ASYMPTOTICS OF EXTREMAL POLYNOMIALS FOR SOME SPECIAL CASES

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Asymptotics of extremal polynomials for some special cases

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We certify that we have read this dissertation and that in our opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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ABSTRACT

ASYMPTOTICS OF EXTREMAL POLYNOMIALS FOR SOME SPECIAL CASES

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We study the asymptotics of orthogonal and Chebyshev polynomials on fractals. We consider generalized Julia sets in the sense of Brück-Büger and weakly equilibrium Cantor sets which was introduced in [62].

We give characterizations for Parreau-Widom condition and optimal smoothness of the Green function for the weakly equilibrium Cantor sets. We also show that, for small parameters, the corresponding Hausdorff measure and the equilibrium measure of a set from this family are mutually absolutely continuous.

We prove that the sequence of Widom-Hilbert factors for the equilibrium measure of a non-polar compact subset of $\mathbb{R}$ is bounded below by 1. We give a sufficient condition for this sequence to be unbounded above.

We suggest definitions for the Szegő class and the isospectral torus for a generic subset of $\mathbb{R}$.

Keywords: Orthogonal polynomials, Chebyshev polynomials, Cantor sets, Szegő class, isospectral torus, Widom factors, Parreau-Widom sets, Green function, Hausdorff measure, equilibrium measure, almost periodicity.

ÖZET

BAZI ÖZEL DURUMLAR İÇİN AŞIT POLİNOMLARIN ASİMPTOTİKLERİ

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Matematik, Doktora
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Fraktalların üstündeki ortogonal ve Chebyshev polinomlarının asimptotiklerini çalıştık. Brück-Büger genelleştirilmiş Julia kümelerini ve [62] numaralı kaynakta tanımlanan zayıf denge Cantor kümelerini ele aldık.

Zayıf denge Cantor kümeleri için Parreau-Widom koşulunun ve Green fonksiyonunun en yüksek düzgünlüğe sahip olmasının tam karakterizasyonlarını verdik. Bunun yanında, küçük parametreler için, bu kümelere karşılık gelen Hausdorff ölçümü ile denge ölçümünün karşılıklı olarak mutlak sürekli olduğunu gösterdik.

Reel doğrunun kutupsuz altkümelerinin denge ölçümlerine karşılık gelen Widom-Hilbert çarpanlarının dizisinin alttan 1 ile sınırlı olduğunu kanıtladık. Bu dizinin üstten sınırsız olması için yeter koşul verdik.

Reel doğrunun tipik bir altkümesi için, Szegő sınıfı ve eşspektral simit kavramlarına tanımlar önerdik.

Anahtar sözcükler: Ortogonal polinomlar, Chebyshev polinomları, Cantor kümeleri, Szegő sınıfı, eşspektral simit, Widom çarpanları, Parreau-Widom kümeleri, Green fonksiyonu, Hausdorff ölçümü, denge ölçümü, yaklaşık periyodiklik.

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List of Symbols and Abbreviations

$\Lambda_h(E)$	h-Hausdorff measure of E
$\mathcal{R}$	class of regular polynomial sequences
$\mathrm{Cap}(K)$	logarithmic capacity of K
$\mathrm{ess\,supp}(\mu)$	essential support of μ
$\mathrm{IT}(K)$	isospectral torus on K
$\mathrm{supp}(\mu)$	support of μ
$\mathrm{Sz}(K)$	Szegő class of measures on K
μ	Borel measure
$\mu'(x)$	Radon-Nikodym derivative of μ
μ_K	equilibrium measure for K
Ω_K	connected component of $\overline{\mathbb{C}} \setminus K$ that contains the point ∞
$\sigma(H)$	spectrum of H
$F(f)$	Fatou set for f
$F_{(f_n)}$	Generalized Fatou set for (f_n)

g_{Ω_K}	Green function with pole at infinity on Ω_K
H	Jacobi matrix
$I(\mu)$	logarithmic energy associated with μ
$J(f)$	Julia set for f
$J_{(f_n)}$	Generalized Julia set for (f_n)
$K(\gamma)$	Weakly equilibrium Cantor set
$P_n(\cdot;\mu)$	n-th monic orthogonal polynomial for μ
$p_n(\cdot;\mu)$	n-th orthonormal polynomial for μ
$T_{n,K}$	n-th Chebyshev polynomial on K
U^μ	logarithmic potential for μ
$W_n^2(\mu)$	nth Widom-Hilbert factor for μ
$W_n(K)$	n-th Widom factor for the sup-norm on K
Reg	class of regular measures in the sense of Stahl-Totik
DCT	direct Cauchy theorem
DOS	density of states
FTA	fundamental theorem of algebra
IDS	integrated density of states

Chapter 1

Introduction

In this thesis, we present some new results in the spectral theory of orthogonal polynomials, potential theory, approximation theory, the theory of one dimensional complex dynamics and geometric measure theory.

Asymptotics of orthogonal polynomials were studied in detail in [120] by Stahl and Totik. These asymptotics are given in terms of concepts from potential theory such as capacity, equilibrium measure and Green's function. Some properties of regular measures (later on these measures are called regular in the sense of Stahl-Totik) were investigated. Orthogonal polynomials (more precisely the norm of monic orthogonal polynomials in the corresponding Hilbert space divided by the n-th power of logarithmic capacity of the support of the measure) for regular measures obey some asymptotics. Orthogonal polynomials for measures in the Szegő class or in the isospectral torus associated with a finite union of intervals (see e.g. [47] for more information), or more generally a Parreau-Widom set, have even stronger asymptotics. It is an interesting problem to define the Szegő class and the isospectral torus on more general sets. See e.g. [77, 88] for previous attempts on this issue.

In the absence of a general theory of measures whose orthogonal polynomials satisfy the strong asymptotics mentioned above, it is natural to consider concrete examples and numerical experiments related to fractal sets. Some properties of orthogonal polynomials and recurrence coefficients have been known for equilibrium measures of polynomial Julia sets for more than 30 years, see e.g. [21, 22]. Here, different aspects of orthogonal polynomials on some special Cantor-type sets which can be classified as generalized Julia sets are investigated. Some asymptotics concerning Chebyshev polynomials are also studied on sets which are obtained by polynomial iterations. We also study the asymptotics of orthogonal polynomials associated with the equilibrium measure of any non-polar subset of $\mathbb{R}$. In addition to these problems, we consider the relation between Hausdorff measures and equilibrium measures on Cantor sets.

The results contained in this thesis can be found in [1–11]. The articles [5–9] are joint work with Goncharov, [10] is a joint work with Hatinoğlu and Goncharov, [11] is a joint work with Goncharov and Şimşek. We use the materials from [10,11] in the first chapter. The results contained in the chapters 2, 3, 4, 5, 6, 7, 8, 9, 10 are based on [5], [7], [8], [1], [2], [11], [3], [4], [9], respectively.

The plan of this thesis is as follows. In the next section, we provide background information. In Section 1.2, we summarize the main results. In chapters 2-9 we prove these results. In Chapter 10, we discuss some open problems.

1.1 Background and notation

1.1.1 Potential theory

Let $K \subset \mathbb{C}$ be a compact set and let $\mathcal{M}(K)$ denote the set of all unit Borel (probability) measures supported on K. For $\mu \in \mathcal{M}(K)$, we define the *logarithmic potential* as

$$U^\mu(z) := \int \log \frac{1}{|z-t|}\, d\mu(t).$$

The logarithmic energy associated with μ is defined as

$$I(\mu) := \int \left(\int \log \frac{1}{|z-t|}\, d\mu(t) \right) d\mu(z).$$

If $\inf_{\mu\in\mathcal{M}(K)} I(\mu) = \infty$, then we say that K is a *polar set* (or equivalently zero capacity set). Conversely, if $\inf_{\mu\in\mathcal{M}(K)} I(\mu) =: V_K < \infty$ then there is a unique probability measure μ_K which assumes the infimum. This measure is called the *equilibrium measure* for K. In this case, we define the *logarithmic capacity* of K as $\mathrm{Cap}(K) := \exp(-V_K)$.

A sequence (μ_n) from $\mathcal{M}(K)$ is *weak-star convergent* to $\mu \in \mathcal{M}(K)$ if

$$\lim_{n\to\infty} \int f(t)\, d\,\mu_n(t) = \int f(t)\, d\,\mu(t),$$

for every $f \in C(K)$ and we write $\mu_n \to \mu$ in this case.

For a non-polar compact set $K \subset \mathbb{C}$, let Ω_K be the connected component of $\overline{\mathbb{C}} \setminus K$ that contains the point ∞, where $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Then the *Green function with pole at infinity* is a real and non-negative valued function defined as

$$g_{\Omega_K}(z) := \begin{cases} -U^{\mu_K}(z) + V_K & \text{if } z \in \Omega_K, \\ \lim\limits_{z'\to z} \sup\limits_{z'\in\Omega_K} g_{\Omega_K}(z) & \text{if } z \in \partial\Omega_K. \end{cases}$$

A compact set $K \subset \mathbb{C}$ is said to be *regular with respect to the Dirichlet problem* if $g_{\Omega_K}(z)$ is continuous. We have to note that $\mathrm{Cap}(K) = \mathrm{Cap}(\partial\Omega_K)$ and $\mathrm{supp}(\mu_K) \subset \partial\Omega_K$ where $\mathrm{supp}(\cdot)$ denotes the support.

The next result is due to Brolin [36].

Lemma 1.1.1. *Let K and L be two non-polar compact subsets of $\mathbb{C}$ such that $K \subset L$. Let $(\mu_n)_{n=1}^{\infty}$ be a sequence of probability measures supported on L that converges to a measure μ supported on K. Suppose that the following two conditions hold :*

(a) $\liminf\limits_{n\to\infty} U^{\mu_n}(z) \geq V_K$ *on* K.

(b) $\mathrm{supp}(\mu_K) = K$.

Then $\mu = \mu_K$.

For a more complete description of potential theory, we refer the reader to [106, 110].

1.1.2 Chebyshev polynomials

Let $K \subset \mathbb{C}$ be compact set containing infinitely many points. We use $\| \cdot \|_{L^\infty(K)}$ to denote the sup-norm on K. The unique monic polynomial $T_{n,K}$ of degree n satisfying

$$\|T_{n,K}\|_{L^\infty(K)} = \min\{\|Q_n\|_{L^\infty(K)} : Q_n \text{ monic of degree } n\}$$

is called the n-th *Chebyshev polynomial* on K.

If $\mathrm{Cap}(K) > 0$ then we define the *n-th Widom factor for the sup-norm* on K by

$$W_n(K) := \|T_{n,K}\|_{L^\infty(K)}/\mathrm{Cap}(K)^n.$$

A sequence $(a_n)_{n=1}^\infty$ with $a_n > 0$ has *subexponential growth* if $a_n = \exp(n \cdot \varepsilon_n)$ with $\varepsilon_n \to 0$ as $n \to \infty$. It is due to Schiefermayr [111] that $W_n(K) \geq 2$ if $K \subset \mathbb{R}$. It is also known that (see [54, 121])

$$\|T_{n,K}\|_{L^\infty(K)}^{1/n} \to \mathrm{Cap}(K) \tag{1.1}$$

as $n \to \infty$. This implies that $(1/n) \log W_n(K) \to 0$ as $n \to \infty$. Thus, the sequence of Widom factors for the sup-norm has subexponential growth for each non-polar compact set K. See [129, 131, 134] for further discussion.

1.1.3 Orthogonal polynomials and Jacobi Matrices

For a unit Borel measure μ with an infinite compact support on $\mathbb{C}$, using the Gram-Schmidt process for the set $\{1, z, z^2, \dots\}$ in $L^2(\mu)$, one can find the sequence of polynomials $(p_n(\cdot; \mu))_{n=0}^\infty$ with positive leading coefficients κ_n satisfying

$$\int p_m(z;\mu)\overline{p_n(z;\mu)}\, d\mu(z) = \delta_{mn}$$

where $p_n(\cdot;\mu)$ is of degree n. Here, $p_n(\cdot;\mu)$ is called the *n-th orthonormal polynomial* for μ. We denote the n-th monic orthogonal polynomial $p_n(\cdot;\mu)/\kappa_n$ by $P_n(\cdot;\mu)$.

If μ is supported on $\mathbb{R}$ and we assume that $P_{-1}(\cdot;\mu) := 0$ and $P_0(\cdot;\mu) := 1$ then there are two bounded sequences $(a_n)_{n=1}^\infty$, $(b_n)_{n=1}^\infty$ such that the polynomials $(P_n(\cdot;\mu))_{n=0}^\infty$ satisfy a three-term recurrence relation

$$P_{n+1}(x;\mu) = (x - b_{n+1})P_n(x;\mu) - a_n^2\, P_{n-1}(x;\mu), \qquad n \in \mathbb{N}_0,$$

where $a_n > 0$, $b_n \in \mathbb{R}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The norms of orthogonal polynomials can be written in terms of the recurrence coefficients for $n \geq 1$:

$$\|P_n(\cdot;\mu)\|_{L^2(\mu)} = a_1 \cdots a_n. \tag{1.2}$$

Conversely, if two bounded sequences $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ are given with $a_n > 0$ and $b_n \in \mathbb{R}$ for each $n \in \mathbb{N}$ then we can define the corresponding *Jacobi matrix* H as the following:

$$H = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \dots \\ a_1 & b_2 & a_2 & 0 & \dots \\ 0 & a_2 & b_3 & a_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Here, H is a self-adjoint bounded operator acting in $l^2(\mathbb{N})$. The (scalar valued) spectral measure μ of H for the cyclic vector $(1,0,\dots)^T$ is the measure that has $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ as recurrence coefficients. Due to this one to one correspondence between probability measures and Jacobi matrices, we also denote the Jacobi matrix associated with μ by $H(\mu)$ or $H((a_n, b_n)_{n=1}^\infty)$. For a discussion of the spectral theory of orthogonal polynomials on $\mathbb{R}$ we refer the reader to [116, 135].

Let $c = (c_n)_{n=-\infty}^\infty$ be a two sided sequence taking values on $\mathbb{C}$ and $c^j = (c_{n+j})_{n=-\infty}^\infty$ for $j \in \mathbb{Z}$. Then c is called *almost periodic* if $\{c^j\}_{j\in\mathbb{Z}}$ is precompact in $l^\infty(\mathbb{Z})$. A one-sided sequence $d = (d_n)_{n=1}^\infty$ is called almost periodic if it is the restriction of a two sided almost periodic sequence to $\mathbb{N}$. Each one sided almost periodic sequence has only one extension to $\mathbb{Z}$ which is almost periodic, see Section 5.13 in [116]. Hence one-sided and two sided almost periodic sequences are essentially the same objects. A Jacobi matrix $H(\mu)$ is called almost periodic if the sequences of recurrence coefficients $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ for μ are almost periodic.

A sequence $s = (s_n)_{n=1}^{\infty}$ is called *asymptotically almost periodic* if there is an almost periodic sequence $d = (d_n)_{n=1}^{\infty}$ such that $d_n - s_n \to 0$ as $n \to \infty$. In this case d is unique and it is called the almost periodic limit. See [100, 116, 123] for more details on almost periodic functions.

The next theorem which is due to Stahl-Totik (Theorem 3.1.1. in [120]) illustrates how asymptotics of orthogonal polynomials are related to potential theoretic tools.

Theorem 1.1.2. *Let μ be a probability measure with $\operatorname{supp}(\mu) = K$. Then the following are pairwise equivalent.*

(i) $\lim_{n\to\infty} \kappa_n^{1/n} = \frac{1}{\mathrm{Cap(K)}}$ *where κ_n is the leading coefficient of the n-th orthonormal polynomial $p_n(z;\mu)$.*

(ii) The limit

$$\lim_{n\to\infty} |p_n(z;\mu)|^{1/n} = \exp(g_{\Omega_K}(z))$$

holds true locally uniformly at the complement of the convex hull of K.

(iii) The limit

$$\limsup_{n\to\infty} |p_n(z;\mu)|^{1/n} = 1$$

holds true on $\partial\Omega_K$ except possibly on a polar set.

We say that μ is *regular in the sense of Stahl-Totik* and write $\mu \in \mathbf{Reg}$ if μ satisfies one of the conditions $(i), (ii)$ and (iii) in Theorem 1.1.2.

The n-th monic orthogonal polynomial for μ satisfies the following property:

$$\|P_n(\cdot;\mu)\|_{L^2(\mu)} = \min\{\|Q_n\|_{L^2(\mu)} : Q_n \text{ monic polynomial of degree } n\} \tag{1.3}$$

By (1.3) and using the assumption that μ is a unit measure, we have

$$\|P_n(\cdot;\mu)\|_{L^2(\mu)} \le \|T_{n,\operatorname{supp}(\mu)}\|_{L^2(\mu)} \le \|T_{n,\operatorname{supp}(\mu)}\|_{L^\infty(\operatorname{supp}(\mu))}$$

for each $n \in \mathbb{N}$. Thus, by (1.1) it follows that

$$\limsup_{n\to\infty} \|P_n(\cdot;\mu)\|_{L^2(\mu)}^{1/n} \le \mathrm{Cap}(\operatorname{supp}(\mu)). \tag{1.4}$$

It is due to H. Widom [140] (see also [114]) that for any non-polar compact subset K of $\mathbb{R}$, we have $\mu_K \in \mathbf{Reg}$. For a probability measure μ let $\operatorname{supp}(\mu) = K \subset \mathbb{R}$ where K is compact and non-polar. Furthermore, let $f(t)$ be the Radon-Nikodym derivative of μ with respect to μ_K. If $f(t) > 0$ μ_K-almost everywhere on K then $\mu \in \mathbf{Reg}$, see Chapter 4 in [120]. The condition $f > 0$ almost everywhere is called the *Erdős-Turan criterion* of regularity. There are many other criteria for regularity of measures. See [114, 120] for a more general exposition of known results.

Suppose that μ is a unit Borel measure on $\mathbb{C}$ and $\operatorname{Cap}(\operatorname{supp}(\mu)) > 0$. Let us define *n-th Widom-Hilbert factor* as

$$W_n^2(\mu) := \frac{||P_n(\cdot;\mu)||_{L^2(\mu)}}{\operatorname{Cap}(\operatorname{supp}(\mu))^n}.$$

By (1.4) and part (i) of Theorem 1.1.2, regularity of μ is equivalent to the condition

$$\lim_{n\to\infty} W_n^2(\mu)^{1/n} = 1 \tag{1.5}$$

In general, for $1 \le p \le \infty$, we can define $W_n^p(\mu)$ as $\frac{\inf_{\mathcal{M}_n} ||Q||_{L^p(\mu)}}{\operatorname{Cap}(\operatorname{supp}(\mu))^n}$ where $||\cdot||_{L^p(\mu)}$ is the standard norm in the space $L^p(\mu)$ and $\mathcal{M}_n$ is the set of all monic polynomials of degree n. In the case $p = \infty$, we do not use the superscript ∞. Since $\mu(\mathbb{C}) = 1$, by Hölder's inequality, $W_n^p(\mu) \le W_n^r(\mu)$ for $1 \le p \le r \le \infty$.

As in the case $p = \infty$, the value W_n^p is invariant under dilation and translation. Indeed, the map $\varphi(z) = w = az + b$ with $a \neq 0$ transforms μ_0 into μ with $d\mu(w) = d\mu_0(\frac{w-b}{a})$. If $Q_n(\mu_0, z) = z^n + \cdots$ realizes the infimum of norm in $L^p(\mu_0)$ then $Q_n(\mu, w) = a^n Q_n(\mu_0, \frac{w-b}{a})$ does so in the space $L^p(\mu)$. Therefore, $||Q_n(\mu,\cdot)||_{L^p(\mu)} = |a|^n \cdot ||Q_n(\mu_0,\cdot)||_{L^p(\mu)}$. On the other hand, $\operatorname{Cap}(aK + b) = |a| \operatorname{Cap}(K)$. From here, $W_n^p(\mu) = W_n^p(\mu_0)$.

1.1.4 Parreau-Widom Sets and Szegő Class

Let us introduce another concept which is important for the recent theory of orthogonal and Chebyshev polynomials. Let $K \subset \mathbb{R}$ be a non-polar compact set that is regular with respect to the Dirichlet problem. Furthermore, let $\{c_j\}_j$ denote the set of critical points of g_{Ω_K}. Then K is said to be a *Parreau-Widom* set if $\sum_j g_{\Omega_K}(c_j) < \infty$. The set of critical points of a regular set is countable and Parreau-Widom sets have positive Lebesgue measure. For different aspects of Parreau-Widom sets, see [43, 67, 142].

A measure μ supported on $\mathbb{R}$ can be written as

$$d\mu(x) = \mu'(x)dx + d\mu_{\mathrm{s}}(x)$$

by Lebesgue's decomposition theorem, where $\mu'(x)$ is the absolutely continuous part and $d\mu_{\mathrm{s}}$ denotes the singular part with respect to the Lebesgue measure.

Following [44], let us define the Szegő class of measures on a given Parreau-Widom set K. By $\operatorname{ess\,supp}(\cdot)$ we denote the *essential support* of the measure, that is the set of accumulation points of the support. We have $\operatorname{Cap}(\operatorname{supp}(\mu)) = \operatorname{Cap}(\operatorname{ess\,supp}(\mu))$, see Section 1 of [116]. A measure μ is in the *Szegő class* of K if

(i) $\operatorname{ess\,supp}(\mu) = K$.

(ii) $\int_K \log \mu'(x)\, d\mu_K(x) > -\infty$. (Szegő condition)

(iii) the isolated points $\{x_n\}$ of $\operatorname{supp}(\mu)$ satisfy $\sum_n g_{\Omega_K}(x_n) < \infty$.

By Theorem 2 in [43] and its proof, (ii) can be replaced by one of the following conditions in our terms on the recurrence coefficients associated with μ so that the definition includes the same family of measures:

(ii′) $\limsup_{n\to\infty} W_n^2(\mu) > 0$. (Widom condition) (ii$''$) $\liminf_{n\to\infty} W_n^2(\mu) > 0$. (Widom condition 2)

We denote the Szegő class of K by $\mathrm{Sz}(K)$. If we combine (ii$'$) with (1.4) and the equivalence of regularity with the condition given in (1.5) we see that $\mu \in \mathrm{Sz}(K)$ implies the regularity of μ in the sense of Stahl-Totik.

Let μ be a probability measure which is purely absolutely continuous with $\mathrm{supp}(\mu) = [-1,1]$ and suppose that $E(\mu') := \int_{[-1,1]} \log \mu'(x)\, d\mu_{[-1,1]}(x) = \int_{-1}^{1} \frac{\log \mu'(x)}{\pi\sqrt{1-x^2}}\, dx > -\infty$, which means that this integral converges for it cannot be $+\infty$. By setting $z = \infty$, in part (ii) of Theorem 1.1.2 we get (see, e.g. [135], p.26, (10.4) in [127])

$$\lim_n W_n^2(\mu) = \sqrt{\pi}\, \exp(E(\mu')/2),$$

Thus, if μ is an absolutely continuous measure and satisfies $\mu \in \mathrm{Sz}([-1,1])$ then the sequence of Widom-Hilbert factors converges to some positive value. The inverse implication is also valid: If $\mathrm{supp}(\mu) = [-1,1]$, μ is purely absolutely continuous and $\lim_n W_n^2(\mu)$ exists in $(0,\infty)$ then $\mu \in \mathrm{Sz}([-1,1])$ (see Theorem 2, [43]).

1.1.5 Generalized Julia sets and $K(\gamma)$

Let $(f_n)_{n=1}^\infty$ be a sequence of rational functions with $\deg f_n \geq 2$ in $\overline{\mathbb{C}}$ and $F_n := f_n \circ f_{n-1} \circ \ldots \circ f_1$. The function F_n is used in the text always for the n-th composition and ρ_n for the leading coefficient of F_n for $n \geq 1$. The domain of normality for $(F_n)_{n=1}^\infty$ in the sense of Montel is called the *Fatou set* for (f_n). The complement of the Fatou set in $\overline{\mathbb{C}}$ is called the *Julia set* for (f_n). We denote them by $F_{(f_n)}$ and $J_{(f_n)}$ respectively. These generalized Fatou and Julia sets were considered first in [56]. In particular, if $f_n = f$ for some fixed rational function f for all n then $F(f)$ and $J(f)$ are used instead. To distinguish this last case, the word *autonomous* is used in the literature.

The next result on the Chebyshev polynomials for autonomous Julia sets is due to Kamo-Borodin [72]:

Theorem 1.1.3. *Let $f(z) = z^m + a_{m-1}z^{m-1} + \ldots + a_0$ be a nonlinear complex polynomial and $T_k(z)$ be a Chebyshev polynomial on $J(f)$. Then $(T_k \circ f^{(n)})(z)$ is also a Chebyshev polynomial on $J(f)$ for each $n \in \mathbb{N}$. In particular, this implies that there exists a complex number τ such that $f^{(n)}(z) - \tau$ is a Chebyshev polynomial on $J(f)$ for all $n \in \mathbb{N}$.*

Let $f_n(z) = \sum_{j=0}^{d_n} a_{n,j} \cdot z^j$ where $d_n \geq 2$ and $a_{n,d_n} \neq 0$ for all $n \in \mathbb{N}$. Following [38], we say that (f_n) is a *regular polynomial sequence* if the following properties are satisfied for some positive constants A_1, A_2, A_3:

- $|a_{n,d_n}| \geq A_1$ for all $n \in \mathbb{N}$.
- $|a_{n,j}| \leq A_2 |a_{n,d_n}|$ for $j = 0, 1, \ldots, d_n - 1$ and $n \in \mathbb{N}$.
- $\log |a_{n,d_n}| \leq A_3 \cdot d_n$, holds for all $n \in \mathbb{N}$.

If (f_n) is a regular polynomial sequence then we write $(f_n) \in \mathcal{R}$. In this case, by [38], $J_{(f_n)}$ is a compact set in $\mathbb{C}$ that is regular with respect to the Dirichlet problem. Let $\mathcal{K}_{(f_n)} := \{z \in \mathbb{C} : (F_n(z))_{n=1}^{\infty}$ is bounded$\}$ and $\mathcal{A}_{(f_n)}(\infty) := \{z \in \overline{\mathbb{C}} :\ F_n(z)$ goes locally uniformly to ∞ as $n \to \infty\}$.

We remark that, for a sequence $(f_n) \in \mathcal{R}$, the degrees of polynomials need not be the same and they do not have to be bounded above either. Julia sets $J_{(f_n)}$ when $(f_n) \in \mathcal{R}$ were introduced and considered in [40] and all results given in the next theorem are from Section 2 and Section 4 of the paper [38]. While (1.7) is contained in the proof of Theorem 4.2 in [38], (1.8) follows by comparing the right parts of these two equations, using that $g_{\Omega_{J_{(f_n)}}}$ has a logarithmic singularity at infinity and $F_k(z)$ goes locally uniformly to ∞ for such z.

Theorem 1.1.4. *Let $(f_n) \in \mathcal{R}$. Then the following propositions hold:*

(a) The set $\mathcal{A}_{(f_n)}(\infty)$ is an open connected set containing ∞. Moreover, for every $R > 1$ satisfying the inequality

$$A_1 R \left(1 - \frac{A_2}{R-1}\right) > 2, \tag{1.6}$$

the compositions $F_n(z)$ goes locally uniformly to infinity whenever $z \in \triangle_R$ where $\triangle_R = \{z \in \overline{\mathbb{C}} : |z| > R\}$.

(b) $\mathcal{A}_{(f_n)}(\infty) = \cup_{k=1}^{\infty} {F_k}^{-1}(\triangle_R)$ *and* $f_n(\overline{\triangle_R}) \subset \triangle_R$ *if* $R > 1$ *satisfies* (1.6). *Furthermore,* $\mathcal{A}_{(f_n)}(\infty)$ *is a domain containing* $\triangle_R$ *and we have* $J_{(f_n)} = \partial \mathcal{A}_{(f_n)}(\infty)$.

(c) The Green function for the complement of the set is given by

$$g_{\Omega_{J_{(f_n)}}}(z) = \begin{cases} \lim_{k\to\infty} \frac{1}{d_1 \cdots d_k} \log |F_k(z)| & \textit{if } z \in \mathcal{A}_{(f_n)}(\infty), \\ 0 & \textit{otherwise.} \end{cases} \tag{1.7}$$

Moreover,

$$g_{\Omega_{J_{(f_n)}}}(z) = \lim_{k\to\infty} \frac{1}{d_1 \cdots d_k} g_{\Omega_{J_{(f_n)}}}(F_k(z)) \tag{1.8}$$

where $z \in \mathcal{A}_{(f_n)}(\infty)$. *In both* (1.7) *and* (1.8), *limits hold locally uniformly in* $\mathcal{A}_{(f_n)}(\infty)$.

(d) The logarithmic capacity of the compact set $J_{(f_n)}$ *is given by the expression*

$$\mathrm{Cap}(J_{(f_n)}) = \exp\left(- \lim_{k\to\infty} \sum_{j=1}^{k} \frac{\log |a_{j,d_j}|}{d_1 \cdots d_j} \right).$$

(e) $F_k^{-1}(F_k(J_{(f_n)})) = J_{(f_n)}$ *and* $J_{(f_n)} = F_k^{-1}(J_{(f_{k+n})})$ *for all* $k \in \mathbb{N}$. *Here we use the notation* $(f_{k+n}) = (f_{k+1}, f_{k+2}, f_{k+3}, \ldots)$.

(f) $\triangle_R \subset \overline{F_k^{-1}(\triangle_R)} \subset F_{k+1}^{-1}(\triangle_R) \subset \mathcal{A}_{(f_n)}(\infty)$ *for all* $k \in \mathbb{N}$ *and each* $R > 1$ *satisfying* (1.6).

(g) $\partial \mathcal{A}_{(f_n)}(\infty) = J_{(f_n)}$ *and* $\mathcal{K}_{(f_n)} = \overline{\mathbb{C}} \setminus \mathcal{A}_{(f_n)}(\infty)$. *Thus,* $\mathcal{K}_{(f_n)}$ *is a compact subset of* $\mathbb{C}$ *and* $J_{(f_n)}$ *has no interior points.*

The construction in this paragraph follows [62]. Let, here and in the sequel, $\gamma_0 := 1$ and $\gamma = (\gamma_k)_{k=1}^{\infty}$ be a sequence satisfying $0 < \gamma_k < 1/4$ for all $k \in \mathbb{N}$ provided that $\sum_{k=1}^{\infty} 2^{-k} \log(1/\gamma_k) < \infty$. We define $(f_n)_{n=1}^{\infty}$ by $f_1(z) := 2z(z-1)/\gamma_1 + 1$ and $f_n(z) := z^2/(2\gamma_n) + 1 - 1/(2\gamma_n)$ for $n > 1$. Let $E_0 := [0,1]$ and $E_n := F_n^{-1}([-1,1])$. Then, E_n is a union of 2^n disjoint non-degenerate compact intervals in $[0,1]$ and $E_n \subset E_{n-1}$ for all $n \in \mathbb{N}$. It turns out that, $K(\gamma) := \cap_{s=0}^{\infty} E_s$ is a non-polar Cantor set in $[0,1]$ where $\{0,1\} \subset K(\gamma)$.

Let us look more carefully at the construction. We denote the connected components of E_n by $I_{j,n}$ and the length of $I_{j,n}$ by $l_{j,n}$ for $j = 1, \ldots, 2^n$, call these intervals as *basic intervals of n-th level*, define $a_{j,n}$ and $b_{j,n}$ by $[a_{j,n}, b_{j,n}] := I_{j,n}$. Let $I_{1,0} := E_0$ and $a_{j_1,n} > a_{j_2,n}$ if $j_1 > j_2$. Then we have $I_{2j-1,n+1} \cup I_{2j,n+1} \subset I_{j,n}$ for all $n \in \mathbb{N}_0$ where $a_{2j-1,n+1} = a_{j,n}$ and $b_{2j,n+1} = b_{j,n}$. Denoting the gap $(b_{2j-1,n+1}, a_{2j,n+1})$ by $C_{j,n}$, for $1 \leq j \leq 2^n$ and $n \in \mathbb{N}_0$, it follows that

$$K(\gamma) = [0,1] \setminus (\cup_{n=0}^{\infty} \cup_{1 \leq j \leq 2^n} C_{j,n}).$$

Using Theorem 11 in [59], we see that $\mu_{E_n}(I_{j,n}) = 1/2^n$ for all $1 \leq j \leq 2^n$ and $n \in \mathbb{N}_0$. Furthermore, $\mu_{E_k}(I_{j,n}) = 1/2^n$ for $k > n$ since $I_{j,n} \cap E_k$ consists of 2^{k-n} basic disjoint intervals of k-th level. Since $(E_k)_{k=0}^{\infty}$ is a decreasing sequence of sets with $\cap_{s=0}^{\infty} E_s = K(\gamma)$, by part (ii) of Theorem A.16 in [114], it follows that

$$\mu_{K(\gamma)}(I_{j,n}) = \mu_{K(\gamma)}(I_{j,n} \cap K(\gamma)) = 1/2^n. \tag{1.9}$$

The last in particular implies that $\mu_{K(\gamma)}([0,r]) \in \mathbb{Q}$ for all $r \in \mathbb{R}$ with $r \notin K(\gamma)$.

It follows from the definition of equilibrium measure that $\operatorname{supp}(\mu_{K(\gamma)}) \subset K(\gamma)$. We also have $K(\gamma) \subset \operatorname{supp}(\mu_{K(\gamma)})$ since for any $x \in K(\gamma)$ and $\epsilon > 0$ the open ball $B(x,\epsilon)$ centered at x with radius ϵ contains a basic interval $I_{j,n}$. From the above paragraph $\mu_{K(\gamma)}(I_{j,n} \cap K(\gamma)) > 0$ and therefore $K(\gamma) = \operatorname{supp}(\mu_{K(\gamma)})$.

There are other ways to define $K(\gamma)$. Let us consider the original definition given in [62]. We define the basic intervals using a different notation since we use both of these notations in the subsequent chapters. Let $r_0 := 1$, $r_n := \gamma_n r_{n-1}^2$ and define

$$\Phi_1(x) := x - 1 \text{ and } \Phi_{2^{n+1}}(x) := \Phi_{2^n}(x) \cdot (\Phi_{2^n}(x) + r_n) \tag{1.10}$$

for $s \in \mathbb{N}_0$ in a recursive fashion. Thus, $\Phi_2(x) = x \cdot (x-1)$ for each γ, whereas, for $n \geq 2$, the polynomial Φ_{2^n} essentially depends on the parameter γ. Then for $n \in \mathbb{N}_0$ we have

$$E_n = \{x \in \mathbb{R} : \Phi_{2^{n+1}}(x) \leq 0\} = \left(\frac{2}{r_n}\Phi_{2^n} + 1\right)^{-1}([-1,1]) = \cup_{j=1}^{2^n} I_{j,n}, .$$

In addition, using Green function $g_{\Omega_{K(\gamma)}}$ (see Corollary 1 and Section 6 in [62]), one can easily find $\mathrm{Cap}(K(\gamma)) = \exp\left(\sum_{k=1}^{\infty} 2^{-k}\log\gamma_k\right)$. Also, we discuss a modification of $K(\gamma)$ in Chapter 4.

1.1.6 Hausdorff measure

A function $h : \mathbb{R}_+ \to \mathbb{R}_+$ is called a *dimension function* if it is increasing, continuous and $h(0) = 0$. Given a Borel set $E \subset \mathbb{C}$, its *h-Hausdorff measure* is defined as

$$\Lambda_h(E) = \lim_{\delta\to 0^+} \inf\left\{\sum h(r_j) : E \subset \bigcup B(z_j, r_j) \text{ with } r_j \leq \delta\right\},$$

For a dimension function h, a Borel set $K \subset \mathbb{C}$ is an h-set if $0 < \Lambda_h(K) < \infty$. To denote the Hausdorff measure for $h(t) = t^\alpha$, Λ_α is used. *Hausdorff dimension* of K is defined as $\inf\{\alpha \geq 0 : \Lambda_\alpha(K) = 0\}$ and denoted by $\mathrm{HD}(\cdot)$. *Hausdorff dimension of a unit Borel measure* μ supported on $\mathbb{C}$ is defined by $\dim(\mu) := \inf\{\mathrm{HD}(K) : \mu(K) = 1\}$. We use $|\cdot|$ in order to denote the Lebesgue measure on the real line.

The next theorem follows immediately from the definition of Λ_h. It is a simple part of Frostman's theorem (see e.g. T.D.1 in [57]).

Theorem 1.1.5. *Let h be a dimension function. If μ is a positive Borel measure such that*

$$\mu(B(z, r)) \leq h(r)$$

for all z and r, then the following is valid for any Borel set E

$$\mu(E) \leq \Lambda_h(E).$$

For the converse relation we use a simple version of T.7.6.1.(a) in [105]. Here, $b(1)$ is the Besicovitch covering number corresponding to the line (one can take $b(1) = 5$).

Theorem 1.1.6. *Assume that μ is a Borel probability measure on $\mathbb{R}$ and A is a bounded Borel subset of $\mathbb{R}$. If there exists a constant C such that*

$$h(r) \leq C \cdot \mu(B(x, r))$$

for all $x \in A$ and $r > 0$, then for any Borel set $E \subset A$

$$\Lambda_h(E) \leq b(1)\, C \cdot \mu(E).$$

The set $K(\gamma)$ is weakly equilibrium in the following sense. Given $s \in \mathbb{N}$, we uniformly distribute the mass 2^{-s} on each $I_{j,s}$ for $1 \leq j \leq 2^{-s}$. Let us denote by λ_s the normalized in this sense Lebesgue measure on E_s, so $d\lambda_s = (2^s l_{j,s})^{-1} dt$ on $I_{j,s}$.

Theorem 1.1.7. *([62],T.4)The sequence $(\lambda_s)_{s=1}^{\infty}$ is weak star convergent to $\mu_{K(\gamma)}$.*

1.1.7 Smoothness of Green's functions

Let $K \subset \mathbb{R}$ be a non-polar compact set. Then the Green function g_{Ω_K} is said to be *Hölder continuous* with exponent β if there exists a number $A > 0$ such that $g_{\Omega_K}(z) \leq A(\mathrm{dist}(z, K))^{\beta}$ holds for all z satisfying $\mathrm{dist}(z, K) \leq 1$. This exponent is at most 1/2. If the Green function of a set $K \subset \mathbb{R}$ is Hölder continuous with the exponent 1/2 then the Green function is said to be *optimally smooth.*

Smoothness properties of Green functions are examined for a variety of sets. For the complement of autonomous Julia sets, see [75] and for the complement of $J_{(f_n)}$ see [37, 38]. When K is a symmetric Cantor-type set in $[0, 1]$, it is possible to give a sufficient and necessary condition in order the Green function for the complement of the Cantor set is Hölder continuous with the exponent 1/2. See Chapter 5 in [128] for details. For applications of smoothness of Green functions, we refer the reader to [33].

By the next theorem, which was proven in [124], it is possible to associate the density properties of equilibrium measures with the smoothness properties of Green's functions.

Theorem 1.1.8. *Let $K \subset \mathbb{C}$ be a non-polar compact set which is regular with respect to the Dirichlet problem. Let $z_0 \in \partial\Omega$ where Ω is the unbounded component* *of* $\overline{\mathbb{C}} \setminus K$. *Then for every* $0 < r < 1$ *we have*

$$\int_0^r \frac{\mu_K(D_t(z_0))}{t} dt \leq \sup_{|z-z_0|=r} g_\Omega(z) \leq 3 \int_0^{4r} \frac{\mu_K(D_t(z_0))}{t} dt.$$

1.2 An overview of the results

1.2.1 Orthogonal polynomials

1.2.1.1 Orthogonal polynomials on fractals

In this subsection, we discuss the results concerning orthogonal polynomials for the equilibrium measure of some Cantor-type sets, and more generally fractals on $\mathbb{C}$. Note that equilibrium measures distinguish two different cases on $\mathbb{R}$: When $|E| > 0$, μ_E has a non-trivial absolutely continuous part (see e.g. [101]) and if $|E| = 0$ then μ_E is a singular continuous measure. Some of the results given below are valid for both of these cases.

In [21], Barnsley et al. showed that for a non-linear polynomial f, a subsequence of orthogonal polynomials with respect to $\mu_{J(f)}$ can be written explicitly. They used the invariance of the equilibrium measure of $J(f)$ with respect to f. In Chapter 4, using a different technique, we find a subsequence of orthogonal polynomials for $\mu_{J_{(f_n)}}$ provided that $(f_n) \in \mathcal{R}$.

In [22], it was shown that if the Julia set $J(f)$ for a monic non-linear polynomial f lies on $\mathbb{R}$, then the recurrence coefficients for $\mu_{J(f)}$ can be found by some simple formulas. It was shown in [20] that for a variety of measures with a Cantor support (i.e. the support is a Cantor set) orthogonal polynomials can be calculated explicitly. Later on, in [85], Mantica developed a technique which is numerically stable, in order to calculate the recurrence coefficients recursively for a fairly large family of measures. After that, many calculations and conjectures have been made regarding the recurrence coefficients for measures having a Cantor support.

In Chapter 3 and Chapter 4, we study orthogonal polynomials and recurrence coefficients for a family of measures that have not been considered before. In Chapter 4, we show that , if $(f_n) \in \mathcal{R}$ and $J_{(f_n)} \subset \mathbb{R}$, then the recurrence coefficients for $\mu_{J_{(f_n)}}$ can be calculated recursively in a long but explicit way. For any $\gamma = (\gamma_k)_{k=1}^{\infty}$ with $0 < \gamma_k < 1/4$ for all $k \in \mathbb{N}$, by Theorem 3.4.3, the recurrence coefficients for $\mu_{K(\gamma)}$ can be calculated in a much simpler way. Moreover, in Chapter 7, we provide evidence for numerical stability of the algorithm given in Theorem 3.4.3.

One of the most interesting problems concerning orthogonal polynomials on $\mathbb{R}$ is the character of periodicity of recurrence coefficients. We show in Chapter 3 that for $\mu_{K(\gamma)}$, $b_n = 0,5$ for all $n \in \mathbb{N}$. Given $0 < \gamma_k \leq 1/6$ for all $k \in \mathbb{N}$ and $a_0 := 0$, we prove that (see Theorem 3.4.7) $\lim_{s\to\infty} a_{j\cdot 2^s+n} = a_n$ for $j \in \mathbb{N}$ and $n \in \mathbb{N}_0$. In particular, $\liminf a_n = 0$ holds. Depending on numerical evidence, it is conjectured in Chapter 7 that the recurrence coefficients for $\mu_{K(\gamma)}$ are always asymptotically almost periodic.

If $\gamma_s \leq 1/6$ for each $s \in \mathbb{N}$ then $K(\gamma)$ has zero Lebesgue measure, $\mu_{K(\gamma)}$ is purely singular continuous and $\liminf a_n = 0$ for $\mu_{K(\gamma)}$ by Remark 3.4.8.

1.2.1.2 Widom-Hilbert factors

The study of Widom factors for the Hilbert norm goes back to [141]. It turns out that the Szegő condition and the Widom condition are equivalent on Parreau-Widom sets, by [43]. Moreover, by [43], the sequence of Widom-Hilbert factors for measures from the Szegő class is bounded above. This makes Widom-Hilbert factor a central concept in order to define the Szegő class on more general sets.

It is an open problem how to characterize measures satisfying the Widom condition with a generalization of the Szegő condition on a prescribed set which is not necessarily Parreau-Widom. In Chapter 8 we prove the following theorem which gives a particular answer to this problem:

Theorem 1.2.1. *Let K be a non-polar compact subset of $\mathbb{R}$. Then $W_n^2(\mu_K) \geq 1$ for all $n \in \mathbb{N}$.*

If we also assume in Theorem 1.2.1 that K is regular with respect to the Dirichlet problem then $\operatorname{supp}(\mu_K) = K$ holds. Thus, for any regular non-polar compact subset of $\mathbb{R}$, one can find a measure satisfying the Widom condition whose support is equal to the set. Actually, using this fact, we suggest a definition for the Szegő class in Chapter 9 which makes sense on any regular non-polar compact set on $\mathbb{R}$.

According to this definition, the Szegő class of a non-polar regular compact set K on $\mathbb{R}$ is non-empty since $\mu_K \in \mathrm{Sz}(K)$. If K is taken to to be equal to a Parreau-Widom set then this definition coincides with the definition suggested in [44].

The other main problem is the boundedness of Widom-Hilbert factors. In Chapter 8, we prove the following result:

Corollary 1.2.2. *Let K be a non-polar compact subset of $\mathbb{R}$ and $(a_n)_{n=1}^{\infty}$ be the sequence of recurrence coefficients for μ_K. If $\liminf_{n\to\infty} a_n = 0$ then $(W_n^2(\mu_K))_{n=1}^{\infty}$ and $(W_n(K))_{n=1}^{\infty}$ are unbounded.*

Corollary 1.2.2 cannot be applied to sets having positive measure since in this case we always have $\liminf_{n\to\infty} a_n > 0$, see Remark 3.4.8 in Chapter 3. There are some sets for which the assumptions in Corollary 1.2.2 hold, see e.g. Chapter 3 and [22]. Apart from these particular examples, there is no criterion on an arbitrary set K on $\mathbb{R}$ (except having positive Lebesgue measure) determining if $\liminf_{n\to\infty} a_n = 0$ for μ_K.

Widom-Hilbert factors for $\mu_{K(\gamma)}$ are studied in Chapter 3. For $\gamma = (\gamma_s)_{s=1}^{\infty}$ satisfying $\gamma_s \leq 1/6$, it is proved that

$$\liminf_{s\to\infty} W_{2^s}^2\left(\mu_{K(\gamma)}\right) = \liminf_{n\to\infty} W_n^2\left(\mu_{K(\gamma)}\right).$$

It is also shown that for $\gamma_{2k} = 1/6$ and $\gamma_{2k-1} = 1/k$, we have $\lim_{n\to\infty} W_n^2\left(\mu_{K(\gamma)}\right) = \infty$.

1.2.1.3 Spacing of zeros

For a measure μ which is supported on $\mathbb{R}$, let $Z_n(\mu) := \{x : P_n(x;\mu) = 0\}$. We define $U_n(\mu)$ by

$$U_n(\mu) := \inf_{\substack{x,x'\in Z_n(\mu)\\ x\neq x'}} |x - x'|.$$

In [77] Krüger and Simon gave a lower bound for $U_n(\mu)$ where μ is the Cantor-Lebesgue measure of the (translated and scaled) Cantor ternary set. It seems that, [71, 77] are the only work, before ours, concerning the spacing of the zeros of orthogonal polynomials for a singular continuous measure supported on a zero Lebesgue measure set.

Let $\gamma = (\gamma_k)_{k=1}^{\infty}$ and $n \in \mathbb{N}$ with $n > 1$ be given and define $\delta_k = \gamma_0 \cdots \gamma_k$ for all $k \in \mathbb{N}_0$. Let s be the integer satisfying $2^{s-1} \leq n < 2^s$. By Theorem 6.4.6,

$$\delta_{s+2} \leq U_n(\mu_{K(\gamma)}) \leq \frac{\pi^2}{4} \cdot \delta_{s-2}$$

holds. In particular, if there is a number c such that $0 < c < \gamma_k < 1/4$ holds for all $k \in \mathbb{N}$ then we have

$$c^2 \cdot \delta_s \leq U_n(\mu_{K(\gamma)}) \leq \frac{\pi^2}{4c^2} \cdot \delta_s. \tag{1.11}$$

Especially, (1.11) is interesting since it depicts the asymptotic behavior of $(U_n(\cdot))_{n=2}^{\infty}$ very accurately for singular continuous measures.

1.2.2 Chebyshev polynomials

The smallest closed disk $\overline{B(a,r)}$ containing K is called the *Chebyshev disk* for K and the center a of this disk is called the *Chebyshev center* of K. Let p be a polynomial of degree $s \geq 2$. If K has the origin as its Chebyshev center then, by [94], p is the Chebyshev polynomial of degree s on $p^{-1}(K)$. This last result was used in many papers to obtain a subsequence of Chebyshev polynomials on some sets. We prove a result (Lemma 5.2.2) in Chapter 5 which can be seen as an extension of this theorem.

On autonomous polynomial Julia sets, a subsequence of Chebyshev polynomials can be found explicitly by [72]. In Chapter 5, we show that this is also true for generalized Julia sets.

1.2.3 Other results

Let $\epsilon_k = 1/4 - \gamma_k$ for all $k \in \mathbb{N}$. We prove in Chapter 4 that, $K(\gamma)$ is a Parreau-Widom set if and only if $\sum_{k=1}^{\infty} \sqrt{\epsilon_k} < \infty$. We remark that since autonomous totally disconnected Julia sets on $\mathbb{R}$ have zero Lebesgue measure (see e.g. Section 1.19. in [80]), such sets can not be Parreau-Widom.

In [128] (this was the first example of a Cantor type set such that the Green function is optimally smooth), it was shown that for the Green function for the complement of symmetric Cantor sets, optimal smoothness can be characterized in terms of the lengths of the basic intervals in the construction. It is known that the Green function for the complement of the Cantor ternary set is not optimally smooth, by [13], but the supremum of the exponents making the Green function Hölder continuous is unknown. In Chapter 4, we show that $g_{\Omega_{K(\gamma)}}$ is optimally smooth if and only if $\sum_{k=1}^{\infty} \epsilon_k < \infty$.

Let $\gamma = (\gamma_k)_{k=1}^{\infty}$ with $0 < \gamma_k < 1/32$ satisfy $\sum_{k=1}^{\infty} \gamma_k < \infty$. This implies that $K(\gamma)$ has Hausdorff dimension 0. In Chapter 2, we construct a dimension function h_γ that makes $K(\gamma)$ an h-set. We also show that there is a $C > 0$ such that for any Borel set B, $C^{-1} \cdot \mu_{K(\gamma)}(B) < \Lambda_{h_\gamma}(B) < C \cdot \mu_{K(\gamma)}(B)$ and in particular the equilibrium measure and Λ_{h_γ} restricted to $K(\gamma)$ are mutually absolutely continuous. In [64], it was shown recently by Goncharov and Ural that indeed these two measures coincide. To the best of our knowledge, this is the first example of a subset of $\mathbb{R}$ such that the equilibrium measure is a Hausdorff measure restricted to the set. For a general treatment of the relation between these measures, see [82] among others.

Chapter 2

Two Measures on Cantor Sets

2.1 Introduction

The relation between the α dimensional Hausdorff measure Λ_α and the harmonic measure ω on a finitely connected domain Ω is understood well. Due to Makarov [81], we know that, for a simply connected domain, $\dim \omega = 1$. Pommerenke [103] gives a full characterization of parts of $\partial\Omega$ where ω is absolutely continuous or singular with respect to a linear Hausdorff measure. Later similar facts were obtained for finitely connected domains. In the infinitely connected case there are only particular results. Model example here is $\Omega = \overline{\mathbb{C}} \setminus K$ for a Cantor-type set K. On most of such sets we have the strict inequality $\dim \omega < \alpha_K$ (see, e.g. [24], [83], [138], [143]), where α_K stands for the Hausdorff dimension of K. This inequality implies that $\Lambda_{\alpha_K} \perp \omega$ on K. These results motivate the problem to find a Cantor set for which its harmonic measure and the corresponding Hausdorff measure are not mutually singular.

In this chapter, it is shown that the set $K(\gamma)$ is dimensional and the corresponding Hausdorff measure Λ_h restricted to $K(\gamma)$ and $\mu_{K(\gamma)}$ are mutually absolutely continuous.

Suppose we are given a non polar compact set K that coincides with its exterior boundary. Then for the equilibrium measure μ_K on K we have the representation $\mu_K(\cdot) = \omega(\infty, \cdot, \overline{\mathbb{C}} \setminus K)$ in terms of the value of the harmonic measure at infinity (see e.g. [106], T.4.3.14). Moreover, since measures $\omega(z_1, \cdot, \overline{\mathbb{C}} \setminus K)$ and $\omega(z_2, \cdot, \overline{\mathbb{C}} \setminus K)$ are mutually absolutely continuous (see e.g. [106] Cor.4.3.5), our result concerning absolutely continuity is valid even if, instead of $\mu_{K(\gamma)}$, we take the harmonic measure at any other point.

2.2 Dimension function of $K(\gamma)$

In this chapter, we make the assumption

$$\sum_{s=1}^{\infty} \gamma_s < \infty. \tag{2.1}$$

Let $M := 1 + \exp\left(16 \sum_{s=1}^{\infty} \gamma_s\right)$, so $M > 2$, and $\delta_s := \gamma_1 \gamma_2 \ldots \gamma_s$. By Lemma 6 in [62],

$$\delta_s < l_{j,s} < M \cdot \delta_s \text{ for } 1 \le j \le 2^s. \tag{2.2}$$

We construct a dimension function for $K(\gamma)$, following Nevanlinna [93]. Let $\eta(\delta_s) = s$ for $s \in \mathbb{Z}_+$ with $\delta_0 := 1$. We define $\eta(t)$ for (δ_{s+1}, δ_s) by

$$\eta(t) = s + \frac{\log \frac{\delta_s}{t}}{\log \frac{\delta_s}{\delta_{s+1}}}.$$

This makes η continuous and monotonically decreasing on $(0, 1]$. In addition, we have $\lim_{t \to 0} \eta(t) = \infty$. Also observe that, for the derivative of η on (δ_{s+1}, δ_s), we have

$$\frac{d\eta}{dt} = \frac{-1}{t \log \frac{1}{\gamma_{s+1}}} \ge \frac{-1}{t \log 32} \quad \text{and} \quad \frac{d\eta}{d \log t} \ge \frac{-1}{\log 32}.$$

Define $h(t) = 2^{-\eta(t)}$ for $0 < t \le 1$ and $h(t) = 1$ for $t > 1$. Then h is a dimension function with $h(\delta_s) = 2^{-s}$ and

$$\frac{d \log h}{d \log t} \le \frac{\log 2}{\log 32} < 1.$$

Therefore if $m > 1$ and $r \leq 1$ we get the following inequality:

$$\log \frac{h(r)}{h\left(\frac{r}{m}\right)} < \int_{r/m}^{r} d \log t = \log m.$$

Finally, we obtain

$$h(r) < m \cdot h\left(\frac{r}{m}\right) \quad \text{for} \quad m > 1 \quad \text{and} \quad 0 < r \leq 1. \tag{2.3}$$

Let us show that $K(\gamma)$ is an h-set for the given function h.

Theorem 2.2.1. *Let γ satisfy* (2.1)*. Then* $1/8 \leq \Lambda_h(K(\gamma)) \leq M/2$.

Proof. First, observe that, by (2.2), for each $s \in \mathbb{N}$ the set $K(\gamma)$ can be covered by 2^s intervals of length $M \cdot \delta_s$. Since $M/2 > 1$, we have by (2.3),

$$\Lambda_h(K(\gamma)) \leq \limsup_{s \to \infty} (2^s \cdot h(M/2 \cdot \delta_s)) \leq \limsup_{s \to \infty} (2^s \cdot M/2 \cdot h(\delta_s)) = M/2.$$

We proceed to show the lower bound. Let (J_ν) be an open cover of $K(\gamma)$. Then, by compactness, there are finitely many intervals $(J_\nu)_{\nu=1}^m$ that cover $K(\gamma)$. Since $K(\gamma)$ is totally disconnected, we can assume that these intervals are disjoint. Each J_ν contains a closed subinterval $J'_\nu = [a_\nu, b_\nu]$ whose endpoints belong to $K(\gamma)$ and covers all points of $K(\gamma)$ in J_ν. Since the intervals $(J'_\nu)_{\nu=1}^m$ are disjoint, all a_ν, b_ν are endpoints of some basic intervals. Let n be the minimal number such that all $(a_\nu)_{\nu=1}^m, (b_\nu)_{\nu=1}^m$ are the endpoints of $n-$th level. Thus, each $I_{j,n}$ for $1 \leq j \leq 2^n$ is contained in some J'_ν. Let N_ν be the number of n-th level intervals in J'_ν. Clearly, $\sum_{\nu=1}^m N_\nu = 2^n$.

For a fixed $\nu \in \{1, 2, \ldots, m\}$, let q_ν be the smallest number such that J'_ν contains at least one basic interval I_{j,q_ν}. Clearly, $q_\nu \leq n$ and $l_{j,q_\nu} \leq d_\nu$ where d_ν is the length of J_ν. Therefore, by (2.2),

$$h(d_\nu) \geq h(l_{j,q_\nu}) \geq h(\delta_{q_\nu}) = 2^{-q_\nu}.$$

Let us cover J'_ν by the smallest set G_ν which is a finite union of adjacent intervals of the level q_ν. Observe that G_ν consists of at least one and at most four such intervals. Each interval of the $q_\nu-$th level contains 2^{n-q_ν} subintervals of the $n-$th level. This gives at most $2^{n-q_\nu+2}$ intervals of level n in the set G_ν. Hence

$$N_\nu \leq 2^{n-q_\nu+2}.$$

Therefore,

$$\sum_{\nu=1}^{m} h(d_\nu) \geq \sum_{\nu=1}^{m} 2^{-q_\nu} \geq 2^{-n-2} \sum_{\nu=1}^{m} N_\nu = 1/4.$$

Since $h(d) < 2 \cdot h(d/2)$ from (2.3), finally we obtain the desired bound. $\square$

Similar arguments apply to a part of $K(\gamma)$ on any basic interval.

Corollary 2.2.2. *Let γ satisfy* (2.1)*. Then $2^{-s-3} \leq \Lambda_h(K(\gamma) \cap I_{j,s}) \leq M \cdot 2^{-s-1}$ for each $s \in \mathbb{N}$ and $1 \leq j \leq 2^s$.*

Remark 2.2.3. A set E is called *dimensional* if there is a dimension function h that makes E an $h-$set. It should be noted that not all sets are dimensional. If we replace the condition $h(0) = 0$ by $h(0) \geq 0$, then any sequence gives a trivial example of a dimensionless set. Best in [32] presented an example of a dimensionless Cantor set provided $h(0) = 0$. The author considered dimension functions with the additional condition of concavity, but did not use it in his construction.

2.3 Harmonic Measure and Hausdorff measure for $K(\gamma)$

In the main theorem of this chapter and below, by Λ_h we mean the Hausdorff measure restricted to the compact set $K(\gamma)$ corresponding to the function h.

Theorem 2.3.1. *Let γ satisfy* (2.1)*. Then measures $\mu_{K(\gamma)}$ and Λ_h are mutually absolutely continuous.*

Proof. Let us fix any open interval I of length $2r$ and show that

$$\mu_{K(\gamma)}(I) \leq 8\, h(r). \tag{2.4}$$

Then, by Theorem 1.1.5, $\mu_{K(\gamma)}(E) \leq 8\,\Lambda_h(E)$ for any Borel set E and $\mu_{K(\gamma)} \ll \Lambda_h$.

First suppose that the endpoints of I do not belong to $K(\gamma)$. Then there exists $I' = [a, b] \subset I$ which contains all points in $K(\gamma) \cap I$. Let us take, as above, minimal n and q such that both a and b are the endpoints of $n-$th level and I' contains at least one basic interval $I_{j,q}$. All points in $K(\gamma) \cap I$ can be covered by 4 adjacent intervals of the level q and this cover G contains $4 \cdot 2^{s-q}$ intervals of the level s for $s \geq q$. Hence $\int \chi_G \, d\lambda_s = 4 \cdot 2^{-q}$. The characteristic function $\chi_{I'}$ is continuous on E_s for $s \geq n$. By Theorem 1.1.7,

$$4 \cdot 2^{-q} \geq \lim_{s\to\infty} \int \chi_{I'} \, d\lambda_s = \int \chi_{I'} \, d\mu_{K(\gamma)} = \mu_{K(\gamma)}(I).$$

On the other hand, I contains some basic interval $I_{j,q}$. Therefore $2r > l_{j,q}$ and, by (2.2),

$$h(2r) \geq h(l_{j,q}) \geq h(\delta_q) = 2^{-q}.$$

Combining these inequalities with (2.3) gives (2.4):

$$8\,h(r) > 4\,h(2r) \geq 4 \cdot 2^{-q} \geq \mu_{K(\gamma)}(I).$$

Now let us consider the case when at least one of the endpoints of $I = (z - r, z + r)$ is contained in $K(\gamma)$. Since the set is totally disconnected, we can take two real null sequences $(\alpha_n)_{n=1}^{\infty}$ and $(\beta_n)_{n=1}^{\infty}$ such that the endpoints of $I_n = (z - r - \alpha_n, z + r + \beta_n)$ do not belong to $K(\gamma)$ for each n. Arguing as above, we see that

$$\mu_{K(\gamma)}(I) \leq \mu_{K(\gamma)}(I_n) \leq 8\,h(r + \varepsilon_n) \leq 8(1 + \varepsilon_n/r)\,h(r),$$

where $\varepsilon_n = \max\{\alpha_n, \beta_n\}$. Since $\varepsilon_n \to 0$ as $n \to \infty$, we have (2.4) for this case as well.

We proceed to show that $\Lambda_h \ll \mu_{K(\gamma)}$. Let us fix $x \in K(\gamma)$ and $r > 0$. In order to use Theorem 1.1.6, let us show that

$$h(r) \leq 2M \cdot \mu_{K(\gamma)}(I), \tag{2.5}$$

where $I = (x - r, x + r)$. Clearly, it is enough to consider only $r < 1$. Let us fix two consecutive basic intervals containing our point: $x \in I_{i,\,s} \subset I_{j,\,s-1}$ with $l_{i,\,s} \le r < l_{j,\,s-1}$. Then $I \supset I_{i,\,s}$ and $\mu_{K(\gamma)}(I) \ge 2^{-s}$, by (1.9). On the other hand, by (2.2) and (2.3),

$$h(r) < h(l_{j,\,s-1}) < h(M\,\delta_{s-1}) < M\,h(\delta_{s-1}) = M\,2^{-s+1}.$$

This gives (2.5) and completes the proof. $\square$

Example 2.3.2. The sequence γ with $\gamma_s = \exp(-8s+4)$ for $s \in \mathbb{N}$ satisfies all required conditions. In particular, $\mathrm{Cap}(\mathrm{K}(\gamma)) = \exp(-12)$. Here, $\delta_s = \exp(-4s^2)$ and $\eta(t) = s + \frac{-4s^2 - \log t}{8s+4}$ for $\delta_s \le t < \delta_{s-1}$. The Hausdorff measure Λ_h corresponding to the function $h(t) = 2^{-\eta(t)}$ and $\mu_{K(\gamma)}$ are mutually absolutely continuous.

Chapter 3

Orthogonal polynomials for the weakly equilibrium Cantor sets

3.1 Introduction

This chapter is concerned with the spectral theory of orthogonal polynomials for $\mu_{K(\gamma)}$ with a special emphasis on purely singular continuous case. It should be noted that Cantor sets appear as supports of spectral measures for some important Schrödinger operators used in physics (see e.g. the review [112] and [16]). Nonetheless, here, our motivation is purely mathematical. We are interested in the following two problems related to orthogonal polynomials on Cantor-type sets. What can be said about the periodicity of corresponding Jacobi parameters? What is the notion of the Szegő class of measures on Cantor sets?

Concerning the first problem, the fundamental conjecture (see [86] and also Conjecture 3.1 in [77]) is that, for a large class of measures supported on Cantor sets, including the self-similar measures generated by linear iterated function systems (IFS), the corresponding Jacobi matrices are asymptotically almost periodic. Confirmation of this hypothesis may allow to extend the methods used in [45, 46] for the finite gap sets to the Cantor sets with zero Lebegue measure.

Concerning the second question, we mention that the Szegő theorem was generalized recently in [43] to the class of Parreau-Widom sets.

There are two main directions in the development of the theory of orthogonal polynomials for purely singular continuous measures. The first deals with a renormalization technique suggested by Mantica in [85], which enables to efficiently compute Jacobi parameters (see e.g. [68, 77, 85]) for balanced measures via a linear IFS. Moreover, possible extensions of the notion of isospectral torus for singular continuous measures can be found in [77, 88].

On the other hand, there is a theory of orthogonal polynomials for equilibrium measures of real polynomial Julia sets (see e.g [21–23, 27]). This includes simple formulas for orthogonal polynomials and recurrence coefficients, and almost periodicity of Jacobi matrices for certain Julia sets.

We remark that, in the construction of $K(\gamma)$, the technique of inverse polynomial images was central. Thus, our results can be compared with [31, 59]. Furthermore, similarities between the results obtained here and for orthogonal polynomial Julia sets are not mere coincidence. As soon as $\inf \gamma_k > 0$, $K(\gamma)$ can be considered as a generalized polynomial Julia set in the sense of Brück-Büger as we see in Chapter 4.

This chapter is organized as follows. In Section 2 we show that the 2^s degree orthogonal polynomial P_{2^s} coincides with the corresponding Chebyshev polynomial. In Sections 3 and 4 we suggest a procedure to find P_n for $n \neq 2^s$. This allows to analyze the asymptotic behavior of the Jacobi parameters $(a_n)_{n=1}^{\infty}$. Note that, if one can obtain a stronger version of Theorem 3.4.7 by showing that the limit of a_{j2^s+n} hold uniformly in n and j as in [23], this would imply that the Jacobi matrices considered here are almost periodic provided that $\sup \gamma_k \leq 1/6$.

Since $\mathrm{Cap}(K(\gamma))$ is known, we estimate (Section 5) the Widom-Hilbert factors and check the Widom condition.

We assume that $0^0 := 1$.

3.2 Orthogonal Polynomials

By Lemma 6 in [62],

$$\gamma_1 \dots \gamma_s < l_{i,s} < \exp\left(16 \sum_{k=1}^{s} \gamma_k\right) \gamma_1 \dots \gamma_s, \quad 1 \le i \le 2^s,$$

provided $\gamma_k \le 1/32$ for all k. Then the Lebesgue measure $|E_s|$ of the set E_s does not exceed $(\sqrt{e}/16)^s$. Here, $|K(\gamma)| = 0$ and, by [43], $K(\gamma)$ is not a Parreau-Widom set. In Section 4 we show that $|K(\gamma)| = 0$ as well if $\gamma_k \le 1/6$ for all k.

On the other hand, by choosing $(\gamma_k)_{k=1}^{\infty}$ sufficiently close to $1/4$, we can obtain Cantor sets with positive Lebesgue measure. What is more, in the limit case, when all $\gamma_k = 1/4$, we get $E_s = [0,1]$ for all s and $K(\gamma) = [0,1]$ (see Example 1 in [62]).

For brevity, in this chapter, instead of $\|\cdot\|_{L^2(\mu_{K(\gamma)})}$, $P_n(\cdot;\mu_{K(\gamma)})$ and $W_n^2(\mu_{K(\gamma)})$ and we use $\|\cdot\|$ and P_n, W_n^2 respectively. The main result of this section is that, for $n = 2^s$ with $s \in \mathbb{N}_0$, the polynomial P_n coincides with the corresponding Chebyshev polynomial for $K(\gamma)$. The next two theorems will play a crucial role.

Theorem 3.2.1 ([62], Prop.1)**.** *For each $s \in \mathbb{N}_0$ the polynomial $\Phi_{2^s} + r_s/2$ is the Chebyshev polynomial for $K(\gamma)$.*

Remark 3.2.2. Only the values $s \in \mathbb{N}$ were considered in [62]. But, clearly, for $s = 0$ the polynomial $\Phi_1(x) + 1/2 = x - 1/2$ is Chebyshev.

Remark 3.2.3. Since real polynomials are considered here and the alternating set for $\Phi_{2^s} + r_s/2$ consists of $2^s + 1$ points, the Chebyshev property of this polynomial follows by the Chebyshev alternation theorem.

Theorem 3.2.4 ([110], III.T.3.6)**.** *Let $K \subset \mathbb{R}$ be a non-polar compact set. Then the normalized counting measures on the zeros of the Chebyshev polynomials converge to the equilibrium measure of K in the weak-star topology.*

For $s \in \mathbb{N}$, the polynomial $\Phi_{2^s} + r_s\,/\,2$ has simple real zeros $(x_k)_{k=1}^{2^s}$ which are symmetric about $x = 1/2$. Let us denote by ν_s the normalized counting measure at these points, that is $\nu_s = 2^{-s} \sum_{k=1}^{2^s} \delta_{x_k}$.

Lemma 3.2.5. *Let $s > m$ with $s, m \in \mathbb{N}_0$. Then* $\int \left(\Phi_{2^m} + \frac{r_m}{2}\right) d\nu_s = 0$.

Proof. For $m = 0$ we have the result by symmetry. Suppose $m \geq 1$. By (1.10), at the points $(x_k)_{k=1}^{2^s}$ we have

$$\Phi_{2^s} + \frac{r_s}{2} = (\Phi_{2^{s-1}})^2 + r_{s-1}\Phi_{2^{s-1}} + \frac{r_s}{2} = 0.$$

The discriminant of the equation is positive. Therefore, the roots satisfy

$$(\Phi_{2^{s-1}} + \alpha_{s-1}^1)(\Phi_{2^{s-1}} + \alpha_{s-1}^2) = 0,$$

where $\alpha_{s-1}^1 + \alpha_{s-1}^2 = r_{s-1}$ and $0 < \alpha_{s-1}^1, \alpha_{s-1}^2 < r_{s-1}$. Thus, a half of the points satisfy $\Phi_{2^{s-1}} + \alpha_{s-1}^1 = 0$ while the other half satisfy $\Phi_{2^{s-1}} + \alpha_{s-1}^2 = 0$.

Rewriting the equation $\Phi_{2^{s-1}} + \alpha_{s-1}^1 = 0$, we see that

$$\Phi_{2^{s-2}}^2 + r_{s-2}\Phi_{2^{s-2}} + \alpha_{s-1}^1 = 0.$$

Since $r_{s-2}^2 > 4r_{s-1} > 4\alpha_{s-1}^1$, this yields

$$(\Phi_{2^{s-2}} + \alpha_{s-2}^1)(\Phi_{2^{s-2}} + \alpha_{s-2}^2) = 0$$

with $\alpha_{s-2}^1 + \alpha_{s-2}^2 = r_{s-2}$ and $0 < \alpha_{s-2}^1, \alpha_{s-2}^2 < r_{s-2}$. By the same argument, the second half of the roots satisfy

$$(\Phi_{2^{s-2}} + \alpha_{s-2}^3)(\Phi_{2^{s-2}} + \alpha_{s-2}^4) = 0$$

with $\alpha_{s-2}^3 + \alpha_{s-2}^4 = r_{s-2}$ and $0 < \alpha_{s-2}^3, \alpha_{s-2}^4 < r_{s-2}$.

Since at each step $r_{i-1}^2 > 4r_i$ we can continue this procedure until obtaining $\Phi_{2^{m+1}}$. So we can decompose the Chebyshev nodes $(x_k)_{k=1}^{2^s}$ into 2^{s-m-1} groups. All 2^{m+1} nodes from the $i-$th group G_i satisfy

$$\Phi_{2^{m+1}} + \alpha_{m+1}^i = 0, \;\; 0 < \alpha_{m+1}^i < r_{m+1}.$$

By using these 2^{s-m-1} equations we finally obtain

$$(\Phi_{2^m} + \alpha_m^{2i-1})(\Phi_{2^m} + \alpha_m^{2i}) = 0$$

where $\alpha_m^{2i-1} + \alpha_m^{2i} = r_m$. Thus, given fixed i with $1 \le i \le 2^{s-m-1}$, for 2^m points from the group G_i we have $\Phi_{2^m} = -\alpha_m^{2i-1}$, whereas for the other half, $\Phi_{2^m} = -\alpha_m^{2i}$. Consequently,

$$\int \left(\Phi_{2^m} + \frac{r_m}{2}\right) d\nu_s = \int \Phi_{2^m} d\nu_s + \frac{r_m}{2} = \frac{\sum_{i=1}^{2^{s-m-1}} 2^m(-\alpha_m^{2i-1} - \alpha_m^{2i})}{2^s} + \frac{r_m}{2} = 0.$$

$\square$

Lemma 3.2.6. *Let* $0 \le i_1 < i_2 < \ldots i_n < s$. *Then*

(a) $$\int \Phi_{2^{i_1}} \Phi_{2^{i_2}} \ldots \Phi_{2^{i_n}} d\nu_s = \int \Phi_{2^{i_1}} d\nu_s \int \Phi_{2^{i_2}} d\nu_s \ldots \int \Phi_{2^{i_n}} d\nu_s = (-1)^n \prod_{k=1}^{n} \frac{r_{i_k}}{2}.$$

(b) $$\int \left(\Phi_{2^{i_1}} + \frac{r_{i_1}}{2}\right) \left(\Phi_{2^{i_2}} + \frac{r_{i_2}}{2}\right) \ldots \left(\Phi_{2^{i_n}} + \frac{r_{i_n}}{2}\right) d\nu_s = 0.$$

Proof. (a) Suppose that $i_1 \ge 1$. As above, we can decompose the nodes $(x_k)_{k=1}^{2^s}$ into 2^{s-i_1-1} equal groups such that the nodes from the $j-$th group satisfy an equation

$$(\Phi_{2^{i_1}} + \alpha_{i_1}^{2j-1})(\Phi_{2^{i_1}} + \alpha_{i_1}^{2j}) = 0$$

with $\alpha_{i_1}^{2j-1} + \alpha_{i_1}^{2j} = r_{i_1}$. If, on some set, $(\Phi_{2^k} + \alpha)(\Phi_{2^k} + \beta) = 0$ with $\alpha + \beta = r_k$, then $\Phi_{2^{k+1}} = \Phi_{2^k}^2 + \Phi_{2^k} r_k = -\alpha\beta$. Hence, for each $i \in \mathbb{N}$, the polynomial $\Phi_{2^{k+i}}$ is constant on this set. Therefore the function $\Phi_{2^{i_2}} \ldots \Phi_{2^{i_n}}$ takes the same value for all x_k from the $j-$th group. This allows to apply the argument of Lemma 3.2.5:

$$\int \Phi_{2^{i_1}} \Phi_{2^{i_2}} \ldots \Phi_{2^{i_n}} d\nu_s = -\frac{r_{i_1}}{2} \int \Phi_{2^{i_2}} \Phi_{2^{i_3}} \ldots \Phi_{2^{i_n}} d\nu_s.$$

This equality is valid also for $i_1 = 0$ since

$$\int \left(\Phi_1 + \frac{1}{2}\right) \Phi_{2^{i_2}} \ldots \Phi_{2^{i_n}} d\nu_s = 0,$$

by symmetry. Proceeding this way, the result follows, since $-r_m/2 = \int \Phi_{2^m} d\nu_s$, by Lemma 3.2.5.

(b) Opening the parentheses yields

$$\int \Phi_{2^{i_1}}\Phi_{2^{i_2}}\dots\Phi_{2^{i_n}}d\nu_s+\sum_{k=1}^{n}\frac{r_{i_k}}{2}\int\prod_{j\neq k}\Phi_{2^{i_j}}d\nu_s+\dots+\prod_{k=1}^{n}\frac{r_{i_k}}{2}.$$

By Lemma 3.2.5 and part (a), this is

$$\prod_{k=1}^{n}\frac{r_{i_k}}{2}\cdot\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}=0.$$

□

Remark 3.2.7. We can use $\mu_{K(\gamma)}$ instead of ν_s in Lemma 3.2.5 and Lemma 3.2.6, since, by Theorem 3.2.4, $\nu_s\to\mu_{K(\gamma)}$ in the weak-star topology.

Theorem 3.2.8. *The monic orthogonal polynomial P_{2^s} with respect to the equilibrium measure $\mu_{K(\gamma)}$ coincides with the corresponding Chebyshev polynomial $\Phi_{2^s}+r_s/2$ for all $s\in\mathbb{N}_0$.*

Proof. For $s=0$ we have the result by symmetry. Let $s\geq 1$. Each polynomial $T(x)$ of degree less than 2^s is a linear combination of polynomials of the type

$$\left(\Phi_{2^{s-1}}(x)+\frac{r_{s-1}}{2}\right)^{n_{s-1}}\dots\left(\Phi_2(x)+\frac{r_1}{2}\right)^{n_1}\left(x-\frac{1}{2}\right)^{n_0}$$

with $n_i\in\{0,1\}$. By Lemma 3.2.6, $\Phi_{2^s}+r_s\,/\,2$ is orthogonal to all polynomials of degree less than 2^s, so it is P_{2^s}. □

By (1.10), we immediately have

Corollary 3.2.9. $P_{2^{s+1}}=P_{2^s}^2-(1-2\,\gamma_{s+1})\,r_s^2/4$ *for* $s\in\mathbb{N}_0$.

3.3 Some products of orthogonal polynomials

So far we only obtain orthogonal polynomials of degree 2^s. We try to find P_n for other degrees. By Corollary 3.2.9, since $\int P_{2^{s+1}}d\mu_{K(\gamma)}=0$, we have

$$||P_{2^s}||^2=\int P_{2^s}^2d\mu_{K(\gamma)}=(1-2\,\gamma_{s+1})\,r_s^2/4 \tag{3.1}$$

and

$$P_{2^{s+1}} = P_{2^s}^2 - ||P_{2^s}||^2,\ \forall s \in \mathbb{N}_0. \tag{3.2}$$

Our next goal is to evaluate $\int A\, d\mu_{K(\gamma)}$ for *A-polynomial* of the form

$$A = (P_{2^{s_n}})^{i_n}(P_{2^{s_{n-1}}})^{i_{n-1}} \ldots (P_{2^{s_1}})^{i_1}, \tag{3.3}$$

where $s_n > s_{n-1} > \ldots > s_1 > 0$ and $i_1, i_2, \ldots, i_n \in \{1, 2\}$.

The next lemma is basically a consequence of (3.2).

Lemma 3.3.1. *Let A be a polynomial satisfying* (3.3). *Then the following propositions hold:*

(a) If $i_n = 2$ then $\displaystyle\int A\, d\mu_{K(\gamma)} = \|P_{2^{s_n}}\|^2 \int P_{2^{s_{n-1}}}^{i_{n-1}} \cdots P_{2^{s_1}}^{i_1}\, d\mu_{K(\gamma)}.$

(b) Suppose that $n = k + m$ with $i_n = i_{n-1} = \ldots = i_{k+1} = 1$ and $i_k = 2$. In addition, let $s_{k+j} = s_k + j$ for $1 \le j \le m$. Then

$$\int A\, d\mu_{K(\gamma)} = \|P_{2^{s_n}}\|^2 \int P_{2^{s_{k-1}}}^{i_{k-1}} \cdots P_{2^{s_1}}^{i_1}\, d\mu_{K(\gamma)}.$$

(c) If $i_k = 1$ and $s_k \ge s_{k-1} + 2$ for some $k \in \{2, 3, \ldots, n\}$, then $\displaystyle\int A\, d\mu_{K(\gamma)} = 0.$

(d) If $i_1 = 1$ then $\displaystyle\int A\, d\mu_{K(\gamma)} = 0.$

Proof. (a) Using (3.2), we have $P_{2^{s_n}}^2 = P_{2^{s_n+1}} + \|P_{2^{s_n}}\|^2$. The result easily follows since the degree of $P_{2^{s_{n-1}}}^{i_{n-1}} \cdots P_{2^{s_1}}^{i_1}$ is less than 2^{s_n+1}.

(b) Here $A = P_{2^{s_n}}\, P_{2^{s_{n-1}}} \cdots P_{2^{s_{k+1}}} P_{2^{s_k}}^2 \cdot Q$ with $Q = P_{2^{s_{k-1}}}^{i_{k-1}} \cdots P_{2^{s_1}}^{i_1}$. Observe that $\deg Q < 2^{s_{k-1}+2} \le 2^{s_{k+1}}$. We apply (3.2) repeatedly. First, since $s_{k+1} = s_k + 1$, we have $P_{2^k}^2 = P_{2^{s_{k+1}}} + ||P_{2^{s_k}}||^2$. Similarly, $P_{2^{s_k+1}} P_{2^{s_k}}^2 = P_{2^{s_k+2}} + ||P_{2^{s_{k+1}}}||^2 + P_{2^{s_{k+1}}}\, ||P_{2^{s_k}}||^2$. After m steps we write A in the form

$(P_{2^{s_{n+1}}} + \|P_{2^{s_n}}\|^2 + \mathcal{L})\,Q$, where $\mathcal{L}$ is a linear combination of the polynomials $P_{2^{s_n}},\ P_{2^{s_n}}P_{2^{s_n-1}}, \cdots, P_{2^{s_n}}P_{2^{s_n-1}}\cdots P_{2^{s_{k+1}}}$. Here, $2^{s_n} > 2^{s_n-1} + \cdots 2^{s_{k+1}} +$ deg Q. By orthogonality, all terms vanish after integration, except $\|P_{2^{s_n}}\|^2\,Q$, which is the desired conclusion.

(c) Let us take the maximal k with such property. Repeated application of (a) and (b) enables us to reduce $\int A\,d\mu_{K(\gamma)}$ to $C\int A_1\,d\mu_{K(\gamma)}$ with $C>0$ and $A_1 = P_{2^{s_m}}\cdots P_{2^{s_k}}\cdot R$, where $R = P^{i_{k-1}}_{2^{s_{k-1}}}\cdots P^{i_1}_{2^{s_1}}$ with deg $R < 2^{s_{k-1}+2} \le 2^{s_k}$. Comparing the degrees gives the result.

(d) Take the largest k with $i_1 = i_2 = \cdots = i_k = 1$. Then, as above, $\int A\,d\mu_{K(\gamma)} = C\cdot\int P_{2^{s_k}}\cdots P_{2^{s_1}}\,d\mu_{K(\gamma)} = 0$, since the degree of the first polynomial exceeds the common degree of others.

$\square$

Theorem 3.3.2. *For $A-$polynomial given in* (3.3)*, let* $c_k = (i_k-1)^{s_k-s_{k-1}-1}$ *and* $c = \prod_{k=1}^n c_k$. *Here,* $s_0 := -1$ *and* $i_{n+1} := 2$. *Then* $\int A\,d\mu_{K(\gamma)} = c\cdot\prod_{k=1}^n ||P_{2^k}||^{2(i_{k+1}-1)}$.

Proof. First we remark that $c\in\{0,1\}$. Clearly, $c_1 = (i_1-1)^{s_1} = 0$ if and only if $i_1 = 1$. For $k>1$ we get $c_k = 0$ if and only if $i_k = 1$ and $s_k > s_{k-1}+1$. Therefore, $c = 0$ just in the cases (c) and (d) above.

Let us show that the procedures $(a)-(d)$ of Lemma 3.3.1 allow to find $\int A\,d\mu_{K(\gamma)}$ for all values of $(i_k)_{k=1}^n$ and $(s_k)_{k=1}^n$ stated after (3.3). Consider the string $\mathcal{I} = \{i_n, i_{n-1}, \cdots, i_1\}$. If $i_1 = 1$ then $c = 0$ and $\int A\,d\mu_{K(\gamma)} = 0$, by (d), so the result follows. Suppose $i_1 = 2$. Then we can decompose $\mathcal{I}$ into substrings of the types $\{2\}$, $\{1,2\}, \cdots, \{1,\cdots,1,2\}$. The number and the ordering of such substrings may be arbitrary. We go over substrings of $\mathcal{I}$ in left-to-right order. If we meet $\{i_k\}$ with $i_k = 2$ then we use (a). Observe that here $i_{k+1} = 2$ if $k\neq n$. Hence this substring contributes a term $||P_{2^k}||^2$ into the product representing $\int A\,d\mu_{K(\gamma)}$. For a general substring $\{i_k, \cdots, i_{k-m}\}$ with $i_k = \cdots = i_{k-m+1} = 1$, $i_{k-m} = 2$ we also have $i_{k+1} = 2$ provided that $k \neq n$. Consider the corresponding values s_j for $k-m \leq j \leq k$. Suppose that these numbers are consecutive, that is $s_{j+1} = s_j + 1$ for $k-m \leq j \leq k-1$. Then we use the procedure (b). In this case, $i_{k+1} - 1 = 1$ and $i_{j+1} - 1 = 0$ for $k-m \leq j \leq k-1$. As above, the substring gives a contribution $||P_{2^k}||^2$ into the common product. Otherwise, $s_{j+1} \geq s_j + 2$ for some j. Then, by (c), $\int A\, d\mu_{K(\gamma)} = 0$. On the other hand, here, $c = c_j = 0$, so the desired representation for $\int A\, d\mu_{K(\gamma)}$ is valid as well. □

Corollary 3.3.3. *For $A-$polynomial given in* (3.3)*, let $A = A_1 \cdot P_{2^{s_1}}^{i_1}$, so A_1 contains all terms of A except the last. Suppose $i_1 = i_2 = 2$. Then $\int A\, d\mu_{K(\gamma)} = ||P_{2^{s_1}}||^2 \int A_1\, d\mu_{K(\gamma)}$.*

We will represent P_n in terms of *B-polynomials* that are defined, for $2^m \leq n < 2^{m+1}$ with $m \in \mathbb{N}_0$, as

$$B_n = (P_{2^m})^{i_m} (P_{2^{m-1}})^{i_{m-1}} \ldots (P_1)^{i_1},$$

where $i_k \in \{0, 1\}$ is the $k-$th coefficient in the binary representation $n = i_m\, 2^m + \cdots + i_0$.

Thus, B_n is a monic polynomial of degree n. The polynomials $B_{(2k+1)\cdot 2^s}$ and $B_{(2j+1)\cdot 2^m}$ are orthogonal for all $j, k, m, s \in \mathbb{N}_0$ with $s \neq m$. Indeed, if $\min\{m, s\} = 0$ then $\int B_{(2k+1)\cdot 2^s}\, B_{(2j+1)\cdot 2^m}\, d\mu_{K(\gamma)} = 0$, since one polynomial is symmetric about $x = 1/2$, whereas another is antisymmetric. Otherwise we use Lemma 3.3.1 (d). By (a), we have

$$||B_n||^2 = \prod_{k=0}^{m} ||P_{2^k}||^{2i_k} = \prod_{k=0, i_k \neq 0}^{m} ||P_{2^k}||^2.$$

Theorem 3.3.4. *For each $n \in \mathbb{N}$, let $n = 2^s(2k+1)$, the polynomial P_n has a unique representation as a linear combination of $B_{2^s}, B_{3\cdot 2^s} \ldots, B_{(2k-1)\cdot 2^s}, B_{(2k+1)\cdot 2^s}$.*

Proof. Consider $Q = a_0 B_{2^s} + a_1 B_{3\cdot 2^s} + \ldots + a_{k-1} B_{(2k-1)\cdot 2^s} + B_{(2k+1)\cdot 2^s}$, where $(a_j)_{j=0}^{k-1}$ are chosen such that Q is orthogonal to all $B_{(2j+1)2^s}$ with $j = 0, 1, \ldots, k-1$. This gives a system of k linear equations with k unknowns $(a_j)_{j=0}^{k-1}$. The determinant of this system is the Gram determinant of linearly independent functions

$(B_{(2j+1)2^s})_{j=0}^{k-1}$. Therefore it is positive and the system has a unique solution. In addition, as was remarked above, Q is orthogonal to all $B_{(2j+1)\cdot 2^m}$ with $m \neq s$. Thus, Q is a monic polynomial of degree n that is orthogonal to all polynomials of degree $< n$, so $Q = P_n$. $\square$

Corollary 3.3.5. *The polynomial $P_{2^s(2k+1)}$ is a linear combination of products of the type $P_{2^{s_m}} P_{2^{s_{m-1}}} \cdots P_{2^s}$, so the smallest degree of $P_{2^{s_j}}$ in every product is 2^s.*

To illustrate the theorem, we consider, for given $s \in \mathbb{N}_0$, the easiest cases with $k \leq 2$. Clearly, $P_{2^s} = B_{2^s}$. Since $B_{3\cdot 2^s} = P_{2^s} P_{2^{s+1}}$, we take $P_{3\cdot 2^s} = a_0 P_{2^s} + P_{2^{s+1}} P_{2^s}$, where a_0 is such that $\int P_{3\cdot 2^s} P_{2^s} \, d\mu_{K(\gamma)} = 0$. By Lemma 3.3.1,

$$P_{3\cdot 2^s} = P_{2^{s+1}} P_{2^s} - \frac{\|P_{2^{s+1}}\|^2}{\|P_{2^s}\|^2} P_{2^s}.$$

Similarly, $B_{5\cdot 2^s} = P_{2^s} P_{2^{s+2}}$ and $P_{5\cdot 2^s} = a_0 P_{2^s} + a_1 P_{2^{s+1}} P_{2^s} + P_{2^s} P_{2^{s+2}}$ with

$$a_0 = \frac{||P_{2^{s+2}}||^2}{||P_{2^s}||^4 - ||P_{2^{s+1}}||^2}, \;\; a_1 = -a_0 \, \frac{||P_{2^s}||^2}{||P_{2^{s+1}}||^2}.$$

Using (3.1), all coefficient can be expressed only in terms of $(\gamma_k)_{k=1}^{\infty}$. As k gets bigger, the complexity of calculations increases.

Remark 3.3.6. In general, the polynomial P_n is not Chebyshev. For example, $P_3 = P_1(P_2 + a_0)$ with $a_0 = -\frac{(1-2\gamma_2)\gamma_1^2}{1-2\gamma_1}$. At least for small γ_1, the polynomial $P_3(x) = (x - 1/2)(x^2 - x + \gamma_1/2 + a_0)$ increases on the first basic interval $I_{1,1} = [0, l_{1,1}]$. Here, $l_{1,1}$ is the first solution of $\Phi_2 = -r_1$, so $l_{1,1} = (1 - \sqrt{1 - 4\gamma_1})/2$. If P_3 is the Chebyshev polynomial then, by the Chebyshev alternation theorem, $P_3(l_{1,1}) = P_3(1)$, but it is not the case.

3.4 Jacobi parameters

We are interested in the analysis of asymptotic behavior of the Jacobi parameters. Since $\mu_{K(\gamma)}$ is symmetric about $x = 1/2$, all b_n are equal to $1/2$.

In the next lemmas we use the equality $\int P_n\, P_m\, P_{n+m}\, d\mu_{K(\gamma)} = ||P_{n+m}||^2$, which follows by orthogonality of P_{n+m} to all polynomials of smaller degree.

Lemma 3.4.1. *For all $s \in \mathbb{N}_0$ and $k \in \mathbb{N}$ we have*

$$P_{2^s(2k+1)} = P_{2^s} \cdot P_{2^{s+1}k} - \frac{\|P_{2^{s+1}k}\|^2}{\|P_{2^s(2k-1)}\|^2}\, P_{2^s(2k-1)}.$$

Proof. Consider the polynomial $Q = P_{2^s} \cdot P_{2^{s+1}k} - \frac{\|P_{2^{s+1}k}\|^2}{\|P_{2^s(2k-1)}\|^2} P_{2^s(2k-1)}$.Since $\deg(P_{2^s} \cdot P_{2^{s+1}k}) > \deg P_{2^s(2k-1)}$, it is a monic polynomial of degree $2^s(2k+1)$. Let us show that Q is orthogonal to P_n for all n with $0 \le n < 2^s(2k+1)$. This will mean that $Q = P_{2^s(2k+1)}$.

Suppose $0 \le n < 2^s(2k-1)$. Then orthogonality follows by comparison of the degrees.

If $n = 2^s(2k-1)$ then $\int Q\, P_n\, d\mu_{K(\gamma)} = 0$ due to the choice of coefficient of the addend in Q and the remark above.

Let $2^s(2k-1) < n < 2^s(2k+1)$. We show that $\int P_{2^s}\, P_{2^{s+1}k}\, P_n\, d\mu_{K(\gamma)} = 0$. We write k in the form $k = 2^q(2l+1)$ with some $q, l \in \mathbb{N}_0$. In turn, $n = 2^m(2p+1)$ with $m \neq s$. By Corollary 3.3.5, $P_{2^{s+1}k}$ is a linear combination of products of $P_{2^{s_j}}$ with $\min s_j = s+1+q$ in every product. Similarly for P_n, but here the smallest degree is 2^m. Therefore, $P_{2^s}\, P_{2^{s+1}k}\, P_n$ is a linear combination of $A-$polynomials and for each $A-$polynomial the exponent of the smallest term is 1. By Lemma 3.3.1(d), the corresponding integral is zero. □

Lemma 3.4.2. *For all $s \in \mathbb{N}_0$ and $k \in \mathbb{N}$ we have*

$$a^2_{2^s(2k+1)}\, a^2_{2^s(2k+1)-1} \cdots a^2_{2^{s+1}k+1} + a^2_{2^{s+1}k}\, a^2_{2^{s+1}k-1} \cdots a^2_{2^{s+1}k-2^s+1} = \|P_{2^s}\|^2.$$

Proof. By Lemma 3.4.1 and the remark above,

$$||P_{2^s(2k+1)}||^2 = \int P^2_{2^s}\, P^2_{2^{s+1}k} d\mu_{K(\gamma)} - \frac{\|P_{2^{s+1}k}\|^4}{\|P_{2^s(2k-1)}\|^2}. \tag{3.4}$$

Let us show that

$$\int P^2_{2^s}\, P^2_{2^{s+1}k} d\mu_{K(\gamma)} = ||P_{2^s}||^2 ||P_{2^{s+1}k}||^2.$$

If $k = 2^m$, we have this immediately, by Lemma 3.3.1(a).

Otherwise, $2^{s+1}k = 2^m(2l+1)$ with $l \in \mathbb{N}$ and $m \geq s+1$. Then, by Corollary 3.3.5, $P_{2^{s+1}k}$ is a linear combination of products $P_{2^{s_q}} \cdots P_{2^{s_j}} \cdots P_{2^m}$ with $s_j > m$ except the last term. From here, $P^2_{2^{s+1}k} = P^2_{2^m} \cdot \sum \alpha_j A_j$, where $\sum \alpha_j A_j$ is a linear combination of $A-$type polynomials with $s_1 > m$ for each A_j. Therefore,

$$||P_{2^{s+1}k}||^2 = \sum \alpha_j \int A_j P^2_{2^m}\, d\mu_{K(\gamma)}.$$

On the other hand,

$$\int P^2_{2^s}\, P^2_{2^{s+1}k} d\mu_{K(\gamma)} = \sum \alpha_j \int A_j P^2_{2^m}\, P^2_{2^s}\, d\mu_{K(\gamma)}.$$

By Corollary 3.3.3, this is $||P_{2^s}||^2 ||P_{2^{s+1}k}||^2$.

Therefore, (3.4) can be written as

$$\frac{\|P_{2^s(2k+1)}\|^2}{\|P_{2^{s+1}k}\|^2} = \|P_{2^s}\|^2 - \frac{\|P_{2^{s+1}k}\|^2}{\|P_{2^s(2k-1)}\|^2},$$

which is the desired result, as $a_n = \|P_n\| \,/\, \|P_{n-1}\|$. □

Theorem 3.4.3. *The recurrence coefficients* $(a_n)_{n=1}^{\infty}$ *can be calculated recursively by using Lemma 3.4.2 and* (3.1).

Proof. We already have $a_1 = ||P_1||$ and $a_2 = ||P_2|| \,/\, ||P_1||$. Suppose, by induction, that all a_i are given up to $i = n$. If $n + 1 = 2^s > 2$ then

$$a_{n+1} = \frac{||P_{2^s}||}{||P_{2^{s-1}}|| \cdot a_{2^{s-1}+1} \cdot a_{2^{s-1}+2} \cdots a_{2^s-1}},$$

where the norms of polynomials can be found by (3.1).

Otherwise, $n + 1 = 2^s(2k+1)$ for some $s \in \mathbb{N}_0$ and $k \in \mathbb{N}$. By Lemma 3.4.2, we have

$$a^2_{n+1} = a^2_{2^s(2k+1)} = \frac{\|P_{2^s}\|^2 - a^2_{2^{s+1}k} \cdots a^2_{2^{s+1}k-2^s+1}}{a^2_{2^s(2k+1)-1} \cdots a^2_{2^{s+1}k+1}},$$

provided $s \neq 0$. If $s = 0$ then the denominator in the fraction above is absent. This gives a_{n+1}, since the recurrence coefficients are positive. □

In order to illustrate the theorem, let us consider the cases of small s. If $s = 0$ then $n+1 = 2k+1$ and $a_{2k+1}^2 = a_1^2 - a_{2k}^2$. Next, for $s = 1$ and $s = 2$,

$$a_{4k+2}^2 = \frac{||P_2||^2 - a_{4k}^2\, a_{4k-1}^2}{a_{4k+1}^2}, \; a_{8k+4}^2 = \frac{||P_4||^2 - a_{8k}^2\, a_{8k-1}^2\, a_{8k-2}^2\, a_{8k-3}^2}{a_{8k+3}^2\, a_{8k+2}^2\, a_{8k+1}^2}, \text{ etc.}$$

Thus, $a_1 = \frac{\sqrt{1-2\gamma_1}}{2}$, $a_2 = \frac{\sqrt{1-2\gamma_2}}{\sqrt{1-2\gamma_1}}\,\gamma_1$, $a_3^2 = a_1^2 - a_2^2$, $a_4 = \frac{\gamma_1\gamma_2\sqrt{1-2\gamma_3}}{a_3\sqrt{1-2\gamma_2}}$, $a_5^2 = a_1^2 - a_4^2$, etc.

Remark 3.4.4. If $\gamma_n < 1/4$ for $1 \le n \le s$ and $\gamma_n = 1/4$ for $n > s$ then $K(\gamma) = E_s = (2\Phi_{2^s}/r_s + 1)^{-1}[-1,1]$. Here $(\Phi_{2^n} + r_n/2)_{n=0}^{\infty}$ are the Chebyshev polynomials for E_s, as is easy to check. Therefore Theorems 3.2.8 and 3.4.3 are applicable for this case as well. For further information about Jacobi parameters corresponding equilibrium measures of polynomial inverse images, we refer the reader to the article [59].

Remark 3.4.5. Suppose $\gamma_n = 1/4$ for $n \le N$ with $2^s \le N < 2^{s+1}$. Then $a_1 = 1/\sqrt{8}$ and $a_2 = a_3 = \cdots = a_{2^{s+1}-1} = 1/4$. In particular, if $\gamma_n = 1/4$ for all n then $a_n = 1/4$ for all $n \ge 2$, which corresponds to the case of the Chebyshev polynomials of first kind on $[0,1]$.

Lemma 3.4.6. *Suppose $\gamma_s \le 1/6$ for all s. For fixed $s \in \mathbb{N}_0$, let $c = \frac{4\gamma_{s+1}^2}{(1-2\gamma_{s+1})^2}$ and $C = \frac{2}{1+\sqrt{1-4c}}$. Then the following inequalities hold with $k \in \mathbb{N}_0$:*

(a) If $n = 2^s(2k+1)$ then

$$\frac{1}{2}||P_{2^s}||^2 \le C^{-1}\, ||P_{2^s}||^2 \le a_n^2 \cdots a_{n-2^s+1}^2 \le ||P_{2^s}||^2.$$

(b) If $n = 2^s(2k+2)$ then

$$a_n^2 \cdots a_{n-2^s+1}^2 \le C\, \frac{\|P_{2^{s+1}}\|^2}{\|P_{2^s}\|^2} \le 2\frac{\|P_{2^{s+1}}\|^2}{\|P_{2^s}\|^2}.$$

Proof. Note that, if γ_{s+1} increases from 0 to 1/6, then c increases from 0 to 1/4 and C increases from 1 to 2. By (3.1) and the definition of r_s, we get

$$\frac{\|P_{2^{s+1}}\|^2}{\|P_{2^s}\|^2} = \gamma_{s+1}^2\, r_s^2\, \frac{1-2\gamma_{s+2}}{1-2\gamma_{s+1}} = (1-2\gamma_{s+2})\, c\, \|P_{2^s}\|^2 < \|P_{2^s}\|^2/4. \tag{3.5}$$

We proceed by induction. For a fixed $s \in \mathbb{N}_0$, let $k = 0$. Then we have at once

$$a_{2^s}^2 \cdots a_1^2 = \|P_{2^s}\|^2 \quad \text{and} \quad a_{2^{s+1}}^2 \cdots a_{2^s+1}^2 = \frac{\|P_{2^{s+1}}\|^2}{\|P_{2^s}\|^2}.$$

Suppose (a), (b) are satisfied for $k \leq m$. We apply Lemma 3.4.2 with $k = m+1:$

$$a_{2^s(2m+3)}^2 \cdots a_{2^s(2m+2)+1}^2 + a_{2^s(2m+2)}^2 \cdots a_{2^s(2m+2)-2^s+1}^2 = \|P_{2^s}\|^2,$$

where for the addend we can use (b) for $k = m$. Therefore,

$$\|P_{2^s}\|^2 - C\frac{\|P_{2^{s+1}}\|^2}{\|P_{2^s}\|^2} \leq a_{2^s(2m+3)}^2 \cdots a_{2^s(2m+2)+1}^2 \leq \|P_{2^s}\|^2,$$

which is (a) for $k = m+1$, by (3.5).

Next, we claim that

$$a_{2^s(2m+4)}^2 \cdots a_{2^s(2m+2)+1}^2 \leq \|P_{2^{s+1}}\|^2 \tag{3.6}$$

for $m \in \mathbb{N}_0$. If $m = 2l+1$ then we use Lemma 3.4.2 with $s+1$ instead of $s:$

$$a_{2^{s+1}(2k+1)}^2 \cdots a_{2^{s+2}k+1}^2 + \;\; \textit{positive term} \;\; = \|P_{2^{s+1}}\|^2,$$

which implies (3.6), if we take $k = l+1$, as $2(2k+1) = 2m+4,\ 4k = 2m+2$.

Suppose m is even. Lemma 3.4.2 now gives

$$\textit{positive term} \;\; + a_{2^{s+2}k}^2 \cdots a_{2^{s+2}k-2^{s+1}+1}^2 = \|P_{2^{s+1}}\|^2,$$

where we take $k = m/2+1$. Thus, (3.6) holds true in both cases.

Putting together (a) for $k = m+1$ and (3.6) we get (b) for $k = m+1$. $\square$

Theorem 3.4.7. *Let $\gamma_s \leq 1/6$ for all s. Then $\lim_{s\to\infty} a_{j\cdot 2^s+n} = a_n$ for $j \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Here, $a_0 := 0$. In particular, $\liminf a_n = 0$.*

Proof. We first show that $\lim_{s\to\infty} a_{j\cdot 2^s} = 0$ for all $j \in \mathbb{N}$. Let $j = 2^l(2k+1)$ where $k, l \in \mathbb{N}_0$. For $s > 0$, the Jacobi parameters admit the following inequality by Lemma 3.4.6(a):

$$a^2_{2^{s+l}(2k+1)} \dots a^2_{2^{s+l+1}k+1} \leq \|P_{2^{s+l}}\|^2. \tag{3.7}$$

If $i < s + l$ where $i \in \mathbb{N}_0$, we have $2^{s+l}(2k+1) - 2^i = 2^i(2^{s+l-i}(2k+1) - 1)$. Since $2^{s+l-i}(2k+1) - 1$ is a positive odd number, by Lemma 3.4.6(a), we have the inequalities

$$\frac{1}{2}\|P_{2^i}\|^2 \leq a^2_{2^{s+l}(2k+1)-2^i} \dots a^2_{2^{s+l}(2k+1)-2^{i+1}+1} \ \text{ for } \ i = 0, \dots, s+l-1.$$

We multiply these $s + l$ inequalities side by side:

$$2^{-s-l}||P_1||^2 \cdots ||P_{2^{s+l-1}}\|^2 \leq a^2_{2^{s+l}(2k+1)-1} \cdots a^2_{2^{s+l+1}k+1}$$

and use (3.7):

$$a^2_{j\cdot 2^s} = a^2_{2^{s+l}(2k+1)} \leq \frac{2^{s+l}\|P_{2^{s+l}}\|^2}{\|P_{2^{s+l-1}}\|^2\|P_{2^{s+l-2}}\|^2 \dots \|P_1\|^2}.$$

By (3.5), the fraction above is bounded by 2^{-s-l+2}. Thus, $\lim_{s\to\infty} a_{j\cdot 2^s} = 0$.

If $n = 1$ then $a^2_{j\cdot 2^s+1} = a^2_1 - a^2_{j\cdot 2^s} \to a^2_1$, which is our claim.

Suppose, by induction, that $\lim_{s\to\infty} a_{j\cdot 2^s+n} = a_n$ for $n = 0, 1, \dots, m$ and all $j \in \mathbb{N}$. Let $m+1 = 2^p(2q+1)$ where $p, q \in \mathbb{N}_0$. If $q = 0$ then $j \cdot 2^s + m + 1 = j \cdot 2^{s-p} + 1$, so we get the case with $n = 1$. Thus, we can suppose $q \in \mathbb{N}$. Then $j \cdot 2^s + m + 1 = 2^p(2^{s+l-p}(2k+1) + 2q + 1)$ and, for large enough s, we can apply Lemma 3.4.2:

$$a^2_{j\cdot 2^s+m+1}\, a^2_{j\cdot 2^s+m} \cdots a^2_{j\cdot 2^s+m-2^p+1} + a^2_{j\cdot 2^s+m-2^p} \cdots a^2_{j\cdot 2^s+m+1-2^{p+1}} = \|P_{2^p}\|^2.$$

Here all indices, except the first, are of the form $j \cdot 2^s + n$ with $n < m + 1$. Therefore, by induction hypothesis, $a^2_{j\cdot 2^s+n} \to a_n$ as $s \to \infty$ and

$$(\lim_{s\to\infty} a^2_{j\cdot 2^s+m+1})\, a^2_m \cdots a^2_{m-2^p+1} + a^2_{m-2^p} \cdots a^2_{m+1-2^{p+1}} = \|P_{2^p}\|^2.$$

On the other hand, if we apply Lemma 3.4.2 to the number $m+1$, then we get the same equality with a^2_{m+1} instead of $\lim_{s\to\infty} a^2_{j\cdot 2^s+m+1}$. Since all a_k are positive, we have the desired result. $\square$

Remark 3.4.8. Since $\liminf a_n = 0$, by [53], $\mu_{K(\gamma)}$ is purely singular. In particular, this implies that $\mu_{K(\gamma)}$ is purely singular continuous since the equilibrium measure can not have isolated point in its support. Moreover, absence of a non-trivial absolutely continuous part of the equilibrium measure, by [101], guarantees that the support has zero Lebesgue measure. Thus $|K(\gamma)| = 0$ if $\gamma_s \le 1/6$ for all $s \in \mathbb{N}$.

3.5 Widom factors

Recall that $\mathrm{Cap}(K(\gamma)) = \exp\left(\sum_{k=1}^{\infty} 2^{-k} \log \gamma_k\right)$. In terms of $(\gamma_k)_{k=1}^{\infty}$ we can rewrite $\|P_{2^s}\|$ as

$$\frac{\sqrt{1-2\gamma_{s+1}}}{2} \exp\left(2^s \sum_{k=1}^{s} 2^{-k} \log \gamma_k\right).$$

Therefore,

$$W_{2^s}^2 = \frac{\sqrt{1-2\gamma_{s+1}}}{2\exp\left(\sum_{k=s+1}^{\infty} 2^{s-k} \log \gamma_k\right)} \ge \sqrt{2}, \tag{3.8}$$

since $\gamma_s \le 1/4$. The limit values $\gamma_s = 1/4$ for all s give the Widom-Hilbert Factors for the equilibrium measure on $[0,1]$.

Clearly, (3.8) implies that $\limsup W_n^2 > 0$. If $\gamma_s \le 1/6$ for all s then

$$W_{2^s}^2 \ge \sqrt{6}. \tag{3.9}$$

Let us show that, in this case, $\liminf W_n^2 > 0$.

Theorem 3.5.1. *Let $(W_n^2)_{n=1}^{\infty}$ be Widom-Hilbert factors for $\mu_{K(\gamma)}$ where $\gamma_s \le 1/6$ for all s. Then*

(a) $\liminf\limits_{s\to\infty} W_{2^s}^2 = \liminf\limits_{n\to\infty} W_n^2.$

(b) $\limsup\limits_{n\to\infty} W_n^2 = \infty.$

Proof. (a) We show that $W_n^2 > W_{2^s}^2$ for $2^s < n < 2^{s+1}$. Let $n = 2^s + 2^{s_1} + \ldots + 2^{s_m}$ with $s > s_1 > s_2 > \ldots > s_m \geq 0$. Then we decompose the product $a_1 \cdots a_n$ into groups $(a_1 \cdots a_{2^s}) \cdot (a_{2^s+1} \cdots a_{2^s+2^{s_1}}) \cdots (a_{2^s+\ldots+2^{s_{m-1}}+1} \cdots a_n)$. For the first group we have $a_1 \cdots a_{2^s} = ||P_{2^s}||$. For the second group we use Lemma 3.4.6(a) with $n = 2^s + 2^{s_1} : a_{2^s+1} \cdots a_{2^s+2^{s_1}} \geq ||P_{2^{s_1}}||/\sqrt{2}$. Similar estimation is valid for all other groups. Therefore,

$$\begin{aligned} W_n^2 &= \frac{a_1 \cdots a_{2^s}}{\mathrm{Cap}(K(\gamma))^{2^s}} \frac{a_{2^s+1} \cdots a_{2^s+2^{s_1}}}{\mathrm{Cap}(K(\gamma))^{2^{s_1}}} \cdots \frac{a_{2^s+\ldots+2^{s_{m-1}}+1} \cdots a_n}{\mathrm{Cap}(K(\gamma))^{2^{s_m}}} \\ &\geq W_{2^s}^2 W_{2^{s_1}}^2 \cdots W_{2^{s_m}}^2 (\sqrt{2})^{-m}, \end{aligned}$$

which exceeds $W_{2^s}^2(\sqrt{3})^m$, by (3.9). From here, $\min_{2^s \leq n < 2^{s+1}} W_n^2 = W_{2^s}^2$ and the result follows.

(b) Applying the procedure above to $W_{2^s-1}^2$ and taking the limit gives the desired result.

□

In order to illustrate the behavior of Widom-Hilbert factors, let us consider some examples. Suppose $\gamma_s \leq 1/6$ for all s.

Example 3.5.2. If $\gamma_s \to 0$ then $W_n^2 \to \infty$. Indeed, $W_{2^s}^2 \geq \frac{1}{\sqrt{6}} \exp(\frac{1}{2} \log \frac{1}{\gamma_{s+1}})$.

Example 3.5.3. There exists $\gamma_s \nrightarrow 0$ with $W_n^2 \to \infty$. Indeed, we can take $\gamma_{2k} = 1/6$, $\gamma_{2k-1} = 1/k$.

Example 3.5.4. If $\gamma_s \geq c > 0$ for all s then $\liminf_{n\to\infty} W_n^2 \leq 1/2c$.

Example 3.5.5. There exists γ with $\inf \gamma_s = 0$ and $\liminf_{n\to\infty} W_n^2 < \infty$. Here we can take $\gamma_s = 1/6$ for $s \neq s_k$ and $\gamma_{s_k} = 1/k$ for a sparse sequence $(s_k)_{k=1}^\infty$. Then $(W_{2^{s_k}}^2)_{k=1}^\infty$ is bounded.

Chapter 4

Orthogonal polynomials on generalized Julia sets

4.1 Introduction

Potential theoretical tools for Julia sets of polynomials were developed in [36] by Brolin. Mañé and Rocha, in [84], showed that Julia sets are uniformly perfect in the sense of Pommerenke and in particular they are regular with respect to the Dirichlet problem.

In this chapter, we consider orthogonal polynomials with respect to the equilibrium measure of $J_{(f_n)}$ where $(f_n) \in \mathcal{R}$.

We have to note that for the sequences $(f_n) \in \mathcal{R}$ satisfying the additional condition $d_n = d$ for some $d \geq 2$, there is a nice theory concerning various properties of Julia sets. For details, see [49, 90].

When we say that $(f_n) \in \mathcal{R}$ then the sequences (d_n), $(a_{n,j})$, $(A_i)_{i=1}^3$ will be used just as in the definition given in the first chapter and $F_n(z)$ will stand for $f_n \circ \ldots \circ f_1(z)$. Thus F_n is a polynomial with the leading coefficient $(a_{1,d_1})^{d_2 \cdots d_l}(a_{2,d_2})^{d_3 \cdots d_l} \cdots a_{l,d_l}$ of degree $d_1 \cdots d_n$. We consider and count multiple roots of a polynomial separately.

In Section 2, we give background information about the properties of autonomous polynomial Julia sets regarding orthogonal polynomials. In Section 3, we prove that for certain degrees, orthogonal polynomials associated with the equilibrium measure of $J_{(f_n)}$ are given explicitly in terms of the compositions F_n. In Section 4, we show that the recurrence coefficients can be calculated provided that $J_{(f_n)}$ is real. These two results generalize Theorem 3 in [21] and Theorem 1 in [22] respectively. In addition to these results we discuss resolvent functions and a general method to construct real Julia sets. Techniques that we use here are rather different compared to those of autonomous setting. This is mostly due to the fact that, in the generalized case, Julia sets do not have complete invariance but we only have the properties given in part (e) of Theorem 1.1.4.

In Section 6, we consider a quadratic family of polynomials (f_n) such that the set $K_1(\gamma) = J_{(f_n)}$ is a modification of the set $K(\gamma)$. In terms of the parameter γ we give a criterion for the Green function $g_{\Omega_{K_1(\gamma)}}$ to be optimally smooth. In the last section, a criterion is presented for $K_1(\gamma)$ to be a Parreau-Widom set.

4.2 Preliminaries

Polynomial Julia sets are one of the most studied objects in one dimensional complex dynamics.For classical results related to potential theory, see [36]. For a more general exposition we refer to the monograph [92] and the survey [80].

Before going any further, we want to mention the results from [21] and [22] concerning orthogonal polynomials for the autonomous Julia sets. Let $f(z) = z^n + k_1 z^{n-1} + \ldots + k_n$ be a nonlinear monic polynomial of degree n and let P_j denote the j-th monic orthogonal polynomial associated to the equilibrium measure of $J(f)$. Then we have,

(a) $P_1(z) = z + k_1/n$.

(b) $P_{ln}(z) = P_l(f(z))$, for $l = 0, 1, \ldots$

(c) $P_{n^l}(z) = f^l(z) + k_1/n$ for $l = 0, 1, \ldots$, where f^l is the l-th iteration of the function f.

In Theorem 4.3.2, we recover parts (a) and (c) in a more general setting. Even without having the analogous equations to part (b), recurrence coefficients appear as the outcome of Theorem 4.4.1.

4.3 Orthogonal polynomials

Let $(f_n) \in \mathcal{R}$. Then, by the fundamental theorem of algebra (FTA), $F_k(z) - a = 0$ has $d_1 \cdots d_k$ solutions counting multiplicities. For a fiven k, let us define the normalized counting measure as $\nu_k^a = \frac{1}{d_1 \cdots d_k} \sum_{l=1}^{d_1 \cdots d_k} \delta_{z_l}$ where $z_1, \ldots, z_{d_1 \cdots d_k}$ are the roots of $F_k(z) - a$. In [36] and later on in [37], it was shown that $\nu_k^a \to \mu_{J_{(f_n)}}$ for a proper a where in the first article $f_n = f$ with a monic nonlinear polynomial f and in the second one $f_n(z) = z^2 + c_n$. Our technique used below is the same in essence with the proofs in [36, 37]. Due to some minor changes and for the convenience of the reader, we include the proof of the theorem.

Theorem 4.3.1. *Let $(f_n) \in \mathcal{R}$. Then for $a \in \mathbb{C} \setminus \overline{\mathbb{D}}$ satisfying the condition*

$$|a| A_1 \left(1 - \frac{A_2}{|a| - 1}\right) > 2, \tag{4.1}$$

we have $\nu_k^a \to \mu_{J_{(f_n)}}$.

Proof. Choose a number $a \in \mathbb{C} \setminus \overline{\mathbb{D}}$ satisfying (4.1). Let $K := J_{(f_n)}$ and $L := \{z \in \mathbb{C} : |z| \leq |a|\}$. Then, by part (b) of Theorem 1.1.4, $K \subsetneq L$. Moreover, since K is regular with respect to the Dirichlet problem and K is equal to the boundary of the component of $\overline{\mathbb{C}} \setminus K$ that contains ∞, we have (see e.g. Theorem 4.2.3. of [106]) that $\mathrm{supp}(\mu_K) = K$.

Observe that, ${F_k}^{-1}(a) \cap \mathcal{A}_{(f_n)}(\infty)$ is contained in L for all $k \in \mathbb{N}$ by part (b) of Theorem 1.1.4. Thus, $(\nu_k^a)_{k=1}^{\infty}$ has a convergent subsequence $(\nu_{k_l}^a)_{l=1}^{\infty}$ by Helly's selection principle (see e.g. Theorem 0.1.3. in [110]). Let us denote the limit by μ. The set $\cup {F_k}^{-1}(a)$ can not accumulate to a point z in $\mathcal{A}_{(f_n)}(\infty)$, since this would contradict with the fact that $F_k(z)$ goes locally uniformly to ∞ by part (a) of Theorem 1.1.4. Thus, $\text{supp}(\mu) \subset \partial \mathcal{A}_{(f_n)}(\infty) = K$.

Now, we want to show that $\liminf_{l\to\infty} U^{k_l}(z) \geq V_K$ for all $z \in K$. Let $z \in K$ where U^k denote the logarithmic potential for ν_k^a. We have

$$|F_{k_l}(z) - a| = |(a_{1,d_1})^{d_2\cdots d_{k_l}}||(a_{2,d_2})^{d_3\cdots d_{k_l}}|\cdots|a_{k_l,d_{k_l}}| \prod_{j=1}^{d_1\cdots d_{k_l}} |z - z_{j,k_l}|,$$

for some $z_{j,k_l} \in L$. Thus,

$$U^{k_l}(z) = \frac{\sum_{j=1}^{d_1\cdots d_{k_l}} \log|z - z_{j,k_l}|}{-d_1\cdots d_{k_l}} = \sum_{j=1}^{k_l} \frac{\log|a_{j,d_j}|}{d_1\cdots d_j} - \frac{\log|F_{k_l}(z) - a|}{d_1\cdots d_{k_l}}. \tag{4.2}$$

Using part (d) of Theorem 1.1.4 and the fact that $|F_k(z)| \leq |a|$ for $z \in K$, we see that the following inequality follows from (4.2):

$$\liminf_{l\to\infty} U^{k_l}(z) \geq \liminf_{l\to\infty} \left(\sum_{j=1}^{k_l} \frac{\log|a_{j,d_j}|}{d_1\cdots d_j} - \frac{\log|2a|}{d_1\cdots d_{k_l}} \right) \geq V_K.$$

Hence, by Lemma 1.1.1, we have $\nu_{k_l}^a \to \mu_K$. Since $(\nu_{k_l}^a)$ is an arbitrary convergent subsequence, $\nu_k^a \to \mu_K$ also holds. $\square$

In the next theorem, we use algebraic properties of polynomials as well as analytic properties of the corresponding Julia sets. Let $f(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots a_0$ be a nonlinear polynomial of degree n and let $z_1, z_2, \ldots, z_n$ be roots of f counting multiplicities. Then, for $k = 1, 2, \ldots n-1$ we have the following Newton's identities:

$$s_k(f(z)) + \frac{a_{n-1}}{a_n} s_{k-1}(f(z)) + \ldots + \frac{a_{n-k+1}}{a_n} s_1(f(z)) = -k\frac{a_{n-k}}{a_n}, \tag{4.3}$$

where $s_k(f(z)) := \sum_{j=1}^{n} (z_j)^k$.

For the proof of (4.3), see [91] among others. Note that, none of these equations include the term a_0. This implies that the values $(s_k)_{k=1}^{n-1}$ are invariant under translation of the function f, i.e.

$$s_k(f(z)) = s_k(f(z) + c) \tag{4.4}$$

for any $c \in \mathbb{C}$. Let $(P_j)_{j=1}^{\infty}$ denote the sequence of monic orthogonal polynomials associated to $\mu_{J_{(f_n)}}$ where $\deg P_j = j$. Now we are ready to prove our first main result.

Theorem 4.3.2. *For $(f_n) \in \mathcal{R}$, we have the following identities:*

(a) $P_1(z) = z + \dfrac{1}{d_1}\dfrac{a_{1,d_1-1}}{a_{1,d_1}}$.

(b) $P_{d_1\cdots d_l}(z) = \dfrac{1}{(a_{1,d_1})^{d_2\cdots d_l}(a_{2,d_2})^{d_3\cdots d_l}\cdots a_{l,d_l}}\left(F_l(z) + \dfrac{1}{d_{l+1}}\dfrac{a_{l+1,d_{l+1}-1}}{a_{l+1,d_{l+1}}}\right)$.

Proof. (a) Let $(f_n) \in \mathcal{R}$ be given and $a \in \mathbb{C} \setminus \overline{\mathbb{D}}$ satisfy (4.1). Fix an integer m greater than 1. By FTA, The solutions of the equation $F_m(z) = a$ satisfy an equation of the form

$$\big(F_{m-1}(z) - \beta_{m-1}^{1}\big)\ldots(F_{m-1}(z) - \beta_{m-1}^{d_m}) = 0,$$

where $\beta_{m-1}^{1}, \ldots, \beta_{m-1}^{d_m} \in \mathbb{C}$. The $d_1 \cdots d_{m-1}$ roots of the equation $F_{m-1} - \beta_{m-1}^{j} = 0$ are the solutions of an equation

$$(F_{m-2}(z) - \beta_{m-2}^{1,j})\ldots(F_{m-2}(z) - \beta_{m-2}^{d_{m-1},j}) = 0,$$

with some $\beta_{m-2}^{1,j}, \ldots, \beta_{m-2}^{d_{m-1},j}$. Continuing this way, the points satisfying the equation $F_m(z) = a$ can be grouped into $d_2 \cdots d_m$ parts of size d_1 such that each part consists of the roots of an equation

$$f_1(z) - \beta_1^{j} = 0,$$

for $j \in \{1, \ldots, d_2 \cdots d_m\}$ and $\beta_1^j \in \mathbb{C}$. If for each j, we denote the normalized counting measure on the roots of $f_1(z) - \beta_1^j$ by λ_j, then

$$\nu_m^a = \frac{1}{d_2 \cdots d_m}\sum_{j=1}^{d_2\cdots d_m}\lambda_j.$$

Hence, by (4.3) and (4.4),

$$\int z\, d\nu_m^a = \frac{1}{d_2\cdots d_m}\sum_{j=1}^{d_2\cdots d_m}\int z\, d\lambda_j = \frac{1}{d_2\cdots d_m}\sum_{j=1}^{d_2\cdots d_m}\frac{s_1(f_1(z) - \beta_1^j)}{d_1}$$

$$= \frac{1}{d_1 \cdots d_m} \sum_{j=1}^{d_2 \cdots d_m} s_1(f_1(z)) = -\frac{1}{d_1} \frac{a_{1,d_1-1}}{a_{1,d_1}}.$$

Since ν_m^a converges to the equilibrium measure of $J(f_n)$ by Theorem 4.3.1, the result follows.

(b) Let $m, l \in \mathbb{N}$ where $m > l+1$. As above, the roots of the equation $F_m(z) = a$ where $a \in \mathbb{C} \setminus \overline{\mathbb{D}}$ satisfies (4.1), can be grouped into $d_{l+2} \cdots d_m$ parts of size $d_1 \cdots d_{l+1}$ such that each part obeys an equation of the form

$$F_{l+1}(z) - \beta_{l+1}^j = 0,$$

for $j = 1, 2, \ldots, d_{l+2} \cdots d_m$. Recall that $F_{l+1}(z) = f_{l+1}(t)$ with $t = F_l(z)$.

By FTA, we have $f_{l+1}(t) - \beta_{l+1}^j = (t - \beta_l^{1,j}) \cdots (t - \beta_l^{d_{l+1},j})$ for some $\beta_l^{1,j}, \ldots, \beta_l^{d_{l+1},j}$. By (4.3) and (4.4), for $k \in \{1, \ldots, d_{l+1} - 1\}$ and $j, j' \in \{1, \ldots, d_{l+2} \cdots d_m\}$, we have

$$s_k(f_{l+1}(t) - \beta_{l+1}^j) := \sum_{r=1}^{d_{l+1}} (\beta_l^{r,j})^k = \sum_{r=1}^{d_{l+1}} (\beta_l^{r,j'})^k = s_k(f_{l+1}(t) - \beta_{l+1}^{j'}).$$

Now we can rewrite $F_{l+1}(z) - \beta_{l+1}^j = 0$ as $(F_l(z) - \beta_l^{1,j}) \cdots (F_l(z) - \beta_l^{d_{l+1},j}) = 0$ for j as above. Let us denote the normalized counting measures on the roots of $F_l(z) - \beta_l^{r,j} = 0$ by $\lambda_{r,j}$ for $r = 1, \ldots, d_{l+1}$ and $j = 1, \ldots, d_{l+2} \cdots d_m$. Clearly, this yields

$$\nu_m^a = \frac{1}{d_{l+2} \cdots d_m} \sum_{j=1}^{d_{l+2} \cdots d_m} \frac{1}{d_{l+1}} \sum_{r=1}^{d_{l+1}} \lambda_{r,j} = \frac{1}{d_{l+1} \cdots d_m} \sum_{j=1}^{d_{l+2} \cdots d_m} \sum_{r=1}^{d_{l+1}} \lambda_{r,j}. \tag{4.5}$$

Thus, by using (4.5), (4.3) and (4.4), we deduce that

$$\begin{aligned}
\int F_l(z)\, d\nu_m^a &= \frac{1}{d_{l+1}\cdots d_m} \sum_{j=1}^{d_{l+2}\cdots d_m} \sum_{r=1}^{d_{l+1}} \int F_l(z)\, d\lambda_{r,j} \\
&= \frac{1}{d_{l+1}\cdots d_m} \sum_{j=1}^{d_{l+2}\cdots d_m} \sum_{r=1}^{d_{l+1}} \beta_l^{r,j} \\
&= \frac{1}{d_{l+1}\cdots d_m} \sum_{j=1}^{d_{l+2}\cdots d_m} s_1(f_{l+1}(t) - \beta_{l+1}^j) \\
&= \frac{1}{d_{l+1}\cdots d_m} \sum_{j=1}^{d_{l+2}\cdots d_m} s_1(f_{l+1}(t)) \\
&= -\frac{1}{d_{l+1}} \frac{a_{l+1,d_{l+1}-1}}{a_{l+1,d_{l+1}}}.
\end{aligned}$$

To shorten notation, we write c instead of $\frac{1}{d_{l+1}} \frac{a_{l+1,d_{l+1}-1}}{a_{l+1,d_{l+1}}}$. Thus, we have

$$\int (F_l(z) + c)\, d\nu_m^a = 0. \tag{4.6}$$

Let us show that the integrand is orthogonal to z^k with $1 \le k \le d_1 \cdots d_l - 1$ as well. For the same $\lambda_{r,j}$, as above, we have

$$\int (F_l(z) + c)\, \overline{z^k}\, d\lambda_{r,j} = \frac{1}{d_1 \cdots d_l} \left(\beta_l^{r,j} + c\right) \cdot \overline{s_k\left(F_l(z) - \beta_l^{r,j}\right)}.$$

By (4.4), $\overline{s_k\left(F_l(z) - \beta_l^{r,j}\right)} = \overline{s_k\left(F_l(z)\right)}$, so it does not depend on r or j. This and the representation (4.5) imply that

$$\begin{aligned}
\int (F_l(z) + c)\, \overline{z^k}\, d\nu_m^a &= \frac{1}{d_{l+1}\cdots d_m} \sum_{j=1}^{d_{l+2}\cdots d_m} \sum_{r=1}^{d_{l+1}} \int (F_l(z) + c)\, \overline{z^k}\, d\lambda_{r,j} \\
&= \frac{\overline{s_k\left(F_l(z)\right)}}{d_1 \dots d_l} \int (F_l(z) + c)\ d\nu_m^a,
\end{aligned}$$

where the last term is equal to 0, by (4.6). It follows that $(F_l(z) + c) \perp z^k$ for $k \le \deg F_l - 1$ in $L^2(\mu_{J_{(f_n)}})$, since ν_m^a converges to the equilibrium measure of $J(f_n)$. This completes the proof of the theorem.

$\square$

4.4 Moments and resolvent functions

In this section we consider Julia sets that are subsets of the real line.

If μ is a probability measure which has infinite compact support in $\mathbb{R}$, then the monic orthogonal polynomials $(P_n)_{n=1}^{\infty}$ satisfy a recurrence relation

$$P_{n+1}(x) = (x - b_{n+1})P_n(x) - a_n^2 P_{n-1}(x),$$

assuming that $P_0 = 1$ and $P_{-1} = 0$. If the moments $c_n = \int x^n d\mu$ are known for all $n \in \mathbb{N}_0$ then we have the formula

$$p_n(x) = \frac{1}{\sqrt{D_n D_{n-1}}} \begin{vmatrix} c_0 & c_1 & \dots & c_n \\ c_1 & c_2 & \dots & c_{n+1} \\ \vdots & \vdots & & \vdots \\ c_{n-1} & c_n & \dots & c_{2n-1} \\ 1 & x & \dots & x^n \end{vmatrix} \tag{4.7}$$

where p_n is the n-th orthonormal polynomial and D_n is the determinant for the matrix M_n with the entries $(M_n)_{i,j} = c_{i+j}$ for $i, j = 0, 1, \dots n$. From (4.7), one can also calculate recurrence coefficients $(a_n, b_n)_{n=1}^{\infty}$. See [135] for a detailed description of the orthogonal polynomials on the real line. In the next theorem, we show that the moments for the equilibrium measure of $J_{(f_n)}$ can be calculated recursively whenever $(f_n) \in \mathcal{R}$. Note that $c_0 = 1$ since the equilibrium measure is of unit mass.

Theorem 4.4.1. *Let $(f_n) \in \mathcal{R}$ and $l > 0$ be an integer. Furthermore, let*

$$\frac{F_l(z)}{p_l} = z^{d_1 \cdots d_l} + a_{d_1 d_2 \cdots d_l - 1} z^{d_1 d_2 \cdots d_l - 1} + \dots + a_1 z + a_0,$$

where p_l is the leading coefficient for F_l. Then, each moment $c_k = \int x^k d\mu_{J_{(f_n)}}$ for $k \in \{1, 2 \dots, (d_1 d_2 \cdots d_l) - 1\}$ is equal to $\frac{s_k(F_l(z))}{d_1 \cdots d_l}$ where $s_k(F_l(z))$ can be calculated recursively by Newton's identities.

Proof. Let m be an integer greater than l. Consider the roots of the equation $F_m(z) = a$ where $a \in \triangle_1$ satisfies the condition (4.1). Then, following the proof of Theorem 4.3.2, we can divide these roots into $d_{l+1} \cdots d_m$ parts of size $d_1 \cdots d_l$ such that the nodes in each of the groups constitute the roots of an equation of the form

$$F_l(z) - \beta^j = 0,$$

for $j = 1, 2, \ldots, d_{l+1} \cdots d_m$. If for each j we denote the normalized counting measure on the roots of $F_l(z) - \beta^j$ by λ_j, then by (4.3) and (4.4), this leads to

$$\begin{aligned}
\int x^k \, d\nu_m^a &= \frac{1}{d_{l+1} \cdots d_m} \sum_{j=1}^{d_{l+1} \cdots d_m} \int x^k \, d\lambda_j \\
&= \frac{1}{d_{l+1} \cdots d_m} \sum_{j=1}^{d_{l+1} \cdots d_m} \frac{s_k(F_l(z) - \beta^j)}{d_1 \cdots d_l} \\
&= \frac{1}{d_{l+1} \cdots d_m} \sum_{j=1}^{d_{l+1} \cdots d_m} \frac{s_k(F_l(z))}{d_1 \cdots d_l} = \frac{s_k(F_l(z))}{d_1 \cdots d_l},
\end{aligned}$$

for $k = 1, 2 \ldots, (d_1 d_2 \cdots d_l) - 1$. Since the weak star limit of the sequence $(d\nu_m^a)$ is the equilibrium measure of the Julia set by Theorem 4.3.1, we have $\int x^k d\mu_{J_{(f_n)}} = \frac{s_k(F_l(z))}{d_1 \ldots d_l}$ which concludes the proof. □

In Sections 3-5 of Chapter 3, orthogonal polynomials and recurrence coefficients are discussed for the quadratic case. It would be interesting to obtain similar results for $\mu_{J_{(f_n)}}$ if we only assume that $(f_n) \in \mathcal{R}$ and $J_{(f_n)} \subset \mathbb{R}$.

Let $J_{(f_n)} \subset [-M, M]$ for some $M \in \mathbb{R}$ where $(f_n) \in \mathcal{R}$. If we denote the Jacobi operator associated with $\mu_{J_{(f_n)}}$ by $H_{(f_n)}$ then the *resolvent function* $R_{(f_n)}$ is defined as

$$R_{(f_n)}(z) := \int \frac{d\,\mu_{J_{(f_n)}}(x)}{x - z} = \langle (H_{(f_n)} - z)^{-1} \delta_1, \delta_1 \rangle$$

for $z \in \mathbb{C} \setminus J_{(f_n)}$. Note that $R_{(f_n)}$ is an analytic function. If $f_n = f$ for a nonlinear polynomial f for all $n \in \mathbb{N}$ then the resolvent function satisfies a functional equation:

$$R_{(f)}(z) = \frac{f'(z)}{\deg f} R_{(f)}(f(z)). \tag{4.8}$$

See e.g. [29] for a discussion of resolvent functions and operators associated with the equilibrium measure of autonomous polynomial Julia sets. It is well known that (see e.g. p. 53 in [116]) for $z \in \mathbb{C} \setminus \overline{B(0,M)}$

$$R_{(f_n)}(z) = -\sum_{n=0}^{\infty} c_n z^{-(n+1)} \tag{4.9}$$

where c_n is the n-th moment for $\mu_{J_{(f_n)}}$ and the series at (4.9) is absolutely convergent in the corresponding domain.

We define the ∂ operator as

$$\partial = \frac{\partial_x - i\partial_y}{2}.$$

If g is a harmonic function on a simply connected domain $D \subset \mathbb{C}$ then (see e.g. Theorem 1.1.2 in [106]) there is an analytic function h on D such that $g = \operatorname{Re} h$ holds. Moreover, we have $h'(z) = 2\partial g(z)$. Furthermore,

$$g_{\Omega_{J_{(f_n)}}}(z) = \log\left(\operatorname{Cap}(J_{(f_n)})^{-1}\right) - U^{\mu_{J_{(f_n)}}}(z)$$

holds where $U^{\mu_{J_{(f_n)}}}$ is the logarithmic potential for $\mu_{J_{(f_n)}}$. In addition, for each $z_0 \in \mathbb{C} \setminus J_{(f_n)}$, there is a $\delta > 0$ and an analytic function h (which may depend on z_0) such that (see e.g. p. 87 in [55]) $h'(z) = R_{(f_n)}(z)$ and $\operatorname{Re} h = U^{\mu_{J_{(f_n)}}}$ for $z \in D_\delta(z_0)$. By harmonicity of $U^{\mu_{J_{(f_n)}}}$ this implies

$$2\partial g_{\Omega_{J_{(f_n)}}}(z) = -2\partial U^{\mu_{J_{(f_n)}}}(z) = -R_{(f_n)}(z) \tag{4.10}$$

for all $z \in \mathbb{C} \setminus J_{(f_n)}$. The next theorem follows from the discussion above.

Theorem 4.4.2. *Let $J_{(f_n)} \subset \mathbb{R}$ provided that $(f_n) \in \mathcal{R}$. Then the following functional equation holds where the limit exists locally uniformly in $\mathbb{C} \setminus J_{(f_n)}$:*

$$R_{(f_n)}(z) = \lim_{k\to\infty} \frac{R_{(f_n)}(F_k(z))F_k'(z)}{d_1 \cdots d_k}. \tag{4.11}$$

Proof. If we apply ∂ to both sides of (1.8), it is permitted to change the differentiation and limit since (see e.g. p. 16 in [18]) $g_{\Omega_{J_{(f_n)}}}$ is harmonic in $\mathcal{A}_{(f_n)}(\infty) \setminus \infty$. Note that $\mathcal{A}_{(f_n)}(\infty) \setminus \infty = \mathbb{C} \setminus J_{(f_n)}$ here since $J_{(f_n)}$ lies on $\mathbb{R}$. Hence, we have

$$\partial g_{\Omega_{J_{(f_n)}}}(z) = \lim_{k\to\infty} \frac{\partial g_{\Omega_{J_{(f_n)}}}(F_k(z))F_k'(z)}{d_1 \cdots d_k} \tag{4.12}$$

where the limit on the right side of (4.12) holds locally uniformly. Using (4.10) and (4.12), we have (4.11) immediately. □

Remark 4.4.3. Provided that $f_n = f$ for a fixed nonlinear polynomial f in Theorem 4.4.2, (4.11) reduces to (4.8) if we put $f(z)$ instead of z in both sides of (1.8) and follow the steps of the proof of Theorem 4.4.2.

4.5 Construction of real Julia sets

Let f be a nonlinear real polynomial with real and simple zeros $x_1 < x_2 < \ldots < x_n$ and distinct extremas $y_1 < \ldots < y_{n-1}$ with $|f(y_i)| > 1$ for $i = 1, 2, \ldots, n-1$. Then we say that f is an *strictly admissible* polynomial. We list useful features of preimages of strictly admissible polynomials.

Theorem 4.5.1. *[59] Let f be an strictly admissible polynomial of degree n. Then*

$$f^{-1}([-1,1]) = \cup_{i=1}^{n} E_i$$

where E_i is a closed non-degenerate interval containing exactly one root x_i of f for each i. These intervals are pairwise disjoint and $\mu_{f^{-1}([-1,1])}(E_i) = 1/n$.

We say that an strictly admissible polynomial f satisfies *the property* (A) if

(a) $f^{-1}([-1,1]) \subset [-1,1]$,

(b) $f(\{-1,1\}) \subset \{-1,1\}$,

(c) $f(a) = 0$ implies $f(-a) = 0$.

Clearly, (c) implies that f is even or odd.

Lemma 4.5.2. *Let g_1 and g_2 be strictly admissible polynomials satisfying (A). Then $g_3 := g_2 \circ g_1$ is also an strictly admissible polynomial that satisfies (A).*

Proof. Let $\deg g_k = n_k$. Moreover, let $(x_{j,1})_{j=1}^{n_1}$, $(x_{j,2})_{j=1}^{n_2}$ be the zeros and $(y_{j,1})_{j=1}^{n_1-1}$ and $(y_{j,2})_{j=1}^{n_2-1}$ be the critical points of g_1, g_2 respectively. Then the equation $g_3(z) = 0$ implies that $g_1(z) = x_{j,2}$ for some $j \in \{1, \ldots, n_2\}$. By (a) and (b), the equation $g_1(z) = \beta$ has n_1 distinct roots for $|\beta| \leq 1$ and the sets of roots of $g_1(z) = \beta_1$ and $g_1(z) = \beta_2$ are disjoint for different $\beta_1, \beta_2 \in [-1, 1]$. Therefore, g_3 has $n_1 n_2$ distinct zeros. Similarly, $(g_3)'(z) = g_2'(g_1(z))g_1'(z) = 0$ implies $g_1'(z) = 0$ or $g_1(z) = y_{j,2}$ for some $j \in \{1, \ldots, n_2 - 1\}$. The equation $g_1'(z) = 0$ has $n_1 - 1$ distinct solutions in $(-1, 1)$. For each of them $|g_1(z)| > 1$ and ${g_2}'(g_1(z)) \neq 0$. On the other hand, for each $j \leq n_2 - 1$, the equation $g_1(z) = y_{j,2}$ has n_1 distinct solutions with ${g_1}'(y_{j,2}) \neq 0$. Thus, the total number of solutions for the equation ${g_3}'(z) = 0$ is $n_1 - 1 + n_1(n_2 - 1) = n_1 n_2 - 1$ which is required. Hence, g_3 is strictly admissible. It is straightforward that for the function g_3 parts (a) and (b) are satisfied. The part (c) is also satisfied for g_3, since arbitrary compositions of even and odd functions are either even or odd. □

Lemma 4.5.3. *Let $(f_n) \in \mathcal{R}$ be a sequence of strictly admissible polynomials satisfying (A). Then F_n is an strictly admissible polynomial with the property (A). Besides, $F_{n+1}^{-1}([-1, 1]) \subset F_n^{-1}([-1, 1]) \subset [-1, 1]$ and $K = \cap_{n=1}^{\infty} F_n^{-1}([-1, 1])$ is a Cantor set in $[-1, 1]$.*

Proof. All statements except the last one follow directly from Lemma 4.5.2 and the representation $F_n(z) = f_n \circ F_{n-1}(z)$. Let us show that K is totally disconnected.

If K is polar then (see e.g. Corollary 3.8.5. of [106]) it is totally disconnected. If K is non-polar, then (see e.g. Theorem A.16. of [114]), $\mu_{F_n^{-1}([-1,1])} \to \mu_K$. Suppose that K is not totally disconnected. Then K contains an interval E such that $E \subset F_n^{-1}([-1, 1])$ for all n. Since we have $\mu_{F_n^{-1}([-1,1])}(E) \leq 1/(d_1 \ldots d_n)$ by Theorem 4.5.1, convergence of $(\mu_{F_n^{-1}([-1,1])})$ implies that $\mu_K(E) = 0$. Thus all interior points of E in $\mathbb{R}$ are outside of the support of μ_K. This is impossible by Theorem 4.2.3. of [106] since $K = \partial(\overline{\mathbb{C}} \setminus K)$ and $\mathrm{Cap}(E) > 0$. □

Here we consider strictly admissible polynomials as polynomials of complex variable.

Lemma 4.5.4. *Let f be an strictly admissible polynomial satisfying* (A). *Then* $|f(z)| > 1 + 2\epsilon$ *provided* $|z| > 1 + \epsilon$ *for* $\epsilon > 0$. *If* $|z| = 1$ *then* $|f(z)| > 1$ *unless* $z = \pm 1$.

Proof. Let $\deg f = n$ and $x_1 < x_2 < \ldots < x_n$ be the zeros of f. By (c), $x_k = -x_{n+1-k}$ for $k \leq n$. In particular, if n is odd, then $x_{(n+1)/2} = 0$.

Let $x_i \neq 0$ and $\epsilon > 0$. Then, by the law of cosines, the polynomial $P_{x_i}(z) := z^2 - x_i^2$ attains minimum of its modulus on the set $\{z : |z| = 1 + \epsilon\}$ at the point $z = 1 + \epsilon$. Therefore $|P_{x_i}(z)|/|P_{x_i}(\pm 1)| > 1 + 2\epsilon$ for any z with $|z| = 1 + \epsilon$. Using the symmetry of the roots of f about $x = 0$, we see that $|f(z)| = |f(z)/|f(\pm 1)| > 1 + 2\epsilon$ for such z.

If $|z| = 1$ then $|P_{x_i}(z)|$ attains its minimum at the points ± 1. Hence we have $|f(z)| = |f(z)|/|f(\pm 1)| > 1$ if $|z| = 1$ and $z \neq \pm 1$. □

In the next theorem we use the argument of Theorem 1 in [62].

Theorem 4.5.5. *Let* $(f_n) \in \mathcal{R}$ *be a sequence of strictly admissible polynomials satisfying* (A). *Then* $K = \cap_{n=1}^{\infty} F_n^{-1}([-1, 1]) = J_{(f_n)}$.

Proof. Let us prove first the inclusion $J_{(f_n)} \subset K$. Let $R > 1$ be any number satisfying $A_1 R(1 - (A_2/(R - 1))) > 2$. Then by part (b) of Theorem 1.1.4, we have $\mathcal{A}_{(f_n)}(\infty) = \cup_{k=1}^{\infty} {F_k}^{-1}(\triangle_R)$ and $f_n(\overline{\triangle}_R) \subset \triangle_R$ for all n. If we show that $|F_n(z)| > 1+\epsilon$ for some $n \in \mathbb{N}$ and for some positive ϵ, this implies that $F_{n+k}(z) \in \triangle_R$ for some positive k by Lemma 4.5.4 and thus $z \notin J_{(f_n)}$.

Let $|z| = 1 + \epsilon$ where $\epsilon > 0$. Then by Lemma 4.5.4, $|F_1(z)| > 1 + 2\epsilon$. Hence, $z \notin J_{(f_n)}$.

Let $|z| = 1$ where $z \neq \pm 1$. Then using Lemma 4.5.4, we see that $|F_1(z)| > 1$. Thus, $z \notin J_{(f_n)}$.

If we let $z \in [-1, 1] \backslash K$, then there exists a number $N \in \mathbb{N}$ such that $|F_N(z)| > 1$. As a result, $z \notin J_{(f_n)}$.

Letting $z = x + iy$ where $x \notin K$, $|y| > 0$ and $|z| < 1$ implies that there exists a positive number N such that $|F_N(x)| > 1$. Since all of the zeros of F_n are on the real line by Lemma 4.5.3, we have $|F_n(z)| > |F_n(x)| > 1$. Hence $z \notin J_{(f_n)}$.

Let $z = x + iy$ where $x \in K$, $|y| > 0$ and $|z| < 1$. Since K is a Cantor set by Lemma 4.5.3, there exists a number $N \in \mathbb{N}$ such that $n > N$ implies that each connected component of $F_n^{-1}([-1, 1])$ has length less than $y^2/8$. Let $x_1 < x_2 \ldots < x_{d_1 \ldots d_{N+1}}$ be the roots of the polynomial F_{N+1} and E_j denote the connected component of $F_{N+1}^{-1}([-1, 1])$ containing x_j for $j = 1, 2, \ldots, d_1 \ldots d_{N+1}$. Furthermore, let $E_s = [a, b]$ be the component containing the point x. Observe that $|F_{N+1}(a)| = |F_{N+1}(b)| = 1$. So, in order to show $z \notin J_{(f_n)}$, it is enough to show that $|F_{N+1}(z)| > |F_{N+1}(a)|$.

If $j < s$, then $|a - x_j| \leq |x - x_j| < |z - x_j|$.

If $j = s$, then $|a - x_j| < y^2/8 < |y| \leq |z - x_j|$.

If $j > s$, then

$$
\begin{aligned}
|a - x_j| &= \sqrt{|x_j - a|^2} \\
&\leq \sqrt{|x_j - x|^2 + |x - a|^2 + 2|x_j - x||x - a|} \\
&< \sqrt{|x_j - x|^2 + \frac{y^4}{64} + \frac{y^2}{2}} \\
&< \sqrt{|x_j - x|^2 + y^2} = |z - x_j|.
\end{aligned}
$$

Therefore, $|F_n(z)| > 1$. Thus, we have $J_{(f_n)} \subset K$ and $\overline{\mathbb{C}} \setminus K \subset \mathcal{A}_{(f_n)}(\infty)$.

For the inverse inclusion, observe that $K \subset \{z : |F_n(z)| \leq 1 \text{ for all } n\}$ where $\{z : |F_n(z)| \leq 1 \text{ for all } n\} \cap \mathcal{A}_{(f_n)}(\infty) = \emptyset$. Since K is contained in the real line and $\overline{\mathbb{C}} \setminus K \subset \mathcal{A}_{(f_n)}(\infty)$ by the first part of the proof, we have $K \subset \partial \mathcal{A}_{(f_n)}(\infty) = J_{(f_n)}$. $\square$

Corollary 4.5.6. *Orthogonal polynomials associated to the equilibrium measure of K and the corresponding recurrence coefficients (Jacobi coefficients) can be calculated by Theorem 4.3.2 and Theorem 4.4.1.*

4.6 Smoothness of Green's functions

For some generalized Julia sets a deeper analysis can be done. In this section we consider a modification $K_1(\gamma)$ of the set $K(\gamma)$ that will quite correspond to Theorem 4.5.5. We give a necessary and sufficient condition on the parameters that makes the Green function $g_{\Omega_{K_1(\gamma)}}$ optimally smooth. Although smoothness properties of Green functions are interesting in their own rights, in our case the optimal smoothness of $g_{\Omega_{K_1(\gamma)}}$ is necessary for $K_1(\gamma)$ to be a Parreau-Widom set.

Let $\gamma := (\gamma_n)_{n=1}^{\infty}$ be given such that $0 < \gamma_n < 1/4$ for all n, $\epsilon_n := 1/4 - \gamma_n$. Take $f_n(z) = \frac{1}{2\gamma_n}(z^2-1)+1$ for $n \in \mathbb{N}$. Thus, $F_1(z) = \frac{1}{2\gamma_1}(z^2-1)+1$ and similarly $F_n(z) = \frac{1}{2\gamma_n}(F_{n-1}^2(z)-1)+1$ for $n \geq 2$. It is easy to see that, as a polynomial of real variable, F_n is strictly admissible, it satisfies (A) and, in addition, all minimums of F_n are the same and equal to $1 - \frac{1}{2\gamma_n}$. Then $K_1(\gamma) = \cap_{n=1}^{\infty} F_n^{-1}([-1,1])$ is a stretched version of the set $K(\gamma)$ from [62]. Here,

$$g_{\Omega_{K_1(\gamma)}}(z) = \lim_{n\to\infty} 2^{-n} \log |F_n(z)|.$$

Since the leading coefficient of F_n is $2^{1-2^n}\gamma_n\,\gamma_{n-1}^2 \cdots \gamma_1^{2^{n-1}}$, the logarithmic capacity of $K_1(\gamma)$ is $2\exp(\sum_{n=1}^{\infty} 2^{-n} \log \gamma_n)$.

If, in addition, for some $0 < c < 1/4$ we have $\gamma_n \geq c$ for all n, then $(f_n) \in \mathcal{R}$ and, by Theorem 4.5.5, $K_1(\gamma) = J_{(f_n)}$. Without this condition the sequence (f_n) is not regular, the set $K_1(\gamma)$ is not uniformly perfect (at least if we assume that $\gamma_n \leq 1/32$ for all $n \in \mathbb{N}$, see Theorem 3 in [62]), but polynomials from Theorem 4.3.2 are still orthogonal, by Chapter 3.

In the limit case, when all $\gamma_n = 1/4$, F_n is the Chebyshev polynomial (of the first kind) T_{2^n} and $K_1(\gamma) = [-1,1]$.

Let $I_{1,0} := [-1,1]$. The set $F_n^{-1}([-1,1])$ is a disjoint union of 2^n non-degenerate closed intervals $I_{j,n} = [a_{j,n}, b_{j,n}]$ with length $l_{j,n}$ for $1 \leq j \leq 2^n$. We call them *basic intervals of $n-$th level.* The inclusion $F_{n+1}^{-1}([-1,1]) \subset F_n^{-1}([-1,1])$ implies that $I_{2j-1,n+1} \cup I_{2j,n+1} \subset I_{j,n}$ where $a_{2j-1,n+1} = a_{j,n}$ and $b_{2j,n+1} = b_{j,n}$. We denote the gap $(b_{2j-1,n+1}, a_{2j,n+1})$ by $C_{j,n}$ and the length of the gap by $h_{j,n}$. Thus,

$$K_1(\gamma) = [-1,1] \setminus \left(\bigcup_{n=0}^{\infty} \bigcup_{1 \le j \le 2^n} C_{j,n} \right).$$

Let us consider the parameter function $v_\gamma(t) = \sqrt{1 - 2\gamma(1-t)}$ for $|t| \le 1$ with $0 < \gamma \le 1/4$. This increasing and concave function is an analog of u from [62]. By means of v_γ we can write the endpoints of the basic intervals of $n-$th level, which are the solutions of $F_k(x) = -1$ for $1 \le k \le n$ together with the points ± 1. Namely, $F_n(x) = -1$ gives $F_{n-1}(x) = \pm v_{\gamma_n}(-1)$, then $F_{n-2}(x) = \pm v_{\gamma_{n-1}}(\pm v_{\gamma_n}(-1))$, etc. The iterates eventually give 2^n values

$$x = \pm v_{\gamma_1} \circ (\pm v_{\gamma_2} \circ (\cdots \pm v_{\gamma_{n-1}} \circ (\pm v_{\gamma_n}(-1) \cdots), \tag{4.13}$$

which are the endpoints $\{b_{2j-1,n}, a_{2j,n}\}_{j=1}^{2^{n-1}}$. The remaining 2^n points can be found similarly, as the solutions of $F_k(x) = -1$ for $1 \le k < n$ and ± 1.

As in Lemma 2 in [62], $\min_{1 \le j \le 2^n} l_{j,n}$ is realized on the first and the last intervals. Since the rightmost solution of $F_n(x) = -1$, namely $a_{2^n,n}$, is given by (4.13) with all signs positive, we have

$$l_{1,n} = l_{2^n,n} = 1 - v_{\gamma_1}(v_{\gamma_2}(\cdots v_{\gamma_{n-1}}(v_{\gamma_n}(-1) \cdots). \tag{4.14}$$

The next lemma shows that $l_{1,n}$ can be evaluated in terms of δ_n.

Lemma 4.6.1. *For each γ with $0 < \gamma_k \le 1/4$ and for all $n \in \mathbb{N}$ we have*

$$2\,\delta_n \le l_{1,n} \le (\pi^2/2)\,\delta_n.$$

Proof. Clearly, $1 - v_\gamma(t) = \frac{2}{1+v_\gamma(t)}\,\gamma(1-t)$. Repeated application of this to (4.14) gives the representation $l_{1,n} = 2\,\varkappa_n(\gamma)\,\delta_n$, where $\varkappa_n(\gamma)$ is equal to

$$\frac{2}{1 + v_{\gamma_1}(v_{\gamma_2}(\cdots v_{\gamma_n}(-1)\cdots))}\ \frac{2}{1 + v_{\gamma_2}(\cdots v_{\gamma_n}(-1)\cdots)} \cdots \frac{2}{1 + v_{\gamma_n}(-1)}.$$

Since $v_{1/4}(t) \le v_\gamma(t) \le 1$, we have $1 \le \varkappa_n(\gamma) \le \varkappa_n(1/4)$, where the last denotes the value of $\varkappa_n$ in the case when all $\gamma_k = 1/4$. This gives the left part of the inequality. Let C_{2^n} be the distance between 1 and the rightmost extrema of T_{2^n}. Hence, see e.g. p.7. of [109], $C_{2^n} = 1-\cos(\pi/2^n) < \pi^2/(2\cdot 4^n)$. On the other hand, $C_{2^n} = 2\,\varkappa_n(1/4)\,4^{-n}$. Therefore, $\varkappa_n(1/4) < \pi^2/4$, and the lemma follows. $\square$

For the case $\gamma_n \leq 1/32$ for all n, smoothness of the Green's function for $\overline{\mathbb{C}} \setminus K(\gamma)$ and related properties are examined in [61], [62]. The next theorem is complementary to Theorem 1 of [61] and examines the smoothness of the Green function as $\gamma_n \to 1/4$.

Theorem 4.6.2. *The function $g_{\Omega_{K_1(\gamma)}}$ is Hölder continuous with the exponent $1/2$ if and only if $\sum_{k=1}^{\infty} \epsilon_k < \infty$.*

Proof. Let us assume that $\sum_{k=1}^{\infty} \epsilon_k < \infty$. Then $\prod_{k=1}^{\infty}(1-4\epsilon_k) = a$ for some $0 < a < 1$, $\delta_n = 4^{-n}\prod_{k=1}^{n}(1-4\epsilon_k) > a\,4^{-n}$ and, by Lemma 4.6.1, $2a \cdot 4^{-n} \leq l_{1,n}$ for all $n \in \mathbb{N}$.

Let z_0 be an arbitrary point of $K_1(\gamma)$. We claim that $\mu_{K_1(\gamma)}(D_t(z_0)) \leq \frac{4\sqrt{2}}{\sqrt{a}}\sqrt{t}$ for all $t > 0$. It is evident for $t \geq 1/32$, as $\mu_{K_1(\gamma)}$ is a probability measure. Let $0 < t < 1/32$. Fix n with $l_{1,n} < t \leq l_{1,n-1}$. We have $t > 2a \cdot 4^{-n}$.

On the other hand, $D_t(z_0)$ can contain points from at most 4 basic intervals of level $n-1$. Since $\mu_{F_n^{-1}([-1,1])} \to \mu_{K_1(\gamma)}$, by [114], we have $\mu_{K_1(\gamma)}(I_{j,k}) = 1/2^k$ for all $k \in \mathbb{N}$ and $1 \leq j \leq 2^k$. Therefore, $\mu_{K_1(\gamma)}(D_t(z_0)) \leq 2^{3-n} < 8\sqrt{t/2a}$, which is our claim. The optimal smoothness of $g_{\Omega_{K_1(\gamma)}}$ follows from Theorem 1.1.8.

Conversely, suppose that, on the contrary, $\sum_{k=1}^{\infty} \epsilon_k = \infty$. This is equivalent to the condition $4^n\delta_n \to 0$ as $n \to \infty$. Thus, for any $\sigma > 0$, there is a number N such that $n > N$ implies that $4^n\delta_n < \sigma$. For any $t \leq l_{1,N+1}$, there exists $m \geq N+1$ such that $l_{1,m+1} < t \leq l_{1,m}$. Then, $\mu_{K_1(\gamma)}(D_t(0)) \geq \mu_{K_1(\gamma)}(I_{1,m+1}) = 2^{-m-1}$. On the other hand, by Lemma 4.6.1, $t \leq 2\pi^2\sigma\,4^{-m-1}$. Therefore, for any $t \leq l_{1,N+1}$ we have $\frac{\sqrt{t}}{\pi\sqrt{2\,\sigma}} \leq \mu_{K_1(\gamma)}(D_t(0))$. Hence, the inequality

$$\frac{\sqrt{2}}{\pi\sqrt{\sigma}}\,\sqrt{r} \leq \int_0^r \frac{\mu_{K_1(\gamma)}(D_t(0))}{t}dt,$$

holds for $r \leq l_{1,N+1}$. By Theorem 1.1.8, $g_{\Omega_{K_1(\gamma)}}(-r) \geq \frac{\sqrt{2}}{\pi\sqrt{\sigma}}\sqrt{r}$. Since σ is here as small as we wish, the Green function is not optimally smooth. $\square$

4.7 Parreau-Widom sets

Our aim is to give a criterion when $K_1(\gamma)$ is a Parreau-Widom set.

We begin with a technical lemma.

Lemma 4.7.1. *Given $p \in \mathbb{N}$, let $b_0 = 1$ and $b_{k+1} = b_k(1 + 4^{-p+k}\, b_k)$ for $0 \leq k \leq p-1$. Then $b_p < 2$.*

Proof. We have $b_1 = 1 + 4^{-p}$, $b_2 = 1 + (1+4)\,4^{-p} + 2\cdot 4\cdot 4^{-2p} + 4\cdot 4^{-3p}, \cdots$, so $b_k = \sum_{n=0}^{N_k} a_{n,k} 4^{-np}$ with $N_k = 2^k - 1$ and $a_{0,k} = 1$. Let $a_{n,k} := 0$ if $n > N_k$. The definition of b_{k+1} gives the recurrence relation

$$a_{n,k+1} = a_{n,k} + 4^k \sum_{j=1}^{n} a_{n-j,k}\, a_{j-1,k} \ \text{ for } \ 1 \leq n \leq N_{k+1}. \tag{4.15}$$

If $N_k < n \leq N_{k+1}$, that is $n = N_k + m$ with $1 \leq m \leq N_k + 1$, then the formula takes the form $a_{n,k+1} = 4^k \sum_{j=m}^{n-m+1} a_{n-j,k}\, a_{j-1,k}$, since $a_{n-j,k} = 0$ for $j < m$ and $a_{j-1,k} = 0$ for $j > n-m+1$. In particular, $a_{N_{k+1},k+1} = 4^k\, a_{N_k,k}^2$ and $a_{1,k+1} = a_{1,k} + 4^k$. Therefore, $a_{1,k} = 1 + 4 + \cdots + 4^{k-1} < 4^k/3$. Let us show that $a_{n,k} < C_n\, 4^{nk}$ with $C_n = 4^{1-n}/3$ for $n \geq 2$. This gives the desired result, as $b_p = \sum_{n=0}^{N_p} a_{n,p} 4^{-np} < 1 + 1/3 \cdot \sum_{n=1}^{N_p} 4^{1-n} < 2$.

By induction, suppose the inequality $a_{j,k} < C_j\, 4^{jk}$ is valid for $1 \leq j \leq n-1$ and for all $k > 0$. We consider $j = n$. The bound $a_{n,i} < C_n\, 4^{ni}$ is valid for $i = 1$, as $a_{n,1} = 0$ for $n \geq 2$. Suppose it is valid as well for $i \leq k$.

We use (4.15) repeatedly, in order to reduce the second index, and, after this, the induction hypothesis:

$$a_{n,k+1} = \sum_{q=1}^{k} 4^q \sum_{j=1}^{n} a_{n-j,q}\, a_{j-1,q} < \sum_{q=1}^{k} 4^{nq} \sum_{j=1}^{n} C_{n-j}\, C_{j-1} < \sum_{q=1}^{k} 4^{nq}$$
$$< C_n\, 4^{n(k+1)},$$

where $C_0 := 1$. Therefore the desired bound is valid for all positive n and k. $\square$

Theorem 4.7.2. *$K_1(\gamma)$ is a Parreau-Widom set if and only if* $\sum_{k=1}^{\infty} \sqrt{\epsilon_k} < \infty$.

Proof. Let $E_n = \{z \in \mathbb{C} : |F_n(z)| \leq 1\}$. Then $g_{\Omega_{E_n}}(z) = 2^{-n} \log |F_n(z)|$ for $z \notin E_n$. Clearly, the critical points of $g_{\Omega_{E_n}}$ coincide with the critical points of F_n and thus they are real. Let $Y_n = \{x : F_n^{'}(x) = 0\},\ Z_n = \{x : F_n(x) = 0\}$. We see at once that $Y_n \cap Z_n = \emptyset$ and $Z_k \cap Z_n = \emptyset$ for $n \neq k$. Since $F_n^{'} = F_{n-1}F_{n-1}^{'}/\gamma_n$, we have $Y_n = Y_{n-1} \cup Z_{n-1}$, so $Y_n = Z_{n-1} \cup Z_{n-2} \cup \cdots \cup Z_0$, where $Z_0 = \{0\}$. Therefore, the critical points of $g_{\Omega_{E_n}}$ are also critical for $g_{\Omega_{E_{n+1}}}$. Of course, $g_{\Omega_{E_n}} \nearrow g_{\Omega_{K_1(\gamma)}}$, so the set of critical points for $g_{\Omega_{K_1(\gamma)}}$ is $\cup_{n=0}^{\infty} Z_n$. It follows that $PW(K_1(\gamma)) = \sum_{n=1}^{\infty} \sum_{z \in Z_{n-1}} g_{\Omega_{K_1(\gamma)}}(z)$. In addition, for each $k \geq n$ the function F_k is constant on the set Z_{n-1} which contains 2^{n-1} points. By Theorem 1.1.4(c), the Green function is also constant on this set. Let $s_n = 2^{n-1} g_{\Omega_{K_1(\gamma)}}(z)$, where z is any point from Z_{n-1}. Then

$$PW(K_1(\gamma)) = \sum_{n=1}^{\infty} s_n.$$

We can certainly assume that $\sum_{n=1}^{\infty} \epsilon_n < \infty$. Indeed, it is immediate if $\sum_{n=1}^{\infty} \sqrt{\epsilon_n} < \infty$. On the other hand, if $z \in Z_{n-1}$, that is $F_{n-1}(z) = 0$, then $F_n(z) = 1 - \frac{1}{2\gamma_n} = -1 - \frac{8\epsilon_n}{1-4\epsilon_n}$. Since $g_{\Omega_{E_n}} \nearrow g_{\Omega_{K_1(\gamma)}}$, we have $s_n > 1/2 \log |F_n(z)| > 1/2 \log(1 + 8\epsilon_n) > 2\epsilon_n$, as $\log(1+t) > t/2$ for $0 < t < 2$. Therefore the supposition $PW(K_1(\gamma)) < \infty$ implies that $\sum_{n=1}^{\infty} \epsilon_n < \infty$.

Let $a = \prod_{n=1}^{\infty} (1 - 4\epsilon_n)$. Since $\epsilon_n \in (0, \frac{1}{4})$ is a term of convergent series, we have $0 < a < 1$.

Our aim is to evaluate s_n from both sides for large enough n. From now on we consider only n such that $\epsilon_n \leq a/36$. Then $1 - 4\epsilon_n > 8/9$ and for $\sigma_n := \frac{8\epsilon_n}{1-4\epsilon_n}$

we have $0 < \sigma_n < 1/4$. Given n, we fix $p = p(n) \in \mathbb{N}$ with

$$a \cdot 4^{-1-p} < \sigma_n \le a \cdot 4^{-p}. \tag{4.16}$$

This gives

$$\frac{1}{\sqrt{a}}\sqrt{\sigma_n} \le 2^{-p} < \frac{2}{\sqrt{a}}\sqrt{\sigma_n}. \tag{4.17}$$

Clearly, $\sum_{n=1}^{\infty}\sqrt{\epsilon_n} < \infty$ if and only if $\sum_{n=1}^{\infty}\sqrt{\sigma_n} < \infty$.

Consider the function $f(t) = \frac{1}{2\beta}(t^2-1)+1$ for $t > 1$, where $\beta = 1/4 - \epsilon$ with $\epsilon < 1/36$. If $t = 1+\sigma$ with small σ, then we use the representation $f(t) = 1+\sigma_1$ where $4\sigma < \sigma_1 = 4\sigma\,\frac{1+\sigma/2}{1-4\epsilon}$. On the other hand, for large t we have $t^2 \le f(t) < \frac{1}{2\beta}\,t^2 < \frac{9}{4}\,t^2$.

Let us fix $z \in Z_{n-1}$. Then, as above, $|F_n(z)| = 1+\sigma_n$. Clearly, $F_{n+1}(z) = f(F_n(z))$ with $\beta = \gamma_{n+1}$. Hence, $F_{n+1}(z) = 1+\sigma_{n+1}$ with $4\sigma_n < \sigma_{n+1} = 4\sigma_n\,\frac{1+\sigma_n/2}{1-4\epsilon_{n+1}}$. We continue in this fashion to obtain $F_{n+p}(z) = 1+\sigma_{n+p}$ with

$$4^p\,\sigma_n < \sigma_{n+p} = 4^p\,\sigma_n \cdot \prod_{k=n}^{n+p-1}\frac{1+\sigma_k/2}{1-4\epsilon_{k+1}} < a^{-1}\,4^p\,\sigma_n \cdot \prod_{k=n}^{n+p-1}(1+\sigma_k/2). \tag{4.18}$$

After that we use the second estimation for f. This gives $F_{n+p}^2(z) \le F_{n+p+1}(z) < \frac{9}{4}\,F_{n+p}^2(z)$ and, for each $k \in \mathbb{N}$,

$$F_{n+p}^{2^k}(z) \le F_{n+p+k}(z) < (9/4)^{2^k-1}\,F_{n+p}^{2^k}(z).$$

From this, we have

$$2^{-n-p}\,\log F_{n+p}(z) \le g_{\Omega_{E_{n+p+k}}}(z) \le 2^{-n-p}[\log(9/4) + \log F_{n+p}(z)].$$

Recall that $g_{\Omega_{E_{n+p+k}}}(z) \nearrow g_{\Omega_{K_1(\gamma)}}(z)$ as $k \to \infty$. Also $s_n = 2^{n-1}g_{\Omega_{K_1(\gamma)}}(z)$ and $F_{n+p}(z) = 1+\sigma_{n+p}$. Hence,

$$2^{-p-1}\,\log(1+\sigma_{n+p}) \le s_n \le 2^{-p-1}[\log(9/4) + \log(1+\sigma_{n+p})]. \tag{4.19}$$

We are able to show the statement of the theorem. Suppose that $K_1(\gamma)$ is a Parreau-Widom set, so the series $\sum_{n=1}^{\infty} s_n$ converges. By (4.19) and (4.18), we have $s_n \geq 2^{-p-1} \log(1+4^p \sigma_n)$. By (4.16), $4^p \sigma_n < 1$ and $\log(1+4^p \sigma_n) > 4^p \sigma_n/2$. Therefore, $s_n \geq 2^p \sigma_n/4$. Finally, we use (4.17) to obtain $s_n \geq \sqrt{a\,\sigma_n}/8$, which implies the convergence of $\sum_{n=1}^{\infty} \sqrt{\sigma_n}$ and $\sum_{n=1}^{\infty} \sqrt{\epsilon_n}$.

Conversely, suppose that $\sum_{n=1}^{\infty} \sqrt{\sigma_n} < \infty$. By (4.19), $s_n \leq 2^{-p} \log(3/2) + 2^{-p-1}\sigma_{n+p}$. Here, by (4.17), the series $\sum_{n=1}^{\infty} 2^{-p(n)}$ converges. For the addend, (4.18) implies

$$2^{-p-1}\sigma_{n+p} < (2a)^{-1} \cdot 2^p\, \sigma_n \prod_{k=n}^{n+p-1} (1+\sigma_k/2).$$

From (4.17) it follows that $2^p\, \sigma_n \leq \sqrt{a\sigma_n}$, a term of convergent series. Let us show that

$$\prod_{k=n}^{n+p-1} (1+\sigma_k/2) < 2. \tag{4.20}$$

This will give the convergence of $\sum_{n=1}^{\infty} s_n$, which is the desired conclusion.

We use notations of Lemma 4.7.1. By (4.16), we have $1+\sigma_n/2 \leq 1+ a\, 4^{-p}/2 < b_1$. Then

$$1+\sigma_{n+1}/2 < 1+ \frac{a}{1-4\epsilon_{n+1}}\, 4^{-p+1}(1+\sigma_n/2) < 1+4^{-p+1}\, b_1 = b_2/b_1$$

and $(1+\sigma_n/2)(1+\sigma_{n+1}/2) < b_2$. Similarly, by (4.18) and (4.16),

$$1+\sigma_{n+k+1}/2 < 1+ \frac{a}{(1-4\epsilon_{n+1})\cdots(1-4\epsilon_{n+k})}\, 4^{-p+k}\, b_k < b_{k+1}/b_k$$

for $k \leq p-2$. Lemma 4.7.1 now yields (4.20). $\square$

Chapter 5

Chebyshev polynomials on generalized Julia sets

5.1 Introduction

In Section 2, we review some facts about generalized Julia sets and Chebyshev polynomials. In the last section, we present a result which can be seen as a generalization of Theorem 1.1.3. Polynomials considered in these sections are always nonlinear complex polynomials unless stated otherwise. For a deeper discussion of Chebyshev polynomials, we refer the reader to [97, 98, 118].

5.2 Preliminaries

The next result is a consequence of Theorem 1.1.4.

Proposition 5.2.1. *Let* $(f_n) \in \mathcal{R}$. *Then*

$$\lim_{k\to\infty}\left(\sup_{a\in\overline{\mathbb{C}}\setminus F_k^{-1}(\triangle_R)} \operatorname{dist}(a, \mathcal{K}_{(f_n)})\right) = 0,$$

where R *be a real number satisfying* (1.6).

Proof. Using part (gf) of Theorem 1.1.4, we have $\overline{\mathbb{C}} \setminus F_{k+1}^{-1}(\triangle_R) \subset \overline{\mathbb{C}} \setminus F_k^{-1}(\triangle_R)$ which implies that

$$(a_k) := \left(\sup_{a \in \overline{\mathbb{C}} \setminus F_k^{-1}(\triangle_R)} \operatorname{dist}(a, \mathcal{K}_{(f_n)}) \right)$$

is a decreasing sequence.

Suppose that $a_k \to \epsilon$ as $k \to \infty$ for some $\epsilon > 0$. Then, by compactness of the set $\overline{\mathbb{C}} \setminus F_k^{-1}(\triangle_R)$, there exists a number $b_k \in \overline{\mathbb{C}} \setminus F_k^{-1}(\triangle_R)$ for each k such that $\operatorname{dist}(b_k, \mathcal{K}_{(f_n)}) \geq \epsilon$. But since $\cap_{k=1}^{\infty} \overline{\mathbb{C}} \setminus F_k^{-1}(\triangle_R) = \mathcal{K}_{(f_n)}$ by parts (b) and (g) of Theorem 1.1.4, (b_k) should have an accumulation point b in $\mathcal{K}_{(f_n)}$ with $\operatorname{dist}(b, \mathcal{K}_{(f_n)}) > \epsilon/2$ which is clearly impossible. This completes the proof. $\square$

The next result which is vital for the proof of Lemma 5.3.1 is from [94]:

Theorem 5.2.2. *Let $L \subset \mathbb{C}$ be a compact set with* $\operatorname{card} L \geq 2$ *having the origin as its Chebyshev center. Let $L_p = p^{-1}(L)$ for some monic complex polynomial p with* $\deg p = n$*. Then p is the unique Chebyshev polynomial of degree n on L_p.*

5.3 Results

First, we begin with a lemma which is also interesting in its own right.

Lemma 5.3.1. *Let f and g be two non-constant complex polynomials and K be a compact subset of $\mathbb{C}$ with* $\operatorname{card} K \geq 2$*. Furthermore, let α be the leading coefficient of f. Then the following propositions hold.*

(a) The Chebyshev polynomial of degree $\deg f$ *on the set $(g \circ f)^{-1}(K)$ is of the form $f(z)/\alpha - \tau$ where $\tau \in \mathbb{C}$.*

(b) If g is given as a linear combination of monomials of even degree and $K = \overline{B(0, R)}$ for some $R > 0$ then the $\deg f$*-th Chebyshev polynomial on $(g \circ f)^{-1}(K)$ is $f(z)/\alpha$.*

Proof. Let $K_1 := g^{-1}(K)$. Then $(g\circ f)^{-1}(K) = f^{-1}(K_1) = (f/\alpha)^{-1}(K_1/\alpha)$ where $K_1/\alpha - \tau = \{z : z = z_1/\alpha - \tau \text{ for some } z_1 \in K_1\}$. By the fundamental theorem of algebra, $\text{card}(K_1/\alpha) = \text{card}K_1 \geq \text{card}K$ and K_1 is compact by the continuity of $g(z)$. The set K_1/α is also compact since the compactness of a set is preserved under a linear transformation. Let τ be the Chebyshev center for K_1/α. Then $K_1/\alpha - \tau$ is a compact set with the Chebyshev center as the origin. Note that, $\text{card}(K_1/\alpha - \tau) = \text{card}(K_1/\alpha)$ and $(f/\alpha)^{-1}(K_1/\alpha) = (f/\alpha - \tau)^{-1}(K_1/\alpha - \tau)$. Using Theorem 5.2.2, for $p(z) = f(z)/\alpha - \tau$ and $L = K_1/\alpha - \tau$, we see that $p(z)$ is the $\deg f$-th Chebyshev polynomial on $L_p = (g\circ f)^{-1}(K)$. This proves the first part of the lemma.

Suppose further that $g(z) = \sum_{j=0}^{n} a_j \cdot z^{2j}$ for some $n \geq 1$ and $(a_0, \ldots, a_n) \in \mathbb{C}^{n+1}$ with $a_n \neq 0$. Let $K = \overline{B(0,R)}$ for some $R > 0$. Then the Chebyshev center for $K_1/\alpha = g^{-1}(K)/\alpha = g^{-1}(\overline{B(0,R)})/\alpha$ is the origin since $g(z)/\alpha = g(-z)/\alpha$ for all $z \in \mathbb{C}$. Thus, $f(z)/\alpha$ is the $\deg f$-th Chebyshev polynomial for $(g\circ f)^{-1}(K)$ under these extra assumptions. □

The next theorem shows that it is possible to obtain similar results to Theorem 1.1.3 in a richer setting.

Theorem 5.3.2. *Let $(f_n) \in \mathcal{R}$. Then the following hold:*

(a) For each $m \in \mathbb{N}$, the $\deg F_m$-th Chebyshev polynomial on $J_{(f_n)}$ is of the form $F_m(z)/\rho_m - \tau_m$ where $\tau_m \in \mathbb{C}$.

(b) If, in addition, each f_n is given as a linear combination of monomials of even degree then $F_m(z)/\rho_m$ is the $\deg F_m$-th Chebyshev polynomial on $J_{(f_n)}$ for all m.

Proof. Let $m \in \mathbb{N}$ be given and $R > 1$ satisfy (1.6). For each natural number $l > m$, define $g_l := f_l \circ \ldots \circ f_{m+1}$. Then $F_l = g_l \circ F_m$ for each such l. Using part (a) of Lemma 5.3.1 for $g = g_l$, $f = F_m$ and $K = \overline{B(0,R)}$, we see that the $(d_1 \cdots d_m)$-th Chebyshev polynomial on $(g_l \circ F_m)^{-1}(\overline{B(0,R)})$ is of the form

$F_m(z)/\rho_m - \tau_l$ where $\tau_l \in \mathbb{C}$. Let $C_l := ||F_m(z)/\rho_m - \tau_l||_{L^\infty((g_l \circ F_m)^{-1}(K))}$. Note that, by part (f) of Theorem 1.1.4,

$$F_t^{-1}(\overline{B(0,R)}) \subset F_s^{-1}(\overline{B(0,R)}) \subset \overline{B(0,R)} \tag{5.1}$$

provided that $s < t$. This implies that $(C_j)_{j=m+1}^\infty$ is a decreasing sequence of positive numbers and hence has a limit C. The last follows from the observation that the norms of the Chebyshev polynomials of same degree on a decreasing sequence of compact sets constitute a decreasing sequence on $\mathbb{R}$.

Let $P_{d_1 \cdots d_m}(z) = \sum_{j=0}^{d_1 \cdots d_m} a_j z^j$ be the $(d_1 \cdots d_m)$-th Chebyshev polynomial on $\mathcal{K}_{(f_n)}$. Since $\mathcal{K}_{(f_n)} \subset (g_l \circ F_m)^{-1}(\overline{B(0,R)})$ for each l, we have $C_0 := ||P_{d_1 \cdots d_m}||_{\mathcal{K}_{(f_n)}} \leq C$. Suppose that $C_0 < C$.

Let $\epsilon = \min\{C - C_0, 1\}$. Using the compactness of $\overline{B(0,R)}$ let us choose a $\delta > 0$ such that for all $|z_1 - z_2| < \delta$ and $z_1, z_2 \in \overline{B(0,R)}$ we have $|P_{d_1 \cdots d_m}(z_1) - P_{d_1 \cdots d_m}(z_2)| < \epsilon/2$. By Proposition 5.2.1, there exists a real number $N_0 > m$ such that $N > N_0$ with $N \in \mathbb{N}$ implies that

$$\sup_{z \in \overline{\mathbb{C}} \setminus F_N^{-1}(\triangle_R)} \mathrm{dist}(z, \mathcal{K}_{(f_n)}) < \delta.$$

Therefore, for any $z \in F_{N_0+1}^{-1}(\overline{B(0,R)})$, there exists a number $z' \in \mathcal{K}_{(f_n)}$ with $|z - z'| < \delta$. Hence, for each $z \in F_{N_0+1}^{-1}(\overline{B(0,R)})$, we have

$$|P_{d_1 \cdots d_m}(z)| < |P_{d_1 \cdots d_m}(z')| + \frac{\epsilon}{2} < C \leq \left|\left| \frac{F_m}{\rho_m} - \tau_{N_0+1} \right|\right|_{L^\infty\left(F_{N_0+1}^{-1}(\overline{B(0,R)})\right)},$$

where in the first inequality, we use $z, z' \in \overline{B(0,R)}$. This contradicts with the fact that $F_m(z)/\rho_m + \tau_{N_0+1}$ is the $(d_1 \cdots d_m)$-th Chebyshev polynomial on $F_{N_0+1}^{-1}(\overline{B(0,R)})$. Thus $C_0 = C$.

Using the triangle inequality in (5.2) and (5.3), the monotonicity of $(C_l)_{l=m+1}^\infty$

in (5.4) and (5.1) in (5.5), we have

$$
\begin{aligned}
|t_l| &= \left|\left|\frac{F_m}{\rho_m} + \frac{F_m}{\rho_m} - \tau_l\right|\right|_{L^\infty\left(F_l^{-1}(\overline{B(0,R)})\right)} \\
&\leq \left|\left|\frac{F_m}{\rho_m} - \tau_l\right|\right|_{L^\infty\left(F_l^{-1}(\overline{B(0,R)})\right)} + \left|\left|\frac{F_m}{\rho_m}\right|\right|_{L^\infty\left(F_l^{-1}(\overline{B(0,R)})\right)} \qquad (5.2) \\
&\leq C_l + |\tau_{m+1}| + \left|\left|\frac{F_m}{\rho_m} - \tau_{m+1}\right|\right|_{L^\infty\left(F_l^{-1}(\overline{B(0,R)})\right)} \qquad (5.3) \\
&\leq C_{m+1} + |\tau_{m+1}| + \left|\left|\frac{F_m}{\rho_m} - \tau_{m+1}\right|\right|_{L^\infty\left(F_l^{-1}(\overline{B(0,R)})\right)} \qquad (5.4) \\
&\leq 2C_{m+1} + |\tau_{m+1}|. \qquad (5.5)
\end{aligned}
$$

for $l \geq m+1$. This shows that $(\tau_l)_{l=m+1}^\infty$ is a bounded sequence. Thus $(\tau_l)_{l=m+1}^\infty$ has at least one convergent subsequence $(\tau_{l_k})_{k=1}^\infty$ with a limit τ_m. Therefore,

$$
C \leq \lim_{k\to\infty} \left|\left|\frac{F_m}{\rho_m} - \tau_m\right|\right|_{L^\infty\left(F_{l_k}^{-1}(\overline{B(0,R)})\right)} \leq \lim_{k\to\infty} (C_{l_k} + |\tau_{l_k} - \tau_m|) = C. \qquad (5.6)
$$

By the uniqueness of Chebyshev polynomials and (5.6), $F_m(z)/\rho_m - \tau_m$ is the $(d_1 \cdots d_m)$-th Chebyshev polynomial on $\mathcal{K}_{(f_n)}$. By the maximum principle, for any polynomial Q, we have

$$
||Q||_{L^\infty\left(\mathcal{K}_{(f_n)}\right)} = ||Q||_{L^\infty\left(\partial\mathcal{K}_{(f_n)}\right)} = ||Q||_{L^\infty\left(J_{(f_n)}\right)}.
$$

Hence the Chebyshev polynomials on $\mathcal{K}_{(f_n)}$ and $J_{(f_n)}$ should coincide. This proves the first assertion.

Suppose that the assumption given in part (b) is satisfied. Then by the part (b) of Lemma 5.3.1, for $g = g_l$, $f = F_m$ and $K = \overline{B(0,R)}$, the $(d_1 \cdots d_m)$-th Chebyshev polynomial on $(g_l \circ F_m)^{-1}(\overline{B(0,R)})$ is of the form $F_m(z)/\rho_m - \tau_l$ where $\tau_l = 0$ for $l > m$. Thus, arguing as above, we can reach the conclusion that $F_m(z)/\rho_m$ is the $(d_1 \cdots d_m)$-th Chebyshev polynomial for $J_{(f_n)}$ provided that the assumption in the part (b) holds. This completes the proof. $\square$

This theorem gives the total description of 2^n degree Chebyshev polynomials for the most studied case, i.e., $f_n(z) = z^2 + c_n$ with $c_n \in \mathbb{C}$ for all n. If $(c_n)_{n=1}^\infty$ is bounded then the logarithmic capacity of $J_{(f_n)}$ is 1. Moreover, by [39], we know that if $|c_n| \leq 1/4$ for all n then $J_{(f_n)}$ is connected. If $|c_n| < c < 1/4$, then $J_{(f_n)}$ is a quasicircle and hence a Jordan curve. See [37], for the definition of a quasicircle and proof of the above fact.

Chapter 6

Spacing properties of the zeros of orthogonal polynomials

6.1 Introduction

In the last ten years, there has been an explosion of interest in spacing of the zeros of orthogonal polynomials on the real line. For probability measures having a non-trivial absolutely continuous part (with respect to the Lebesgue measure on $\mathbb{R}$), there are many results (see e.g. [17,78,79,113,132,137]) concerning the fine structure of the zeros of orthogonal polynomials. Breuer in [35] gives an example of a measure such that it is purely singular on $(-2, 2)$ yet it has a strong clock behavior (see e.g. Section 1 in [17] for a discussion on clock behavior) in $(-2, 2)$ (follows from Theorem 1.2 in [35] and Theorem 23.1 in [115]). Simon and Krüger, in [77], discuss the zero spacing of the orthogonal polynomials for the Cantor-Lebesgue measure of the Cantor ternary set. In this chapter, our main aim is to give some examples of singular continuous measures for which the minimal distance between the consecutive zeros of the associated orthogonal polynomials can be computed accurately.

Throughout this chapter, measures that we consider are probability measures, unless specified otherwise, with a compact support in $\mathbb{C}$. Let μ be a measure with $\mathrm{Card}(\mathrm{supp}(\mu)) \geq n > 1$ for some $n \in \mathbb{N}$. Then using the Gram-Schmidt process for $\{1, x, \ldots, x^{n-1}\}$ in $L^2(\mu)$, the monic orthogonal polynomial $P_m(\cdot; \mu)$ of degree m can be found provided that $m < n$.

The plan of the chapter is as follows. In Section 2, we briefly summarize some known facts on the orthogonal polynomials associated with discrete measures. In Section 3, we discuss spacing of the zeros of orthogonal polynomials for fairly general measures. The only new result in that section is Theorem 6.3.2. In the last section, we focus on the zero spacing of orthogonal polynomials for $\mu_{K(\gamma)}$.

6.2 Preliminaries

Let $(f_n)_{n=1}^{\infty}$ be a sequence of nonlinear polynomials. For each $n \in \mathbb{N}$ we use the following notation: $f_n(z) = \sum_{j=0}^{d_n} a_{n,j} \cdot z^j$ where $d_n \geq 2$, $a_{n,j} \in \mathbb{C}$ for $j = 0, \ldots, d_n$ and $a_{n,d_n} \neq 0$. The composition $f_n \circ f_{n-1} \cdots \circ f_1$ will be denoted by F_n and ρ_n is used to denote the leading coefficient of F_n. The normalized counting measure on the roots (counting multiplicity) of $F_n(z) - a = 0$ is denoted by ν_n^a where $a \in \mathbb{C}$.

The next result is a more general version of Theorem 4.3.2 and implies Theorem 3.2.8. In these theorems the limit sequence is the equilibrium measure of some prescribed sets.

Theorem 6.2.1. *Let $(f_n)_{n=1}^{\infty}$ be a sequence of nonlinear polynomials. Suppose further that there is an $a \in \mathbb{C}$ such that $\nu_n^a \to \mu$ as $n \to \infty$ where μ is a probability measure and* $\mathrm{supp}(\mu)$ *is an infinite compact set in $\mathbb{C}$. Then we have the following identities:*

(a) $P_1(z; \mu) = z + \dfrac{1}{d_1} \dfrac{a_{1,d_1-1}}{a_{1,d_1}}.$

(b) $P_{d_1 \cdots d_l}(z; \mu) = \dfrac{1}{\rho_l} \left(F_l(z) + \dfrac{1}{d_{l+1}} \dfrac{a_{l+1,d_{l+1}-1}}{a_{l+1,d_{l+1}}} \right)$ *for all $l \in \mathbb{N}$.*

Proof. The proof is almost the same as the proof of Theorem 4.3.2. In the proof of Theorem 4.3.2, omit the first line, replace the equilibrium measure of $J_{(f_n)}$ by μ where it is necessary. Then we have the proof of this theorem. $\square$

In the last section, we focus on a concrete family of measures but the techniques used in the last two sections are applicable to some extent for many other measures supported on $\mathbb{R}$ provided that the associated orthogonal polynomials satisfy (a) and (b) of the above theorem.

If μ is a measure with an infinite support on $\mathbb{R}$ then the zeros of $P_n(\cdot;\mu)$ are simple and real. We enumerate the zeros $(x_{j,n}(\mu))_{j=1}^n$ of $P_n(\cdot;\mu)$ so that they satisfy

$$x_{1,n}(\mu) < x_{2,n}(\mu) < \cdots < x_{n,n}(\mu).$$

Define $x_{0,n}(\mu)$ as the leftmost point and $x_{n+1,n}(\mu)$ as the rightmost point of $\mathrm{supp}(\mu)$, respectively. Then (see e.g. (2) in p. 358 of [52] and Theorem 2.3 in [136]), for $1 \le i \le n$ we have $x_{i,n}(\mu) \in (x_{0,n}, x_{n+1,n})$. The next theorem (see for example Proposition 1.1 in [19] for a proof of it) will be used many times in the subsequent sections.

Theorem 6.2.2. *Let λ be a measure with $\mathrm{supp}(\lambda) = \{c_{i,r}\}_i \subset \mathbb{R}$ where $r \in \mathbb{N}$ with $r > 1$ and $i = 1, \ldots, r$ provided that $c_{1,r} < c_{2,r} < \cdots < c_{r,r}$. Then, the zeros of $P_s(\cdot;\lambda)$ lie in $(c_{1,r}, c_{r,r})$ and they are real and simple where $1 \le s < r$. Moreover, in each interval $[c_{j,r}, c_{j+1,r}]$ there is at most one zero of $P_s(\cdot;\lambda)$ where $j \in \{1, \ldots, r-1\}$.*

We can reduce the infinite support case to the finite case by a classical technique. By doing that, we can use results such as Theorem 6.2.2 which are valid for discrete measures. Assume that μ is a measure with an infinite support on $\mathbb{R}$ and let $r \in \mathbb{N}$. Then there is a unique measure $\mu^{(r)}$ with $\mathrm{supp}\left(\mu^{(r)}\right) = \{x : P_r(x;\mu) = 0\}$ such that for any polynomial π with $\deg \pi \le 2r-1$ we have

$$\int \pi(x)\, d\mu(x) = \int \pi(x)\, d\mu^{(r)}(x).$$

In particular,

$$P_s(\cdot;\mu) = P_s\left(\cdot;\mu^{(r)}\right)$$

holds for all $1 \leq s < r$ provided that $r > 1$. See Theorem 2.5 in [136] for the proof.

6.3 Some general results

If $\mathrm{supp}(\mu)$ is a Cantor set on $\mathbb{R}$ then the maximal distance between the consecutive zeros of any associated orthogonal polynomial is not so interesting since this value is bounded below (see e.g. (iii) in p. 358 of [52]) by the half of the length of the largest gap of $\mathrm{supp}(\mu)$. We only discuss $U_n(\cdot)$ here. By $d(A, B)$ we denote the Euclidean distance between the sets $A, B \subset \mathbb{C}$. The next proposition is a direct consequence of Theorem 3.3.3 in [122].

Proposition 6.3.1. *Let μ be a measure with an infinite support on $\mathbb{R}$. Then for any fixed $l, m, n \in \mathbb{N}$ with $l > m > n > 1$, we have*

$$d(Z_l(\mu), Z_m(\mu)) = \inf_{\substack{1 \leq i \leq l \\ 1 \leq j \leq m}} |x_{i,l}(\mu) - x_{j,m}(\mu)| \leq U_n(\mu).$$

Theorem 6.3.2. *Let μ be a measure with an infinite support on $\mathbb{R}$ and let $(f_n)_{n=1}^{\infty}$ be a sequence of nonlinear polynomials. Assume further that there exists an $s_0 \in \mathbb{N}$ such that $f_l'(0) = 0$ for all $l > s_0$ and $P_{d_1 \cdots d_m}(\cdot; \mu) = F_m(\cdot)/\rho_m$ holds for all $m \in \mathbb{N}$. Then for all $k, k' \in \mathbb{N}_0$ with $k > k'$ the following holds:*

$$d(Z_{d_1 \cdots d_{s_0+k}}(\mu), Y_{d_1 \cdots d_{s_0+k}}(\mu)) \leq d(Z_{d_1 \cdots d_{s_0+k}}(\mu), Z_{d_1 \cdots d_{s_0+k'}}(\mu)).$$

Proof. Let us fix k and k'. If $k > k' + 1$ then for any $z \in \mathbb{C}$,

$$\begin{aligned}
P'_{d_1 \cdots d_{s_0+k}}(z; \mu) &= F'_{s_0+k}(z)/\rho_{s_0+k} \\
&= ((f_{s_0+k} \circ f_{s_0+k-1} \circ \cdots \circ f_{s_0+k'+1}) \circ F_{s_0+k'})'(z)/\rho_{s_0+k} \\
&= \frac{F'_{s_0+k'}(z) \cdot (f_{s_0+k} \circ f_{s_0+k-1} \circ \cdots \circ f_{s_0+k'+1})'(F_{s_0+k'}(z))}{\rho_{s_0+k}}
\end{aligned}$$

$$= \frac{(F'_{s_0+k'} \cdot (f'_{s_0+k'+1} \circ F_{s_0+k'}) \cdot (f_{s_0+k} \circ \cdots \circ f_{s_0+k'+2})'(f_{s_0+k'+1} \circ F_{s_0+k'}))(z)}{\rho_{s_0+k}}$$

holds. Since $f'_{s_0+k'+1}(0) = 0$ and $P_{d_1 \cdots d_{s_0+k'}}(z;\mu) = F_{s_0+k'}(z)/\rho_{s_0+k'}$ hold, we have

$$Z_{d_1 \cdots d_{s_0+k'}}(\mu) \subset Y_{d_1 \cdots d_{s_0+k}}(\mu).$$

If $k = k'+1$ then

$$P'_{d_1 \cdots d_{s_0+k}}(z;\mu) = \frac{F'_{s_0+k'}(z) \cdot (f'_{s_0+k'+1} \circ F_{s_0+k'})(z)}{\rho_{s_0+k}}.$$

Thus, $Z_{d_1 \cdots d_{s_0+k'}}(\mu) \subset Y_{d_1 \cdots d_{s_0+k}}(\mu)$ holds similarly. Hence, we have

$$d(Z_{d_1 \cdots d_{s_0+k}}(\mu), Y_{d_1 \cdots d_{s_0+k}}(\mu)) \leq d(Z_{d_1 \cdots d_{s_0+k}}(\mu), Z_{d_1 \cdots d_{s_0+k'}}(\mu)).$$

$\square$

The next proposition gives an upper bound for $U_n(\mu)$. It is a straightforward consequence of classical results and we give the proof for the convenience of the reader.

Proposition 6.3.3. *Let μ be a measure with an infinite support on $\mathbb{R}$ and let $n \in \mathbb{N}$ be given. Then for any $r \in \mathbb{N}$ satisfying $r > 1$ and $r \geq n$, we have*

$$U_r(\mu) \leq \inf_{0 \leq i \leq n-1} |x_{i+2,n}(\mu) - x_{i,n}(\mu)|. \tag{6.1}$$

Proof. For $r = n$, (6.1) follows from the definition of $U_r(\mu)$. So, let us pick an $r \in \mathbb{N}$ with $r > n$.

Let $j \in \{0, \ldots, n-1\}$ be chosen so that

$$|x_{j+2,n}(\mu) - x_{j,n}(\mu)| = \inf_{0 \leq i \leq n-1} |x_{i+2,n}(\mu) - x_{i,n}(\mu)|.$$

There are two cases to consider.

First, assume that $x_{j,n}(\mu) = x_{0,n}(\mu)$ or $x_{j+2,n}(\mu) = x_{n+1,n}(\mu)$ holds. Let $x_{j,n}(\mu) = x_{0,n}(\mu)$. Using Theorem 6.2.2 for $\lambda = \mu^{(r)}$, we have

$$x_{0,n}(\mu) < x_{1,r}(\mu) < x_{1,n}(\mu) < x_{r,r}(\mu) < x_{n+1,n}(\mu).$$

If we use Theorem 6.2.2, for $\lambda = \mu^{(r)}$ we see that $[x_{1,r}(\mu), x_{2,r}(\mu)]$ may contain at most one element from $\{x_{1,n}(\mu), \ldots, x_{n,n}(\mu)\}$. Therefore, $U_r(\mu) \leq |x_{2,r}(\mu) -$

$x_{1,r}(\mu)| \leq |x_{2,n}(\mu) - x_{0,n}(\mu)|$. For the case, $j+2 = n+1$, a similar argumentation shows that $U_r(\mu) \leq |x_{r,r}(\mu) - x_{r-1,r}(\mu)| \leq |x_{n+1,n}(\mu) - x_{n-1,n}(\mu)|$.

Now, let us assume that $x_{j,n}(\mu) \neq x_{0,n}(\mu)$ and $x_{j+2,n} \neq x_{n+1,n}$. Using Theorem 6.2.2 for $\lambda = \mu^{(r)}$, we have

$$x_{1,r}(\mu) < x_{j,n}(\mu) < x_{j+1,n}(\mu) < x_{j+2,n}(\mu) < x_{r,r}(\mu).$$

Thus, there is a $k_1 \in \mathbb{N}$ with $1 < k_1 < r$ such that $x_{k_1,r}(\mu) \in [x_{j,n}(\mu), x_{j+1,n}(\mu)]$ because otherwise there is an $i \in \{1, \ldots, r-1\}$ such that $[x_{j,n}(\mu), x_{j+1,n}(\mu)] \subset (x_{i,r}(\mu), x_{i+1,r}(\mu))$ and this would imply that $[x_{i,r}(\mu), x_{i+1,r}(\mu)]$ contains two zeros of $P_n(\cdot; \mu^{(r)})$ which is impossible again by Theorem 6.2.2. On the other hand $x_{k_1,r}(\mu)$ can not be the only zero of $P_r(\cdot; \mu)$ in $[x_{j,n}(\mu), x_{j+2,n}(\mu)]$. This would imply that $(x_{k_1-1,r}(\mu), x_{k_1+1,r}(\mu))$ contains at least three zeros of $P_n(\cdot; \mu)$ but this is impossible by Theorem 6.2.2 as $[x_{k_1-1,r}(\mu), x_{k_1+1,r}(\mu)] = [x_{k_1-1,r}(\mu^{(r)}), x_{k_1+1,r}(\mu^{(r)})]$ may contain at most 2 zeros of $P_n(\cdot; \mu^{(r)})$ if we let $\lambda = \mu^{(r)}$. Hence there is a $k_2 \in \mathbb{N}$ with $1 < k_2 < r$ and $k_2 \neq k_1$ such that $x_{k_2,r}(\mu) \in [x_{j,n}(\mu), x_{j+2,n}(\mu)]$. Thus, $U_r(\mu) \leq |x_{k_2,r}(\mu) - x_{k_1,r}(\mu)| \leq |x_{j+2,n}(\mu) - x_{j,n}(\mu)|$. This shows that (6.1) holds. □

6.4 Zero spacing of orthogonal polynomials for a special family

In this section, we study the spacing of the zeros of orthogonal polynomials for $\mu_{K(\gamma)}$.

For a given sequence $\gamma = (\gamma_k)_{k=1}^{\infty}$, f_1 has two inverse branches $v_{1,1}, v_{2,1} : [-1,1] \to [0,1]$ with $v_{1,1}(t) = 1/2 - (1/2)\sqrt{1 - 2\gamma_1 + 2\gamma_1 t}$ and $v_{2,1}(t) = 1 - v_{1,1}(t)$ where $(f_n)_{n=1}^{\infty}$ is defined as in the beginning of this section. For each $n > 1$, f_n has two inverse branches $v_{1,n}, v_{2,n} : [-1,1] \to [-1,1]$ such that $v_{1,n}(t) = \sqrt{1 - 2\gamma_n + 2\gamma_n t}$ and $v_{2,n}(t) = -v_{1,n}(t)$. Note that $v_{1,n}([-1,1]) \cap v_{2,n}([-1,1]) = \emptyset$ for all $n \in \mathbb{N}$. By the fundamental theorem of algebra, for each $a \in \mathbb{C}$, $F_n(z) = a$ has at most 2^n different solutions and therefore $\{v_{i_1,1} \circ \cdots \circ v_{i_n,n}\}_{i_n \in \{1,2\}}$ gives the total set of inverse branches of $F_n = f_n \circ \cdots \circ f_1$ on $[-1, 1]$. In addition to this, for each $I_{j,n}$ there is a unique choice of $i_l \in \{1, 2\}$ for $l = 1, \ldots, n$ giving $(v_{i_1,1} \circ \cdots \circ v_{i_n,n})([-1, 1]) = I_{j,n}$ and in particular $I_{1,n} = (v_{1,1} \circ v_{1,2} \cdots \circ v_{1,n})([-1, 1])$.

Now, let $u(t) = 1/2-(1/2)\sqrt{1-4t}$ for $0 \leq t \leq 1/4$. Then $(v_{i_1,1} \circ \cdots \circ v_{i_n,n})(t) = g_1(\gamma_1 \cdot g_2(\gamma_2 \ldots \gamma_{n-1} \cdot g_n(\tilde{t})))$ for all $t \in [-1, 1]$ where $g_l = u$ if $i_l = 1$ and $g_l = 1-u$ if $i_l = 2$ for $l = 1, \ldots, n$, and $\tilde{t} = (\gamma_n - \gamma_n t)/2$. This last representation of inverse branches, which was used also in Section 3 of [62], simplifies the calculations since we have only two functions u and $1 - u$ instead of 2^n different functions. The function u has a couple of nice properties that we will exploit many times. The last two of them are from [62].

Proposition 6.4.1. *The following hold:*

(a) u and u' are strictly increasing. In particular, u is strictly convex.

(b) $U_n := u(\gamma_1 \cdot u(\gamma_2 \ldots \gamma_{n-1} \cdot u(\gamma_n))) = (1 - \cos(\pi/2^n))/2$ for all $n \in \mathbb{N}$ where we take $\gamma_k = 1/4$ for all k. The number $(1 - \cos(\pi/2^n))/2$ is the leftmost critical point of $F_n(z)$ and $F_n(z)/\rho_n = 2^{-2^n} T_{2^n}(2z - 1)$ by Example 1 of [62] where T_{2^n} is the 2^n-th monic Chebyshev polynomial of the first kind.

(c) $u(at) \leq au(t)$ for all $0 \leq t \leq 1/4$ and $0 \leq a \leq 1$.

(d) $t \leq u(t)$ for $0 \leq t \leq 1/4$.

The next two lemmas easily follow from the properties mentioned in this section and the theorems from the previous one.

Lemma 6.4.2. *Let $\gamma = (\gamma_k)_{k=1}^{\infty}$ be given. For all $n \in \mathbb{N}$, we have*

$$d\left(Z_{2^n}\left(\mu_{K(\gamma)}\right), Y_{2^n}\left(\mu_{K(\gamma)}\right)\right) \geq$$
$$\inf_{\substack{i_j \in \{1,2\} \\ t \in \{-1,1\}}} |(v_{i_1,1} \circ \cdots \circ v_{i_n,n})(t) - (v_{i_1,1} \circ \cdots \circ v_{i_n,n})(0)| =$$

$$\inf_{\substack{g_i \in \{u, 1-u\} \\ \tilde{t} \in \{0, \gamma_n\}}} |g_1(\gamma_1 \cdot g_2(\ldots \gamma_{n-1} \cdot g_n(\tilde{t}))) - g_1(\gamma_1 \cdot g_2(\ldots \gamma_{n-1} \cdot g_n(\gamma_n/2)))|.$$

Proof. Let us choose an $n \in \mathbb{N}$. Note that, $|F_n(z)| > 1$ for all z satisfying $P'_{2^n}(z;\mu_{K(\gamma)}) = F'_n(z)/\rho_n = 0$. Moreover $F_n(I_{j,n}) = [-1,1]$ and thus $Y_{2^n}\left(\mu_{K(\gamma)}\right) \cap I_{j,n} = \emptyset$ for each $1 \leq j \leq 2^n$. This implies that $d(x_{j,2^n}(\mu_{K(\gamma)}), Y_{2^n}(\mu_{K(\gamma)})) \geq d\left(x_{j,2^n}\left(\mu_{K(\gamma)}\right), \{a_{j,n}, b_{j,n}\}\right)$. Hence, the first inequality holds. The second one follows from the definition of g_i. □

Lemma 6.4.3. *Let $\gamma = (\gamma_k)_{k=1}^{\infty}$ and $r \in \mathbb{N}$ be given. Then, for any $k \in \mathbb{N}$ with $r \geq 2^k$,*

$$\begin{aligned} U_r\left(\mu_{K(\gamma)}\right) &\leq \inf_{0\leq i\leq 2^k-1} \left|x_{i+2,2^k}\left(\mu_{K(\gamma)}\right) - x_{i,2^k}\left(\mu_{K(\gamma)}\right)\right| \\ &\leq \left|x_{2,2^k}\left(\mu_{K(\gamma)}\right) - x_{0,2^k}\left(\mu_{K(\gamma)}\right)\right| \\ &\leq l_{1,k-1}, \end{aligned}$$

holds.

Proof. By using Proposition 6.3.3 for $\mu = \mu_{K(\gamma)}$ and $n = 2^k$, it can be seen that the first inequality holds. The last one holds true since $\left[x_{0,2^k}\left(\mu_{K(\gamma)}\right), x_{2,2^k}\left(\mu_{K(\gamma)}\right)\right] = \left[0, x_{2,2^k}\left(\mu_{K(\gamma)}\right)\right] \subset I_{1,k-1}$ for all $k \in \mathbb{N}$. □

Now, let us prove an auxiliary result which is an analogue of Lemma 4.6.1.

Lemma 6.4.4. *Let $\gamma = (\gamma_k)_{k=1}^{\infty}$ be given. Then*

$$\delta_s \leq l_{1,s} \leq \frac{\pi^2}{4} \cdot \delta_s \tag{6.2}$$

holds for all $s \in \mathbb{N}_0$.

Proof. For $s = 0$, (6.2) holds trivially. So, let $s \geq 1$. Observe that

$$\begin{aligned} l_{1,s} &= u(\gamma_1 \cdot u(\gamma_2 \ldots \gamma_{s-1} \cdot u(\gamma_s))) - u(\gamma_1 \cdot u(\gamma_2 \ldots \gamma_{s-1} \cdot u(0))) \\ &= u(\gamma_1 \cdot u(\gamma_2 \ldots \gamma_{s-1} \cdot u(\gamma_s))) \qquad (6.3) \\ &= u((4\gamma_1) \cdot (1/4) \cdot u((4\gamma_2) \cdot (1/4) \ldots \gamma_{s-1} \cdot u((4\gamma_s) \cdot (1/4)))) \\ &\leq 4^s \delta_s U_s. \qquad (6.4) \end{aligned}$$

The three equalities above are straightforward and the last inequality follows from the parts (b) and (c) of Proposition 6.4.1. Since $1 - \cos x \leq x^2/2$ for all $x \in [0, \infty)$, we have

$$U_s = (1 - \cos(\pi/2^s))/2 \leq (4^{-s}\pi^2)/4.$$

Using this and (6.4), the right part of (6.2) follows. Using (d) in Proposition 6.4.1, it is elementary to see that $\delta_s \leq u(\gamma_1 \cdot u(\gamma_2 \ldots \gamma_{s-1} \cdot u(\gamma_s)))$. This and (6.3) give the left part of (6.2). $\square$

The next lemma will allow us to find a lower bound for $U_n(\mu(K(\gamma)))$.

Lemma 6.4.5. *Let $\gamma = (\gamma_k)_{k=1}^{\infty}$ be given. Then for any choice of $g_i \in \{u, 1-u\}$, for $i = 1, \ldots, n$, we have*

$$\inf_{\tilde{t} \in \{0, \gamma_n\}} |g_1(\gamma_1 \cdot g_2(\ldots \gamma_{n-1} \cdot g_n(\tilde{t}))) - g_1(\gamma_1 \cdot g_2(\ldots \gamma_{n-1} \cdot g_n(\gamma_n/2)))| \geq$$

$$\geq u(\gamma_1 \cdot u(\gamma_2 \cdot \ldots \gamma_{n-1} \cdot u(\gamma_n/2))) \geq l_{1,n+1} \geq \delta_{n+1}. \tag{6.5}$$

Proof. Let $n \in \mathbb{N}$ be given. Then $l_{1,n+1} \geq \delta_{n+1}$ by Lemma 6.4.4. Since $u(t) \leq 1/2$, we have $\gamma_n \cdot u(\gamma_{n+1}) \leq \gamma_n/2$. Using the part (c) of Proposition 6.4.1 we see that

$$u(\gamma_1 \cdot u(\gamma_2 \ldots u(\gamma_n/2))) \geq u(\gamma_1 \cdot u(\gamma_2 \ldots \gamma_n \cdot u(\gamma_{n+1}))) = l_{1,n+1}$$

holds and thus the second inequality in (6.5) follows.

In order to prove the first inequality in (6.5), it suffices to show that for a given $c \in [0, \gamma_n/2]$, and $g_i \in \{u, 1-u\}$, $i = 1, \ldots, n$, the following inequality holds:

$$|g_1(\gamma_1 \ldots \gamma_{n-1} \cdot g_n(c + \gamma_n/2)) - g_1(\gamma_1 \ldots \gamma_{n-1} \cdot g_n(c))| \geq u(\gamma_1 \cdot u(\gamma_2 \cdot \ldots \gamma_{n-1} \cdot u(\gamma_n/2))). \tag{6.6}$$

Let

$$q_{n+1-k} := g_k(\gamma_k \cdot g_{k+1}(\ldots \gamma_{n-1} \cdot g_n(c + \gamma_n/2))),$$

$$t_{n+1-k} := g_k(\gamma_k \cdot g_{k+1}(\ldots \gamma_{n-1} \cdot g_n(c))),$$

$$r_{n+1-k} := u(\gamma_k \cdot u(\gamma_{k+1} \ldots \gamma_{n-1} \cdot u(\gamma_n/2))),$$

$$s_k := |q_k - t_k|,$$

for $k = 1, \ldots, n$. If $n = 1$ then $s_n \geq r_n$ holds since from strict convexity of u we have

$$|g_1(c + \gamma_1/2) - g_1(c)| = u(c + \gamma_1/2) - u(c) \geq u(\gamma_1/2) - u(0). \tag{6.7}$$

Suppose that $n > 1$. We want to show that $s_n \geq r_n$. Let us proceed by induction. For $k = 1$, $s_k \geq r_k$ since

$$|g_n(c + \gamma_n/2) - g_n(c)| \geq u(\gamma_n/2)$$

holds. Suppose that the induction hypothesis holds for all $k = 1, \ldots m$ provided that $m \leq n - 1$. Using, strict convexity of u in (6.8) and the fact that u is increasing in (6.9) we have

$$\begin{aligned}
s_{m+1} &= |q_{m+1} - t_{m+1}| \\
&= |g_{n-m}(\gamma_{n-m} \cdot q_m) - g_{n-m}(\gamma_{n-m} \cdot t_m)| \\
&= |u(\gamma_{n-m} \cdot q_m) - u(\gamma_{n-m} \cdot t_m)| \\
&\geq u(\gamma_{n-m} \cdot |q_m - t_m|) && (6.8) \\
&= u(\gamma_{n-m} \cdot s_m) \\
&\geq u(\gamma_{n-m} \cdot r_m) && (6.9) \\
&= r_{m+1}.
\end{aligned}$$

Hence, $s_n \geq r_n$ holds if we take $m = n - 1$ above. This gives (6.6) and completes our proof. □

Eventually, we are ready to prove the main result of the chapter.

Theorem 6.4.6. *Let $\gamma = (\gamma_k)_{k=1}^{\infty}$ and $n \in \mathbb{N}$ with $n > 1$ be given. Furthermore, let s be the integer satisfying $2^{s-1} \leq n < 2^s$. Then*

$$\delta_{s+2} \leq U_n\left(\mu_{K(\gamma)}\right) \leq \frac{\pi^2}{4} \cdot \delta_{s-2} \tag{6.10}$$

holds. In particular, if $\inf_k \gamma_k = c > 0$ then we have

$$c^2 \cdot \delta_s \leq U_n\left(\mu_{K(\gamma)}\right) \leq \frac{\pi^2}{4c^2} \cdot \delta_s. \tag{6.11}$$

Proof. First, let us prove (6.10). Recall that for all $m > 1$, $f'_m(0) = 0$ and $d_i = 2$ for all $i \in \mathbb{N}$. Using Theorem 6.3.2 for $\mu = \mu_{K(\gamma)}$, $s_0 = s$, $k = 1$ and $k' = 0$, we have

$$d\left(Z_{2^{s+1}}\left(\mu_{K(\gamma)}\right), Y_{2^{s+1}}\left(\mu_{K(\gamma)}\right)\right) \leq d\left(Z_{2^{s+1}}\left(\mu_{K(\gamma)}\right), Z_{2^s}\left(\mu_{K(\gamma)}\right)\right).$$

Using Proposition 6.3.1, for $\mu = \mu_{K(\gamma)}$, $l = 2^{s+1}$, $m = 2^s$, it can be seen that

$$d\left(Z_{2^{s+1}}\left(\mu_{K(\gamma)}\right), Z_{2^s}\left(\mu_{K(\gamma)}\right)\right) \leq U_n\left(\mu_{K(\gamma)}\right).$$

Hence $d\left(Z_{2^{s+1}}\left(\mu_{K(\gamma)}\right), Y_{2^{s+1}}\left(\mu_{K(\gamma)}\right)\right) \leq U_n\left(\mu_{K(\gamma)}\right)$ holds. By Lemma 6.4.2 and Lemma 6.4.5, the term on the left part of this last inequality is bounded below by δ_{s+2}. This gives the first inequality in (6.10).

Using Lemma 6.4.3 for $r = n$ and $k = s - 1$ and then Lemma 6.4.4, we deduce that

$$U_n\left(\mu_{K(\gamma)}\right) \leq l_{1,s-2} \leq (\pi^2/4)\delta_{s-2}.$$

This completes the proof of (6.10).

Combining the first inequality of (6.10) and the fact that $c^2 \leq \gamma_{s+1} \cdot \gamma_{s+2}$, the first inequality in (6.11) follows. Observe that $1/c^2 \geq 1/(\gamma_{s-1} \cdot \gamma_{s-2})$. Hence, the second inequality in (6.10) implies that of (6.11). So, we are done. $\square$

Remark 6.4.7. If there is a c with $0 < c \leq \gamma_k \leq 1/6$ for all $k \in \mathbb{N}$, then $\mu_{K(\gamma)}$ is purely singular continuous. Moreover (6.11) is satisfied and $|K(\gamma)| = 0$ holds.

If $\gamma = (\gamma_k)_{k=1}^{\infty}$ satisfies $\sum_{k=1}^{\infty} \sqrt{(1 - 4\gamma_k)} < \infty$ then there is a c such that $0 < c \leq \gamma_k < 1/4$ holds for all $k \in \mathbb{N}$ and $K(\gamma)$ is a Parreau-Widom set by Chapter 4. Thus, (see [43, 101] for the proof) $\mu_{K(\gamma)}$ and the Lebesgue measure restricted to $K(\gamma)$ are mutually absolutely continuous . Yet, (6.11) gives a pretty accurate description of $\left(U_n\left(\mu_{K(\gamma)}\right)\right)_{n=2}^{\infty}$.

For a $\gamma = (\gamma_k)_{k=1}^{\infty}$ with $\sum_{k=1}^{\infty} \gamma_k < \infty$ and $\gamma_k \leq 1/32$ for all $k \in \mathbb{N}$, the Hausdorff measure Λ_h defined in Chapter 2 and $\mu_{K(\gamma)}$ are mutually absolutely continuous. In this case, $K(\gamma)$ is of Hausdorff dimension zero and we only have (6.10).

It seems that similar results to Theorem 6.4.6 can be obtained for the equilibrium measure of the Julia set $J(f) \subset \mathbb{R}$ of a quadratic polynomial of the form $f(z) = z^2 - c$ with $c > 2$. If we let $f_n := f$ for all $n \in \mathbb{N}$ then by [21], $F_n(z) = P_{2^n}(z; \mu_{J(f)})$ and $f_n'(0) = 0$ holds. Besides, the inverse branches of f are $f_{\pm}(t) = \pm\sqrt{t+c}$ and possibly they lead to similar calculations.

Chapter 7

Asymptotic properties of Jacobi matrices for $\mu_{K(\gamma)}$

7.1 Introduction

In the previous chapters, we study the orthogonal polynomials for $\mu_{K(\gamma)}$ in detail but we do not have complete characterizations of most of the properties mentioned above in terms of the parameters. The results and conjectures of this chapter are meant to suggest some formulations of theorems for further work on these sets as well as other Cantor sets. Here, except Theorem 7.4.1, we use numerical methods in order to analyze the asymptotic behavior of Jacobi matrices for $\mu_{K(\gamma)}$.

The plan of the chapter is as follows. In Section 2, we review the previous results on $K(\gamma)$ and provide evidence for the numerical stability of the algorithm obtained in Theorem 3.4.3 for calculating the recurrence coefficients. In Section 3, we discuss the behavior of recurrence coefficients in different aspects and propose some conjectures about the character of periodicity of the Jacobi matrices. In Section 4, the properties of Widom factors are investigated. We also prove that the sequence of Widom-Hilbert factors for the equilibrium measure of autonomous quadratic Julia sets is unbounded above as soon as the Julia set is totally disconnected. In the last section, we study local behavior of the spacing properties of the zeros of orthogonal polynomials for the equilibrium measures of weakly equilibrium Cantor sets and make a few comments on possible consequences of our numerical experiments.

7.2 Preliminaries and numerical stability of the algorithm

We show in Section 3 of Chapter 3 that for all $s \in \mathbb{N}_0$ we have

$$||P_{2^s}\left(\cdot;\mu_{K(\gamma)}\right)||_{L^2\left(\mu_{K(\gamma)}\right)} = \sqrt{(1-2\,\gamma_{s+1})\,r_s^2/4}. \tag{7.1}$$

The diagonal elements, the b_n's of $H\left(\mu_{K(\gamma)}\right)$, are equal to $0,5$ by Section 4 in Chapter 3. For the outdiagonal elements by Theorem 3.4.3, we have the following relations:

$$a_1 = \|P_1\left(\cdot;\mu_{K(\gamma)}\right)\|_{L^2\left(\mu_{K(\gamma)}\right)}, \tag{7.2}$$

$$a_2 = \|P_2\left(\cdot;\mu_{K(\gamma)}\right)\|_{L^2\left(\mu_{K(\gamma)}\right)}/\|P_1\left(\cdot;\mu_{K(\gamma)}\right)\|_{L^2\left(\mu_{K(\gamma)}\right)}. \tag{7.3}$$

If $n+1 = 2^s > 2$ then

$$a_{n+1} = \frac{||P_{2^s}\left(\cdot;\mu_{K(\gamma)}\right)||_{L^2\left(\mu_{K(\gamma)}\right)}}{||P_{2^{s-1}}\left(\cdot;\mu_{K(\gamma)}\right)||_{L^2\left(\mu_{K(\gamma)}\right)}\cdot a_{2^{s-1}+1}\cdot a_{2^{s-1}+2}\cdots a_{2^s-1}}. \tag{7.4}$$

If $n+1 = 2^s(2k+1)$ for some $s \in \mathbb{N}$ and $k \in \mathbb{N}$, then

$$a_{n+1} = \sqrt{\frac{\|P_{2^s}\left(\cdot;\mu_{K(\gamma)}\right)\|^2_{L^2\left(\mu_{K(\gamma)}\right)} - a^2_{2^{s+1}k}\cdots a^2_{2^{s+1}k-2^s+1}}{a^2_{2^s(2k+1)-1}\cdots a^2_{2^{s+1}k+1}}}, \tag{7.5}$$

If $n+1 = (2k+1)$ for $k \in \mathbb{N}$ then

$$a_{n+1} = \sqrt{\|P_1\left(\cdot;\mu_{K(\gamma)}\right)\|^2_{L^2\left(\mu_{K(\gamma)}\right)} - a^2_{2k}}. \tag{7.6}$$

The relations (7.1), (7.2), (7.3), (7.4), (7.5), (7.6) completely determine $(a_n)_{n=1}^\infty$ and naturally define an algorithm. This is the main algorithm that we use and we call it Algorithm 1.

We want to examine numerical stability of Algorithm 1 since roundoff errors can be huge due to the recursive nature of it. Before this, let us list some remarkable properties of $K(\gamma)$ which follow from the previous results. Part (f) can be found in [10].

Theorem 7.2.1. *For a given $\gamma = (\gamma_s)_{s=1}^\infty$, the following propositions hold:*

(a) *If $\sum_{s=1}^\infty \gamma_s < \infty$ and $\gamma_s \leq 1/32$ for all $s \in \mathbb{N}$ then $K(\gamma)$ is of Hausdorff dimension zero.*

(b) *If $\gamma_s \leq 1/6$ for each $s \in \mathbb{N}$ then $K(\gamma)$ has zero Lebesgue measure, $\mu_{K(\gamma)}$ is purely singular continuous and $\liminf a_n = 0$ for $\mu_{K(\gamma)}$.*

(c) *Let $\tilde{f} := (\tilde{f}_s)_{s=1}^\infty$ be a sequence of functions such that $\tilde{f}_s = f_s$ for $1 \leq s \leq k$ for some $k \in \mathbb{N}$ and $\tilde{f}_s(z) = 2z^2 - 1$ for $s > k$. Then $\cap_{n=1}^\infty \tilde{F}_n^{-1}([-1,1]) = E_k$ where $\tilde{F}_n := \tilde{f}_n \circ \cdots \circ \tilde{f}_1$.*

(d) *$g_{\Omega_{K(\gamma)}}$ is Hölder continuous with exponent $1/2$ if and only if $\sum_{s=1}^\infty \epsilon_s < \infty$.*

(e) *$K(\gamma)$ is a Parreau-Widom set if and only if $\sum_{s=1}^\infty \sqrt{\epsilon_s} < \infty$.*

(f) *If $\sum_{s=1}^\infty \epsilon_s < \infty$ then there is $C > 0$ such that for all $n \in \mathbb{N}$ we have*

$$W_n^2(\mu_{K(\gamma)}) = \frac{\| P_n\left(\cdot; \mu_{K(\gamma)}\right) \|_{L^2\left(\mu_{K(\gamma)}\right)}}{(\mathrm{Cap}(K(\gamma)))^n} = \frac{a_1 \dots a_n}{(\mathrm{Cap}(K(\gamma)))^n} \leq Cn.$$

(g) $\mathrm{Cap}(K(\gamma)) = \exp\left(\sum_{k=1}^\infty 2^{-k} \log \gamma_k\right)$.

(h) *Let $v_{1,1}(t) = 1/2 - (1/2)\sqrt{1 - 2\gamma_1 + 2\gamma_1 t}$ and $v_{2,1}(t) = 1 - v_{1,1}(t)$. For each $n > 1$, let $v_{1,n}(t) = \sqrt{1 - 2\gamma_n + 2\gamma_n t}$ and $v_{2,n}(t) = -v_{1,n}(t)$. Then the zero set of $P_{2^s}\left(\cdot; \mu_{K(\gamma)}\right)$ is $\{v_{i_1,1} \circ \cdots \circ v_{i_s,s}(0)\}_{i_s \in \{1,2\}}$ for all $s \in \mathbb{N}$.*

(i) $\mathrm{supp}(\mu_{K(\gamma)}) = \mathrm{ess\,supp}(\mu_{K(\gamma)}) = K(\gamma)$. *If $K(\gamma) = [0,1] \setminus \cup_{k=1}^\infty (c_i, d_i)$ where $c_i \neq d_j$ for all $i, j \in \mathbb{N}$ then $\mu_{K(\gamma)}([0, e_i]) \subset \{m 2^{-n}\}_{m,n \in \mathbb{N}}$ where $e_i \in (c_i, d_i)$.*

Moreover for each $m \in \mathbb{N}$ and $n \in \mathbb{N}$ with $m2^{-n} < 1$ there is an $i \in \mathbb{N}$ such that $\mu_{K(\gamma)}([0, e_i]) = m2^{-n}$.

We consider 4 different models depending on γ in the whole chapter. They are:

1. $\gamma_s = 1/4 - (1/(50 + s)^4)$.
2. $\gamma_s = 1/4 - (1/(50 + s)^2)$.
3. $\gamma_s = 1/4 - (1/(50 + s)^{(5/4)}$.
4. $\gamma_s = 1/4 - (1/50)$.

Model 1 represents an example where $K(\gamma)$ is Parreau-Widom and Model 2 gives a non Parreau-Widom set with fast growth of γ. Model 3 produces a non Parreau-Widom $K(\gamma)$ with relatively slow growth of γ but still $g_{\Omega_{K(\gamma)}}$ is optimally smooth. Model 4 yields a set which is neither Parreau-Widom nor the Green's function for the complement of it is optimally smooth. We used Matlab in all of the experiments.

If f is a nonlinear polynomial having real coefficients with real and simple zeros $x_1 < x_2 < \ldots < x_n$ and distinct extremas $y_1 < \ldots < y_{n-1}$ where $|f(y_i)| > 1$ for $i = 1, 2, \ldots, n-1$, we say that f is an *admissible* polynomial. Clearly, for any choice of γ, f_n is admissible for each $n \in \mathbb{N}$ and this implies by Lemma 4.5.2 that F_n is also admissible. By the remark after Theorem 4 and Theorem 11 in [59] it follows that the Christoffel numbers (see p. 565 in [59] for the definition) for the 2^n-th orthogonal polynomial of μ_{E_n} are equal to $1/2^n$. Let $\mu_{K(\gamma)}^n$ be the measure which assigns $1/2^n$ mass to each zero of $P_{2^n}(\cdot; \mu_{K(\gamma)})$. By Remark 3.4.8, the recurrence coefficients $(a_k)_{k=1}^{2^n-1}$, $(b_k)_{k=1}^{2^n}$ for μ_{E_n} are exactly those of $\mu_{K(\gamma)}$. This implies that (see e.g. Theorem 1.3.5 in [116]) the Christoffel numbers corresponding to 2^n-th orthogonal polynomial for $\mu_{K(\gamma)}$ are also equal to $1/2^n$.

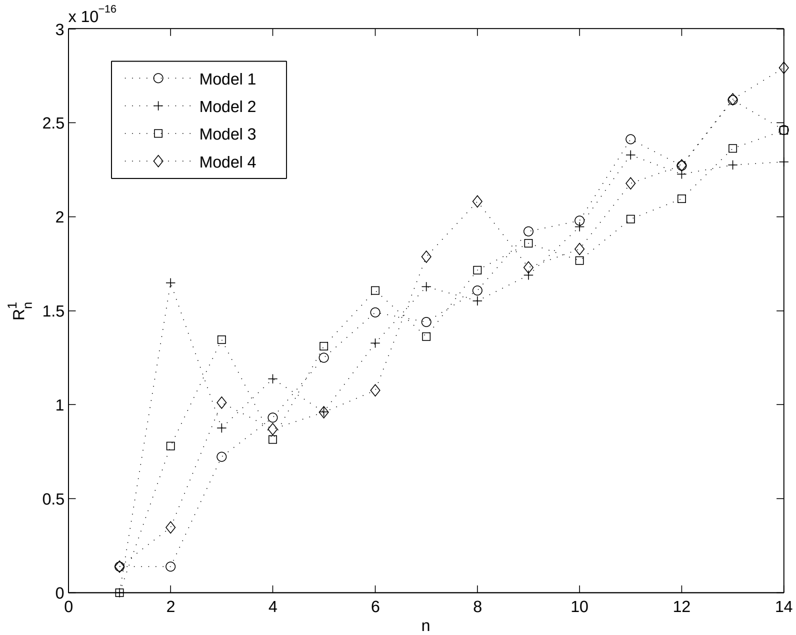

Figure 7.1: Errors associated with eigenvalues.

Let

$$H\left(\mu^n_{K(\gamma)}\right) = \begin{pmatrix} b_1 & a_1 & & & \\ a_1 & b_2 & a_2 & & \\ & a_2 & \ddots & \ddots & \\ & & \ddots & \ddots & a_{2^n-1} \\ & & & a_{2^n-1} & b_{2^n} \end{pmatrix},$$

where the coefficients $(a_k)_{k=1}^{2^n-1}$, $(b_k)_{k=1}^{2^n}$ are the Jacobi parameters for $\mu_{K(\gamma)}$. Then the set of eigenvalues of $H\left(\mu^n_{K(\gamma)}\right)$ is exactly the zero set of $P_{2^n}\left(\cdot;\mu_{K(\gamma)}\right)$. Moreover, by [60], the square of first component of normalized eigenvectors gives one of the Christoffel numbers, which in our case is equal to $1/2^n$. For each $n \in \{1,\ldots,14\}$, using gauss.m, we computed the eigenvalues and first component of normalized eigenvectors of $H\left(\mu^n_{K(\gamma)}\right)$ where the coefficients are obtained from Algorithm 1. We compared these values with the zeros obtained by part (h) of Theorem 7.2.1 and $1/2^n$ respectively. For each n, let $\{t^n_k\}_{k=1}^{2^n}$ be the set of eigenvalues for $H\left(\mu^n_{K(\gamma)}\right)$ and $\{q^n_k\}_{k=1}^{2^n}$ be the set of zeros where we enumerate these sets so that the smaller the index they have, the value will be smaller. Let $\{w^n_k\}_{k=1}^{2^n}$ be the set of squared first component of normalized eigenvectors. We

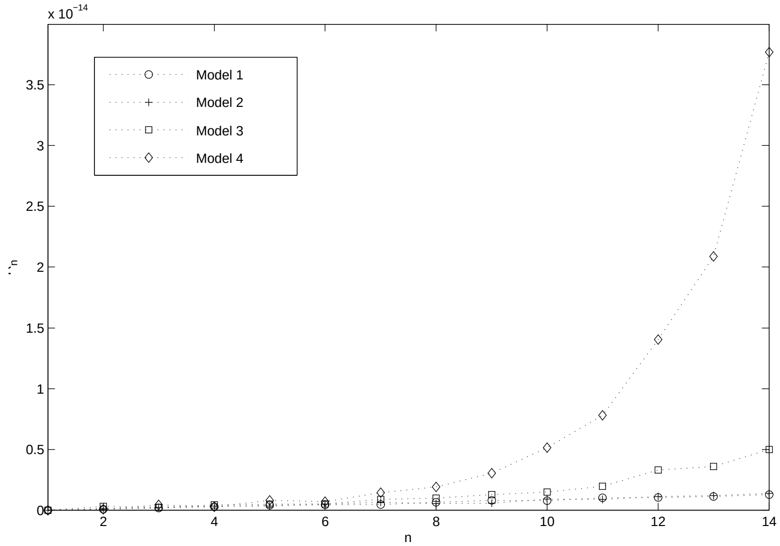

Figure 7.2: Errors associated with eigenvectors.

plotted (see Figure 7.1 and Figure 7.2) $R_n^1 := (1/2^n)(\sum_{k=1}^{2^n} |t_k^n - q_k^n|)$ and $R_n^2 := (1/2^n)(\sum_{k=1}^{2^n} |(1/2^n) - w_k^n|)$. This numerical experiment shows the reliability of Algorithm 1. One can compare these values with Fig. 2 in [87].

7.3 Recurrence Coefficients

Let $J(f)$ be the (autonomous) Julia set for $f(z) = z^2 - c$ for some $c > 2$. Since $(f_n)_{n=1}^{\infty}$ is a sequence of quadratic polynomials, it is natural to ask that to what extent $H\left(\mu_{J(f)}\right)$ and $H\left(\mu_{K(\gamma)}\right)$ have similar behavior. Compare for example Theorem 3.4.7 and Section 3 in [30].

The recurrence coefficients for $\mu_{J(f)}$ can be ordered according to their indices, see (IV.136)-(IV.138) in [29]. We obtain similar results for $\mu_{K(\gamma)}$ in our numerical experiments in each 4 models. That is the numerical experiments suggest that $\min\limits_{i\in\{1,\dots,2^n\}} a_i = a_{2^n}$ for $n \le 14$ and it immediately follows from (7.2) and (7.6) that $\max\limits_{n\in\mathbb{N}} a_n = a_1$. Thus, we make the following conjecture:

Conjecture 7.3.1. *For $\mu_{K(\gamma)}$ we have $\min\limits_{i\in\{1,\dots,2^n\}} a_i = a_{2^n}$ and in particular $\liminf\limits_{s\to\infty} a_{2^s} = \liminf\limits_{n\to\infty} a_n$.*

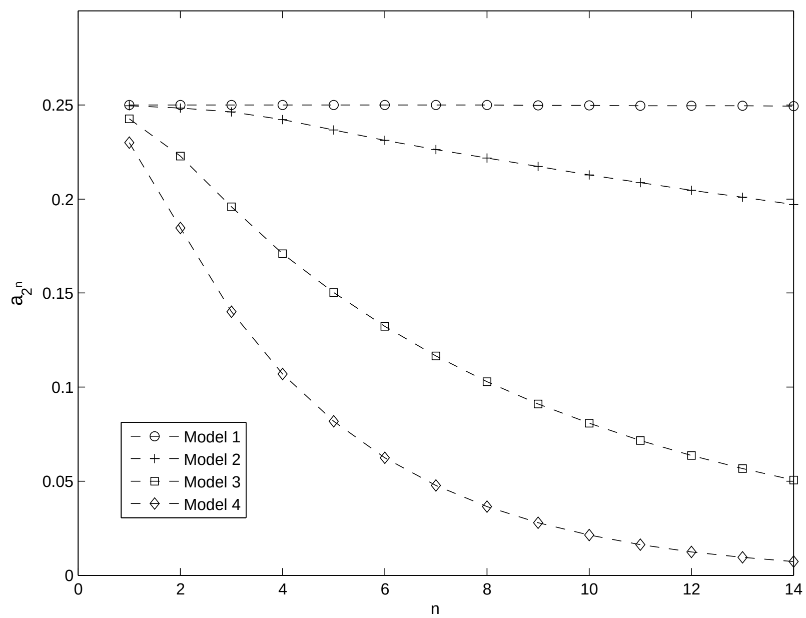

Figure 7.3: The values of outdiagonal elements of Jacobi matrices at the indices of the form 2^s.

By part (e) of Theorem 7.2.1, $\liminf a_n > 0$ for $\mu_{K(\gamma)}$ provided that $\sum_{s=1}^{\infty} \sqrt{\epsilon_s} < \infty$. It also follows from Remark 3.4.8 and [53] that if the a_n's associated with $\mu_{K(\gamma)}$ satisfy $\liminf a_n = 0$ then $K(\gamma)$ has zero Lebesgue measure. Hence asymptotic behavior of the a_n's is also important for understanding the Hausdorff dimension of $K(\gamma)$. We computed $v_n := a_{2^n}/a_{2^{n+1}}$ (see Figure 7.3 and Figure 7.4) for $n = 1, \ldots, 13$ in order to find for which γ's $\liminf a_n = 0$. We assume here Conjecture 7.3.1 is correct.

In Model 1, v_n is very close to 1 which is expected since for this case $\liminf a_n > 0$. In other models, it seems that $(v_n)_{n=1}^{13}$ seems to behave like a constant. Thus, this experiment may be read as unless $\sum_{s=1}^{\infty} \sqrt{\epsilon_s} < \infty$ is satisfied $\liminf a_n = 0$. So, we conjecture:

Conjecture 7.3.2. *For a given* $\gamma = (\gamma_k)_{k=1}^{\infty}$, *let* $\epsilon_k := 1 - 4\gamma_k$ *for each* $k \in \mathbb{N}$. *Then* $K(\gamma)$ *is of positive Lebesgue measure if and only if* $\sum_{s=1}^{\infty} \sqrt{\epsilon_s} < \infty$ *if and only if* $\liminf a_n > 0$.

A more interesting problem is whether $H\left(\mu_{K(\gamma)}\right)$ is almost periodic or at least

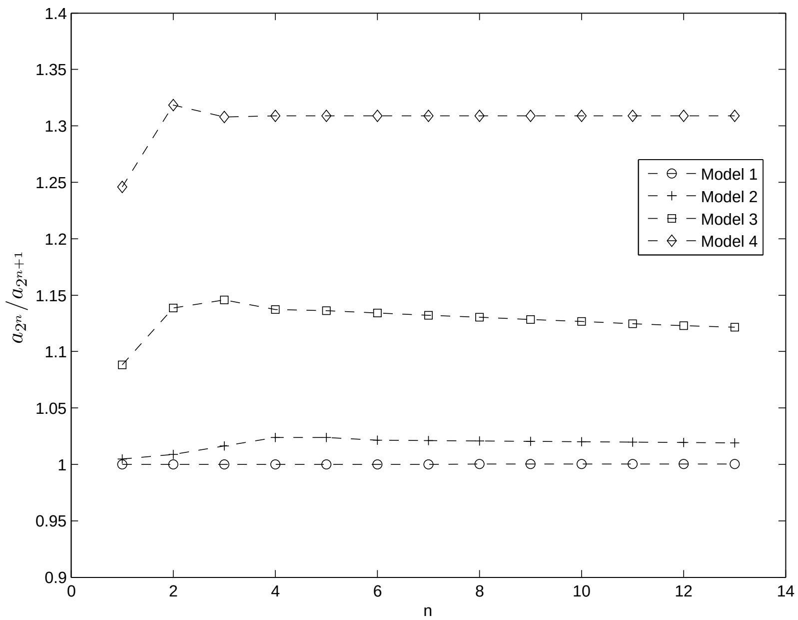

Figure 7.4: The ratios of outdiagonal elements of Jacobi matrices at the indices of the form 2^s.

asymptotically almost periodic. Since $(b_n)_{n=1}^{\infty}$ is a periodic sequence, we only need to deal with $(a_n)_{n=1}^{\infty}$.

For a measure μ with an infinite compact support $\operatorname{supp}(\mu)$, let δ_n be the normalized counting measure on the zeros of $P_n(\cdot;\mu)$. If there is a ν such that $\delta_n \to \nu$ then ν is called the *density of states* (DOS) measure for $H(\mu)$. Besides, $\int_{-\infty}^{x} d\nu$ is called the *integrated density of states* (IDS). For $H\left(\mu_{K(\gamma)}\right)$ the density of states measure is automatically (see Theorem 1.7 and Theorem 1.12 in [116] and also [140]) $\mu_{K(\gamma)}$. Therefore, if x is chosen from one of the gaps (by a gap of a compact set on $K \subset \mathbb{R}$ we mean a bounded component of $R \setminus K$) of $\operatorname{supp}\left(\mu_{K(\gamma)}\right)$, that is $x \in (c_i, d_i)$ (see part (i) of Theorem 7.2.1) then the value of the IDS is equal to $m2^{-n}$ which does not exceed 1 and also for each $m, n \in \mathbb{N}$ with $m2^{-n} < 1$ there is a gap (c_j, d_j) such that the IDS takes the value $m2^{-n}$.

For an almost periodic sequence $c = (c_n)_{n=1}^{\infty}$ the $\mathbb{Z}$-module of the real numbers modulo 1 generated by ω satisfying

$$\{\omega : \lim_{n\to\infty} \frac{1}{N} \sum_{n=1}^{N} \exp\left(2\pi i n\omega\right) c_n \neq 0\}$$

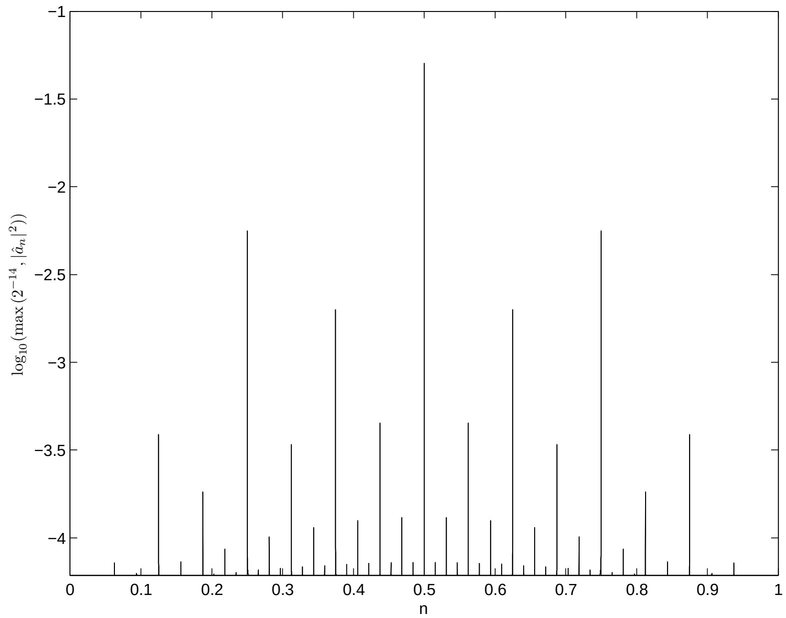

Figure 7.5: Normalized power spectrum of the a_n's for Model 1.

is called the *frequency module* for c and it is denoted by $\mathcal{M}(c)$. The frequency module is always countable and c can be written as a uniform limit of Fourier series where the frequencies are chosen among $\mathcal{M}(c)$. For an almost periodic Jacobi matrix H with coefficients $a = (a_n)_{n=1}^{\infty}$ and $b = (b_n)_{n=1}^{\infty}$, the frequency module $\mathcal{M}(H)$ is the module generated by $\mathcal{M}(a)$ and $\mathcal{M}(b)$. It was shown in Theorem III.1 in [51] that for an almost periodic H, the values of IDS in gaps belong to $\mathcal{M}(H)$. Moreover, (see e.g. Theorem 2.4 in [58]), an asymptotically almost periodic Jacobi matrix has the same density of states measure with the almost periodic limit of it.

In order to examine almost periodicity of the a_n's for $\mu_{K(\gamma)}$ we computed the discrete Fourier transform $(\widehat{a}_n)_{n=1}^{2^{14}}$ for the first 2^{14} coefficients for each model where frequencies run from 0 to 1. We normalized $|\widehat{a}|^2$ dividing it by $\sum_{n=1}^{2^{14}} |\widehat{a}_n|^2$. We plotted (see Figure 7.5) this normalized power spectrum while we did not plot the peak at 0, by detrending the transform.

There are only a small number of peaks in each case compared to 2^{14} frequencies which points out almost periodicity of coefficients. We consider only Model 1 here although we have similar pictures for the other models. The highest 10 peaks are at $0.5, 0.25, 0.75, 0.375, 0.625$, $0.4375, 0.5625, 0.125, 0.875, 0.3125$. All these values are of the form $m2^{-n}$ where $n \leq 4$. This is an important indicator of almost periodicity as these frequencies are exactly the values of IDS for $H\left(\mu_{K(\gamma)}\right)$ in the gaps which appear earlier in the construction of the Cantor set. The following conjecture follows naturally from the above discussion.

Conjecture 7.3.3. *For any γ, $(a_n)_{n=1}^{\infty}$ for $H\left(\mu_{K(\gamma)}\right)$ is asymptotically almost periodic where the almost periodic limit has frequency module equal to $\{m2^{-n}\}_{m,n\in\{N_0\}}$ modulo 1.*

7.4 Widom factors

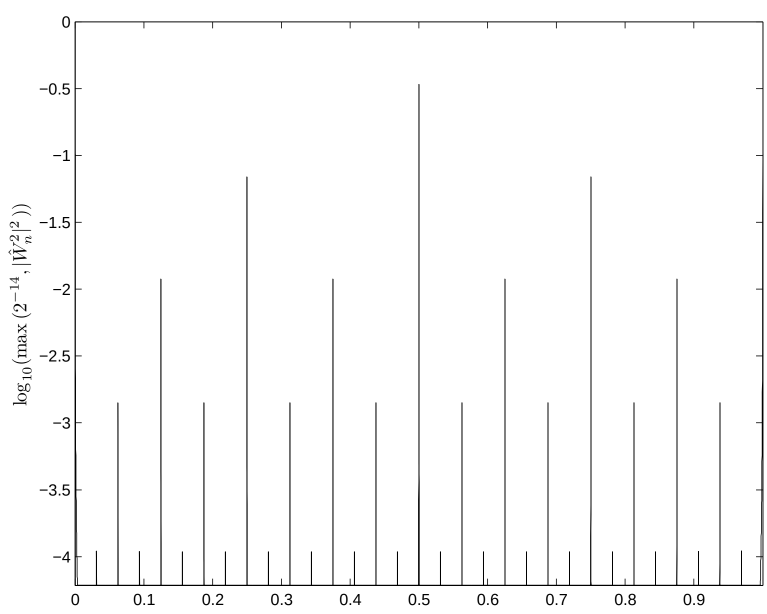

Figure 7.6: Normalized power spectrum of the $W_n^2\left(\mu_{K(\gamma)}\right)$'s for Model 1.

For many compact subsets of $\mathbb{C}$ (see e.g. [12, 48, 133, 141]) the sequence of Widom factors for the sup-norm is bounded. In particular, this is valid for Parreau-Widom sets on $\mathbb{R}$, see [48]. It would be interesting to find (if any) a non Parreau-Widom set K on $\mathbb{R}$ such that it is regular with respect to the Dirichlet problem and $(W_n(K))_{n=1}^{\infty}$ is bounded. Note that if K is a non-polar compact subset of $\mathbb{R}$ which is regular with respect to the Dirichlet problem then by Theorem

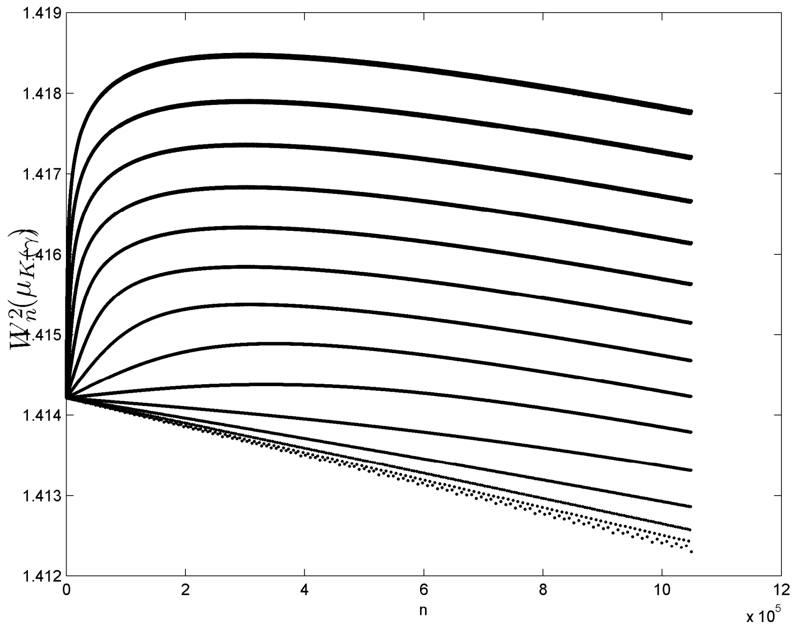

Figure 7.7: Widom-Hilbert factors for Model 1

4.2.3 in [106] and Theorem 5.5.13 in [116] we have $\text{supp}(\mu_K) = K$. In this case, we have $W_n^2(\mu_K) \leq W_n(K)$ since $\|P_n(\cdot;\mu_K)\|_{L^2(\mu_K)} \leq \|T_n\|_{L^2(\mu_K)} \leq \|T_n\|_{L^\infty(K)}$. Therefore, it is possible to formulate the above problem in a weaker form: Is there a non Parreau-Widom set $K \subset \mathbb{R}$ which is regular with respect to the Dirichlet problem such that $(W_n^2(\mu_K))_{n=1}^\infty$ is bounded?

It is always true for a polynomial autonomous Julia set $J(f)$ on $\mathbb{R}$ that $\text{supp}\left(\mu_{J(f)}\right) = J(f)$ since $J(f)$ is regular with respct to the Dirichlet problem by [84]. Now, let us show that $\left(W_n^2\left(\mu_{J(f)}\right)\right)_{n=1}^\infty$ is unbounded when $f(z) = z^2 - c$ and $c > 2$. These quadratic Julia sets are zero Lebesgue measure Cantor sets on $\mathbb{R}$ and therefore not Parreau-Widom. See [36] for a deeper discussion on this particular family.

Theorem 7.4.1. *Let $f(z) = z^2 - c$ for $c \geq 2$. Then $\left(W_n^2\left(\mu_{J(f)}\right)\right)_{n=1}^\infty$ is bounded if and only if $c = 2$.*

Proof. If $c = 2$ then $J(f) = [-2, 2]$. This implies that $\left(W_n^2\left(\mu_{J(f)}\right)\right)_{n=1}^\infty$ is bounded since $J(f)$ is Parreau-Widom.

Let $c \neq 2$. Then $\lim_{n\to\infty} a_{2^n} = 0$ (see e.g. Section IV.5.2 in [29]) where the a_n's are the recurrence coefficients for $\mu_{J(f)}$ and $\mathrm{Cap}(J(f)) = 1$ by [36]. Since $P_{2^{n+1}}\left(\cdot; \mu_{J(f)}\right) = P_{2^n}^2\left(\cdot; \mu_{J(f)}\right) - c$ by Theorem 3 in [21], we have $W_{2^n}^2\left(\mu_{J(f)}\right) = \|P_{2^n}\left(\cdot; \mu_{J(f)}\right)\|_{L^2\left(\mu_{J(f)}\right)} = \sqrt{c}$ for all $n \geq 1$. Moreover,

$$W_{2^n-1}^2\left(\mu_{J(f)}\right) = \frac{W_{2^n}^2\left(\mu_{J(f)}\right)}{a_{2^n}} = \frac{\sqrt{c}}{a_{2^n}}.$$

Hence $\lim_{n\to\infty} W_{2^n-1}^2\left(\mu_{J(f)}\right) = \infty$ as $\lim_{n\to\infty} a_{2^n} = 0$. This completes the proof.

□

In Chapter 3, it is shown that $\left(W_n^2\left(\mu_{K(\gamma)}\right)\right)_{n=1}^{\infty}$ is unbounded if $\gamma_k \leq 1/6$ for all $k \in \mathbb{N}$. We want to examine the behavior of $\left(W_n^2\left(\mu_{K(\gamma)}\right)\right)_{n=1}^{\infty}$ provided that $K(\gamma)$ is not Parreau-Widom. By Chapter 3, $\left(W_{2^n}\left(\mu_{K(\gamma)}\right)\right) \geq \sqrt{2}$ for all $n \in \mathbb{N}_0$ for any choice of γ. Hence,we also have

$$W_{2^n-1}^2\left(\mu_{K(\gamma)}\right) = W_{2^n}^2\left(\mu_{K(\gamma)}\right)\frac{\mathrm{Cap}\left(\mu_{K(\gamma)}\right)}{a_{2^n}} \geq \frac{\sqrt{2}\mathrm{Cap}\left(\mu_{K(\gamma)}\right)}{a_{2^n}} \tag{7.7}$$

for all $n \in \mathbb{N}$.

If we assume that Conjecture 7.3.1 and Conjecture 7.3.2 are correct then $\liminf_{n\to\infty} a_{2^n} = 0$ as soon as $K(\gamma)$ is not Parreau-Widom. If $\liminf_{n\to\infty} a_{2^n} = 0$ then $\limsup_{n\to\infty} W_{2^n-1}\left(\mu_{K(\gamma)}\right) = \infty$ by (7.7). Thus, the numerical experiments indicate the following:

Conjecture 7.4.2. *$K(\gamma)$ is a Parreau-Widom set if and only if $\left(W_n^2\left(\mu_{K(\gamma)}\right)\right)_{n=1}^{\infty}$ is bounded if and only if $(W_n(K(\gamma)))_{n=1}^{\infty}$ is bounded.*

Let K be a union of finitely many compact non-degenerate intervals on $\mathbb{R}$. Since $\mu_K \in \mathrm{Sz}(K)$, by Corollary 6.7 in [46], $(W_n^2(\mu_K))_{n=1}^{\infty}$ is asymptotically almost periodic. We plotted (see Figure 7.7) the Widom-Hilbert factors for Model 1 for the first 2^{20} values and it seems that $\limsup W_n^2\left(\mu_{(K(\gamma))}\right) \neq \sup W_n^2\left(\mu_{(K(\gamma))}\right)$. For Model 1, we plotted (see Figure 7.6) the power spectrum for $(W_n^2(\mu_K))_{n=1}^{2^{14}}$ where we normalized $|\widehat{W}^2|^2$ dividing it by $\sum_{n=1}^{2^{14}} |\widehat{W}_n^2(\mu_K)|^2$. Frequencies run from 0 to 1 here and we did not plot the big peak at 0.

Clearly, there are only a few peaks as in (see Figure 7.5) which is an important indicator of almost periodicity. The highest 10 peaks are at $0.5, 0.00006103515625, 0.25, 0.75, 0.125,\ 0.875, 0.375, 0.625, 0.062.5, 0.9375$. These values are quite different than those of peaks in Figure 7.5. This may be an indicator of a different frequency module of the almost periodic limit. By Conjecture 7.4.2, $\left(W_n^2\left(\mu_{K(\gamma)}\right)\right)_{n=1}^{\infty}$ is unbounded and cannot be asymptotically almost periodic if $K(\gamma)$ is not Parreau-Widom. We make the following conjecture:

Conjecture 7.4.3. $\left(W_n^2\left(\mu_{K(\gamma)}\right)\right)_{n=1}^{\infty}$ *is asymptotically almost periodic if and only if $K(\gamma)$ is Parreau-Widom. If $K(\gamma)$ is Parreau-Widom then the almost periodic limit's frequency module includes the module generated by $\{m2^{-n}\}_{m,n\in\{N_0\}}$ modulo 1.*

7.5 Spacing properties of orthogonal polynomials and further discussion

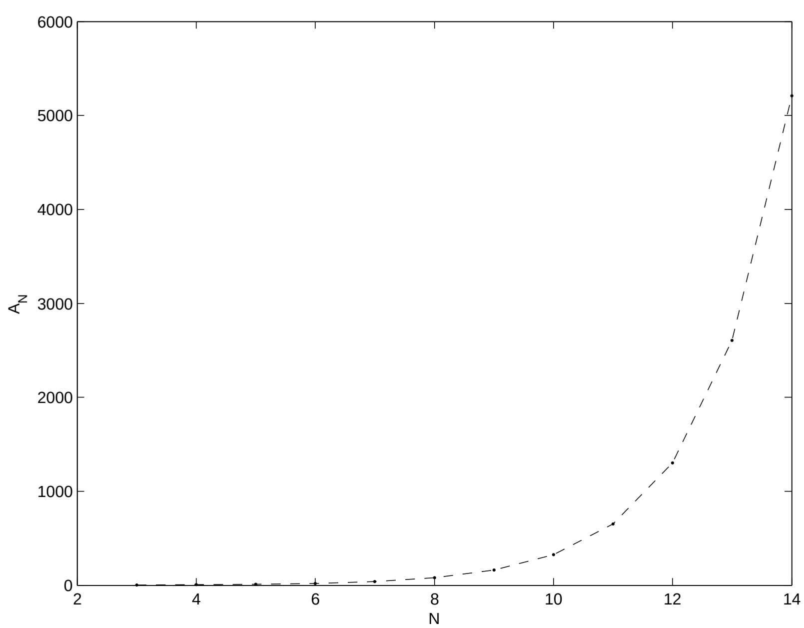

Figure 7.8: Maximal ratios of the distances between adjacent zeros

For a given $\gamma = (\gamma_k)_{k=1}^{\infty}$ let us enumerate the elements of the zeros of $P_n(\cdot;\mu_{K(\gamma)})$ by $x_{1,N} < \cdots < x_{N,N}$. The behavior of $\left(U_N\left(\mu_{K(\gamma)}\right)\right)_{N=1}^{\infty}$, in other

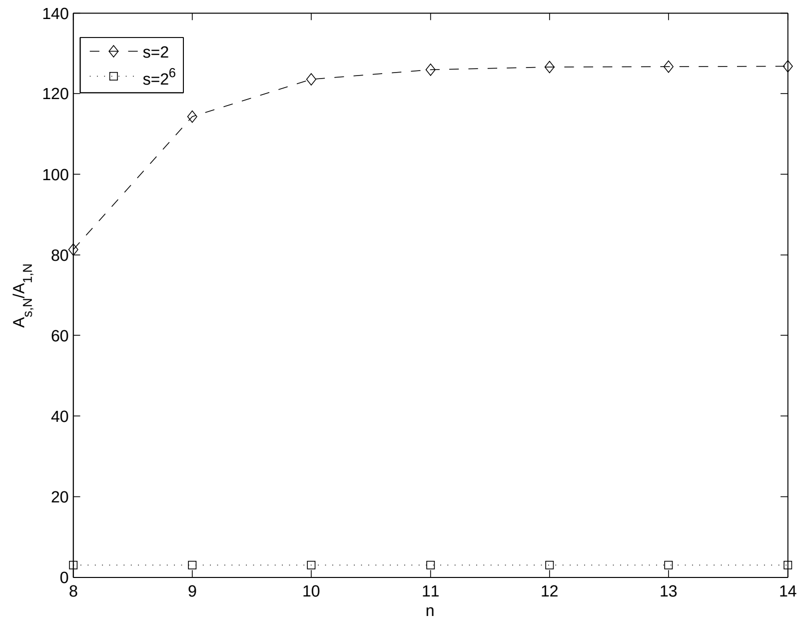

Figure 7.9: Ratios of the distances between prescribed adjacent zeros

words, the global behavior the spacing of the zeros, is investigated in Chapter 6. Here, we numerically study some aspects of the local behavior of the zeros.

We consider only Model 1 since the calculations give similar results for the other models. For $N = 2^3, 2^4, \dots, 2^{14}$, let $A_{n,N} := |x_{2n,N} - x_{2n-1,N}|$ where $n \in \{1, \dots, N/2\}$. We computed (see Figure 7.8) $A_N := \max_{n,m \in \{1,\dots,N/2\}} \dfrac{A_{n,N}}{A_{m,N}}$ for each such N.

$(A_{2^n})_{n=3}^{14}$ increases fast and this indicates that $(A_{2^n})_{n=2}^{\infty}$ is unbounded.

For $N = 2^{14}$ and $s = 2$, $s = 6$ we plotted (see Figure 7.9) $A_{s,N}/A_{1,N}$. These ratios tend to converge fast.

In the next conjecture, we exclude the case of small γ for the following reason: Let $\gamma = (\gamma_k)_{k=1}^{\infty}$ satisfies $\sum_{k=1}^{\infty} \gamma_k = M < \infty$ with $\gamma_k \leq 1/32$ for all $k \in \mathbb{N}$ and $\delta_k := \gamma_1 \cdots \gamma_k$. Then $A_{j,2^k} \leq \exp(16M)\delta_{k-1}$ for all $k > 1$ by Lemma 6 in [62]. By Lemma 4 and Lemma 6 in [62] we conversely have $A_{j,2^k} \geq (7/8)\delta_{k-1}$. Therefore $A_{2^k} \leq (8/7)\exp(16M)$. Hence, $(A_{2^n})_{n=2}^{\infty}$ is bounded.

Conjecture 7.5.1. *For each $\gamma = (\gamma_k)_{k=1}^{\infty}$ with $\inf_k \gamma_k > 0$, $(A_{2^k})_{k=1}^{\infty}$ is an unbounded sequence. If $s = 2^k$ for some $k \in \mathbb{N}$, there is a $c_0 \in \mathbb{R}$ depending on k such that*

$$\lim_{n\to\infty} \frac{A_{s,2^n}}{A_{1,2^n}} = c_0.$$

For the parameters $c > 3$, $H\left(\mu_{J(f)}\right)$ is almost periodic where $f(z) = z^2 - c$, see [27]. It was conjectured in [28] that $H\left(\mu_{J(f)}\right)$ is always almost periodic as soon as $c > 2$. For $c = 2$, $H\left(\mu_{J(f)}\right)$ is not almost periodic since $a_1 = 2$ and $a_n = 1$ for $n \geq 2$ but it is asymptotically almost periodic. Therefore if this conjecture is true then we have the following: $H\left(\mu_{J(f)}\right)$ is almost periodic if and only if $J(f)$ is non Parreau-Widom.

We did not make any distinction between asymptotic almost periodicity and almost periodicity in Section 3 and Section 4 since these two cases are indistinguishable numerically. But we remark that if $\liminf a_n \neq 0$ then the asymptotics $\lim_{j\to\infty} a_{j\cdot 2^s+n} = a_n$ cease to hold immediately. We do not expect $H\left(\mu_{K(\gamma)}\right)$ to be almost periodic for the Parreau-Widom case for that reason. For a parameter $\gamma = (\gamma_s)_{s=1}^{\infty}$ such that $\lim_{j\to\infty} a_{j\cdot 2^s+n} = a_n$ holds for each s and n it is likely that $H\left(\mu_{K(\gamma)}\right)$ is almost periodic. These asymptotics hold only for the non Parreau-Widom case but it is unclear that if these hold for all parameters making $K(\gamma)$ non Parreau-Widom.

Chapter 8

Orthogonal polynomials associated with equilibrium measures on $\mathbb{R}$

8.1 Introduction and results

It was recently proved that there are Cantor sets for which $(W_n(K))_{n=1}^{\infty}$ is bounded, see Theorem 1.4 and Remarks just below the theorem in [48]. In the other direction, for each sequence $(c_n)_{n=1}^{\infty}$ of positive real numbers with subexponential growth, there is a Cantor set $K(\gamma)$ such that $W_n(K(\gamma)) \geq c_n$ for all $n \in \mathbb{N}$, see Theorem 4.4 in [63].

Throughout the chapter, by a measure we mean a unit Borel measure with an infinite compact support on $\mathbb{R}$.

The behavior of $(W_n^2(\mu_K))_{n=1}^{\infty}$ is unknown for many cases and the main aim of this chapter is to study the upper and lower bounds of this sequence for general compact sets on $\mathbb{R}$. We remark that by Lemma 1.2.7 in [120] we have $\mathrm{Cap}(\mathrm{supp}(\mu_K)) = \mathrm{Cap}(K)$, and we use these expressions interchangeably.

Recall that μ satisfies the Szegő condition on K if $\int \log \mu'(t)\, d\mu_K(t) > -\infty$. In this case we write $\mu \in \mathrm{Sz}(K)$. It is known that $\mu_K \in \mathrm{Sz}(K)$, see Proposition 2 and (4.1) in [43]. By [43], this implies that there is an $M > 0$ such that $1/M < W_n^2(\mu_K) < M$ holds for all $n \in \mathbb{N}$. In the inverse direction, one can find a Cantor set $K(\gamma)$ such that $W_n^2\left(\mu_{K(\gamma)}\right) \to \infty$ as $n \to \infty$, see Chapter 3.

First, we restrict our attention to union of several intervals. Let T_N be a real polynomial of degree N with $N \geq 2$ such that it has N real and simple zeros $x_1 < \cdots < x_n$ and $N-1$ critical points $y_1 < \cdots < y_{n-1}$ with $|T_N(y_i)| \geq 1$ for each $i \in \{1, \ldots, N-1\}$. We call such a polynomial admissible. If $K = T_N^{-1}([-1,1])$ for an admissible polynomial T_N then K is called a T-set. A T-set is of the form $\cup_{i=1}^n [\alpha_i, \beta_i]$ with $n \leq N$ where N is the degree of the associated admissible polynomial. For applications of T-sets to polynomial inequalities and spectral theory of orthogonal polynomials, we refer the reader to [95, 130] and Chapter 5 in [116]. We have the following characterization for T-sets, see Lemma 2.2 in [126]:

Theorem 8.1.1. *Let $K = \cup_{j=1}^n [\alpha_j, \beta_j]$ be a disjoint union of n intervals. Then K is a T-set if and only if $\mu_K([\alpha_j, \beta_j]) \in \mathbb{Q}$. If $K = T_N^{-1}[-1,1]$ for some admissible polynomial T_N then for each $j \in \{1, \ldots, n\}$ there is an $l \in \mathbb{N}$ such that $\mu_K([\alpha_j, \beta_j]) = l/N$.*

If $K = T_N^{-1}[-1,1]$ for an admissible polynomial T_N then (see Theorem 9 and Lemma 3 in [59]) since $\mu_K \in \mathrm{Sz}(K)$, there is a sequence $(a_n')_{n=1}^\infty$ with $a_k' = a_{k+N}'$ for each $k \in \mathbb{N}$ such that $a_n - a_n' \to 0$ as $n \to \infty$ where $(a_n)_{n=1}^\infty$ is the sequence of recurrence coefficients for μ_K. In this case we call $(a_n')_{n=1}^\infty$ the periodic limit for $(a_n)_{n=1}^\infty$ and $(a_n)_{n=1}^\infty$ asymptotically periodic. Our first theorem is about $(W_n^2(\mu_K))_{n=1}^\infty$ when K is a T-set.

Theorem 8.1.2. *Let $K = T_N^{-1}[-1,1]$ where T_N is an admissible polynomial with leading coefficient c. Furthermore, let $(a_n)_{n=1}^\infty$ be the sequence of recurence coefficients for μ_K and $(a_n')_{n=1}^\infty$ be the periodic limit of it. Then*

(a) $\liminf\limits_{n\to\infty} W_n^2(\mu_K) = \sqrt{2}$.

(b) $W_n^2(\mu_K) \geq 1$ *for each* $n \in \mathbb{N}$.

(c) $\inf_l \dfrac{a_1' \cdots a_l'}{\mathrm{Cap}(K)^l} = \dfrac{a_1' \cdots a_N'}{\mathrm{Cap}(K)^N} = 1.$

An arbitrary compact set K on $\mathbb{R}$ can be approximated in an appropriate way by T-sets, see Section 5.8 in [116] and Section 2.4 in [125]. We rely upon these techniques in order to prove Theorem 1.2.1.

Remark 8.1.3. Theorem 1.2.1 can be seen as an analogue of Schiefermayr's Theorem (Theorem 2 in [111]). It is unclear whether 1 on the right side of the inequality in Theorem 1.2.1 can be improved. This constant can be at most $\sqrt{2}$ by part (a) of Theorem 8.1.2. It suffices to find a bigger lower bound for $W_n^2(\mu_K)$ in part (b) of Theorem 8.1.2 to improve the result.

Regularity of μ_K in the sense of Stahl-Totik follows as a corollary of Theorem 1.2.1 since the inequality $\liminf_{n\to\infty} (W_n^2(\mu_K))^{1/n} \geq 1$ directly follows. On the other hand, regularity of a measure μ in the sense of Stahl-Totik does not even imply that $\limsup_{n\to\infty} W_n^2(\mu) > 0$, see e.g. Example 1.4 in [114]. Hence, the implications of Theorem 1.2.1 are profoundly different than those of $\mu_K \in \mathbf{Reg}$. Corollary 1.2.2 which gives a sufficient condition for unboundedness of $(W_n^2(\mu_K))_{n=1}^{\infty}$ is a corollary of Theorem 1.2.1.

In Section 2, we present some aspects of Widom's theory and give proofs for the theorems.

8.2 Proofs

Let $K = \cup_{j=1}^{p}[\alpha_j, \beta_j]$ be a disjoint union of several intervals, $E_j := [\alpha_j, \beta_j]$ for each $j \in \{1, \ldots, p\}$ and $\{c_j\}_{j=1}^{p-1}$ (for $p = 1$ there are no critical points) be the set of critical points of g_{Ω_K}. Then (see e.g. p. 186 in [96]), we have

$$\mu_K'(t) = \frac{1}{\pi} \frac{|q(t)|}{\sqrt{\prod_{j=1}^{p} |(t-\alpha_j)(t-\beta_j)|}}, \quad t \in K \tag{8.1}$$

where $q(t) = 1$ if $p = 1$ and $q(t) = \prod_{j=1}^{p-1}(t - c_j)$ if $p > 1$.

Let $\partial g_{\Omega_K}/\partial n_+$ and $\partial g_{\Omega_K}/\partial n_-$ denote the normal derivatives of g_{Ω_K} in the positive and negative direction respectively. These functions are well defined on K except the end points of the intervals. Moreover by symmetry of K with respect to $\mathbb{R}$, we have $\partial g_{\Omega_K}/\partial n_+ = \partial g_{\Omega_K}/\partial n_-$, see p. 121 in [110]. Let $\partial g_{\Omega_K}/\partial n := \partial g_{\Omega_K}/\partial n_+$. Then, $(\partial g_{\Omega_K}/\partial n)(t) = \pi\, \mu_K'(t)$, see (5.6.7) in [116]. This is why we can state the functions and theorems in [141] in terms of μ_K instead of $\partial g_{\Omega_K}/\partial n$. Similarly, instead of harmonic measure at infinity we use the equilibrium measure, since these two measures are the same, see Theorem 4.3.14 in [106]. The concepts that we describe below can be found in [15, 141] but with somewhat a different terminology.

Let $\mu \in \mathrm{Sz}(K)$ and h be the harmonic function in $\overline{\mathbb{C}} \setminus K$ having boundary values (nontangential limit exists a.e.) $\log \mu'(t)$. Then following Section 5 and Section 14 of [141], we define the multivalued analytic function R in $\overline{\mathbb{C}} \setminus K$ by $R(z) = \exp\left(h(z) + i\tilde{h}(z)\right)$ where $\tilde{h}$ is a harmonic conjugate of h and

$$R(\infty) = \exp\left(\int \log \mu'(t) d\mu_K(t)\right).$$

Now, R has no zeros or poles. Moreover, $\log|R(z)|$ is single-valued on $\overline{\mathbb{C}} \setminus K$ and has boundary values $\log \mu'(t)$ on K.

Let F be a multivalued meromorphic function having finitely many zeros and poles in $\overline{\mathbb{C}} \setminus K$ for which $|F(z)|$ is single-valued. Then,

$$\gamma_j(F) := (1/2\pi) \underset{E_j}{\triangle} \arg F,$$

for each $j \in \{1, \ldots, p\}$. Here, $\underset{E_j}{\triangle} \arg F$ denotes the increment of the argument of F in going around a positively oriented curve F_j enclosing E_j. The curve is taken so close to E_j that it does not intersect with or enclose any points of E_k with $k \neq j$. A multiple-valued function U in $\overline{\mathbb{C}} \setminus K$ with a single-valued absolute value is of class Γ_γ if $\gamma = (\gamma_1, \ldots, \gamma_p) \in [0, 1)^p$ and $\gamma_j(U) = \gamma_j \mod 1$ for each $j \in \{1, \ldots, p\}$.

Let $H^2(\overline{\mathbb{C}} \setminus K, \mu', \Gamma_\gamma)$ denote the space of multi-valued analytic functions F

from Γ_γ in $\overline{\mathbb{C}} \setminus K$ such that $|F(z)^2 R(z)|$ has a harmonic majorant. Then

$$\nu(\mu', \Gamma_\gamma) := \inf_F \int_E |F(t)|^2 \mu'(t) dt.$$

where $F \in H^2(\overline{\mathbb{C}} \setminus K, \mu', \Gamma_\gamma)$ and $|F(\infty)| = 1$.

For the class associated with $(-n\mu_E(E_1) \mod 1, \ldots, -n\mu_E(E_p) \mod 1)$ we use Γ_n.

Before giving the proofs, we state some results from [141] in a unified way. The part (a) is Theorem 12.3, part (c) is Theorem 9.2 (see p. 223 for the explanation of why it is applicable) and part (b) is given in p. 216 in [141].

Theorem 8.2.1. *Let $K = \cup_{j=1}^p [\alpha_j, \beta_j]$ be a disjoint union intervals and let $\mu \in \mathrm{Sz}(K)$. Then*

(a) $\left(W_n^2(\mu)\right)^2 \sim \nu(\mu', \Gamma_n)$ where $a_n \sim b_n$ means that $\frac{a_n}{b_n} \to 1$ as $n \to \infty$.

(b) $\left(W_n^2(\mu)\right)^2 \geq \frac{\nu(\mu', \Gamma_n)}{2}$ for all $n \in \mathbb{N}$.

(c) The limit points of $\left(\left(W_n^2(\mu)\right)^2\right)_{n=1}^{\infty}$ are bounded below by

$$2\pi R(\infty) \mathrm{Cap}(K) \exp(-\mathrm{PW}(K)).$$

Proof of Theorem 8.1.2. Let $\{\alpha_j\}_j$ and $\{\beta_j\}_j$ be the set of left and right endpoints of the connected components of K respectively so that $\alpha_1 < \beta_1 < \cdots < \alpha_p < \beta_p$. Moreover let $E_j := [\alpha_j, \beta_j]$ for each $j \in \{1, \ldots, p\}$ and $\{c_j\}_j$ be the set of critical points of g_{Ω_K}.

(a) First, let us show that $\liminf_{n\to\infty} \left(W_n^2(\mu_K)\right)^2 \geq 2$. Since $\mu_K \in \mathrm{Sz}(K)$, Theorem 8.2.1 is applicable. We need to compute

$$\log R(\infty) = \int \log \mu_K'(t)\, d\mu_K(t).$$

Using (8.1), we can write

$$\log R(\infty) = -\log \pi + D_1 + D_2 + D_3$$

where

$$D_1 = -\frac{1}{2}\sum_{j=1}^{p}\int \log|t-\alpha_j|\,d\mu_K(t),$$

$$D_2 = -\frac{1}{2}\sum_{j=1}^{p}\int \log|t-\beta_j|\,d\mu_K(t),$$

$$D_3 = \sum_{j=1}^{p-1}\int \log|t-c_j|\,d\mu_K(t), \text{ if } p \geq 2$$

and $D_3 = 0$ if $p = 1$.

Since K is regular with respect to the Dirichlet problem, g_{Ω_K} can be extended to $\overline{\mathbb{C}}$ by taking $g_{\Omega_K}(z) = 0$ for $z \in K$ so that g_{Ω_K} is continuous everywhere in $\mathbb{C}$. Besides,

$$g_{\Omega_K}(z) = -U^{\mu_K}(z) - \log \mathrm{Cap}(K) \tag{8.2}$$

holds in $\mathbb{C}$ where $U^{\mu_K}(z) = -\int \log|z-t|\,d\mu_K(t)$. See p. 53-54 in [110].

By (8.2), for any $z \in K$ we have $\int \log|z-t|\,d\mu_K(t) = \log \mathrm{Cap}(K)$. Hence, $D_1 + D_2 = 2p(-1/2)\log \mathrm{Cap}(K) = -\log(\mathrm{Cap}(K)^p)$.

For $p \geq 2$, $\int \log|t-c_j|\,d\mu_K(t) = g_{\Omega_K}(c_j) + \log \mathrm{Cap}(K)$ by (8.2). Thus,

$$D_3 = \mathrm{PW}(K) + \log\left(\mathrm{Cap}(K)^{p-1}\right). \tag{8.3}$$

But since $\mathrm{PW}(K) + \log(\mathrm{Cap}(K)^{p-1}) = 0$ for $p = 1$, (8.3) is valid for $p \geq 1$. Therefore,

$$\log R(\infty) = -\log\pi + \mathrm{PW}(K) - \log \mathrm{Cap}(K).$$

Using part (c) of Theorem 8.2.1, we have

$$\liminf_{n\to\infty}\left(W_n^2(\mu_K)\right)^2 \geq \frac{2\pi\exp(\mathrm{PW}(K))\mathrm{Cap}(K)}{\pi\exp(\mathrm{PW}(K))\mathrm{Cap}(K)} \geq 2.$$

In order to complete the proof, it is enough to show that

$$\liminf_{n\to\infty}\left(W_n^2(\mu_K)\right)^2 \leq 2. \tag{8.4}$$

On $[-1,1]$, we have the formula $p_l(x;\mu_{[-1,1]}) = \sqrt{2}S_l(x)$ where S_l is the l-th Chebyshev polynomial on $[-1,1]$ of the first kind, see (1.89b) in [109]. By Theorem 1 and Theorem 11 in [59] this gives,

$$p_{lN}(x;\mu_K) = p_l\left(T_N(x);\mu_{[-1,1]}\right) = \sqrt{2}S_l(T_N(x)),$$

for each $l \in \mathbb{N}$. The leading coefficient of $p_{lN}(x; \mu_K)$ is $\sqrt{2} \cdot 2^{l-1} \cdot c^l$ or in other words $\|P_{lN}(\cdot; \mu_K)\|_{L^2(\mu_K)} = (\sqrt{2} \cdot 2^{l-1} \cdot c^l)^{-1}$. By (5.2) in [59], $\mathrm{Cap}(K)^{lN} = (2c)^{-l}$ since (see e.g. p. 135 in [106]) $\mathrm{Cap}[-1, 1] = 1/2$. Therefore, $W_{lN}^2(\mu_K) = \sqrt{2}$ for each $l \in \mathbb{N}$ and (8.4) holds. This completes the proof of part (a).

(b) By Theorem 8.1.1, $(lN + s)\mu_K(E_j) = s \cdot \mu_K(E_j) \mod 1$ for all $l \in \mathbb{N}$, $s \in \{0, \ldots, N-1\}$ and $j \in \{1, \ldots, N\}$. Hence $\Gamma_{lN+s} = \Gamma_s$ where l and s are as above. Therefore, $(\nu(\mu'_K, \Gamma_n))_{n=1}^{\infty}$ is a periodic sequence of period N. This implies that $\inf_{n\in\mathbb{N}} \nu(\mu'_K, \Gamma_n) = \liminf_{n\to\infty} \nu(\mu'_K, \Gamma_n)$. By part (a) of Theorem 8.2.1 and part (a) of this theorem, we have

$$\liminf_{n\to\infty} \nu(\mu'_K, \Gamma_n) = \liminf_{n\to\infty} \left(W_n^2(\mu_K)\right)^2 = 2. \tag{8.5}$$

From (8.5), it follows that, $\inf_{n\in\mathbb{N}} \nu(\mu'_K, \Gamma_n) = 2$. By part (b) of Theorem 8.2.1, we get $(W_n^2(\mu_K))^2 \geq 1$ for each $n \in \mathbb{N}$ which gives the desired result.

(c) Equality on the right can be found in the literature, see e.g. (2.23) in [50]. As we see, in the proof of part (b), $(W_n^2(\mu_K))_{n=1}^{\infty}$ is asymptotically periodic with the periodic limit $\left(\sqrt{\nu(\mu'_K, \Gamma_n)}\right)_{n=1}^{\infty}$. The periodic limit can be written in the form

$$\left(d\frac{a'_1 \cdots a'_n}{\mathrm{Cap}(K)^n}\right)_{n=1}^{\infty},$$

by Corollary 6.7 of [46] where $d \in \mathbb{R}^+$. Since $W_{lN}^2(\mu_K) = \sqrt{2}$ by the proof of part (a) and $\frac{a'_1 \cdots a'_{lN}}{\mathrm{Cap}(K)^{lN}} = 1$ holds for all $l \in \mathbb{N}$, we obtain $d = \sqrt{2}$. Besides,

$$\liminf_{l\to\infty} \sqrt{2}\frac{a'_1 \cdots a'_l}{\mathrm{Cap}(K)^l} = \liminf_{l\to\infty} W_l^2(\mu_K) = \sqrt{2} \tag{8.6}$$

holds by part (a). Using periodicity and (8.6), we have

$$\inf_{l\in\mathbb{N}} \frac{a'_1 \cdots a'_l}{\mathrm{Cap}(K)^l} = \liminf_{l\to\infty} \frac{a'_1 \cdots a'_l}{\mathrm{Cap}(K)^l} = 1.$$

This concludes the proof.

□

Proof of Theorem 1.2.1. By Theorem 5.8.4 in [116], there is a sequence $(F_s)_{s=1}^{\infty}$ of T-sets such that

$$K \subset \cdots \subset F_{s+1} \subset F_s \subset \cdots \subset \mathbb{R} \tag{8.7}$$

and

$$\cap_{s=1}^{\infty} F_s = K \tag{8.8}$$

hold. Moreover, (8.7) and (8.8) imply that

$$\mu_{F_s} \to \mu_K \tag{8.9}$$

in weak star sense, and

$$\operatorname{Cap}(F_s) \to \operatorname{Cap}(K)$$

as $s \to \infty$.

Let $n \in \mathbb{N}$. Then for each $s \in \mathbb{N}$, we have

$$\|P_n(\cdot;\mu_{F_s})\|_{L^2(\mu_{F_s})} \leq \|P_n(\cdot;\mu_K)\|_{L^2(\mu_{F_s})} \tag{8.10}$$

by minimality of $P_n(x;\mu_{F_s})$ in $L^2(\mu_{F_s})$. It follows from monotonicity (see e.g. Theorem 5.1.2 in [106]) of capacity that

$$\operatorname{Cap}(K) \leq \operatorname{Cap}(F_s) \text{ for each } s \in \mathbb{N}. \tag{8.11}$$

Hence,

$$\begin{aligned}
\left(W_n^2(\mu_K)\right)^2 &= \frac{\int P_n^2(t;\mu_K)\,d\mu_K(t)}{\operatorname{Cap}(K)^{2n}} \\
&= \frac{\lim_{s\to\infty}\int P_n^2(t;\mu_K)\,d\mu_{F_s}(t)}{\operatorname{Cap}(K)^{2n}} && (8.12)\\
&\geq \liminf_{s\to\infty} \frac{\int P_n^2(t;\mu_{F_s})\,d\mu_{F_s}(t)}{\operatorname{Cap}(F_s)^{2n}} && (8.13)\\
&= \liminf_{s\to\infty}\left(W_n^2(\mu_{F_s})\right)^2 \\
&\geq 1. && (8.14)
\end{aligned}$$

In order to obtain (8.12), we use (8.9). The inequality (8.13) follows from (8.10) and (8.11), and (8.14) is obtained by using part (b) of Theorem 8.1.2. Thus, the proof is complete. $\square$

Proof of Corollary 1.2.2. Let $\left(a_{n_j}\right)_{j=1}^{\infty}$ be a subsequence of $(a_n)_{n=1}^{\infty}$ such that $a_{n_j} \to 0$ as $j \to \infty$. By (1.2) and Theorem 1.2.1, for each $j > 1$, we have

$$W_{n_j-1}^2(\mu_K) = W_{n_j}^2(\mu_K)\frac{\mathrm{Cap}(K)}{a_{n_j}} \geq \frac{\mathrm{Cap}(K)}{a_{n_j}} \tag{8.15}$$

Since $a_{n_j} \to 0$ as $j \to \infty$, the right hand side of (8.15) goes to infinity as $j \to \infty$. Hence $\lim_{j\to\infty} W_{n_j-1}^2(\mu_K) = \infty$ and in particular $(W_n^2(\mu_K))_{n=1}^{\infty}$ is unbounded. Since $\mathrm{supp}(\mu_K) \subset K$, $\|T_{n,\mathrm{supp}(\mu_K)}\|_{L^\infty(\mathrm{supp}(\mu_K))} \leq \|T_{n,K}\|_{L^\infty(K)}$ holds for all $n \in \mathbb{N}$. Thus, we have $W_n^2(\mu_K) \leq W_n(K)$ for each $n \in \mathbb{N}$. This implies that $(W_n(K))_{n=1}^{\infty}$ is also unbounded. $\square$

Chapter 9

Orthogonal polynomials on Cantor sets of zero Lebesgue measure

9.1 Introduction

Throughout the chapter, by a measure μ we mean a unit Borel measure with an infinite compact support on $\mathbb{R}$. In this chapter, we use H^+ instead of H in order to denote one sided Jacobi matrices.

Let δ_n be the normalized counting measure on the zeros of the n-th monic orthogonal polynomial for μ. If there is a measure ν such that $\delta_n \to \nu$ in weak star sense then ν is called the density of states (DOS) measure for μ or for $H^+(\mu)$. Moreover, for a regular measure μ with a non-polar compact support K, δ_n converges to the equilibrium measure of K as $n \to \infty$, see e.g. Theorem 1.7 in [114]. We say that $H^+(\mu)$ is regular if $\mu \in \mathbf{Reg}$, see Theorem 1.1.2.

Similarly, one can also define two sided Jacobi matrices. For two bounded sequences $(a_n)_{n=-\infty}^{\infty}$ and $(b_n)_{n=-\infty}^{\infty}$ with $a_n \geq 0$ and $b_n \in \mathbb{R}$ for $n \in \mathbb{Z}$, the corresponding Jacobi matrix is defined as follows:

$$H = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & a_{-1} & b_0 & a_0 & & & \\ & & a_0 & b_1 & a_1 & & \\ & & & a_1 & b_2 & a_2 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

For the restriction $H^+\left((a_k, b_k)\right)_{k=n+1}^{\infty}$ of H we use $H^+_{\restriction n}$. In order to denote $H^+\left((a_{n-k}, b_{n-k+1})\right)_{k=1}^{\infty}$ we use $H^-_{\restriction n}$. If $n = 0$ we omit the subscript. In order these restrictions to be well defined, $a_k \neq 0$ or $a_{n-k} \neq 0$ should be valid respectively for $k \in \mathbb{N}$. By the DOS measure for H we mean the DOS measure for H^+.

There are some classes of measures for which the asymptotics stronger than those of regular measures. Let $K = \cup_{i=1}^{n}[\alpha_i, \beta_i]$ (we call these sets finite gap sets) where each $[\alpha_i, \beta_i]$ is a non-degenerate compact interval on $\mathbb{R}$. The Szegő class of measures and the measures associated with the isospectral torus on K are typical examples of such class of measures, see [15], [45], [46], [141]. If we replace a finite gap set by a Parreau-Widom set, the concepts of Szegő class and isospectral torus on this set still make sense but the theory is more complicated and less complete in this generality. We refer the reader to [43], [44], [99], [119], [139] for some results about orthogonal polynomials and Jacobi matrices on Parreau-Widom sets.

If $L \subset \mathbb{R}$ is a zero Lebesgue measure Cantor set with positive capacity then we do not have a general theory of measures which are analogues of Szegő class and isospectral torus on L. There are still interesting results -both analytic and numerical- and conjectures regarding what these two classes of objects may mean in this case.

Orthogonal polynomials on zero Lebesgue measure sets are our main focus here. We review related results, conjectures and suggested definitions. We prove a couple of theorems on Parreau-Widom sets which also have some implications on the zero measure case. We define Szegő class and isospectral torus on more general sets and in our approach the concept of equilibrium measure is central.

The plan of the chapter is as follows. In Section 2 and Section 3 we discuss the results which are valid on finite gap sets and Parreau-Widom sets concerning Szegő class and isospectral torus. We also prove some new results in these sections, see Theorem 9.2.3, Theorem 9.2.5 and Theorem 9.3.1. In Section 4, we propose definitions for Szegő class and isospectral torus which are compatible with the definitions in finite gap sets and previously suggested definitions for zero measure case. In Section 5, we review some conjectures and results concerning orthogonal polynomials where the support of the prescribed measures are of zero Lebesgue measure.

To denote the essential spectrum we use σ_{ess}. Here, the essential spectrum is used for the set of all accumulation points of the spectrum.

9.2 Isospectral Torus

Let K be a Parreau-Widom set. We can uniformize $\overline{\mathbb{C}}\backslash K$ by the universal covering map $\mathbf{x} : \mathbb{D} \to \overline{\mathbb{C}} \setminus K$. There is a Fuchsian group Γ_K of Möbius transformations on $\mathbb{D}$ such that

$$\mathbf{x}(z) = \mathbf{x}(w) \iff \exists \zeta \in \Gamma_K \text{ such that } z = \zeta(w).$$

We fix $\mathbf{x}$ uniquely by requiring

$$\mathbf{x}(0) = \infty, \quad \lim_{z \to \infty} z\mathbf{x}(z) > 0.$$

A meromorphic function f in $\mathbb{D}$ is called *bounded characteristic* if it can be represented as a ratio of two bounded functions $f = g_1/g_2$. If the denominator g_2 is an outer function then we say that f belongs to the *Smirnov class* in $\mathbb{D}$. We say that a meromorphic function F in $\overline{\mathbb{C}} \setminus K$ belongs to the Smirnov class $N_+(\overline{\mathbb{C}} \setminus K)$ if $F \circ \mathbf{x}$ belongs to the Smirnov class in $\mathbb{D}$. Let, $F(x \pm i0) := \lim_{\epsilon \downarrow 0} F(x \pm i\epsilon)$.

We say that the *Direct Cauchy Theorem* (DCT) holds in $\overline{\mathbb{C}} \setminus K$ if

$$\frac{1}{2\pi i}\int_K \frac{F(x+i0)}{x-z}dx - \frac{1}{2\pi i}\int_K \frac{F(x-i0)}{x-z}dx = F(z)$$

holds for all $z \in \overline{\mathbb{C}} \setminus K$ and for all $F \in N_+(\overline{\mathbb{C}} \setminus K)$ such that

$$\int_K |F(x+i0)|dx + \int_K |F(x-i0)|dx < \infty \text{ and } F(\infty) = 0.$$

See [142] for information on DCT.

There are various ways to define the isospectral torus on a finite gap set K, see Section 3 in [47]. In order to discuss what this concept may mean on more general sets, we first give background information on related concepts.

Let us recall that on a Parreau-Widom set K, the Lebesgue measure $dx_{\restriction K}$ restricted to K and $d\mu_K$ are mutually absolutely continuous, see [119]. In particular this implies that $|K| > 0$. Regularity of K with respect to the Dirichlet problem implies by Theorem 4.2.3 in [106] and Theorem 5.5.13 in [116] that $\text{supp}(\mu_K) = K$. Therefore,

$$|(x-a, x+a) \cap K| > 0 \tag{9.1}$$

holds for all $x \in K$ and $a > 0$.

As it was shown in [99], [119], [139] there are good and bad (these terms were used in [139] in this fashion) Parreau-Widom sets. In this context, good and bad Parreau-Widom sets are classified in terms DCT, see [44] and [142] for more on this issue.

For a given Jacobi operator $H = H((a_n, b_n)_{n=-\infty}^{\infty})$ let us denote $(a_n, b_n)_{n=-\infty}^{\infty}$ by c_H for simplicity. We define the shift operator S by

$$S(c_H) = S((a_n, b_n)_{n=-\infty}^{\infty}) = (a_{n+1}, b_{n+1})_{n=-\infty}^{\infty}$$

Recall that a Jacobi operator H is called almost periodic if

$$\{S^k(c_H) : \ k \in \mathbb{Z}\} \tag{9.2}$$

is precompact in $l^\infty(\mathbb{Z}) \times l^\infty(\mathbb{Z})$ which can be equipped with the metric

$$d_l((a_n,b_n)_{n=-\infty}^\infty, (a'_n,b'_n)_{n=-\infty}^\infty) = \max\left(\sup_{n\in\mathbb{Z}} |a_n - a'_n|, \sup_{n\in\mathbb{Z}} |b_n - b'_n|\right).$$

We denote the closure of the set given in (9.2) by Θ_{c_H} and call it the *hull* of c_H. The hull can be made into a compact abelian group. Moreover, there is a unique invariant measure $m_{\Theta_{c_H}}$ which is also ergodic on Θ_{c_H}, see Section 5.3 in [123] for more details. This is equivalent to saying that the set of two sided Jacobi operators corresponding to the elements of the hull is a family of uniquely ergodic elements and actually from (1) of Theorem 9.2.1 below this family is strictly ergodic. See e.g. [25] for more on ergodic Jacobi operators.

A characterization of almost periodicity is the following, see Appendix to Section 5.13 in [116]: For every $\varepsilon > 0$ there is an $L \in \mathbb{N}$ such that for every $m \in \mathbb{Z}$ there is an r such that $|m-r| < L$ implies that $d_l\left(S^m(c_H), S^r(c_H)\right) < \varepsilon$.

We list some properties of almost periodic sequences and Jacobi operators.

Theorem 9.2.1. *1. If H is almost periodic and $c_{H'} \in \Theta_{c_H}$ then $\Theta_{c_H} = \Theta_{c_{H'}}$. Hence $\{S^n(x)\}_{n\in\mathbb{Z}}$ is dense in Θ_{c_H} for all $x \in \Theta_{c_H}$.*

2. Let $(a_n)_{n=-\infty}^\infty$ be a real-valued almost periodic sequence with $\liminf_{n\to\infty} a_n = c$. Then for every $x = (x_n)_{n=-\infty}^\infty$ in the closure of $\{(a_{n+k})_{n=-\infty}^\infty : k \in \mathbb{Z}\}$ we have,

$$\inf_{n\in\mathbb{Z}} x_n = \liminf_{n\to\infty} x_n = \liminf_{n\to-\infty} x_n = c$$

3. Let H be almost periodic and $f : (a'_n,b'_n)_{n=-\infty}^\infty \to a'_0$ for $(a'_n,b'_n)_{n=-\infty}^\infty \in \Theta_{c_H}$. If

$$A(\Theta_{c_H}) := \int \log f((a'_n,b'_n)_{n=-\infty}^\infty) dm_{\Theta_{c_H}} > -\infty$$

then $a'_n > 0$ for all $n \in \mathbb{Z}$ for $m_{\Theta_{c_H}}$ almost every c'.

4. If H is almost periodic then there is a $\tilde{\nu}$ such that for all $c_{H'} \in \Theta_{c_H}$ the DOS measure for $H'(a'_n,b'_n)$ is $\tilde{\nu}$ provided that $a'_n > 0$ for all $n \in \mathbb{N}$. Moreover,

$$\sigma(H') = \sigma_{\text{ess}}((H')^+) = \text{supp}(\tilde{\nu}). \tag{9.3}$$

Proof. From the above discussion, (1) and (2) easily follow. For (3), see Section 1 in [89]. In (4), $\sigma(H') = \operatorname{supp}(\tilde{\nu})$ can be found in Section 2 in [65]. By Proposition 1.8 in [114] we have $\operatorname{supp}(\tilde{\nu}) \subset \sigma_{\text{ess}}((H')^+)$ in (4). But since $\sigma_{\text{ess}}((H')^+) \subset \sigma(H')$ by Lemma 3.7 in [123] we have (9.3). □

For a given $c = (a_n, b_n)_{n=-\infty}^{\infty}$ with $a_n \neq 0$ for $n \geq 0$, let

$$M_n(z) = \begin{pmatrix} (z - b_n)/a_n & -a_{n-1}/a_n \\ 1 & 0 \end{pmatrix},$$

and

$$M^{(n)}(z) = M_n(z) \cdots M_1(z),$$

for $z \in \mathbb{C}^+$. If H is almost periodic and $A(\Theta_{c_H}) > -\infty$ then there is a function $\tilde{\gamma}$ (see e.g. (3.6) in [42]) such that

$$\tilde{\gamma}(z) = \lim_{n \to \infty} \frac{1}{n} \log \|M^{(n)}(z)\|$$

for $m_{\Theta_{c_H}}$ almost every $c_{H'}$. In particular, $\lim_{n\to\infty} \log{(a_1 \cdots a_n)^{1/n}} = \lim_{n\to\infty}(1/n)\sum_{k=1}^{n} \log a_k = A(\Theta_{c_H})$ for $m_{\Theta_{c_H}}$ almost every $c_{H'}$. Moreover, we also have $\tilde{\gamma}(z) = \int \log|z - t| d\tilde{\nu}(t) - A(\Theta_{c_H}) > 0$. In addition,

$$\gamma(x) := \lim_{y \to 0^+} \tilde{\gamma}(x + iy)$$

exists for all $x \in \mathbb{R}$, see Section 2 in [69]. Let $\gamma := \tilde{\gamma}$ on $\mathbb{C}^+$. Then the function γ which is defined on $\mathbb{C} \cup \mathbb{R}$ is called the *Lyapunov exponent*. See Section 7 of [114] and [74] for more on these concepts.

The next result concerning regular measures will be used when we discuss Jacobi matrices on Julia sets. We have $\operatorname{Cap}(\operatorname{supp}(\mu)) = \operatorname{Cap}(\operatorname{ess\,supp}(\mu))$, see Section 1 of [114]. Thus, for an almost periodic H, we have $\operatorname{Cap}(\sigma(H)) = \operatorname{Cap}(\sigma(H^+))$ by part (4) of Theorem 9.2.1.

Proposition 9.2.2. *Let $H((a_n, b_n)_{n=-\infty}^{\infty})$ be almost periodic and $\operatorname{Cap}(\sigma(H)) > 0$. Moreover, let H^+ be regular. Then $A(\Theta_{c_H}) > -\infty$ and $\gamma = g_{\overline{\mathbb{C}} \setminus \sigma(H)}$ on $\mathbb{C}^+$. As a corollary $\gamma = 0$ quasi-everywhere on $\sigma(H)$. Besides, for $m_{\Theta_{c_H}}$ almost every $c_{H'}$, $(H')^+$ is regular in the sense of Stahl-Totik.*

Proof. We have $\lim_{n\to\infty}(a_1\cdots a_n)^{1/n} = \operatorname{Cap}(\sigma(H))$ by regularity and almost periodicity of H^+. Let $H^\varepsilon = H((a_n+\varepsilon, b_n)_{n=-\infty}^{\infty})$ for some $\varepsilon > 0$. Then H^ϵ is also almost periodic and $\liminf_{n\to\infty}(a_n+\varepsilon) > 0$. This implies that $A(\Theta_{c_{H^\varepsilon}}) > -\infty$. Besides, $m_{\Theta_{c_{H^\varepsilon}}}$ is the corresponding image measure, that is

$$\int f d\, m_{\Theta_{c_{H^\varepsilon}}} = \int f\circ(H+\epsilon H_0) d\, m_{\Theta_{c_H}}$$

where H_0 is the Jacobi matrix with $a_n \equiv 1$ and $b_n \equiv 0$. This implies that $m_{\Theta_{c_{H^\varepsilon}}} \to m_{\Theta_{c_H}}$ as $\varepsilon \to 0$, see Section 2 in [69]. By Lemma 5.1 in [69], $\limsup_{\epsilon\to 0^+} A(\Theta_{c_{H^\varepsilon}}) \leq A(\Theta_{c_H})$. Since $\liminf_{n\to\infty}(a_n+\varepsilon) > 0$, by Theorem 7.1 (f) in [114], $\exp^{A(\Theta_{c_{H^\varepsilon}})} = \lim_{n\to\infty}((a_1+\varepsilon)\cdots(a_n+\varepsilon))^{1/n}$ holds. Thus,

$$\exp^{A(\Theta_{c_H})} \geq \operatorname{Cap}(\sigma(H)) > 0. \tag{9.4}$$

Moreover, by part (4) of Theorem 9.2.1, we have $\tilde{\nu} = \mu_{\sigma(H)}$.

Now, consider the constant function

$$h(z) := \gamma(z) - g_{\sigma(H)}(z) = -A(\Theta_{c_H}) + \log \operatorname{Cap}(\sigma(H))$$

on $\mathbb{C}^+$. Since $g_{\sigma(H')}$ can be as close to 0 as we wish and $\gamma > 0$, h should be non-negative on $\mathbb{C}^+$. This gives

$$\exp^{A(\Theta_{c_H})} \leq \operatorname{Cap}(\sigma(H)). \tag{9.5}$$

By (9.4) and (9.5), $h(z) = 0$ and $\gamma = g_{\sigma(H)}$ on $\mathbb{C}^+$. Since $\lim_{y\to 0^+} g_{\sigma(H)}(x+iy) = 0$ quasi-everywhere on $\sigma(H)$ (see e.g. p. 53-54 in [110]) we have $\gamma = 0$ quasi-everywhere on $\sigma(H)$.

By (9.4) and (9.5), $A(\Theta_{c_H}) = \log \operatorname{Cap}(\sigma(H))$. Thus $\lim_{n\to\infty}(a_1\cdots a_n)^{1/n} = \operatorname{Cap}(\sigma(H))$ for $m_{\Theta_{c_H}}$ almost every $c_{H'}$. This proves the last statement. □

If we are given H, then H is called *reflectionless* on $A \subset \mathbb{R}$ if

$$\lim_{\epsilon\to 0^+} \langle \delta_n, (H - z - i\epsilon)^{-1}\delta_n\rangle = 0$$

for almost every $z \in A$ and for all $n \in \mathbb{Z}$.

If $H((a_n, b_n)_{n=-\infty}^{\infty})$ is almost periodic, $A(\Theta_{c_H}) > -\infty$ and $\gamma = 0$ quasi-everywhere on $\sigma(H)$ then for $\mu_{\Theta_{c_H}}$ almost every $c_{H'}$, H' is reflectionless on $\sigma(H)$ in view of Kotani's theory, see [76], [14]. Moreover if $|\sigma(H)| > 0$ then being reflectionless implies the existence of a non-trivial absolutely continuous part, see e.g. Theorem 7.4.2 in [116]. In this case, we have $inf_{n\in\mathbb{Z}} a_n > 0$ by the following argument: Let us assume that $inf_{n\in\mathbb{Z}} a_n = 0$. Then by part (2) of Theorem 9.2.1, $\liminf_{n\to\infty} a_n' = \liminf_{n\to-\infty} a_n' = 0$ for almost every $c_{H'(a_n',b_n')_{n=-\infty}^{\infty}}$ in Θ_{c_H}. This implies by [53] that, $(H')^+$ and $(H')^-$ does not have an absolutely continuous spectrum for almost every $c_{H'(a_n',b_n')_{n=-\infty}^{\infty}}$ in Θ_{c_H}. Since $\sigma_{\rm ac}(H') = \sigma_{\rm ac}((H')^+) \cup \sigma_{\rm ac}((H')^-)$ holds true (see e.g. Lemma 3.11 in [123]), for almost every $c_{H'(a_n',b_n')_{n=-\infty}^{\infty}}$, H' has a non-trivial absolutely continuous part. This is a contradiction.

Now let us sketch the proof of a result which is a generalization of Proposition 4.1 in [77] (we skip one of the equivalent statements). The proof is almost identical.

Theorem 9.2.3. *Let K be a Parreau-Widom set such that DCT holds on $\overline{\mathbb{C}} \setminus K$ and H be a Jacobi operator with $\sigma(H) = K$. Then the following conditions are equivalent.*

(i) H is reflectionless on K.

(ii) H is almost periodic and H^+ and H^- are regular.

Proof. (i) $\Longrightarrow$ (ii) Almost periodicity of reflectionless operators in this case is a result of Sodin and Yuditskii [119], see also Section 1 in [44]. Regularity of H^+ and H^- can be found in Corollary 2.1 in [44].

(ii) $\Longrightarrow$ (i) By Proposition 9.2.2, we have $\gamma = 0$ almost everywhere on K. Then the proof is the same with the proof of (2) $\Longrightarrow$ (1) in Proposition 4.1 of [77]. We note that Remling's Theorem (Theorem 1.1 in [44] and see also Theorem 1.4 in [107]) is applicable on Parreau-Widom sets since (9.1) holds. $\square$

Two sided Jacobi operators satifying the hypotheses of Theorem 9.2.3 are called the *isospectral torus* of K which we denote by $\mathrm{IT}(K)$. Under some conditions on K, it is homeomorphic to a torus, see [108].

There are Parreau-Widom sets such that the equivalence in Theorem 9.2.3 does not hold. The following remarkable result was proven by Volberg and Yudistkii, see Theorem 1.7 and Theorem 1.8 in [139].

Theorem 9.2.4. *Let $K = [b_0, a_0] \setminus \cup_{j\geq 1}(e_j, f_j)$ be a Parreau-Widom set such that DCT does not hold on $\overline{\mathbb{C}} \setminus K$. Suppose that the following conditions hold:*

(i) Every reflectionless operator H on K with $\sigma(H) = K$ has purely absolutely continuous spectrum.

(ii) The frequencies $\{\mu_K([b_0, e_j])\}_{j\geq 1}$ are rationally independent.

Then none of the reflectionless Jacobi operators on K whose spectrum is K are almost periodic.

Examples of Parreau-Widom sets satisfying the hypotheses of Theorem 9.2.4 can be found in Section 1.3 of [139]. The following result is a corollary of Theorem 9.2.4.

Theorem 9.2.5. *Let K be a Parreau-Widom set satisfying all assumptions of Theorem 9.2.4. Then there is no H with $\sigma(H) = K$ such that it is almost periodic and H^+ is regular.*

Proof. Assume to the contrary that such an H exists. Then $\gamma(z) = 0$ almost everywhere (quasi-everywhere implies almost everywhere, see p. 198 in [106]) on K by Proposition 9.2.2. Then by Kotani's theory, this implies that for $\mu_{\Theta_{c_H}}$ almost every $c_{H'}$, H' is reflectionless on $\sigma(H)$. By part (4) of Theorem 9.2.1, $\sigma(H) = \sigma(H')$ holds for each H' associated with an element in Θ_{c_H}. This is impossible by Theorem 9.2.4 since all Jacobi operators associated with an element in Θ_{c_H} are almost periodic. $\square$

Comparing how different Jacobi operators may behave depending on DCT, the sets that satisfy the assumptions in Theorem 9.2.4 can be considered as the representative of the bad sets in terms of spectral theory of orthogonal polynomials.

9.3 Szegő Class

Recall that, a Jacobi operator $H^+(a_n, b_n)_{n=1}^{\infty}$ is called asymptotically almost periodic if there is an almost periodic Jacobi operator $H(a'_n, b'_n)_{n=-\infty}^{\infty}$ such that $\limsup_{n\to\infty}(|a_n - a'_n| + |b_n - b'_n|) = 0$. In this case, we call $H(a'_n, b'_n)_{n=-\infty}^{\infty}$ the almost periodic limit.

Let K be a Parreau-Widom set such that DCT holds on $\overline{\mathbb{C}} \setminus K$. If $\mu \in \mathrm{Sz}(K)$ then by Theorem 1.2 in [44], $H^+(\mu)$ is asymptotically almost periodic. By Proposition 2.1 in [44], if $H \in \mathrm{IT}(K)$ then the spectral measure of H^+ belongs to the Szegő class of K.

For the Szegő class of a Parreau-Widom set, we have a result similar to Theorem 9.2.5.

Theorem 9.3.1. *Let K be a Parreau-Widom set satisfying all assumptions of Theorem 9.2.4. If $\mu \in \mathrm{Sz}(K)$ then $H^+(\mu)$ cannot be asymptotically almost periodic.*

Proof. Assume that there is a measure $\mu \in \mathrm{Sz}(K)$ such that $H^+(\mu) = H^+(a_n, b_n)_{n=1}^{\infty}$ is asymptotically almost periodic with the almost periodic limit $H'(a'_n, b'_n)_{n=-\infty}^{\infty}$.

Since μ has a non-trivial absolutely continuous part, by [53] there is a $d > 0$ such that $\inf_{n\in\mathbb{N}} a_n = d$. This implies that $\liminf_{n\to\infty} a'_n \geq d$. By part (2) of Theorem 9.2.1, we also have $inf_{n\in\mathbb{Z}} a'_n \geq d > 0$. Thus $(H')^+$ is well defined. Moreover, by Theorem 7.1 (f) in [114], there is a positive number A such that

$$A = \lim_{n\to\infty} (a_1' \cdots a_n')^{1/n}. \tag{9.6}$$

By Lemma 3.9 in [123], $H^+(\mu)$ and $(H')^+$ have the same essential spectra which implies that $\sigma(H') = K$ by part (4) of Theorem 9.2.1. Now, let us show that $(H')^+$ is regular which contradicts with Theorem 9.2.5.

By regularity of μ and (9.6), there is a $C > 0$ such that

$$C = \lim_{n\to\infty} \left(\frac{a_1 \cdots a_n}{a_1' \cdots a_n'} \right)^{1/n}. \tag{9.7}$$

If we let $r_n := a_n/a_n'$ for $n \in \mathbb{N}$ in (9.7) and take the logarithm of both sides then we obtain

$$\log C = \lim_{k\to\infty} \frac{1}{k} \sum_{n=1}^{k} \log r_n.$$

But since $\log r_n \to 0$ as $n \to \infty$ and the partial sums $\frac{1}{k}\sum_{n=1}^{k} \log r_n$, for k=1,..., are Cesaro means of $(\log r_n)$ we have $\log C = 0$ and thus $C = 1$. Since $H^+(\mu)$ and $(H')^+$ have the same essential spectra, $C = 1$ implies the regularity of the latter.

$\square$

Since $\mu_K \in \mathrm{Sz}(K)$, Theorem 9.3.1 implies that there are some Parreau-Widom sets K for which $H^+(\mu_K)$ is not asymptotically almost periodic.

9.4 The Szegő class and the isospectral torus of a generic set

Before surveying the known examples for zero measure case, we first give definitions of Szegő class and isospectral torus on generic sets (we discuss below what we mean by a generic set). For previous suggestions for the definition of isospectral torus, we refer the reader to [77], [88]. Our approach will be very similar to that of [77].

Our definitions should be compatible with the definitions in Parreau-Widom case. For this, we impose the conditions of non-polarity and regularity with respect to the Dirichlet problem to the sets we consider. The sets satisfying these mild assumptions are our generic sets. For the Szegő class we suggest the following definition.

Definition 9.4.1. Let K be a non-polar compact subset of $\mathbb{R}$ which is regular with respect to the Dirichlet problem. Then $\mu \in \mathrm{Sz}(K)$ if

(i) $\operatorname{ess\,supp}(\mu) = K$.

(ii) $\inf_{n\in\mathbb{N}} W_n^2(\mu) > 0$.

(iii) the isolated points $\{x_n\}$ of $\operatorname{supp}(\mu)$ satisfy $\sum_k g_K(x_n) < \infty$.

We remark that, if K is taken to be equal to a Parreau-Widom set then this definition coincides with the definition given in Section 3. We choose the property $\inf_{n\in\mathbb{N}} W_n^2(\mu) > 0$ instead of $\sup_{n\in\mathbb{N}} W_n^2(\mu) > 0$ in order to have a smaller class of measures.

As a corollary of Theorem 1.2.1 we have the following result.

Corollary 9.4.2. *Let K be a non-polar subset of $\mathbb{R}$ which is regular with respect to the Dirichlet problem. Then $\mu_K \in \mathrm{Sz}(K)$.*

Proof. Since K is regular, combining Theorem 4.2.3 in [106] and Theorem 5.5.13 in [116] we obtain $\operatorname{supp}(\mu_K) = K$. The condition $\inf_{n\in\mathbb{N}} W_n^2(\mu) > 0$ holds by Theorem 1.2.1. The condition concerning isolated points automatically holds since μ_K does not have isolated points. $\square$

This result implies that the Szegő class of a generic set is always non-empty.

Next, let us discuss the isospectral torus.

Definition 9.4.3. Let K be a non-polar compact subset of $\mathbb{R}$ which is regular with respect to the Dirichlet problem. Then a two-sided Jacobi operator H is in $\mathrm{IT}(K)$ if

(i) $\sigma(H) = K$.

(ii) H is almost periodic.

(iii) There is a $c_{H'} \in \Theta_{c_H}$ such that $(H')^+$ is regular in the sense of Stahl-Totik.

This definition is almost identical with the definition suggested in p. 83 of [77]. Our definition allows that $a_n = 0$ for some n. Thus, there may be some $c_{\tilde{H}} \in \Theta_{c_H}$ such that $\left(\tilde{H}\right)^+$ is not regular. Nevertheless by Proposition 9.2.2 for $m_{\Theta_{c_H}}$ almost every $c_{H'}$, we have that H' is two-sided regular with $a_n \neq 0$ for all $n \in \mathbb{N}$. Here, the emphasis is on the hull rather than each element of the hull. This will allow us to cover some interesting examples (see Section 5) of operators in the definition of the isospectral torus which are not covered by the definition in [77].

We remark that this definition coincides with the definition discussed in Section 2, as easy to check, if K is taken to be a finite gap set. Moreover, $\mathrm{IT}(K) = \emptyset$ by Theorem 9.2.5 if K satisfies the assumptions of Theorem 9.2.4.

There are some interesting examples of almost periodic Jacobi operators discussed in the literature many times, see e.g. [16]. In our discussion of the concepts of Szegő class and isospectral torus, we fix the set first instead of choosing an operator. Hence, we disregard these kind of examples. We focus on three cases in the next section: Cantor ternary set, polynomial Julia sets and generalized polynomial Julia sets.

9.5 Three examples

9.5.1 Cantor ternary set

For a general treatment of iterated function systems, we refer the reader to [20], [70].

Let $w_1(x) = x/3$ and $w_2(x) = (x+2)/3$. Then the Cantor ternary set K_0 is the unique solution of $K = \cup_{j\in 1,2} w_j(K)$ among the subsets of $[-1,1]$. The Cantor-Lebesgue measure η_{K_0} is the unique unit Borel measure satisfying

$$\int_{K_0} f\,d\mu = \frac{1}{2}\sum_{j=1}^{2}\int_{K_0} (f\circ w_j)\,d\mu$$

for all $f \in C(K_0)$. We have $\operatorname{supp}(\eta_{K_0}) = K_0$ and $|K_0| = 0$. It is easy to verify that η_{K_0} is purely singular continuous.

In [77] scaled and translated versions of K_0 and η_{K_0} were under investigation. We do not make any distinctions below for the usual Cantor set and scaled and translated versions since these results are invariant under such operations.

Some results concerning the moments $\int t^n\,d\eta_{K_0}(t)$ can be found in [20]. A numerically stable algorithm for calculating the recurrence coefficients for η_{K_0} was given in [85] by Mantica. In [68] some properties regarding orthogonal polynomials associated with η_{K_0} were tested numerically. It was conjectured in [86] that these coefficients are asymptotically almost periodic. This conjecture was repeated in [77] (Conjecture 3.1) by checking the behavior of the first 100000 coefficients.

It is due to Białas-Cież and Volberg that the Cantor ternary set is regular with respect to the Dirichlet problem, see [34]. Regularity of η_{K_0} in the sense of Stahl-Totik was proven in [77]. If the conjecture on the recurrence coefficients regarding almost periodicity is correct then there is an almost periodic Jacobi operator H (which is the almost periodic limit of $H^+(\eta_{K_0})$) such that H^+ is regular and $\sigma(H) = K_0$. Thus, we have $\mathrm{IT}(K_0) \neq \emptyset$.

In [77], the behavior of $(W_n^2(\mu))_n$ was numerically examined. It was conjectured that (Conjecture 3.2)

$$0 < \inf_{n\in\mathbb{N}} W_n^2(\eta_{K_0}) \le \sup_{n\in\mathbb{N}} W_n^2(\eta_{K_0}) < \infty. \tag{9.8}$$

Hence $\eta_{K_0} \in \mathrm{Sz}(K_0)$ provided that (9.8) holds.

By Theorem 6.4 in [77], for $\beta > 0$, the spectral measure for $H^+(\eta_{K_0}) + \beta\langle\delta_1, \cdot\rangle\delta_1$ is purely discrete. This shows that, unlike the absolutely continuous part (see e.g. Section 7.3 in [116]), the singular continuous part of the spectral measure of a Jacobi operator is not preserved under finite rank perturbations.

9.5.2 Polynomial Julia sets

For some of the results on the orthogonal polynomials associated with equilibrium measures, see [22], [23], [29]. There are some other results when the measure is not the equilibrium measure, see e.g. [30], [73].

We want to consider the simplest case that is $f(z) = z^2 - c$ with $c > 2$. For such an f, $J(f)$ is a Cantor set on $\mathbb{R}$ and $|J(f)| = 0$, see e.g. Section 12 in [36]. The recurrence coefficients for $\mu_{J(f)}$ can be calculated recursively by the following formulas, see p. 89 of [29]:

$$\begin{aligned} a_1 &= \sqrt{c}, n \ge 1 \\ a_{2n}^2 a_{2n-1}^2 &= a_n^2, \\ a_{2n}^2 + a_{2n+1}^2 &= c \\ b_n &= 0. \end{aligned}$$

Let $c \ge 3$. Then $H^+\left(\mu_{J(f)}\right)$ is almost periodic, see p. 93 of [29]. Note that, the DOS measure ν for $\mu_{J(f)}$ is again $\mu_{J(f)}$ by regularity of equilibrium measures, see e.g. Theorem 1.12 in [114]. Since $J(f)$ is regular with respect to the Dirichlet problem, $\mathrm{supp}\left(\mu_{J(f)}\right) = J(f)$. Let us denote the almost periodic extension of this operator to $l^2(\mathbb{Z})$ by $H(\mu_{J(f)})$. In this extension we have $a_0 = 0$. Some properties of $H^-(\mu_{J(f)})$ were studied in [117]. We have $\sigma(H(\mu_{J(f)})) = J(f)$ by part (4) of

Theorem 9.2.1. Since $H^+\left(\mu_{J(f)}\right)$ is regular in the sense of Stahl-Totik we have $H(\mu_{J(f)}) \in \mathrm{IT}(J(f))$. By Corollary 9.4.2, we also have $\mu_{J(f)} \in \mathrm{Sz}(J(f))$.

Another interesting property of this family of Jacobi operators is that the spectral measure for $H^+_{\restriction 1}\left(\mu_{J(f)}\right)$ is purely discrete, see Theorem 3 in [23]. Thus, the singular continuous part of the spectrum is not preserved under such an operation. We remark that if μ is a measure with a non-trivial absolutely continuous part, then absolutely continuous spectra of $H^+(\mu)$ and $H^+_{\restriction 1}(\mu)$ coincide, see Section 7.3 in [116].

9.5.3 Generalized polynomial Julia sets

Depending on numerical evidence, we conjecture in Chapter 7 that $H^+(\mu_{K(\gamma)})$ is asymptotically almost periodic for all γ. If this conjecture is correct, then $\mathrm{IT}(K(\gamma)) \neq \emptyset$ by a similar argument used in Section 5.1 in order to show $\mathrm{IT}(K_0) \neq \emptyset$.

Chapter 10

Some Open Problems and Conjectures

It was conjectured in p. 123 of [26] that if f is a non-linear polynomial such that $J(f)$ is a totally disconnected subset of $\mathbb{R}$ then the recurrence coefficients for $\mu_{J(f)}$ are almost periodic. This is still an open problem. In Chapter 7, we conjecture that the recurrence coefficients for $\mu_{K(\gamma)}$ are asymptotically almost periodic for any γ. It may be hoped that a more general and slightly weaker version of Bellissard's conjecture can be valid.

Problem 10.0.1. Let (f_n) be a regular polynomial sequence such that $J_{(f_n)}$ is a Cantor-type subset of the real line. Prove that the recurrence coefficients for $\mu_{J(f_n)}$ are asymptotically almost periodic.

Problem 10.0.2. Let K be a non-polar compact subset of $\mathbb{R}$. Is there a general relation between the zero spacing of orthogonal polynomials for μ_K and smoothness of g_{Ω_K}? Is there a relation between the zero spacing of μ_K and the Markov factors?

As mentioned in Chapter 1, the Szegő condition and the Widom condition are equivalent on Parreau-Widom sets. Let K be a Parreau-Widom set. Let μ be a measure such that $\operatorname{ess\,supp}(\mu) = K$ and the isolated points $\{x_n\}$ of $\operatorname{supp}(\mu)$ satisfy $\sum_n g_{\Omega_K}(x_n) < \infty$. Then, the Szegő condition is equivalent to the condition

$$\int_K \log(d\mu/d\mu_K)\, d\mu_K(x) > -\infty. \tag{10.1}$$

This condition is also equivalent to the Widom condition under these assumptions.

We show in Chapter 8 that $\inf_{n\in\mathbb{N}} W_n(\mu_K) \geq 1$ for non-polar compact $K \subset \mathbb{R}$. Thus, the Szegő condition in the form (10.1) and the Widom condition are related on arbitrary non-polar compact sets.

Problem 10.0.3. Let K be a non-polar compact subset of $\mathbb{R}$ which is regular with respect to the Dirichlet problem. Let μ be a measure such that $\operatorname{ess\,supp}(\mu) = K$. Assume that the isolated points $\{x_n\}$ of $\operatorname{supp}(\mu)$ satisfy $\sum_n g_{\Omega_K}(x_n) < \infty$. If the condition (10.1) is valid for μ is it necessarily true that the Widom condition or the Widom condition 2 holds? Conversely, does the Widom condition imply (10.1)?

Problem 10.0.4. Is it possible to find a regular non-polar compact subset K of $\mathbb{R}$ which is not Parreau-Widom but $(W_n(K))_{n=1}^\infty$ is bounded? If K has zero Lebesgue measure then is it true that $(W_n(K))_{n=1}^\infty$ is unbounded? We can ask the same problems if we replace $(W_n(K))_{n=1}^\infty$ by $(W_n^2(\mu_K))_{n=1}^\infty$ above.

To our knowledge, in all known cases when $(W_n^2(\mu_K))_{n=1}^\infty$ is bounded, $(W_n(K))_{n=1}^\infty$ is also bounded. Thus, it is plausible to make the following conjecture (see also Conjecture 7.4.2):

Conjecture 10.0.5. *Let K be a non-polar compact subset of $\mathbb{R}$. Then $(W_n^2(\mu_K))_{n=1}^\infty$ is bounded if and only if $(W_n(K))_{n=1}^\infty$ is bounded.*

For $K(\gamma)$, it is true that $\mu_{K(\gamma)}([0,c]) \in \mathbb{Q}$ if $c \in \mathbb{R} \setminus K(\gamma)$, see Chapter 1.

Problem 10.0.6. Let K be a regular non-polar compact subset of $\mathbb{R}$ and α be the leftmost end point of K. Let $\mu_K([\alpha, c]) \in \mathbb{Q}$ for all $c \in \mathbb{R} \setminus K$. What can we say about K? Is it necessarily a polynomial generalized Julia set? Does this imply that there is a sequence of admissible polynomials $(f_n)_{n=1}^\infty$ such that $(F_n^{-1}[-1,1])_{n=1}^\infty$ is a decreasing sequence of sets such that $K = \cap_{n=1}^\infty F_n^{-1}[-1,1]$?

Problem 10.0.7. Let K be a non-polar compact subset of $\mathbb{R}$ such that μ_K is equal to a Hausdorff measure restricted to K. Is it necessarily true that the Hausdorff dimension of K is 0?

For polynomial Julia sets which are totally disconnected there is a formula for $\dim(\mu_{J(f)})$, see e.g.p. 23 in [82] and p.176-177 in [104].

Problem 10.0.8. Is it possible to find simple formulas for $\dim\left(\mu_{J_{(f_n)}}\right)$ where (f_n) is a regular polynomial sequence?

In order $(W_n(K))_{n=1}^{\infty}$ to be bounded, there are sufficient conditions given in terms of the smoothness of the outer boundary of K in [12, 133]. There is also an old and open question (we consider this as an open problem since we could not find any concrete examples in the literature although in [102], Pommerenke says that "D. Wrase in Karlsruhe has shown that an example constructed by J. Clunie [Ann. of Math., 69 (1959), 511-519] for a different purpose has the required property.") proposed by Ch.Pommerenke in [102] which is in the inverse direction: Find (if it is possible) a continuum K with $\mathrm{Cap}(K) = 1$ such that $(W_n(K))_{n=1}^{\infty}$ is unbounded. To answer this question in the affirmative, it is natural to consider a continuum with a nonrectifiable outer boundary. Thus, we make the following conjecture:

Conjecture 10.0.9. *Let $f(z) = z^2 + 1/4$. Then, $(W_n(J(f))_{n=1}^{\infty}$ is unbounded.*

By Theorem 1 in [66], for $f(z) = z^2 + 1/4$, $J(f)$ has Hausdorff dimension greater than 1 and in this case (see e.g. p. 130 in [41]) $J(f)$ is not a quasicircle. Hence, Theorem 2 of [12] is not applicable for $J(f)$ since it requires even stronger assumptions on the outer boundary.

Problem 10.0.10. Theorem 9.2.5 implies the existence of positive measure sets with empty isospectral torus. In Section 9.5.2, a zero Lebesgue measure set with non-empty isospectral torus is included. Is there a zero measure non-polar compact subset K of $\mathbb{R}$ which is regular with respect to the Dirichlet problem satisfying $\mathrm{IT}(K) = \emptyset$? If this is the case, is there a general condition on zero measure sets which is similar to DCT condition?

Problem 10.0.11. What measures other than the equilibrium measure belong to the Szegő class of a generic set? Is there a generic set K with $|K| = 0$ such that $H^+(\mu_K)$ is not asymptotically almost periodic? If there is, does it imply that $\mathrm{IT}(K) = \emptyset$?

Problem 10.0.12. What is the value of $\liminf a_n$ where $(a_n)_{n=1}^{\infty}$ is the sequence of recurrence coefficients for μ_{K_0}. If this value is 0 then by Corollary 1.2.2, $(W_n^2(\mu_{K_0}))_{n=1}^{\infty}$ is unbounded. It may be also true that $\liminf_{n\to\infty} a_n \neq 0$ but $(W_n^2(\mu_{K_0}))_{n=1}^{\infty}$ is unbounded.

A couple of algorithms for computing the recurrence coefficients associated with μ_{K_0} were discussed in [87]. Studying these coefficients, at least numerically, can give some ideas about general behavior of $(W_n^2(\mu_K))_{n=1}^{\infty}$ and $H^+(\mu_K)$ for generic zero Lebesgue measure sets. Moreover, a comparison of these results with the results obtained for the Cantor-Lebesgue measure in [68], [77], [85] is of particular interest since μ_{K_0} and η_{K_0} are mutually singular by [83].

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