SUBGAME PERFECT IMPLEMENTATION OF MEN-OPTIMAL MATCHINGS

A Master's Thesis

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To Can, Mutlu and Altuğ

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Graduate School of Economics and Social Sciences
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by

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ABSTRACT

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July 2017

In this study, we explore monotonicities and implementability of different matching rules. We find a self-monotonicity of the stable rule and an h-monotonicity for the men-optimal rule, which does not satisfy Maskin-monotonicity. We then offer a sequential matching mechanism that implements the men-optimal rule in subgame perfect equilibrium, when there is no other matching that weakly Pareto-dominates the men-optimal matching for men. In our mechanism, women propose to men in an arbitrary hierarchy order, and each man either accepts or rejects the proposals he receives, where accepting means permanent matching with the proposing woman.

Keywords: Implementation, Men-Optimal Stable Matching, Matching Theory, Self-Monotonicity, Subgame Perfect Equilibrium.

ÖZET

ERKEK-OPTIMAL EŞLEŞMELERIN ALT-OYUN-YETKIN UYGULANMASI

Teoman, Ece
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Tez Yöneticisi: Prof. Dr. Semih Koray

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Bu araştırmada değişik eşleşme kurallarının tekdüzeliğini ve uygulanabilirliğini inceliyoruz. Kararlı kuralın bir öz-tekdüzeliğini ve Maskin-tekdüzeliği sağlamayan erkek-optimal eşleşme kuralının bir h-tekdüzeliğini buluyoruz. Erkek-optimal eşleşmeyi zayıf anlamda Pareto-domine eden bir eşleşmenin bulunmadığı durumlarda erkek-optimal eşleşmeyi alt-oyun-yetkin dengeyle uygulayan bir ardışık eşleşme mekanizması öneriyoruz. Mekanizmamızda kadınlar rastgele bir hiyerarşi sıralamasıyla erkeklere teklifte bulunuyor ve erkekler, kabul etmek kalıcı bir şekilde eşleşmek anlamına gelmek üzere, aldıkları teklifleri kabul veya reddediyorlar.

Anahtar Kelimeler: Alt-Oyun-Yetkin Denge, Erkek-Optimal Kararlı Eşleşme, Eşleşme Teorisi, Öz-Tekdüzelik, Uygulama.

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CHAPTER 1

INTRODUCTION

Matching problems are concerned with assigning members of one group to one or more members of another group by taking their preferences into consideration, where the said groups are finite and disjoint. Such problems can involve matchings, which are one-to-one as in kidney exchange, or one-to-many as in assigning students to schools, or many-to-many as in matching firms and consultants. Marriage problems, introduced by Gale and Shapley (1962), exemplify two-sided and one-to-one matchings, where we have two disjoint groups, men and women, and they have preferences over the set of possible mates.

In our setting, we approach matching problems as social choice problems. Members of the society, men and women, have preferences on possible matchings, which are directly induced by individuals' preferences on their possible mates. In other words, the set of outcomes consists of matchings in our context, and we have different *matching rules* corresponding to different social properties. For instance, *the stable rule* assigns the set of all stable matchings, while *the men-optimal rule* assigns the men-optimal stable matching to each preference profile. In this study, we are mainly concerned with these two rules.

All the results we obtain for the men-optimal rule naturally also apply to *the women-optimal rule* with obvious modifications.

Kara and Sönmez (1996) is among the first studies to explore the monotonicity and implementability of different matching rules. Their focus is on the Pareto and Individually Rational Rule, while they also consider monotonic subsolutions of this rule as well as their implementability. Their results include that the stable rule is Maskin-monotonic and it is Nash implementable, when the society has at least three individuals. They make use of Danilov's monotonicity (Danilov, 1992) and the results from Yamato (1992) to prove their implementability result. They also show that the stable rule is the minimal subsolution of the Pareto and Individually Rational Rule, which is Nash implementable (and thus also Maskin-monotonic). It then follows as a corollary that the men-optimal rule (as well as the women-optimal rule) is neither monotonic nor Nash implementable. In another study, Alcalde (1996) shows that the Individually Rational Rule is implementable, and he offers a class of game forms to implement it.

In order to explore whether the stable rule (and its refinements) are implementable according to other solution concepts (possibly certain refinements of the Nash equilibrium notion), we first deal with self-monotonicities of these rules. As is introduced in Koray (2002), a self-monotonicity of a social choice rule (SCR) is a strongest Maskin-type monotonicity satisfied by that rule. We find a self-monotonicity of the stable rule in Chapter 3 and an h-monotonicity of the men-optimal rule in Chapter 4. Although the intuition is that stronger self-monotonicities are associated with a wider variety of implementabilities of the SCR considered, the precise relationship between these two notions in the particular context of the stable rule and its refinements is yet to be found. Along the same line, we then turn to the explicit construction of an extensive-form mechanism, which implements the men-optimal rule in subgame perfect equilibrium (SPE).

Kara and Sönmez (1996) show that the men-optimal rule does not satisfy Maskin-monotonicity. We exemplify some preference profile types, possibly shedding some further light to why and how the men-optimal rule fails to be Maskin-monotonic.

Suh and Wen (2008) study a sequential matching mechanism that implements the men-optimal rule under a certain condition imposed on the preference profiles. Their mechanism works under the Eeckhout condition introduced by Eeckhout (2000). They also introduce a weaker condition α^M. The α^M condition turns out to be necessary and sufficient for the men-optimal stable matching to be Pareto-optimal for men, thereby ruling out the preference profiles that prevent the men-optimal rule to satisfy Maskin-monotonicity.

We offer an alternative extensive form mechanism that implements the men-optimal rule in SPE under the α^M condition. Our mechanism appears to be simpler. Under some randomly chosen priority order, women move first. Each woman proposes to available and acceptable men in an order chosen by herself. Each man, who receives a proposal, irreversibly either accepts or rejects that proposal. Each woman continues to propose until she gets matched with some man or stays single having been rejected by every available and acceptable man. The SPE outcome of this mechanism is the men-optimal stable matching under the α^M condition. So our mechanism captures the controversial nature between the preferences of men and women over possible matchings.

Sequential matching mechanisms and implementation by SPE are also used for one-to-many and many-to-many matchings. Alcalde et al. (1998), Alcalde and Romero-Medina (2000) and Alcalde and Romero-Medina (2005) study sequential matching mechanisms and in all of these studies, the SPE outcome favors the first-mover in the mechanism.

CHAPTER 2

PRELIMINARIES

In this chapter, we provide definitions and notation for notions needed for our general setup. We will define more specific concepts in the related chapters.

A *matching problem* consists of two finite, nonempty and disjoint sets, M and W, and a preference profile $(R_i)_{i \in N}$ where $N = M \cup W$. In our context, we refer to M as the set of men and to W as the set of women. We provide the formal definitions below.

Definition 1. A simple graph G is an ordered pair (V, E), where V is a nonempty set of vertices and E is a subset of $\{\{x, y\} \mid x, y \in V, x \neq y\}$, referred to as the set of edges. Given a graph G, we denote its set of vertices and its set of edges by $V(G)$ and $E(G)$, respectively. We also write
$\Gamma_G(x) = \{y \in V(G) \mid xy \in E(G)\}$ for each $x \in V(G)$.

Definition 2. Let M and W be nonempty sets with $M \cap W = \emptyset$. A graph G is said to be bipartite with bipartition (M, W) if $V(G) = M \cup W$ and
$E(G) \subset \{mw \mid m \in M, w \in W\}$.

Definition 3. Let G stand for an (M, W)-bipartite graph. A subgraph μ of G with $V(\mu) = V(G)$ is said to be a matching if

$\forall x \in V(\mu) = V(G) : deg_\mu x \in \{0, 1\}$. Given a matching μ in G, we write $\mu(x) = x$ if $x \in V(G)$ with $deg_\mu x = 0$, while we write $\mu(m) = w$ and $\mu(w) = m$ in case $mw \in E(\mu)$.

Definition 4. For any $x \in V(G)$, let R_x be a linear order on $\Gamma_G(x) \cup \{x\}$. We refer to $R = (R_x)_{x \in V(G)}$ as a linear order profile for G.

Definition 5. Let R be a linear order profile for G. We say that a matching μ in G is *individually rational (IR)* wrt R, if for each $i \in M \cup W$ we have $\mu(i)\, R_i\, i$. We say that a pair $(m, w) \in M \times W$ *blocks a matching* μ *wrt* R if $mw \notin E(G)$ and we have $m\, R_w\, \mu(w)$ and $w\, R_m\, \mu(m)$. A matching μ is a *stable matching wrt* R if it is individually rational and there is no pair blocking μ wrt R.

Definition 6. Let R be a linear order profile for G and write $\mathcal{M}$ for the set of all matchings in G. For each $x \in V(G)$ and any $\mu, \mu' \in \mathcal{M}$, we say that $\mu\, \bar{R}_x\, \mu'$ if and only if $\mu(x)\, R_x\, \mu'(x)$. $\bar{R}_x$ is a complete preorder on $\mathcal{M}$ for each $x \in M \cup W$. We also write $\mu\, \bar{P}_x\, \mu'$ if and only if $\mu\, \bar{R}_x\, \mu'$ and $\mu'\, \not{\bar{R}}_x\, \mu$.

Individuals' preferences on the possible mates uniquely induce preferences on the set of all possible matchings as we assume that each $x \in M \cup W$ only cares about whom she or he is matched with. For a fixed M and W, given a bipartite graph G, we denote the set of all possible matchings by $\mathcal{M}$ and, given a preference profile R for G, the set of stable matchings by $\mathcal{SM}$.

Remark 1. Throughout this study, we use $\bar{R}$ to denote the complete preorder profile induced on the set of matchings by the linear order profile R for G.

Assumption 1. Being self-matched is the least desirable alternative for each individual.

Definition 7. Given a set $N = \{1, \dots, n\}$ of individuals and a finite nonempty set of alternatives, A, denoting the set of all linear orders on A by $\mathcal{L}(A)$, a map

$F : D \longrightarrow 2^A$ is called a *social choice rule (SCR)*, where $\emptyset \neq D \subset \mathcal{L}(A)^N$. We refer to a member R of $\mathcal{L}(A)^N$ as a preference profile. Given $R \in \mathcal{L}(A)^N$, $i \in N$, $a \in A$, the set $L_i(a, R) = \{b \in A \mid aR_ib\}$ is called *the lower contour set* of a for i at R.

Definition 8. Let $F : D \longrightarrow 2^A$ be an SCR, where $\emptyset \neq D \subset \mathcal{L}(A)^N$. Let Γ stand for an extensive form game with player set N, where the players' preferences over the set $T(\Gamma)$ of terminal nodes of the game tree of Γ are not specified. Also let $h : T(\Gamma) \longrightarrow A$ be an (outcome) function. The pair (Γ, R) is referred to as an extensive mechanism. For any $R \in D$, denote $\Gamma(R)$ for the extensive form game in which the players' preferences on $t(\Gamma)$ are induced by R via outcome function h. We say that (Γ, h) implements F in subgame perfect Nash equilibrium σ, if $\forall R \in D : F(R) = h(\sigma(\Gamma(R)))$, where $\sigma(\Gamma(R))$ stands for the set of terminal nodes the subgame perfect equilibria of $\Gamma(R)$ lead to by abuse of notation.

CHAPTER 3

THE STABLE RULE

In this chapter, we show that the stable rule satisfies Maskin-monotonicity which is a necessary condition for Nash implementability (Maskin, 1999). We then find a self-monotonicity for the stable rule in an attempt to utilize the connection between self-monotonicities of an SCR F and the implementability of F in different game-theoretic solution concepts as introduced in Koray (2002).

3.1 Maskin-Monotonicity of the Stable Rule

Definition 9. The *stable rule* F is a social choice rule that assigns to each preference profile R the set of matchings that are stable with respect to R, i.e., $F(R) = \mathcal{SM}(R)$ for every $R \in \mathcal{L}(A)^N$.

Definition 10. Let $F : \mathcal{L}(A)^N \longrightarrow 2^A$ be an SCR. We say that F is Maskin-monotonic if, for all $R, R' \in \mathcal{L}(A)^N$, $\mu \in F(R)$, one has

$$[\forall i \in N : L_i(\mu, R) \subset L_i(\mu, R')] \Rightarrow \mu \in F(R').$$

Remark 2. When we deal with the stable rule as an SCR, note that any alternative set is the set $\mathcal{M}$ of matchings. Thus, the domain of our SCR consists formally of complete preorder profiles on $\mathcal{M}$. We refer to a complete preorder profile $\bar{R}$ on $\mathcal{M}$ as admissible if and only if $\bar{R}$ is induced by some preference profile $R = (R_i)_{i \in N}$, where R_i is a linear order on $\Gamma_G(i) \cup \{i\}$ for each $i \in N$. When F denotes the stable rule, we will write both $F(R)$ and $F(\bar{R})$ for the image of F depending upon the context.

Lemma 1. *Let G be an (M, W)-bipartite graph and set $N = M \cup W$. The stable rule F is Maskin-monotonic.*

Proof. Take two complete preorder profiles $\bar{R}, \bar{R}'$ on $\mathcal{M}$ which are admissible and let R, R' be the linear order profiles inducing $\bar{R}, \bar{R}'$, respectively. Take some $\mu \in \mathcal{M}$ such that $\mu \in F(\bar{R})$ and assume that $L_i(\mu, \bar{R}) \subset L_i(\mu, \bar{R}')$ for all $i \in N$. Now we will show that $\mu \in F(\bar{R}')$ as well, i.e. μ is stable under R'.

(i) μ is individually rational with respect to R'.

Consider the matching $\bar{\mu} \in \mathcal{M}$ such that $\forall i \in N : \bar{\mu}(i) = i$. Since μ is a stable matching under R, it is also individually rational w.r.t. R. So, we have for all $i \in N$, $\mu(i)\ R_i\ \bar{\mu}(i)$ and thus $\mu\ \bar{R}_i\ \bar{\mu}$. Hence, $\bar{\mu} \in L_i(\mu, \bar{R})$. Since we also have $L_i(\mu, \bar{R}) \subset L_i(\mu, \bar{R}')$ for every $i \in N$, $\bar{\mu} \in L_i(\mu, \bar{R}')$ as well. This implies that for all $i \in N$, $\mu\ \bar{R}'_i\ \bar{\mu}$, and equivalently $\mu(i)\ R'_i\ \bar{\mu}(i) = i$. Hence, μ is individually rational w.r.t. R'.

(ii) There is no pair $(m, w) \in M \times W$ that blocks μ under R'.
Suppose that there is a pair $(m, w) \in E(G) \setminus E(\mu)$ which blocks μ under R'. So we have $w\ P'_m\ \mu(m)$ and $m\ P'_w\ \mu(w)$, equivalently $\mu'\ \bar{P}'_m\ \mu$ and $\mu'\ \bar{P}'_w\ \mu$ where, $\mu' \in \mathcal{M}$ is such that $mw \in E(\mu')$.

We picked $\bar{R}, \bar{R}'$ and μ such that $L_i(\mu, \bar{R}) \subset L_i(\mu, \bar{R}')$ for all $i \in N$. Since

$\mu' \notin L_m(\mu, \bar{R}')$, μ cannot be in $L_m(\mu, \bar{R})$. Similarly, $\mu' \notin L_w(\mu, \bar{R})$. So, $\mu' \, \bar{P}_m \, \mu$ and $\mu' \, \bar{P}_w \, \mu$, or equivalently, $w = \mu'(m) \, P_m \, \mu(m)$ and $m = \mu'(w) \, P_w \, \mu(w)$. But then (m, w) blocks μ under R, a contradiction. Thus, μ is a stable matching under R', i.e. $\mu \in F(\bar{R}')$. So, the stable rule F satisfies Maskin-monotonicity. □

3.2 A Self-monotonicity of the Stable Rule

Definition 11. Let F be the stable rule. Define $GrF = \{(\mu, R) \in A \times \mathcal{L}(A)^N \mid \mu \in F(R)\}$. Let $h : GrF \longrightarrow (2^A)^N$ be a map (to which we refer as a monotonicity-potential of F). We say that F is *h-monotonic* if for any $R, R' \in \mathcal{L}(A)^N$ and any $\mu \in F(R)$, we have

$$[\forall i \in N : h_i(\mu, R) \subset L_i(\mu, R')] \Rightarrow \mu \in F(R').$$

Definition 12. Let $h : GrF \to 2^{A^N}$ be a monotonicity-potential of F. We say that h is a self-monotonicity of F if F is h-monotonic and there is no monotonicity-potential h' of F such that F is h'-monotonic with $h' \subsetneqq h$.

Our candidate for a self-monotonicity of the stable rule is given by $h : GrF \to 2^{A^N}$, where

$$h_i(\mu, \bar{R}) = \begin{cases} \{\mu' \in \mathcal{M} \mid \mu' \, \bar{P}_{\mu'(i)} \, \mu\} & \text{if i} \in M \\ L_i(\mu, \bar{R}) & \text{if i} \in W \end{cases} \quad \text{at each}(\mu, \bar{R}) \in GrF.$$

We should note, at this point, that the roles of M and W can be switched in the above correspondence, rendering another self-monotonicity of the stable rule as well.

Proposition 1. *The correspondence $h : GrF \to 2^{A^N}$ defined above is a* *self-monotonicity of the stable rule F.*

Before proceeding with the proof of *Proposition 1* let us prove the following claim.

Claim. *For any $\bar{R}$ and for each $i \in N$, we have $h_i(\mu, \bar{R}) \subset L_i(\mu, \bar{R})$ for any $\mu \in F(\bar{R})$.*

Proof. Since we set $h_i(\mu, \bar{R}) = L_i(\mu, \bar{R})$ for all $i \in W$, we have
$h_i(\mu, \bar{R}) \subset L_i(\mu, \bar{R})$ for all $i \in W$ by construction.
Now suppose that for some $\bar{R}$, some $\mu \in F(\bar{R})$ and some $i \in M$,
$h_i(\mu, \bar{R}) \not\subset L_i(\mu, \bar{R})$. Then there is some $\mu' \in \mathcal{M}$ with $\mu' \, \bar{P}_{\mu'(i)} \, \mu$ and $\mu' \notin L_i(\mu, \bar{R})$. Since $\mu' \notin L_i(\mu, \bar{R})$, $\mu' \, \bar{P}_i \, \mu$ and equivalently $\mu'(i) \, P_i \, \mu(i)$. But we also have $\mu' \, \bar{P}_{\mu'(i)} \, \mu$, so $\mu'(\mu'(i)) = i \, P_{\mu'(i)} \, \mu(i)$. Thus, $(i, \mu'(i))$ blocks μ under R, a contradiction to $\mu \in F(\bar{R})$. So, for all $i \in M$ we have $h_i(\mu, \bar{R}) \subset L_i(\mu, \bar{R})$ for any $\bar{R}$ and any $\mu \in F(\bar{R})$.
Hence, we proved our claim that for all $i \in N$, for any $\bar{R}$ and for any $\mu \in F(\bar{R})$ we have $h_i(\mu, \bar{R}) \subset L_i(\mu, \bar{R})$.

□

Proof. We will show that F is h-monotonic and that there is no other monotonicity-potential h' with $h' \subsetneq h$ such that F is h'-monotonic.

- F is h-monotonic:

Take some $\bar{R}, \bar{R}'$ on $\mathcal{M}$ and let R, R' be the linear order profiles inducing $\bar{R}, \bar{R}'$, respectively. Take some $\mu \in \mathcal{M}$ with $\mu \in F(\bar{R})$ and $h_i(\mu, \bar{R}) \subset L_i(\mu, \bar{R}')$ for all $i \in N$. We will show that $\mu \in F(\bar{R}')$.

We will first show that μ is a stable matching under R'.

(i) μ is IR with respect to R'.

This is trivially satisfied due to our choice of preference profiles.

(ii) There is no pair (m, w) blocking μ under R'.

Suppose that there is $(m, w) \in M \times W$ with $mw \in E(G) \setminus E(\mu)$, which blocks μ under R'. So we have $m \; P'_w \; \mu(w)$ and $w \; P'_m \; \mu(m)$. Since we have $h_w(\mu, \bar{R}) = L_w(\mu, \bar{R}) \subset L_w(\mu, \bar{R}')$, any $\mu' \in \mathcal{M}$ with $mw \in E(\mu')$ does not belong to $L_w(\mu, \bar{R}')$ and consequently $\mu' \notin L_w(\mu, \bar{R})$ either. Then we have $\mu' \; \bar{P}_w \; \mu$ and equivalently $\mu'(w) = m \; P_w \; \mu(w)$, so $\mu' \in h_m(\mu, \bar{R})$. Since we took $\bar{R}, \bar{R}'$ and μ such that $h_m(\mu, \bar{R}) \subset L_m(\mu, \bar{R}')$, for the matching μ' with $mw \in E(\mu')$ we have $\mu \; \bar{R}'_m \; \mu'$, so $\mu' \; \not{\bar{P}}'_m \; \mu$ and equivalently $w \; \not{P}'_m \; \mu(m)$, a contradiction. Thus, there is no pair blocking μ under R'.

- There is no other monotonicity-potential h' with $h' \subsetneqq h$ such that F is $h' - monotonic$.

Let h' be a monotonicity-potential with $h' \subsetneqq h$. Suppose that F is $h' - monotonic$. That is, for any $\bar{R}, \bar{R}'$ on $\mathcal{M}$ and $\mu \in \mathcal{M}$ with $\mu \in F(\bar{R})$, if we have $h'_i(\mu, \bar{R}) \subset L_i(\mu, \bar{R}')$ for all $i \in N$, then we also have $\mu \in F(\bar{R}')$.
Due to our construction of h, we should consider the implications of $h' \subsetneqq h$ in two cases.

Case 1: There exist some man $m \in M$ and some $(\mu, \bar{R}) \in GrF$ such that there is some other matching μ' in G with $\mu' \in h_m(\mu, \bar{R}) \setminus h'_m(\mu, \bar{R})$.
Now we pick some other profile $\bar{R}'$ such that $L_i(\mu, \bar{R}) \cap h'_i(\mu, \bar{R}) \subset L_i(\mu, \bar{R}')$ is satisfied for all $i \in N$, $\mu' \notin L_m(\mu, \bar{R}')$ and $\mu' \notin L_{\mu'(m)}(\mu, \bar{R})$. Set $\mu'(m) = w$. Since μ' does not belong to $h'_m(\mu, \bar{R})$, it does not have to be in the lower contour of μ in a new profile and we took $\bar{R}'$ such that $\mu' \notin L_m(\mu, \bar{R}')$. On the other hand, we took $\mu' \in h_m(\mu, \bar{R})$ and it means that we have $\mu' \; \bar{P}_w \; \mu$, equivalently $\mu'(w) = m \; P_w \; \mu(w)$. We also took $h' \subsetneqq h$, so we have

$h'_w(\mu, \bar{R}) \subset h_w(\mu, \bar{R}) = L_w(\mu, \bar{R})$. Now we know $\mu' \notin L_w(\mu, \bar{R})$ and it cannot belong to $h'_w(\mu, \bar{R})$ because of the relation between the two sets shown above. It implies that w can rank μ' above μ in a new profile which satisfies our hypothesis and we picked $\bar{R}'$ such that $\mu' \notin L_w(\mu, \bar{R}')$. Hence, the hypothesis of $h' - monotonicity$ is satisfied for $\mu, \bar{R}$ and $\bar{R}'$, however, we have $m \in M$ and $w \in W$ with $\mu' \notin L_m(\mu, \bar{R}')$ and $\mu' \notin L_w(\mu, \bar{R}')$. Thus, (m, w) blocks μ under R' and $\mu \notin F(\bar{R}')$.

Case 2: There is some woman $w \in W$ and $\exists(\mu, \bar{R}) \in GrF$ such that $\exists \mu' \in \mathcal{M}$ with $\mu' \in h_m(\mu, \bar{R}) \setminus h'_m(\mu, \bar{R})$. Since $h_w(\mu, \bar{R}) = L_i(\mu, \bar{R})$, we also have $\mu' \notin L_w(\mu, \bar{R}) \cap h'_w(\mu, \bar{R})$.

Consider a complete preorder profile $\bar{R}'$ such that $L_w(\mu, \bar{R}) \cap h'_w(\mu, \bar{R}) \subset L_w(\mu, \bar{R}')$ is satisfied, $\mu' \notin L_w(\mu, \bar{R}')$ and $\mu' \notin L_{\mu'(w)}(\mu, \bar{R}')$. It is clear from above that μ' can be ranked above μ by w in the new profile. Also notice that $\mu' \in L_w(\mu, \bar{R})$ implies that $\mu \, \bar{R}_w \, \mu'$ and hence $\mu' \notin (h_{\mu'(w)})(\mu, \bar{R})$. That is, μ' can be ranked above μ by $\mu'(w)$ in the new profile. Thus, the changes in the positions of μ' by w and $\mu'(w)$ under $\bar{R}'$ do not contradict with the requirement of $h' - monotonicity$. However, $\mu \notin F(\bar{R}')$ because under $\bar{R}'$ we have $\mu' \, \bar{P}_w \, \mu$ and $\mu' \, \bar{P}_{\mu(w)} \, \mu$ implying that $(\mu'(w), w)$ blocks μ under $\bar{R}'$.

Therefore, F is not $h' - monotonic$.

$\square$

In this study, we, unfortunately, cannot trace the connection between $h-$ and self-monotonicites of the stable rule and its σ-implementability for different solution concepts σ suggested by Koray (2002) any further and leave it as an open question yet to be worked on.

CHAPTER 4

THE MEN-OPTIMAL RULE

In this chapter, we explore the monotonicity structure of the men-optimal rule F_M and we define a sequential mechanism which implements F_M in subgame perfect Nash equilibrium.

Definition 13. Let G be an (M, W)-bipartite graph and R a linear order profile for G. We say that a stable matching μ wrt R is *men-optimal* if, for any stable matching μ' in G wrt R, one has $\forall m \in M : \mu \, \bar{R}_m \, \mu'$.

Assuming that the preference profile R is well-understood, we denote the men-optimal stable matching by μ_M. The women-optimal stable matching is defined similarly and denoted as μ_W.

Definition 14. The men-optimal rule F_M is the social choice rule that assigns the men-optimal matching to each preference profile, i.e.$F_M(R) = \mu_M(R)$ for each $R \in \mathcal{L}(A)^N$.

4.1 Monotonicity of the Men-Optimal Rule

Kara and Sönmez (1996) argue in a corollary to their main result, that the men-optimal rule does not satisfy Maskin-monotonicity. Here we present an example to demonstrate why.

Example 1. The following preference profile for $|M| = |W| = 3$ is an example, where there is a matching μ that weakly Pareto-dominates μ_M for men where $E(\mu_M)(R) = \{m_1w_1, m_2w_3, m_3w_2\}$ and $E(\mu) = \{m_1w_1, m_2w_2, m_3w_3\}$. This example sheds some light on why the men-optimal rule does not satisfy Maskin-monotonicity and it is suggestive concerning the monotonicty structure of the men-optimal rule.

R_{m_1}	R_{m_2}	R_{m_3}	R_{w_1}	R_{w_2}	R_{w_3}
w_3	w_2	w_3	m_2	$\boxed{m_3}$	$\boxed{m_2}$
w_2	$\boxed{w_3}$	$\boxed{w_2}$	$\boxed{m_1}$	m_1	m_3
$\boxed{w_1}$	w_1	w_1	m_3	m_2	m_1
m_1	m_2	m_3	w_1	w_2	w_3

Now consider a new preference profile R' where $R'_i = R_i$ for all $i \in N \setminus \{m_1\}$ and R'_{m_1} is such that w_1 is ranked first and nothing else is changed. It is straightforward that R and R' satisfy the hypothesis of Maskin-monotonicity. However, even though m_1's mate does not change under R', he alters the rest of the matching by changing the position of his men-optimal mate and now $\mu_M(R') = \mu \neq \mu_M(R)$.

R'_{m_1}	R'_{m_2}	R'_{m_3}	R'_{w_1}	R'_{w_2}	R'_{w_3}
$\boxed{w_1}$	$\boxed{w_2}$	$\boxed{w_3}$	m_2	m_3	m_2
w_2	w_3	w_2	$\boxed{m_1}$	m_1	$\boxed{m_3}$
w_3	w_1	w_1	m_3	$\boxed{m_2}$	m_1
m_1	m_2	m_3	w_1	w_2	w_3

This example gives us an idea about the preference profiles under which men-optimal rule fails to satisfy Maskin-monotonicity. We propose an h-monotonicity of the men-optimal rule that uses the insight from the above example. Since the rule does not satisfy Maskin-monotonicity, we know that its self-monotonicity can be a superset of lower contour set for each individual.

Proposition 2. *Let F_M be the M-optimal stable matching rule. For any $(\mu, \bar{R}) \in GrF_M$, define h for each $m \in M$ by $h_m(\mu, \bar{R}) = L_m(\mu, \bar{R}) \cup \{\mu' \in \mathcal{M} \mid \mu'$ weakly M-Pareto dominates μ w.r.t. $R\}$ and for each $w \in W$ by $h_w(\mu, \bar{R}) = L_w(\mu, \bar{R})$. Now F_M is h-monotonic.*

Proof. Take any $(\mu, \bar{R}) \in GrF_M$ and some $\bar{R}'$ with $h_i(\mu, \bar{R}) \subset L_i(\mu, \bar{R}')$ for all $i \in N$. Suppose that $\mu \neq F_M(\bar{R}')$. Since we know that $F_M(\bar{R}') \neq \emptyset$, there must exist a matching $\bar{\mu} \in \mathcal{SM}$ such that $\bar{\mu} = F_M(\bar{R}')$. By the definition of M-optimal matching, $\mu' \; \bar{R}' \; \mu$ for all $m \in M$. Since $\mu \neq F_M(\bar{R}') = \bar{\mu}$, for some $m' \in M$, $\mu(m') \neq \bar{\mu}(m')$ and it implies that, for that man $m' \in M$ we have $\bar{\mu'} \; \bar{P'}_{m'} \; \mu$. Now since we have

$\forall m \in M : \bar{\mu} \; \bar{R'}_m \; \mu$ and
$\exists m' \in M : \bar{\mu} \; \bar{P'}_{m'} \; \mu$,

$\bar{\mu}$ weakly M-Pareto dominates μ w.r.t. R' and clearly $\bar{\mu} \notin L'_m(\mu, \bar{R}')$. So, $\bar{\mu}$ cannot belong to $L'_m(\mu, \bar{R})$ either and we can conclude that $\bar{\mu}$ M-Pareto dominates μ w.r.t. R as well. Hence, $\bar{\mu} \in h'_m(\mu, \bar{R})$ and due to our choice of $\bar{R}'$, $\bar{\mu} \in L'_m(\mu, \bar{R}')$, a contradiction. So, F_M is h-monotonic.

$\square$

Intuitively, h does not seem to be sufficiently tight to be a self-monotonicity of F_M. Finding the self-monotonicites of F_M stays as an open problem for further research.

4.2 A Matching Mechanism and Implementation of F_M

We now introduce our sequential matching mechanism. Throughout this study, we have an equal number of men and women, and our mechanism works for such societies.

We start with an arbitrary hierarchy order on the set of women and women take turns to make proposals to men, according to this order. A woman whose turn comes proposes to a man from the set of unmatched men and the man who receives her proposal either accepts or rejects it. If she is rejected, then she proposes to another man who is available and she continues to make proposals until some man accepts her or the set of available men gets exhausted. If she is rejected by every man she proposes to, then she is self-matched. In case of acceptance by a man, they get matched irreversibly. In either case, the next woman on the hierarchy order takes her turn. A woman can propose to the same man only once. This procedure ends up with a matching.

The following condition is introduced by Eeckhout (2000), and he shows that it is a sufficient condition for the existence of a unique stable matching.

Condition 1. A linear order profile R is said to satisfy the Eeckhout

Condition if it is possible to rename the agents as $M = \{m_1, \ldots, m_n\}$, $W = \{w_1, \ldots, w_n\}$ such that

(i) $w_i \, P_{m_i} \, w_j$ for all $j > i$, and

(ii) $m_i \, P_{w_i} \, m_j$ for all $j > i$.

Under this condition, the unique stable matching is μ where $\mu(m_i) = w_i$ for all $i \in \{1, 2, \ldots, n\}$.

Suh and Wen (2008) show that under the Eeckhout condition, their mechanism implements the men-optimal rule. It is easy to see that the SPE outcome of our mechanism is the men-optimal stable matching under the Eeckhout condition, i.e., under the Eeckhout condition, our mechanism implements the men-optimal rule, in SPE.

It is straightforward to see this result but we sketch the proof in order to be illustrative. Under any preference profile R that satisfies the Eeckhout condition and any hierarchy order among women, m_1 likes w_1 the most and w_1 top-ranks m_1. Regardless of w_1's position in the hierarchy order, she will propose to m_1 when he is available and m_1 will reject every proposal he receives before w_i and accept her proposal in the SPE. Knowing m_1 and w_1's equilibrium strategies, w_2 will propose to m_2 when it is her turn, and m_2 will wait until w_2 (by rejecting other women) and accept w_2's proposal. Applying this reasoning to the rest of men and women, we see that the SPE outcome of the mechanism is the men-optimal matching μ_M.

In their study, Suh and Wen (2008) provides a less restrictive condition called α^M Condition, under which the men-optimal stable matching is Pareto-optimal for men. Same as Eeckhout (2000), under α^M condition, men-optimal stable matching is μ where $\mu(m_i) = w_i$ for all $i \in \{1, 2, \ldots, n\}$. It is easily seen that Eeckhout condition implies α^M condition but the reverse is not true. They define α^W condition similarly and α^M together with α^W imply Eeckhout condition.

Condition 2. A linear order profile R is said to satisfy the α^M Condition if it is possible to rename the agents as $M = \{m_1, \ldots, m_n\}$, $W = \{w_1, \ldots, w_n\}$ such that

(i) for any $m_i \in M$, $w_i \, P_{m_i} \, w_j$ for all $j > i$.

(ii) for any $m_i \in M$, if we have $w_l \, P_{m_k} \, w_k$ for some $l < k$, then $m_l \, P_{w_l} \, m_k$.

Claim. *Under our matching mechanism, there is no man or woman who stays single in the equilibrium, i.e. $\nexists i \in N$ such that $\mu(i) = i$ where μ is the SPE outcome of the mechanism.*

Proof. Suppose there are some $i \in N$ such that $\mu(i) = i$ in the equilibrium. Since the number of men and women are equal, number of men and women who stay single is also the same. Take some $m \in M$ who stays single. Consider the decision node where m receives the last proposal that he receives in the equilibrium path. If m chooses to Accept, then he is matched with some woman $w \in W$ where $w \, P_m \, m$. Hence, rejecting the last proposal m receives in the equilibrium path cannot be subgame perfect and it contradicts with μ being the SPE outcome of the mechanism. □

Theorem 1. *Under α^M Condition, the unique Subgame Perfect Equilibrium outcome of our matching mechanism is the men-optimal matching μ_M.*

Proof. We will prove by induction on the number of men and women.

<u>Base case:</u> $n = 1$.

We have only one possible preference profile which is $w_1 \, P_{m_1} \, m_1$ and $m_1 \, P_{w_1} \, w_1$ and it satisfies α^M trivially. The men-optimal stable matching under this profile is $\mu_M(m_1) = w_1$ and the hierarchy order is also trivial where w_1 moves first. So,

w_1 proposes to m_1 and since m_1 prefers being matched with w_1 to staying single, he accepts. Hence, subgame perfect equilibrium outcome is μ_M.

Induction hypothesis: When we have $|M| = |W| = n$, SPE outcome of the mechanism is μ_M under any preference profile satisfying α^M and any hierarchy order.

Now assume we have $|M| = |W| = n + 1$. If the first woman in the hierarchy order is matched with her mate under μ_M, then the following subgame starting with the second woman is basically a new game with n men and n women whose preferences satisfy α^M, thus SPE outcome will be μ_M by the induction hypothesis.

Let w_i be the first woman in the hierarchy order. We first show that m_i will accept w_i's proposal if w_i proposes to m_i. Suppose to the contrary that w_i's proposal gets rejected by m_i. The subgame starting with m_i's rejection leads to a matching μ as a subgame perfect equilibrium outcome. Then it must be the case that $\mu(m_i)\ P_{m_i}\ w_i$. We now claim that also for any $m_j \in M \setminus \{m_i\}$, $\mu(m_j)\ R_{m_j}\ w_j$.

Suppose that there is some $m_k \in M \setminus \{m_i\}$ such that $w_k\ P_{m_k}\ \mu(m_k)$. This means that w_k does not propose to m_k when it is her turn, for otherwise m_k would accept the proposal and get matched with w_k rather than $\mu(m_k)$. In case m_k is still unmatched at w_k's proposal stage, w_k proposes to another $m_l \in M \setminus \{m_i\}$ and gets matched with him under μ, where $m_l\ P_{w_k}\ m_k$. If, on the other hand, m_k gets matched with $\mu(m_k)$ at some earlier stage and is thus not available to w_k at her proposal stage, it is precisely because he foresees that he will not receive a proposal from w_k even if he keeps himself available to w_k by rejecting all proposals before w_k's proposal stage. Hence, in this case as well, $m_l = \mu(w_k)$ will be such that $m_l\ P_{w_k}\ m_k$. But then by α^M, $w_l\ P_{m_l}\ w_k = \mu(m_l)$.

Now, however, m_l is in the same position as m_k in the sense that he prefers his mate w_l under μ_M to his mate w_k under μ. Repeating the above argument by substituting m_l for m_k, we obtain yet another man m_s who prefers μ_M to μ. As this procedure can be repeated indefinitely, our supposition requires infinitely many men, contradicting that M is finite. Therefore, we conclude that, for all $m \in M$, $\mu(m)\, R_m\, \mu_M(m)$. As $\mu(m_i)\, P_{m_i}\, w_i = \mu_M(m_i)$, this means that μ Pareto-dominates μ_M for men. As we know from Suh and Wen (2008) that α^M is sufficient for the Pareto optimality of μ_M for men, this yields the desired contradiction. So, if w_i proposes to m_i, then m_i accepts the proposal.

Now suppose that w_i proposes to some other men before proposing to m_i. Consider the such last man and call him m_j. We have two possible cases.

(i) $m_i\, P_{w_i}\, m_j$: In this case, if m_j rejects w_i's proposal, then w_i will propose to m_i and m_i will accept her proposal. If m_j accepts w_i's offer, then w_i is matched with someone who is less preferable for her where she could be matched with m_i is she proposes to him before proposing to m_j. In this case, w_i's action is not subgame perfect. So, w_i does not propose to some other man who is less preferable than m_i before proposing to m_i and get matched.

(ii) $m_j\, P_{w_i}\, m_i$: By α^M, we have $w_j\, P_{m_j}\, w_i$. If m_j accepts w_i's proposal, he is matched with w_i and if he rejects it, then w_i will propose to m_i and get accepted which results in μ_M as the SPE outcome where m_j is matched with w_j. Thus, m_j rejects w_i's offer. Now consider the previous man who receives w_i's proposal, say m_k. Again we have $w_k\, P_{m_k}\, w_i$ by α^M. When m_k receives w_i's proposal, knowing that m_j will reject her and m_i will accept her, m_k also rejects w_i and gets matched with w_k. The same reasoning applies to every man w_i finds more preferable than m_i and proposes to before proposing to m_i.

Hence, w_i proposes to and gets accepted by m_i in the subgame perfect equilibrium. In the following subgame, every woman is matched with her mate under μ_M by the induction hypothesis and thus, subgame perfect equilibrium outcome is μ_M.

$\square$

CHAPTER 5

CONCLUSION

The main result of this study is the construction of a new extensive form mechanism that implements the men-optimal stable rule in SPE under condition α^M introduced by Suh and Wen (2008), who also proved the subgame perfect implementability of the same rule under α^M via a different mechanism. It is, in general, not easy to make a simplicity comparison between two different mechanisms that do the same job. In our context, it might be worth, however, to note that, under the Eeckhout (2000) condition, the fact that our mechanism subgame perfect implements the men-optimal stable matching follows immediately.

We know from Suh and Wen (2008) that a preference profile satisfies α^M if and only if the men-optimal rule is Pareto optimal for men under that profile. In other words, the α^M condition rules out the possibility that some men may improve upon the men-optimal matching without hurting other men. In our case as well as in Suh and Wen (2008), it is this fact, which makes the mechanism do the desired trick. The question of whether the domain of preference profiles allowing the subgame perfect implementability of the men-optimal rule can be further expanded, however, remains still open.

We then turned to the notions of h-monotonicity and self-monotonicty of an SCR introduced by Koray (2002) to deal with that problem. Roughly speaking, an h-monotonicity is a generalized Maskin-monotonicity, and a self-monotonicity is a strongest h-monotonicity of an SCR. It is known that the h-monotonicities of a game-theoretic solution concept σ are inherited by the σ-implementable SCRs via the game form employed.

We also already know from Kara and Sönmez (1996) that the stable rule is Danilov-monotonic and thus Nash implementable, while the men-optimal rule is not even Maskin-monotonic. This gives rise to two types of open research questions. One is concerned with finding the set of game-theoretic solution concepts according to which the stable rule is implementable in addition to the Nash equilibrium notion. In an attempt to deal with that problem, we find a self-monotonicity of the stable rule.

The other problem concerns the implementability of the men-optimal rule, which seems to be tougher as the men optimal rule is "less monotonic" than the stable rule itself. We proceed in this case in a similar manner and find an h-monotonicity of this refinement of the stable rule. The gap between generalized monotonicities and implementability of the refinements of the stable rule in different equilibrium notions remains, however, yet to be filled in.

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