REPRESENTATIONS OF SYMMETRIC GROUPS AND STRUCTURES OF LIE ALGEBRA

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Representations of Symmetric Groups and Structures of Lie Algebra
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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

REPRESENTATIONS OF SYMMETRIC GROUPS AND STRUCTURES OF LIE ALGEBRA

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The aim of this thesis construct structure of Free Lie Algebra $L(V)$ generated by finite dimensional vector space V and decompose into irreducible components of a given degree n. To splits into irreducible component, representation of $\mathrm{GL}(V)$ is main tool. However, representation of symmetric groups is used to split since representations of $\mathrm{GL}(V)$ and representations of symmetric group have duality, called Schur duality. After decomposing, Kraśkiewicz-Weyman theory and formula using character theory are used to determine the multiplicity of irreducible component.

Keywords: Free Lie Algebras, Representation of $\mathrm{GL}(V)$,Symmetric Groups.

ÖZET

SİMETRİK GRUPLARIN TEMSİLLERİ VE LİE CEBİR YAPILARI

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Bu tezin amacı sonlu boyutlu vektör uzayı V tarafndan üretilen serbest Lie cebir yaplarını oluşturmak ve verilen derece n'e göre indirgenemez bileşenlerine ayırmaktır. İndirgenemez bileşenlerine ayırmak için $\mathrm{GL}(V)$ temsilleri ana araçtır. Ancak, $\mathrm{GL}(V)$ ve simetrik grupların temsilleri dualite, Schur dualitesi olarak adlandırılır, gösterdiginden simetrik grupların temsilleri kullanılmıştır. İndirgenemez bileşenlerine ayırdıktan sonra, Kraśkiewicz-Weyman teorisini ve karakter teoriyi kullanan formül ile indirgenemez bileşenlerin çarpanlarına karar verilir.

Anahtar sözcükler: Serbest Lie Cebirleri, $\mathrm{GL}(V)$ Temsilleri, Simetrik Gruplar.

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I would like to dedicate this thesis to my parents Fatma Acar and Hacı Emin Acar who have always been my nearest.

Contents

Chapter 1

Introduction

The thesis focused on structure of Free Lie Algebra $L(V)$ generated by finite dimensional vector space V. The algebra splits into components of degree $n = 1, 2, \ldots$.

$$L(V) = \oplus_{n=1,2,\ldots} L_n(V)$$

where $L_n(V)$ homogeneous components of degree n. For example,

$$L_1(V) = V \quad L_2(V) = \Lambda^2(V) \tag{1.1}$$

. Clearly, 1.1 are irreducible representations of GL(V). For a small degree n, $L_n(V)$ are irreducible and Lie elements of degree n form a finite dimensional representation of GL(V). Our primary aim is to find a decomposition of $L_n(V)$ into irreducible components of degree n.

The problem we dealt with as mentioned above is to find irreducible components and calculating multiplicities of the components of Free Lie Algebra $L_n(V)$. The problem is actually an old one. Many people have attended to this problem. Indeed, calculating multiplicities of the irreducible component of Free Lie Algebra $L_n(V)$ is really difficult without computer. When analyzing the paper [10], we found many wrong calculations for Free Lie Algebra of degree 10.

When trying to decompose of Free Lie Algebra $L_n(V)$, we use representations of symmetric groups. While studying representations of the symmetric Groups, Young diagram is the main tool. To calculate the multiplicity, Kraśkiewicz-Weyman theory is applicable. However, for higher degrees, the application of this theory is not possible by hand. If this theory can be embedded into a computer programme, then it can be useful for higher degrees. In general, 4.1.4 is used for calculating multiplicity using Maple 18. Furthermore, after calculating the multiplicity, when analyzing [6], we understood that the multiplicity of irreducible components of Free Lie Algebra is related to double cosets and then using GAP , we started calculating non-empty double cosets. Moreover, for the prime degree, we develop a new formula which only depends on our calculations. Unfortunately, we only have formula for the prime degree. The formula for even degree and odd degree can be considered in future studies.

Chapter 2

Summary of Representation Theory of Finite Groups

2.1 Preliminaries

In this chapter, main theorems and definitions used in the study are presented. These theorems and definitions are mostly found in [4]. Moreover, only complex representations are considered, that is the ground field is $\mathbb{C}$.
Let G be a finite group and V be a finite dimensional vector space. It is said to;

$$\rho : G \mapsto GL_n(\mathbb{C})$$

be *representation* of the group G over the finite dimensional complex vector space V. Instead of using ρ, V is called representation of group G and it is denoted by $G : V$.
A representation V is called *irreducible representation* if it has no proper nonzero invariant subspace apart from $\{0\}$ and itself, W of V.
A FG-module V is called *completely reducible* if $V = U_1 \oplus U_2 \oplus \ldots U_r$ where each U_i is irreducible.

Theorem 2.1.1. *If G be a finite group and V be a finite dimensional complex vector space, then every $\mathbb{C}G$-module is completely reducible.*

Moreover, the following theorem is useful to find the dimension of irreducible components.

Theorem 2.1.2. *Let $U_1, U_2, \ldots, U_k$ be the all non-isomorphic irreducible $\mathbb{C}G$-modules. Then,*

$$\sum_{i=1}^{k} (\dim U_i)^2 = |G| \tag{2.1}$$

Especially, every finite abelian group has the 1-dimensional irreducible $\mathbb{C}G$-module. Another important issue in representation theory is characters. A *character* of the representation $G : V$ is a function such that $\chi_V : G \to \mathbb{C}$ defined by $\chi_V(g) = Tr(g_V)$, the trace of g on V. Moreover, this function is complex-valued function. The followings are the elementary properties of character function:

(1) $\chi_V(e) = \dim(V)$

(2) $\chi_V(g) = \overline{\chi_V(g^{-1})}$

(3) $\chi_V \otimes \chi_W(g) = \chi_V(g)\chi_W(g)$,

(4) $\chi_V \oplus \chi_W(g) = \chi_V(g) + \chi_W(g)$.

(5) Characters are the same value on the conjugacy classes.

(6) The isomorphic $\mathbb{C}G$-modules V and W have the same character.

Theorem 2.1.3. *The number of the conjugacy classes of the group G is equal to the number of the irreducible characters of the group G.*

Let φ and ϕ be the functions of the group G to $\mathbb{C}$. Then, the inner product is defined

$$\langle \varphi, \phi \rangle = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \overline{\phi(g)}$$

The inner product can be defined over the characters. Then, character functions has the following orthogonality relations:

Theorem 2.1.4. *Let G be a finite group and $V_1, V_2, \ldots, V_k$ be the complete set of the finite dimensional complex representations and χ_i is the corresponding character of V_i. Let $g_1, g_2, \ldots, g_k$ be the representative of the conjugacy classes of the group G and $C_G(g_i)$ be the centralizer of the element g_i in the group G. Then,*

(i) $\langle \chi_r, \chi_s \rangle = \delta_{rs}$

(ii) The row orthogonality relation is

$$\sum_{i=1}^{k} \frac{\chi_r(g_i)\overline{\chi_s(g_i)}}{|C_G(g_i)|} = \delta_{rs} \quad \textit{for all } r,s$$

(iii) The column orthogonality relation is

$$\sum_{i=1}^{k} \chi_r(g_i)\overline{\chi_s(g_i)} = \delta_{rs}|C_G(g_i)| \quad \textit{for all } r,s$$

From (i), it can be said that the irreducible characters $\chi_1, \chi_2, \ldots, \chi_k$ form an orthonormal set.
Let G be a finite group and the vector space $\mathbb{C}G$-module with the natural multiplication, that is $vg \in \mathbb{C}G$ $v \in \mathbb{C}G$ and $g \in G$, is called *regular $\mathbb{C}G$-module* and the dimension of the regular $\mathbb{C}G$-module is $|G|$. The character values of the regular representation is

$$\chi_{reg}(1) = |G|$$

and

$$\chi_{reg}(g) = 0 \quad g \neq 1$$

2.2 Induced Representations

Let H be a subgroup of the finite group G. In the process of induction, aim is to construct $\mathbb{C}G$-representation using $\mathbb{C}H$-representation.

Definition 2.2.1. *Suppose that H is a subgroup of the finite group G and U be the submodule of the $\mathbb{C}H$-module. Then, $U \uparrow G$ denote the $\mathbb{C}G$-representation and it is called $\mathbb{C}G$-representation induced from U.*

Let's try to explain the induced representation with an example.

Example 2.2.2.

$$G = D_6 = \langle a, b | a^3 = b^2 = 1, b^{-1}ab = a^2 \rangle$$

and

$$H = \langle a | a^3 = 1 \rangle$$

Clearly, H is a subgroup of the group G. Then, $\mathbb{C}H$-representation is constructed. Since H is a cyclic group of order 3, $\omega = e^{\frac{2\pi i}{3}}$ and the $\mathbb{C}H$-representation are the following:

$$U_1 = span\{1 + a + a^2\}$$

$$U_2 = span\{1 + \omega^2 a + \omega a^2\}$$

$$U_3 = span\{1 + \omega a^2 + \omega a\}$$

Then, the $\mathbb{C}G$-representation is in the following form:

$$U_1 \uparrow G = span\{1 + a + a^2, b + ba + ba^2\}$$

$$U_2 \uparrow G = span\{1 + \omega^2 a + \omega a^2, b + \omega^2 ab + \omega a^2 b\}$$

$$U_3 \uparrow G = span\{1 + \omega a^2 + \omega a, b + \omega a^2 b + \omega ab\}$$

In below example is related to find irreducible representations and their dimensions.

Example 2.2.3.

$$Let \;\; G = D_6 = \langle a, b | a^3 = b^2 = 1, b^{-1}ab = a^2 \rangle$$

Firstly, conjugacy classes should be decided to find how many irreducible components the group G has. The conjugacy classes are $\{1\}, \{a, a^2\}, \{b, ab, a^2b\}$. So, the group G has 3 irreducible components in the representation. Then, the dimensions of the irreducible components should be decided. One of the irreducible component should be 1-dimensional, say U_1, because of trivial representation. Then, using the equation 2.1, it can be concluded that the other representations should be 1-dimensional, say U_2, and 2-dimensional U_3. Hence ,

$$D_6 \cong U_1 \oplus U_2 \oplus U_3$$

Chapter 3

Young Diagram and Representations of Symmetric Groups S_n

3.1 Young Diagram and Irreducible Representations of S_n

This section mainly depends on [3]. The *symmetric group* S_n is the group of all permutations of n numbers. The degree of S_n is n and the order of S_n is n!.

Definition 3.1.1. $\lambda = (\lambda_1, \lambda_2, \ldots)$ *is called partition on n if λ_i's are the non-negative integers with $\lambda_1 \geq \lambda_2 \geq \ldots$ and*

$$\sum_{i=1} \lambda_i = n$$

The number of the partitions of n is equal to the number of the conjugacy classes of S_n. Therefore, the number of the partitions of n is equal to the number of the irreducible representations.

Definition 3.1.2. *Let λ be a partition of n, denoted by $\lambda \mapsto n$, the diagram $[\lambda]$ is*

$\{(i,j) : i,j \in \mathbb{Z}, 1 \leq i, 1 \leq j \leq \lambda_i\}$. *Moreover, a λ-tableau is form of the diagram with filling the numbers $\{1,2,\ldots,n\}$ where the entries weakly increase along each row and strictly increase each column.*

In below example, diagram and tableau are shown to understand the difference between them.

Example 3.1.3. *Let $\lambda = (3,2,1)$ be partition of 6. Then,* $\begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2} \\ \cline{1-1}\end{array}$ *is the diagram. If the diagram is filled with numbers {1,2,3,4,5,6} with no repeats and obeying the putting the numbers, it is called young tableaux such as* $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$ *,* $\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$

After giving young diagram definition, the representation of S_n can be analyzed. Up to this point, it is only known that the number of the irreducible representation is equal to the number of the partition. Additionally, the multiplicity of the irreducible ones is also the crucial point. At this point, using young diagrams provide the simple solution for finding multiplicity. The multiplicity of the irreducible representation is exactly equal to the number of young diagram that can be drawn corresponding λ. Let's analyze this phenomenon with an example.

Example 3.1.4. *In this example which is taken from [5], try to write S_4 representation.* $[4], [3,1], [2^2], [2,1^2], [1^4]$ *are the partition of S_4. Then,*

(1) There is only one young tableau corresponding to $[4]$*, that is* $\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline\end{array}$

(2) $[3,1]$ *has 3 different young tableaux and they are* $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$ *,* $\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}$ *,* $\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$.

(3) There are 2 young tableaux for $[2^2]$ $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$ *,* $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$.

(4) Like $[3,1]$*,* $[2,1^2]$ *has 3 young tableaux and they are* $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$ *,* $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$ *,* $\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$.

(5) $[1^4]$ *has only one young tableaux and it is* $\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array}$.

Hence,

$$S_4 \cong [4] + 3[3,1] + 2[2^2] + 3[2,1^2] + [1^4]$$

3.2 Models of Symmetric Groups

This section mainly depends on [8]. K-linear representation which each irreducible representation appears with multiplicity 1 is called *model of the group G over K* . In this chapter, it is mentioned about the model of S_n over $\mathbb{C}$.
Let σ_k be involution, i.e. ${\sigma_k}^2 = 1$ with k transpositions and C_{σ_k} be the centralizer of σ_k in S_n. Now, it should be set

$$I_\sigma{}^+ = \{i : \sigma(i) = i\}, I_\sigma{}^- = \{i : \sigma(i) \neq i\}$$

where σ is an involution. Then, for each element δ in centralizer can be written as the product of two elements in $I_\sigma{}^+$ and $I_\sigma{}^-$, that is $\delta = \delta^+\delta^-$ where δ^+ in $I_\sigma{}^+$, δ^- in $I_\sigma{}^-$. Actually , this determines the group of the centralizer of σ_k in S_n. Moreover, let $t_k = t_{\sigma_k}$ 1-dimensional representation of C_{σ_k} and $t_k(\delta)$ takes the sign value of δ^- and $T_{\sigma_k} = T_k$ is the corresponding induced representation of S_n.

Theorem 3.2.1. *(1) Representations T_k have multiplicity one for k=0,1,$\ldots$,$\frac{n}{2}$*

(2) Each irreducible representation of symmetric group is contained in some T_k.

(3) T_i and T_j with $i \neq j$, do not have common component.

Hence, it can be said that $\sum T_k$ contains each irreducible representation of S_n. The model of the S_n is used to calculate the intertwining number. In this project, the intertwining number of the two induced representation is provide to develop new formula. Hence, it will be used the induced representation of t_k is denoted by T_k.

3.3 Representations of GL(V)

This section mainly depends on [2]. The irreducible polynomial representations of $GL_m(V)$ where V is a m-dimensional complex space is studied in this section. The theorems and definitions mentioned below depend on [2].
Let R be a commutative ring and E be a R-module. Then, for each partition λ, an R-module can be formed which is denoted by E^λ. Let R be $\mathbb{C}$ and if $\lambda = (n)$, it will get the $Sym^n(E)$. If $\lambda = (1^n)$, it will get the exterior power $\Lambda^n(E)$. Let E^λ be a finite-dimensional representation of $GL(E)$.

Definition 3.3.1. *A representation V, the complex vector space with finite dimension of $GL(E)$, is called polynomial if the map;*

$$\rho : GL(E) \mapsto GL(V)$$

is given by the polynomials. Like polynomials, if the map is given by rational functions, then it is called rational.

After giving these definitions, the aim is to show that the representations E^λ is the same as the irreducible representations of $GL(E)$, where λ of m have at most m rows. In case of λ having more than m rows, E^λ is 0.
Choose a basis for E, actually the bases elements identify $GL(E)$ and H be a subgroup of diagonal matrices of GL(E), then $x = diag(x_1, ..., x_m)$ in H for the diagonal matrix with these entries.

Definition 3.3.2. *A vector v in a representation is V called weight vector with weight $\alpha = (\alpha_1, ..., \alpha_m)$.$\alpha_i$'s are integers, if $xv = {x_1}^{\alpha_1} .... {x_m}^{\alpha_m}$ for each x in H.*

Hence, $H : V$ acts on by commuting the diagonal matrices. Then, any V can be written in the direct sum of its weight spaces,i.e.

$V = \oplus V_\alpha$ where $V_\alpha = v \in V : xv = (\prod {x_i}^{\alpha_i})v$ for all x $\in H$

Definition 3.3.3. *Let B, Borel group of all upper triangular matrices, be a subgroup of GL(E). A weight vector v in a representation V is called a highest weight vector if $B.v = C^*$*

Theorem 3.3.4. *In case of λ having at most m rows,the representations of E^λ of $GL_m(\mathbb{C})$ is the irreducible representation with highest weight $\lambda = (\lambda_1, ..., \lambda_m)$.*

Chapter 4

Lie Algebras

Definition 4.0.1. *A vector space L over the field K with a bilinear map $L \times L \to L$, denoted by $(x, y) \to [x, y]$ is called Lie Algebra L over K and satisfies the following properties:*

(1) [x,x]=0 for all $x \in L$

(2) [x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0 for all x,y,z $\in L$

4.1 Free Lie Algebra

Let X be a set with elements $X = \{x_1, x_2, \ldots, x_n\}$. Then, L_n is called *Free Lie Algebra*, Lie Algebra over the field K, generated by X without any imposed relations.
An associative with unit algebra $U(L)$ is called *universal enveloping algebra of L* if there is a Lie algebra homomorphism $i : L \to UL$ such that for any Lie algebra homomorphism $f : L \to A$ into a unitary associative algebra A there is a unique factorization

$$L \to U(L) \to_F A$$

where F a unique map such that $f = F \circ i$ and a morphism of unitary algebras. This definition is taken from [9].

4.1.1 Models of GL(V)

This section mainly depends on [8].

Theorem 4.1.1. *Let V be a finite dimensional vector space over the field of characteristic zero K. Let $L_2(V)$ be free nilpotent Lie algebra of class 2 generated by the vector space V. The representation of GL(V) in the $U(L_2(V))$ is isomorphic to the direct sum of all finite irreducible representations of GL(V).*

Proof. $L_2(V) = V \oplus \Lambda^2 V$ where $\Lambda^2 V$ is 2nd exterior power of V. Moreover,

$$U(L_2(V)) \cong \oplus_{2i+j=n} Sym^i \Lambda^2 V \otimes Sym^j V$$

Then, rewrite this sum:

$$Sym^i \Lambda^2 V \otimes Sym^j V = V^{\otimes n} \otimes_{C(\sigma_i)} t_{\sigma_i} = V^{\otimes n} \otimes_{S_n} (KS_n \otimes_{C(\sigma_i)} t_{\sigma_i}) = V^{\otimes n} \otimes_{S_n} T_{\sigma_i}$$

where σ_i is the involution of i-transpotions, $C(\sigma_i)$ is the centralizer of the involution, t_{σ_i} is character of the corresponding T_{σ_i} irreducible representation. Then, using 3.2.1, it can be obtained that

$$\begin{aligned} U(L_2(V)) &\cong \oplus_{n=0}^{\infty} \oplus_{2i+j=n} Sym^i \Lambda^2 V \otimes Sym^j V \\ &= \oplus_{n=0}^{\infty} \oplus_{i=0}^{\frac{n}{2}} V^{\otimes n} \otimes_{S_n} T_{\sigma_i} \\ &= \oplus_{n=0}^{\infty} \oplus_{\chi} V^{\otimes n} \otimes_{S_n} T_{\chi} \end{aligned}$$

where χ is irreducible characters of S_n. Hence, GL(V)-modules $V^{\otimes n} \otimes_{S_n} T_\chi$ are irreducible and they present all finite irreducible GL(V)-modules. □

4.1.2 Transformation of Cyclic Words into Lie Elements

Let G be a finite group and V be $\mathbb{C}G$-module. Then,

$$T(V) = \sum_{n=0}^{\infty} V^{\otimes n}$$

be the tensor algebra of the module V and,

$$L(V) = \sum_{n=0}^{\infty} L_n(V)$$

be the Lie subalgebra generated by the commutators of the elements in V and it is graded by element's degrees. In this chapter, main approacch is writing Lie elements in terms of cyclic words. To do this, define two transformations over $\mathbb{C}$, which are :

$$l_n : C_n \to L_n, \ c_n : L_n \to C_n$$

Now, consider two element c_n and l_n in the group algebra of $\mathbb{C}S_n$:

$$l_n = \frac{1}{n} \sum_{\sigma \in S_n} \varepsilon^{maj\sigma} \sigma, \ c_n = \frac{1}{n} \sum_{k=0}^{n-1} \varepsilon^{-k} \tau^k$$

where ε is a n-th root of unity, τ is a long cycle in S_n and $maj\sigma$ is defined as;

$$maj\sigma = \sum_{\sigma_k \geq \sigma_{k+1}} k \ mod \ n$$

The isomorphism between L_n and C_n can be constructed by using the properties of c_n and l_n as stated below :

Lemma 4.1.2. *c_n and l_n have the following properties:*

(1) $c_n l_n = l_n$

(2) $l_n c_n = c_n$

(3) ${c_n}^2 = c_n$

(4) ${l_n}^2 = l_n$

Hence, it can be said that $V^n c_n = C_n$ or in other words $c_n : V^n$ produces the cyclic words. Moreover, $L_n : V^n$ acts on:

$$(x_1 x_2 \ldots x_n) l_n = \frac{1}{n} \sum_I \varepsilon^{maj\sigma} X_I$$

where $X_I = (x_1 x_2 \ldots x_n)$. Therefore, $V^n l_n = L_n$ with adding phase factor root of unity of degree n.

4.1.3 Spectrum of $L(V)$

This section mainly depends on [5]. Let V be a finite dimensional vector space over $\mathbb{C}$, which is characteristic zero. Also, it can be taken any field with characteristic zero. Let T be the tensor algebra over V, and it is denoted by:

$$T = \oplus_{n \geq 0} T_n$$

T is a GL(V)-module. Hence, each T_n is a GL(V)-submodule. Moreover, T_n is called *nth tensor representation.* In 1903,1923 Schur stated the following theorem:

Theorem 4.1.3. *T_n are reducible GL(V)-submodule and irreducible components has one to one correspondence by partitions of n, which is the dimension of vector space V. So, it is denoted by:*

$$T_n = \oplus t_\lambda [\lambda]$$

where $[\lambda]$ is irreducible GL(V)-submodule related to λ and t_λ represents the multiplicity. T can be considered as Lie algebra by setting:

$$[x, y] = x \otimes y - y \otimes x.$$

So, it can be constructed Lie subalgebra L generated by V such that:

$$L = \oplus L_n \textit{ where } n \geqslant 1 \textit{ and } L_n = T_n \cap L.$$

L_n, $GL(V)$-submodule of T_n, is called *nth Lie representation.* Hence, each L_n can be written by:

$$L_n \cong \oplus I_\lambda [\lambda]$$

where λ is a partition of n.

In brief, T is a GL(V) module and it is written by direct sum of irreducible components of T_n. In 1901, Schur showed that T_n's are reducible. Hence, T_n can be represented by direct sum of irreducible components having one to one correspondence by partition of n, say λ. T can be considered in a different manner.T is Lie algebra and constructed Lie subalgebra L generated by V. L can be written as direct sum of L_n and each L_n is intersection of T_n and L. Since L_n has intersection part of T_n, L_n is also written as direct sum of partitions of n.

Then, L_n and T_n has multiplicities I_λ and t_λ, respectively, I_λ definitely satisfies $0 \leq I_\lambda \leq t_\lambda$. This brings up the question of what is the value of I_λ. Answer is in Theorem 4.1.4

Theorem 4.1.4.

$$I_\lambda = \frac{1}{n} \sum_{d|n} \mu(d) \chi_\lambda(\tau^{n/d})$$

where μ is mobius function, χ_λ is the character of S_n-module related to λ, τ is a long cycle in S_n. Since Wever's publication, it is known which module present in decomposition of L_n, whose multiplicity is greater than 0.

Chapter 5

Multiplicities of the Irreducible Component of L_n

5.1 Kraśkiewicz-Weyman Theorem

This section mainly depends on [7]. In 1974, Klyachko proved that almost every irreducible GL(V)-module appears in Lie representation which is stated in 5.1.1

Theorem 5.1.1. *Let $n \geq 3$ and λ be a partition of n, $\lambda \vdash n$ and there exists an irreducible GL(V)-submodule of L_n with isomorphism type corresponding to λ iff λ has more dim(V) part and $\lambda \neq (1^n), (n), (2^2) or (2^3)$.*

After Klyachko's publication, the theorem has attracted most people's attention. In 1987, Kraśkiewicz-Weyman approach to combinatorial way of this multiplicities problem and their work is actually combinatorial interpretation of the multiplicities of irreducible GL(V)-modules appear in L_n. Before stating Kraśkiewicz-Weyman theorem, major index should be defined.

Definition 5.1.2. *Let λ be a partition of n , and let T be a standart tableau. An entry i is called descent if $i + 1$ seems the row which is below to the row occurring i. Descent set is denoted by $D(T)$, the summation of the elements in $D(T)$ is* *called major index.*

In 1987, Kraśkiewicz-Weyman theorem has given useful combinatorial way to find I_λ.

Theorem 5.1.3. *Let a,$n \in \mathbb{N}$ be fixed coprime numbers and let λ be a partition of n with at most dim(V) parts. The irreducible GL(V)-module corresponding to λ occurs in L_n with multiplicity which is equal to the number of standard tableaux of shape λ with major index congruent to a modulo n.*

Let's explain the theory with an example.

Example 5.1.4. *Let $\lambda = (3,2) \vdash 5$ and now try to write all possible standart tableaux.* $\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}$. *The major index of the Young diagrams corresponding to λ is stated in below:*

. $\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}$ *Descent set is $D(T) = \{3\}$. Hence, major index is 3.*
$3 \equiv 3 \mod 5$. *Therefore, $I_\lambda = 1$. Let $a = 3$ be fixed number is coprime to 5.*

. $\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array}$ *Descent set is $D(T) = \{1,4\}$. Hence, major index is 5.*
Since $5 \equiv 0 \mod 5$, *I_λ is still 1.*

. $\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array}$ *Descent set is $D(T) = \{2,4\}$. Thus, major index is 6.*
Since $6 \equiv 1 \mod 5$, *I_λ is still 1.*

. $\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array}$ *Descent set is $D(T) = \{2\}$. Hence, major index is 2.*
Since $2 \equiv 2 \mod 5$, *I_λ is still 1 since a is 3.*

. $\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}$ *Descent set is $D(T) = \{1,3\}$. Hence, major index is 4.*
Since $4 \equiv 4 \mod 5$, *I_λ is still 1 because of* $a = 3$

Moreover, it can be shown that $\lambda = (1^n), (n), (2^2) \text{or} (2^3)$ has multiplicity 0 by theorem 5.1.3. Let's show that (2^2) has 0 multiplicity:

- $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$ major index is 2.

- $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$ major index is 4.Hence, there is no standart tableau, whose major index is coprime to 4. Thus, (2^2) has multiplicity 0.

Let's analyze multiplicity of (2^3):

- $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline \end{array}$ major index is 6.

- $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & 6 \\ \hline \end{array}$ major index is 10.

- $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & 6 \\ \hline \end{array}$ major index is 8.

- $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline \end{array}$ major index is 9.

- $\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline \end{array}$ major index is 12.

Hence, (2^3) has multiplicity 0. For the (1^n), major index is $n(n+1)/2$, then major index is divisible by n. Hence, multiplicity of (1^n) is zero. On the other hand, (n) has 0 major index. Therefore, (n) has multiplicity 0.
Kraśkiewicz-Weyman theorem, that is explained above, is useful for small degree of Lie Algebra. As the number increases, number of the tableaaux also increases which makes Kraśkiewicz-Weyman theorem is not feasible.

5.2 Kirillov's Formula

In this chapter, intertwining number is mentioned and a relation between intertwining numbers and multiplicity of irreducible representation of Free Lie Algebra is constructed. This chapter is mostly based on [6].
Computing intertwining number of two induced representation is crucial for representation theory. Computing intertwining number is identical to calculating non-empty double coset for finite groups. Before starting the Kirillov's formula, let's give some useful information about double cosets.

Definition 5.2.1. *Let H and K be two subgroups of finite group G, then the set HgK is called double cosets of H and K in G containing the element $g \in G$*

$$HgK = \{hgk | h \in H, k \in K\}$$

Lemma 5.2.2. *Let H and K be subgroubs of the finite group G.The properties of the double coset HgK of the group G is the following:*

(1) Every element $g \in G$ is contained HgK.

(2) The double cosets are either equal or disjoint,i.e. distinct double cosets give a partition of the group G.

(3) If g=e, identity element of G, then $HgK = HK$,double coset of H and K containing the identitiy element.

(4) HK is just a subset of the group G. It may not be a subgroup of the group G.

Example 5.2.3. *Let $H = \{(1), (12)\}$ and $K = \{(1), (13)\}$ be subgroup of S_3. But, $HK = \{(1), (12), (13), (132)\}$ is not a subgroup of S_3.*

One may think HgK as a union of some of the right cosets of H in group G. Then, the question is how many right cosets of H is contained in double coset HgK. Let's assume that $Hgk_1 = Hgk_2$ where $k_1 \in K, k_2 \in K$

$$H = Hgk_2{k_1}^{-1}g^{-1} \iff gk_2{k_1}^{-1}g^{-1} \in H \iff k_2{k_1}^{-1} \in g^{-1}Hg$$

$\iff k_2{k_1}^{-1} \in g^{-1}Hg \cap K \iff (g^{-1}Hg \cap K)k_1k_2^{-1} = g^{-1}Hg \cap K \iff (H^g \cap K)k_1 = (H^g \cap K)k_2$

Therefore, $\mid K : K \cap H^g \mid$ many right cosets of H are contained in HgK. Hence, the number of double cosets is calculated as shown in Lemma 5.2.4:

Lemma 5.2.4. $|HgK| = \frac{|H||K|}{|H \cap gKg^{-1}|}$

As seen from the formula, calculating double cosets is really difficult because of calculating gKg^{-1}. However, calculating double cosets for group with prime cardinality is not diffucult since $|H \cap gKg^{-1}|$ is equal to 1. Then, the number of double cosets is just the multiplication of the order of subgroups H and K. Let's try to obtain double cosets of S_3.

Example 5.2.5. *Let* $G = \{(1),(12),(13),(23),(123),(132)\}$ *be group,* $H = \{(1),(123),(132)\}$ *and* $K = \{(1)\}$ *be subgroups of the group* G*. Then:*

$$H(1)K = \{(1),(123),(132)\}$$

$$H(12)K = \{(12),(23),(13)\}$$

are the double cosets of the group G*.*

After the definition of double coset and how one find the number of double cosets, let's return main topic and explain how it is used for Kirillov's formula. Let H_1 and H_2 be two subgroup of finite group G and U_1 and U_2 be representations of these subgroups in spaces V_1 and V_2, resprectively. An operator $K(g)$ on G with values in $Hom(V_1, V_2)$ having the property:

$$K(h_1g_1k_1) = U_1(h_1)K(g)U_2(h_2) \tag{5.1}$$

Equation 5.1 is called Kirillov's equation.
Moreover, intertwinig number of two induced representation is actually equal to the dimension of space of homomorphisms. That is:

$$i({U_1}_{H_1}^G, {U_2}_{H_2}^G) = dim(Hom_G({U_1}^G, {U_2}^G))$$

In addition, it is known that:

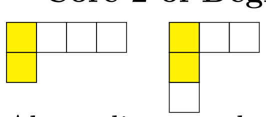
$$Hom_G(U_1^G, U_2^G) \cong \oplus Hom_{H_1^x \cap H_2^y}(U_1^x, U_2^y) = \sum_{xy^{-1} \in D} H_1 \setminus G / H_2$$

Here, double cosets is crucial to calculate the intertwining number. In the multiplicities of component L_n case, subgroup H_1 is taken the cyclic group generated by the long cycle and subgroup H_2 is the centralizer of the involution. U_1, representation of subgroup H_1, is taken to 1-dimensional representation and U_2 is 1-dimensional sign representation of subgroup H_2. Actually, sign induced representation comes from the model of S_n, the induced representation of U_2 comes from the Lie algebra. In this case, $Hom_{H_1^x \cap H_2^y}(U_1^x, U_2^y)$ is either 0 or 1 as the representations are 1-dimensional linear representatio. Hence, number of dimension yields the number of component in L_n. In other words, $i({U_1}_{H_1}^G, {U_2}_{H_2}^G)$ is equal to number of double coset which satisfies the Kirillov's equation. Double cosets which does not satisfy the Kirillov's equation is called *empty double coset*. Therefore, number of irreducible component of L_n is equal to the non-empty double cosets of S_n.

[10] published during the second world war, most of the multiplities of degree 10 was written wrong. It is really understandable because without computer calculating multiplities are really difficult. For our calculations, GAP and MAPLE 18 are used and their result document is given in the appendix part. In [5], multiplicities of L_n is calculated up to 6. In this thesis, up to degree 15 double cosets are calculated by using GAP and multiplicities of irreducible component of Free Lie Algebra are calculated up to degree 12 by using MAPLE 18. Detailed calculations are given only for L_5 and L_7.

For L_5, S_5 should be analyzed. It has 7 conjugacy classes, that is $[1^5], [1,4], [1,2^2], [1^2,3], [1^3,2], [2,3], [5]$. By theorem 5.1.1, it is known that $[1^5]$ and $[5]$ has multiplicity 0. The other ones are classified by the core. MAPLE 18 calculation gives $L_5 = [4,1] + [3,2] + [3,1^2] + [2^2,1] + [,21^3]$. Using this fact, number of empty double cosets S_5 can be calculated:

Core 2 of Degree 5:

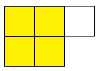
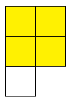
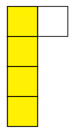


Above diagrams have core 2. For these diagrams, centralizer of the involution is $\mathbb{Z}_2 \times S_3$ and the cylic group is always C_5 . By using Theorem 5.2.4, S_5 has

$$|C_5 \backslash S_5 / \mathbb{Z}_2 \times S_3| = \frac{5!}{5.3!.2} = 2 \text{ double cosets}$$

On the other hand, calculation done by MAPLE 18, results $[1,4]$ and $[1^2,3]$ having multiplicity 1. Hence, L_5 has no empty double cosets for core 2.

Core 4 of Degree 5:

 ,  , and 

Above young diagrams correspond to the core 4. By MAPLE 18 calculations, these diagrams have multiplicity 1, hence there should be 3 non-empty double cosets. On the other hand, by calculation of number of double cosets give:

$$|C_5 \backslash S_5 / \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2| = \frac{5!}{5.8} = 3$$

Hence, there is no non-empty double cosets.

To calculate the multiplicities of component L_7, conjugacy classes of S_7 should be analyzed. It has 15 conjugacy classes. They are:

$$[1^7], [2, 1^5], [3, 1^4], [4, 1^3], [2^2, 1^4], [5, 1^2], [3, 2, 1^2], [6, 1], [4, 2, 1], [2^3, 1], [3^2, 1], [3, 2^2],$$

[5,2],[4,3],[7].

By Theorem 5.1.1, [7] and $[1^7]$ has multiplicity 0. Then using Theorem 4.1.4 to calculate te multiplicities by using MAPLE 18. Then, for L_7, Theorem 4.1.4 gives the following equation for each young diagram of S_7.

$$\frac{1}{7}[\mu(1)\chi_\lambda(\tau^7) + \mu(7)\chi_\lambda(\tau)]$$

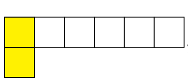
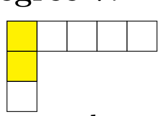
where $\tau = (1, 2, 3, 4, 5, 6, 7)$, $\tau^7 = (1), \mu(1) = 1$ and $\mu(7) = -1$. Then using MAPLE 18, multiplities of all component of L_7 are calculated. Hence,

$$L_7 \cong [6, 1] + 2[5, 1^2] + 2[5, 2] + 3[4, 1^3] + 5[4, 2, 1] + 2[4, 3] + 2[3, 1^4] + 5[3, 2, 1^2]$$

$+3[3^2, 1] + 3[3, 2^2] + 2[2^3, 1] + 2[2^2, 1^3] + [2, 1^5]$

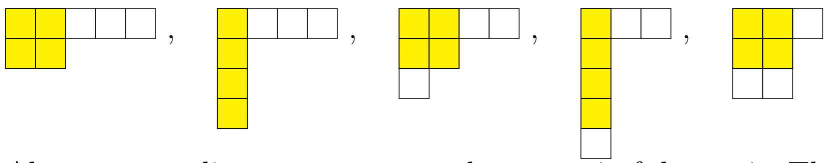
The number of double cosets is analyzed by classifying the core.

Core 2 of Degree 7:

, 

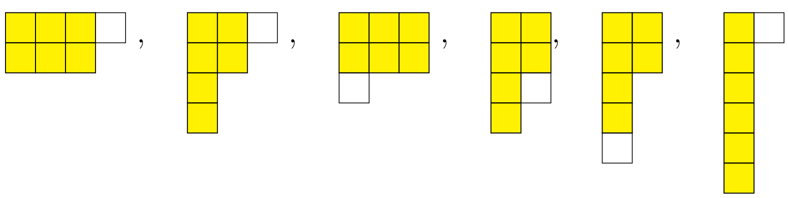
Above young diagrams have core 2 and their multiplicies are 1 and 2, respectively. When calculating number of double cosets, C_7 and centralizer of the involution C_{σ_2} are considered as subgroups of L_7, where σ_2 is corresponding to the one involution. As a result of GAP calculations, the order of the double cosets corresponding these diagrams is 1680 and this gives 3 different double cosets. Hence, there is one empty double coset for core 2.

Core 4 of Degree 7:



Above young diagrams correspond to core 4 of degree 7. Their multiplicities are 2,3,5,2 and 3, respectively. Let C_7 and centralizer of involution having 2 transpositions C_{σ_4} be the subgroups of S_7. According to GAP calculations, there are 15 double cosets and each of them has 336 elements. Hence, there are 14 different double cosets which is exactly the summation of the multiplicities. Therefore, there is no empty double cosets for core 4.

Core 6 of Degree 7:



Above young diagrams correspond to core 6 of degree 7 and their multiplicities are 2,5,3,2,2 and 1, respectively. When calculating the number of double cosets, C_7 and C_{σ_6} are subgroups of group S_7. Using GAP, there are 15 double cosets and order of them is 336. Hence,like core 4, L_7 has no empty double coset since 15 is the sum of the multiplicities of the young diagram for core 6.

Like L_7, the representation of L_n is calculated up to 12. Only results are presented here. Details of the calculations can be found in appendix part.

Moreover, as mentioned above, Kirillov's theorem declares that the intertwining number of the two induced representations is equal to the non-empty double cosets. Although, in general calculating the non-empty double cosets is not easy, in this thesis, only 1-dimensional representations are considered. Hence, the situation gets much more easy because of only dealing with scalars. When calculating the number of the non-empty double cosets, if the chosen subgroups do not intersect, then the number of non-empty double cosets is exactly the same as the number of the double cosets. Hence, for prime degree, we get the following formula:

Theorem 5.2.6. *Number of the irreducible component of the prime degree p of Free Lie Algebra is equal to:*

$$\frac{1}{p}\sum_{k=1}^{\frac{p}{2}}\binom{p}{2k}\frac{2k!}{k!2^k}$$

Proof. In prime degree of the Free Lie Algebra, number of the irreducible components is equal to the number of the non-empty double cosets. To be non-empty, double cosets should satisfy the Kirillov's equation. The subgroups for the double cosets in the prime degree of Free Lie Algebra are cyclic group generated by long cycle C_p and the centralizer of the involutions C_{σ_k} where k is the number of transpositions. When counting the non-empty double cosets, all double cosets should be non-empty in case the intersection of the two subgroups do not intersect. In prime degree ,C_p and C_{σ_k} have trivial intersection, then all double cosets must be non-empty double cosets. Therefore,the number of the irreducible components of the prime degree of Free Lie Algebra is exactly equal to the sum of the double cosets with respect to core. Because of considering all core, summation is over σ_k and the number of the double coset is equal to

$$\frac{1}{p}\sum_{k=1}^{\frac{p}{2}}\binom{p}{2k}\frac{2k!}{k!2^k}$$

□

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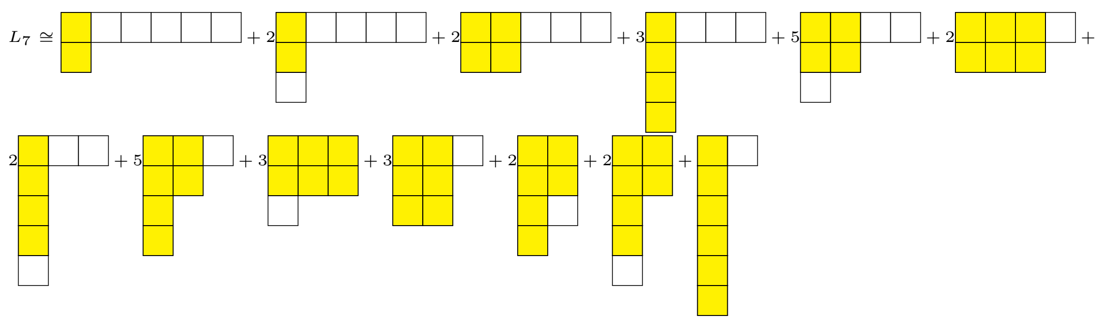
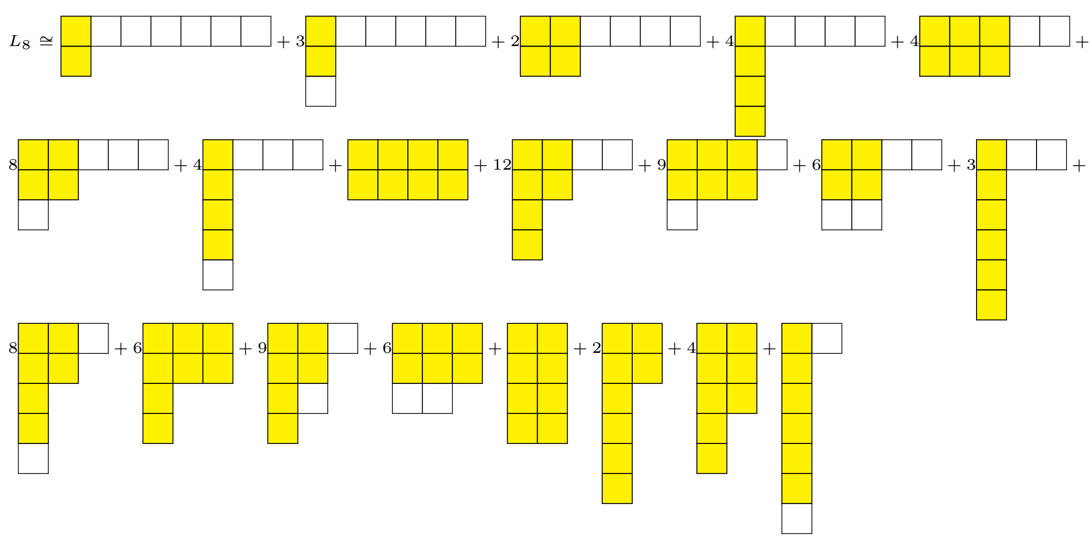
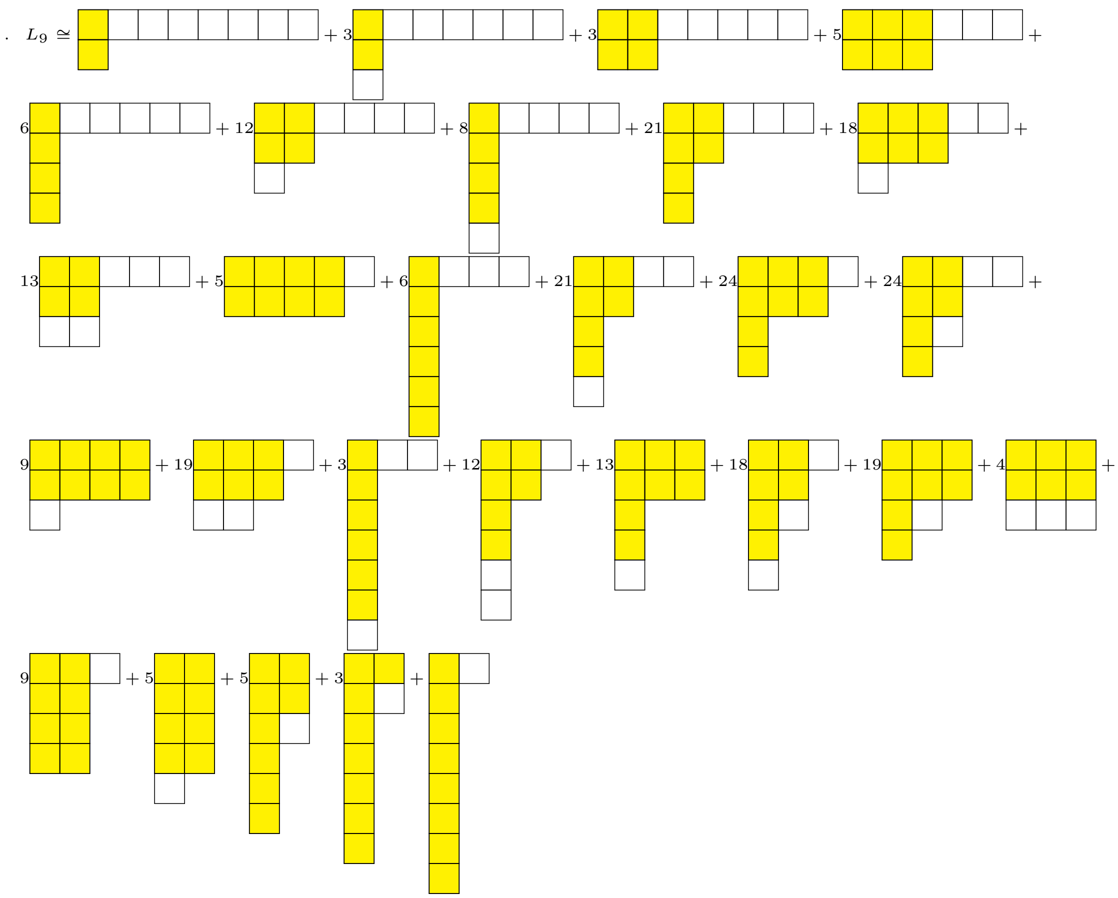
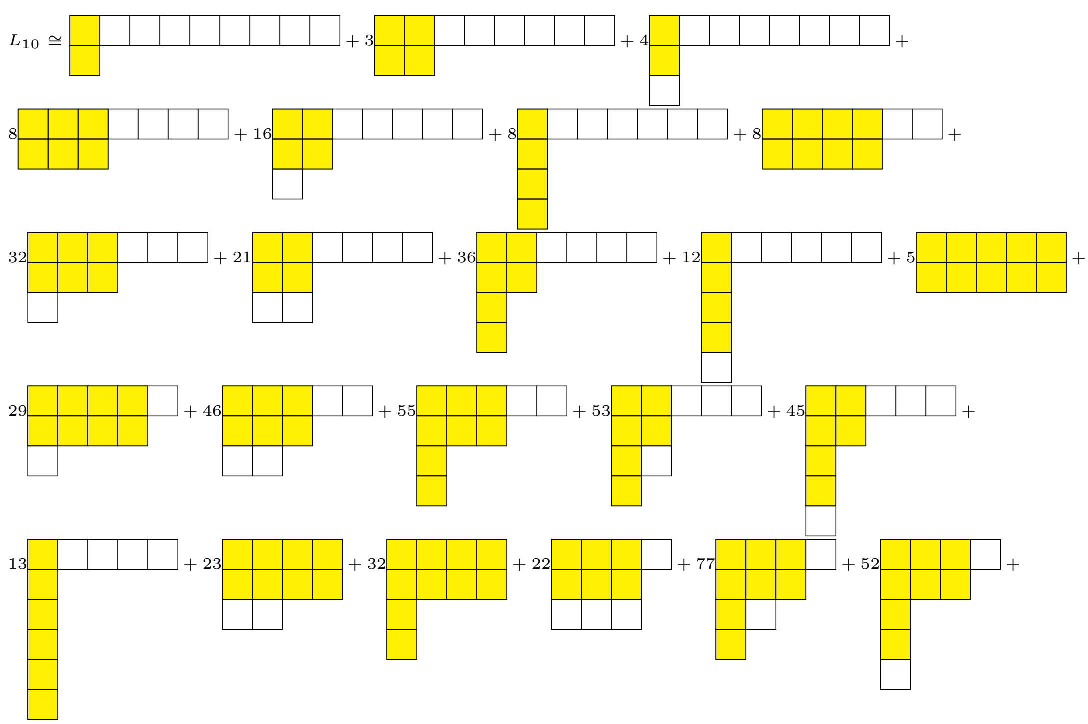
Appendix A

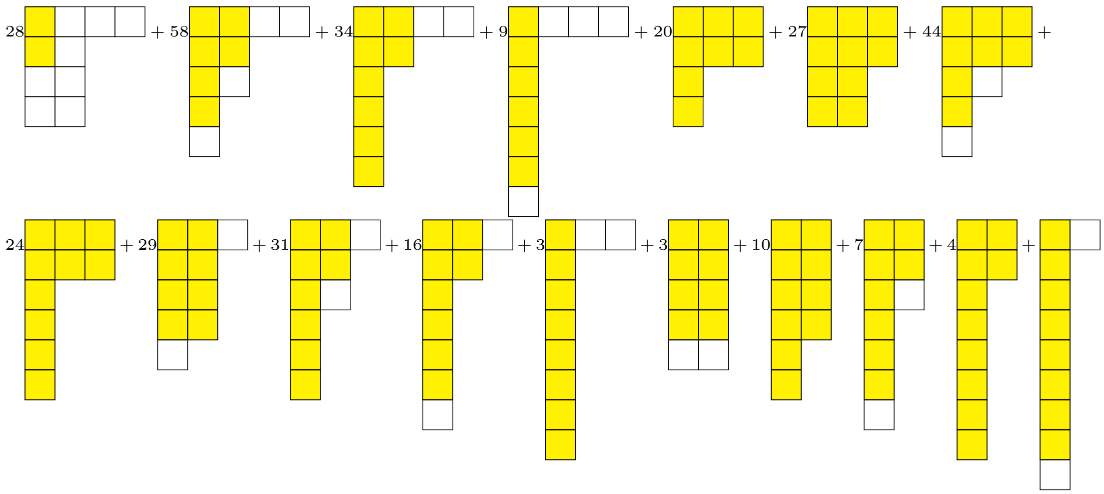
Decomposition of Free Lie Algebra into Irreducible Components up to degree 12 and the respective cores

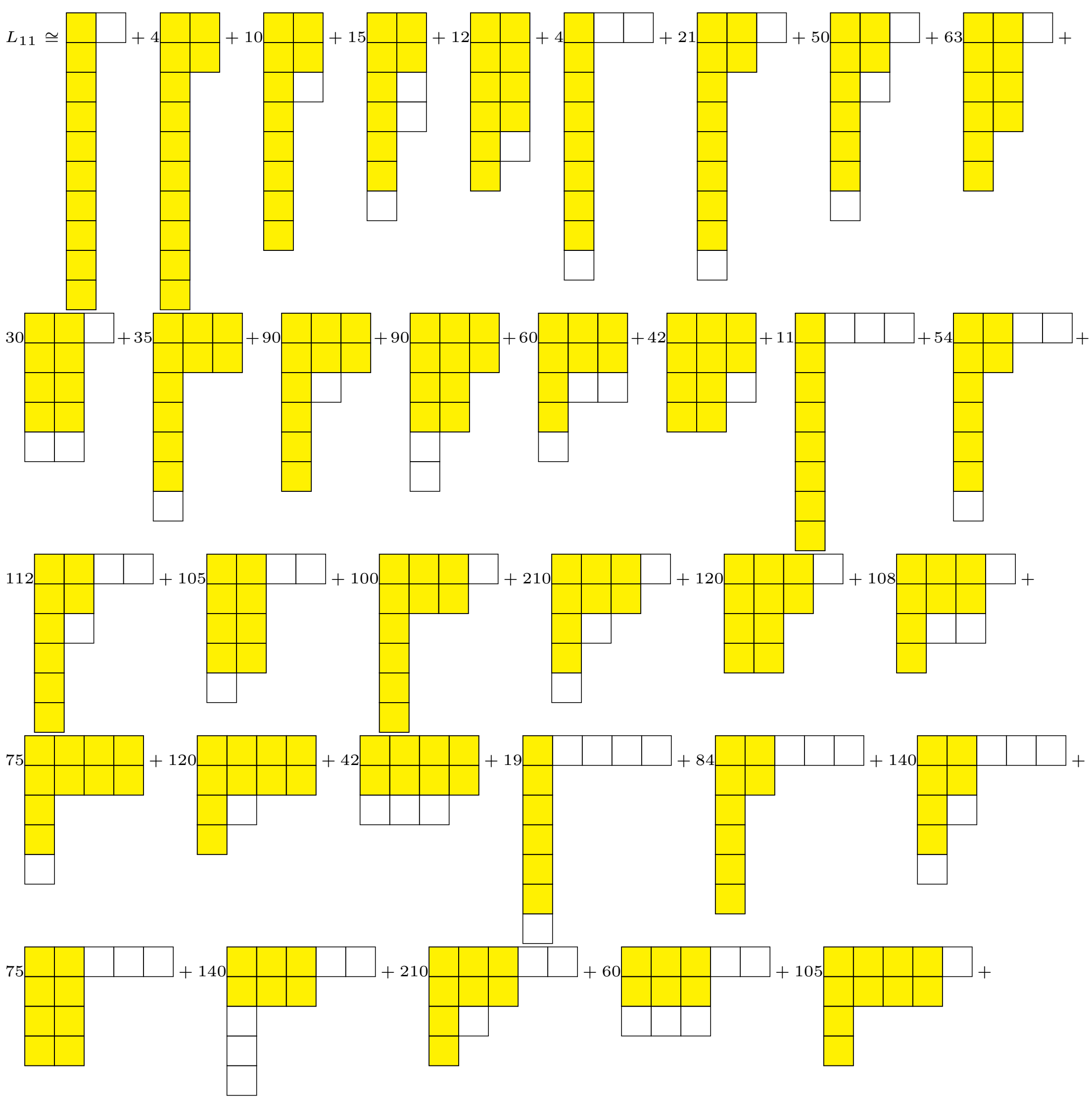


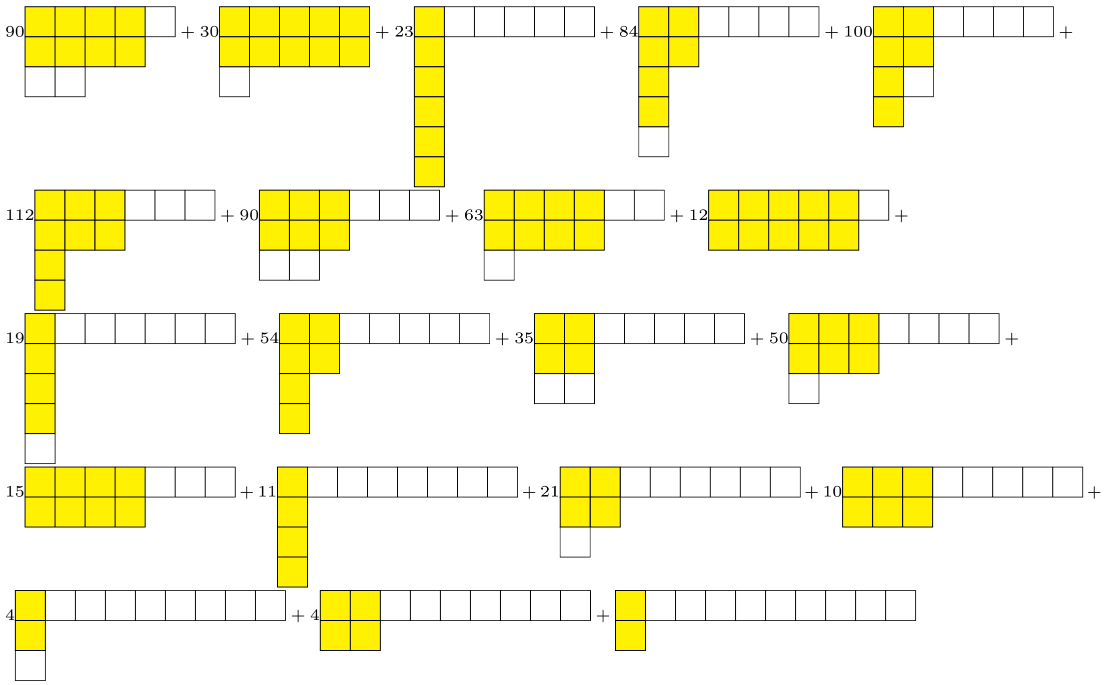


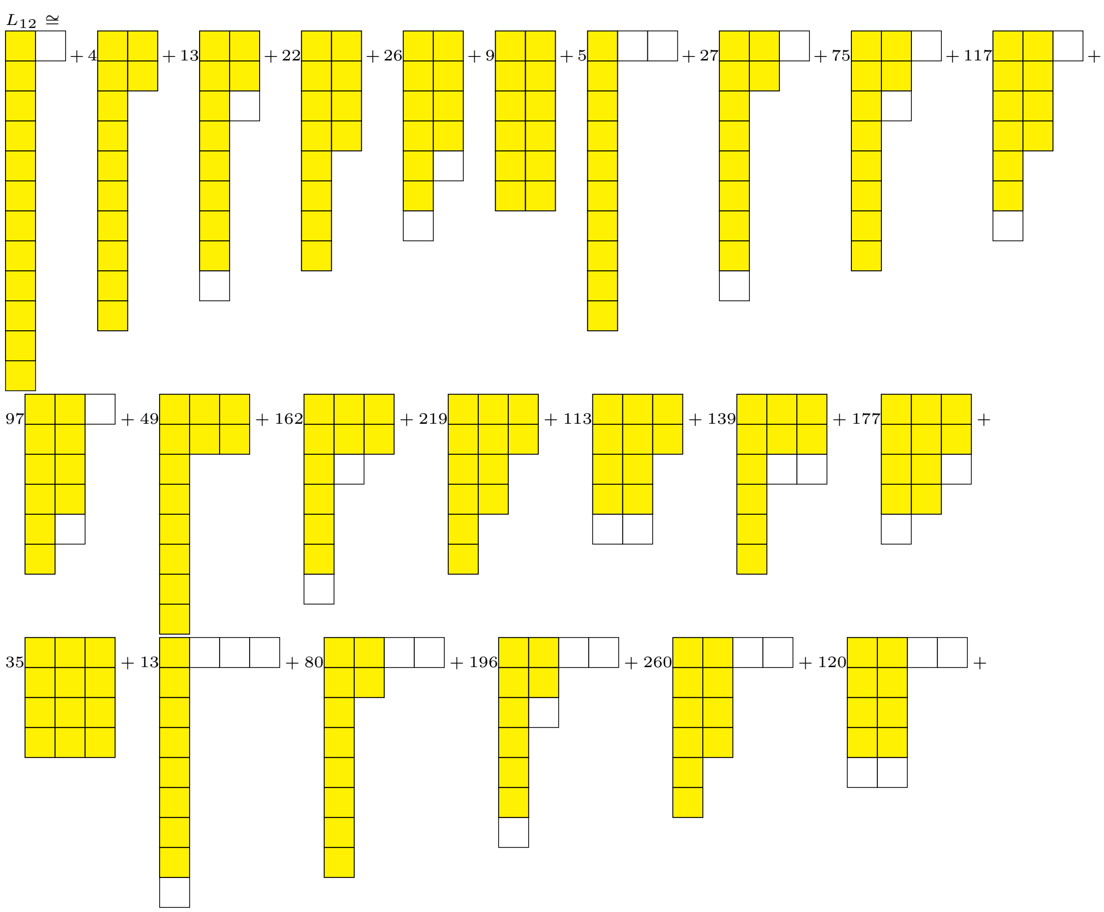

$L_9 \cong$ + 3 + 3 + 5 +
6 + 12 + 8 + 21 + 18 +
13 + 5 + 6 + 21 + 24 + 24 +
9 + 19 + 3 + 12 + 13 + 18 + 19 + 4 +
9 + 5 + 5 + 3 +

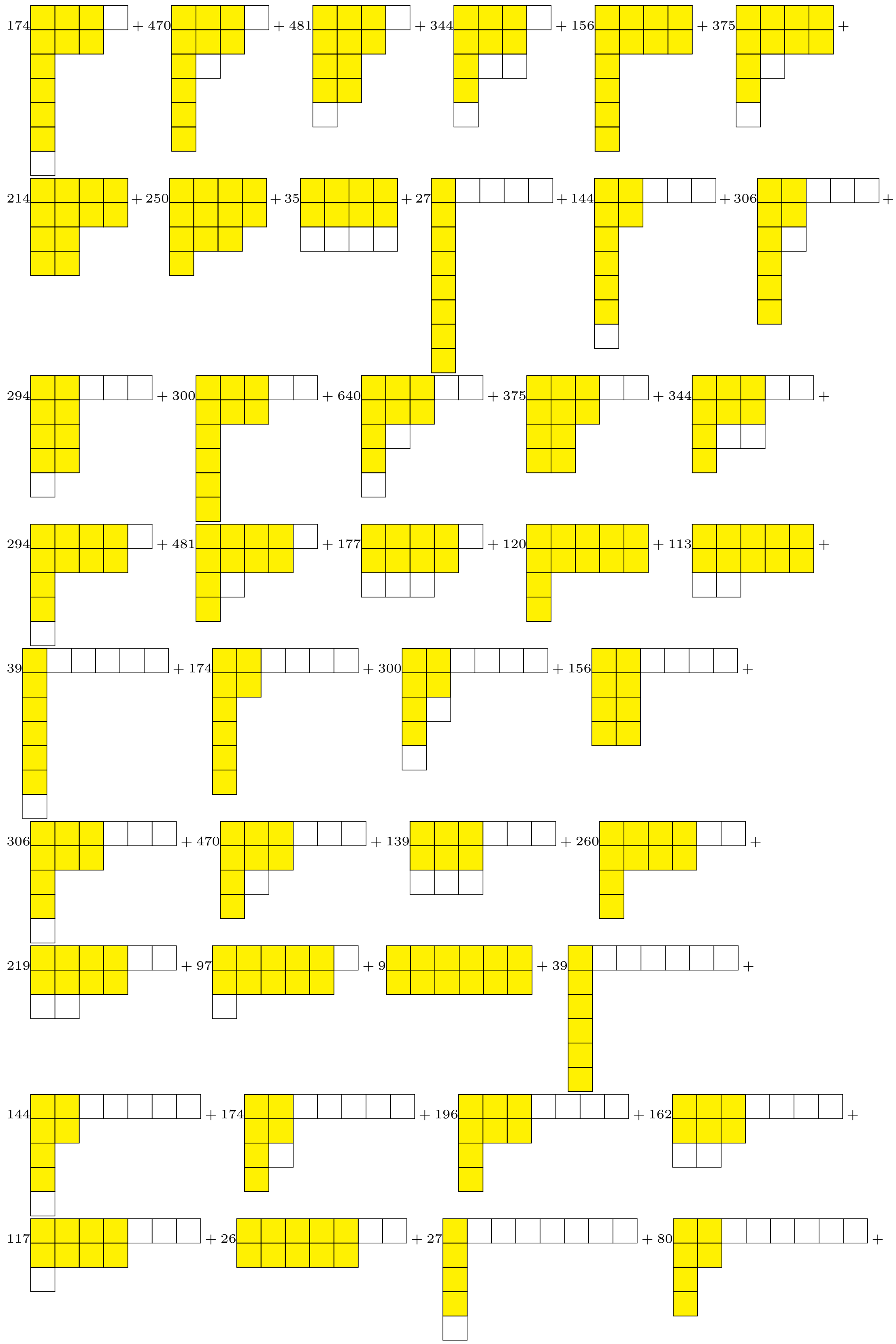

$L_{10} \cong$ + 3 + 4 +
8 + 16 + 8 + 8 +
32 + 21 + 36 + 12 + 5 +
29 + 46 + 55 + 53 + 45 +
13 + 23 + 32 + 22 + 77 + 52 +

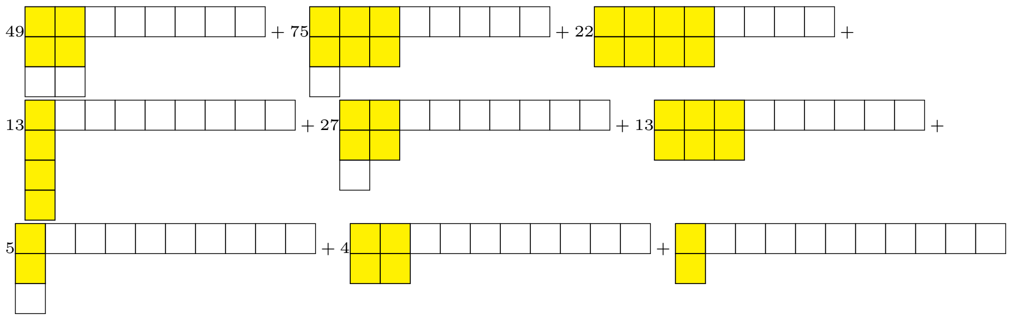

28 + 58 + 34 + 9 + 20 + 27 + 44 +
24 + 29 + 31 + 16 + 3 + 3 + 10 + 7 + 4 +


$L_{11} \cong$ + 4 + 10 + 15 + 12 + 4 + 21 + 50 + 63 +
30 + 35 + 90 + 90 + 60 + 42 + 11 + 54 +
112 + 105 + 100 + 210 + 120 + 108 +
75 + 120 + 42 + 19 + 84 + 140 +
75 + 140 + 210 + 60 + 105 +


90 + 30 + 23 + 84 + 100 +
112 + 90 + 63 + 12 +
19 + 54 + 35 + 50 +
15 + 11 + 21 + 10 +
4 + 4 +


$L_{12} \cong$
+ 4 + 13 + 22 + 26 + 9 + 5 + 27 + 75 + 117 +
97 + 49 + 162 + 219 + 113 + 139 + 177 +
35 + 13 + 80 + 196 + 260 + 120 +


174 + 470 + 481 + 344 + 156 + 375 +
214 + 250 + 35 + 27 + 144 + 306 +
294 + 300 + 640 + 375 + 344 +
294 + 481 + 177 + 120 + 113 +
39 + 174 + 300 + 156 +
306 + 470 + 139 + 260 +
219 + 97 + 9 + 39 +
144 + 174 + 196 + 162 +
117 + 26 + 27 + 80 +


49
+ 75
+ 22
+
13
+ 27
+ 13
+
5
+ 4
+

Appendix B

Code for Partitions of Symmetric Groups of Degree 7 to 12 in GAP

In GAP, conjugacy classes of S_7 to the S_{12} is calculated.
CONJUGACY CLASSES OF SYMMETRIC GROUP OF ORDER 7:
There are 15 conjugacy classes.

(), (1,2), (1,2)(3,4), (1,2)(3,4)(5,6), (1,2,3), (1,2,3)(4,5), (1,2,3)(4,5)(6,7),(1,2,3)(4,5,6),(1,2,3,4), (1,2,3,4)(5,6), (1,2,3,4)(5,6,7), (1,2,3,4,5),(1,2,3,4,5)(6,7), (1,2,3,4,5,6), (1,2,3,4,5,6,7)

CONJUGACY CLASSES OF SYMMETRIC GROUP OF ORDER 8:
There are 22 conjugacy classes.

(), (1,2), (1,2)(3,4), (1,2)(3,4)(5,6), (1,2)(3,4)(5,6)(7,8), (1,2,3), (1,2,3)(4,5), (1,2,3)(4,5)(6,7), (1,2,3)(4,5,6), (1,2,3)(4,5,6)(7,8), (1,2,3,4), (1,2,3,4)(5,6), (1,2,3,4)(5,6)(7,8), (1,2,3,4)(5,6,7), (1,2,3,4)(5,6,7,8), (1,2,3,4,5), (1,2,3,4,5)(6,7), (1,2,3,4,5)(6,7,8), (1,2,3,4,5,6), (1,2,3,4,5,6)(7,8), (1,2,3,4,5,6,7), (1,2,3,4,5,6,7,8)

CONJUGACY CLASSES OF SYMMETRIC GROUP OF ORDER 9:
There are 30 conjugacy classes.

(), (1,2), (1,2)(3,4), (1,2)(3,4)(5,6), (1,2)(3,4)(5,6)(7,8), (1,2,3), (1,2,3)(4,5), (1,2,3)(4,5)(6,7), (1,2,3)(4,5)(6,7)(8,9), (1,2,3)(4,5,6), (1,2,3)(4,5,6)(7,8), (1,2,3)(4,5,6)(7,8,9), (1,2,3,4), (1,2,3,4)(5,6), (1,2,3,4)(5,6)(7,8), (1,2,3,4)(5,6,7), (1,2,3,4)(5,6,7)(8,9), (1,2,3,4)(5,6,7,8), (1,2,3,4,5), (1,2,3,4,5)(6,7), (1,2,3,4,5)(6,7)(8,9), (1,2,3,4,5)(6,7,8), (1,2,3,4,5)(6,7,8,9), (1,2,3,4,5,6), (1,2,3,4,5,6)(7,8), (1,2,3,4,5,6)(7,8,9), (1,2,3,4,5,6,7), (1,2,3,4,5,6,7)(8,9), (1,2,3,4,5,6,7,8), (1,2,3,4,5,6,7,8,9)

CONJUGACY CLASSES OF SYMMETRIC GROUP OF ORDER 10:
There are 42 conjugacy classes.

(), (1,2), (1,2)(3,4), (1,2)(3,4)(5,6), (1,2)(3,4)(5,6)(7,8), (1,2)(3,4)(5,6)(7,8)(9,10), (1,2,3), (1,2,3)(4,5), (1,2,3)(4,5)(6,7), (1,2,3)(4,5)(6,7)(8,9), (1,2,3)(4,5,6), (1,2,3)(4,5,6)(7,8), (1,2,3)(4,5,6)(7,8)(9,10), (1,2,3)(4,5,6)(7,8,9), (1,2,3,4), (1,2,3,4)(5,6), (1,2,3,4)(5,6)(7,8), (1,2,3,4)(5,6)(7,8)(9,10), (1,2,3,4)(5,6,7), (1,2,3,4)(5,6,7)(8,9), (1,2,3,4)(5,6,7)(8,9,10), (1,2,3,4)(5,6,7,8), (1,2,3,4)(5,6,7,8)(9,10), (1,2,3,4,5), (1,2,3,4,5)(6,7), (1,2,3,4,5)(6,7)(8,9), (1,2,3,4,5)(6,7,8), (1,2,3,4,5)(6,7,8)(9,10), (1,2,3,4,5)(6,7,8,9), (1,2,3,4,5)(6,7,8,9,10), (1,2,3,4,5,6), (1,2,3,4,5,6)(7,8), (1,2,3,4,5,6)(7,8)(9,10), (1,2,3,4,5,6)(7,8,9), (1,2,3,4,5,6)(7,8,9,10), (1,2,3,4,5,6,7), (1,2,3,4,5,6,7)(8,9), (1,2,3,4,5,6,7)(8,9,10), (1,2,3,4,5,6,7,8), (1,2,3,4,5,6,7,8)(9,10), (1,2,3,4,5,6,7,8,9), (1,2,3,4,5,6,7,8,9,10)

CONJUGACY CLASSES OF SYMMETRIC GROUP OF ORDER 11:
There are 56 conjugacy classes.

(), (1,2), (1,2)(3,4), (1,2)(3,4)(5,6), (1,2)(3,4)(5,6)(7,8), (1,2)(3,4)(5,6)(7,8)(9,10), (1,2,3), (1,2,3)(4,5), (1,2,3)(4,5)(6,7), (1,2,3)(4,5)(6,7)(8,9), (1,2,3)(4,5)(6,7)(8,9)(10,11), (1,2,3)(4,5,6), (1,2,3)(4,5,6)(7,8), (1,2,3)(4,5,6)(7,8)(9,10), (1,2,3)(4,5,6)(7,8,9), (1,2,3)(4,5,6)(7,8,9)(10,11), (1,2,3,4), (1,2,3,4)(5,6), (1,2,3,4)(5,6)(7,8), (1,2,3,4)(5,6)(7,8)(9,10), (1,2,3,4)(5,6,7), (1,2,3,4)(5,6,7)(8,9), (1,2,3,4)(5,6,7)(8,9)(10,11), (1,2,3,4)(5,6,7)(8,9,10), (1,2,3,4)(5,6,7,8), (1,2,3,4)(5,6,7,8)(9,10), (1,2,3,4)(5,6,7,8)(9,10,11), (1,2,3,4,5), (1,2,3,4,5)(6,7), (1,2,3,4,5)(6,7)(8,9), (1,2,3,4,5)(6,7)(8,9)(10,11), (1,2,3,4,5)(6,7,8), (1,2,3,4,5)(6,7,8)(9,10), (1,2,3,4,5)(6,7,8)(9,10,11), (1,2,3,4,5)(6,7,8,9), (1,2,3,4,5)(6,7,8,9)(10,11), (1,2,3,4,5)(6,7,8,9,10), (1,2,3,4,5,6), (1,2,3,4,5,6)(7,8), (1,2,3,4,5,6)(7,8)(9,10), (1,2,3,4,5,6)(7,8,9), (1,2,3,4,5,6)(7,8,9)(10,11), (1,2,3,4,5,6)(7,8,9,10), (1,2,3,4,5,6)(7,8,9,10,11), (1,2,3,4,5,6,7), (1,2,3,4,5,6,7)(8,9), (1,2,3,4,5,6,7)(8,9)(10,11), (1,2,3,4,5,6,7)(8,9,10), (1,2,3,4,5,6,7)(8,9,10,11), (1,2,3,4,5,6,7,8), (1,2,3,4,5,6,7,8)(9,10), (1,2,3,4,5,6,7,8)(9,10,11), (1,2,3,4,5,6,7,8,9), (1,2,3,4,5,6,7,8,9)(10,11), (1,2,3,4,5,6,7,8,9,10), (1,2,3,4,5,6,7,8,9,10,11)

CONJUGACY CLASSES OF SYMMETRIC GROUP OF ORDER 12:
There are 77 conjugacy classes.

(), (1,2), (1,2)(3,4), (1,2)(3,4)(5,6), (1,2)(3,4)(5,6)(7,8), (1,2)(3,4)(5,6)(7,8)(9,10), (1,2)(3,4)(5,6)(7,8)(9,10)(11,12), (1,2,3), (1,2,3)(4,5), (1,2,3)(4,5)(6,7), (1,2,3)(4,5)(6,7)(8,9), (1,2,3)(4,5)(6,7)(8,9)(10,11), (1,2,3)(4,5,6), (1,2,3)(4,5,6)(7,8), (1,2,3)(4,5,6)(7,8)(9,10), (1,2,3)(4,5,6)(7,8)(9,10)(11,12), (1,2,3)(4,5,6)(7,8,9), (1,2,3)(4,5,6)(7,8,9)(10,11), (1,2,3)(4,5,6)(7,8,9)(10,11,12), (1,2,3,4), (1,2,3,4)(5,6), (1,2,3,4)(5,6)(7,8), (1,2,3,4)(5,6)(7,8)(9,10), (1,2,3,4)(5,6)(7,8)(9,10)(11,12), (1,2,3,4)(5,6,7), (1,2,3,4)(5,6,7)(8,9), (1,2,3,4)(5,6,7)(8,9)(10,11), (1,2,3,4)(5,6,7)(8,9,10), (1,2,3,4)(5,6,7)(8,9,10)(11,12), (1,2,3,4)(5,6,7,8), (1,2,3,4)(5,6,7,8)(9,10), (1,2,3,4)(5,6,7,8)(9,10)(11,12), (1,2,3,4)(5,6,7,8)(9,10,11), (1,2,3,4)(5,6,7,8)(9,10,11,12), (1,2,3,4,5), (1,2,3,4,5)(6,7), (1,2,3,4,5)(6,7)(8,9), (1,2,3,4,5)(6,7)(8,9)(10,11), (1,2,3,4,5)(6,7,8), (1,2,3,4,5)(6,7,8)(9,10), (1,2,3,4,5)(6,7,8)(9,10)(11,12), (1,2,3,4,5)(6,7,8)(9,10,11), (1,2,3,4,5)(6,7,8,9), (1,2,3,4,5)(6,7,8,9)(10,11), (1,2,3,4,5)(6,7,8,9)(10,11,12), (1,2,3,4,5)(6,7,8,9,10), (1,2,3,4,5)(6,7,8,9,10)(11,12), (1,2,3,4,5,6), (1,2,3,4,5,6)(7,8), (1,2,3,4,5,6)(7,8)(9,10), (1,2,3,4,5,6)(7,8)(9,10)(11,12), (1,2,3,4,5,6)(7,8,9), (1,2,3,4,5,6)(7,8,9)(10,11), (1,2,3,4,5,6)(7,8,9)(10,11,12), (1,2,3,4,5,6)(7,8,9,10), (1,2,3,4,5,6)(7,8,9,10)(11,12), (1,2,3,4,5,6)(7,8,9,10,11), (1,2,3,4,5,6)(7,8,9,10,11,12), (1,2,3,4,5,6,7), (1,2,3,4,5,6,7)(8,9), (1,2,3,4,5,6,7)(8,9)(10,11), (1,2,3,4,5,6,7)(8,9,10), (1,2,3,4,5,6,7)(8,9,10)(11,12), (1,2,3,4,5,6,7)(8,9,10,11), (1,2,3,4,5,6,7)(8,9,10,11,12), (1,2,3,4,5,6,7,8), (1,2,3,4,5,6,7,8)(9,10), (1,2,3,4,5,6,7,8)(9,10)(11,12), (1,2,3,4,5,6,7,8)(9,10,11), (1,2,3,4,5,6,7,8)(9,10,11,12), (1,2,3,4,5,6,7,8,9), (1,2,3,4,5,6,7,8,9)(10,11), (1,2,3,4,5,6,7,8,9)(10,11,12), (1,2,3,4,5,6,7,8,9,10), (1,2,3,4,5,6,7,8,9,10)(11,12), (1,2,3,4,5,6,7,8,9,10,11), (1,2,3,4,5,6,7,8,9,10,11,12)

Appendix C

Number of Double Cosets Symmetric Groups Degree 7 to 12

NUMBER OF DOUBLE COSETS SYMMETRIC GROUP S_7 RESPECTIVE CORES:
Core 2: The subgroups calculating double cosets are cylic group generated by long cycle C_7 and centralizer of the involution consisting of 1 transposition C_{σ_2}. There are 3 double cosets.
Core 4:The subgroups calculating double cosets are cylic group generated by long cycle C_7 and centralizer of the involution consisting of 2 transpositions C_{σ_4}. There are 15 double cosets.
Core 6:The subgroups calculating double cosets are cylic group generated by long cycle C_7 and centralizer of the involution consisting of 3 transpositions C_{σ_6}. There are 15 double cosets.
NUMBER OF DOUBLE COSETS SYMMETRIC GROUP S_8 RESPECTIVE CORES:
Core 2: The subgroups calculating double cosets are cylic group generated by long cycle C_8 and centralizer of the involution consisting of 1 transposition C_{σ_2}. There are 4 double cosets.
Core 4:The subgroups calculating double cosets are cylic group generated by long cycle C_8 and centralizer of the involution consisting of 2 transpositions C_{σ_4}. There are 29 double cosets.
Core 6:The subgroups calculating double cosets are cylic group generated by long cycle C_8 and centralizer of the involution consisting of 3 transpositions C_{σ_6}. There are 56 double cosets. Core 8:The subgroups calculating double cosets are cylic group generated by long cycle C_8 and centralizer of the involution consisting of 4 transpositions C_{σ_8}. There are 18 double cosets.
NUMBER OF DOUBLE COSETS SYMMETRIC GROUP S_9 RESPECTIVE CORES:
Core 2: The subgroups calculating double cosets are cylic group generated by long cycle C_9 and centralizer of the involution consisting of 1 transposition C_{σ_2}. There are 4 double cosets.
Core 4:The subgroups calculating double cosets are cylic group generated by long cycle C_9 and centralizer of the involution consisting of 2 transpositions C_{σ_4}. There are 29 double cosets.
Core 6:The subgroups calculating double cosets are cylic group generated by long cycle C_9 and centralizer of the involution consisting of 3 transpositions C_{σ_6}, 6 means there are three transpositions. There are 56 double cosets. Core 8:The subgroups calculating double cosets are cylic group generated by long cycle C_9 and centralizer of the involution consisting of 4 transpositions C_{σ_8}. There are 18 double cosets.
NUMBER OF DOUBLE COSETS SYMMETRIC GROUP S_{10} RESPECTIVE CORES:
Core 2: The subgroups calculating double cosets are cylic group generated by long cycle C_{10} and centralizer of the involution consisting of 1 transposition C_{σ_2}. There are 5 double cosets.
Core 4:The subgroups calculating double cosets are cylic group generated by long cycle C_{10} and centralizer of the involution consisting of 2 transpositions C_{σ_4}. There are 66 double cosets.
Core 6:The subgroups calculating double cosets are cylic group generated by long cycle C_{10} and centralizer of the involution consisting of 3 transpositions C_{σ_6}, 6 means there are three transpositions. There are 322 double cosets.
Core 8:The subgroups calculating double cosets are cylic group generated by long cycle C_{10} and centralizer of the involution consisting of 4 transpositions C_{σ_8}. There are 485 double cosets.
Core 10:The subgroups calculating double cosets are cylic group generated by long cycle C_{10} and centralizer of the involution consisting of 5 transpositions $C_{\sigma_{10}}$. There are 105 double cosets
NUMBER OF DOUBLE COSETS SYMMETRIC GROUP S_{11} RESPECTIVE CORES:
Core 2: The subgroups calculating double cosets are cylic group generated by long cycle C_{11} and centralizer of the involution consisting of 1 transposition C_{σ_2}. There are 5 double cosets.
Core 4:The subgroups calculating double cosets are cylic group generated by long cycle C_{11} and centralizer of the involution consisting of 2 transpositions C_{σ_4}. There are 90 double cosets.
Core 6:The subgroups calculating double cosets are cylic group generated by long cycle C_{11} and centralizer of the involution consisting of 3 transpositions C_{σ_6}, 6 means there are three transpositions. There are 630 double cosets.
Core 8:The subgroups calculating double cosets are cylic group generated by long cycle C_{11} and centralizer of the involution consisting of 4 transpositions C_{σ_8}. There are 1575 double cosets.

Core 10:The subgroups calculating double cosets are cylic group generated by long cycle C_{11} and centralizer of the involution consisting of 5 transpositions $C_{\sigma_{10}}$. There are 945 double cosets

NUMBER OF DOUBLE COSETS SYMMETRIC GROUP S_{12} RESPECTIVE CORES:

Core 2: The subgroups calculating double cosets are cylic group generated by long cycle C_{12} and centralizer of the involution consisting of 1 transposition C_{σ_2}. There are 6 double cosets.

Core 4:The subgroups calculating double cosets are cylic group generated by long cycle C_{12} and centralizer of the involution consisting of 2 transpositions C_{σ_4}. There are 128 double cosets.

Core 6:The subgroups calculating double cosets are cylic group generated by long cycle C_{12} and centralizer of the involution consisting of 3 transpositions C_{σ_6}, 6 means there are three transpositions. There are 1170 double cosets.

Core 8:The subgroups calculating double cosets are cylic group generated by long cycle C_{12} and centralizer of the involution consisting of 4 transpositions C_{σ_8}. There are 4365 double cosets.

Core 10:The subgroups calculating double cosets are cylic group generated by long cycle C_{12} and centralizer of the involution consisting of 5 transpositions $C_{\sigma_{10}}$. There are 5238 double cosets.

Core 12:The subgroups calculating double cosets are cylic group generated by long cycle C_{12} and centralizer of the involution consisting of 6 transpositions $C_{\sigma_{12}}$. There are 902 double cosets.

Appendix D

Code for Multiplicities of Irreducible Components of Free Lie Algebra Degree 7 to Degree 12

The above code calculate the irreducible component of Free Lie Algebra L_7:

```
with(combinat):(Chi([1,1,1,1,1,1,1],[1,1,1,1,1,1,1])- Chi([1,1,1,1,1,1,1],[7]))/7; 0 with(combinat):(Chi([1,1,1,1,1,2],[1,1,1,1,1,1,1])-
Chi([1,1,1,1,1,2],[7]))/7; 1 with(combinat):(Chi([1,1,1,1,3],[1,1,1,1,1,1,1])- Chi([1,1,1,1,3],[7]))/7; 2
with(combinat):(Chi([1,1,1,4],[1,1,1,1,1,1,1])- Chi([1,1,1,4],[7]))/7; 3 with(combinat):(Chi([1,1,1,2,2],[1,1,1,1,1,1,1])-
Chi([1,1,1,2,2],[7]))/7; 2 with(combinat):(Chi([1,1,5],[1,1,1,1,1,1,1])- Chi([1,1,5],[7]))/7; 2
with(combinat):(Chi([1,1,2,3],[1,1,1,1,1,1,1])- Chi([1,1,2,3],[7]))/7; 5 with(combinat):(Chi([1,6],[1,1,1,1,1,1,1])- Chi([1,6],[7]))/7;
1 with(combinat):(Chi([1,2,4],[1,1,1,1,1,1,1])- Chi([1,2,4],[7]))/7; 5 with(combinat):(Chi([1,2,2,2],[1,1,1,1,1,1,1])-
Chi([1,2,2,2],[7]))/7; 2 with(combinat):(Chi([1,3,3],[1,1,1,1,1,1,1])- Chi([1,3,3],[7]))/7; 3
with(combinat):(Chi([2,2,3],[1,1,1,1,1,1,1])- Chi([2,2,3],[7]))/7; 3 with(combinat):(Chi([2,5],[1,1,1,1,1,1,1])- Chi([2,5],[7]))/7; 2
with(combinat):(Chi([3,4],[1,1,1,1,1,1,1])- Chi([3,4],[7]))/7; 2 with(combinat):(Chi([7],[1,1,1,1,1,1,1])- Chi([7],[7]))/7; 0
```

The above code calculate the irreducible component of Free Lie Algebra L_8:

```
with(combinat):(Chi([1,1,1,1,1,1,1,1],[1,1,1,1,1,1,1,1])- Chi([1,1,1,1,1,1,1,1],[2,2,2,2]))/8; 0
with(combinat):(Chi([1,1,1,1,1,1,2],[1,1,1,1,1,1,1,1])- Chi([1,1,1,1,1,1,2],[2,2,2,2]))/8; 1
with(combinat):(Chi([1,1,1,1,1,3],[1,1,1,1,1,1,1,1])- Chi([1,1,1,1,1,3],[2,2,2,2]))/8; 3
with(combinat):(Chi([1,1,1,1,4],[1,1,1,1,1,1,1,1])- Chi([1,1,1,1,4],[2,2,2,2]))/8; 4
with(combinat):(Chi([1,1,1,1,2,2],[1,1,1,1,1,1,1,1])- Chi([1,1,1,1,2,2],[2,2,2,2]))/8; 2
with(combinat):(Chi([1,1,1,5],[1,1,1,1,1,1,1,1])- Chi([1,1,1,5],[2,2,2,2]))/8; 4 with(combinat):(Chi([1,1,1,2,3],[1,1,1,1,1,1,1,1])-
Chi([1,1,1,2,3],[2,2,2,2]))/8; 8 with(combinat):(Chi([1,1,6],[1,1,1,1,1,1,1,1])- Chi([1,1,6],[2,2,2,2]))/8; 3
with(combinat):(Chi([3,5],[1,1,1,1,1,1,1,1])- Chi([3,5],[2,2,2,2]))/8; 4 with(combinat):(Chi([4,4],[1,1,1,1,1,1,1,1])-
Chi([4,4],[2,2,2,2]))/8; 1 with(combinat):(Chi([1,1,2,4],[1,1,1,1,1,1,1,1])- Chi([1,1,2,4],[2,2,2,2]))/8; 12
with(combinat):(Chi([1,1,2,2,2],[1,1,1,1,1,1,1,1])- Chi([1,1,2,2,2],[2,2,2,2]))/8; 4 with(combinat):(Chi([1,1,3,3],[1,1,1,1,1,1,1,1])-
Chi([1,1,3,3],[2,2,2,2]))/8; 6 with(combinat):(Chi([1,7],[1,1,1,1,1,1,1,1])- Chi([1,7],[2,2,2,2]))/8; 1
with(combinat):(Chi([1,2,2,3],[1,1,1,1,1,1,1,1])- Chi([1,2,2,3],[2,2,2,2]))/8; 9 with(combinat):(Chi([1,3,4],[1,1,1,1,1,1,1,1])-
Chi([1,3,4],[2,2,2,2]))/8; 9 with(combinat):(Chi([1,2,5],[1,1,1,1,1,1,1,1])- Chi([1,2,5],[2,2,2,2]))/8; 8
with(combinat):(Chi([2,2,2,2],[1,1,1,1,1,1,1,1])- Chi([2,2,2,2],[2,2,2,2]))/8; 1 with(combinat):(Chi([2,2,4],[1,1,1,1,1,1,1,1])-
Chi([2,2,4],[2,2,2,2]))/8; 6 with(combinat):(Chi([2,3,3],[1,1,1,1,1,1,1,1])- Chi([2,3,3],[2,2,2,2]))/8; 6
with(combinat):(Chi([2,6],[1,1,1,1,1,1,1,1])- Chi([2,6],[2,2,2,2]))/8; 2 with(combinat):(Chi([8],[1,1,1,1,1,1,1,1])-
Chi([8],[2,2,2,2]))/8; 0
```

The above code calculate the irreducible component of Free Lie Algebra L_9:

```
with(combinat):(Chi([1,1,1,1,1,1,1,1,1],[1,1,1,1,1,1,1,1,1])- Chi([1,1,1,1,1,1,1,1,1],[3,3,3]))/9; 0
with(combinat):(Chi([1,1,1,1,1,1,1,2],[1,1,1,1,1,1,1,1,1])- Chi([1,1,1,1,1,1,1,2],[3,3,3]))/9; 1
```

with(combinat):(Chi([1,1,1,1,1,1,3],[1,1,1,1,1,1,1,1,1])- Chi([1,1,1,1,1,1,3],[3,3,3]))/9; 3
with(combinat):(Chi([1,1,1,1,1,4],[1,1,1,1,1,1,1,1,1])- Chi([1,1,1,1,1,4],[3,3,3]))/9; 6
with(combinat):(Chi([1,1,1,1,1,2,2],[1,1,1,1,1,1,1,1,1])- Chi([1,1,1,1,1,2,2],[3,3,3]))/9; 3
with(combinat):(Chi([1,1,1,1,5],[1,1,1,1,1,1,1,1,1])- Chi([1,1,1,1,5],[3,3,3]))/9; 8
with(combinat):(Chi([1,1,1,1,2,3],[1,1,1,1,1,1,1,1,1])- Chi([1,1,1,1,2,3],[3,3,3]))/9; 12
with(combinat):(Chi([1,1,1,6],[1,1,1,1,1,1,1,1,1])- Chi([1,1,1,6],[3,3,3]))/9; 6 with(combinat):(Chi([1,1,1,2,4],[1,1,1,1,1,1,1,1,1])- Chi([1,1,1,2,4],[3,3,3]))/9; 21 with(combinat):(Chi([1,1,1,2,2,2],[1,1,1,1,1,1,1,1,1])- Chi([1,1,1,2,2,2],[3,3,3]))/9; 5 with(combinat):(Chi([1,1,1,3,3],[1,1,1,1,1,1,1,1,1])- Chi([1,1,1,3,3],[3,3,3]))/9; 13 with(combinat):(Chi([1,1,7],[1,1,1,1,1,1,1,1,1])- Chi([1,1,7],[3,3,3]))/9; 3 with(combinat):(Chi([1,1,2,2,3],[1,1,1,1,1,1,1,1,1])- Chi([1,1,2,2,3],[3,3,3]))/9; 18 with(combinat):(Chi([3,6],[1,1,1,1,1,1,1,1,1])- Chi([3,6],[3,3,3]))/9; 5 with(combinat):(Chi([1,1,3,4],[1,1,1,1,1,1,1,1,1])- Chi([1,1,3,4],[3,3,3]))/9; 24 with(combinat):(Chi([1,1,2,5],[1,1,1,1,1,1,1,1,1])- Chi([1,1,2,5],[3,3,3]))/9; 21 with(combinat):(Chi([1,2,2,2,2],[1,1,1,1,1,1,1,1,1])- Chi([1,2,2,2,2],[3,3,3]))/9; 5 with(combinat):(Chi([1,2,2,4],[1,1,1,1,1,1,1,1,1])- Chi([1,2,2,4],[3,3,3]))/9; 24 with(combinat):(Chi([1,2,3,3],[1,1,1,1,1,1,1,1,1])- Chi([1,2,3,3],[3,3,3]))/9; 19 with(combinat):(Chi([1,2,6],[1,1,1,1,1,1,1,1,1])- Chi([1,2,6],[3,3,3]))/9; 12 with(combinat):(Chi([1,3,5],[1,1,1,1,1,1,1,1,1])- Chi([1,3,5],[3,3,3]))/9; 18 with(combinat):(Chi([1,4,4],[1,1,1,1,1,1,1,1,1])- Chi([1,4,4],[3,3,3]))/9; 9 with(combinat):(Chi([1,8],[1,1,1,1,1,1,1,1,1])- Chi([1,8],[3,3,3]))/9; 1 with(combinat):(Chi([1,2,3,3],[1,1,1,1,1,1,1,1,1])- Chi([1,2,3,3],[3,3,3]))/9; 19 with(combinat):(Chi([3,3,3],[1,1,1,1,1,1,1,1,1])- Chi([3,3,3],[3,3,3]))/9; 4 with(combinat):(Chi([2,7],[1,1,1,1,1,1,1,1,1])- Chi([2,7],[3,3,3]))/9; 3 with(combinat):(Chi([2,3,4],[1,1,1,1,1,1,1,1,1])- Chi([2,3,4],[3,3,3]))/9; 19 with(combinat):(Chi([2,2,2,3],[1,1,1,1,1,1,1,1,1])- Chi([2,2,2,3],[3,3,3]))/9; 9 with(combinat):(Chi([2,2,5],[1,1,1,1,1,1,1,1,1])- Chi([2,2,5],[3,3,3]))/9; 13 with(combinat):(Chi([4,5],[1,1,1,1,1,1,1,1,1])- Chi([4,5],[3,3,3]))/9; 5 with(combinat):(Chi([9],[1,1,1,1,1,1,1,1,1])- Chi([9],[3,3,3]))/9; 0

The above code calculate the irreducible component of Free Lie Algebra L_{10}:

with(combinat): (1/10)*(Chi([1,9],[1,1,1,1,1,1,1,1,1,1])-Chi([1,9],[2,2,2,2,2])-Chi([1,9],[5,5])+Chi([1,9],[10])); 1 with(combinat): (1/10)*(Chi([2,8],[1,1,1,1,1,1,1,1,1,1])-Chi([2,8],[2,2,2,2,2])-Chi([2,8],[5,5])+Chi([2,8],[10])); 3 with(combinat): (1/10)*(Chi([1,1,8],[1,1,1,1,1,1,1,1,1,1])-Chi([1,1,8],[2,2,2,2,2])-Chi([1,1,8],[5,5])+Chi([1,1,8],[10])); 4 with(combinat): (1/10)*(Chi([3,7],[1,1,1,1,1,1,1,1,1,1])-Chi([3,7],[2,2,2,2,2])-Chi([3,7],[5,5])+Chi([3,7],[10])); 8 with(combinat): (1/10)*(Chi([1,2,7],[1,1,1,1,1,1,1,1,1,1])-Chi([1,2,7],[2,2,2,2,2])-Chi([1,2,7],[5,5])+Chi([1,2,7],[10])); 16 with(combinat): (1/10)*(Chi([1,1,1,7],[1,1,1,1,1,1,1,1,1,1])-Chi([1,1,1,7],[2,2,2,2,2])-Chi([1,1,1,7],[5,5])+Chi([1,1,1,7],[10])); 8 with(combinat): (1/10)*(Chi([4,6],[1,1,1,1,1,1,1,1,1,1])-Chi([4,6],[2,2,2,2,2])-Chi([4,6],[5,5])+Chi([4,6],[10])); 8 with(combinat): (1/10)*(Chi([1,3,6],[1,1,1,1,1,1,1,1,1,1])-Chi([1,3,6],[2,2,2,2,2])-Chi([1,3,6],[5,5])+Chi([1,3,6],[10])); 32 with(combinat): (1/10)*(Chi([2,2,6],[1,1,1,1,1,1,1,1,1,1])-Chi([2,2,6],[2,2,2,2,2])-Chi([2,2,6],[5,5])+Chi([2,2,6],[10])); 21 with(combinat): (1/10)*(Chi([1,1,2,6],[1,1,1,1,1,1,1,1,1,1])-Chi([1,1,2,6],[2,2,2,2,2])-Chi([1,1,2,6],[5,5])+Chi([1,1,2,6],[10])); 36 with(combinat): (1/10)*(Chi([1,1,1,1,6],[1,1,1,1,1,1,1,1,1,1])-Chi([1,1,1,1,6],[2,2,2,2,2])-Chi([1,1,1,1,6],[5,5])+Chi([1,1,1,1,6],[10])); 12 with(combinat): (1/10)*(Chi([5,5],[1,1,1,1,1,1,1,1,1,1])-Chi([5,5],[2,2,2,2,2])-Chi([5,5],[5,5])+Chi([5,5],[10])); 5 with(combinat): (1/10)*(Chi([1,4,5],[1,1,1,1,1,1,1,1,1,1])-Chi([1,4,5],[2,2,2,2,2])-Chi([1,4,5],[5,5])+Chi([1,4,5],[10])); 29 with(combinat): (1/10)*(Chi([2,3,5],[1,1,1,1,1,1,1,1,1,1])-Chi([2,3,5],[2,2,2,2,2])-Chi([2,3,5],[5,5])+Chi([2,3,5],[10])); 46 with(combinat): (1/10)*(Chi([1,1,3,5],[1,1,1,1,1,1,1,1,1,1])-Chi([1,1,3,5],[2,2,2,2,2])-Chi([1,1,3,5],[5,5])+Chi([1,1,3,5],[10])); 55 with(combinat): (1/10)*(Chi([1,2,2,5],[1,1,1,1,1,1,1,1,1,1])-Chi([1,2,2,5],[2,2,2,2,2])-Chi([1,2,2,5],[5,5])+Chi([1,2,2,5],[10])); 53 with(combinat): (1/10)*(Chi([1,1,1,2,5],[1,1,1,1,1,1,1,1,1,1])-Chi([1,1,1,2,5],[2,2,2,2,2])-Chi([1,1,1,2,5],[5,5])+Chi([1,1,1,2,5],[10])); 45 with(combinat): (1/10)*(Chi([1,1,1,1,1,5],[1,1,1,1,1,1,1,1,1,1])-Chi([1,1,1,1,1,5],[2,2,2,2,2])-Chi([1,1,1,1,1,5],[5,5])+Chi([1,1,1,1,1,5],[10])); 13 with(combinat): (1/10)*(Chi([2,4,4],[1,1,1,1,1,1,1,1,1,1])-Chi([2,4,4],[2,2,2,2,2])-Chi([2,4,4],[5,5])+Chi([2,4,4],[10])); 23 with(combinat): (1/10)*(Chi([1,1,4,4],[1,1,1,1,1,1,1,1,1,1])-Chi([1,1,4,4],[2,2,2,2,2])-Chi([1,1,4,4],[5,5])+Chi([1,1,4,4],[10])); 32 with(combinat): (1/10)*(Chi([3,3,4],[1,1,1,1,1,1,1,1,1,1])-Chi([3,3,4],[2,2,2,2,2])-Chi([3,3,4],[5,5])+Chi([3,3,4],[10])); 22 with(combinat): (1/10)*(Chi([1,2,3,4],[1,1,1,1,1,1,1,1,1,1])-Chi([1,2,3,4],[2,2,2,2,2])-Chi([1,2,3,4],[5,5])+Chi([1,2,3,4],[10])); 77 with(combinat): (1/10)*(Chi([1,1,1,3,4],[1,1,1,1,1,1,1,1,1,1])-Chi([1,1,1,3,4],[2,2,2,2,2])-Chi([1,1,1,3,4],[5,5])+Chi([1,1,1,3,4],[10])); 52 with(combinat): (1/10)*(Chi([2,2,2,4],[1,1,1,1,1,1,1,1,1,1])-Chi([2,2,2,4],[2,2,2,2,2])-Chi([2,2,2,4],[5,5])+Chi([2,2,2,4],[10])); 28 with(combinat): (1/10)*(Chi([1,1,2,2,4],[1,1,1,1,1,1,1,1,1,1])-Chi([1,1,2,2,4],[2,2,2,2,2])-Chi([1,1,2,2,4],[5,5])+Chi([1,1,2,2,4],[10])); 58 with(combinat): (1/10)*(Chi([1,1,1,1,2,4],[1,1,1,1,1,1,1,1,1,1])-Chi([1,1,1,1,2,4],[2,2,2,2,2])-Chi([1,1,1,1,2,4],[5,5])+Chi([1,1,1,1,2,4],[10])); 34 with(combinat): (1/10)*(Chi([1,1,1,1,1,1,4],[1,1,1,1,1,1,1,1,1,1])-Chi([1,1,1,1,1,1,4],[2,2,2,2,2])-Chi([1,1,1,1,1,1,4],[5,5])+Chi([1,1,1,1,1,1,4],[10])); 9 with(combinat): (1/10)*(Chi([1,3,3,3],[1,1,1,1,1,1,1,1,1,1])-Chi([1,3,3,3],[2,2,2,2,2])-Chi([1,3,3,3],[5,5])+Chi([1,3,3,3],[10])); 20 with(combinat): (1/10)*(Chi([2,2,3,3],[1,1,1,1,1,1,1,1,1,1])-Chi([2,2,3,3],[2,2,2,2,2])-Chi([2,2,3,3],[5,5])+Chi([2,2,3,3],[10])); 27 with(combinat): (1/10)*(Chi([1,1,2,3,3],[1,1,1,1,1,1,1,1,1,1])-Chi([1,1,2,3,3],[2,2,2,2,2])-Chi([1,1,2,3,3],[5,5])+Chi([1,1,2,3,3],[10])); 44 with(combinat): (1/10)*(Chi([1,1,1,1,3,3],[1,1,1,1,1,1,1,1,1,1])-Chi([1,1,1,1,3,3],[2,2,2,2,2])-Chi([1,1,1,1,3,3],[5,5])+Chi([1,1,1,1,3,3],[10])); 24 with(combinat): (1/10)*(Chi([1,2,2,2,3],[1,1,1,1,1,1,1,1,1,1])-Chi([1,2,2,2,3],[2,2,2,2,2])-Chi([1,2,2,2,3],[5,5])+Chi([1,2,2,2,3],[10])); 29 with(combinat):

(1/10)*(Chi([1,1,1,2,2,3],[1,1,1,1,1,1,1,1,1,1])-Chi([1,1,1,2,2,3],[2,2,2,2,2])-Chi([1,1,1,2,2,3],[5,5])+Chi([1,1,1,2,2,3],[10])); 31
with(combinat):
(1/10)*(Chi([1,1,1,1,1,2,3],[1,1,1,1,1,1,1,1,1,1])-Chi([1,1,1,1,1,2,3],[2,2,2,2,2])-Chi([1,1,1,1,1,2,3],[5,5])+Chi([1,1,1,1,1,2,3],[10]));
16 with(combinat): (1/10)*(Chi([1,1,1,1,1,1,1,3],[1,1,1,1,1,1,1,1,1,1])-Chi([1,1,1,1,1,1,1,3],[2,2,2,2,2])-
Chi([1,1,1,1,1,1,1,3],[5,5])+Chi([1,1,1,1,1,1,1,3],[10])); 3 with(combinat):
(1/10)*(Chi([2,2,2,2,2],[1,1,1,1,1,1,1,1,1,1])-Chi([2,2,2,2,2],[2,2,2,2,2])-Chi([2,2,2,2,2],[5,5])+Chi([2,2,2,2,2],[10])); 3
with(combinat):
(1/10)*(Chi([1,1,2,2,2,2],[1,1,1,1,1,1,1,1,1,1])-Chi([1,1,2,2,2,2],[2,2,2,2,2])-Chi([1,1,2,2,2,2],[5,5])+Chi([1,1,2,2,2,2],[10])); 10
with(combinat):
(1/10)*(Chi([1,1,1,1,2,2,2],[1,1,1,1,1,1,1,1,1,1])-Chi([1,1,1,1,2,2,2],[2,2,2,2,2])-Chi([1,1,1,1,2,2,2],[5,5])+Chi([1,1,1,1,2,2,2],[10])); 7
with(combinat): (1/10)*(Chi([1,1,1,1,1,1,2,2],[1,1,1,1,1,1,1,1,1,1])-Chi([1,1,1,1,1,1,2,2],[2,2,2,2,2])-
Chi([1,1,1,1,1,1,2,2],[5,5])+Chi([1,1,1,1,1,1,2,2],[10])); 4 with(combinat):
(1/10)*(Chi([1,1,1,1,1,1,1,1,2],[1,1,1,1,1,1,1,1,1,1])-Chi([1,1,1,1,1,1,1,1,2],[2,2,2,2,2])-
Chi([1,1,1,1,1,1,1,1,2],[5,5])+Chi([1,1,1,1,1,1,1,1,2],[10])); 1 with(combinat):
(1/10)*(Chi([1,1,1,1,1,1,1,1,1,1],[1,1,1,1,1,1,1,1,1,1])-Chi([1,1,1,1,1,1,1,1,1,1],[2,2,2,2,2])-
Chi([1,1,1,1,1,1,1,1,1,1],[5,5])+Chi([1,1,1,1,1,1,1,1,1,1],[10])); 0 with(combinat):
(1/10)*(Chi([10],[1,1,1,1,1,1,1,1,1,1])-Chi([10],[2,2,2,2,2])-Chi([10],[5,5])+Chi([10],[10])); 0

The above code calculate the irreducible component of Free Lie Algebra L_{11}:

with(combinat):(1*Chi([1,1,1,1,1,1,1,1,1,2],[1,1,1,1,1,1,1,1,1,1,1])+(-1)*Chi([1,1,1,1,1,1,1,1,1,2],[11]))/11; 1
with(combinat):(1*Chi([1,1,1,1,1,1,1,2,2],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([1,1,1,1,1,1,1,2,2],[11]))/11; 4
with(combinat):(1*Chi([1,1,1,1,1,2,2,2],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([1,1,1,1,1,2,2,2],[11]))/11; 10
with(combinat):(1*Chi([1,1,1,2,2,2,2],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([1,1,1,2,2,2,2],[11]))/11; 15
with(combinat):(1*Chi([1,1,1,2,2,2,2],[1,1,1,1,1,1,1,1,1,1,1])+(-1)*Chi([1,1,1,2,2,2,2],[11]))/11; 15
with(combinat):(1*Chi([1,1,1,1,1,1,1,1,3],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([1,1,1,1,1,1,1,1,3],[11]))/11; 4
with(combinat):(1*Chi([1,1,1,1,1,1,2,3],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([1,1,1,1,1,1,2,3],[11]))/11; 21
with(combinat):(1*Chi([1,1,1,1,2,2,3],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([1,1,1,1,2,2,3],[11]))/11; 50
with(combinat):(1*Chi([1,1,2,2,2,3],[1,1,1,1,1,1,1,1,1,1,1])+(-1)*Chi([1,1,2,2,2,3],[11]))/11; 63
with(combinat):(1*Chi([2,2,2,2,3],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([2,2,2,2,3],[11]))/11; 30
with(combinat):(1*Chi([1,1,1,1,1,3,3],[1,1,1,1,1,1,1,1,1,1,1])+(-1)*Chi([1,1,1,1,1,3,3],[11]))/11; 35
with(combinat):(1*Chi([1,1,1,2,3,3],[1,1,1,1,1,1,1,1,1,1,1])+(-1)*Chi([1,1,1,2,3,3],[11]))/11; 90
with(combinat):(1*Chi([1,2,2,3,3],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([1,2,2,3,3],[11]))/11; 90
with(combinat):(1*Chi([1,1,3,3,3],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([1,1,3,3,3],[11]))/11; 60
with(combinat):(1*Chi([2,3,3,3],[1,1,1,1,1,1,1,1,1,1,1])+(-1)*Chi([2,3,3,3],[11]))/11; 42
with(combinat):(1*Chi([1,1,1,1,1,1,1,4],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([1,1,1,1,1,1,1,4],[11]))/11; 11
with(combinat):(1*Chi([1,1,1,1,1,2,4],[1,1,1,1,1,1,1,1,1,1,1])+(-1)*Chi([1,1,1,1,1,2,4],[11]))/11; 54
with(combinat):(1*Chi([1,1,1,2,2,4],[1,1,1,1,1,1,1,1,1,1,1])+(-1)*Chi([1,1,1,2,2,4],[11]))/11; 112
with(combinat):(1*Chi([1,2,2,2,4],[1,1,1,1,1,1,1,1,1,1,1])+(-1)*Chi([1,2,2,2,4],[11]))/11; 105
with(combinat):(1*Chi([1,1,1,1,3,4],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([1,1,1,1,3,4],[11]))/11; 100
with(combinat):(1*Chi([1,1,2,3,4],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([1,1,2,3,4],[11]))/11; 210
with(combinat):(1*Chi([2,2,3,4],[1,1,1,1,1,1,1,1,1,1,1])+(-1)*Chi([2,2,3,4],[11]))/11; 120
with(combinat):(1*Chi([1,3,3,4],[1,1,1,1,1,1,1,1,1,1,1])+(-1)*Chi([1,3,3,4],[11]))/11; 108
with(combinat):(1*Chi([1,1,1,4,4],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([1,1,1,4,4],[11]))/11; 75
with(combinat):(1*Chi([1,2,4,4],[1,1,1,1,1,1,1,1,1,1,1])+(-1)*Chi([1,2,4,4],[11]))/11; 120
with(combinat):(1*Chi([3,4,4],[1,1,1,1,1,1,1,1,1,1,1])+(-1)*Chi([3,4,4],[11]))/11; 42
with(combinat):(1*Chi([1,1,1,1,1,1,5],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([1,1,1,1,1,1,5],[11]))/11; 19
with(combinat):(1*Chi([1,1,1,1,2,5],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([1,1,1,1,2,5],[11]))/11; 84
with(combinat):(1*Chi([1,1,2,2,5],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([1,1,2,2,5],[11]))/11; 140
with(combinat):(1*Chi([2,2,2,5],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([2,2,2,5],[11]))/11; 75
with(combinat):(1*Chi([1,1,1,3,5],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([1,1,1,3,5],[11]))/11; 140
with(combinat):(1*Chi([1,2,3,5],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([1,2,3,5],[11]))/11; 210
with(combinat):(1*Chi([3,3,5],[1,1,1,1,1,1,1,1,1,1,1])+(-1)*Chi([3,3,5],[11]))/11; 60
with(combinat):(1*Chi([1,1,4,5],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([1,1,4,5],[11]))/11; 105
with(combinat):(1*Chi([2,4,5],[1,1,1,1,1,1,1,1,1,1,1])+(-1)*Chi([2,4,5],[11]))/11; 90
with(combinat):(1*Chi([1,5,5],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([1,5,5],[11]))/11; 30
with(combinat):(1*Chi([1,1,1,1,1,6],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([1,1,1,1,1,6],[11]))/11; 23
with(combinat):(1*Chi([1,1,1,2,6],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([1,1,1,2,6],[11]))/11; 84
with(combinat):(1*Chi([1,2,2,6],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([1,2,2,6],[11]))/11; 100
with(combinat):(1*Chi([1,1,3,6],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([1,1,3,6],[11]))/11; 112
with(combinat):(1*Chi([2,3,6],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([2,3,6],[11]))/11; 90
with(combinat):(1*Chi([1,4,6],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([1,4,6],[11]))/11; 63
with(combinat):(1*Chi([5,6],[1,1,1,1,1,1,1,1,1,1,1])+(-1)*Chi([5,6],[11]))/11; 12
with(combinat):(1*Chi([1,1,1,1,7],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([1,1,1,1,7],[11]))/11; 19
with(combinat):(1*Chi([1,1,2,7],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([1,1,2,7],[11]))/11; 54

with(combinat):(1*Chi([2,2,7],[1,1,1,1,1,1,1,1,1,1,1])+(-1)*Chi([2,2,7],[11]))/11; 35
with(combinat):(1*Chi([1,3,7],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([1,3,7],[11]))/11; 50
with(combinat):(1*Chi([4,7],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([4,7],[11]))/11; 15
with(combinat):(1*Chi([1,1,1,8],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([1,1,1,8],[11]))/11; 11
with(combinat):(1*Chi([1,2,8],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([1,2,8],[11]))/11; 21
with(combinat):(1*Chi([3,8],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([3,8],[11]))/11; 10
with(combinat):(1*Chi([1,1,9],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([1,1,9],[11]))/11; 4
with(combinat):(1*Chi([2,9],[1,1,1,1,1,1,1,1,1,1,1])+(-1)*Chi([2,9],[11]))/11; 4
with(combinat):(1*Chi([1,10],[1,1,1,1,1,1,1,1,1,1,1])+(-1)*Chi([1,10],[11]))/11; 1
with(combinat):(1*Chi([11],[1,1,1,1,1,1,1,1,1,1,1])+(-1)* Chi([11],[11]))/11; 0
with(combinat):(1*Chi([1,1,1,1,1,1,1,1,1,1,1],[1,1,1,1,1,1,1,1,1,1,1])+(-1)*Chi([1,1,1,1,1,1,1,1,1,1,1],[11]))/11; 0
with(combinat):(1*Chi([1,2,2,2,2,2],[1,1,1,1,1,1,1,1,1,1,1])+(-1)*Chi([1,2,2,2,2,2],[11]))/11; 12

The above code calculate the irreducible component of Free Lie Algebra L_{12}:

with(combinat):(1/12)*(Chi([1,1,1,1,1,1,1,1,1,1,1,1],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([1,1,1,1,1,1,1,1,1,1,1,1],[2,2,2,2,2,2])-
Chi([1,1,1,1,1,1,1,1,1,1,1,1],[3,3,3,3])+Chi([1,1,1,1,1,1,1,1,1,1,1,1],[6,6])); 0
(1/12)*(Chi([1,1,1,1,1,1,1,1,1,1,2],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([1,1,1,1,1,1,1,1,1,1,2],[2,2,2,2,2,2])-
Chi([1,1,1,1,1,1,1,1,1,1,2],[3,3,3,3])+Chi([1,1,1,1,1,1,1,1,1,1,2],[6,6])); 1 (1/12)*(Chi([1,1,1,1,1,1,1,1,2,2],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([1,1,1,1,1,1,1,1,2,2],[2,2,2,2,2,2])- Chi([1,1,1,1,1,1,1,1,2,2],[3,3,3,3])+Chi([1,1,1,1,1,1,1,1,2,2],[6,6])); 4
(1/12)*(Chi([1,1,1,1,1,1,2,2,2],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([1,1,1,1,1,1,2,2,2],[2,2,2,2,2,2])-
Chi([1,1,1,1,1,1,2,2,2],[3,3,3,3])+Chi([1,1,1,1,1,1,2,2,2],[6,6])); 13 (1/12)*(Chi([1,1,1,1,2,2,2,2],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([1,1,1,1,2,2,2,2],[2,2,2,2,2,2])- Chi([1,1,1,1,2,2,2,2],[3,3,3,3])+Chi([1,1,1,1,2,2,2,2],[6,6])); 22
(1/12)*(Chi([1,1,2,2,2,2,2],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([1,1,2,2,2,2,2],[2,2,2,2,2,2])-
Chi([1,1,2,2,2,2,2],[3,3,3,3])+Chi([1,1,2,2,2,2,2],[6,6])); 26 (1/12)*(Chi([2,2,2,2,2,2],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([2,2,2,2,2,2],[2,2,2,2,2,2])- Chi([2,2,2,2,2,2],[3,3,3,3])+Chi([2,2,2,2,2,2],[6,6])); 9
(1/12)*(Chi([1,1,1,1,1,1,1,1,1,3],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([1,1,1,1,1,1,1,1,1,3],[2,2,2,2,2,2])-
Chi([1,1,1,1,1,1,1,1,1,3],[3,3,3,3])+Chi([1,1,1,1,1,1,1,1,1,3],[6,6])); 5 (1/12)*(Chi([1,1,1,1,1,1,1,2,3],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([1,1,1,1,1,1,1,2,3],[2,2,2,2,2,2])- Chi([1,1,1,1,1,1,1,2,3],[3,3,3,3])+Chi([1,1,1,1,1,1,1,2,3],[6,6])); 27
(1/12)*(Chi([1,1,1,1,1,2,2,3],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([1,1,1,1,1,2,2,3],[2,2,2,2,2,2])-
Chi([1,1,1,1,1,2,2,3],[3,3,3,3])+Chi([1,1,1,1,1,2,2,3],[6,6])); 75 (1/12)*(Chi([1,1,1,2,2,2,3],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([1,1,1,2,2,2,3],[2,2,2,2,2,2])- Chi([1,1,1,2,2,2,3],[3,3,3,3])+Chi([1,1,1,2,2,2,3],[6,6])); 117
(1/12)*(Chi([1,2,2,2,2,3],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([1,2,2,2,2,3],[2,2,2,2,2,2])-
Chi([1,2,2,2,2,3],[3,3,3,3])+Chi([1,2,2,2,2,3],[6,6])); 97 with(combinat):(1/12)*(Chi([1,1,1,1,2,3,3],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([1,1,1,1,2,3,3],[2,2,2,2,2,2])- Chi([1,1,1,1,2,3,3],[3,3,3,3])+Chi([1,1,1,1,2,3,3],[6,6])); 162
(1/12)*(Chi([1,1,1,1,1,1,3,3],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([1,1,1,1,1,1,3,3],[2,2,2,2,2,2])-
Chi([1,1,1,1,1,1,3,3],[3,3,3,3])+Chi([1,1,1,1,1,1,3,3],[6,6])); 49 (1/12)*(Chi([1,1,2,2,3,3],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([1,1,2,2,3,3],[2,2,2,2,2,2])- Chi([1,1,2,2,3,3],[3,3,3,3])+Chi([1,1,2,2,3,3],[6,6])); 219
(1/12)*(Chi([2,2,2,3,3],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([2,2,2,3,3],[2,2,2,2,2,2])- Chi([2,2,2,3,3],[3,3,3,3])+Chi([2,2,2,3,3],[6,6])); 113
(1/12)*(Chi([1,1,1,3,3,3],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([1,1,1,3,3,3],[2,2,2,2,2,2])-
Chi([1,1,1,3,3,3],[3,3,3,3])+Chi([1,1,1,3,3,3],[6,6])); 139 (1/12)*(Chi([1,2,3,3,3],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([1,2,3,3,3],[2,2,2,2,2,2])- Chi([1,2,3,3,3],[3,3,3,3])+Chi([1,2,3,3,3],[6,6])); 177 (1/12)*(Chi([3,3,3,3],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([3,3,3,3],[2,2,2,2,2,2])- Chi([3,3,3,3],[3,3,3,3])+Chi([3,3,3,3],[6,6])); 35 (1/12)*(Chi([1,1,1,1,1,1,1,1,4],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([1,1,1,1,1,1,1,1,4],[2,2,2,2,2,2])- Chi([1,1,1,1,1,1,1,1,4],[3,3,3,3])+Chi([1,1,1,1,1,1,1,1,4],[6,6])); 13
ith(combinat):(1/12)*(Chi([1,1,1,1,1,1,2,4],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([1,1,1,1,1,1,2,4],[2,2,2,2,2,2])-
Chi([1,1,1,1,1,1,2,4],[3,3,3,3])+Chi([1,1,1,1,1,1,2,4],[6,6])); 80 (1/12)*(Chi([1,1,1,1,2,2,4],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([1,1,1,1,2,2,4],[2,2,2,2,2,2])- Chi([1,1,1,1,2,2,4],[3,3,3,3])+Chi([1,1,1,1,2,2,4],[6,6])); 196
(1/12)*(Chi([1,1,2,2,2,4],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([1,1,2,2,2,4],[2,2,2,2,2,2])-
Chi([1,1,2,2,2,4],[3,3,3,3])+Chi([1,1,2,2,2,4],[6,6])); 260 (1/12)*(Chi([2,2,2,2,4],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([2,2,2,2,4],[2,2,2,2,2,2])- Chi([2,2,2,2,4],[3,3,3,3])+Chi([2,2,2,2,4],[6,6])); 120
(1/12)*(Chi([1,1,1,1,1,3,4],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([1,1,1,1,1,3,4],[2,2,2,2,2,2])-
Chi([1,1,1,1,1,3,4],[3,3,3,3])+Chi([1,1,1,1,1,3,4],[6,6])); 174 (1/12)*(Chi([1,1,1,2,3,4],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([1,1,1,2,3,4],[2,2,2,2,2,2])- Chi([1,1,1,2,3,4],[3,3,3,3])+Chi([1,1,1,2,3,4],[6,6])); 470
(1/12)*(Chi([1,2,2,3,4],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([1,2,2,3,4],[2,2,2,2,2,2])- Chi([1,2,2,3,4],[3,3,3,3])+Chi([1,2,2,3,4],[6,6])); 481
(1/12)*(Chi([1,1,3,3,4],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([1,1,3,3,4],[2,2,2,2,2,2])- Chi([1,1,3,3,4],[3,3,3,3])+Chi([1,1,3,3,4],[6,6])); 344
(1/12)*(Chi([1,1,3,3,4],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([1,1,3,3,4],[2,2,2,2,2,2])- Chi([1,1,3,3,4],[3,3,3,3])+Chi([1,1,3,3,4],[6,6])); 344
(1/12)*(Chi([1,1,1,1,4,4],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([1,1,1,1,4,4],[2,2,2,2,2,2])-
Chi([1,1,1,1,4,4],[3,3,3,3])+Chi([1,1,1,1,4,4],[6,6])); 156 (1/12)*(Chi([1,1,2,4,4],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([1,1,2,4,4],[2,2,2,2,2,2])- Chi([1,1,2,4,4],[3,3,3,3])+Chi([1,1,2,4,4],[6,6])); 375 (1/12)*(Chi([2,2,4,4],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([2,2,4,4],[2,2,2,2,2,2])- Chi([2,2,4,4],[3,3,3,3])+Chi([2,2,4,4],[6,6])); 214 (1/12)*(Chi([1,3,4,4],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([1,3,4,4],[2,2,2,2,2,2])- Chi([1,3,4,4],[3,3,3,3])+Chi([1,3,4,4],[6,6])); 250 (1/12)*(Chi([4,4,4],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([4,4,4],[2,2,2,2,2,2])- Chi([4,4,4],[3,3,3,3])+Chi([4,4,4],[6,6])); 35 (1/12)*(Chi([1,1,1,1,1,1,1,5],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([1,1,1,1,1,1,1,5],[2,2,2,2,2,2])- Chi([1,1,1,1,1,1,1,5],[3,3,3,3])+Chi([1,1,1,1,1,1,1,5],[6,6])); 27
(1/12)*(Chi([1,1,1,1,1,2,5],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([1,1,1,1,1,2,5],[2,2,2,2,2,2])-
Chi([1,1,1,1,1,2,5],[3,3,3,3])+Chi([1,1,1,1,1,2,5],[6,6])); 144 (1/12)*(Chi([1,1,1,2,2,5],[1,1,1,1,1,1,1,1,1,1,1,1])-

Chi([1,1,1,2,2,5],[2,2,2,2,2,2])- Chi([1,1,1,2,2,5],[3,3,3,3])+Chi([1,1,1,2,2,5],[6,6])); 306
(1/12)*(Chi([1,2,2,2,5],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([1,2,2,2,5],[2,2,2,2,2,2])- Chi([1,2,2,2,5],[3,3,3,3])+Chi([1,2,2,2,5],[6,6])); 294
(1/12)*(Chi([1,1,1,1,3,5],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([1,1,1,1,3,5],[2,2,2,2,2,2])-
Chi([1,1,1,1,3,5],[3,3,3,3])+Chi([1,1,1,1,3,5],[6,6])); 300 (1/12)*(Chi([1,1,2,3,5],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([1,1,2,3,5],[2,2,2,2,2,2])- Chi([1,1,2,3,5],[3,3,3,3])+Chi([1,1,2,3,5],[6,6])); 640 (1/12)*(Chi([2,2,3,5],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([2,2,3,5],[2,2,2,2,2,2])- Chi([2,2,3,5],[3,3,3,3])+Chi([2,2,3,5],[6,6])); 375 (1/12)*(Chi([1,3,3,5],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([1,3,3,5],[2,2,2,2,2,2])- Chi([1,3,3,5],[3,3,3,3])+Chi([1,3,3,5],[6,6])); 344 (1/12)*(Chi([1,1,1,4,5],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([1,1,1,4,5],[2,2,2,2,2,2])- Chi([1,1,1,4,5],[3,3,3,3])+Chi([1,1,1,4,5],[6,6])); 294 (1/12)*(Chi([1,2,4,5],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([1,2,4,5],[2,2,2,2,2,2])- Chi([1,2,4,5],[3,3,3,3])+Chi([1,2,4,5],[6,6])); 481 (1/12)*(Chi([3,4,5],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([3,4,5],[2,2,2,2,2,2])- Chi([3,4,5],[3,3,3,3])+Chi([3,4,5],[6,6])); 177 (1/12)*(Chi([1,1,5,5],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([1,1,5,5],[2,2,2,2,2,2])- Chi([1,1,5,5],[3,3,3,3])+Chi([1,1,5,5],[6,6])); 120 (1/12)*(Chi([2,5,5],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([2,5,5],[2,2,2,2,2,2])- Chi([2,5,5],[3,3,3,3])+Chi([2,5,5],[6,6])); 113 (1/12)*(Chi([1,1,1,1,1,1,6],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([1,1,1,1,1,1,6],[2,2,2,2,2,2])- Chi([1,1,1,1,1,1,6],[3,3,3,3])+Chi([1,1,1,1,1,1,6],[6,6])); 39
(1/12)*(Chi([1,1,1,1,2,6],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([1,1,1,1,2,6],[2,2,2,2,2,2])-
Chi([1,1,1,1,2,6],[3,3,3,3])+Chi([1,1,1,1,2,6],[6,6])); 174 (1/12)*(Chi([1,1,2,2,6],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([1,1,2,2,6],[2,2,2,2,2,2])- Chi([1,1,2,2,6],[3,3,3,3])+Chi([1,1,2,2,6],[6,6])); 300 (1/12)*(Chi([2,2,2,6],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([2,2,2,6],[2,2,2,2,2,2])- Chi([2,2,2,6],[3,3,3,3])+Chi([2,2,2,6],[6,6])); 156 (1/12)*(Chi([1,1,1,3,6],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([1,1,1,3,6],[2,2,2,2,2,2])- Chi([1,1,1,3,6],[3,3,3,3])+Chi([1,1,1,3,6],[6,6])); 306 (1/12)*(Chi([1,2,3,6],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([1,2,3,6],[2,2,2,2,2,2])- Chi([1,2,3,6],[3,3,3,3])+Chi([1,2,3,6],[6,6])); 470 (1/12)*(Chi([3,3,6],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([3,3,6],[2,2,2,2,2,2])- Chi([3,3,6],[3,3,3,3])+Chi([3,3,6],[6,6])); 139 (1/12)*(Chi([1,1,4,6],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([1,1,4,6],[2,2,2,2,2,2])- Chi([1,1,4,6],[3,3,3,3])+Chi([1,1,4,6],[6,6])); 260 (1/12)*(Chi([2,4,6],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([2,4,6],[2,2,2,2,2,2])- Chi([2,4,6],[3,3,3,3])+Chi([2,4,6],[6,6])); 219 (1/12)*(Chi([1,5,6],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([1,5,6],[2,2,2,2,2,2])- Chi([1,5,6],[3,3,3,3])+Chi([1,5,6],[6,6])); 97 (1/12)*(Chi([6,6],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([6,6],[2,2,2,2,2,2])- Chi([6,6],[3,3,3,3])+Chi([6,6],[6,6])); 9 (1/12)*(Chi([1,1,1,1,1,7],[1,1,1,1,1,1,1,1,1,1,1,1])-
Chi([1,1,1,1,1,7],[2,2,2,2,2,2])- Chi([1,1,1,1,1,7],[3,3,3,3])+Chi([1,1,1,1,1,7],[6,6])); 39
(1/12)*(Chi([1,1,1,2,7],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([1,1,1,2,7],[2,2,2,2,2,2])- Chi([1,1,1,2,7],[3,3,3,3])+Chi([1,1,1,2,7],[6,6])); 144
(1/12)*(Chi([1,2,2,7],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([1,2,2,7],[2,2,2,2,2,2])- Chi([1,2,2,7],[3,3,3,3])+Chi([1,2,2,7],[6,6])); 174
(1/12)*(Chi([1,1,3,7],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([1,1,3,7],[2,2,2,2,2,2])- Chi([1,1,3,7],[3,3,3,3])+Chi([1,1,3,7],[6,6])); 196
(1/12)*(Chi([2,3,7],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([2,3,7],[2,2,2,2,2,2])- Chi([2,3,7],[3,3,3,3])+Chi([2,3,7],[6,6])); 162
(1/12)*(Chi([1,4,7],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([1,4,7],[2,2,2,2,2,2])- Chi([1,4,7],[3,3,3,3])+Chi([1,4,7],[6,6])); 117
(1/12)*(Chi([5,7],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([5,7],[2,2,2,2,2,2])- Chi([5,7],[3,3,3,3])+Chi([5,7],[6,6])); 26
(1/12)*(Chi([1,1,1,1,8],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([1,1,1,1,8],[2,2,2,2,2,2])- Chi([1,1,1,1,8],[3,3,3,3])+Chi([1,1,1,1,8],[6,6])); 27
(1/12)*(Chi([1,1,2,8],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([1,1,2,8],[2,2,2,2,2,2])- Chi([1,1,2,8],[3,3,3,3])+Chi([1,1,2,8],[6,6])); 80
(1/12)*(Chi([2,2,8],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([2,2,8],[2,2,2,2,2,2])- Chi([2,2,8],[3,3,3,3])+Chi([2,2,8],[6,6])); 49
(1/12)*(Chi([1,3,8],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([1,3,8],[2,2,2,2,2,2])- Chi([1,3,8],[3,3,3,3])+Chi([1,3,8],[6,6])); 75
(1/12)*(Chi([4,8],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([4,8],[2,2,2,2,2,2])- Chi([4,8],[3,3,3,3])+Chi([4,8],[6,6])); 22
(1/12)*(Chi([1,1,1,9],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([1,1,1,9],[2,2,2,2,2,2])- Chi([1,1,1,9],[3,3,3,3])+Chi([1,1,1,9],[6,6])); 13
(1/12)*(Chi([1,2,9],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([1,2,9],[2,2,2,2,2,2])- Chi([1,2,9],[3,3,3,3])+Chi([1,2,9],[6,6])); 27
(1/12)*(Chi([3,9],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([3,9],[2,2,2,2,2,2])- Chi([3,9],[3,3,3,3])+Chi([3,9],[6,6])); 13
(1/12)*(Chi([1,1,10],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([1,1,10],[2,2,2,2,2,2])- Chi([1,1,10],[3,3,3,3])+Chi([1,1,10],[6,6])); 5
(1/12)*(Chi([2,10],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([2,10],[2,2,2,2,2,2])- Chi([2,10],[3,3,3,3])+Chi([2,10],[6,6])); 4
(1/12)*(Chi([1,11],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([1,11],[2,2,2,2,2,2])- Chi([1,11],[3,3,3,3])+Chi([1,11],[6,6])); 1
(1/12)*(Chi([12],[1,1,1,1,1,1,1,1,1,1,1,1])- Chi([12],[2,2,2,2,2,2])- Chi([12],[3,3,3,3])+Chi([12],[6,6])); 0