MINIMAL SURFACES ON THREE-DIMENSIONAL WALKER MANIFOLDS

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

MINIMAL SURFACES ON THREE-DIMENSIONAL WALKER MANIFOLDS

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Lorentzian Geometry has shown to be very useful in a wide range of studies including many diverse research fields, especially in the theory of general relativity and mathematical cosmology. A Walker manifold descends from the structure of Lorentzian manifolds which is characterized by admitting a parallel degenerate distribution.

In the present thesis, we investigate and derive the equations of minimal surfaces on three-dimensional Walker manifolds, with a particular interest on those surfaces which are represented by the graph of a smooth function. Our study is closely related with (Lorentzian) isothermal coordinates which provide an easier approach for deriving such equations, and they are locally defined for any surface on the underlying manifold. By using the well-known property of vanishing mean curvature for minimal surfaces, together with the geometric restrictions posed by the chosen coordinates, we obtain a class of graphs of functions which are minimal under certain conditions on the corresponding function.

Keywords: Walker manifold, minimal surface, isothermal coordinates.

ÖZET

ÜÇ BOYUTLU WALKER MANİFOLDLARDA MINIMAL YÜZEYLER

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Matematik, Yüksek Lisans
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Lorentz Geometrisi'nin özellikle genel görelilik ve matematiksel kozmolojiyi içeren üzere, birçok farklı, gelişen araştırma alanlarında yararlı olduğu görülmüştür. Paralel dejenere dağılımı karakterize kabul eden Walker manifoldları, Lorentz manifold yapısından türemiştir.

Bu tezde, üç boyutlu Walker manifoldlar üzerindeki minimal yüzeyler üzerine çalıştık ve bu yapılar üzerindeki minimal yüzeyler için denklemler türettik. Özellikle, düzgün bir fonksiyon grafiği tarafından temsil edilen yüzeyler üzerinde çalıştık. Çalışmamız, bu denklemleri türetmek için kolay bir yöntem sağlayan Lorentz izotermal (eşışıl) koordinatlarla yakından ilgilidir ve bu koordinatlar manifold üzerindeki her yüzey için bölgesel olarak tanımlanmıştır. Minimal yüzeylerde ortalama eğriliğin sıfır olduğu gerçeğini ve seçilen koordinatlarla konumlanmış geometrik sınırlamayı kullanarak, karşılık gelen fonksiyonun belli şartlardaki minimal fonksiyon grafik sınıflarını elde ettik.

Anahtar sözcükler: Walker manifoldu, minimal yüzeyler, izotermal koordinatlar.

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“What you seek is seeking you”

∼ Rumi

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Chapter 1

Introduction

The study of indefinite metrics has given rise to the pseudo-Riemannian (also known as semi-Riemannian) geometry which has shown to have a lot of applications especially in physical contexts such as general relativity and string theory [1, 2]. Astonishingly, it seems like part of the universe can even be represented by the models established via the Lorentzian geometry which represents a particular case of pseudo-Riemannian geometry. More explicitly, Lorentzian manifolds are smooth manifolds which are furnished with the Lorentzian metric, i.e. the pseudo-Riemannian metric that has signature $(1, q)$. Those manifolds are exploited as cosmological models to predict phenomena happening on the planets scale such as the Big Bang and the universe expansion [2] (More detailed information about Lorentzian Geometry can be found in Chapter 2.) One of these interesting manifolds are the pseudo-Riemannian manifolds which admit a *parallel degenerate line field*. They appear naturally in many diverse physical settings as well. Namely, among many other applications they establish the underlying structure of pp-waves models [3, 4] and are used in the study of hh-spaces in general relativity [5]. Walker studied these manifolds and in 1950, he described their local structure by deriving a canonical form [6] which played a central role for further investigations [7, 8, 9]. Motivated by his work, a semi-Riemannian manifold which admits a parallel degenerate line field is called a *Walker manifold*.

Walker's result aroused curiosity among researchers to investigate more about the Walker geometry which has provided a rich source of interesting geometric topics. For instance, curvature properties and a complete characterization of locally symmetric or locally conformally flat three-dimensional Walker manifolds have been studied in detail by Chaichi, García-Río and Vázquez-Abal in [7]. Calvaruso and Van der Veken obtained a classification of parallel surfaces with a Walker manifold as the ambient space [9]. Also in [8], Calvaruso and De Leo investigated Ricci solitons on three-dimensional Walker manifolds where they showed that the existence of non-trivial Ricci solitons depends on the defining function f of the manifold.

On the other hand, minimal surfaces, i.e. the surfaces with vanishing mean curvature everywhere, have been an interesting area of research since the eighteenth century. Indeed, the first research goes back to J.L. Lagrange (1768) who considered the variational problem of finding a surface of least area bounded by a given closed countour. Later, G. Monge (1776) was the one who discovered that the area-minimizing condition on a surface leads to the condition $H = 0$, which justifies the notion *minimal* of the surfaces whose mean curvature vanishes identically [10]. During the nineteenth century, major achivements in classifying and constructing minimal surfaces on Euclidean 3-space were accomplished by K. Weierstrass, Lie, Riemann, Schwarz and others whose works resulted in the establishment of a strong connection between minimal surfaces and *holomorphic functions* [10]. A completely different way of approach to minimal surfaces was presented by Bernstein [11, 12] when he considered those surfaces purely from the point of view of partial differential equations.

Furthermore, interesting flourishing of the minimal surface theory has arisen in the last two decades, partly due to generalization directions which involve evolving the theory into Riemannian spaces as well as applying it to higher dimensions (which in turn gave rise to widening the surface classes). For example, the Weirestrass results on 3-dimesional Euclidean space have been extended to 3-dimensional Minkowski space by Kobayashi for spacelike minimal surfaces [13], and by Konderak for the case of timelike minimal surfaces [14]. Moreover, these theorems were particularly obtained for simply connected immersed minimal surfaces in the three-dimensional Heisenberg group $\mathbb{H}^3$ and in the product $\mathbb{H}^2 \times \mathbb{R}$ of the hyperbolic plane with the real line [15]. An another significant result was obtained recently by Lira and Mercuri [16] by making a further extension of Weierstrass representation formulas for immersed minimal surfaces in Riemannian and Lorentzian three-dimensional manifolds.

Motivated by the works and results mentioned above, in this thesis we will study the minimal surfaces on three-dimensional Walker manifolds, with a particular interest on surfaces which are represented by the graph of a function. As a brief outline, we begin the second chapter by an introduction of semi-Riemannian geometry and its properties. We particularly study the Lorentzian geometry by introducing the Lorentz-Minkowski space. We then study more closely the surfaces on this space, and we define the mean and Gauss curvatures together with some examples. In chapter 3, we finally define the Walker manifolds and next we explain their geometric structure. Finally, we present our main result; we derive the equations of minimal surfaces that are represented by the graph of a function on a 3-dimensional Walker manifold.

Chapter 2

Preliminaries

In this chapter, before studying the surfaces in three-dimensional Minkowski space, we will first recall some basic definitions and results related to semi-Riemannian Geometry. O'Neill and Tilla put forward these basic definitions in their classical books [2, 17] which were nicely summarized by López in his paper [18].

2.1 Semi-Riemannian Geometry

Let V be an arbitrary vector space of dimension $n \geq 1$ over $\mathbb{R}$. Then a bilinear form on V is an $\mathbb{R}$-bilinear function $l : V \times V \longrightarrow \mathbb{R}$. The form l is symmetric if $l(w, v) = l(v, w)$ for all v and w in V.

Definition 2.1.1. A symmetric bilinear form l on V is called

(1) *positive definite* provided $v \neq 0$ implies $l(v, v) > 0$

(2) *negative definite* provided $v \neq 0$ implies $l(v, v) < 0$

(3) *nondegenerate* provided $l(v, w) = 0$ for all $w \in V$ implies $v = 0$

Definition 2.1.2. The *index* ν of a symmetric bilinear form l on V is the dimension of a subspace $W \subset V$ such that

(1) $l|_W$ is negative definite.

(2) If $W' \subset V$ is another subspace such that $l|_{W'}$ is negative definite, then $\dim W' \leq \dim W$.

A nondegenerate, symmetric, bilinear form g is generally called an *inner product*. If g is an inner product on V, the *norm* of a vector v is given by $|v| = \sqrt{|g(v, v)|}$, and two vectors v and w are said to be *orthogonal* if $g(v, w) = 0$. If W is a subspace of V, then the *orthogonal complement* $W^\perp$ of W in V is defined by

$$W^\perp = \{v \in V : g(v, w) = 0 \, for \, all \, w \in W\}$$

A vector v such that $|g(v, v)| = 1$ is called a *unit vector*. A basis $\{e_1, ..., e_n\}$ for V whose elements are mutually orthogonal unit vectors is called an *orthonormal basis* for V.

Definition 2.1.3. A *semi-Riemannian metric* g for a manifold M is a smooth symmetric, nondegenerate tensor field of type $(0, 2)$ on M, which assigns to each point $p \in M$ an inner product g_p on the tangent space T_pM.

Here the index of g_p is the same for all p, and that same index value ν of g_p on M is called the *index* of M.

If g has components g_{ij} in local coordinates $(x^1, ..., x^n)$ on $U \subset M$, then the metric g is represented by

$$g = \sum_{i,j=1}^{n} g_{ij}(x) dx^i dx^j \qquad (metric\ tensor)$$

with

$$g_{ij} = g_{ji} \quad (symmetric) \quad \text{and} \quad \det(g_{ij}) \neq 0 \quad (non\text{-}degenerate)$$

Here,

$$g_{ij} = \langle \partial_i, \partial_j \rangle \quad , \quad 1 \leq i, j \leq n$$

where ∂_i denotes the vector field $\partial/\partial x_i$ on U. Because g is non-degenerate, the matrix $(g_{ij}(p))$ is invertible for each p in U. The inverse matrix is denoted by $(g^{ij}(p))$.

The *signature* of g will be denoted by (s, r), where s is the number of negative eigenvalues and $r = n - s$ is the number of postive eigenvalues with $n = dimM$.

Definition 2.1.4. A *semi-Riemannian manifold* is a smooth manifold furnished with a semi-Riemannian metric g.

If the metric g has signature $(0, n)$, then M is a *Riemannian manifold*; that means each g_p is a positive definite inner product on the tangent space $T_p(M)$ at p in M. However, when g fails to be positive definite, we have other types of manifolds:

Definition 2.1.5. Given a manifold M of dimension $n \geq 2$, the pair (M, g) is called a *Lorentzian manifold* when g is a *Lorentzian metric.* A pseudo-Riemannian metric g is called a Lorentzian metric if it has signature $(1, n-1)$.

2.2 Introduction to Minkowski space

With the definitions given in the previous section, now we are going to define the *Minkowski space* and introduce its basic properties with a special focus on the surfaces of this space [18].

Definition 2.2.1. The *Minkowski space* is the metric space $\mathbb{E}^3_1 = (\mathbb{E}^3, \langle , \rangle)$ where the metric $\langle , \rangle = dx_1^2 + dx_2^2 - dx_3^2$ is called the *Minkowski metric.* By $\mathbb{E}^3$ we denote the three dimensional Euclidean space.

A vector $v \in \mathbb{E}_1^3$ is classified as *timelike*, *spacelike* or *lightlike* if $\langle v, v \rangle$ is negative, positive or zero, respectively:

(1) $\langle v, v \rangle < 0$ (*timelike*),
(2) $\langle v, v \rangle > 0$ (*spacelike*),
(3) $\langle v, v \rangle = 0$ and $v \neq 0$ (*lightlike*).

The category into which a given vector falls is called its *causal character*.

Now, let W be a subspace of $\mathbb{E}_1^3$, and g be the metric on $\mathbb{E}_1^3$. There are three mutually exclusive possibilities for W:

(1) $g|_W$ is positive definite; that is, W is an inner product space. Then W is said to be *spacelike*.

(2) $g|_W$ is non-degenerate of index 1. Then W *timelike*.

(3) $g|_W$ is degenerate. Then W *lightlike*.

The set of all lightlike vectors of $\mathbb{E}_1^3$ is called the *lightcone of* $\mathbb{E}_1^3$:

$$\mathcal{C} = \{(x, y, z) \in \mathbb{E}_1^3 : x^2 + y^2 + z^2 = 0\} - \{(0, 0, 0)\}.$$

The set of timelike vectors is

$$\mathcal{T} = \{(x, y, z) \in \mathbb{E}_1^3 : x^2 + y^2 + z^2 < 0\}.$$

The existence of timelike and lightlike vectors in $\mathbb{E}_1^3$ poses some different geometric properties when compared with Euclidean 3-space $\mathbb{E}^3$:

Proposition 2.2.2. *Let $\mathbb{E}_1^3$ be the Minkowski 3-space.*

(1) Two lightlike vectors $u, v \in \mathbb{E}_1^3$ are linearly independent if and only if $\langle u, v \rangle = 0$.

(2) If $u, v \in \mathbb{E}^3_1$ are two non-spacelike vectors with $\langle u, v\rangle = 0$, then they are lightlike vectors.

(3) If u and v are two timelike vectors, then $\langle u, v\rangle \neq 0$.

If u is a timelike vector, the *timelike cone* of u is

$$\mathcal{C}(u) = \{v \in \mathcal{T} : \langle u, v\rangle < 0\}$$

and its *opposite* timelike cone is

$$\mathcal{C}(-u) = -\mathcal{C}(u) = \{v \in \mathcal{T} : \langle u, v\rangle > 0\}$$

Since $u^\perp$ is spacelike, $\mathcal{T}$ is the disjoint union of these two timelike cones.

Some properties of the timelike cones are the following:

Proposition 2.2.3. *(1) Two timelike vectors u and v lie in the same timelike cone if and only if $\langle u, v\rangle < 0$.*

(2) $u \in \mathcal{C}(v)$ if and only if $\mathcal{C}(u) = \mathcal{C}(v)$

(3) The timelike cones are convex cones, that is, if $v, w \in \mathcal{C}(u)$ and $a \geq 0, b \geq 0$ (not both zero), then $av + bw \in \mathcal{C}(u)$.

An important difference between $\mathbb{E}^3$ and $\mathbb{E}^3_1$ is also the *Cauchy-Schwarz inequality.* The inequality is basically reversed in Minkowski space [18]:

Proposition 2.2.4. *Let u and v be timelike vectors in $\mathbb{E}^3_1$. Then*

(1) $|\langle u, v\rangle| \geq |u||v|$, with equality if and only if u and v are proportional.

(2) If u and v are in the same timecone of V, there exists a unique number $\lambda \geq 0$, called the hyperbolic angle between u and v, such that

$$|\langle u, v\rangle| = -|u||v| \cosh \lambda$$

This result affects also the *triangle inequality* in $\mathbb{E}^3_1$.

Corollary 2.2.5. *If u and v are timelike vectors in the same timecone, then*

$$|\langle u, v\rangle| \leq |u + v|$$

with equality if and only if u and v are proportional.

Finally, we conclude this section by the following definition:

Definition 2.2.6. A Lorentz manifold M is *time-orientable* if there exists a timelike vector field X on M.

2.3 Surfaces in Minkowski Space

In this section we will recall some basic facts of local surface theory in Minkowski 3-space, and then introduce the surfaces in $\mathbb{E}^3_1$ with a special interest on those that have some geometric restrictions [18].

Definition 2.3.1. A *semi-Riemannian surface M* is a semi-Riemannian manifold of dimension 2 in the 3-dimesional Minkowski space $\mathbb{E}^3_1$.

Let M be a smooth, connected, semi-Riemannian surface. Let $\chi : M \to \mathbb{E}^3_1$ be an *immersion*, which is defined as a differentiable map whose differential map $d\chi_p : T_pM \to \mathbb{E}^3$ is injective for all $p \in M$.

A plane in $\mathbb{E}^3_1$ is called *spacelike*, *timelike* or *lightlike* if and only if its Euclidean unit normals are *timelike*, *spacelike* or *lightlike* respectively. Knowing this, we can define the causality of a surface M in $\mathbb{E}^3_1$:

Definition 2.3.2. Let M be a surface. An immersion $\chi : M \to \mathbb{E}^3_1$ is called *spacelike*, *timelike* or *lightlike* if all tangent planes T_pM are *spacelike*, *timelike* or *lightlike* respectively.

Remark 2.3.3. Non-degenerate surfaces can be spacelike or timelike.

Given an immersed surface in $\mathbb{E}_1^3$, it is not neccessarily one of the above types of surfaces because the causal character may change in different points of the same surface. For instance, the sphere $\mathbb{S}^2$ is composed of three regions with different casual characters. Precisely, $\{(x,y,z) \in \mathbb{S}^2 : |z| > \frac{1}{\sqrt{2}}\}$ is spacelike, $\{(x,y,z) \in \mathbb{S}^2 : |z| < \frac{1}{\sqrt{2}}\}$ is timelike, and $\{(x,y,z) \in \mathbb{S}^2 : |z| = \frac{1}{\sqrt{2}}\}$ is lightlike.

In the cases M is a spacelike (resp. timelike) surface with $p \in M$, then we can express $\mathbb{E}_1^3$ as the decomposition $T_pM \oplus (T_pM)^\perp$, where $(T_pM)^\perp$ is a timelike (resp. spacelike) subspace of dimension 1.

Definition 2.3.4. A *Gauss map* is a differntiable map $N : M \to \mathbb{E}_1^3$ which assigns to every point $p \in M$ a vector $N(p) \in (T_pM)^\perp$ such that $|N(p)| = 1$.

Recall that a surface is *orientable* if there is a family of coordinate charts where the change of parameters has positive Jacobian. This is equivalent to the existence of a Gauss map, also called as an *orientation* of M in the case when M is a non-degenerate surface.

An important result concerning the surfaces in $\mathbb{E}_1^3$ is the following:

Proposition 2.3.5. *Let M be a compact surface and let $\chi : M \to \mathbb{E}_1^3$ be a spacelike, timelike or lightlike immersion. Then $\partial M \neq 0$.*

This result shows that the causal character of an immersion imposes conditions on the surface M, and thus it disregards the study of closed surfaces in Minkowski space.

As examples, we will study the causal character of a *pseudosphere* and of the *graph* of a function which we will encounter in the next chapter [18].

(1) Consider the pseudosphere of center p_0 and radius r:

$$\mathbb{S}_1^2(r; p_0) = \{p \in \mathbb{E}_1^3 : \langle p - p_0, p - p_0 \rangle = r^2\}$$

The tangent plane at p is $T_pM = Span\{p - p_0\}^\perp$ and the normal vector is $N(p) = (p - p_0)/r$.

$N(p)$ is a spacelike vector, thus the surface is *timelike*. If p_0 is the origin and $r = 1$, the surface is also called the *De Sitter space* which is denoted by $\mathbb{S}^2_1$:

$$\mathbb{S}^2_1 = \{(x, y, z) \in \mathbb{E}^3_1 : x^2 + y^2 - z^2 = 1\}.$$

(2) Let $f : \Omega \subset \mathbb{E}^2 \to \mathbb{E}^1$ be a smooth function defined on a domain $\Omega \subset \mathbb{E}^2$. Define the *graph* of f by

$$graph(f) = \{(x, y, f(x, y))|(x, y) \in \Omega\}$$

and let S be the image of the immersion

$$\varphi : \Omega \to \mathbb{E}^3_1 \qquad \varphi(x, y) = (x, y, f(x, y))$$

Taking partial derivatives, we have $\varphi_x = (1, 0, f_x)$ and $\varphi_y = (0, 1, f_y)$. Then the matrix of the induced metric with respect to $\{\varphi_x, \varphi_y\}$ is

$$\begin{pmatrix} 1 - f_x^2 & -f_x f_y \\ -f_x f_y & 1 - f_y^2 \end{pmatrix}$$

whose determinant is

$$1 - f_x^2 - f_y^2 = 1 - |\nabla f|^2$$

This implies that the immersion is spacelike if $|\nabla f|^2 < 1$, timelike if $|\nabla f|^2 > 1$ and lightlike if $|\nabla f| = 1$.

Here we should note an important difference from the Euclidean case; given a function f, one can consider the graph of f on each plane, and for each plane there is a different causal character of the same function f. For example, if the surface is a graph on the timelike plane (yz-plane) of equation $x = 0$ given by $Q = \{(f(y, z), y, z)|(y, z) \in \Omega\}$, the matrix of the metric is

$$\begin{pmatrix} 1 + f_y^2 & f_y f_z \\ f_y f_z & f_z^2 - 1 \end{pmatrix}$$

Its determinant is $-f_y^2 + f_z^2 - 1$, which is different than $1 - |\nabla f|^2$ (the determinant when the surface is a graph on xy-plane). Thus, the same function f may give surfaces with different causal character. For example, if we take $f(x, y) = 0$ on $\Omega = \mathbb{E}^2$, then S becomes a spacelike (horizontal) plane, but Q is a timelike (vertical) plane.

In Euclidean space, we know that an immersion $\chi : M \to \mathbb{E}^2$ of a surface M is locally a graph on one of the three coordinate planes. But if we put the Lorentzian metric on $\mathbb{E}^3$, we can know precisely on what coordinate plane a given spacelike or timelike surface is a graph [18]:

Proposition 2.3.6. *A spacelike (resp.timelike) surface is locally the graph of a function defined in the plane of equation $z = 0$ (resp. $x = 0$ or $y = 0$).*

We conclude this section with the following result for spacelike surfaces:

Theorem 2.3.7. *Let M be a surface and let $\chi : M \to \mathbb{E}^3_1$ be a spacelike immersion. Then M is orientable.*

2.4 Mean and Gauss Curvature

In this section we will give some basic definitions related to the curvature of a surface, and then finally define the two most important curvature functions of a surface: mean and Gauss curvatures [2, 18].

Let M be a connected smooth manifold of dimension n. Let $C^\infty(M)$ denote the set of all smooth functions on M, and $\mathfrak{X}(M)$ the set of all vector fields tangent to M.

Definition 2.4.1. For an ordered pair $X, Y \in \mathfrak{X}(M)$, the *Lie bracket* is a vector field $[X, Y] \in \mathfrak{X}(M)$ such that for a smooth function f

$$[X, Y](f) = X(Y(f)) - Y(X(f)).$$

Definition 2.4.2. A *connection* on M is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ such that

(1) $\nabla_X Y$ is $C^\infty(M)$-linear in X

(2) $\nabla_X Y$ is $\mathbb{R}$-linear in Y

(3) $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$, for all $f \in C^\infty(M)$

Here we denote $\nabla_X Y := \nabla(X, Y)$. The vector field $\nabla_X Y$ is also known as the *covariant derivative* of Y along X.

Definition 2.4.3. Let (M, g) be a Lorentzian manifold. Then there is a unique connection ∇, called the *Levi-Civita connection*, on M such that:

(4) $[X, Y] = \nabla_X Y - \nabla_Y X$

(5) $Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

Now, let $\chi : M \to \mathbb{E}^3_1$ be a spacelike or timelike immersion of a surface M. We will denote by ∇^0 the Levi-Civita connection on $\mathbb{E}^3_1$. If $X, Y \in \mathfrak{X}(M)$, we have the decomposition

$$\nabla^0_X Y = (\nabla^0_X Y)^\top + (\nabla^0_X Y)^\perp \tag{2.4.1}$$

where $\top$ and $\perp$ denote the tangent and normal components of $\nabla^0_X Y$ respectively with respect to M. Let ∇ denote the induced connection on M by the immersion χ, that is,

$$\nabla_X Y = (\nabla^0_X Y)^\top$$

and we define the second fundamental form of X as the tensorial, symmetric map

$$\mathit{\Pi} : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)^\perp \qquad , \qquad \mathit{\Pi}(X, Y) = (\nabla^0_X Y)^\perp$$

So, (2.4.1) becomes $\nabla^0_X Y = (\nabla_X Y) + \mathit{\Pi}(X, Y)$, known as *Gauss formula.*

We recall from the previous section that for a given surface X in $\mathbb{E}^3_1$, the Gauss map is the differentiable map $N : X \to \mathbb{S}^2$ that maps every point $p \in X$ to a unit vector $N(p)$ orthogonal at X at p.

In Minkowski space, two surfaces play the same role as spheres in $\mathbb{E}^3$: the *pseudohyperbolic surface* and the *pseudosphere*. The pseudohyperbolic surface of radius $r > 0$ is

$$\mathbb{H}^{2,1}(r) = \{p \in \mathbb{E}^3_1\,; \langle p, p\rangle = -r^2\}$$

which is a spacelike surface. From the Euclidean viewpoint, $\mathbb{H}^{2,1}(r)$ is the hyperboloid of two sheets $x_1^2 + x_2^2 + x_3^2 = -r^2$ which is obtained by rotating the hyperbola $x_1^2 - x_3^2 = -r^2$ in the plane $x_2 = 0$ about the x_3-axis.
The other surface is the Lorentz sphere (pseudosphere) $\mathbb{S}^{2,1}(r)$:

$$\mathbb{S}^{2,1}(r) = \{p \in \mathbb{E}^3_1\,; \langle p, p\rangle = r^2\}$$

which is a timelike surface, and is obtained by rotating the hyperbola $x_1^2 - x_3^2 = 1$ in the plane $x = 0$ about the x_3-axis.

Definition 2.4.4. Let $X \in T_pM$ and define $(A_N(X))(p) = -(\nabla_X N)(p)$ where N denotes a (local) unit normal vector field on the surface M. Note that $\langle A_N(X), N\rangle = -\langle \nabla_X N, N\rangle = -(1/2)X(\langle N, N\rangle) = 0$ so that $(A_N(X))(p) \in T_pM$. Hence, A is a linear transformation on T_pM, called the *Weingarten map*:

$$A_p : T_pM \to T_pM \quad , \quad A_p = (A_N(X))_p.$$

It is also known as the derivative of the Gauss map, that is, it measures the variation of the normal near p or how the surface "curves" near p. For that reason it is also called the *shape operator*.

So, let M be a nondegenerate connected surface in $\mathbb{E}^3_1$. Consider a unit normal vector field N of M. If M is spacelike, respectively timelike, the Gauss map is given by $N : M \to \mathbb{H}^{2,1}(1)$, respectively $N : M \to \mathbb{S}^{2,1}(1)$. Now we can define the mean curvature and the Gauss curvature in $\mathbb{E}^3_1$. Firstly, we recall that in Euclidean space the Weingarten map is diagonalizable because it is a self-adjoint endomorphism with respect to a Riemannian metric, and *principal curvatures* are defined as the eigenvalues of the Weingarten map.

In Minkowski space the Weingarten map A is self-adjoint with respect to the induced metric $\langle , \rangle$. If the metric is Riemannian then A is diagonalizable, but it might not be diagonalizable if the metric is Lorentzian. This means that while principal curvatures are well-defined on a spacelike surface, they are not defined on a timelike surface.

Now, we will define the Gauss and mean curvature in terms of trace and determinant of Weingarten map A.

Proposition 2.4.5. *Consider the Weingarten map A of a non-degenerate surface $\mathbb{E}^3_1$. Then*

$$K = \epsilon \, det(A) \qquad and \qquad H = \epsilon \, \frac{1}{2} trace(A)$$

where $\epsilon = \langle N, N \rangle$, and K and H denote the Gauss and mean curvature respectively.

Definition 2.4.6. Consider $\chi : M \to \mathbb{E}^3_1$ a non-degenerate immersion and $p \in M$. If the Weingarten map A_p at the point p is diagonalizable, the eigenvalues of A_p are called the *principal curvatures* at p and we denote them by $\lambda_1(p)$ and $\lambda_2(p)$.

Corollary 2.4.7. *Assume that A_p is diagonalizable in a non-degenerate surface of $\mathbb{E}^3_1$. Then*

$$H(p) = \epsilon \, \frac{\lambda_1(p) + \lambda_2(p)}{2} \quad , \quad K(p) = \epsilon \, \lambda_1(p)\lambda_2(p)$$

In Euclidean space, an *umbilic point* is defined to be the point where two principal curvatures coincide. But in Minkowski space we cannot use the same definition since principal curvatures are not always defined in a given surface:

Definition 2.4.8. Let $\chi : M \to \mathbb{E}^3_1$ be a spacelike or timelike immersion. A point $p \in M$ is called *umbilic* if there exists $\tau(p) \in \mathbb{R}$ such that

$$\langle \mathit{\Pi}(u, v), N(p) \rangle = \tau(p) \langle u, v \rangle, \quad u, v \in T_pM$$

A surface is called *totally umbilical* if all point are umbilic.

Hence an umbilic is a point where the second and the first fundamental forms are proportional. This is also equivalent to saying that $\langle A_p u, v \rangle = \tau(p)\langle u, v \rangle$, and in particular A_p must be diagonalizable because $\langle Ae_1, e_2 \rangle = 0$. So, we say that p is an umbilic point if and only if $\lambda_1(p) = \lambda_2(p)$. In Euclidean space, we have the inequality $H^2 - K \geq 0$ where equality holds only at an umbilic. But in $\mathbb{E}^3_1$, Weingarten map needs to be diagonalizable:

Proposition 2.4.9. *Assume that M is a non-degenerate surface of $\mathbb{E}_1^3$, $p \in M$ and A_p is diagonalizable. Then,*

$$H(p)^2 - \epsilon\, K(p) \geq 0,$$

and equality holds if and only if p is umbilic. In particular, in a timelike surface, if $H(p)^2 - \epsilon\, K(p) < 0$, then p is not umbilic.

The diagonalizability of the Weingarten map depends on the existence of real roots of its characteristic polynomial $P(\tau)$. We can easily compute $P(\tau)$ and get $P(\tau) = \tau^2 - 2H\epsilon\tau + \epsilon K$ and its discriminant as $D = 4(H^2 - \epsilon K)$. Thus:

(1) If $H^2 - \epsilon K > 0$, there are two different real roots of $\tau(p)$ and the Weingarten map is diagonalizable.

(2) If $H^2 - \epsilon K < 0$, A is not diagonalizable.

(3) If $H^2 - \epsilon K = 0$, there is a double root of $\tau(p)$. Here there are two cases: if $\epsilon = -1$, the root $\tau = -H$ is an eigenvalue of A and the point is umbilic. And if $\epsilon = 1$, then the matrix might or might not be diagonalizable.

We conclude this section by showing how to compute the curvatures by using a local parametrization and then applying it to a surface given as a graph of a function.

Let $\varphi(u, v)$ be a local parametrization of a (spacelike or timelike) surface:

$$\varphi : U \subset \mathbb{R}^2 \rightarrow \mathbb{E}_1^3 \qquad , \qquad \varphi = \varphi(u, v)$$

and let $B = \{\varphi_u, \varphi_v\}$ be a local basis of the tangent plane at each point of $\varphi(U)$. $\{E, F, G\}$ and $\{L, M, N\}$ will represent the coefficients of the first and second fundamental forms and we will denote $W = EG - F^2$. The surface is spacelike if $W > 0$ and it is timelike if $W < 0$. Take the unit normal vector field:

$$\mathbf{N} = \frac{\varphi_u \times \varphi_v}{|\varphi_u \times \varphi_v|}$$

Here $|\varphi_u \times \varphi_v| = \sqrt{-\epsilon(EG - F^2)} = \sqrt{-\epsilon W}$ and

$$E = \langle \varphi_u, \varphi_u \rangle \qquad\qquad L = \langle \mathbf{N}, \varphi_{uu} \rangle$$
$$F = \langle \varphi_u, \varphi_v \rangle \qquad\qquad M = \langle \mathbf{N}, \varphi_{uv} \rangle$$
$$G = \langle \varphi_v, \varphi_v \rangle \qquad\qquad N = \langle \mathbf{N}, \varphi_{vv} \rangle$$

Then, the Weingarten map A is expressed as:

$$A = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

and therefore the mean curvature and the Gauss curvature are

$$H = \epsilon \, \frac{1}{2} \frac{LG - 2MF + NE}{EG - F^2} \quad , \quad K = \epsilon \, \frac{LN - M^2}{EG - F^2}. \tag{2.4.2}$$

Finally, we will consider an example [18]:

Example 2.4.10. (Graph) Let f be a smooth function and consider the surface M given by $z = f(x, y)$. Let $\varphi(x, y) = (x, y, f(x, y))$ be the local parametrization of the surface. After some simple calculations, we get:

$$E = 1 - f_x^2$$
$$F = -f_x f_y$$
$$G = 1 - f_y^2$$

Thus $\quad EG - F^2 = 1 - f_x^2 - f_y^2 = 1 - |\nabla f|^2.$

If the surface is spacelike (resp. timelike) we have $|\nabla f|^2 < 1$ (resp.> 1) on the domain of $\varphi(u, v)$. Then, by using the formulas in (2.4.2), we get

$$(1 - f_y^2) f_{xx} + 2 f_x f_y + (1 - f_x^2) f_{yy} = -2H(-\epsilon(1 - |\nabla f|^2))^{3/2}$$

$$K = -\frac{f_{xx} f_{yy} - f_{xy}^2}{(1 - f_x^2 - f_y^2)^2}$$

Chapter 3

Minimal Surfaces on Three-Dimensional Walker Manifolds

In this chapter, we will define the Walker manifolds and we will discuss some of their geometric properties as a first step before investigating the minimal surfaces on these manifolds [1, 7].

3.1 Walker Manifolds

It is a well-known result that a Riemannian manifold can be decomposed locally as a direct product whenever there exists a parallel line field. The same holds true for semi-Riemannian manifolds as long as the line field is non-degenerate, which means that it is generated by a non-null vector field [1]. However, in the semi-Riemannian setting, different geometrical consequences arise when manifolds admit parallel degenerate line fields. So our first objective is to understand these kind of manifolds.

Let M be a semi-Riemannian manifold whose tangent bundle is decomposed as $TM = V_1 \otimes V_2$ where V_1 and V_2 are smooth subbundles which are called *distributions*. This gives rise to two complementary projections π_1 and π_2 of TM onto V_1 and V_2 respectively. If $\nabla \pi_1 = 0$ then V_1 is said to be a *parallel distribution*. Equivalently, this means that $\nabla V_1 \subset V_1$.

Now, let V_1 be a parallel distribution. Then the metric restricted to V_1 has constant rank, and we say that V_1 is a *null parallel distribution* whenever the restricted metric vanishes identically. Again, assume that V_1 is parallel with dimension 1, so that it is a *line field*. If V_1 is not null, then through parallel translation there is a parallel vector field which spans V_1 whenever M is simply connected. However, V_1 being null does not imply that it should be spanned by a parallel vector field [1].

Definition 3.1.1. Lorentzian manifolds which admit null parallel distributions are called *Walker manifolds*, and in the case where the distributions are spanned by a null vector they are called *strict Walker manifolds*.

Walker has derived adapted coordinates to a parallel plane field [6]. Hence, the metric of a three-dimensional Walker manifold (M, g_f) with coordinates (x, y, z) is expressed as

$$g_f = \varepsilon dy^2 + f dz^2 + dx \otimes dz + dz \otimes dx$$

and its matrix form as

$$g_f = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & f \end{pmatrix}, \quad \text{with its inverse} \quad g_f^{-1} = \begin{pmatrix} -f & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{3.1.1}$$

for some function $f(x, y, z)$, where $\varepsilon = \pm 1$ and thus $\mathcal{D} = \langle \frac{\partial}{\partial x} \rangle$ as the parallel degenerate line field. Notice that when $\varepsilon = 1$ and $\varepsilon = -1$ the Walker manifold has signature (2,1) and (1,2) respectively, and therefore is Lorentzian in both cases. In our work we will always take $\varepsilon = -1$.

Remark 3.1.2. If there exists a parallel null vector $U = \frac{\partial}{\partial x}$, the coordinates above are affected in a way that $f(x, y, z) = f(y, z)$ [6].

To construct an orthonormal frame for (M, g), first we need to define the vector product on a Walker manifold.

Let $u, v \in M$. The vector product $u \times v$ of u and v is the unique vector defined as

$$u \times v = (u_1v_2 - u_2v_1 - (u_2v_3 - u_3v_2)f, u_1v_3 - u_3v_1, u_2v_3 - u_3v_2) \qquad (3.1.2)$$

which satisfies

$$\langle u \times v, w \rangle = det(u, v, w) = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \qquad (3.1.3)$$

Here $w \in \{E_1, E_2, E_3\}$, the standard basis of the real vector space $\mathbb{E}^3$.

Having defined the vector product in (M, g_f), now we are able to construct an orthonormal frame on the manifold. So, let $u = (0, 1, 0)$ and $v = (0, 0, \frac{1}{\sqrt{|f|}})$. Then

$$u \times v = (-\sqrt{f}, 0, \frac{1}{\sqrt{f}}).$$

Notice that $|u| = |v| = |u \times v| = 1$. Here we assume that $f > 0$, and the case $f < 0$ can be studied in a similar way. So, we get the following orthonormal frame

$$e_1 = -\sqrt{f}\partial_x + \frac{1}{\sqrt{f}}\partial_z, \quad e_2 = \partial_y, \quad e_3 = \frac{1}{\sqrt{f}}\partial_z \qquad (3.1.4)$$

where

$$\langle e_1, e_1 \rangle = 1, \quad \langle e_2, e_2 \rangle = -1, \quad \langle e_3, e_3 \rangle = 1.$$

3.2 Connections and Curvature

We denote by ∇ the Levi-Civita connection of (M, g_f) that we defined in the previous chapter (see Def. 2.4.2). In local coordinates it can be described using the *Christoffel symbols* which are the real-valued functions Γ_{ij}^k such that

$$\nabla_{\partial_i}\partial_j = \sum_k \Gamma_{ij}^k \partial_k \quad (1 \leq i, j \leq n) \qquad (3.2.1)$$

where

$$\Gamma_{ij}^{k} = \frac{1}{2}\sum_{l=1}^{n} g^{kl}\left(\frac{\partial g_{jl}}{\partial x^{i}} + \frac{\partial g_{il}}{\partial x^{j}} - \frac{\partial g_{ij}}{\partial x^{l}}\right), \tag{3.2.2}$$

and $\partial_i, \partial_j, \partial_k$ represent the coordinate vector fields $\partial/\partial x^i, \partial/\partial x^j, \partial/\partial x^k$, respectively, and $g^{ij} = (g_{ij})^{-1}$ [2].

Using (3.2.2), we calculate the possible non-zero Christoffel symbols of a Walker metric g_f as

$$\begin{aligned}
\Gamma_{13}^{1} = \Gamma_{31}^{1} &= \frac{1}{2} f_x,\\
\Gamma_{23}^{1} = \Gamma_{32}^{1} &= \frac{1}{2} f_y,\\
\Gamma_{33}^{1} &= \frac{1}{2}\left(f_z + f f_x\right),\\
\Gamma_{33}^{2} &= \frac{1}{2} f_y,\\
\Gamma_{33}^{3} &= -\frac{1}{2} f_x.
\end{aligned}$$

Having calculated the Christoffel symbols, we get the following possibly non-vanishing components of the Levi-Civita connection on (M, g_f) [7]:

$$\begin{aligned}
\nabla_{\partial_x}\partial_z &= \frac{1}{2} f_x \partial_x,\\
\nabla_{\partial_y}\partial_z &= \frac{1}{2} f_y \partial_x,\\
\nabla_{\partial_z}\partial_z &= \frac{1}{2}(f f_x + f_z)\partial_x + \frac{1}{2} f_y \partial_y - \frac{1}{2} f_x \partial_z.
\end{aligned}$$

Let R denote the *curvature tensor* of (M, g_f) defined as

$$R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]. \tag{3.2.3}$$

Again by [7], R is completely determined by the following possibly non-zero components:

$$\begin{aligned}
R(\partial_x, \partial_z)\partial_x &= \frac{1}{2} f_{xx}\partial_x, \\
R(\partial_x, \partial_z)\partial_y &= \frac{1}{2} f_{xy}\partial_x, \\
R(\partial_y, \partial_z)\partial_x &= \frac{1}{2} f_{xy}\partial_x, \\
R(\partial_y, \partial_z)\partial_y &= \frac{1}{2} f_{yy}\partial_x, \\
R(\partial_x, \partial_z)\partial_z &= \frac{1}{2} f f_{xx}\partial_x + \frac{1}{2} f_{xy}\partial_y - \frac{1}{2} f_{xx}\partial_z, \\
R(\partial_y, \partial_z)\partial_z &= \frac{1}{2} f f_{xy}\partial_x + \frac{1}{2} f_{yy}\partial_y - \frac{1}{2} f_{xy}\partial_z.
\end{aligned}$$

We recall that in the Lorentzian setting, the curvature of a 3-dimensional manifold is completely determined by the *Ricci tensor* defined by

$$\rho(X, Y)_P = \sum_{i=1}^{3} \varepsilon_i g(R(X, E_i)Y, E_i)_p$$

for $p \in M$ and X, Y tangent vectors to M at p, where $\{E_1, E_2, E_3\}$ is an orthonormal frame on (M, g_f) and $\varepsilon_i = g_p(E_i, E_i) = \pm 1$.

Then, for a 3-dimensional Walker manifold with respect to the coordinate basis $\{\partial_x, \partial_y, \partial_z\}$ and the orthonormal frame $\{E_1, E_2, E_3\}$, we get the Ricci tensor as

$$\rho = \begin{pmatrix} \frac{1}{2}f_{xx} & \frac{1}{2}f_{xy} & \frac{1}{2}f_{yy} \\ 0 & 0 & -\frac{1}{2}f_{xy} \\ 0 & 0 & \frac{1}{2}f_{xx} \end{pmatrix}$$

with eigenvalues $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = \frac{1}{2}f_{xx}$.

Recall that *strict* 3-dimensional Walker manifolds are 3-dimensional Walker manifolds whose parallel null vector field is spanned by ∂_x. Hence in a strict Walker 3-manifold, f is independent of the parameter x, i.e. $f = f(y, z)$ [6].

Therefore, for a strict 3-dimensional Walker manifold, the Levi-Civita connection and Riemannian curvature are determined by the following possibly non-vaishing coefficients respectively:

$$\nabla_{\partial_y}\partial_z = \frac{1}{2} f_y \partial_x,$$
$$\nabla_{\partial_z}\partial_z = \frac{1}{2} f_z \partial_x - \frac{\varepsilon}{2} f_y \partial_y,$$

and

$$R(\partial_y, \partial_z)\partial_y = \frac{1}{2} f_{yy} \partial_x,$$
$$R(\partial_y, \partial_z)\partial_z = \frac{1}{2} f_{yy} \partial_y.$$

Moreover, the Ricci tensor ρ for a strictly 3-dimensional Walker manifold M becomes

$$\rho = \begin{pmatrix} 0 & 0 & \frac{1}{2} f_{yy} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

3.3 Minimal Surfaces

In this section, we are going to investigate the surfaces with vanishing mean curvature on a 3-dimensional Walker manifold. In particular, we will consider the surfaces which are the *graph* of a smooth function h.

We recall that different from Euclidean case, in Minkowski space given a function h one can consider the graph of h on each plane, and for each plane there is a different causal character of the same function h. Therefore, we are going to study each case seperately and derive the equations of graphs with vanishing mean curvature.

Throughout this work, by a surface X we mean a connected, oriented, smooth 2-manifold, and the metric that we use on X, i.e. the Walker metric, is non-degenerate and Lorentzian (indefinite).

In order to study the surfaces on a manifold, whether it be Riemannian or Lorentzian, we need to introduce *(Lorentz) isothermal coordinates* which in most cases turn out to be very useful by simplifying the calculations to a great extent.

First, we recall that two metrics g and $\tilde{g}$ on X are conformally equivalent if and only if there is a *positive* C^∞ function μ such that $g = \mu\tilde{g}$. We observe that while oriented angles on X are preserved by conformally Riemannian metrics, conformally equivalent Lorentzian metrics preserve the casual character of any tangent vector on X.

Definition 3.3.1. (1) For a Riemannian metric g on X, coordinates (x, y) are called *isothermal coordinates* if and only if g is equivalent to $dx^2 + dy^2$.

(2) For a Lorentzian metric g on X, coordinates (x, y) are called *Lorentz isothermal coordinates* if and only if g is equivalent to $-dx^2 + dy^2$. Moreover, by taking $x = u - v$ and $y = u + v$ we get the so called *proper null coordinates* (u, v) in which case g becomes equivalent to $dudv$.

For the proof of local existence of such coordinates on both Riemann and Lorentz surfaces, see [19, 20].

Observe that if we choose isothermal coordinates, then the mean curvature simplifies to

$$H = \frac{L + N}{2E} \tag{3.3.1}$$

and

$$H = \frac{L - N}{2E} \tag{3.3.2}$$

for spacelike and timelike surfaces respectively.

3.3.1 Graphs on xy-plane

Let $h : U \subset \mathbb{E}^2 \to \mathbb{E}^1$ be a smooth function defined on a domain $U \subset \mathbb{E}^2$, and let X be the image of the immersion

$$\varphi : U \to (M, g_f) \quad , \quad \varphi(u,v) = (u, v, h(u,v)) \tag{3.3.3}$$

By Proposition 2.3.6 we know that φ is a spacelike immersion. This means that the shape operator of the surface X is diagonalizable, hence we can use isothermal coordinates as in the Riemannian case. With straightforward calculations we obtain the coefficients of the first and second fundamental form as

$$\begin{aligned}
E &= h_u(fh_u + 2) \\
F &= h_v(fh_u + 1) \\
G &= \varepsilon + fh_v^2 \\
L &= h_{uu}(f - h_u)/\sqrt{1 + h_u^2 + h_v^2} \\
M &= h_{uv}(f - h_u)/\sqrt{1 + h_u^2 + h_v^2} \\
N &= h_{vv}(f - h_u)/\sqrt{1 + h_u^2 + h_v^2}
\end{aligned}$$

and using (3.3.1) we get

$$H = \frac{(h_{uu} + h_{vv})(f - h_u)}{2E\sqrt{1 + h_u^2 + h_v^2}} \tag{3.3.4}$$

It is clear that the mean curvature vanishes if and only if $h_{uu} + h_{vv} = 0$ or $f - h_u = 0$. On the other hand, by choosing isothermal coordinates we have $E = G$ and $F = 0$, and thus we have two cases:

<u>*Case A*</u> : Assume that $h_{uu} + h_{vv} = 0$. Then by choosing isothermal coordinates we have

$$h_v(fh_u + 1) = 0 \tag{3.3.5}$$

$$fh_u^2 + 2h_u = \varepsilon + fh_v^2 \tag{3.3.6}$$

From (3.3.5) we extract two other subcases, that is, $h_v = 0$ and $fh_u + 1 = 0$.

Case A.1 : If $h_v = 0$, then $h_{vv} = 0$ which implies that $g_{uu} = 0$. Then $h = au + b$ where a and b are real constants. If $a = 0$, then (3.3.6) does not hold. But if $a \neq 0$, then by (3.3.6) we obtain $f = -\frac{1+2a}{a^2}$, a constant function, which is a trivial case.

Case A.2 : If $fh_u + 1 = 0$, then

$$h_u = -\frac{1}{f} \tag{3.3.7}$$

$$h_{uu} = \frac{f_u}{f^2} \tag{3.3.8}$$

By using (3.3.6) and (3.3.7), we get

$$h_v = \sqrt{f-1}/f \tag{3.3.9}$$

$$h_{vv} = \frac{f_v(2 - f_v^2)}{f^2\sqrt{f-1}} \tag{3.3.10}$$

Integrating (3.3.9) and (3.3.10) gives respectively

$$h(u,v) = -\int \frac{1}{f}\,du + K_1(v) \tag{3.3.11}$$

$$h(u,v) = \int \frac{\sqrt{f-1}}{f}\,dv + K_2(u) \tag{3.3.12}$$

Finally, we have assumed that $h_{uu} + h_{vv} = 0$, that is, $h_{uu} = -h_{vv}$. Then, by substituting (3.3.8) and (3.3.10) into the equation we get

$$\frac{f_u}{f^2} = \frac{f_v(f_v^2 - 2)}{f^2\sqrt{f-1}}$$

which simplifies to

$$f_u\sqrt{f-1} = f_v(f_v^2 - 2). \tag{3.3.13}$$

Therefore, whenever f satisfies (3.3.13), the immersion

$$\varphi(u,v) = \left(u, v, -\int \frac{1}{f}\,du + K_1(v)\right)$$

or

$$\varphi(u,v) = \left(u, v, \int \frac{\sqrt{f-1}}{f}\,dv + K_2(u)\right)$$

is spacelike minimal, and the surface is also called as a *maximal* surface.

Case B : Now we assume that $f - h_u = 0$. Again we have two subcases:

Case B.1 : If $h_v = 0$, then (3.3.6) becomes $h_u^3 + 2h_u = -1$, and after integrating we get

$$2h(u) = -\left(\int h_u^3\, du + u + C(v)\right).$$

Thus,

$$\varphi(u,v) = \left(u, v, -\frac{1}{2}\left(\int h_u^3\, du + u + C(v)\right)\right)$$

gives a maximal surface.

Case B.2 : If $fh_u + 1 = 0$, then $h_u = -\frac{1}{f}$. But we already have the condition $h_u = f$. Hence $f^2 = -1$, that is, $f = \pm i$, a constant function, again contradicting the assumed non-triviality of the manifold.

We summarize those results in the following theorem:

Theorem 3.3.2. *Let (M, g_f) be a 3-dimensional, non-trivial Walker manifold, and let X be a surface in (M, g_f) given as the graph of a smooth function $h(u,v) : U \subseteq \mathbb{E}^2 \to \mathbb{E}^1$ on **xy**-plane. Then X is minimal when h is given by one of*

$$h(u,v) = -\int \frac{1}{f}\, du + K_1(v)$$
$$h(u,v) = \int \frac{\sqrt{f-1}}{f}\, dv + K_2(u)$$
$$h(u) = -\frac{1}{2}\left(\int h_u^3\, du + u + C(v)\right)$$

where f satisfies

$$f_u\sqrt{f-1} = f_v(f_v^2 - 2)$$

3.3.2 Graphs on xz-plane

Let us now consider the smooth function $h : U \subset \mathbb{E}^2 \to \mathbb{E}^1$ in the second coordinate, that is, let X be the image of the immersion

$$\varphi : U \to (M, g_f) \quad , \quad \varphi(u,v) = (u, h(u,v), v) \tag{3.3.14}$$

Again by Proposition 2.3.6 we know that φ is a timelike immersion. Since for *timelike* surfaces the shape operator may or may not be diagonalizable, we are going to use the *Lorentz* isothermal coordinates.

We calcualate the coefficients of first and second fundamental form, and we find

$$\begin{aligned}
E &= \varepsilon h_u^2 \\
F &= \varepsilon h_v h_u + 1 \\
G &= \varepsilon g_v^2 + f \\
L &= \varepsilon(-h_{uu})/\sqrt{1 + h_u^2 + h_v^2} \\
M &= \varepsilon(-h_{uv})/\sqrt{1 + h_u^2 + h_v^2} \\
N &= \varepsilon(-h_{vv})/\sqrt{1 + h_u^2 + h_v^2}
\end{aligned}$$

and by (3.3.2) we get the mean curvature as

$$H = \frac{h_{uu} - h_{vv}}{2\varepsilon g_u^2 \sqrt{h_u^2 + h_v^2}} \tag{3.3.15}$$

Hence $H = 0$, if and only if, $h_{uu} - h_{vv} = 0$. Furthermore, Lorentz isothermal coordinates necessitate that $E = -G$ and $F = 0$, which as a result put the following restrictions on h:

$$h_u h_v = 1 \tag{3.3.16}$$

$$h_u^2 + h_v^2 = f \tag{3.3.17}$$

Now, writing $h_v = 1/h_u$ gives us $h_{vv} = -h_{uv}/h_u^2$.

Then if we substitute into $h_{uu} - h_{vv} = 0$, we get

$$h_{uu}\, h_u^2 = -h_{uv} \tag{3.3.18}$$

Here we can take integrate and obtain

$$h_u^3 = -3 \int h_{uv}\, du + A(v) \tag{3.3.19}$$

Or we can substitute $h_u^2 = f - h_v^2$ into (3.3.18) and get the following integral equation:

$$h(u,v) = -\iint (f - h_v)^2 h_{vv}\, dvdu + C(v) + B(u) \tag{3.3.20}$$

On the other hand, we have the equality $h_{uv} = h_{vu}$ of second order partial derivatives. After simple calcualtions, we get

$$\begin{aligned} h_{uv} &= -h_{vv}/h_v^2 \\ h_{uv} &= -h_{vv}/h_v^2 \end{aligned}$$

And by using the assumption we made eariler $h_{uu} = h_{vv}$, we get the following

$$\begin{aligned} h_{uv} = h_{vu} &\iff h_v^2 = h_u^2 \\ &\iff h_v = h_u \\ &\iff h_v^2 = 1 \qquad (since\ h_u h_v = 1) \\ &\iff h_v = h_u = 1 \\ &\iff h(u,v) = u + v + C \quad , \quad C - constant \end{aligned}$$

Then by using (3.3.17), we obtain $f = h_v^2 + h_u^2 = 2$, which implies the absence of a non-trivial case:

Theorem 3.3.3. *Let (M, g_f) be a 3-dimensional, non-trivial Walker manifold, and let $h(u,v) : U \subseteq \mathbb{E}^2 \to \mathbb{E}^1$ be a smooth function. Then the image of the immersion*

$$\varphi(u,v) = (u, h(u,v), v)$$

cannot be a minimal surface for any nonconstant function h.

3.3.3 Graphs on yz-plane

Finally we will consider the case where X is the image of the immersion

$$\varphi : U \to (M, g_f) \quad , \quad \varphi(u,v) = (h(u,v), u, v) \tag{3.3.21}$$

for a given smooth function $h : U \subset \mathbb{E}^2 \to \mathbb{E}^1$.

By the same argument as used in the prevoius case, since φ is a timelike immersion we will choose the coordinates to be *Lorentz* isothermal.

In a similar way, after obtaining the coefficients of first and second fundamental form as

$$\begin{aligned}
E &= \varepsilon \\
F &= h_u \\
G &= f + 2h_v \\
L &= (-h_{uu}h_v)/\sqrt{1 + h_u^2 + h_v^2} \\
M &= (-h_{uv}h_v)/\sqrt{1 + h_u^2 + h_v^2} \\
N &= (-h_{vv}h_v)/\sqrt{1 + h_u^2 + h_v^2}
\end{aligned}$$

we get the mean curvature as

$$H = \frac{-h_v(h_{uu} - h_{vv})}{2\varepsilon\sqrt{h_u^2 + h_v^2}} \tag{3.3.22}$$

If $h_v = 0$ then $h \equiv 0$, because by choosing Lorentz isothermal coordinates $F = h_u = 0$. Therefore, $h_v \neq 0$. This implies that mean curvature vanishes if and only if $h_{uu} - h_{vv} = 0$. Also the Lorentz isothermal coordinates impose the following conditions

$$h_u = 0 \tag{3.3.23}$$

$$f + 2h_v = 1 \tag{3.3.24}$$

This means that $h_{uu} = 0$ which implies that $h_{vv} = 0$ also. Then we have

$$h(v) = av + b \tag{3.3.25}$$

where a and b are real constants. Here we have two cases:

Case i : If $a = 0$ then $h = b$. In this case we get $f = 1$ by using (3.3.24).

Case ii : If $a \neq 0$ then $h = av + b$ and $f = 1 - 2a$.

In either case, we obtain f as a constant function, and thus, we similarly conclude that a 3-dimensional Walker manifold does not admit a minimal surface of this form:

Theorem 3.3.4. *Let (M, g_f) be a 3-dimensional, non-trivial Walker manifold, and let $h(u, v) : U \subseteq \mathbb{E}^2 \to \mathbb{E}^1$ be a smooth function. Then the image of the immersion*

$$\varphi(u, v) = (h(u, v), u, v)$$

cannot be a minimal surface for any nonconstant function h.

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