MONOID ACTIONS, THEIR CATEGORIFICATION AND APPLICATIONS

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We certify that we have read this dissertation and that in our opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

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ABSTRACT

MONOID ACTIONS, THEIR CATEGORIFICATION AND APPLICATIONS

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We study actions of monoids and monoidal categories, and their relations with (co)homology theories. We start by discussing actions of monoids via bi-actions. We show that there is a well-defined functorial reverse action when a monoid action is given, which corresponds to acting by the inverses for group actions. We use this reverse actions to construct a homotopical structure on the category of monoid actions, which allow us to build the Burnside ring of a monoid. Then, we study categorifications of the previously introduced notions. In particular, we study actions of monoidal categories on categories and show that the ideas of action reversing of monoid actions extends to actions of monoidal categories. We use the reverse action for actions of monoidal categories, along with homotopy theory, to define homology, cohomology, homotopy and cohomotopy theories graded over monoidal categories. We show that most of the existing theories fits into our setting; and thus, we unify the existing definitions of these theories. Finally, we construct the spectral sequences for the theories graded over monoidal categories, which are the strongest tools for computation of cohomology and homotopy theories in existence.

Keywords: monoid, monoidal category, action, reversility, Burnside ring, stabilization, (co)homology, (co)homotopy, spectral sequence.

ÖZET

MONOİD ETKİLERİ, KATEGORİFİKASYONLARI VE UYGULAMALARI

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Monoidlerin ve monoidsel kategorilerin etkilerini, ve bu etkilerin (ko)homoloji teorilerine uygulamalarını çalıştık. Başlangıçta, monoidlerin çift taraflı etkilerini tanımladık. Bir monoid etkisinin tersinin funktöriyel olarak tanımlanabileceğini gösterdik. Öyle ki, monoid bir grup olduğunda bu ters etki, grup elemanlarının tersleri ile etki edildiğinde oluşan etkiye tekabül ediyor. Daha sonra, bu ters etkiyi kullanarak monoid etkilerinin kategorisinde homotopi teorisi tanımladık, ve bu homotopy teorisi monoidler için Burnside halkasını tanımlayabilmemizi sağladı. Devamında, başlangıçta ortaya koyduğumuz bazı tanımları kategorifiye ettik. Özellikle, monoidsel kategorilerin kategorilere olan etkilerini çalışıp, ilk bölümdeki ters etkinin monoidsel kategorilerin etkileri için de tanımlanabileceğini gösterdik. Bu ters etkileri, homotopi teorisi ile birlikte kullanarak, monoidsel kategorilerde indekslenen homoloji, kohomoloji, homotopi ve kohomotopi teorilerini tanımladık. Bu tanımlar, hali hazırda literatürde bulunan tüm tanımları ortak bir şekilde genelleştirmekte ve hepsini bir arada çalışabilmek için birleşmiş bir ortam oluşturmaktadır. En sonunda ise kohomoloji ve homotopi teorileri için en güçlü hesaplama aracı olan spektral dizileri, daha öncesinde tanımladığımız monoidsel kategorilerde indekslenen teoriler için inşa ettik.

Anahtar sözcükler: monoid, monoidsel kategori, etki, tersinirlik, Burnside halkası, stabilizasyon, (ko)homoloji, (ko)homotopi, spektral dizi.

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Contents

Chapter 1

Introduction

This thesis essentially consist of three different parts. In the first part, we studied monoid actions on sets via bi-actions. In the second part, we categorified the notions of first part and studied actions of monoidal categories on categories, again via bi-actions. We later use these ideas to study stabilization and (co)homology and (co)homotopy theories graded over monoidal categories. The strongest tools of computation for (co)homology and (co)homotopy theories are spectral sequences. In the third part, we construct spectral sequences for the homology and cohomology theories that are graded over an arbitrary monoidal category.

We start the first part by discussing monoid actions, which appear quite often as mathematical models of progressive processes. In computer science, for example, automata, or so called state machines, is just an monoid action on a finite set. In physics a (continuous) dynamical system can be seen as a monoid action on a space (by continuous maps).

One of the main objectives of the first part is determining the reversible parts of a monoid action. Observe that, once we are given a left group action on a set, we can define a right action on the same set by applying inverses of elements from left. This action is called the inverse or reverse action of the given one. Hence, once we are given a set with a left action of a monoid on it, one can expect that on symmetric parts acting from right reverses the original action. From this observation, we introduced a new approach to the notion of monoid action on a set, so that the monoid acts from both sides but the equivariance of maps is defined in the diagonal. This is particularly good for determining the subsets of a set where the monoid acts by isomorphisms. We construct the notion of the *reverse action* of a given monoid action on a set, as analogues to the group actions. This new action will be defined on a different set, it agrees with the above construction when the monoid is a group.

We show that the category of monoid actions possesses a homotopical category structure in the sense of [1], defined via the maximal reversible parts, see Section 4.6. We show that, on the subcategory of finite sets with an action of a monoid, this homotopical structure is nice enough to have a convenient homotopy category, as it admits a 3-arrow calculus; and thus, due to 27.5 of [1], it is saturated. Using its homotopy category, we define the Burnside ring of a monoid, with an injective Burnside character. This Burnside ring generalized the one defined in [2], and has nice use in automata theory.

In the second part of the thesis, motivated by the definitions of first part, we study actions of monoidal categories on categories. We introduced the reverse action for a monoidal category action on a category, defined parallel to the reverse action for monoid actions.

The main focus of the second part is to obtain of reverse action of a certain types of monoidal category actions, which is called suspension and looping actions. In these cases, reversing actions will correspond to deloopings and desuspensions. When such an action of a monoidal category on a homotopical category is given, the reverse action corresponds to the stabilization with respect to this action and we obtain a bi-functorial stabilization for a general homotopical category with an action of a monoidal category. We also introduce 'environment independent' definitions for (co)homology and (co)homotopy theories graded over a monoidal category. Basically, we just need two pointed homotopical categories $\mathcal{M}$ and $\mathcal{S}$, a monoidal category $\mathcal{I}$, and a suspension or a looping $\mathcal{I}$-action on $\mathcal{M}$.

If $\mathcal{S}$ is considered with the trivial action, then we define these theories via the reverse action of the action induced on the functor category $[\mathcal{M}, \mathcal{S}]$.

In this setting, the morphisms in the monoidal category can be considered as (co)homology operations. In other words, one can consider (co)homology theories graded over a monoidal category as usual (co)homology theories which are graded over a monoid together with a selected set of operations which satisfy certain relations.

The most advanced technologies for computation of (co)homology and (co)homotopy theories are spectral sequences. In the third part of this thesis, we construct a general very abstract form of spectral sequences and discuss under what conditions these spectral sequences converge. By using these abstract form, we construct (co)homology and (co)homotopy spectral sequences that can be used for computation of (co)homology and (co)homotopy theories graded over a monoidal category.

Chapter 2

Category Theoretical Preliminaries

In this section we discuss some category theoretical preliminaries according to our needs in later chapters.

2.1 Homotopical Categories

We follow [3] and [1] as a general reference for this section. We start by stating some basic definitions of homotopical algebra. A convenient way of presenting homotopy theory in a category is to have the concept of weak equivalences. Let $\mathcal{M}$ be a category. A class of morphisms $\mathcal{W}$ in $\mathcal{M}$ is said to have *the* 2*-out-of-*3 *property*, if for every pair of composable morphisms f, g in $\mathcal{M}$ with any two of the morphisms $\{f, g, g \circ f\}$ are in $\mathcal{W}$, then so is the third. If further $\mathcal{W}$ contains every isomorphism in $\mathcal{M}$, then $\mathcal{M}$ is called *a category with weak equivalences.* Every category can be made into a category with weak equivalences in two trivial ways; the first one by choosing isomorphisms as weak equivalences and the second one by choosing all morphisms as weak equivalences. Some other well known examples are weak homotopy equivalences and homotopy equivalences in the category of topological spaces. A slightly modified and stronger version of 2-out-of-3 property is the so called 2-*out-of*-6 *property*. A class of morphisms $\mathcal{W}$ in $\mathcal{M}$ is said to have the 2-out-of-6 property, if given composable morphisms f, g and h in M such that $h \circ g$ and $g \circ f$ are in W, then the morphisms f, g, h and $h \circ g \circ f$ are also in W. If further $\mathcal{W}$ has all the identity maps in $\mathcal{M}$, then $\mathcal{M}$ is called a *homotopical category*. It is easy to see that every homotopical category is a category with weak equivalences. The converse is not true though, and examples are plentiful.

Although very plain and simple, the 2-out-of-6 property becomes extremely useful combined with other easy properties, while trying to understand the homotopy category. All the examples mentioned in the previous paragraph also satisfy the 2-out-of-6 property.

2.1.1 Homotopy category of a category with weak equivalences

When a category with weak equivalences $\mathcal{M}$ is given, the primary concern of homotopy theory (or in particular homotopical algebra) is to formally invert these weak equivalences; that is, define a new category having the same class of objects as $\mathcal{M}$, but all of the weak equivalences become isomorphisms. This new category is called the homotopy category of $\mathcal{M}$ and denoted by $Ho(\mathcal{M})$. This assignment has to be done by a universal property. To be precise the homotopy category of $\mathcal{M}$ (with respect to the class of weak equivalences $\mathcal{W}$) is equipped with a functor

$$Q : \mathcal{M} \to Ho(\mathcal{M})$$

such that Q sends weak equivalences to isomorphisms and given any other functor $F : \mathcal{M} \to \mathcal{N}$ sending weak equivalences to isomorphisms, there exists a unique functor $\overline{F} : Ho(\mathcal{M}) \to \mathcal{N}$ such that $\overline{F} \circ Q = F$. This universal property is especially important in defining homotopy limits and homotopy colimits.

The class of weak equivalences in a category with weak equivalences is called *saturated* if every morphism that become isomorphism in the homotopy category is already a weak equivalence. This is a very nice property that a category with weak equivalences can have, since it means we know exactly which morphisms are inverted and which are not. The 2-out-of-6 property is very useful to show saturation in a category with weak equivalences. For example, if a homotopical category admits a 3-arrow calculus in the sense of 27.3 of [1], or it admits a calculus of fractions in the sense of [4], then we have the saturation. For the latter, saturation is in fact is the same as having the 2-out-of-6 property.

2.1.2 Homotopy limits and homotopy colimits

Let $\mathcal{M}$ be a category with weak equivalences $\mathcal{W}$. If $F : \mathcal{M} \to \mathcal{N}$ is a functor, then the *right Kan extension* of F along $G : \mathcal{M} \to \mathcal{K}$ is a functor

$$Ran_G F : \mathcal{K} \to \mathcal{N}$$

together with a natural transformation $\nu_{G,F}$ from $Ran_G F \circ G$ to F, which is universal as a morphism from $\tilde{F} : [\mathcal{K}, \mathcal{N}] \to [\mathcal{M}, \mathcal{N}]$ induced from $F \circ G$ to F. The definition of left Kan extensions can be obtained by replacing all categories with their opposite in the definition of right Kan extension.

Definition 2.1.1. Let $\mathcal{M}$ be a category with weak equivalences $\mathcal{W}$ and let $F : \mathcal{M} \to \mathcal{N}$ be a functor,

- the left derived functor L_F of F is defined as the right Kan extension of F along $Q : \mathcal{M} \to Ho(\mathcal{M})$
- the right derived functor R_F of F is defined as the left Kan extension of F along $Q : \mathcal{M} \to Ho(\mathcal{M})$

There are other equivalent definitions of homotopy colimit and homotopy limit, but we will use the derived functor perspective.

Definition 2.1.2. Let $\mathcal{M}$ be a category with weak equivalences and let $\phi : C \to \mathcal{M}$ be a functor from the small category C.

- The homotopy colimit of ϕ is defined as the image of image of ϕ under the left derived functor of the colimit functor

$$\operatorname{colim}_C : [C, \mathcal{M}] \to \mathcal{M}.$$

- The homotopy limit of ϕ is defined as image of image of ϕ under the right derived functor of the limit functor

$$\lim_C : [C, \mathcal{M}] \to \mathcal{M}.$$

The homotopy colimit of a diagram over the category

$$1 \longleftarrow 0 \longrightarrow 2$$

is known as *homotopy pushout.* Dually, the homotopy limit of a diagram over the category

$$1 \longrightarrow 0 \longleftarrow 2$$

is known as *homotopy pullback.* These special homotopy colimits and homotopy limits are particularly important for defining cofiber and fiber sequences.

2.1.3 Fiber, cofiber and exact sequences

An object $\varnothing$ in a category $\mathcal{M}$ is called an initial object if for every object A in $\mathcal{M}$, there exist a unique morphism $\varnothing \to A$. Dually, an object $*$ is called terminal object in $\mathcal{M}$ if for every object A, there exist a unique morphism $A \to *$. If these two objects are the same, then it is called a zero object.

Let $\mathcal{M}$ be a category with weak equivalences and with a zero object $*$.

Definition 2.1.3. A sequence of morphisms

$$A \xrightarrow{i} B \xrightarrow{p} C$$

in $\mathcal{M}$ is called a cofiber sequence if the diagram

$$\begin{array}{ccc} A & \xrightarrow{\text{i}} & B \\ \downarrow & & \downarrow \text{p} \\ * & \longrightarrow & C \end{array}$$

is a homotopy push-out and it is called a fiber sequence if it is a homotopy pull-back.

In the category Top_*, pointed topological spaces, with the usual weak homotopy equivalences, the definition of fiber and cofiber sequences given above coincide with the usual definitions of fiber and cofiber sequences in Top_*.

Definition 2.1.4. A sequence

$$A \xrightarrow{i} B \xrightarrow{p} C$$

in $\mathcal{M}$ is called exact if the sequence

$$Y \xrightarrow{f} B \xrightarrow{g} X$$

is simultaneously fiber and cofiber sequence whenever

$$Y \xrightarrow{f} B \xrightarrow{p} C$$

is a fiber and

$$A \xrightarrow{i} B \xrightarrow{g} X$$

is a cofiber sequence.

Proposition 2.1.1. *Let $\mathcal{A}b$ be the category of abelian groups with the isomorphisms are considered as weak equivalences. Then the above definition coincides with the definition of exact sequences of abelian groups.*

Proof. First, observe that when weak equivalences are all isomorphisms, then homotopy pullback and homotopy pushouts are just ordinary pullback and pushout. Let

$$A \xrightarrow{i} B \xrightarrow{p} C$$

be a sequence fitting in the following diagram.

$$\begin{array}{ccccc} & & ker(j) & \longrightarrow & * \\ & & \downarrow & & \downarrow \\ A & \xrightarrow{i} & B & \xrightarrow{j} & C \\ \downarrow & & \downarrow & & \\ * & \longrightarrow & coker(i) & & \end{array}$$

By definition of the pull-backs and pushouts, the sequence

$$ker(j) \longrightarrow B \xrightarrow{j} C$$

is a fiber and the sequence

$$A \xrightarrow{i} B \longrightarrow coker(i)$$

is a cofiber sequence. If the vertical sequence

$$ker(j) \longrightarrow B \longrightarrow coker(i)$$

is simultaneously fiber and cofiber sequence; the following diagram

$$\begin{array}{ccc} ker(j) & \longrightarrow & * \\ \downarrow & & \downarrow \\ B & \longrightarrow & coker(i) \end{array}$$

is simultaneously pull-back and pushout. Hence, $ker(j) = im(i)$, i.e.,

$$A \xrightarrow{i} B \xrightarrow{p} C$$

is an exact sequence. Converse is evident from above; and thus, for abelian group the definition of exact sequence is the same as the exact sequence of Definition 2.1.4. □

2.2 Monoidal categories

In this section we start with discussing basic definitions on monoidal categories. Basically a monoidal category is a category obtained by generalizing the idea of monoids to categories. It is a category with an associative monoid operation, together with unit object equipped with certain coherence conditions. The definitions given here are taken from [5], and we refer to there for further reading on monoidal categories.

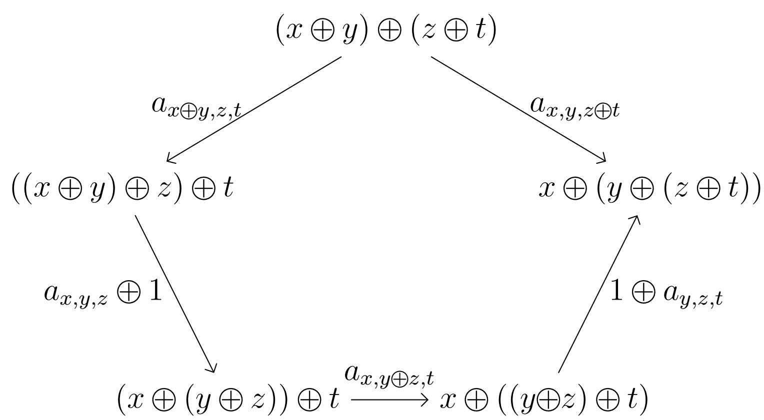
The formal definition of a monoidal category is as follows:

Definition 2.2.1. A monoidal category is a category $\mathcal{I} = (\mathcal{I}, \oplus, 1, \lambda, \rho, a)$ consisting

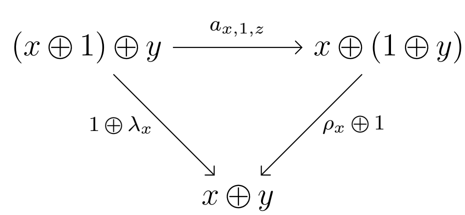
- a functor $\oplus : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$ called the monoidal product
- $1 \in \mathcal{I}$ unit object
- natural isomorphisms $\lambda_x : 1 \oplus x \to x$ and $\rho_x : x \oplus 1 \to x$
- a natural isomorphism $a_{x,y,z} : (x \oplus y) \oplus z \to x \oplus (y \oplus z)$ which gives the associativity

such that two coherence diagrams are commutative

- Pentagon coherence diagram:



- Triangle coherence diagram:



Examples 2.2.1. Some well known examples are as follows:

- *SET*: The category of sets, is an example with union as the monoid operation, empty set as unit.

- Top, $\mathcal{T}op_*$: The category of topological spaces (resp. pointed topological spaces), is an example with cartesian product (resp. smash product) as the monoid operation, the point $*$ (resp. S^0) as the unit.

- Any monoid can be seen as a monoidal category, with objects as elements and morphisms as the identity maps.

- Given a category A, the endofunctor category on A, $End(A)$ is a monoidal category with composition(either from left or right). Furthermore, $End(A)$ is a strict monoidal category, which means all morphisms giving associativity and units are identity for every object.

Definition 2.2.2. A braided monoidal category is a monoidal category $\mathcal{I}$ with a natural isomorphism $B_{x,y} : x \oplus y \to y \oplus x$, such that the following diagrams commute:

$$\begin{array}{ccccc} (x \oplus y) \oplus z & \xrightarrow{a_{x,y,z}} & x \oplus (y \oplus z) & \xrightarrow{B_{x,y\oplus z}} & (y \oplus z) \oplus x \\ \downarrow{\scriptstyle B_{x,y} \oplus id} & & & & \downarrow{\scriptstyle a_{y,z,x}} \\ (y \oplus x) \oplus z & \xrightarrow{a_{y,x,z}} & y \oplus (x \oplus z) & \xrightarrow{id \oplus B_{x,y}} & y \oplus (z \oplus x) \end{array}$$

and

$$\begin{array}{ccccc} x \oplus (y \oplus z) & \xrightarrow{a^{-1}_{x,y,z}} & (x \oplus y) \oplus z & \xrightarrow{B_{x\oplus y,z}} & z \oplus (x \oplus y) \\ \downarrow{\scriptstyle id \oplus B_{y,z}} & & & & \downarrow{\scriptstyle a^{-1}_{z,x,y}} \\ x \oplus (z \oplus y) & \xrightarrow{a^{-1}_{y,x,z}} & (x \oplus z) \oplus y & \xrightarrow{B_{x,z} \oplus id} & (z \oplus x) \oplus y \end{array}$$

commute, where a denote the associativity isomorphism. If further we have $B_{y,x} \circ B_{x,y} = id$ for all objects x, y, then $\mathcal{I}$ is called a symmetric monoidal category.

2.2.1 Monoidal functors

A monoidal functor is a functor that preserves the underlying monoidal product, analogues to a homomorphism preserving the monoid operation. However, the definition of a monoidal functor a little bit more complicated since it should preserves the coherence conditions as well. The following definition of monoidal functor is also called strong monoidal functor. Here in this thesis, all monoidal functors are assumed to be strong.

Definition 2.2.3. Given two monoidal categories

$$\mathcal{I} = (\mathcal{I}, \oplus, 1_{\mathcal{I}}, \lambda, \rho, a) \ \text{ and } \mathcal{J} = (\mathcal{J}, \odot, 1_{\mathcal{J}}, \hat{\lambda}, \hat{\rho}, \hat{a}),$$

a functor $F : \mathcal{I} \to \mathcal{J}$ together with natural isomorphisms

$$\psi_{x,y} : F(x) \odot F(y) \to F(x \oplus y)$$

and a morphism

$$\phi : 1_j \to F(1_{\mathcal{I}}),$$

is called a monoidal functor, if for every three objects x, y and z in $\mathcal{I}$ the diagrams

$$\begin{array}{ccc}
(F(x) \odot F(y)) \odot F(z) & \xrightarrow{\hat{a}_{F(x),F(y),F(z)}} & F(x) \odot (F(y) \odot F(z)) \\
\downarrow{\scriptstyle \psi_{x,y} \odot 1_J} & & \downarrow{\scriptstyle 1_J \odot \psi_{y,z}} \\
F(x \oplus y) \odot F(z) & & F(x) \odot F(y \oplus z) \\
\downarrow{\scriptstyle \psi_{x \oplus y, z}} & & \downarrow{\scriptstyle \psi_{x, y \oplus z}} \\
F((x \oplus y) \oplus z) & \xrightarrow{F(a_{x,y,z})} & F(x \oplus (y \oplus z))
\end{array}$$

and

$$\begin{array}{ccc}
F(x) \odot 1_{\mathcal{J}} & \xrightarrow{\phi \odot 1_{\mathcal{J}}} & F(x) \odot F(1_{\mathcal{I}}) \\
\downarrow{\scriptstyle \hat{\rho}_{F(x)}} & & \downarrow{\scriptstyle \psi_{x,1_{\mathcal{I}}}} \\
F(x) & \xrightarrow{F(\rho_x)} & F(x \oplus 1_{\mathcal{I}})
\end{array}
\qquad
\begin{array}{ccc}
1_{\mathcal{J}} \odot F(x) & \xrightarrow{1_{\mathcal{J}} \odot \phi} & F(1_{\mathcal{I}}) \odot F(x) \\
\downarrow{\scriptstyle \hat{\lambda}_{F(x)}} & & \downarrow{\scriptstyle \psi_{1_{\mathcal{I}},x}} \\
F(x) & \xrightarrow{F(\lambda_x)} & F(1_{\mathcal{I}} \oplus x)
\end{array}$$

are commutative.

The natural transformation ψ and the morphism ϕ in the above definition are called *coherence maps*. If the coherence maps are chosen to be natural transformations instead of isomorphisms, then in this case such a functor is called a *lax monoidal functor*, and if the coherence maps are identity maps, then it is called *strict monoidal functor*.

A braided monoidal functor between braided monoidal categories is a monoidal functor that respect the braiding and a symmetric monoidal functor between symmetric monoidal categories is a monoidal functor such that the obvious diagram with the symmetry isomorphism commutes.

Chapter 3

Preliminaries on Equivariant homotopy theory

A group G equipped with a topology is called a topological group. Given a topological group and a space X, a continuous action of G on X is a continuous map

$$\cdot : G \times X \longrightarrow X$$

which is also a group action. Such a space is called a G-space. A map $f : X \to Y$ of such objects is a G-equivariant map; i.e., it satisfies $f(g \cdot x) = g \cdot f(x)$ for every g in G and x in X. Such a map is called a G-map.

In this chapter we will mention some basic notions in equivariant homotopy theory which are preliminaries to our work. We use [6] as a general source for the theorems, definitions and notations that we use in this chapter.

Equivariant homotopy theory concerns with the homotopy theory of G-spaces; i.e., homotopy theory in the category of G-spaces. A G-homotopy between G-maps $f, g : A \to B$ is a G-equivariant homotopy with respect to the unit interval with trivial action; that is, a G-map

$$H : A \times I \longrightarrow B$$

which is also a homotopy in the usual sense.

We denote by $GTop$ (resp. $GTop_*$) the category of left G-spaces (resp. pointed left G-spaces), with G-maps (resp. pointed G-maps). For a given subgroup H of G, the H-fixed points of a G-space X is defined as

$$X^H = \{x \in X : hx = x \ \forall h \in H\}.$$

Much of the equivariant homotopy theory of G-spaces (unpointed or pointed) can be done by using the ordinary homotopy theory of fixed points of topological spaces. This is due to the Elmendorfs theorem, see [7]. In fact, the category $GTop$ (resp. $GTop_*$) is a homotopical category. A morphism $f : X \to Y$ is a weak equivalence provided that for any closed subgroup H of G, the map restricted to the fixed points

$$f^H : X^H \longrightarrow Y^H$$

is a weak homotopy equivalence in the usual sense.

The homotopy theory of G-spaces well studied and it is known that most of the classical theorems, such as 'Whitehead theorem', 'Cellular approximation theorem', etc., has equivariant versions, see e.g. Chapter I of [6]. The equivariant homology and cohomology theories are also defined and proven to be very useful in this setting, especially for equivariant surgery and Smith theory. On the other hand, the standard cohomology theories that are graded over integers miss very crucial theorems such as Poincaré duality. Hence, the cohomology theories for equivariant spaces ought to have richer structures and they should be graded over representations, see Chapter X of [6], i.e., stable homotopy theory has also richer structure.

3.1 Equivariant spectra and cohomology

Let G be a group. A complete G-universe is a countably infinite dimensional real inner product space U such that G acts on U through isometries and U contains the trivial representation, all irreducible representations and each of its finite dimensional representations infinitely often.

For any G-space X and G-representation V, define

$$\Sigma^V X = X \wedge S^V$$

and

$$\Omega^V X = Hom(S^V, X),$$

where S^V is the one point compactification of the representation V. Given a complete G-universe U, the set of G-stable maps between a finite G-CW-complex A and a G-space B is defined as

$$\{A, B\}_G = \mathrm{colim}_V[\Sigma^V A, \Sigma^V B],$$

where colimit runs over all finite dimensional sub-inner product spaces of U. The equivariant version of the Freudenthal suspension theorem, as given in Chapter IX in [6], is given as follows:

Theorem 3.1.1 (Equivariant Freudenthal suspension theorem)**.** *For a finite group G and A, B as above, there is a representation V_0 such that, for any representation V, the map*

$$\Sigma^V : [\Sigma^{V_0} A, \Sigma^{V_0} B] \to [\Sigma^{V_0 \oplus V} A, \Sigma^{V_0 \oplus V} B]$$

is an isomorphism.

The following definition is the equivariant version of the classical notion of spectra. This definition is the version of [6] Chapter XII, however, alternative equivalent definitions exist.

Definition 3.1.1. A G-prespectrum indexed on U is a sequence of based G-spaces E_V, for $V \subset U$ and a sequence of based G-maps, called the structure maps,

$$\sigma_{V,W} : \Sigma^{W-V} E_V \to E_W$$

for each W containing V, such that

- $\sigma_{V,V} : E_V \to E_V$ is the identity map

- for $V \subset W \subset Z$ in A, the diagram

$$\begin{array}{ccc} \Sigma^{Z-W}\Sigma^{W-V}E_V & \longrightarrow & \Sigma^{Z-W}E_W \\ \downarrow & & \downarrow \\ \Sigma^{Z-V}E_V & \longrightarrow & E_Z \end{array}$$

  is commutative.

Here $W - V$ denote the orthogonal complement of V inside W.

It is well-known that the loop space functor Ω is the adjoint of Σ. This is also true equivariantly. Given a prespectrum E, if the adjoints of the structure maps are homeomorphisms, then E is called a *G-spectrum*. A map of G-spectra (or G-prespectra) is a sequence of maps that preserve the structure, i.e., commute with the structure maps. The category of G-prespectra and G-spectra indexed over A are denoted by GP_A and GS_A respectively. There exist a spectrification functor

$$L : GP_A \to GS_A,$$

which is the left adjoint of the forgetful functor

$$l : GS_A \to GP_A,$$

see for example [8]. For a given G-prespectrum E, the G-spectrum LE is given by

$$LE_V = colim_W \Omega^{W-V} E_W,$$

where the colimit is taken over finite dimensional subspaces $W \subset U$ that contain V.

As in the standard non-equivariant case, the most important applications of these objects are representation of cohomology theories. The axiomatic definitions of classical non-equivariant cohomology theories is due to Eilenberg-Steenrod, see [9]. For the equivariant case, the axiomatic characterization is given in [6].

3.1.1 Axiomatic definition of equivariant cohomology

The axiomatic characterization of G-cohomology theories given in this section is due to Chapter XIII, 1 of [6]. This characterization is one of our fundamental motivations for definitions given in Chapter 6 of this thesis.

Let G be a group and U be a complete G-universe. The class of all representations of G that can be embeddable in U forms a category, which is denoted by $\mathfrak{R}O(G,U)$. The morphisms of this category is chosen as G-linear isometric isomorphism of representations. Given objects V, W of $\mathfrak{R}O(G,U)$, two maps $f, g : V \to W$ are said to be equivalent if the induced maps in $[S^V, S^W]_G$ are stably homotopic. The equivariant cohomology theories are graded over the equivalence classes in $\mathfrak{R}O(G,U)$ with respect to this equivalence relation, which is a monoidal category with the direct sum operation. This category is denoted by $RO(G,U)$.

The following definition is given as Definition 1.1 in [6] Chapter XIII:

Definition 3.1.2. An $RO(G,U)$ graded equivariant cohomology theory is a functor h_G from the product category $RO(G,U) \times GTop_*^{\mathrm{op}}$ to the category of abelian groups $\mathcal{A}b$ (with notation h_G^V for $h_G(V)$); such that for each object W of $RO(G,U)$ we have an isomorphism

$$\sigma^W : h_G^V(X) \to h_G^{V\oplus W}(\Sigma^W X)$$

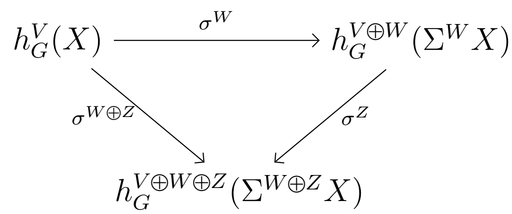
subject to the following axioms:

1. For each representation V, the functor h_G^V sends cofiber sequences in $GTop_*$ to exact sequences in $\mathcal{A}b$ and wedge products in $GTop_*$ to direct products in S.

2. If $\alpha : W \to W'$ is a morphism in $RO(G,U)$, then we have a commutative diagram as follows:

$$\begin{array}{ccc} h_G^V(X) & \xrightarrow{\quad\sigma^W\quad} & h_G^{V\oplus W}(\Sigma^W X) \\ {\scriptstyle \sigma^{W'}}\big\downarrow & & \big\downarrow{\scriptstyle h_G(id\oplus\alpha)(id)} \\ h_G^{V\oplus W'}(\Sigma^{W'} X) & \xrightarrow{h_G(V\oplus W)(\Sigma^\alpha)} & h_G^{V\oplus W'}(\Sigma^W X) \end{array}$$

3. $\sigma(0) = id$ and for each pair of representations (W, Z) we have a commutative diagram as follows:



This definition lead experts of the area to cohomology theories graded over the Picard groups-Picard categories. For Picard graded cohomology theories we refer to [10]. In our point of view, given in Chapter 5 and 6 of this thesis, h_G is an action of the monoidal category $RO(G, U)$, and cohomology theories arise from reversing that action. Our definitions combine all other definitions of cohomology theories graded over different objects, and the extra axioms follow immediately from the reversing process.

Chapter 4

Actions of monoids

Much of the content of this chapter is going to appear in [11]. A monoid is a set I together with a binary operation

$$* : I \times I \longrightarrow I$$

such that $*$ is associative and there is an identity element with respect to $*$. One most known example is the set of natural numbers $\mathbb{N}$, with respect to addition. Any group with the group operation is also a monoid. A homomorphism of monoids is a function that preserve the monoid operation. It is called a monoid isomorphism if it is bijection.

Given two sets A and B, denote the set of functions from A to B by $[A, B]$ and we denote the set of endofunctions on A, $[A, A]$ by $\mathrm{End}(A)$. One can obtain two distinct monoid structures on $\mathrm{End}(A)$.

1. The monoid operation on $\mathrm{End}(A)$ is the composition when endofunctions are applied from right. We denote this monoid by $\mathrm{End}_r(A)$
2. The monoid operation on $\mathrm{End}(A)$ is the composition when endofunctions are applied from left. We denote this monoid by $\mathrm{End}_l(A)$

In the former case we write fg for the composition of f and g in $\mathrm{End}_r(A)$. If a

is in A then

$$(a)(fg) = ((a)f)g.$$

In the latter case we write $f \circ g$ for the composition of f and g in $\mathrm{End}_l(A)$ and if a is in A then

$$(f \circ g)(a) = f(g(a)).$$

Note that the identity map on $End(A)$ is not a monoid isomorphism.

4.1 Actions of monoids on sets

In this section an action of a monoid on a set will be seen as a biaction. The usual left action is an action with the right component is trivial and the usual right action is an action with the left component is trivial.

Definition 4.1.1. Suppose that I is a monoid and A is a set. An action α of I on A is a pair of monoid homomorphisms (α_l, α_r) such that

$$\alpha_l : I \to \mathrm{End}_l(A) \text{ and } \alpha_r : I \to \mathrm{End}_r(A),$$

and α_l commutes with α_r; that is, for every i, j in I and a in A we have

$$(\alpha_l(i)(a))\alpha_r(j) = \alpha_l(i)((a)\alpha_r(j)).$$

Notation. We also use the classical notation $\cdot$ for such an action, i.e. instead of

$$\alpha_l(i)(a)\alpha_r(j)$$

we write

$$i \cdot a \cdot j.$$

When we say $\cdot$ is an action of I on A or $(A, \cdot)$ is a I-set, we understand $\cdot$ is an action from both sides as in the above definition. If an element $i \in I$ acts as a bijection on A from the left (resp. from the right), then we write $(i)^{-1}$, which satisfies

$$i \cdot (i)^{-1} \cdot a = a = (i)^{-1} \cdot i \cdot a$$

$$(\text{resp.} \quad a \cdot i \cdot (i)^{-1} = a = a \cdot (i)^{-1} \cdot i \quad).$$

Definition 4.1.2. Suppose that we have I-actions $\cdot$ on A and $\star$ on B such that both actions operate from left and right. There is an induced I-action $[\cdot, \star]$ on $[A, B]$ such that for f in $[A, B]$ and i in I the function $i[\cdot, \star]f$ is given by the composition

$$A \xrightarrow{-\cdot i} A \xrightarrow{f} B \xrightarrow{i \star -} B$$

and $f[\cdot, \star]i$ is given by the composition

$$A \xrightarrow{i \cdot -} A \xrightarrow{f} B \xrightarrow{- \star i} B.$$

Observe that the left action in the definition commutes with the right action, so that this action is well defined.

4.1.1 Equivariant functions and fixed point sets

Let $(A, \cdot)$ be a I-set. As in the case of a group action it is also possible to talk about centralizers, equivariant functions and fix point sets of the monoid actions. We will start with the definition of the centralizer.

Definition 4.1.3. The the centralizer of I in A, $C_A(I)$, is defined as

$$C_A(I) = \{a \in A : \forall i \in I, i \cdot a = a \cdot i\}.$$

Suppose that we have I-sets $(A, \cdot)$ and $(B, \star)$. Consider the I-action $[\cdot, \star]$ on $[A, B]$.

Definition 4.1.4. A function $f : A \to B$ will be called I-equivariant if and only if it belongs to the centralizer of I in $([A, B], [\cdot, \star])$. Hence the set of I-equivariant functions from A to B will be equal to $C_{[A,B]}(I)$. We denote this set by $\mathrm{Map}_I(A, B)$.

Remark 1. A function $f : A \to B$ is a I-equivariant function if and only if we have the identity

$$f(i \cdot a) \star i = i \star f(a \cdot i)$$

for all i in I and a in A.

Definition 4.1.5. The set of fix points of I on A, $\mathrm{Fix}_I(A)$, is defined as

$$\mathrm{Fix}_I(A) = \mathrm{Map}_I(*, A)$$

where $*$ denotes a set with single element with the trivial I action.

4.1.2 Properties of the equivariant functions

Let $(A, \cdot)$, $(B, \star)$, $(C, \bullet)$, and $(D, \boxdot)$ be four I-sets. Assume

$$f : A \to B \quad \text{and} \quad h : C \to D$$

be two functions. The functions f and h induces a function

$$[B, C] \to [A, D]$$

which sends $g : B \to C$ to the composition

$$A \xrightarrow{\quad f \quad} B \xrightarrow{\quad g \quad} C \xrightarrow{\quad h \quad} D.$$

We have the following proposition:

Proposition 4.1.1. *If $f : A \to B$ and $h : C \to D$ are two I-equivariant functions then the induced function $[B, C] \to [A, D]$ by f and h is I-equivariant.*

Proof. Since f and h are I-equivariant, we have

$$h(i \bullet g(f(i \cdot a) \star i)) \boxdot i = i \boxdot h(g(i \star f(a \cdot i)) \boxdot i)$$

for all a in A, i in I and g in $[B, C]$. Hence we have

$$h \circ ((i[\star, \bullet]g) \circ f)[\cdot, \boxdot]i = i[\cdot, \boxdot]h \circ ((g[\star, \bullet]i) \circ f)$$

for all i in I and g in $[B, C]$. This means the induced function from $[B, C]$ to $[A, D]$ is I-equivariant. $\square$

This proposition shows that compositions by equivariant functions induces an equivariant function between function sets.

Proposition 4.1.2. *Let I be a monoid or a monoid, and A, B be two I-sets. Then we have a bijection*

$$\mathcal{M}ap_I(A,B) \cong \mathcal{F}ix_I([A,B]).$$

Proof. More generally for an I-set A we have a bijection from $C_I(A)$ to $C_I([*,A])$ sending z in $C_I(A)$ to the function from $*$ to A which sends the unique point in $*$ to z. □

Given a function $f : A \to [B,C]$ we define

$$\bar{f} : A \times B \to C$$

by $\bar{f}(a,b) = f(a)(b)$ for all a in A and b in B.

Proposition 4.1.3. *Let A, B and C be three I-sets with I-actions $\cdot$, $\star$ and $\bullet$ respectively. Then the function*

$$\mathcal{M}ap_I(A,[B,C]) \to \mathcal{M}ap_I(A \times B, C)$$

defined by $f \mapsto \bar{f}$ is a bijection.

Proof. We only need to show that $f : A \to [B,C]$ is a I-equivariant function if and only if $\bar{f} : A \times B \to C$ is a I-equivariant function. We know that the statement $f : A \to [B,C]$ is a I-equivariant function means

$$f[\cdot,[\star,\bullet]]i = i[\cdot,[\star,\bullet]]f$$

for all i in I. In other words it means

$$(f(i \cdot a)(i \star b)) \bullet i = i \bullet (f(a \cdot i)(b \star i))$$

for all a in A, b in B and i in I. Hence it is equivalent to

$$\bar{f}(i \cdot a, i \star b) \bullet i = i \bullet \bar{f}(a \cdot i, b \star i)$$

for all a in A, b in B and i in I. Therefore the statement $f : A \to [B,C]$ is a I-equivariant function is equivalent to

$$\bar{f}[\cdot \times \star, \bullet]i = i[\cdot \times \star, \bullet]\bar{f}$$

which means $\bar{f} : A \times B \to C$ is a I-equivariant function. □

Remark 2. If A, B and C be three I-sets, then there is an obvious bijection

$$\mathcal{M}ap_I(A, B \times C]) \to \mathcal{M}ap_I(A, B) \times \mathcal{M}ap_I(A, C).$$

4.2 Categories of I-sets

The class of I-sets with equivariant functions do not form a category since the composition of equivariant functions does not always have to be equivariant. In this section we will construct categories whose class of objects are a subclass of the "sets with an action of I" defined as in the sense of previous section. These categories will contain the usual category of left and right actions of I as a full-subcategory. In each case the morphisms will be I-equivariant functions.

4.2.1 Semi-reversible actions and actions reversible-on-one-side

Let $(A, \cdot)$ be an I-set. First note that if the left component acts by bijections on A then for all a in A then we have the equality

$$(i)^{-1} \cdot (a \cdot j) = ((i)^{-1} \cdot (a)) \cdot j.$$

Similarly in the case when the right component acts by bijections then we have

$$i \cdot (a \cdot (j)^{-1}) = (i \cdot a) \cdot (j)^{-1}.$$

Definition 4.2.1. A I-set $(A, \cdot)$ is called "semi-reversible" if $i \cdot -$ or $- \cdot i$ is an action by automorphism of A for all i in I.

We say the left component (resp. the right component) of $\cdot$ is reversible if $i \cdot -$ (resp. $- \cdot i$) is an automorphism of A for all i.

Definition 4.2.2. A I-set $(A, \cdot)$ is called "reversible on one side" if either the left component or the right component of $\cdot$ is reversible.

Remark 3. If an action is reversible on one side then it is semi-reversible, but not the contrary.

4.2.2 Compositions of equivariant functions

The following technical result shows composition of equivariant functions are equivariant under mild assumptions.

Lemma 4.2.1. *Let S be a set and for every s in S, $(B_s, \star_s)$ be a semi-reversible I-set. Define B as the product*

$$B = \prod_{s \in S} B_s$$

with the I-action given by $\star_s$ on the s^{th} component. Assume $(A, \cdot)$ and $(C, \bullet)$ are I-sets and $f : A \to B$, $g : B \to C$ are I-equivariant functions. Then $g \circ f$ is I-equivariant.

Proof. We want to show

$$i \bullet (g \circ f)(a \cdot i) = (g \circ f)(i \cdot a) \bullet i$$

for any a in A and i in I. Let us denote the left-hand side of above equality by LHS and the right-hand side by RHS. Let f_s denote the s^{th} component of f.

Denote by $\beta(s)_l$ as the map

$$\beta(s)_l(: I \to End_l(A)$$

given by $i \mapsto i \star_s -$ and $\beta(s)_r$ as the map

$$\beta(s)_r : I \to End_r(A)$$

given by $i \mapsto - \star_s i$. Similarly, denote by α_l, α_r the maps into the monoid of endomorphisms of A associated to the right and the left actions on A and denote by γ_l, γ_r the maps the monoid of endomorphisms of C associated to the right and the left actions on C. Since $(B_s, \beta(s))$ is semi-reversible, given any s in S and i in I there exists $x(s,i)$ in $\{l, r\}$ such that $\beta(s)_{x(s,i)}(i)$ is an automorphism of B_s. Since $\beta(s)_{x(s,i)}(i)^{-1} \circ \beta(s)_{x(s,i)}(i)$ is identity, we have

$$\begin{aligned}
\text{LHS} &= \gamma_l(i)(g(f((a)\alpha_r(i)))\\
&= \gamma_l(i)(g((f_s((a)\alpha_r(i)))_{s\in S}))\\
&= \gamma_l(i)(g((E(a,i)_s)_{s\in S}))
\end{aligned}$$

where

$$E(a,i)_s = \begin{cases} (\beta(s)_l(i)^{-1} \circ \beta(s)_l(i))(f_s((a)\alpha_r(i))) & \text{if } x(s,i) = l \\ (f_s((a)\alpha_r(i)))(\beta(s)_r(i)^{-1}\beta(s)_r(i)) & \text{if } x(s,i) = r \end{cases}$$

We have

$$\text{LHS} = \gamma_l(i)(g((F(a,i)_s)_{s\in S}))$$

if $F(a,i)_s$ is defined as follows:

$$F(a,i)_s = \begin{cases} \beta(s)_l(i)^{-1}(\beta(s)_l(i)(f_s((a)\alpha_r(i)))) & \text{if } x(s,i) = l \\ ((f_s((a)\alpha_r(i)))\beta(s)_r(i)^{-1})\beta(s)_r(i) & \text{if } x(s,i) = r \end{cases}$$

Since f is I-equivariant means f_s is I-equivariant for all s in S, we have

$$\text{LHS} = \gamma_l(i)(g((G(a,i)_s)_{s\in S}))$$

where

$$G(a,i)_s = \begin{cases} \beta(s)_l(i)^{-1}((f_s(\alpha_l(i)(a)))\beta(s)_r(i)) & \text{if } x(s,i) = l \\ ((f_s((a)\alpha_r(i)))\beta(s)_r(i)^{-1})\beta(s)_r(i) & \text{if } x(s,i) = r \end{cases}$$

By the above equality

$$\text{LHS} = (g(\beta(s)_l(i)((H(a,i)_s))_{s\in S}))\gamma_r(i)$$

with

$$H(a,i)_s = \begin{cases} \beta(s)_l(i)^{-1}(f_s(\alpha_l(i)(a))) & \text{if } x(s,i) = r \\ (f((a)\alpha_r(i)))\beta(s)_r(i)^{-1} & \text{if } x(s,i) = r \end{cases}$$

Since g is I-equivariant

$$\begin{aligned} \text{LHS} &= (g((J(a,i)_s)_{s\in S}))\gamma_r(i) \\ &= (g((K(a,i)_s)_{s\in S}))\gamma_r(i) \end{aligned}$$

where

$$J(a,i)_s = \begin{cases} \beta(s)_l(i)(\beta(s)_l(i)^{-1}(f_s(\alpha_l(i)(a)))) & \text{if } x(s,i) = l \\ \beta(s)_l(i)((f_s((a)\alpha_r(i)))\beta(s)_r(i)^{-1}) & \text{if } x(s,i) = r \end{cases}$$

and

$$K(a,i)_s = \begin{cases} f_s(\alpha_l(i)(a)) & \text{if } x(s,i) = l \\ (\beta(s)_l(i)(f_s((a)\alpha_r(i))))\beta(s)_r(i)^{-1} & \text{if } x(s,i) = r \end{cases}$$

Since f_s is I-equivariant for all $s \in S$, we have

$$\begin{aligned}
\text{LHS} &= (g((L(a,i)_s)_{s\in S}))\gamma_r(i) \\
&= (g((f_s(\alpha_l(i)(a)))_{s\in S})\gamma_r(i) \\
&= (g(f(\alpha_l(i)(a)))\gamma_r(i) \\
&= \text{RHS}
\end{aligned}$$

where

$$L(a,i)_s = \begin{cases} f_s(\alpha_l(i)(a)) & \text{if } x(s,i) = l \\ ((f_s(\alpha_l(i)(a)))\beta(s)_r(i))\beta(s)_r(i)^{-1} & \text{if } x(s,i) = r \end{cases}$$

This means

$$i \bullet (g \circ f)(a \cdot i) = (g \circ f)(i \cdot a) \bullet i;$$

and hence, we are done. □

4.2.3 Definitions of categories of I-sets

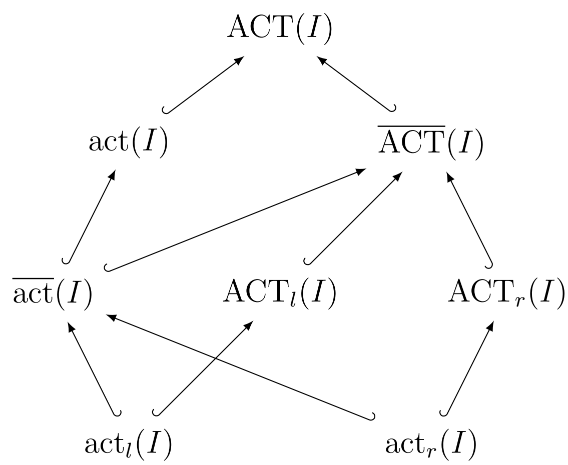
Let I be a monoid, considering the usual definition one-sided of actions we let $\mathrm{ACT}_l(I)$, $\mathrm{ACT}_r(I)$, $\mathrm{act}_l(I)$, $\mathrm{act}_r(I)$ to denote the category of left I-sets, right I-sets, finite left I-sets and finite right I-sets respectively, with I-equivariant maps. Now we construct four new categories as follows:

Definition 4.2.3. We have the following categories:

- $\mathrm{ACT}(I)$: objects are I-sets which are products of semi-reversible I-sets.
- $\mathrm{act}(I)$: objects are finite I-sets which are products of semi-reversible I-sets.
- $\overline{\mathrm{ACT}}(I)$: objects are I-sets which are products of I-sets with actions that are reversible on one side.
- $\overline{\mathrm{act}}(I)$: objects are finite I-sets which are products of I-sets with actions that are reversible on one side.

The morphisms of the categories $\mathrm{ACT}(I)$, $\mathrm{act}(I)$, $\overline{\mathrm{ACT}}(I)$, $\overline{\mathrm{act}}(I)$ are I-equivariant functions (defined as in Section 4.1.1).

We can understand the relation of these categories by the following diagram:



Each arrow in the diagram are inclusions, which map an I-set to itself. All of these subcategories are full-subcategories of $\mathrm{ACT}(I)$.

Proposition 4.2.2. *Let $f : A \to B$ be a bijective equivariant function where $(A, \cdot)$ and $(B, \star)$ are semi-reversible finite I-sets. Then the inverse f^{-1} is equivariant.*

Proof. Assume f is equivariant. We want to show

$$i \cdot f^{-1}(b \star i) = f^{-1}(i \star b) \cdot i,$$

for $b \in B$ and $i \in I$. Assume first both $i \cdot -$ and $i \star -$ are isomorphisms. First since both f and $i \cdot -$ are bijective, we can write

$$i \star (b \star i) = (f(i \cdot (i)^{-1} \cdot (f^{-1}(i \star b))) \star i.$$

Since f is equivariant,

$$i \star (b \star i) = i \star f((i)^{-1} \cdot (f^{-1}(i \star b) \cdot i)).$$

and since $i\star$ is bijective, we get

$$b \star i = f((i)^{-1} \cdot ((f^{-1}(i \star b)) \cdot i))$$

which implies

$$i \cdot f^{-1}(b \star i) = f^{-1}(i \star b) \cdot i.$$

The case when both $- \cdot i$ and $- \star i$ are isomorphisms is the same. Assume now both $- \cdot i$ and $i \star -$ are isomorphisms. Since f is an isomorphism, the composition of f^{-1}, $- \cdot i$ and $i \star -$ is an isomorphisms. Since A and B are finite sets, from the equality

$$f(i \cdot a) \star i = i \star f(a \cdot i)$$

we get $i \cdot -$ and $- \star i$ are isomorphisms as well. Hence, f^{-1} is equivariant. The case $- \cdot i$ and $i \star -$ are isomorphisms is the same. Hence this proves the statement. □

Observe that if the semi-reversible actions are isomorphism in the same side, then we do not need the finiteness assumption. However, in general this proposition is not correct when we drop the assumption on finiteness. For example if $I = \mathbb{N}$ and $A = B = \mathbb{N}$ with the action $\cdot$ on A such that $1 \cdot i = i + 1$ with trivial right component, and the action $\star$ on B such that $(i + 1) \star 1 = i$ and $0 \star 1 = 0$ with trivial left component, then the identity function $id : A \to B$ is equivariant but $id : B \to A$ is not.

4.3 Action reversing functors

For a monoid I we define four monoid homomorphisms as follows: The homomorphisms

$$\iota_l : I \to \mathrm{End}_l(I) \quad \text{and} \quad \iota_r : I \to \mathrm{End}_r(I)$$

sends every element to identity endofunction and the homomorphisms

$$\mu_l : I \to \mathrm{End}_l(I) \quad \text{and} \quad \mu_r : I \to \mathrm{End}_r(I)$$

are given by multiplication from left and right respectively.

4.3.1 Reversing actions from left to right

Consider I itself as a I-set with the action (ι_l, μ_r). Let A be a set with a I-action $\cdot$. To indicate the right action on I is trivial let us denote the set of equivariant functions, $\mathcal{M}ap_I(I, A)$, by $\mathrm{Inv}_l^r(A)$. Let $f : I \to A$ be a I-equivariant map, i.e., for every i, j in I we have

$$f(j) \cdot i = i \cdot f(j * i)$$

Definition 4.3.1. Let $(A, \cdot)$ be a I-set. There is a I-action $[\cdot^-]$ on $\mathrm{Inv}_l^r(A)$ as follows: The left component

$$\theta_l : I \to \mathrm{End}_l(\mathrm{Inv}_l^r(A))$$

is trivial, i.e. $i[\cdot^-]f = f$. The right component

$$-[\cdot^-]k : \mathrm{Inv}_l^r(A) \to \mathrm{Inv}_l^r(A)$$

defined as the function that sends f to the composition

$$I \xrightarrow{k * -} I \xrightarrow{f} A$$

so that we have $((f)[\cdot^-]k)(j) = f(k * j)$, for every $j, k \in I$. Since I is semi-reversible, by Lemma 4.2.1 we can say $[\cdot^-]$ is well defined.

The action $[\cdot^-]$ is called the reverse action of $\cdot$ from left to right.

This construction is functorial on $\mathrm{ACT}(I)$ and we denote the functor sending an I-action on a set A to the reverse I-action on $\mathrm{Inv}_l^r(A)$ by

$$\mathrm{Inv}_l^r : \mathrm{ACT}(I) \to \mathrm{ACT}(I).$$

This functor sends a morphism $f : A \to B$ to the morphism which sends $h : I \to A$ to the composition $f \circ h$ from I to B.

Definition 4.3.2. Given I-set A we can define the evaluation function

$$\mathcal{E}_l^r : \mathrm{Inv}_l^r(A) \to A$$

given by $\mathcal{E}_l^r(f) = f(1)$ whenever we have 1.

Lemma 4.3.1. *The evaluation $\mathcal{E}_l^r$ defines a natural transformation from Inv_l^r to id, the identity functor.*

Proof. Let A be an I-set with action $\cdot$. From the equality

$$\begin{aligned} i \cdot \mathcal{E}_l^r(f[\cdot^-]i) &= i \cdot f(i) = f(1) \cdot i \\ &= \mathcal{E}_l^r(f) \cdot i, \end{aligned}$$

we can say $\mathcal{E}_l^r$ is equivariant, so that it defines a natural transformation from Inv_l^r to id. □

4.3.2 Reversing actions from right to left

Consider I as an I-set with the action (μ_l, ι_r), so that an I-equivariant function $f : I \to A$ satisfies

$$f(i * j) \cdot i = i \cdot f(j)$$

for every i, j in I. In this case we denote the set of equivariant functions from I to A, $\mathcal{M}ap_I(I, A)$, by $\mathrm{Inv}_r^l(A)$.

Definition 4.3.3. We define a I-action $[^-\cdot]$ on $\mathrm{Inv}_r^l(A)$ as follows: The left component is given by the function

$$k[^-\cdot]- : \mathrm{Inv}_r^l(A) \to \mathrm{Inv}_r^l(A)$$

defined as the function that sends f to the composition

$$I \xrightarrow{\mu_r(k)} I \xrightarrow{f} A$$

so that we have $(k[^-\cdot]f)(i) = f(i * k)$. The right component of $[^-\cdot]$ is trivial. The action $[^-\cdot]$ is called the reverse action of $\cdot$ from right to left.

Again by Lemma 4.2.1 this construction is well defined. There is again an equivariant evaluation function

$$\mathcal{E}_r^l : \mathrm{Inv}_r^l(A) \to A$$

given by $\mathcal{E}_r^l(f) = f(1)$ provided that we have 1, which is equivariant. Similar to the Lemma 4.3.1, $\mathcal{E}_r^l$ defines a natural transformation from Inv_r^l to id.

The following is an important property of reverse actions when I is a monoid:

Proposition 4.3.2. *Let $(A, \cdot)$ be an I-set such that the right component of the action is reversible, then there is a isomorphism $\mathrm{Inv}_r^l(A) \cong A$ as I-sets. If the left component of the action is reversible, then there is a isomorphism $\mathrm{Inv}_l^r(A) \cong A$ as I-sets.*

Proof. Assume right component is reversible . Define a map $\phi : A \to \mathrm{Inv}_r^l$ such that $\phi(a) = f_a$ for $a \in A$ where

$$f_a(i) = i \cdot a \cdot i^{-1}.$$

This map is well defined since

$$\begin{aligned} f_a(i * j) \cdot i &= (i * j) \cdot a \cdot (i * j)^{-1} \cdot i \\ &= i \cdot (f_a(j)), \end{aligned}$$

where f_a is in Inv_r^l. Since $f_a(1) = a$, ϕ is the inverse of the $\mathcal{E}_r^l$, so that $\mathcal{E}_r^l$ is a bijection and by Proposition 4.2.2 ϕ is equivariant, so that we get an isomorphism of I-sets. The proof of the case when the left component is reversible is the same. ∎

4.4 As idempotent endofunctors on $\overline{\mathrm{act}}(I)$

Let I be a monoid. The following lemma shows that the reversing functors are idempotent.

Theorem 4.4.1. *The evaluations function $\mathcal{E}_l^r$ (resp. $\mathcal{E}_r^l$) defines a natural isomorphism from $\mathrm{Inv}_l^r \circ \mathrm{Inv}_l^r$ to Inv_l^r (resp. from $\mathrm{Inv}_r^l \circ \mathrm{Inv}_r^l$ to Inv_r^l).*

Proof. For any I-set A, consider the function

$$\Phi_A : \mathrm{Inv}_l^r(A) \to \mathrm{Inv}_l^r \circ \mathrm{Inv}_l^r(A)$$

given by

$$\Phi(g)(i)(j) = g(i * j)$$

for $g \in \mathrm{Inv}_l^r(A)$ and $i, j \in I$. It is straightforward to check the equalities

$$k \cdot (\Phi(g)(i)(j * k)) = (\Phi_A(g)(i)(j))k \cdot$$

and

$$\Phi(g)(i * k) = (\Phi(g)(i))[\cdot^{-}]k$$

so that Φ is well defined. Since

$$\begin{aligned} g(k * i * j) = \Phi((g)[\cdot^{-}]k)(i)(j) &= (\Phi(g))[\cdot^{-}]k(i)(j) \\ &= g(k * i * j), \end{aligned}$$

Φ is equivariant. For any $g \in \mathrm{Inv}_l^r(A)$ we have

$$(\mathcal{E}_l^r \circ \Phi)(g)(i) = \Phi(g)(i)(1) = g(i)$$

and for any $h \in \mathrm{Inv}_l^r \circ \mathrm{Inv}_l^r(A)$ we have

$$\begin{aligned} (\Phi \circ \mathcal{E}_l^r)(h)(i)(j) &= \Phi(h(1))(i)(j) \\ &= h(1)(i * j) \\ &= h(i)(j) \end{aligned}$$

so that $\mathcal{E}_l^r$ and Φ are mutual inverses. This completes the proof. The same proof works for $\mathcal{E}_r^l$ as well. □

We denote the composition of two reverse endofunctors on $\overline{\mathrm{act}}(I)$ by INV, in other words we have

$$\mathrm{INV} = \mathrm{Inv}_l^r \circ \mathrm{Inv}_r^l$$

considered as an endofunctor on $\overline{\mathrm{act}}(I)$. An equivariant function f in $\mathrm{INV}(A)$ satisfies

$$f(i * j)(i * k) = f(j)(k)$$

for every i, j and k in I. For any I-set A we have an evaluation function

$$\mathcal{E} : \mathrm{INV}(A) \to A$$

defined by $\mathcal{E}(f) = f(1)(1)$. If γ is the reverse of the reverse action on A, i.e. action on $\mathrm{Inv}_r^l(\mathrm{Inv}_l^r(A))$, then we have

$$\begin{aligned} \mathcal{E}(i \bullet f) \cdot i &= (f \circ \mu_r(i)(1)(1)) \cdot i \\ &= f(i)(1) \cdot i. \end{aligned}$$

By equivariance of $f(i)$ this is equal to

$$i \cdot f(i)(i) = i \cdot f(1)(1) = i \cdot \mathcal{E}(f)$$

hence, $\mathcal{E}$ is equivariant. Then $\mathcal{E}$ defines a natural transformation from $\mathrm{INV} \circ \mathrm{INV}$ to INV. When I is a commutative monoid, we have the following proposition:

Proposition 4.4.2. *If I is a commutative monoid then $\mathcal{E}$ defines a natural isomorphism from* $\mathrm{INV} \circ \mathrm{INV}$ *to* INV*.*

Proof. For any I-set A, the function

$$\Phi_A : \mathrm{INV}(A) \to \mathrm{INV} \circ \mathrm{INV}(A)$$

given by

$$\Phi_A(g)(i)(j)(k)(l) = g(i * k)(j * l)$$

for $g \in \mathrm{INV}(A)$ and $i, j, k, l \in I$. It is straightforward to check that this function is equivariant since on both $\mathrm{INV}(A)$ and $\mathrm{INV} \circ \mathrm{INV}(A)$, the right actions are trivial. We have

$$\mathcal{E}(\Phi_A(g))(k)(l) = g(k)(l)$$

and

$$\Phi_A(\mathcal{E}(g))(k)(l) = g(k)(l)$$

so that $\mathcal{E}$ and Φ_A are mutual inverses. This completes the proof. □

4.4.1 Reverse actions on finite sets

We again use the same notations for the restrictions of Inv_l^r, Inv_r^l and their compositions INV on $\overline{\mathrm{act}}(I)$. Let $(A, \cdot)$ be an I-set such that the right component of the action is trivial. For an element a in A, let Ia denote the orbit set

$$Ia = \{i \cdot a : i \in I\}$$

and for a given $f : I \to A$ in $\mathrm{Inv}_l^r(A)$ let $If(I)$ denote the set

$$If(I) = \{i \cdot f(j) : i,\ j \in I\}.$$

We define a set A^l as the set

$$A^l = \{a \in A : \text{ for all } i \in I,\ i \cdot (-)|_{Ia} \text{ is one-to-one}\}.$$

Note that

Lemma 4.4.3. *Let I be a monoid and let A be a finite set. Let $(A, \cdot)$ be an I-set such that the right component of the action is trivial. Then there is a isomorphism $\mathrm{Inv}_l^r(A) \cong A^l$ as I-sets.*

Proof. Firstly, for an element $a \in A^l$ we define $f_a : I \to A$ with $f_a(i) = (i)^{-1} \cdot (a)$, then since $a \in A^l$, this is a well-defined map. By definition, for every i, j in I we have:

$$\begin{aligned} i \cdot f_a(j * i) &= i \cdot (j * i)^{-1} \cdot a \\ &= (j)^{-1} \cdot a \\ &= f_a(j). \end{aligned}$$

Hence, f_a is equivariant and we have an injective function $A^l \to \mathrm{Inv}_l^r(A)$.

Now suppose that $f : I \to A$ be a function in $\mathrm{Inv}_l^r(A)$. We claim that $f(1)$ is an element of A^l. Assume the contrary; then there exist i, j, k in I such that

$$j \cdot f(1) \neq k \cdot f(1) \ \text{ and } \ (i * j) \cdot f(1) = (i * k) \cdot f(1).$$

Since A is finite, for every $i \in I$ there exist positive integers m, m' with $m > m'$ such that for all a in $If(I)$, we have the identity $i^m \cdot a = i^{m'} \cdot (a)$. Hence, restriction of the left component of the action of $i^{m-m'} \cdot -$ to the set

$$i^{m'} \cdot (If(I)) := \{i^{m'} \cdot (a) : a \in If(I)\}$$

is the identity function. Moreover, for any $v \in I$ we have

$$f(v) = i^{m'} \cdot f(v * i^{m'})$$

so that $im(f)$ is contained in $i^{m'} \cdot (If(I))$.

Let j and k be two elements in I. As above there are integers t, t' with $t > t'$ and $j^{t'} \cdot f(j^t) = j^{t'} \cdot f(j^{t'})$ so that $j \cdot f(1) = f(j^{t-t'-1})$. Similarly there are integers

s, s' with $s > s'$ and $k \cdot f(1) = f(k^{s-s'-1})$. Hence both $k \cdot f(1)$ and $k \cdot f(1)$ are elements of $im(f)$, which means $i^{m-m'}$ acts as identity on both.

By our initial assumption, we have

$$i^{m-m'-1} \cdot ((i * j) \cdot f(1)) = i^{m-m'-1} \cdot ((i * k) \cdot f(1))$$

which implies

$$i^{m-m'} \cdot (j \cdot f(1)) = i^{m-m'} \cdot (k \cdot f(1))$$

As a result we get $j \cdot f(1) = k \cdot f(1)$; i.e. a contradiction, so that $f(1)$ must be an element of A^l. The evaluation function $\mathcal{E}_l^r$ is injective by definition of A^l and $\mathcal{E}_l^r(f_a) = a$. By Proposition 4.2.2 we get an isomorphism as desired. This completes the proof. □

Objects in $\overline{\text{act}}(I)$ are the actions with either left or right component is reversible. Assume A is an I-set with right component is reversible. Then we define A^l as $\text{Inv}_r^l A^l$. We have the following lemma:

Lemma 4.4.4. *There is an isomorphism* $\text{Inv}_l^r(A) \cong A^l$ *as* I*-sets.*

Proof. The proof follows from Lemma 4.4.3 and Proposition 4.3.2. □

For an I-set A we define A^r similarly. We have a similar lemma as follows:

Lemma 4.4.5. *Let* $(A, \cdot)$ *be an* I*-set such that the left action is reversible. Then there is an isomorphism* $\text{Inv}_r^l(A) \cong A^r$*.*

Let $\overline{\mathcal{E}}$ denote the restriction of $\mathcal{E}$ on finite I-sets. Note that $\overline{\mathcal{E}}$ is bijective by the previous propositions. We have the following lemma:

Proposition 4.4.6. $\overline{\mathcal{E}}$ *defines a natural isomorphism from* $\text{INV} \circ \text{INV}$ *to* INV*.*

Proof. This proposition directly follows from Proposition 4.2.2, since $\overline{\mathcal{E}}$ from $\text{INV} \circ \text{INV}$ to INV is bijective, by the Lemma 4.4.3 and Proposition 4.3.2. □

4.5 Equivalence of view points on groups

The following theorem shows that Definition 4.1.1 is equivalent to the usual one for groups.

Theorem 4.5.1. *For a group G, the categories* $\mathrm{act}(G)$*,* $\overline{\mathrm{act}}(G)$*,* $\mathrm{act}_l(G)$ *and* $\mathrm{act}_r(G)$ *are all equivalent to each other as categories and* $\mathrm{ACT}(G)$*,* $\overline{\mathrm{ACT}}(G)$*,* $\mathrm{ACT}_l(G)$*,* $\mathrm{ACT}_r(G)$ *are all equivalent to each other as categories.*

Proof. Here we will only prove the equivalence of $\mathrm{ACT}(G)$ and $\mathrm{ACT}_l(G)$ the rest is either similar or just obtained by restrictions of the equivalences. First note that the functor

$$\mathrm{Inv}_r^l : \mathrm{ACT}(G) \to \mathrm{ACT}(G)$$

factors through the inclusion

$$\mathrm{inc} : \mathrm{ACT}_l(G) \to \mathrm{ACT}(G).$$

We again write

$$\mathrm{Inv}_r^l : \mathrm{ACT}(G) \to \mathrm{ACT}_l(G)$$

for the functor in the factorization, by an abuse of notation. Then this functor sends an object $(A, \cdot)$ in $\mathrm{ACT}(G)$ to the left action $\mu : G \to \mathrm{End}_l(A)$ given by

$$\mu(g)(a) = g \cdot (a \cdot g^{-1})$$

and sends a morphism f from $(A, \cdot)$ to $(B, \star)$ to itself considered as a function from A to B. Now clearly $\mathrm{Inv}_r^l \circ \mathrm{inc}$ is identity on $\mathrm{ACT}_l(G)$. By Proposition 4.3.1 and 4.3.2, $\mathcal{E}_r^l$ defines a natural isomorphism from $\mathrm{inc} \circ \mathrm{Inv}_r^l$ to the identity on $ACT(G)$. Hence, this gives an equivalence between $\mathrm{ACT}(G)$ and $\mathrm{ACT}_l(G)$. □

We define a functor

$$\mathrm{inv}_l^r : \mathrm{ACT}_l(G) \to \mathrm{ACT}_r(G)$$

which sends a left G action

$$\nu : G \times A \to A, \text{ given by } (g, a) \mapsto g \cdot a$$

for $g \in G$ and $a \in A$, to a right G-action

$$\nu^{-1} : A \times G \to A, \text{ given by } (a, g) \mapsto g^{-1} \cdot a$$

for $g \in G$ and $a \in A$, i.e. the reverse action of ν. The following theorem shows that the two definitions we gave for reverse actions agree for group actions.

Theorem 4.5.2. *The diagram*

$$\begin{array}{ccc} \mathrm{ACT}_l(G) & \xrightarrow{\mathrm{inv}_l^r} & \mathrm{ACT}_r(G) \\ {\scriptstyle \mathrm{inc}}\downarrow & & \downarrow{\scriptstyle \mathrm{inc}} \\ \mathrm{ACT}(G) & \xrightarrow{\mathrm{Inv}_l^r} & \mathrm{ACT}(G) \end{array}$$

is commutative up to a natural isomorphism.

Proof. This follows from Proposition 4.3.2, since group actions are reversible on both sides. □

A version of Theorem 4.5.2 is also true for the case of reversing actions from right to left, i.e. the diagram

$$\begin{array}{ccc} \mathrm{ACT}_r(G) & \xrightarrow{\mathrm{inv}_r^l} & \mathrm{ACT}_l(G) \\ {\scriptstyle \mathrm{inc}}\downarrow & & \downarrow{\scriptstyle \mathrm{inc}} \\ \mathrm{ACT}(G) & \xrightarrow{\mathrm{Inv}_r^l} & \mathrm{ACT}(G) \end{array}$$

is commutative up to a natural isomorphism, where inv_r^l is defined similarly.

4.6 Homotopy category of monoid actions and the Burnside ring

In this section we discuss homotopical category structure on $\overline{\mathrm{act}}(I)$ where I is a monoid. We refer [1] for general terminology and homotopical notions in this section. Let A, B be I-sets in $\overline{\text{act}}(I)$ and $f : A \to B$ be an I-equivariant map. We say f is a weak equivalence if the induced function

$$\text{INV}(f) : \text{INV}(A) \to \text{INV}(B)$$

is an isomorphism. We denote the class of weak equivalences by $\mathcal{W}$. It is straightforward to check that these weak equivalences satisfy the 2-out-of-6 property, since isomorphisms do. Hence this makes $\overline{\text{act}}(I)$ into a homotopical category. The homotopical structure on the subcategories of $\overline{\text{act}}(I)$ is defined accordingly.

In order to define the Burnside ring of a monoid I we concentrate on the actions of I on finite sets. Note that the functor

$$\text{INV} : \overline{\text{act}}(I) \to \overline{\text{act}}(I)$$

factors through the inclusion

$$\text{inc} : \text{act}_l(I) \to \overline{\text{act}}(I).$$

We again denote the functor $\overline{\text{act}}(I) \to \text{act}_l(I)$ in the factorization by INV, by an abuse of notation. Note that the functor

$$\text{INV} : \overline{\text{act}}(I) \to \text{act}_l(I)$$

preserve weak equivalences so does the inclusion

$$\text{inc} : \text{act}_l(I) \to \overline{\text{act}}(I).$$

The composition $\text{INV} \circ \text{inc}$ is identity functor on $\text{act}_l(I)$ and there is a natural weak equivalence from $\text{inc} \circ \text{INV}$ to $id_{\overline{\text{act}}(I)}$ given by the evaluation map $\overline{\mathcal{E}}$. Hence $\text{act}_l(I)$ is a left deformation retract of $\overline{\text{act}}(I)$, so that their homotopy categories are naturally equivalent (see [1], 26.3, 26.5 and 29.1). We will continue with the category $\text{act}_l(I)$ to define the Burnside ring. The category $\text{act}_l(I)$ has nice properties such as monomorphisms are stable under pushouts and epimorphisms are stable under pullbacks [12], as it is a topos, so that isomorphisms are also stable under pullbacks and pushouts. In fact assume we have a diagram

$$\begin{array}{ccc} D & \xrightarrow{f'} & B \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{f} & C \end{array}$$

Pullbacks and pushouts are given in a standard way. If D is the pullback of the maps p and f where $\cdot$, $\star$ and $\bullet$ are the actions on A, B and C respectively, then D is given as the set

$$D = \{(a, b) \in A \times B : f(a) = p(b)\}$$

and the action $\boxdot$ on D is given by a pair of actions; i.e. $\boxdot = (\cdot, \star)$ and trivial right action. The maps p' and f' are induced by projections so that they are equivariant.

If the above square is a pushout then

$$C = (A \amalg B)/\sim$$

where $p'(d) \sim f'(d)$ for all d in D. The action γ on C is given by

$$i \bullet [x] = \begin{cases} i \cdot (x) & \text{if } x \in A \\ i \star (x) & \text{if } x \in B \end{cases}$$

for all $i \in I$, with trivial right action. By equivariance of the maps p' and f' in diagram, so that for all $d \in D$ and $i \in I$ we have

$$i \cdot p'(d) = p'(i \boxdot d) \text{ and } i \star f'(d) = f'(i \boxdot d),$$

so that $i \cdot p'(d) \sim i \star f'(d)$; i.e. the action is well defined. The maps p and f are induced by inclusions so that they are also equivariant.

We will show that the category $\mathrm{act}_l(I)$ admits a 3-arrow calculus, for details of 3-arrow calculus we refer [1], 27.3.

4.6.1 3-arrow calculus of $\mathrm{act}_l(I)$ and Saturation

Let us denote the homotopy category of $\mathrm{act}_l(I)$ by $\mathrm{Ho}(\mathrm{act}_l(I))$ and let

$$L : \mathrm{act}_l(I) \to \mathrm{Ho}(\mathrm{act}_l(I))$$

be the localization with respect to the above weak equivalences (see [1] 26.5). We will show that $\mathrm{act}_l(I)$ admits a 3-arrow calculus. To do this we define two subclasses $\mathcal{U}$ and $\mathcal{V}$ of the class weak equivalences $\mathcal{W}$ of $\text{act}_l(I)$ as follows: $\mathcal{U}$ will be the subclass of $\mathcal{W}$ which are also inclusions and $\mathcal{V}$ will be the subclass of $\mathcal{W}$ which are also surjections. Firstly, suppose that we have a zig-zag

$$A' \xleftarrow{u} A \xrightarrow{f} B$$

in $\text{act}_l(I)$ where u is in $\mathcal{U}$. Then we can associate another zig-zag

$$A' \xrightarrow{f'} B' \xleftarrow{u'} B$$

from the pushout

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u'} \\ A' & \xrightarrow{f'} & B' \end{array}$$

so that $f' \circ u = u' \circ f$ and the function u' is an inclusion. Let $(A, \cdot)$, $(A', \diamond)$, $(B, \star)$ and $(B', \square)$ be I-sets with trivial right actions. To be able to see u' is weak equivalence, it is enough to show $\text{Inv}_l^r(B')$ is contained in the image of $\text{Inv}_l^r(u')$. Assume the contrary and let $\sigma : I \to B'$ be a map in $\text{Inv}_l^r(B')$ which is not in the image of $\text{Inv}_l^r(u')$. Then $\sigma(1)$ is not in the image of u' because otherwise σ factors through u' since $\sigma(1) \in (B')^l$, see Lemma 4.4.3. Hence, $\sigma(1)$ is in the image of f'. Thus, there is an element a' in A' such that $f'(a') = \sigma(1)$. Assume first $a' \notin (A')^l$ i.e. there exist i, i_1, i_2 in I such that

$$i_1 \diamond a' \neq i_2 \diamond a' \text{ but } (i * i_1) \diamond a' = (i * i_2) \diamond a'$$

then there exist $b \in B$ such that $u'(b) = f'(i_1 \diamond a')$. But as in the proof of Lemma 4.4.3 there exist an integer m such that

$$\begin{aligned} f'(a') &= i_1^m \diamond f'(i_1 \diamond a') \\ &= i_1^m \diamond u'(b) = u'(i_1^m \square b) \end{aligned}$$

But then this leads us a contradiction unless $a' \in (A')^l$. Thus, σ must be an element in the image of $\text{Inv}_l^r(f')$. Since u is a weak equivalence, any element in $\text{Inv}_l^r(A')$ factors through u, which implies σ is in the image of $\text{Inv}_l^r(f' \circ u)$. But then we get a contradiction again since σ is not in the image of $\text{Inv}_l^r(u' \circ f)$.

Hence, u' is a weak equivalence, i.e. u' is in $\mathcal{U}$. If u is an isomorphism then u' is also an isomorphism since both u and u' fits in above pushout diagram.

Similarly if we have a zig-zag

$$X \xrightarrow{g} Y \xleftarrow{v} Y'$$

in $\mathrm{act}_l(I)$ where v is in $\mathcal{V}$, then we can associate another zig-zag

$$X \xleftarrow{v'} X' \xrightarrow{g'} Y$$

from the pullback diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & Y' \\ {\scriptstyle v'}\downarrow & & \downarrow{\scriptstyle v} \\ X & \xrightarrow{g} & Y \end{array}$$

so that $g \circ v' = v \circ g'$, and the function v' is a surjection. Let $\sigma : I \to X'$, $\bar{\sigma} : I \to X'$ elements in $\mathrm{Inv}_l^r(X')$ with $\sigma(i) = (x_i, y_i)$ and $\bar{\sigma}(i) = (\bar{x}_i, y_i)$ for $i \in I$ $x_i, \bar{x}_i \in X$ and $y \in Y'$, i.e.

$$\mathrm{Inv}_l^r(v')(\sigma) = \mathrm{Inv}_l^r(v')(\bar{\sigma}).$$

Since $\mathrm{Inv}_l^r(g')(\sigma)(i) = x_i$ and $\mathrm{Inv}_l^r(g')(\bar{\sigma})(i) = \bar{x}_i$, we have

$$\mathrm{Inv}_l^r(v)(x_i) = \mathrm{Inv}_l^r(v')(y_i) = \mathrm{Inv}_l^r(v)(\bar{x}_i).$$

We know v is a weak equivalence so that $\mathrm{Inv}_l^r(v)$ is bijection, thus $x_i = \bar{x}_i$, i.e. v' is a weak equivalence. Hence v' is in $\mathcal{V}$. Again if v is an isomorphism then so does v' since both fits into a pullback diagram.

Assume now $w : M \to N$ is a weak equivalence in $\mathrm{act}_l(I)$, then consider the pushout diagram

$$\begin{array}{ccc} \mathrm{INV}(M) & \xrightarrow{\overline{\mathcal{E}}} & M \\ {\scriptstyle w \circ \overline{\mathcal{E}}}\downarrow & & \downarrow{\scriptstyle u} \\ N & \xrightarrow{\tilde{u}} & M' \end{array}$$

From the Lemmas 4.4.3 and 4.4.5 we know $\overline{\mathcal{E}}$ is injective. Moreover, w is a weak equivalence implies $w \circ \overline{\mathcal{E}}$ is injective, due to the commutativity of the following diagram:

$$\begin{array}{ccc} \mathrm{INV}(M) & \xrightarrow{\overline{\mathcal{E}}} & M \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle w} \\ \mathrm{INV}(N) & \xrightarrow{\overline{\mathcal{E}}} & N \end{array}$$

Since the above square is a pushout, both u and $\tilde{u}$ is injective. Hence, there is a unique function $v : M' \to N$ which is surjective. As before, the functions u and v are also equivariant, so that we have a factorization of w as $w = v \circ u$ such that v is in $\mathcal{V}$ and u is in $\mathcal{U}$. Hence $\mathrm{act}_l(I)$ admits a 3-arrow calculus $\{\mathcal{U}, \mathcal{V}\}$. Thus, we have a 3-arrow description of the hom-sets as in 27.2 of [1]. Moreover, by 27.5 of [1], we also get that the homotopical structure of $\mathrm{act}_l(I)$ is saturated; i.e., a function in $\mathrm{act}_l(I)$ is a weak equivalence if and only if its image in $\mathrm{Ho}(\mathrm{act}_l(I))$, under the localization functor, is an isomorphism.

4.7 The Burnside ring of a monoid

In the classical theory of group actions, when a group G is given, the Burnside ring of G, $A(G)$, is defined as the Gröthendieck ring of the semiring of isomorphism classes of finite G-sets where the addition is given by disjoint union and the multiplication is given by cartesian product. The Burnside ring of a group is a very important construction in the group theory, and has several applications, see e.g. [13], [14], [15], [16]. We define the Burnside ring of a monoid by using the homotopical structure on $\mathrm{act}_l(I)$. The isomorphism classes in $\mathrm{Ho}(\mathrm{act}_l(I))$ forms a semiring under disjoint union as addition and cartesian product as multiplication. We call the Gröthendieck ring of this semiring as the Burnside ring of I, and we denote this ring by $\mathrm{Br}(I)$. Most of the properties of this Burnside ring follows from the Section 4.4.1.

By definition the Burnside ring of a group given in this way is equal to the standard one. Hence, it does validate the name "the Burnside ring of a monoid".

Moreover, the following proposition shows that the definitions of the Burnside ring of a commutative monoid is same as the Burnside ring of its Gröthendieck construction. Let us denote by $K(I)$ the Gröthendieck group of a commutative monoid I. Then $\mathrm{Br}(K(I))$ denotes the usual Burnside ring of the group $K(I)$ (see e.g. [13]).

Theorem 4.7.1. *If I is commutative monoid then $\mathrm{Br}(I)$ is isomorphic to $\mathrm{Br}(K(I))$.*

Proof. Define
$$\widetilde{\Lambda} : \mathrm{act}_l(K(I)) \to \mathrm{act}_l(I)$$
as the map induced by the natural map from I to $K(I)$ and let
$$\Lambda : \mathrm{Br}(K(I)) \to \mathrm{Br}(I)$$
denote the induced function on Burnside rings. Here we will define the inverse of Λ. Let A be an I-set with action $\cdot$ and let $[^{-}\cdot^{-}]$ be the action on $\mathrm{INV}(A)$. Lemma 4.4.3 implies that the action on $\mathrm{INV}(A)$ has a group action factorization, i.e. the map
$$\vartheta_l : I \to \mathrm{End}_l(\mathrm{INV}(A))$$
factors through the inclusion
$$\mathrm{Aut}_l(\mathrm{INV}(A)) \hookrightarrow \mathrm{End}_l(\mathrm{INV}(A)).$$
Hence, we can consider $\mathrm{INV}(A)$ as a $K(I)$-set. Define a function
$$\Gamma : \mathrm{Br}(I) \to \mathrm{Br}(K(I))$$
by sending a class $[A]$ of I-set A in $\mathrm{Br}(I)$ to the class $[\mathrm{INV}(A)]$ in $\mathrm{Br}(K(I))$. Notice that
$$\mathrm{INV}(\widetilde{\Lambda}(A)) \cong \widetilde{\Lambda}(A)$$
by Proposition 4.3.2, so that $\Gamma \circ \Lambda$ is identity. The composition $\Lambda \circ \Gamma$ is also identity since by Proposition 4.4.6, the natural transformation $\overline{\mathcal{E}}$ gives a weak equivalence from $\mathrm{INV}(A)$ to A. Hence Γ is a ring isomorphism with the inverse Λ. Since for every I-set M, N we have
$$\mathrm{Inv}_l^r(M \times N) \cong \mathrm{Inv}_l^r(M) \times \mathrm{Inv}_l^r(N)$$

and

$$\mathrm{Inv}_l^r(M \amalg N) \cong \mathrm{Inv}_l^r(M) \amalg \mathrm{Inv}_l^r(N)$$

then these maps are ring isomorphisms. □

4.7.1 Burnside mark homomorphism

Assume I is a monoid and A is a finite left I-set. Let J be a submonoid of I. We define the mark $\hat{m}_J(A)$ of J on A as the number of elements in $\mathrm{INV}(A)$ that are fixed by every element in J,

$$\hat{m}_J(A) = |\mathcal{F}ix_J(\mathrm{INV}(A))|.$$

In other words, i.e. if $\vartheta_l : I \to \mathrm{End}_l(\mathrm{INV}(A))$ denotes the map associated to the action on $\mathrm{INV}(A)$ (which is the action obtained by reversing α twice) then $\hat{m}_J(A)$ is the number of equivariant functions in $\mathrm{INV}(A)$ satisfying

$$f(i * j)(k) = f(i)(k)$$

for every i, k in I and j in J. This defines a semiring homomorphism

$$\hat{m}_J : \mathrm{Isom}(\mathrm{Ho}(\mathrm{act}_l(I))) \to \mathbb{Z}$$

since

$$\mathrm{INV}(A \amalg B) = \mathrm{INV}(A) \amalg \mathrm{INV}(B),$$

so that $\hat{m}_J(A \amalg B) = \hat{m}_J(A) + \hat{m}_J(B)$ and

$$\mathrm{INV}(A \times B) = \mathrm{INV}(A) \times \mathrm{INV}(B)$$

and hence

$$\hat{m}_J(A \times B) = \hat{m}_J(A).\hat{m}_J(B),$$

same as the usual case. The associated ring homomorphism

$$m_J : \mathrm{Br}(I) \longrightarrow \mathbb{Z}$$

is called the mark homomorphism at J. Note that when a finite I-set A is given, the image of ϑ_l in $\mathrm{Aut}_l(\mathrm{INV}(A))$ form a subgroup, and let $\vartheta_l(I)$ denote this subgroup. Let $\vartheta_l(J)$ denote the image of the submonoid J under ϑ_l, which is a subgroup of $\vartheta_l(I)$. Then mark of A at J corresponds to the usual mark of $\vartheta_l(I)$ at the subgroup $\vartheta_l(J)$. We call an I-set $(A, \cdot)$ in $\mathrm{act}_l(I)$ weakly-transitive, if for every pair of elements $f, g \in \mathrm{INV}(A)$, there is an elements i in I such that $\vartheta_l(i)(f) = g$. The set $\mathrm{INV}(A)$ can be expressed as a disjoint union of orbits $\vartheta_l(I)/\vartheta_l(J_t)$ for some family $\{J_t : t \in T\}$ of submonoids. Hence isomorphism classes weakly transitive I-sets generate the additive group of Burnside ring, same as the classical case, see [17].

Let J and J' be two submonoids of I. We say J and J' are weakly conjugate, and we write $J \sim_I J'$, if for every I-set A the subgroup $\vartheta_l(J)$ is conjugate to $\vartheta_l(J')$ in $\vartheta_l(I)$, where ϑ_l is the action on $\mathrm{INV}(A)$. It is straightforward to check that weak conjugation is an equivalence relation. Let $Y(I)$ be the set of weak conjugacy classes of I, i.e. the set of equivalence classes of '$\sim_I$'. Observe that weakly conjugate submonoid have the same mark, i.e. if $J \sim_I J'$ then for any given I-set A we have $m_J(A) = m_{J'}(A)$, which follows from standard group theory facts. Hence one can see the mark homomorphism as a ring homomorphism

$$m : \mathrm{Br}(I) \longrightarrow \bigoplus_{[J] \in Y(I)} \mathbb{Z}$$

into the direct sum of integers $\mathbb{Z}$, so that $m = \bigoplus_{[J] \in Y(I)} m_J$.

Theorem 4.7.2. *The mark homomorphism m is injective.*

Proof. Proof follows from ideas of the proof in the standard case (see [17], Proposition 1.2.2). Let A be an I-set and x be the corresponding element in the Burnside ring, and let ϑ be the action on $\mathrm{INV}(A)$. Then, since there is an induced action of the group $\vartheta_l(I)$, we can write

$$x = \sum_{J \in Y(I)} z_t[\vartheta_l(I)/\vartheta_l(J)]$$

with $z_J \in \mathbb{Z}$. Let K be a monoid such that $\vartheta_l(K)$ is the maximal conjugacy class in $\vartheta_l(I)$ with respect to the inclusion, with $z_K \neq 0$. The rest is same as the proof of Proposition 1.2.2 in [17]. Since $\mathcal{F}ix_K(\vartheta_l(I)/\vartheta_l(J))$ is non-empty if and only if

$\vartheta_l(K)$ is sub-conjugate to $\vartheta_l(J)$, we have $m(x)$ is non-zero due to maximality of $\vartheta_l(K)$. Hence m is injective. □

Some applications of this Burnside ring in 'Computer science' and in particular in 'Automata theory' are discussed in Section 6.3 of [11].

Chapter 5

Actions of monoidal categories

In this chapter we give a "categorification" of definitions in Chapter 4. We replace the concept of monoid with a monoidal category and it will act on a category instead of on a set. We start this section by defining the notion of an action of a monoidal category on a category.

5.1 Actions on categories and functor categories

Let $\mathcal{A}$ and $\mathcal{B}$ be two categories. Then $[\mathcal{A}, \mathcal{B}]$ denotes the category of functors from $\mathcal{A}$ to $\mathcal{B}$ and $\mathrm{End}(\mathcal{A}) = [\mathcal{A}, \mathcal{A}]$. The endofunctor category $\mathrm{End}(\mathcal{A})$ is a (strict) monoidal category with composition (from left or from right). We denote $\mathrm{End}_l(\mathcal{A})$ (resp. $\mathrm{End}_r(\mathcal{A})$) the monoidal category of endofunctors on $\mathcal{A}$ with composition as the monoid operation applied from left (resp. right).

Notation. If f and g are two endofunctors on $\mathcal{A}$ then

- in $\mathrm{End}_l(\mathcal{A})$ the monoidal product is the composition $f \circ g$ of f and g is the endofunctor obtained by applying g first and applying f second
- in $\mathrm{End}_r(\mathcal{A})$ then the monoidal product fg of f and g is the endofunctor obtained f first and applying g second.

Definition 5.1.1. Let $\mathcal{A}$ be a category and $\mathcal{I}$ be a monoidal category. Then a $\mathcal{I}$-action on $\mathcal{A}$ is a pair $\alpha = (\alpha_l, \alpha_r)$ such that

$$\alpha_l : \mathcal{I} \to \mathrm{End}_l(\mathcal{A}) \text{ and } \alpha_r : \mathcal{I} \to \mathrm{End}_r(\mathcal{A})$$

are strong monoidal functors satisfying

$$(\alpha_l(i)(a))\alpha_r(j) = \alpha_l(i)((a)\alpha_r(j)),$$

for every i, j in $\mathcal{I}$ and a in $\mathcal{A}$, either all of i, j, a are objects or one of them is a morphism.

The choice of equality for commutativity of left and right actions make sense since both $\mathrm{End}_l(\mathcal{A})$ and $\mathrm{End}_l(\mathcal{A})$ are strict monoidal categories. A category with an $\mathcal{I}$-action will be called an $\mathcal{I}$-category.

It is less convenient to use classical '$\cdot$' notation for the action for such actions, since we also need to deal with morphisms. Hence, we stick to the notation of above definition during the whole chapter.

5.1.1 Centralizer and Equivariant functors

Let A, B be two categories. Given a functor $\gamma : \mathcal{A} \to \mathrm{End}(\mathcal{B})$ one can obtain functors:

- $\overline{\gamma} : \mathcal{B} \to [\mathcal{A}, \mathcal{B}]$ given by $\overline{\gamma}(b)(a) = \gamma(a)(b)$
- $\widetilde{\gamma} : \mathcal{A} \times \mathcal{B} \to \mathcal{B}$ given by $\widetilde{\gamma}(a, b) = \gamma(a)(b)$
- $\widetilde{\gamma} : \mathcal{B} \times \mathcal{A} \to \mathcal{B}$ given by $\widetilde{\gamma}(b, a) = \gamma(a)(b)$.

The latter two functors are denoted in the same way but it will be clear from the context which one we mean.

Let $\mathcal{I}$ be a monoidal category, $\mathcal{A}$ be a category and let $\alpha = (\alpha_l, \alpha_r)$ be an $\mathcal{I}$-action on $\mathcal{A}$. Then for x in $\{l, r\}$ one can define a functor $\overline{\alpha_x}$ from $\mathcal{A}$ to $[\mathcal{I}, \mathcal{A}]$ as above, by forgetting the monoidal structure on $\mathcal{I}$.

Definition 5.1.2. The centralizer of $\mathcal{I}$ in $\mathcal{A}$ as the category whose objects are pairs (a, σ) where a is an object in $\mathcal{A}$ and σ is a natural isomorphism from $\overline{\alpha_r}(a)$ to $\overline{\alpha_l}(a)$ such that the following diagrams commute:

$$
\begin{array}{ccc}
(a)id_A & \xrightarrow{\quad id \quad} & id_A(a) \\
\downarrow{\scriptstyle \phi_r} & & \downarrow{\scriptstyle \phi_l} \\
(a)\alpha_r(1) & \xrightarrow{\quad \sigma(1) \quad} & \alpha_l(1)(a_2)
\end{array}
$$

and

$$
\begin{array}{ccc}
(a)\alpha_r(i)\alpha_r(j) \xrightarrow{(\sigma(i))\alpha_r(j)} (\alpha_l(i)(a))\alpha_r(j) & = & \alpha_l(i)((a)\alpha_r(j)) \xrightarrow{\alpha_l(i)(\sigma(j))} \alpha_l(i) \circ \alpha_l(j)(a) \\
\downarrow{\scriptstyle \psi_l} & & \downarrow{\scriptstyle \psi_r} \\
(a)\alpha_r(i \oplus j) & \xrightarrow{\qquad \sigma(i \oplus j) \qquad} & \alpha_l(i \oplus j)(a)
\end{array}
$$

for every i, j objects in $\mathcal{I}$ and a, a_1, a_2 objects in $\mathcal{A}$.

A morphism from (a_1, σ_1) to (a_2, σ_2) is a morphism $m : a_1 \to a_2$ in $\mathcal{A}$ such that the diagram

$$
\begin{array}{ccc}
(a_1)\alpha_r(i) & \xrightarrow{\quad (m)\alpha_r(i) \quad} & (a_2)\alpha_r(i) \\
\downarrow{\scriptstyle \sigma_1(i)} & & \downarrow{\scriptstyle \sigma_2(i)} \\
\alpha_l(i)(a_1) & \xrightarrow{\quad \alpha_l(i)(m) \quad} & \alpha_l(i)(a_2)
\end{array}
$$

is commutative for all i object in I.

We denote the centralizer by $\mathcal{C}_{\mathcal{A}}(\mathcal{I})$. The first diagram in the above definition shows that σ is compatible with the monoidal product.

Remark 4. If we choose σ in the above definition a natural transformation instead, we call the centralizer as *pseudo-centralizer* of $\mathcal{I}$ in $\mathcal{A}$, and denote by $\mathcal{C}_{\mathcal{A}}(\mathcal{I})^P$. Suppose that $\mathcal{A}$ is a category with weak equivalences. Let us choose σ in the above definition a natural weak equivalence. In this case we call the centralizer as *the weak centralizer* of $\mathcal{I}$ in $\mathcal{A}$, and denote by $\mathcal{C}_{\mathcal{A}}(\mathcal{I})^W$.

Definition 5.1.3. Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathcal{I}$-categories with $\mathcal{I}$-actions α, β, respectively. Let i, j be objects in $\mathcal{I}$, $m : i \to j$ be a morphism in $\mathcal{I}$ and $f : \mathcal{A} \to \mathcal{B}$ be a functor. The induced $\mathcal{I}$-action

$$[\alpha, \beta] = ([\alpha, \beta]_l, [\alpha, \beta]_r)$$

on $[\mathcal{A}, \mathcal{B}]$ is defined by

- $[\alpha, \beta]_l : I \to \mathrm{End}_l([\mathcal{A}, \mathcal{B}])$ is the monoidal functor such that the the functor $[\alpha, \beta]_l(i)(f)$ is the composition:

$$\mathcal{A} \xrightarrow{\alpha_r(i)} \mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{\beta_l(i)} \mathcal{B}$$

- $[\alpha, \beta]_r : I \to \mathrm{End}_r([\mathcal{A}, \mathcal{B}])$ is the monoidal functor such that the functor $(f)[\alpha, \beta]_r(i)$ is the composition:

$$\mathcal{A} \xrightarrow{\alpha_l(i)} \mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{\beta_r(i)} \mathcal{B}$$

- the natural transformation $[\alpha, \beta]_l(m)(f)(a)$ is given by the morphism in $\mathcal{B}$ by the composition

$$\begin{array}{ccc} \beta_l(i)(f((a)\alpha_r(i))) & \xrightarrow{[\alpha,\beta]_l(m)(f)(a)} & \beta_l(j)(f((a)\alpha_r(j))) \\ & {\scriptstyle \beta_l(m)(f((a)\alpha_r(i)))} \searrow \quad \nearrow {\scriptstyle \beta_l(j)(f((a)\alpha_r(m)))} & \\ & \beta_l(j)(f((a)\alpha_r(i))) & \end{array}$$

  and the natural transformation $(f)[\alpha, \beta]_r(m)(a)$ is given by the morphism in $\mathcal{B}$ by the composition

$$\begin{array}{ccc} (f(\alpha_l(i)(a)))\beta_r(i) & \xrightarrow{(f)[\alpha,\beta]_r(m)(a)} & (f(\alpha_l(j)(a)))\beta_r(j) \\ & {\scriptstyle (f(\alpha_l(i)(a)))\beta_r(m)} \searrow \quad \nearrow {\scriptstyle (f(\alpha_l(m)(a)))\beta_r(j)} & \\ & (f(\alpha_l(i)(a)))\beta_r(j) & \end{array}$$

Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathcal{I}$-categories with actions α and β be $\mathcal{I}$-action respectively. Consider the $\mathcal{I}$-action $[\alpha, \beta]$ on $[\mathcal{A}, \mathcal{B}]$.

Definition 5.1.4. We define $\mathcal{M}ap_{\mathcal{I}}(\mathcal{A}, \mathcal{B})$ namely the set of $\mathcal{I}$-equivariant functors from $\mathcal{A}$ to $\mathcal{B}$ as the centralizer of $\mathcal{I}$ in $[\mathcal{A}, \mathcal{B}]$; i.e.,

$$\mathcal{M}ap_{\mathcal{I}}(\mathcal{A}, \mathcal{B}) = \mathcal{C}_{[\mathcal{A},\mathcal{B}]}(\mathcal{I}).$$

Hence a functor is called a $\mathcal{I}$-equivariant functor from $\mathcal{A}$ to $\mathcal{B}$ if it is in $\mathcal{M}ap_{\mathcal{I}}(\mathcal{A}, \mathcal{B})$.

In particular if $f : \mathcal{A} \to \mathcal{B}$ is $\mathcal{I}$-equivariant functor then

$$\sigma(i) : (f(\alpha_l(i)(a)))\beta_r(i) \to \beta_l(i)(f((a)\alpha_r(i)))$$

is a natural isomorphism for all i object in $\mathcal{I}$ and a object in $\mathcal{A}$, such that the diagram commutes:

$$\begin{array}{ccc}
((f(\alpha_l(j) \circ \alpha_l(i)(a)))\beta_r(i))\beta_r(j) & \xrightarrow{[\alpha,\beta]_l(i)(\sigma(j)) \circ (\sigma(i))[\alpha,\beta]_r(j)} & \beta_l(j) \circ \beta_l(i)(f((a)\alpha_r(i)\alpha_r(j))) \\
\downarrow \cong & & \downarrow \cong \\
(f(\alpha_l(i \oplus j)(a)))\beta_r(i \oplus j) & \xrightarrow{\sigma(i \oplus j)} & \beta_l(i \oplus j)(f((a)\alpha_r(i \oplus j)))
\end{array}$$

i.e., σ is compatible with the monoidal product as in the sense of Definition 5.1.2.

Given a category $\mathcal{A}$ with an $\mathcal{I}$-action, the fixed-point category of the action is defined as

$$\mathcal{F}ix_I(\mathcal{A}) = \mathcal{M}ap_{\mathcal{I}}(*, \mathcal{A}).$$

Here $*$ is the trivial category; which is a category with a single object and single morphism which is the identity morphism of that object. The action on $*$ here is the trivial I-action. As in the previous section, we have an analogues theorem:

Theorem 5.1.1. *Let $\mathcal{I}$ be a monoidal category and $\mathcal{A}$, $\mathcal{B}$ be two categories with $\mathcal{I}$-actions. Then we have an isomorphism*

$$\mathcal{M}ap_{\mathcal{I}}(\mathcal{A}, \mathcal{B}) \cong \mathcal{F}ix_{\mathcal{I}}([\mathcal{A}, \mathcal{B}]).$$

The proof is similar to the one in the previous section since we have an isomorphism of categories $\mathcal{C}_{\mathcal{A}}(\mathcal{I}) \cong \mathcal{C}_{[*,\mathcal{A}]}(\mathcal{I})$ for any $\mathcal{I}$-category $\mathcal{A}$.

Remark 5. Following the remark 4 above, we define the categories $\mathcal{M}ap^{W}_{\mathcal{I}}(\mathcal{A},\mathcal{B})$ as $\mathcal{C}_{[\mathcal{A},\mathcal{B}]}(\mathcal{I})^{W}$ and $\mathcal{M}ap^{Ps}_{\mathcal{I}}(\mathcal{A},\mathcal{B})$ as $\mathcal{C}_{[\mathcal{A},\mathcal{B}]}(\mathcal{I})^{Ps}$. We will call a functor from $\mathcal{A}$ to $\mathcal{B}$ *a weakly $\mathcal{I}$-equivariant functor* if it belongs to $\mathcal{M}ap^{W}_{\mathcal{I}}(\mathcal{A},\mathcal{B})$ and *a pseudo-$\mathcal{I}$-equivariant functor* if it belongs to $\mathcal{M}ap^{Ps}_{\mathcal{I}}(\mathcal{A},\mathcal{B})$.

Let $(\mathcal{A},\alpha)$, $(\mathcal{B},\beta)$, $(\mathcal{C},\gamma)$, and $(\mathcal{D},\delta)$ be four $\mathcal{I}$-categories. Assume $f:\mathcal{A}\to\mathcal{B}$ and $h:\mathcal{C}\to\mathcal{D}$ be two functors. Then f and h induces a functor $[\mathcal{B},\mathcal{C}]\to[\mathcal{A},\mathcal{D}]$ by composition. We have the following proposition:

Proposition 5.1.2. *If $f:\mathcal{A}\to\mathcal{B}$ and $h:\mathcal{C}\to\mathcal{D}$ are two $\mathcal{I}$-equivariant functors then the induced functor $[\mathcal{B},\mathcal{C}]\to[\mathcal{A},\mathcal{D}]$ by f and h is $\mathcal{I}$-equivariant.*

Proof. Since f and h are $\mathcal{I}$-equivariant, on objects we have

$$(h(\gamma_l(i)(g((f(\alpha_l(i)(a)))\beta_r(i)))))\delta_r(i) \cong \delta_l(i)(h((g(\beta_l(i)(f((a)\alpha_r(i)))))\gamma_r(i)))$$

for every object a in $\mathcal{A}$, i in $\mathcal{I}$ and g in $[\mathcal{B},\mathcal{C}]$. Hence we have the natural isomorphism

$$(h\circ(([\beta,\gamma]_l(i)(g))\circ f))[\alpha,\delta]_r(i) \cong [\alpha,\delta]_l(i)(h\circ(((g)[\beta,\gamma]_r(i))\circ f))$$

for every object i in $\mathcal{I}$ and g in $[\mathcal{B},\mathcal{C}]$. This means the induced functor from $[\mathcal{B},\mathcal{C}]$ to $[\mathcal{A},\mathcal{D}]$ is $\mathcal{I}$-equivariant. The coherence conditions are evident. □

Proposition 5.1.3. *The functor $F : [\mathcal{A}\times\mathcal{B},\mathcal{C}] \to [\mathcal{A},[\mathcal{B},\mathcal{C}]]$ defined by $F(f)(a)(b) = f(a,b)$ is an equivariant equivalence of categories with respect to the induced actions.*

Proof. Let $f:\mathcal{A}\times\mathcal{B}\to\mathcal{C}$ be a functor. Let a be an object in $\mathcal{A}$ and b be an object in $\mathcal{B}$. From the definitions we have the identities

$$[\alpha,[\beta,\gamma]]_l(i)(F((f)[\alpha\times\beta,\gamma]_r(i)))(a)(b) = \gamma(i)(f(\alpha(i)(a),\beta(i)(a))$$

and

$$(F([\alpha\times\beta,\gamma]_l(i)(f)))[\alpha,[\beta,\gamma]]_r(i)(a)(b) = \gamma(i)(f(\alpha(i)(a),\beta(i)(a))$$

in $\mathcal{C}$. Hence the identity map from $(F([\alpha \times \beta, \gamma]_l(i)(f)))[\alpha, [\beta, \gamma]]_r(i)(a)(b)$ to $[\alpha, [\beta, \gamma]]_l(i)(F((f)[\alpha \times \beta, \gamma]_r(i)))(a)(b)$ gives us a natural isomorphism. The above coherence condition can easily be checked.

□

5.2 Reverse actions on categories

In this section we define the reverse action of a given action of a monoidal category on a category. We start with defining four monoidal functors from a given monoidal category $\mathcal{I}$ to $\mathrm{End}_x(\mathcal{I})$, where $x \in \{l, r\}$. The functors $\iota_x : \mathcal{I} \to \mathrm{End}_x(\mathcal{I})$ sends every object to identity endofunctor and the functors

$$\mu_l : \mathcal{I} \to \mathrm{End}_l(\mathcal{I}) \quad \text{and} \quad \mu_r : \mathcal{I} \to \mathrm{End}_r(\mathcal{I})$$

are given by applying the monoid operation from left and right respectively.

Just as in the case of monoids, we will consider the monoidal category $\mathcal{I}$ itself as an $\mathcal{I}$-category with the action (ι_l, μ_r) and (μ_l, ι_r) to define the reverse actions. First we define reversing action from left to right, and thus, will consider $\mathcal{I}$ with the action (ι_l, μ_r).

Let α be an $\mathcal{I}$-action on a category $\mathcal{A}$. We define the reverse $\mathcal{I}$-action θ on $\mathcal{M}ap_{\mathcal{I}}(\mathcal{I}, \mathcal{A})$ as follows: Let $f \in \mathcal{M}ap_{\mathcal{I}}(\mathcal{I}, \mathcal{A})$ be a $\mathcal{I}$-equivariant functor. For all $i \in \mathcal{I}$ define $\theta_l(i)$ as the identity functor on $\mathcal{M}ap_{\mathcal{I}}(\mathcal{I}, \mathcal{A})$ and $\theta_r(i)$ as the functor which sends f to the composition

$$\mathcal{I} \xrightarrow{\mu_l(i)} \mathcal{I} \xrightarrow{f} \mathcal{A}$$

On morphisms, if $m : i \to j$ is a morphism in $\mathcal{I}$, then for every k in $\mathcal{I}$, $\theta_l(m)(f)(k)$ is the morphism $f(m)$ in $\mathcal{A}$ and $\theta_r(m)(f)$ is the natural transformation such that for k in $\mathcal{I}$ it is the morphism

$$f(\mu_l(m)(k)) : f \circ \mu_l(i)(k) \to f \circ \mu_l(j)(k)$$

in $\mathcal{A}$. Let (f, σ) be an object in $\mathcal{M}ap_{\mathcal{I}}(\mathcal{I}, \mathcal{A})$. Then for each object i, j in $\mathcal{I}$,

$$\sigma(i) : (f(j))\alpha_r(i) \to \alpha_l(i)(f(j \oplus i))$$

is an isomorphism. For an object k in $\mathcal{I}$, define the morphism

$$\sigma_k(i) : (f \circ \mu_l(k)(j))\alpha_r(i) \to \alpha_l(i)(f \circ \mu_l(k)(j \oplus i))$$

as the composition

$$f(k \oplus j)\alpha_r(i) \xrightarrow{\sigma(i)} \alpha_l(i)(f((k \oplus j) \oplus i)) \xrightarrow{\cong} \alpha_l(i)(f(k \oplus (j \oplus i))).$$

Hence, $(f \circ \mu_l(k), \sigma_k)$ is an object in $\mathcal{M}ap_{\mathcal{I}}(\mathcal{I}, \mathcal{A})$. As a result we defined the categorical version of the first reverse action, which reverses the left component of the action.

Now consider $\mathcal{I}$ with the action (μ_l, ι_r). We define the other reverse action $\mathcal{I}$-action ϑ on $\mathcal{M}ap_{\mathcal{I}}(\mathcal{I}, \mathcal{A})$ as follows: For $f \in \mathcal{M}ap_{\mathcal{I}}(\mathcal{I}, \mathcal{A})$ an $\mathcal{I}$-equivariant functor, define $\vartheta_r(i)$ as the identity functor on $\mathcal{M}ap_{\mathcal{I}}(\mathcal{I}, \mathcal{A})$ and $\vartheta_l(i)$ as the functor which sends f to the composition

$$\mathcal{I} \xrightarrow{\mu_r(i)} \mathcal{I} \xrightarrow{f} \mathcal{A}$$

On morphisms, if $m : i \to j$ is a morphism in $\mathcal{I}$, then for every k in $\mathcal{I}$, $\vartheta_l(m)(f)(k)$ is the morphism $f(m)$ in $\mathcal{A}$ and $(f)\vartheta_r(m)$ is the natural transformation sending an object k in $\mathcal{I}$ to the morphism

$$f((k)\mu_r(m)) : f \circ \mu_r(i)(k) \to f \circ \mu_r(j)(k)$$

in $\mathcal{A}$. Let (f, ς) be an object in $\mathcal{M}ap_{\mathcal{I}}(\mathcal{I}, \mathcal{A})$. Then, by definition for each object j in $\mathcal{I}$,

$$\varsigma(i) : (f(i \oplus j))\alpha_r(i) \to \alpha_l(i)(f(j))$$

is an isomorphism. For an object k in $\mathcal{I}$, we define the morphism

$$\varsigma_k(i) : (f \circ \mu_r(k)(i \oplus j))\alpha_r(i) \to \alpha_l(i)(f \circ \mu_r(k)(j))$$

as the composition

$$(f((i \oplus j) \oplus k))\alpha_r(i) \xrightarrow{\cong} (f(i \oplus (j \oplus k)))\alpha_r(i) \xrightarrow{\varsigma(i)} \alpha_l(i)(f(j \oplus k)).$$

Hence, $(f \circ \mu_l(k), \varsigma_k)$ is an object in $\mathcal{M}ap_{\mathcal{I}}(\mathcal{I}, \mathcal{A})$. As a result we get the second reverse action, which reverses the right component.

Assume now $\mathcal{I}$ is a symmetric monoidal category and $\mathcal{A}$ be an $\mathcal{I}$-category with the action α whose right component is trivial. Consider $\mathcal{I}$ with the action (ι_l, μ_r). Define an action α' on $\mathcal{M}ap_{\mathcal{I}}(\mathcal{I}, \mathcal{A})$ as follows: define α'_l as the trivial action and for $f : \mathcal{I} \to \mathcal{A}$ an object define $(f)\alpha'_r(i)(j)$ as the object $(f(j))\alpha_r(i)$. A morphism $m : i \to j$ is sent to the obvious morphism in $\mathcal{M}ap_{\mathcal{I}}(\mathcal{I}, \mathcal{A})$. Then, we have

$$(((f)\alpha'_r(i))\theta_r(i))(j) = (((f)\alpha'_r(i))(i \oplus j) = (f(i \oplus j))\alpha_r(i)$$

which is isomorphic to $f(j)$. Similarly,

$$(((f)\theta_r(i))\alpha'_r(i))(j) = ((f)\theta_r(i)(j))\alpha_r(i) = (f(i \oplus j))\alpha_r(i)$$

which is again isomorphic to $f(j)$. Hence, we get the following lemma:

Lemma 5.2.1. *If $\mathcal{I}$ is a symmetric monoidal category and α is an $\mathcal{I}$-action on $\mathcal{A}$ with trivial right component, then the reverse action θ, that reverses the left component of the action, is an auto-equivalence of $\mathcal{M}ap_{\mathcal{I}}(\mathcal{I}, \mathcal{A})$; i.e., an equivalence as an endofunctor on $\mathcal{M}ap_{\mathcal{I}}(\mathcal{I}, \mathcal{A})$.*

A similar lemma can be derived for the second reverse action, ϑ. If we drop the assumption of being symmetric monoidal, then this lemma fails. However, if $\mathcal{A}$ as in above lemma, we can assign a reversible action by first reversing the left component, and then reversing the first reverse action θ, then continue reversing the appearing left and right reverse actions ad infinitum. That is, the reversible action corresponds to an inverse limit in the two category of category of categories. For convenience, we will continue with ordinary reverse actions. We mainly use the first reverse action above on actions with the right component is trivial for convenience. Hence, we just write $\mathcal{M}ap_{\mathcal{I}}(\mathcal{I}, \mathcal{A})$ and we understand the first reverse action; i.e., the reverse from left to right. The isomorphism σ will play an important role in the next chapter, as it is the key point of defining homotopical stabilization in our theories.

5.2.0.1 Objects of the reverse actions

We will discuss how the objects in $\mathcal{M}ap_{\mathcal{I}}(\mathcal{I}, \mathcal{A})$ will look like, where $\mathcal{I}$ is equipped with the action (ι_l, μ_r). An object of $\mathcal{M}ap_{\mathcal{I}}(\mathcal{I}, \mathcal{A})$ is a functor $f : \mathcal{I} \to \mathcal{A}$, which is $\mathcal{I}$ equivariant. Thus, there exist a natural isomorphism

$$\sigma(i) : (f(j))\alpha_r(i) \to \alpha_l(i)(f(j \oplus i))$$

such that the following diagram commutes

$$\begin{array}{ccc}
 & \alpha_l(i)((f(k \oplus i))\alpha_r(j)) = (\alpha_l(i)(f(k \oplus i)))\alpha_r(j) & \\
 \nearrow^{(\sigma(i))\alpha_r(j)} & & \searrow^{\tilde{\sigma}(j)} \\
((f(k))\alpha_r(i))\alpha_r(j) & \longrightarrow & \alpha_l(j)(\alpha_l(i)(f(k \oplus i \oplus j))) \\
\downarrow \cong & & \downarrow \cong \\
(f(k))\alpha_r(i \oplus j) & \xrightarrow{\sigma(i \oplus j)} & \alpha_l(i \oplus j)(f(k \oplus i \oplus j))
\end{array}$$

where $\tilde{\sigma}(j)$ is the corresponding natural transformation. Since σ is a natural transformation, for any morphism $m : i \to j$ in $\mathcal{I}$, the following diagram commutes:

$$\begin{array}{ccc}
(f(k))\alpha_r(i) & \xrightarrow{\sigma(i)} & \alpha_l(i)(f(k \oplus i)) \\
\downarrow (id)\alpha_r(m) & & \downarrow \alpha_l(m)(id \oplus m) \\
(f(k))\alpha_r(j) & \xrightarrow{\sigma(j)} & \alpha_l(j)(f(k \oplus j))
\end{array}$$

The map $\alpha_l(m)(id \oplus m)$ factors through $\alpha_l(j)(f(k \oplus i))$, due to Definition 5.1.3. These commuting diagrams will be essential elements of the theories we will build in the next chapter.

5.2.1 Opposite actions on categories

Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. There is an induced functor $F^{\text{op}} : \mathcal{A}^{\text{op}} \to \mathcal{B}^{\text{op}}$ on opposite categories defined as $F^{\text{op}}(a) = F(a)$ for any object a in $\mathcal{A}$ and for any morphism $f : a \to b$, $F^{\text{op}}(f)$ is

$$F(f)^{\text{op}} : F(b) \to F(a).$$

This functor is called the opposite functor.

Now we will define two other actions which can be assigned to an actions of a monoidal category. These actions will be called opposite actions. Let $\alpha = (\alpha_l, \alpha_r)$ be an $\mathcal{I}$-action on a category $\mathcal{A}$. Then

- We define a new $\mathcal{I}^{\text{op}}$ action $\widetilde{\alpha} = (\widetilde{\alpha}_l, \widetilde{\alpha}_r)$ on $\mathcal{A}^{\text{op}}$ as follows: for $x \in \{l, r\}$, for an object i in $\mathcal{I}^{\text{op}}$, and for $m : i \to j$ a morphism in $\mathcal{I}$ we have $\widetilde{\alpha}_x(i) = \alpha_x(i)^{\text{op}}$ and the natural transformation

  $$\widetilde{\alpha}_x(m^{\text{op}}) : \alpha_x(j)^{\text{op}} \to \alpha_x(i)^{\text{op}}$$

  is defined for every object a in $\mathcal{A}^{op}$, as

  $$\alpha_x(m)(a)^{\text{op}} : \alpha_x(j)(a) \to \alpha_x(i)(a)$$

  in A^{op}.

- Assume that $\alpha_x(m)$ is a natural isomorphism for all $x \in \{l, r\}$ and for all $m : i \to j$ morphism in I. Then we can define a new $\mathcal{I}^{\text{op}}$ action $\overline{\alpha} = (\overline{\alpha}_l, \overline{\alpha}_r)$ on $\mathcal{A}$ as follows: for $x \in \{l, r\}$, for an object i in $\mathcal{I}^{\text{op}}$, and for $m : i \to j$ a morphism in $\mathcal{I}$ we have $\overline{\alpha}_x(i) = \alpha_x(i)$ and the natural transformation

  $$\overline{\alpha}_x(m^{\text{op}}) : \alpha_x(j) \to \alpha_x(i)$$

  is defined for every object a in $\mathcal{A}$, as

  $$\alpha_x(m)(a)^{-1} : \alpha_x(j)(a) \to \alpha_x(i)(a)$$

  in $\mathcal{A}$.

These two actions are usually different than reverse actions. Observe that the latter is not always defined. We will use these actions in the definitions (co)homology theories and (co)homotopy theories.

Chapter 6

(Co)homology and (co)homotopy theories graded over monoidal categories

One can consider (co)homology and (co)homotopy theories as measurement tools that can be used to decide how far a map is from being a weak equivalence. The general idea of measurement is 'assigning an object with well-known properties to one with relatively not well-known properties'. For example, abelian groups are well-known algebraic objects whose properties and classification is easy. On the other hand topological spaces are relatively harder to understand. Hence, one universally assign an abelian group to a space which is invariant under certain properties, such as homotopy equivalence. As doing so, we get a nice measurement tool for measuring homotopical properties of spaces.

Before defining (co)homology and (co)homotopy theories, we first define the categories of (co)homology and (co)homotopy functors. Our purpose is to define these functors as general as possible. Thus, these functors will be defined on a homotopical category $\mathcal{M}$ whose properties are to be understood and will assign objects of $\mathcal{M}$ to another homotopical category $\mathcal{S}$, whose objects are structurally better understood up to weak equivalence.

6.1 (Co)homology functors and (co)homotopy functors

Let $\mathcal{M}$ and $\mathcal{S}$ be two categories with weak equivalences and a zero object

1. A functor from $\mathcal{M}$ to $\mathcal{S}$ will be called a homology functor if it sends weak equivalences to weak equivalences, cofiber sequences in $\mathcal{M}$ to exact sequences in $\mathcal{S}$, and coproducts in $\mathcal{M}$ to coproducts in $\mathcal{S}$. The full subcategory of $[\mathcal{M}, \mathcal{S}]$ with objects the homology functors from $\mathcal{M}$ to $\mathcal{S}$ will be denoted by $\mathrm{hml}(\mathcal{M}, \mathcal{S})$.

2. A functor from $\mathcal{M}^{\mathrm{op}}$ to $\mathcal{S}$ will be called a cohomology functor if it sends weak equivalences to weak equivalences, cofiber sequences in $\mathcal{M}$ to exact sequences in $\mathcal{S}$, and coproducts in $\mathcal{M}$ to products in $\mathcal{S}$. The full subcategory of $[\mathcal{M}^{\mathrm{op}}, \mathcal{S}]$ with objects the cohomology functors from $\mathcal{M}^{\mathrm{op}}$ to $\mathcal{S}$ will be denoted by $\mathrm{cohml}(\mathcal{M}, \mathcal{S})$.

3. A functor from $\mathcal{M}$ to $\mathcal{S}$ will be called a homotopy functor if it sends weak equivalences to weak equivalences, fiber sequences in $\mathcal{M}$ to exact sequences in $\mathcal{S}$, and products in $\mathcal{M}$ to products in $\mathcal{S}$. The full subcategory of $[\mathcal{M}, \mathcal{S}]$ with objects the homotopy functors from $\mathcal{M}$ to $\mathcal{S}$ will be denoted by $\mathrm{hty}(\mathcal{M}, \mathcal{S})$.

4. A functor from $\mathcal{M}^{\mathrm{op}}$ to $\mathcal{S}$ will be called a cohomotopy functor if it sends weak equivalences to weak equivalences, fiber sequences in $\mathcal{M}$ to exact sequences in $\mathcal{S}$, and products in $\mathcal{M}$ to products in $\mathcal{S}$. The full subcategory of $[\mathcal{M}^{\mathrm{op}}, \mathcal{S}]$ with objects the cohomotopy functors from $\mathcal{M}^{\mathrm{op}}$ to $\mathcal{S}$ will be denoted by $\mathrm{cohty}(\mathcal{M}, \mathcal{S})$.

The usual definitions of (co)-homology and (co)-homotopy functors (see [9], [18], [19], [20]) are particular cases of these definitions except few cases of cohomotopy groups. For example, take $\mathcal{M}$ as the category of pointed topological spaces and $\mathcal{S}$ as the category of abelian groups then singular homology H_n and cohomology H^n functors are respectively homology and cohomology functors for all $n \geqslant 0$. Take $\mathcal{M}$ as the category of pointed topological spaces and $\mathcal{S}$ as the category of groups then π_n is a homotopy functor for all $n \geqslant 1$. The cohomotopy groups which take a pointed topological space $(X, *)$ and gives out the set of homotopy classes of maps $[(X, *), (S^n, *)]$ is not a cohomotopy functor but the usual definition of cohomotopy groups for spectra is a cohomotopy functor.

6.2 (Co)homology theories and (co)homotopy theories

Let $\mathcal{I}$ be a symmetric monoidal category and $\mathcal{M}$ be a category with weak equivalences. Let $\mathbb{1}_{\mathcal{M}}$ denote the trivial action on $\mathcal{M}$. An action $\Sigma = (\mathbb{1}_{\mathcal{M}}, \Sigma_r)$ of $\mathcal{I}$ on $\mathcal{M}$ is called suspension action if for each i object in $\mathcal{I}$, $x \in \{l, r\}$ the functor $\Sigma_x(i)$ preserves weak equivalences and homotopy colimits. An action $\Omega = (\Omega_l, \mathbb{1}_{\mathcal{M}})$ of $\mathcal{I}$ on $\mathcal{M}$ is called looping action if for each i object in $\mathcal{I}$, $x \in \{l, r\}$ the functor $\Omega_x(i)$ preserves weak equivalences and homotopy limits.

Let $\mathcal{M}$ and $\mathcal{S}$ be two pointed homotopical categories. Let Σ be a suspension action of $\mathcal{I}$ on $\mathcal{M}^{\text{op}}$, Ω be a looping action of $\mathcal{I}$ on $\mathcal{M}$, and $s = (\mathbb{1}_{\mathcal{S}}, \mathbb{1}_{\mathcal{S}})$ be the trivial action on $\mathcal{S}$. Now we define (co)homology and (co)homotopy theories as follows:

1. Suppose that $\overline{s}$ can be defined. The $\mathcal{I}^{\text{op}}$-action $[\widetilde{\Sigma}, \overline{s}]$ on $[\mathcal{M}, \mathcal{S}]$ induces an $\mathcal{I}^{\text{op}}$-action on $\text{hml}(\mathcal{M}, \mathcal{S})$. Then the category of homology theories graded over $\mathcal{I}$ is defined as

$$\text{HML}_{\mathcal{I}}(\mathcal{M}, \mathcal{S}) = \mathcal{M}ap^{W}_{\mathcal{I}^{\text{op}}}(\mathcal{I}^{\text{op}}, \text{hml}(\mathcal{M}, \mathcal{S}))$$

namely the category having the reverse action of $\mathcal{I}^{\text{op}}$ on $\text{hml}(\mathcal{M}, \mathcal{S})$. Here we say that a homology theory is an object in the category of homology theories.

2. The $\mathcal{I}$-action $[\Sigma, s]$ on $[\mathcal{M}^{\mathrm{op}}, \mathcal{S}]$ induces a $\mathcal{I}$-action on $\mathrm{cohml}(\mathcal{M}, \mathcal{S})$. Then the category of cohomology theories graded over $\mathcal{I}$ is defined as

$$\mathrm{COHML}_{\mathcal{I}}(\mathcal{M}, \mathcal{S}) = \mathcal{M}ap^{W}_{\mathcal{I}}(\mathcal{I}, \mathrm{cohml}(\mathcal{M}, \mathcal{S}))$$

namely the category having the reverse action of $\mathcal{I}$ on $\mathrm{cohml}(\mathcal{M}, \mathcal{S})$. Here we say that a cohomology theory is an object in the category of cohomology theories.

3. The $\mathcal{I}$-action $[\Omega, s]$ on $[\mathcal{M}, \mathcal{S}]$ induces an $\mathcal{I}$-action on $\mathrm{hty}(\mathcal{M}, \mathcal{S})$. Then the category of homotopy theories graded over $\mathcal{I}$ is defined as

$$\mathrm{HTY}_{\mathcal{I}}(\mathcal{M}, \mathcal{S}) = \mathcal{M}ap^{W}_{\mathcal{I}}(\mathcal{I}, \mathrm{hty}(\mathcal{M}, \mathcal{S}))$$

namely the category having the reverse action of $\mathcal{I}$ on $\mathrm{hty}(\mathcal{M}, \mathcal{S})$. Here we say that a homotopy theory is an object in the category of homotopy theories.

4. Suppose that $\overline{s}$ can be defined. The $\mathcal{I}^{\mathrm{op}}$-action $[\widetilde{\Omega}, \overline{s}]$ on $[\mathcal{M}^{\mathrm{op}}, \mathcal{S}]$ induces an $\mathcal{I}^{\mathrm{op}}$-action on $\mathrm{cohty}(\mathcal{M}, \mathcal{S})$. Then the category of cohomotopy theories graded over $\mathcal{I}$ is defined as

$$\mathrm{COHTY}_{\mathcal{I}}(\mathcal{M}, \mathcal{S}) = \mathcal{M}ap^{W}_{\mathcal{I}^{\mathrm{op}}}(\mathcal{I}^{\mathrm{op}}, \mathrm{cohty}(\mathcal{M}, \mathcal{S}))$$

namely the category having the reverse action of $\mathcal{I}^{\mathrm{op}}$ on $\mathrm{cohty}(\mathcal{M}, \mathcal{S})$. Here we say that a cohomotopy theory is an object in the category of cohomotopy theories.

Note that $\overline{s}$ can be defined if we choose $\mathcal{I}$ as the core of a monoidal category.

The collection of all functors $\mathcal{M} \to \mathcal{M}$ which preserve weak equivalences and homotopy colimits form a monoidal category by composition as the monoid operation. Let $\mathcal{M}^{\Sigma}$ denote this monoidal category. Then the (co)homology theories could be graded on a monoidal subcategory of this monoidal category. Similarly the collection of all functors $\mathcal{M} \to \mathcal{M}$ which preserve weak equivalences and homotopy limits form a monoidal category and denote this monoidal category by $\mathcal{M}^{\Omega}$. Then the (co)homotopy theories could be graded on a monoidal subcategory of this monoidal category.

6.3 Lifting Problem, Representability and Spectra

Let $\mathcal{I}$ be a symmetric monoidal category and $\mathcal{M}$, $\mathcal{S}$ be two categories with weak equivalences. Assume both categories $\mathcal{M}$ and $\mathcal{S}$ have a zero object. Let s be the trivial $\mathcal{I}$-action on $\mathcal{S}$. Let Σ be a suspension action of $\mathcal{I}$ on $\mathcal{M}^{\mathrm{op}}$ and Ω be a loop action of $\mathcal{I}$ on $\mathcal{M}$. Let T be another category with weak equivalences with a $\mathcal{I}$-action t such that $t(v)$ preserves weak equivalences for every v object in $\mathcal{I}$. Assume that there exist a $\mathcal{I}$-equivariant functor

$$\mathrm{m} : T \to \mathcal{S}.$$

The functor m induces the following functors:

1. Considering the $\mathcal{I}^{\mathrm{op}}$-actions $\widetilde{\Sigma}$, $\overline{s}$, and $\overline{t}$ on $\mathcal{M}$, $\mathcal{S}$, and T respectively, mor induces a $\mathcal{I}^{\mathrm{op}}$-equivariant functor
$$\mathrm{R}_{\mathrm{hml}} : [\mathcal{M}_{\widetilde{\Sigma}}, T_{\overline{t}}] \to [\mathcal{M}_{\widetilde{\Sigma}}, \mathcal{S}_{\overline{s}}].$$
2. Considering the $\mathcal{I}$-actions Σ, s, and t on $\mathcal{M}^{\mathrm{op}}$, $\mathcal{S}$, and T respectively, mor induces a $\mathcal{I}$-equivariant functor
$$\mathrm{R}_{\mathrm{chm}} : [\mathcal{M}^{\mathrm{op}}_{\Sigma}, T_t] \to [\mathcal{M}^{\mathrm{op}}_{\Sigma}, \mathcal{S}_s].$$
3. Considering the $\mathcal{I}$-actions Ω, s, and t on $\mathcal{M}$, $\mathcal{S}$, and T respectively, mor induces a $\mathcal{I}$-equivariant functor
$$\mathrm{R}_{\mathrm{hty}} : [\mathcal{M}_{\Omega}, T_t] \to [\mathcal{M}_{\Omega}, \mathcal{S}_s].$$
4. Considering the $\mathcal{I}^{\mathrm{op}}$-actions $\widetilde{\Omega}$, $\overline{s}$, and $\overline{t}$ on $\mathcal{M}^{\mathrm{op}}$, $\mathcal{S}$, and T respectively, mor induces a $\mathcal{I}^{\mathrm{op}}$-equivariant functor
$$\mathrm{R}_{\mathrm{cht}} : [\mathcal{M}^{\mathrm{op}}_{\widetilde{\Omega}}, T_{\overline{t}}] \to [\mathcal{M}^{\mathrm{op}}_{\widetilde{\Omega}}, \mathcal{S}_{\overline{s}}].$$

In part (1) and (4) above, we further require that $\overline{s}$ and $\overline{t}$ can be defined. Given mor as above, one can state a lifting problem from (co)homology and (co)homotopy functors with values in $\mathcal{S}$ to (co)homology and (co)homotopy functors with values in T by using respectively defined functors R_*.

6.3.1 Representation of (Co)homology and (Co)homotopy Theories

Let $\mathcal{I}$ be a symmetric monoidal category, $\mathcal{M}$, $\mathcal{S}$ be two categories, Σ be a suspension $\mathcal{I}$-action on $\mathcal{M}^{\rm op}$, Ω be a looping $\mathcal{I}$-action on $\mathcal{M}$, and s be the trivial $\mathcal{I}$-action on $\mathcal{S}$ as in the previous section. In part (1) and (4) below, we further assume that $\overline{\Sigma}$, $\overline{\Omega}$ and $\overline{s}$ can be defined. Suppose that we are given a $\mathcal{I}$-equivariant functor

$$\mathrm{mor} : \mathcal{M}^{\rm op} \times \mathcal{M} \to \mathcal{S}$$

where the $\mathcal{I}$-action on $\mathcal{M}^{\rm op} \times \mathcal{M}$ is $\Sigma \times \Omega$. Then the equivariance of $\mathrm{mor}(-,-)$ implies that for all x and y objects in $\mathcal{M}$, we have the isomorphism

$$\mathrm{mor}((x)\Sigma_r(v), y) \cong \mathrm{mor}(x, \Omega_l(v)(y))$$

in $\mathcal{S}$. Observe that in the case when mor is the internal-hom and $\mathcal{M}$ is $\mathcal{S}$-enriched (with the assumption that $\mathcal{S}$ is symmetric monoidal), we have $\Sigma_r(v)$ and $\Omega_l(v)$ are adjoint pairs.

We define representability of (co)homology and (co)homotopy theories as particular instances of lifting problems, by taking T as $\mathcal{M}^{\rm op} \times \mathcal{M}$, t as $\Sigma \times \Omega$, and m as mor. In this section, we use the notation $\mathcal{C}_\alpha$ for any category $\mathcal{C}$ with an action α.

1. The functor $\mathrm{R}_{\rm hml}$ induces a functor

   $$\mathcal{M}ap^{W}_{\mathcal{I}^{\rm op}}(\mathcal{I}^{\rm op}, [\mathcal{M}_{\tilde{\Sigma}}, \mathcal{M}^{\rm op}_{\overline{\Sigma}} \times \mathcal{M}_{\overline{\Omega}}]) \to \mathcal{M}ap^{W}_{\mathcal{I}^{\rm op}}(\mathcal{I}^{\rm op}, [\mathcal{M}_{\tilde{\Sigma}}, \mathcal{S}]).$$

   We also have the following functor:

   $$\mathcal{M}ap^{W}_{\mathcal{I}^{\rm op}}(\mathcal{I}^{\rm op}, \mathcal{M}^{\rm op}_{\overline{\Sigma}}) \times \mathcal{M}ap^{W}_{\mathcal{I}^{\rm op}}(\mathcal{M}_{\tilde{\Sigma}}, \mathcal{M}_{\overline{\Omega}}) \to \mathcal{M}ap^{W}_{\mathcal{I}^{\rm op}}(\mathcal{I}^{\rm op}, [\mathcal{M}_{\tilde{\Sigma}}, \mathcal{M}^{\rm op}_{\overline{\Sigma}} \times \mathcal{M}_{\overline{\Omega}}])$$

   which sends (E, F) to h, where

   $$h(v)(x) = (E(v), F(x))$$

   for v an object in $\mathcal{I}^{\rm op}$ and x an object in $\mathcal{M}$. We say a homology theory in $\mathrm{HML}_{\mathcal{I}}(\mathcal{M}, \mathcal{S})$ is representable if it isomorphic to an object in the image of the composition of the above functors. Given such a mor if h is representable with respect to mor, then we have

$$h(v)(x) = \mathrm{mor}(E(v), F(x)).$$

2. The functor $\mathrm{R}_{\mathrm{chm}}$ induces a functor

$$\mathcal{M}ap^W_{\mathcal{I}}(\mathcal{I}, [\mathcal{M}^{\mathrm{op}}_{\Sigma}, \mathcal{M}^{\mathrm{op}}_{\Sigma} \times \mathcal{M}_{\Omega}]) \to \mathcal{M}ap^W_{\mathcal{I}}(\mathcal{I}, [\mathcal{M}^{\mathrm{op}}_{\Sigma}, \mathcal{S}]).$$

We also have the following functor:

$$\mathcal{M}ap^W_{\mathcal{I}}(\mathcal{M}^{\mathrm{op}}_{\Sigma}, \mathcal{M}^{\mathrm{op}}_{\Sigma}) \times \mathcal{M}ap^W_{\mathcal{I}}(\mathcal{I}, \mathcal{M}_{\Omega}) \to \mathcal{M}ap^W_{\mathcal{I}}(\mathcal{I}, [\mathcal{M}^{\mathrm{op}}_{\Sigma}, \mathcal{M}^{\mathrm{op}}_{\Sigma} \times \mathcal{M}_{\Omega}])$$

which sends (F, E) to $\tilde{h}$, where

$$\tilde{h}(i)(m) = (F(m), E(i))$$

for i an object in $\mathcal{I}^{\mathrm{op}}$ and x an object in $\mathcal{M}$. We say a cohomology theory in $\mathrm{COHML}_{\mathcal{I}}(\mathcal{M}, \mathcal{S})$ is representable if it isomorphic to an object in the image of the composition of the above functors. Given such a mor if h is representable with respect to mor, then we have

$$h(v)(x) = \mathrm{mor}(F(x), E(v)).$$

3. The functor $\mathrm{R}_{\mathrm{hty}}$ induces a functor

$$\mathcal{M}ap^W_{\mathcal{I}}(\mathcal{I}, [\mathcal{M}_{\Omega}, \mathcal{M}^{\mathrm{op}}_{\Sigma} \times \mathcal{M}_{\Omega}]) \to \mathcal{M}ap^W_{\mathcal{I}}(\mathcal{I}, [\mathcal{M}_{\Omega}, \mathcal{S}]).$$

We also have the following functor:

$$\mathcal{M}ap^W_{\mathcal{I}}(\mathcal{I}, \mathcal{M}^{\mathrm{op}}_{\Sigma}) \times \mathcal{M}ap^W_{\mathcal{I}}(\mathcal{M}_{\Omega}, \mathcal{M}_{\Omega}) \to \mathcal{M}ap^W_{\mathcal{I}}(\mathcal{I}, [\mathcal{M}_{\Omega}, \mathcal{M}^{\mathrm{op}}_{\Sigma} \times \mathcal{M}_{\Omega}])$$

which sends (E, F) to π, where

$$\pi(v)(x) = (E(v), F(x))$$

for v an object in $\mathcal{I}^{\mathrm{op}}$ and x an object in $\mathcal{M}$. We say a homotopy theory in $\mathrm{HTY}_{\mathcal{I}}(\mathcal{M}, \mathcal{S})$ is representable if it isomorphic to an object in the image of the composition of the above functors. Given such a mor if π is representable with respect to mor, then we have

$$\pi(v)(x) = \mathrm{mor}(E(v), F(x)).$$

4. The functor $\mathrm{R}_{\mathrm{cht}}$ induces a functor

$$\mathcal{M}ap^W_{\mathcal{I}^{\mathrm{op}}}(\mathcal{I}^{\mathrm{op}}, [\mathcal{M}^{\mathrm{op}}_{\widetilde{\Omega}}, \mathcal{M}^{\mathrm{op}}_{\overline{\Sigma}} \times \mathcal{M}_{\overline{\Omega}}]) \to \mathcal{M}ap^W_{\mathcal{I}^{\mathrm{op}}}(\mathcal{I}^{\mathrm{op}}, [\mathcal{M}^{\mathrm{op}}_{\widetilde{\Omega}}, \mathcal{S}]).$$

   We also have the following functor:

$$\mathcal{M}ap^W_{\mathcal{I}^{\mathrm{op}}}(\mathcal{M}^{\mathrm{op}}_{\widetilde{\Omega}}, \mathcal{M}^{\mathrm{op}}_{\overline{\Sigma}}) \times \mathcal{M}ap^W_{\mathcal{I}^{\mathrm{op}}}(\mathcal{I}^{\mathrm{op}}, \mathcal{M}_{\overline{\Omega}}) \to \mathcal{M}ap^W_{\mathcal{I}^{\mathrm{op}}}(\mathcal{I}^{\mathrm{op}}, [\mathcal{M}^{\mathrm{op}}_{\widetilde{\Omega}}, \mathcal{M}^{\mathrm{op}}_{\overline{\Sigma}} \times \mathcal{M}_{\overline{\Omega}}])$$

   which sends (F, E) to h, where

$$\pi(v)(x) = (F(x), E(v))$$

   for v an object in $\mathcal{I}^{\mathrm{op}}$ and x an object in $\mathcal{M}$. We say a cohomotopy theory in $\mathrm{COHTY}_{\mathcal{I}}(\mathcal{M}, \mathcal{S})$ is representable if it isomorphic to an object in the image of the composition of the above functors. Given such a mor if π is representable with respect to mor, then we have

$$\pi(v)(x) = \mathrm{mor}(E(v), F(x)).$$

It is natural to define an Ω-spectrum as an object in $\mathcal{M}ap^W_{\mathcal{I}}(\mathcal{I}, \mathcal{M}_{\Omega})$. Weak equivalences are defined object-wise.

Following [6] Chapter XII, we simply say spectra for the objects of $\mathcal{M}ap_{\mathcal{I}}(\mathcal{I}, \mathcal{M}_{\Omega})$. Note that an object in $\mathcal{M}ap_{\mathcal{I}}(\mathcal{I}, \mathcal{M}_{\Omega})$ consist of objects $f(v)$ in $\mathcal{M}$ together with natural isomorphisms

$$\sigma(w) : f(v) \to \Omega_l(w)(f(w \otimes v)).$$

6.4 Relation to stabilization and Picard grading

Let $(\mathcal{M}, \wedge, \mathbb{1})$ be a pointed homotopical, closed symmetric monoidal category. Let $\mathcal{I}$ be a monoidal subcategory of $\mathcal{M}$ and $\Omega = (\Omega_l, \mathbb{1}_{\mathcal{M}})$ be a looping action of $\mathcal{I}$ on $\mathcal{M}$ given by $\Omega_l(v)(x) = \mathcal{M}(v, x)$ for v object in $\mathcal{I}$ and x object in $\mathcal{M}$. Let Υ be the reverse action of Ω, which is defined on $\mathcal{M}ap_{\mathcal{I}}(\mathcal{I}, \mathcal{M}_{\Omega})$, see Section 5.2. For an object (E, σ) in $\mathcal{M}ap_{\mathcal{I}}(\mathcal{I}, \mathcal{M}_{\Omega})$ the right action is given by

$$(E)\Upsilon_r(v)(w) = E(v \wedge w),$$

and the left action of Υ is trivial. Since we have

$$f(w) \cong \Omega_l(v)(f(v \wedge w)) = \mathcal{M}(v, f(w \wedge v)),$$

we can consider the action Ω as an action on $\mathcal{M}ap_{\mathcal{I}}(\mathcal{I}, \mathcal{M}_\Omega)$. In fact, we can define $\Omega_l(v)(E) : \mathcal{I} \to \mathcal{M}$ by the identity

$$\Omega_l(v)(E)(w) = \Omega_l(v)(E(w)).$$

By the equivariance of E, we get

$$\Omega_l(i)((E)\Upsilon_r(i))(j) \cong \Omega_l(i)(E(i \wedge j)) \cong E(j)$$

and

$$((\Omega_l(i)(E))\Upsilon_r(i))(j) \cong \Omega_l(i)(E)(i \wedge j) \cong E(j).$$

As a result, we get that the action Υ induces an auto-equivalence on the homotopy category of $\mathcal{M}ap_{\mathcal{I}}(\mathcal{I}, \mathcal{M}_\Omega)$.

An object v in a symmetric monoidal category $\mathcal{C}$ is invertible if there is an object w with an isomorphism $v \wedge w \to s$. The inverse of an object v is given by the dual $Dv = \mathcal{M}(v, s)$, see [21]. The Picard category $\mathcal{P}ic(\mathcal{C})$ of $\mathcal{C}$, see [21], [22] or [23], is defined as the full-subcategory of invertible objects in $\mathcal{C}$. If M_Ω is as above, we have the following theorem:

Theorem 6.4.1. *The evaluation functor*

$$\mathcal{E} : \mathcal{M}ap_{\mathcal{I}}(\mathcal{I}, \mathcal{M}_\Omega) \to \mathcal{M}_\Omega$$

induces an equivalence on the homotopy categories if and only if $\mathcal{I}$ is a subcategory of $\mathcal{P}ic(Ho\mathcal{M})$.

Proof. Assume first $\mathcal{I}$ is a subcategory of $\mathcal{P}ic(Ho\mathcal{M})$. Define a functor

$$\Phi : \mathcal{M}_\Omega \to \mathcal{M}ap_{\mathcal{I}}(\mathcal{I}, \mathcal{M}_\Omega)$$

by $\Phi(x)(v) = \mathcal{M}(Dv, x)$ where $Dv = \mathcal{M}(v, s)$ is the dual of v. Since

$$\mathcal{M}(w, \mathcal{M}(D(v \wedge w), x)) \cong \mathcal{M}(w \wedge D(w \wedge v), x) \cong \mathcal{M}(Dv, x),$$

Φ is well defined. Then we have

$$\mathcal{E} \circ \Phi(x) = \Phi(x)(s) = \mathcal{M}(s, x) \cong x,$$

i.e. $\mathcal{E} \circ \Phi$ is weak equivalent to identity in the homotopy category. For $f : \mathcal{I} \to M$, we have

$$\Phi \circ \mathcal{E}(f)(v) = \Phi(f(s))(v) \cong \Omega(Dv)(f(s)) \cong \Omega(Dv)(\Omega(v)f(v)) \cong f(v).$$

Hence we get $\mathcal{E}$ induces an equivalence of categories $Ho\mathcal{M}ap_{\mathcal{I}}(\mathcal{I}, \mathcal{M}_{\Omega})$ and $Ho\mathcal{M}$.

Conversely assume $\mathcal{E}$ induces an equivalence on homotopy categories. Then for v an object in $\mathcal{I}$ we have

$$\mathcal{M}(v, f_s(v)) = \Omega(v)(f_s(v)) \cong f_s(s) = s$$

and

$$\mathcal{M}(v, f_s(v)) \cong \mathcal{M}(v, \mathcal{M}(Dv, s)) \cong \mathcal{M}(v \wedge Dv, s) \cong D(v \wedge Dv).$$

As a result we get that $v \wedge Dv$ is weak equivalent to s. Hence $\mathcal{I}$ is a subcategory of $\mathcal{P}ic(Ho\mathcal{M})$. □

6.5 Examples of theories graded over monoidal categories

Recall that, if $\mathcal{M}$ is a category with weak equivalences and a point, a sequence of morphisms

$$A \xrightarrow{i} B \xrightarrow{p} C$$

in $\mathcal{M}$ is called a cofiber sequence if the diagram

$$\begin{array}{ccc} A & \xrightarrow{\text{i}} & B \\ \downarrow & & \downarrow \text{p} \\ * & \longrightarrow & C \end{array}$$

is a homotopy push-out. Assume that pushouts always exist in $\mathcal{M}$. When we have such a diagram, we can form the following one

$$\begin{array}{ccccc} A & \xrightarrow{\text{i}} & B & \longrightarrow & * \\ \downarrow & & \downarrow{\scriptstyle p} & & \downarrow \\ * & \longrightarrow & C & \longrightarrow & D \end{array}$$

where both squares are homotopy pushout diagrams. This make D weak equivalent to ΣA, since the bigger square is also a homotopy pushout. Hence, if we have an object s in $\mathcal{I}$ such that $\Sigma(s)(X)$ is weak equivalent to the homotopy pushout of the two unique diagram of maps

$$* \longleftarrow X \longrightarrow *$$

in $\mathcal{M}$, then the homology and cohomology functors gives us long exact sequences, where the suspension isomorphisms are given by the reverse action. Hence we have the following proposition:

Proposition 6.5.1. *If h is a homology (resp. cohomology) theory satisfying the Eilenberg-Steenrod homology (resp. cohomology) axioms, then h is an element in the category of homology (resp. cohomology) theories defined in Section 6.2 provided that $\mathcal{M} = Top_*$ with weak equivalences as usual weak equivalences, $S = \mathcal{A}b$ and $\Sigma = (1, \Sigma)$ to be the usual reduced suspension functor. The converse is also true.*

Proof. This follows from the fact that given a pointed space X in Top_*, the usual reduced suspension ΣX fits into the following homotopy pushout diagram

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma X \end{array}$$

Hence for a homology or cohomology theory h, it follows from the reverse action that we have an isomorphism

$$\sigma(1) : h(n)(X) \to h(n \oplus 1)(X \wedge S^1).$$

The converse is also evident from above. $\square$

6.5.1 Bigraded (co)homology theories

Let $\mathcal{I}$ be the monoidal category with object set is $\mathbb{Z} \times \mathbb{Z}$ and morphisms are only identity morphisms. Assume Σ is a suspension action on a homotopical category $\mathcal{M}$. Let $\mathcal{A}b$ be an abelian category. Then the cohomology theories in $\mathrm{COHML}_{\mathcal{I}}(\mathcal{M}, \mathcal{A}b)$ are bigraded theories. For any integer i, j, and any object in $\mathrm{COHML}_{\mathcal{I}}(\mathcal{M}, \mathcal{A}b)$ we have the diagram

$$\begin{array}{ccc} h^{i,j}(X) & \xrightarrow{\sigma(1,0)} & h^{i+1,j}(\Sigma(1,0)X) \\ \downarrow{\scriptstyle \sigma(1,1)} & & \downarrow{\scriptstyle h(id \oplus \alpha)(id)} \\ h^{i+1,j+1}(\Sigma(1,1)X) & \xrightarrow{h(1,0)(id \wedge S^{\alpha})} & h^{i+2,j+1}(\Sigma(2,1)X) \end{array}$$

which commutes for any object X in $\mathcal{M}$ where

$$\sigma(i,j) : h(*,*)(X) \to h(*+i, *+j)(\Sigma(i,j)X)$$

is a natural isomorphism for every integer i, j. This is due to the construction of $\mathrm{COHML}_{\mathcal{I}}(\mathcal{M}, \mathcal{A}b)$, see also Section 5.2.0.1. This means the suspension coherence condition is satisfied. Hence, these theories corresponds to the bigraded theories given in [24], section 3.4. A particular example is Motivic cohomology theory, see [24] and [25].

6.5.2 $RO(G)$-graded cohomology theories

In this section we show that the definition of $RO(G)$-graded cohomology theories (see Definition 1.1 in [6], Ch. 13, Sec. 1) is a particular case of the definition we have in the previous section once categories $\mathcal{I}$, $\mathcal{M}$, $\mathcal{S}$ and actions Σ, Ω, s are selected in the right way. Let G be a group and U be G-universe which means U is a countable direct sum of G-representations such that U contains the trivial representation and contains each of its sub-representations infinitely many often. Let $\mathcal{I} = \mathcal{RO}(G; U)$ be the monoidal category whose objects are the G-representations embeddable in U and whose morphisms are G-linear isometric isomorphisms and monoidal product is direct sum. Let $\mathcal{M}$ denote the category of pointed G-spaces and G-maps where we take all spaces to be compactly generated and weakly Hausdorff. In $\mathcal{M}$ we will consider weak equivalences as homotopy weak equivalences. Let $\mathcal{S}$ be the category of abelian groups. In $\mathcal{S}$ the weak equivalences will be group isomorphisms. Notice both categories $\mathcal{M}$ and $\mathcal{S}$ have a zero object. Let s be the trivial $\mathcal{I}$-action on $\mathcal{S}$. Given V an object in $\mathcal{I}^{\mathrm{op}}$, S^V will denote the one point compactification of V. We define

$$\Sigma_l(V)(X) = S^V \wedge X$$

and

$$(X)\Sigma_r(V) = X \wedge S^V$$

for V an object in $\mathcal{I}$ and X and object in $\mathcal{M}$. Let Σ be the $\mathcal{I}$-action $(1, \Sigma_r)$ on $\mathcal{M}$. Let

$$\Omega_l(V)(X) = \mathcal{M}(S^V, X)$$

considered with the compact open topology. Let Ω be the $\mathcal{I}$-action $(\Omega_l, 1)$ on $\mathcal{M}$. Note that the functor $\Sigma_r(V)$ preserves weak equivalences and homotopy colimits for each V object in $\mathcal{I}$ and $\Omega_l(V)$ preserves weak equivalences and homotopy limits for each V object in $\mathcal{I}$, see [8]. Hence Σ is a suspension action and Ω is a loop action.

The $\mathcal{I}$-action $[\Sigma, s]$ on $[\mathcal{M}^{\mathrm{op}}, \mathcal{S}]$ induces an $\mathcal{I}$-action on $\mathrm{cohml}(\mathcal{M}, \mathcal{S})$. Take (h, σ) a cohomology theory graded over $\mathcal{I}$ which is an object in the reverse action of $\mathcal{I}$ on $\mathrm{cohml}(\mathcal{M}, \mathcal{S})$. Then we can consider h as a functor from $\mathcal{I} \times \mathcal{M}^{\mathrm{op}}$ to $\mathcal{S}$; in fact it factors through the homotopy category of its domain. For each object W of $\mathcal{I}$ we have an isomorphism

$$\sigma(W) : h(V)(X) \to h(V \oplus W)(X \wedge S^W)$$

such that the following holds:

1. For each representation V, the functor $h(V)$ from $\mathcal{M}^{\mathrm{op}}$ to $\mathcal{S}$, sends cofiber sequences in $\mathcal{M}$ to exact sequences in $\mathcal{S}$ and wedge products in $\mathcal{M}$ to direct products in $\mathcal{S}$.

2. If $\alpha : W \to W'$ is a morphism in $\mathcal{I}$. Then we have a commutative diagram as follows:

$$\begin{array}{ccc} h(V)(X) & \xrightarrow{\sigma(W)} & h(V \oplus W)(X \wedge S^W) \\ {\scriptstyle \sigma(W')}\downarrow & & \downarrow{\scriptstyle h(id \oplus \alpha)(id)} \\ h(V \oplus W')(X \wedge S^{W'}) & \xrightarrow{h(V \oplus W')(id \wedge S^{\alpha})} & h(V \oplus W')(X \wedge S^W) \end{array}$$

   because σ is natural transformation, combined with the fact that $h(i)$ is contravariant for every object i in $\mathcal{I}$, see also Section 5.2.0.1.

3. $\sigma(0) = id$ and for each pair of representations (W, Z) we have a commutative diagram as follows:

$$\begin{array}{ccc} h(V)(X) & \xrightarrow{\sigma(W)} & h(V \oplus W)(X \wedge S^W) \\ & {\scriptstyle \sigma(W \oplus Z)}\searrow \quad \swarrow{\scriptstyle \sigma(Z)} & \\ & h(V \oplus W \oplus Z)(X \wedge S^{W \oplus Z}) & \end{array}$$

   because σ is compatible with the monoidal structure.

Converse is clear from above. Observe that this is just the list of axioms that defines the $RO(G, U)$-graded cohomology theories, as given in Definition 1.1 in [6], Ch. 13, Sec. 1.

Chapter 7

Spectral sequences

Generally speaking a spectral is the process of analyzing an object by first filtering it into pieces and reconstructing it with the information of each piece. In this chapter we give construction of spectral sequences for the homology, cohomology, homotopy and cohomotopy theories defined in the previous chapter. In Section 7.2.1, we use similar techniques for the constructions of 'spectral sequences for stable ∞-categories' given in [26].

However, we will obtain spectral sequences for theories defined in the previous chapter and for ordinary homotopical categories. Moreover, our general construction does not require to have a homology or cohomology theory, but a class of functor satisfying some conditions. It is also not necessary that these functor take values in an abelian category but a category that has kernels and cokernels, satisfying some mild assumptions.

With these constructions one would be able to construct most of the classical spectral sequences as special cases of these spectral sequences. In addition, this will allow us to construct generalized versions of the classical spectral sequences for the homology, cohomology, homotopy and cohomotopy theories graded over several monoidal categories.

7.1 Kernel-Image-Cokernel factorizations

To define spectral sequences we need some extra conditions on the category that the homology, cohomology, homotopy and cohomotopy theories take values. In particular, we will require to have "Kernel-Image-Cokernel factorization" of any morphism in this category, so that we can talk about homology of a sequence of morphism in this category.

Let $[n]$ denote the poset $\{0 \to 1 \to ... \to n\}$ and let $\mathcal{A}$ be a category with weak equivalences. Let us denote the functor category $[[n], \mathcal{A}]$ by $\mathcal{A}^{[n]}$. Then $\mathcal{A}^{[n]}$ is a category with weak equivalence where weak equivalences defined object wise.

Definition 7.1.1. A functor $\Gamma : \mathcal{A}^{[1]} \to \mathcal{A}^{[4]}$ will be called a kernel-image-cokernel factorization if it preserve weak equivalences and for every morphism $f : A \to B$ in $\mathcal{A}$ considered as an object in $\mathcal{A}^{[1]}$ we have $\Gamma(f)$ in the following form

$$K(f) \xrightarrow{k(f)} A \xrightarrow{m(f)} I(f) \xrightarrow{i(f)} B \xrightarrow{c(f)} C(f).$$

such that $f = i(f) \circ m(f)$.

Remark 6. In abelian categories we always have a kernel-image-cokernel factorization.

Let $\mathcal{A}$ be a category with a kernel-image-cokernel factorization Γ.

Definition 7.1.2. Given a sequence of morphisms in $\mathcal{A}$

$$\dots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \dots$$

we define the n^{th} homology with respect to Γ of this sequence as follows:

$$H_n(A_*) = I(c(\partial_{n+1}) \circ k(\partial_n))$$

Observe that if $\mathcal{A}$ is the category of abelian groups and the sequence A_* is a chain complex in $\mathcal{A}$ then

$$I(c(\partial_{n+1}) \circ k(\partial_n)) = im(ker(\partial_n) \hookrightarrow A_n \to A_n/im(\partial_{n+1})) = H_n(A_*).$$

Hence in the usual sense this definition is the homology of a chain complex.

7.2 (Co)homology spectral sequences

In this section we will introduce (Co)homology spectral sequences. The construction will be given in most general sense

7.2.1 Construction of the spectral sequence

The techniques of this construction is influenced by Definition 1.2.2.2 of [26]. Let $\mathcal{M}$ be a homotopical category with a terminal object $*$. Let $ho\mathcal{M}$ denote the homotopy category of $\mathcal{M}$. Assume that homotopy colimits of diagrams in $\mathcal{M}$ can be defined functorially. Recall that the suspension object associated to a given X an object in $\mathcal{M}$ is obtained as the homotopy pushout

$$\begin{array}{ccc} * & \longrightarrow & \Sigma X \\ \uparrow & & \uparrow \\ X & \longrightarrow & * \end{array}$$

Suppose that we are given a sequence of morphisms in $\mathcal{M}$ as follows:

$$\ldots \longrightarrow X_{-2} \longrightarrow X_{-1} \longrightarrow X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \ldots$$

For each i, j in $\mathbb{Z}$ we define $X_{i,i} = *$ and if $i \leqslant j$ we recursively define $X_{j+1,i}$ by the following homotopy pushout diagram:

$$\begin{array}{ccc} X_{j,i} & \longrightarrow & X_{j+1,i} \\ \uparrow & & \uparrow \\ X_j & \longrightarrow & X_{j+1} \end{array}$$

Notice that for each i, j in $\mathbb{Z}$ there is a morphism $X_{i+1,j} \to X_{i+1,j+1}$ unique under the projection to $ho\mathcal{M}$. We obtain the following diagram

$$
\begin{array}{ccccccccccc}
 & & & & & & & & * & \longrightarrow & \dots \\
 & & & & & & & & \uparrow & & \\
 & & & & & & * & \longrightarrow & X_{2,1} & \longrightarrow & \dots \\
 & & & & & & \uparrow & & \uparrow & & \\
 & & & & * & \longrightarrow & X_{1,0} & \longrightarrow & X_{2,0} & \longrightarrow & \dots \\
 & & & & \uparrow & & \uparrow & & \uparrow & & \\
 & & * & \longrightarrow & X_{0,-1} & \longrightarrow & X_{1,-1} & \longrightarrow & X_{2,-1} & \longrightarrow & \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\dots & \longrightarrow & X_{-1} & \longrightarrow & X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \dots
\end{array}
$$

so that each square is a homotopy pushout. For i, j, k, l in $\mathbb{Z}$, if $i \leqslant j, k \leqslant l$ then the morphism from $X_{j,i}$ to $X_{l,k}$ in the above diagram will be denoted by $f(j, i, l, k)$. Let $i \leqslant j \leqslant k$ and let us denote the morphism from $X_{k,j}$ to $\Sigma X_{j,i}$ in the homotopy pushout diagram

$$
\begin{array}{ccccc}
* & \longrightarrow & X_{k,j} & \xrightarrow{d(k,j,i)} & \Sigma X_{j,i} \\
\uparrow & & \uparrow & & \uparrow \\
X_{j,i} & \longrightarrow & X_{k,i} & \longrightarrow & *
\end{array}
$$

by $d(k, j, i)$. We have the following lemma, which is influence by remark 1.2.2.3 of [26]:

Lemma 7.2.1. *If $i \leqslant j \leqslant k \leqslant l$ then $\Sigma d(k, j, i) \circ d(l, k, j)$ factors through $*$.*

Proof. In the homotopy category the composition $\Sigma d(k, j, i) \circ d(l, k, j)$ is equivalent to the composition $h \circ g \circ f \circ e$ in the below diagram where all rectangles are homotopy pushouts.

$$
\begin{array}{ccccccccc}
 & & & & & & * & \longrightarrow & \Sigma\Sigma X_{j,i} \\
 & & & & & & \uparrow & & {\scriptstyle h}\uparrow \\
 & & & & * & \longrightarrow & \Sigma X_{l,i} & \longrightarrow & \Sigma X_{l,j} \\
 & & & & \uparrow & & \uparrow & & {\scriptstyle g}\uparrow \\
 & & * & \longrightarrow & X_{l,k} & \xrightarrow{e} & \Sigma X_{k,i} & \xrightarrow{f} & \Sigma X_{k,j} \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
* & \longrightarrow & X_{k,j} & \longrightarrow & X_{l,j} & \longrightarrow & \Sigma X_{j,i} & \longrightarrow & * \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
X_{j,i} & \longrightarrow & X_{k,i} & \longrightarrow & X_{l,i} & \longrightarrow & * & &
\end{array}
$$

□

A morphism ε in $\mathcal{M}^{[1]}$ can be considered as a commutative diagram in $\mathcal{M}$

$$
\begin{array}{ccc}
B & \xrightarrow{\epsilon_2} & D \\
\uparrow & & \uparrow \\
A & \xrightarrow{\epsilon_1} & C
\end{array}
$$

Let $\mathcal{A}$ be a category with a kernel-image-cokernel factorization Γ as in definition 7.1.1. Let

$$\mathcal{H} = \{h_n : \mathcal{M}^{[1]} \to \mathcal{A}\}_{n\in\mathbb{Z}}$$

be a family of functors with a family of natural isomorphisms

$$\{\sigma_n : h_n \to h_{n+1} \circ \Sigma_*\}_{n\in\mathbb{Z}}.$$

Axiom 7.2.1. The following conditions holds for the family $\mathcal{H}$

1. h_n preserves weak equivalences for all n
2. For every morphism ε in $\mathcal{M}^{[1]}$ given by the commutative diagram in $\mathcal{M}$ as above, we have

$$I(h_n(\varepsilon)) = h_n \left(\begin{array}{c} D \\ \uparrow \\ A \end{array} \right)$$

3. If the morphism ε fits in a diagram

$$\begin{array}{ccccc}
* & \longrightarrow & & \longrightarrow & \Sigma X \\
\uparrow & & & & \uparrow \\
Z_1 & \longrightarrow & B & \xrightarrow{\epsilon_2} & D \\
\uparrow & & & & \uparrow \\
A & \xrightarrow{\epsilon_1} & C & \longrightarrow & Z_2 \\
\uparrow & & & & \uparrow \\
X & \longrightarrow & & \longrightarrow & *
\end{array}$$

where the three rectangles are homotopy pushouts, then the kernel-image-cokernel decomposition of the image of this morphism under h_n is naturally isomorphic to the following sequence:

$$h_n \begin{pmatrix} B \\ \uparrow \\ X \end{pmatrix} \to h_n \begin{pmatrix} B \\ \uparrow \\ A \end{pmatrix} \to h_n \begin{pmatrix} D \\ \uparrow \\ A \end{pmatrix} \to h_n \begin{pmatrix} D \\ \uparrow \\ C \end{pmatrix} \to h_n \begin{pmatrix} \Sigma X \\ \uparrow \\ C \end{pmatrix}$$

for all n.

If these conditions holds for the family $\mathcal{H}$ then we define a spectral sequence as follows:

Theorem 7.2.2. *There is a ((co)homology) spectral sequence with*

$$E^r_{p,q} = h_{p+q} \begin{pmatrix} X_{p+r-1,p-1} \\ \uparrow \\ X_{p,p-r} \end{pmatrix}$$

and the differential $d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ *is the map induced by the morphism in* $\mathcal{M}^{[1]}$ *given by*

$$\begin{array}{ccc}
X_{p+r-1,p-1} & \xrightarrow{d(p-r+1,p-1,p-r-1)} & \Sigma X_{p-1,p-r-1} \\
\uparrow & & \uparrow \\
X_{p,p-r} & \xrightarrow{d(p,p-r,p-2r)} & \Sigma X_{p-r,p-2r}
\end{array}$$

Proof. We have that differential d_r leaving $E^r_{p,q}$ is induced by

$$\begin{array}{ccc} \Sigma X_{p+r-1,p-1} & \xrightarrow{\Sigma d(p-r+1,p-1,p-r-1)} & \Sigma\Sigma X_{p-1,p-r-1} \\ \uparrow & & \uparrow \\ \Sigma X_{p,p-r} & \xrightarrow{\Sigma d(p,p-r,p-2r)} & \Sigma\Sigma X_{p-r,p-2r} \end{array}$$

on h_{p+q+1} On the other hand the differential entering $E^r_{p,q}$ is induced by the morphism in $\mathcal{M}^{[1]}$

$$\begin{array}{ccc} X_{p+2r-1,p+r-1} & \xrightarrow{d(p+1,p+r-1,p-1)} & \Sigma X_{p+r-1,p-1} \\ \uparrow & & \uparrow \\ X_{p+r,p} & \xrightarrow{d(p+r,p,p-r)} & \Sigma X_{p,p-r} \end{array}$$

on h_{p+q+1}. Hence, by lemma 7.2.1 the composition of these morphisms $\Sigma\tilde{\varepsilon} \circ \varepsilon$ factors through the zero object $*$, i.e. the composition is null-homotopic. Since the functors h_n are weak equivalence preserving functors then the composition $d_r \circ d_r$ is zero map in $\mathcal{A}$. Consider the following diagram

$$\begin{array}{ccccccc} * & \longrightarrow & X_{p+2r-1,p+r} & \longrightarrow & \Sigma X_{p+r,p-r} & \longrightarrow & \Sigma X_{p+r,p-1} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ X_{p+r,p+r-1} & \longrightarrow & X_{p+2r-1,p+r-1} & \xrightarrow{\bar{f}} & \Sigma X_{p+r-1,p-r} & \xrightarrow{\bar{g}} & \Sigma X_{p+r-1,p-1} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ X_{p+r,p} & \xrightarrow{f} & X_{p+2r-1,p} & \xrightarrow{g} & \Sigma X_{p,p-r} & \longrightarrow & \Sigma X_{p,p-1} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ X_{p+r,p-1} & \longrightarrow & X_{p+2r-1,p-1} & \longrightarrow & \Sigma X_{p+1,p-r} & \longrightarrow & * \end{array}$$

where all squares are homotopy pushouts. The differential entering $E_{p,q}$ is given by a map

$$d_r : h_{p+q+1}\begin{pmatrix} X_{p+2r-1,p+r-1} \\ \uparrow \\ X_{p+r,p} \end{pmatrix} \to h_{p+q+1}\begin{pmatrix} \Sigma X_{p+r-1,p-1} \\ \uparrow \\ \Sigma X_{p,p-r} \end{pmatrix}$$

induced by the compositions $g \circ f$ and $\bar{g} \circ \bar{f}$. On the other hand from the following diagram

$$
\begin{array}{ccccccc}
* & \longrightarrow & \Sigma\Sigma X_{p-r,p-r-1} & \longrightarrow & \Sigma\Sigma X_{p-1,p-r-1} & \longrightarrow & \Sigma\Sigma X_{p,p-r-1} \\
\uparrow & & \uparrow & & \bar{h}\uparrow & & \uparrow \\
\Sigma\Sigma X_{p-r+1,p-2r} & \longrightarrow & \Sigma\Sigma X_{p-r,p-2r} & \longrightarrow & \Sigma\Sigma X_{p-1,p-2r} & \longrightarrow & \Sigma\Sigma X_{p,p-2r} \\
\uparrow & & h\uparrow & & \bar{e}\uparrow & & \uparrow \\
\Sigma X_{p+r-1,p-r-1} & \longrightarrow & \Sigma X_{p+r-1,p-r} & \longrightarrow & \Sigma X_{p+r-1,p-1} & \longrightarrow & \Sigma X_{p+r-1,p} \\
\uparrow & & e\uparrow & & \uparrow & & \uparrow \\
\Sigma X_{p,p-r-1} & \longrightarrow & \Sigma X_{p,p-r} & \longrightarrow & \Sigma X_{p,p-1} & \longrightarrow & *
\end{array}
$$

where again all squares are homotopy pushouts, it can be seen that the differential leaving $E_{p,q}$ is given by a map

$$
d_r : h_{p+q+1}\begin{pmatrix} \Sigma X_{p+r-1,p-1} \\ \uparrow \\ \Sigma X_{p,p-r} \end{pmatrix} \to h_{p+q+1}\begin{pmatrix} \Sigma\Sigma X_{p-1,p-r-1} \\ \uparrow \\ \Sigma\Sigma X_{p-r,p-2r} \end{pmatrix}
$$

induced by the compositions $h \circ e$ and $\bar{h} \circ \bar{e}$. Hence, Axiom 7.2.1 part 3 implies that the homology of this sequence is given by $I(c(d_r) \circ k(d_r))$. The composition $c(d_r) \circ k(d_r)$ is equal to

$$
h_{p+q+1}\begin{pmatrix} \Sigma X_{p+r-1,p-1} \\ \uparrow \\ \Sigma X_{p,p-r-1} \end{pmatrix} \overset{k(d_r)}{\longrightarrow} h_{p+q+1}\begin{pmatrix} \Sigma X_{p+r-1,p-1} \\ \uparrow \\ \Sigma X_{p,p-r} \end{pmatrix} \overset{c(d_r)}{\longrightarrow} h_{p+q+1}\begin{pmatrix} \Sigma X_{p+r,p-1} \\ \uparrow \\ \Sigma X_{p,p-r} \end{pmatrix}
$$

which is equivalent to

$$
h_{p+q+1}\begin{pmatrix} \Sigma X_{p+r-1,p-1} \\ \uparrow \\ \Sigma X_{p,p-r-1} \end{pmatrix} \to h_{p+q+1}\begin{pmatrix} \Sigma X_{p+r,p-1} \\ \uparrow \\ \Sigma X_{p,p-r} \end{pmatrix}
$$

Hence the image $I(c(d_r) \circ k(d_r))$ is equal to

$$
h_{p+q+1}\begin{pmatrix} \Sigma X_{p+r,p-1} \\ \uparrow \\ \Sigma X_{p,p-r-1} \end{pmatrix} = h_{p+q}\begin{pmatrix} X_{p+r,p-1} \\ \uparrow \\ X_{p,p-r-1} \end{pmatrix} = E^{r+1}_{p,q}
$$

by Axiom 7.2.1 part 2. □

7.2.2 Convergence

Let $\mathcal{M}$ be a homotopical category such that it has all small homotopy colimits. Recall that for every object X in $\mathcal{M}$ we have

$$\begin{array}{ccc} * & \longrightarrow & \Sigma X \\ \uparrow & & \uparrow \\ X & \longrightarrow & * \end{array}$$

Assume that each h_n in the family $\mathcal{H}$ satisfies

$$h_n\begin{pmatrix} \Sigma X \\ \uparrow \\ * \end{pmatrix} = 0$$

for all n positive integer and for every object X in $\mathcal{M}$. Suppose that in the given a sequence

$$\dots \longrightarrow X_{-2} \longrightarrow X_{-1} \longrightarrow X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \dots$$

and in $\mathcal{M}$. Let $X = hocolim_r X_r$. We define the filtration of

$$h_n\begin{pmatrix} X \\ \uparrow \\ X \end{pmatrix}$$

induced by the above sequence as a sequence of object

$$F^p h_n\begin{pmatrix} X \\ \uparrow \\ X \end{pmatrix} = h_n\begin{pmatrix} X \\ \uparrow \\ X_p \end{pmatrix}$$

Assume we have $X_k \simeq *$ for every $k \leqslant -1$ in the above sequence. This implies that $X_{j,i} = *$ whenever $j \leqslant -1$.

Then whenever $r > p$ we have

$$E^r_{p-r,q+r-1} = h_{p+q-1}\begin{pmatrix} X_{p-1,p-r-1} \\ \uparrow \\ X_{p-r,p-2r} \end{pmatrix} = h_{p+q-1}\begin{pmatrix} X_{p-1,p-r-1} \\ \uparrow \\ * \end{pmatrix} = 0$$

whenever $p+q>0$. When $r>p$ we have $X_{p,p-r}=X_p$ since we it is defined as homotopy pushout

$$\begin{array}{ccc} X_{p-r,p-r} & \longrightarrow & X_{p,p-r} \\ \uparrow & & \uparrow \\ X_{p-r} & \longrightarrow & X_p \end{array}$$

Then by definition we have

$$E^r_{p,q} = h_{p+q}\left(\begin{array}{c} X_{p+r-1,p-1} \\ \uparrow \\ X_p \end{array}\right)$$

Let $Y_p = hocolim_r X_{p+r-1,p-1}$, then

$$E^\infty_{p,q} = h_{p+q}\left(\begin{array}{c} Y_p \\ \uparrow \\ X_p \end{array}\right)$$

Since

$$\begin{array}{ccc} * & \longrightarrow & X_{p+r-1,p-1} \\ \uparrow & & \uparrow \\ X_{p-1} & \longrightarrow & X_{p+r-1} \end{array}$$

is a pushout diagram, then the sequence

$$X_{p-1} \to X_{p+r-1} \to X_{p+r-1,p-1}$$

is a homotopy cofiber sequence. This gives rise to a homotopy cofiber sequence

$$X_{p-1} \to hocolim_r X_{p+r-1} \to hocolim_r X_{p+r-1,p-1}$$

which is equivalent to $X_{p-1}\to X\to Y_p$. We have the following diagram

$$\begin{array}{ccccc} & & * & \longrightarrow & \Sigma X \\ & & \uparrow & & \uparrow \\ * & \longrightarrow & Y_p & \longrightarrow & \Sigma X_{p-1} \\ \uparrow & & \uparrow & & \uparrow \\ X_{p-1} & \longrightarrow & X & \longrightarrow & * \end{array}$$

such that every square is a pushout. Then in following the diagram

$$
\begin{array}{ccccc}
* & \longrightarrow & & \longrightarrow & \Sigma Y_p \\
\uparrow & & & & \uparrow \\
\Sigma X_{p-1} & \longrightarrow & \Sigma X & \longrightarrow & \Sigma X \\
\uparrow & & & & \uparrow \\
\Sigma X_{p-1} & \longrightarrow & \Sigma X_p & \longrightarrow & \Sigma X \\
\uparrow & & & & \uparrow \\
Y_p & \longrightarrow & & \longrightarrow & *
\end{array}
$$

all three rectangles are pushouts. Hence we have

$$
E^{\infty}_{p,q} = h_{p+q}\left(\begin{array}{c} Y_p \\ \uparrow \\ X_p \end{array}\right) = C(h_{p+q}(G))
$$

where G is the morphism in $\mathcal{M}^{[1]}$ given by

$$
\begin{array}{ccc}
X & \xrightarrow{=} & X \\
\uparrow & & \uparrow \\
X_{p-1} & \longrightarrow & X_p
\end{array}
$$

Hence we have

$$
E^{\infty}_{p,q} = C(F^{p-1}h_{p+q}(id_X) \to F^p h_{p+q}(id_X))
$$

Then the spectral sequence given above is convergent.

7.3 Spectral sequences for homology theories graded over a monoidal category

In this section we give some examples of spectral sequences for homology and cohomology theories given in section 6. Let $\mathcal{M}$ be a homotopical category with a zero object $*$. Let $\mathcal{A}$ be an abelian category where isomorphism are considered as weak equivalences. Consider $\mathcal{A}$ with its own kernel-image-cokernel decomposition which is given by

$$Ker(f) \xrightarrow{ker(f)} A \xrightarrow{coim(f)} Im(f) \xrightarrow{im(f)} B \xrightarrow{coker(f)} Coker(f)$$

where $ker(f)$ $coim(f)$, $im(f)$ and $coker(f)$ denote the kernel, coimage, image and cokernel of f respectively. Let $\mathcal{I}$ be a monoidal category and Σ be an $\mathcal{I}$-action on $\mathcal{M}$ which is a suspension action and s be an $\mathcal{I}$-action on $\mathcal{A}$ which is a both suspension and loop action such that $\overline{s}$ can be defined. Suppose that there is an object 1 in $\mathcal{I}$ such that for every object X in $\mathcal{M}$ the right action $\Sigma_r(1)(X)$ is defined as the suspension object ΣX of X and $\Sigma_l(1)(X) = X$. We are going to define a spectral sequence for a homology theory $\tilde{h}$ in $\mathrm{HML}_{\mathcal{I}}(\mathcal{M}, \mathcal{A})$. Define a family

$$\mathcal{H} = \{h_n : \mathcal{M}^{[1]} \to \mathcal{A}\}_{n \in \mathbb{Z}}$$

such that for any object i in $\mathcal{I}$ and for any morphism $f : A \to B$ in $\mathcal{M}$ we have

$$h_n \begin{pmatrix} B \\ \uparrow \\ A \end{pmatrix} = Im \begin{pmatrix} \tilde{h}_{i+n}(B) \\ \Big\uparrow \tilde{h}_{i+n} f \\ \tilde{h}_{i+n}(A) \end{pmatrix}$$

where n is the object in $\mathcal{I}$ obtained by n-fold sum of the object 1. The inverse action of Σ on 1 give us another family of natural isomorphisms

$$\{\sigma_n : h_n \to h_{n+1} \circ \Sigma_*\}_{n \in \mathbb{Z}}.$$

Our claim is that the family $\mathcal{H}$ satisfies the Axiom 7.2.1. Since $\tilde{h}$ is a homology theory Axiom 7.2.1 part 1 is satisfied automatically, since $\tilde{h}_i$ preserves weak equivalences for every object i in $\mathcal{I}$. Suppose that ε be a morphism in $\mathcal{M}^{[1]}$ given by a commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\epsilon_1} & D \\ f \Big\uparrow & & \Big\uparrow g \\ A & \xrightarrow[\epsilon_2]{} & C \end{array}$$

in $\mathcal{M}$. Then for any object i in I, in the following commutative diagram

$$\begin{array}{ccccc}
\tilde{h}_i(B) & & \longrightarrow & & \tilde{h}_i(D) \\
\uparrow & & & & \uparrow {\scriptstyle im(\tilde{h}_i(g))} \\
Im(\tilde{h}_i(f)) & \xrightarrow{coim(\tilde{h}_i(\varepsilon))} & Im(\varepsilon) & \xrightarrow{im(\tilde{h}_i(\varepsilon))} & Im(\tilde{h}_i(g)) \\
\uparrow {\scriptstyle coim(\tilde{h}_i(f))} & & & & \uparrow \\
\tilde{h}_i(A) & & \longrightarrow & & \tilde{h}_i(C)
\end{array}$$

the compositions

$$coim(\tilde{h}_i(\varepsilon)) \circ coim(\tilde{h}_i(f)) \text{ and } \; im(\tilde{h}_i(g)) \circ im(\tilde{h}_i(\varepsilon))$$

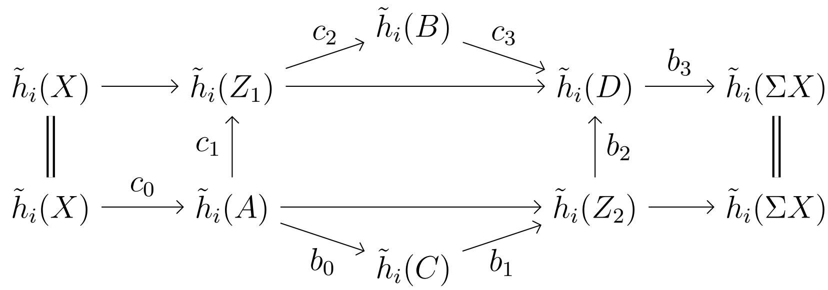
are epimorphisms and monomorpshism respectively. Thus, by the uniqueness of epi-mono factorization up to isomorphism in an abelian category, we have that the image of the morphism

$$\tilde{h}_i(\varepsilon) : Im(\tilde{h}_i(f)) \to Im(\tilde{h}_i(g))$$

in $\mathcal{A}$ is equal to image of the composition $\tilde{h}_i(A) \to \tilde{h}_i(D)$ which implies Axiom 7.2.1 part 2 holds for the family $\mathcal{H}$. Now assume that the above commutative diagram ε fits into the diagram

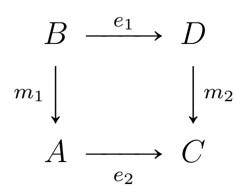
$$\begin{array}{ccccc}
* & & \longrightarrow & & \Sigma X \\
\uparrow & & & & \uparrow \\
Z_1 & \longrightarrow & B & \xrightarrow{\epsilon_2} & D \\
\uparrow & & & & \uparrow \\
A & \xrightarrow{\epsilon_1} & C & \longrightarrow & Z_2 \\
\uparrow & & & & \uparrow \\
X & & \longrightarrow & & *
\end{array}$$

such that each square is a homotopy pushout. From the diagram we obtain the following diagram where the rows are exact sequences:



where c_0, c_1, c_2 and c_3 are corresponding induced morphisms.

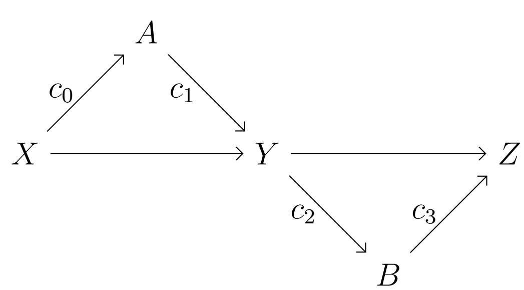
Lemma 7.3.1. *[27] Given a diagram in an abelian category $\mathcal{A}$*



1. *if the diagram is a pullback and e_2 is an epimorphism then e_1 is an epimorphism*
2. *if the diagram is a pushout and m_1 is an monomorphism then m_2 is an monomorphism*

We need the following technical lemma:

Lemma 7.3.2. *Given a diagram in an abelian category $\mathcal{A}$*



such that the row is exact, then

$$ker(Im(c_2 \circ c_1) \to Im(c_3 \circ c_2 \circ c_1)) = Im(c_2 \circ c_1 \circ c_0).$$

Proof. From this diagram one can obtain the following diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & \nearrow & & \searrow & & & \\
 & & & Im(c_2c_1c_0) & & Im(c_3c_2c_1) & & & \\
 & & {\scriptstyle e_{2,1,0}}\nearrow & & {\scriptstyle m_{2,1,0}}\searrow \quad {\scriptstyle e_{3,2,1}}\nearrow & & {\scriptstyle m_{3,2,1}}\searrow & & \\
 & & Im(c_1c_0) & & Im(c_2c_1) & & Im(c_3c_2) & & \\
 & {\scriptstyle e_{1,0}}\nearrow & & {\scriptstyle m_{1,0}}\searrow \quad {\scriptstyle e_{2,1}}\nearrow & & {\scriptstyle m_{2,1}}\searrow \quad {\scriptstyle e_{3,2}}\nearrow & & {\scriptstyle m_{3,2}}\searrow & \\
 & Im(c_0) & & Im(c_1) & & Im(c_2) & & Im(c_3) & \\
{\scriptstyle e_0}\nearrow & & {\scriptstyle m_0}\searrow \quad {\scriptstyle e_1}\nearrow & & {\scriptstyle m_1}\searrow \quad {\scriptstyle e_2}\nearrow & & {\scriptstyle m_2}\searrow \quad {\scriptstyle e_3}\nearrow & & {\scriptstyle m_3}\searrow \\
X & \xrightarrow[c_0]{} & A & \xrightarrow[c_1]{} & Y & \xrightarrow[c_2]{} & B & \xrightarrow[c_3]{} & Z
\end{array}
$$

obtained by the canonical epi-mono factorizations of each map (see [27]), that all maps labeled as e_{-} are epimorphisms and that all maps labeled as m_{-} are monomorphisms. Since the sequence

$$X \xrightarrow{c_1c_0} Y \xrightarrow{c_3c_2} Z$$

is exact, then we have $Im(c_1c_0) = Ker(c_3c_2)$, so that in the following portion of the above diagram

$$
\begin{array}{ccccc}
Im(c_1c_0) & \longrightarrow & Im(c_2c_1c_0) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
Im(c_1) & \longrightarrow & Im(c_2c_1) & \longrightarrow & Im(c_3c_2c_1)
\end{array}
$$

the outer square is a pullback diagram. We want to show the right hand side square is also a pullback so that $Im(c_2c_1c_0)$ is the kernel of the map

$$Im(c_2c_1) \to Im(c_3c_2c_1)$$

in the above diagram. Assume that the following diagram

$$
\begin{array}{ccc}
K & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
Im(c_2c_1) & \longrightarrow & Im(c_3c_2c_1)
\end{array}
$$

is a pullback diagram. Then there is a unique map

$$\kappa : Im(c_2c_1c_0) \to K$$

so that the following diagram is commutative:

$$\begin{array}{ccccccc} Im(c_1c_0) & \longrightarrow & Im(c_2c_1c_0) & \xrightarrow{\kappa} & K & \longrightarrow & 0 \\ \downarrow & & & \searrow & \downarrow & & \downarrow \\ Im(c_1) & & \longrightarrow & & Im(c_2c_1) & \longrightarrow & Im(c_3c_2c_1) \end{array}$$

where both squares (that contain K as a vertex) are pullbacks. Since the maps

$$m_{2,1,0} : Im(c_2c_1c_0) \to Im(c_2c_1) \quad \text{and} \quad K \to Im(c_2c_1)$$

in the diagram are monomorphisms, so does κ. We also have

$$e_{2,1} : Im(c_1) \to Im(c_2c_1)$$

is epimorphism. Since $\mathcal{A}$ is an abelian category, then the map

$$Im(c_1c_0) \to K$$

in the diagram above is an epimorphism (see [27]); as both maps fits into a pullback square as above. Since

$$e_{2,1,0} : Im(c_1c_0) \to Im(c_2c_1c_0)$$

also an epimorphism, then we have κ is an epimorphism, which implies it is an isomorphism. $\square$

Hence we have

$$Ker(\tilde{h}_i(\varepsilon)) = Im \begin{pmatrix} \tilde{h}_i(B) \\ \uparrow \\ \tilde{h}_i(X) \end{pmatrix}$$

If we apply dual procedure to the sequence

$$\tilde{h}_i(A) \xrightarrow{b_0} \tilde{h}_i(C) \xrightarrow{b_1} \tilde{h}_i(Z_6) \xrightarrow{b_2} \tilde{h}_i(D) \xrightarrow{b_3} \tilde{h}_i(\Sigma X)$$

again obtained from the diagram so that the sequence

$$\tilde{h}_i(A) \xrightarrow{b_1 b_0} \tilde{h}_i(Z_6) \xrightarrow{b_3 b_2} \tilde{h}_i(\Sigma X)$$

is exact, we obtain

$$Coker(\tilde{h}_i(\varepsilon)) = Im \begin{pmatrix} \tilde{h}_i(\Sigma X) \\ \uparrow \\ \tilde{h}_i(C) \end{pmatrix}$$

From part 2 of Axiom 7.2.1 we have

$$Im(\tilde{h}_i(\varepsilon)) = Im \begin{pmatrix} \tilde{h}_i(D) \\ \uparrow \\ \tilde{h}_i(A) \end{pmatrix}$$

Hence we have shown that Axiom 7.2.1, Part 3 also holds for the family $\mathcal{H}$. Hence if we are given a sequence of morphisms

$$\ldots \longrightarrow X_{-2} \longrightarrow X_{-1} \longrightarrow X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \ldots$$

in $\mathcal{M}$, we can construct an associated spectral sequence for the homology theory $\tilde{h}$. In the case when $X_n = *$ for $n < 0$, then the spectral sequence converges. More precisely, we have the following theorem:

Theorem 7.3.3. *Let $\mathcal{I}$, $\mathcal{M}$ and $\mathcal{A}$ as above. For any homology theory $\tilde{h}$ in* $\mathrm{HML}_{\mathcal{I}}(\mathcal{M}, \mathcal{A})$ *and for every object $i \in \mathcal{I}$; there is a spectral sequence with*

$$E^1_{p,q}(i) = \tilde{h}_{i+p+q}(X_{p,p-1})$$

which converges to $\tilde{h}_{i+p+q}(hocolim_n X_n)$

This theorem is an straightforward consequence of the definition of h_n's as the image of the homologies. There is also a cohomology version as well. This time for any object i in $\mathcal{I}$ and for any morphism $f : A \to B$ in $\mathcal{M}$ we just define

$$h_n \begin{pmatrix} B \\ \uparrow \\ A \end{pmatrix} = Im \begin{pmatrix} \tilde{h}^{i+n}(B) \\ \big\downarrow \tilde{h}^{i+n} f \\ \tilde{h}^{i+n}(A) \end{pmatrix}$$

where $\tilde{h}$ is a cohomology theory $\mathrm{COHML}_{\mathcal{I}}(\mathcal{M}, \mathcal{A})$. Reversing all arrows in the above construction will give us the cohomology spectral sequence. This cohomology spectral sequence will converge to

$$\tilde{h}^{i+p+q}(hocolim_n X_n),$$

as it is equal to the image of the identity map on $\tilde{h}^{i+p+q}(hocolim_n X_n)$. In other words we have the following theorem:

Theorem 7.3.4. *For any cohomology theory* $\mathrm{COHML}_{\mathcal{I}}(\mathcal{M}, \mathcal{A})$ *and for every object* $i \in \mathcal{I}$*; there is a spectral sequence with*

$$E_1^{p,q}(i) = \tilde{h}^{i+p+q}(X_{p,p-1})$$

which converges to $\tilde{h}^{i+p+q}(hocolim_n X_n)$

7.4 Homotopy spectral sequences

The homotopy theories defined in Section 6.2 with objects in a category with the inverse of a loop space action. For the spectral sequences for these homotopy theories we will associate a loop space object in a category with a weak equivalences and point. The definition of a loop space object is as follows:

Definition 7.4.1. For an object X in $\mathcal{M}$ we define the loop space object ΩX from the the homotopy pullback square

$$\begin{array}{ccc} \Omega X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & X \end{array}$$

Then one can also define objects $X_{j-1,i}$ in M as follows analogues to above construction of the spectral sequences by using homotopy pullbacks. For each integers i, j, define $X_{i,i} = *$ and $X_{j-1,i}$ by the homotopy pullback diagrams

$$\begin{array}{ccc} X_{j-1} & \longrightarrow & X_j \\ \downarrow & & \downarrow \\ X_{j-1,i} & \longrightarrow & X_{j,i} \end{array}$$

So that we obtain the following diagram

$$
\begin{array}{ccccccccccc}
\dots & \longrightarrow & X_{-2} & \longrightarrow & X_{-1} & \longrightarrow & X_0 & \longrightarrow & X_1 & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & X_{-2,1} & \longrightarrow & X_{-1,1} & \longrightarrow & X_{0,1} & \longrightarrow & * & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
\dots & \longrightarrow & X_{-2,0} & \longrightarrow & X_{-1,0} & \longrightarrow & * & & & & \\
 & & \downarrow & & \downarrow & & & & & & \\
\dots & \longrightarrow & X_{-2,-1} & \longrightarrow & * & & & & & & \\
 & & \downarrow & & & & & & & & \\
\dots & \longrightarrow & * & & & & & & & &
\end{array}
$$

If $i \leqslant j \leqslant k$ then the morphism from $\Omega X_{j,k}$ to $X_{i,j}$ in the homotopy pullback diagram

$$
\begin{array}{ccccc}
\Omega X_{j,k} & \longrightarrow & X_{i,j} & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & X_{i,k} & \longrightarrow & X_{j,k}
\end{array}
$$

will be denoted by $\delta(i, j, k)$. Homotopy version of lemma 7.2.1 is as follows:

Lemma 7.4.1. *If $i \leqslant j \leqslant k \leqslant l$ then $\delta(i, j, k) \circ \Omega\delta(j, k, l)$ factors through $*$.*

Proof. As in the proof of 7.2.1 in the homotopy category the composition $\delta(k, j, i) \circ \Omega\delta(l, k, j)$ is equivalent to the composition $e \circ f \circ g \circ h$ in the below diagram where all rectangles are homotopy pullbacks.

$$
\begin{array}{ccccccccc}
\Omega\Omega X_{k,l} & \longrightarrow & * & & & & & & \\
{\scriptstyle h}\big\downarrow & & \big\downarrow & & & & & & \\
\Omega X_{i,k} & \longrightarrow & \Omega X_{i,l} & \longrightarrow & * & & & & \\
{\scriptstyle g}\big\downarrow & & \big\downarrow & & \big\downarrow & & & & \\
\Omega X_{j,k} & \xrightarrow{f} & \Omega X_{j,l} & \xrightarrow{e} & X_{i,j} & \longrightarrow & * & & \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
* & \longrightarrow & \Omega X_{k,l} & \longrightarrow & X_{i,k} & \longrightarrow & X_{j,k} & \longrightarrow & * \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
 & & * & \longrightarrow & X_{i,l} & \longrightarrow & X_{j,l} & \longrightarrow & X_{k,l}
\end{array}
$$

□

Let $\mathcal{A}$ be a category with a kernel-image-cokernel factorization Γ. Let

$$\mathcal{H} = \{h_n : \mathcal{M}^{[1]} \to \mathcal{A}\}_{n\in\mathbb{Z}}$$

be a family of functors with a family of natural isomorphisms

$$\{\sigma_n : h_n \to h_{n+1} \circ \Sigma_*\}_{n\in\mathbb{Z}}.$$

Suppose that we are given a a morphism ε in $\mathcal{M}^{[1]}$ as

$$
\begin{array}{ccc}
B & \longrightarrow & D \\
\big\uparrow & & \big\uparrow \\
A & \longrightarrow & C
\end{array}
$$

Axiom 7.4.1. The following conditions holds for the family $\mathcal{H}$

1. h_n preserves weak equivalences for all n
2. For every morphism ε in $\mathcal{M}^{[1]}$ given by the commutative diagram in $\mathcal{M}$ as above, we have

$$I(h_n(\varepsilon)) = h_n \begin{pmatrix} D \\ \big\uparrow \\ A \end{pmatrix}$$

3. If the morphism ε fits in a diagram

$$\begin{array}{ccccccc}
\Omega X & \longrightarrow & Z_7 & \longrightarrow & Z_8 & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A & \longrightarrow & Z_5 & \longrightarrow & C & \longrightarrow & Z_6 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Z_3 & \longrightarrow & B & \longrightarrow & Z_4 & \longrightarrow & D \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & Z_1 & \longrightarrow & Z_2 & \longrightarrow & X
\end{array}$$

where each square is a homotopy pullback square, then the kernel-image-cokernel decomposition of $h_n(\varepsilon)$ is naturally isomorphic to the following sequence:

$$h_n\begin{pmatrix} \Omega X \\ \downarrow \\ B \end{pmatrix} \to h_n\begin{pmatrix} A \\ \downarrow \\ B \end{pmatrix} \to h_n\begin{pmatrix} A \\ \downarrow \\ D \end{pmatrix} \to h_n\begin{pmatrix} C \\ \downarrow \\ D \end{pmatrix} \to h_n\begin{pmatrix} C \\ \downarrow \\ X \end{pmatrix}$$

for all n.

Theorem 7.4.2. *There is a ((co)homotopy) spectral sequence with*

$$E^r_{p,q} = h_{p+q}\begin{pmatrix} X_{p-r,p} \\ \downarrow \\ X_{p-1,p+r-1} \end{pmatrix}$$

and the differential $d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ is the map induced by the morphism in $\mathcal{M}^{[1]}$ given by

$$\begin{array}{ccc}
\Omega X_{p-r,p} & \xrightarrow{\delta(p,p-r,p-2r)} & X_{p-2r,p-r} \\
\downarrow & & \downarrow \\
\Omega X_{p-1,p+r-1} & \xrightarrow{\delta(p-r+1,p-1,p-r-1)} & X_{p-r-1,p-1}
\end{array}$$

The proof is analogues to the proof of Theorem 7.2.2.

One can obtain dual spectral sequences for homotopy and cohomotopy theories graded over symmetric monoidal categories, by using similar arguments that are used in Section 7.3. These spectral sequences will compute homotopy and cohomotopy groups graded over symmetric monoidal categories of homotopy limits of cofiltered objects.

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