is maximal for G.

Now, assume that F is also $\mathcal{Y}$-self selective at R, for some $\mathcal{Y} \not\subset \mathcal{X}$. Then, by a previous proposition, F is also $\mathcal{Z} = (\mathcal{X} \cup \mathcal{Y})$-self selective at R. Hence, since $\mathcal{X} \subset \mathcal{Z}$, F is more self-selective than G at R. □

CHAPTER 5

CONCLUSION

After the notion of self-selectivity was introduced by Koray (2000), there has been many studies, each exploring different properties of self-selectivity. The one we used mainly in this study was the notion of $\mathcal{X}$-self selectivity, which was first introduced by Koray and Slinko (2008). In that study, they deal with correspondences and their single-valued refinements, and show that for certain kinds of $\mathcal{X}$-self selective SCFs, we again end up with either dictatoriality or π-antidictatoriality, where π is the correspondence which our SCF is a refinement of. We assume that the SCFs we are testing satisfy IIA, which is not an assumption in Koray and Slinko (2008). Furthermore, we do not take our SCF to be a refinement of a certain SCC. However, when we impose the same conditions in Koray and Slinko (2008), we obtain alternative proofs to some of the propositions in that study.

In this study, we explored practical solutions to test the self selectivity of an SCF against a very large or infinite set. To say that a function is $\mathcal{T}$-self selective, we need to test that SCF against all subsets of $\mathcal{T}$, which are infinitely many. We started by showing that testing a neutral SCF which satisfies IIA against the set of all dictatorships is equivalent to test that SCF against all subsets of $\mathcal{T}$. This is an improvement upon practicality, since the number of dictatorships are limited to the number of agents in the society and thus testing self-selectivity of a function against $\mathcal{T}$ becomes easier.

We then showed that if the set of dictatorships is included in $\mathcal{X}$ and also if the SCFs that choose the alternatives which are Pareto optimal but not top-ranked by anyone at R, then $\mathcal{X}$-self selectivity implies $\mathcal{P}$-self selectivity. The reason we do not simply assume that the set $\mathcal{X}$ is not $(P \setminus T)$-complete is that $\mathcal{X}$ becomes P-complete, which is a case we considered later on. In that case, that is if $\mathcal{X}$ is P-complete and F is $\mathcal{X}$-self selective, then F is also $\mathcal{P}$-self selective.

Then, showed that for a neutral SCF that satisfies IIA, $\mathcal{X}$-self selectivity is closed under set intersection and union. The closed under intersection part is obvious, since when a function is $\mathcal{X}$-self selective at a profile R, then it is self-selective to any subset of $\mathcal{X}$ as well. Showing that it is also closed under set union, directly led to another corollary, when combined with Proposition 1. In particular, since we assume our function to be $\mathcal{D}$-self selective, together with the fact that $\mathcal{D}$-self selectivity is closed under set union, we can conclude that testing our SCF against the dictatorships one by one, without testing against other subsets of $\mathcal{D}$, is sufficient to say that our SCF is also $\mathcal{T}$-self selective. This also makes testing $\mathcal{T}$-self selectivity easier, since for a set of test functions including many SCFs, the profile we have over the SCFs can be complicated and may take time to determine what our SCF chooses in that profile. Testing our SCF against just one other SCF eases this process.

This also led to another notion, the degree of self-selectivity. Since $\mathcal{X}$-self selectivity is closed under set union, there must exist a maximal set for each neutral SCF that satisfies IIA such that, that set is maximal for our SCF at a preference profile. Thus, we can compare the self-selectivities of two SCFs. According to our definition, an SCF F is more self-selective than another SCF

G at a profile R if and only if, the maximal set of F at R is a superset of the maximal set of G at R. Thus, it may not be possible to compare the self-selectivities of two SCFs, if one set is not included in the other. According to our definition, since dictatorship and anti-dictatorship are universally self-selective, they are the most self-selective of all SCFs.

For future research, it would be interesting to determine which SCFs are more self-selective for a given profile R. This definition of degree of self-selectivity depends on the given profile. Thus, it should be considered for a given profile, since an SCF F can be more self-selective than another SCF G at one profile, and less self-selective than G at another. However, for a given R, the families of SCFs which are more self-selective than other families of SCFs may be characterized.

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