ON THE NÉRON-SEVERI LATTICE OF DELSARTE SURFACES

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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The Néron-Severi group, $NS(X)$, of a given (non-singular projective) variety, X, is defined in only algebro-geometric terms, however it is also known to be an arithmetic invariant. So it is an important study that helps understanding the geometry of the variety. However, there is no known method to compute it in general. For this reason, one first computes the Picard number $\rho(X) = \text{rnk } NS(X)$ of the variety. There has been many studies which elevated the understanding of $\rho(X)$ in special cases. Yet the difficulty of the computation in the general case still remains.

On the other hand, in the case of Delsarte surfaces, an explicit algorithm to compute $\rho(X)$ is given by Shioda [1], and Degtyarev [2] showed that a generating set for the Néron-Severi group, $NS(X)$ can be computed in some cases. Moreover, Heijne [3] gives a classification of all Delsarte surfaces with only isolated ADE singularities. We give an introduction to Delsarte surfaces, and determine which of the Delsarte surfaces given in [3] fit in the descriptions given in [2].

Keywords: Néron-Severi group, Delsarte surface, Smith normal form.

ÖZET

DELSARTE YÜZEYLERİNİN NÉRON-SEVERI LATİSİ ÜZERİNE

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Verilen bir (tekilsiz izdüşümsel) çeşitlemin (X), Néron-Severi grubu ($NS(X)$), sadece cebirsel geometriye ait terimlerle tarif edilir, ancak aritmetik bir değişmez olarak da bilinir. Bu yüzden çeşitlemin geometrisini anlamaya yardımcı olan önemli bir çalışma alanı oluşturur. Ancak, bu grubu hesaplamak için genel geçer bir yöntem bulunmamaktadır. Bu sebeple önce Néron-Severi grubunun mertebesi olan Picard sayısı ($\rho(X)$) hesaplanır. Özel durumlarda $\rho(X)$'in anlaşılmasını artıran çok sayıda çalışma yapıldı. Yine de bu hesaplamayı yapmanın zorluğu hala geçerliliğini koruyor.

Öte yandan, Shioda [1] Delsarte yüzeyleri üzerinde çalışan, $\rho(X)$'i hesaplayan açık bir algoritma üretmiştir, ve Degtyarev [2] bazı durumlarda Néron-Severi grubu için açık bir üretici küme hesaplanabildiğini göstermiştir. Bunun yanı sıra, Heijne [3] sadece ayrık ADE tekillikleri içeren Delsarte yüzeyleri için bir sınıflandırma yapmıştır. Biz Delsarte yüzeylerini tanıtıp Heijne tarafından verilen yüzeylerden hangilerinin Degtyarev'in çalışmasındaki durumlara uyduğunu belirledik.

Anahtar sözcükler: Néron-Severi grubu, Delsarte yüzeyleri, Smith normal formu.

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Contents

Chapter 1

Introduction

In this thesis, we will be mainly working on complex Delsarte surfaces. A Delsarte surface is an algebraic surface X_A in $\mathbb{P}^3$, defined by a sum of four monomials with properties given in Definition 3.1.1. One can regard these surfaces as a generalization of Delsarte curves (as introduced by Shioda in [1]). Likewise, one can consider Fermat surfaces which are of the form $X_m := \{z_0^m + z_1^m + z_2^m + z_3^m = 0\}$ as covering spaces of X_A (q.v. Chapter 3).

The Néron-Severi group of a nonsingular projective variety X is defined as the group consisting of divisors on X up to algebraic equivalence. It is a desirable work area, on any given algebraic variety, to understand its Picard and/or Néron-Severi groups because of their close relation to many mathematical problems, like calculation of Brauer groups. The Picard number (also called Picard rank) $\rho(X)$ of a given algebraic variety is defined as the rank of its Néron-Severi group, $NS(X)$ (q.v. Chapter 2). The Picard number is not a birational invariant, hence one can prefer to work with the Lefschetz number $\lambda(X) := b_2(X) - \rho(X)$ where $b_2(X)$ is the second Betti number of X. The Lefschetz number is, on the other hand, a birational invariant of the surface [1]. Main reason for us to work with birational invariants is to extend our reach to the singular Delsarte surfaces.

In general, there is no known method for calculating the Picard number of a given algebraic variety, even when one works over $\mathbb{C}$. However, in the case of Delsarte surfaces, Shioda gives an explicit algorithm to compute $\rho(X)$, using the Lefschetz number [1] for fields with arbitrary characteristic. Moreover, Degtyarev also shows that, in some cases, one can compute the generating set for $NS(X)$ of a Delsarte surface [2] explicitly.

There is another useful method for defining Delsarte surfaces. Let $\mathbb{G} \cong \mathbb{Z}^3$ be a multiplicative abelian group with generating set $\{t_i\}_{i=0,1,2,3}$. We have the relation $\prod_{i=0}^{3} t_i = 1$ between these generators. Then every epimorphism $\alpha : \mathbb{G} \twoheadrightarrow G$ where $|G|$ is finite has a corresponding Delsarte surface X_α (q.v. Definition 3.1.2). When one is interested only in birational invariants, this description helps to focus only on the epimorphism between abelian groups above.

1.1 Main Problem

Let $\alpha : \mathbb{G} \twoheadrightarrow G$ be an epimorphism with $|G|$ finite and let X_α be the corresponding Delsarte surface. We consider the Néron-Severi group, $NS(X_\alpha)$, as a subgroup of the homology group $H_2(X_\alpha)/\mathrm{Tors}(H_2(X_\alpha))$ as a consequence of Poincaré duality. This allows one to represent a divisor $D \subset X_\alpha$ by its fundamental class $[D]$. With this in mind, we consider a certain 'obvious' divisor V_α and define $\mathbf{S}_\alpha$ to be the subgroup of $NS(X_\alpha)$ generated by the irreducible components of the divisor V_α. Hence if we consider the inclusion $\iota : V_\alpha \hookrightarrow X_\alpha$, we can also state that $\mathbf{S}_\alpha = \mathrm{Im}[\iota_* : H_2(V_\alpha) \to H_2(X_\alpha)/\mathrm{Tors}(H_2(X_\alpha))]$. Shioda [4] shows that in some cases we have

$$NS(X_\alpha) \otimes \mathbb{Q} = \mathbf{S}_\alpha \otimes \mathbb{Q}. \tag{1.1}$$

This means that components of the divisor V_α generate the Néron-Severi lattice $NS(X_\alpha)$ over $\mathbb{Q}$. This statement is shown by independently calculating the Picard rank of the surface and the rank of the $\mathbf{S}_\alpha$. An interesting question is to find cases in which this property extends to hold over $\mathbb{Z}$. This question leads one to inspect cases which satisfy $\mathbf{T}_\alpha := \mathrm{Tors}(NS(X_\alpha)/\mathbf{S}_\alpha) = 0$ which gives an answer for the question. However, the calculation of this torsion is a valid question even when we do not have (1.1). Although showing that in general we do not have this desired property, Degtyarev reveals some of these special cases in [2] in a theorem. We will state this theorem because it is of vital importance to our work.

Theorem 1.1.1 ([2], Theorem 1.7)**.** *We have* $\boldsymbol{T}_\alpha = 0$ *in each of the following cases:*

1. *Fermat surfaces* X_α*, with* $\alpha = \mathbb{G} \twoheadrightarrow \mathbb{G}/m\mathbb{G}$*;*
2. *Delsarte surfaces* X_α *that are unramified at* ∞*, with* $\alpha(t_0) = 1$ *;*
3. *cyclic Delsarte surfaces* X_α*, with* $\alpha = \mathbb{G} \twoheadrightarrow G$ *where* G *is cyclic.*

We will be studying Delsarte surfaces from point of view of singularities. As with other algebraic varieties with rational double points, there are useful classifications that one can consider for Delsarte surfaces. For this purpose we relate to a work of Heijne [3], in the Appendix of which all Delsarte surfaces with only isolated ADE singularities are listed, along with their calculated Picard numbers. We take the information on corresponding matrices of these surfaces, only. Our main goal is to determine the special cases with $\mathbf{T}_\alpha = 0$, especially the cyclic Delsarte surfaces out of these surfaces. We intend to construct an explicit relation between these matrices and the latter definition of Delsarte surfaces (q.v. 3.1.2) by calculating a kernel in the matrix form for each surface and using the Smith normal form (q.v. Chapter 4). Next theorem can be seen as the main result of this work.

Theorem 1.1.2. *The Delsarte surfaces marked with * in Appendix A are the Delsarte surfaces with only isolated ADE singularities satisfying* $\boldsymbol{T}_\alpha = 0$*.*

The groups $\mathbb{G}$ and G are paired with modules in order to be compatible with our calculations(cf. Shioda [1]), then we calculate the Smith normal forms to discern which surfaces in Appendix A are the cases that we are interested in.

1.2 Contents of the Thesis

In chapter 2, we give information and definitions about ADE singularities and the Picard number of surface. Moreover, we recall how one calculates and uses the Smith normal form of a matrix for our purposes. In chapter 3, we give some properties of the Delsarte surfaces while making connection between the two definitions mentioned above. We also introduce the 'obvious' divisor V_α. In chapter 4, we prove our main result and give an example to depict how this method works. The explicit results for all 83 surfaces are given in Appendix A.

Chapter 2

Preliminaries

In this thesis we only consider algebraic varieties defined over the field $\mathbb{C}$. As a result we assume the surfaces are oriented with the canonical complex orientation.

2.1 Singularities

There are different ways to define ADE singularities, however we will be using the same definition Heijne used in [3].

Definition 2.1.1. An ADE singularity is a singular point P on a complex surface S which is locally isomorphic to one of the following types of singularities:

- $z_0^{n+1} + z_1^2 + z_2^2$; i.e of type A_n, $n \geq 1$,
- $z_0^{n-1} + z_0 z_1^2 + z_2^2$; i.e of type D_n, $n \geq 4$,
- $z_0^3 + z_1^4 + z_2^2$; i.e of type E_6,
- $z_0^3 + z_0 z_1^3 + z_2^2$; i.e of type E_7,
- $z_0^3 + z_1^5 + z_2^2$; i.e of type E_8.

Using this definition, Heijne manages to reduce the number of Delsarte surfaces with only isolated ADE singularities to 83.

2.2 The Néron-Severi Group

Our main interest revolves around the Néron-Severi group of an algebraic variety (i.e. a Delsarte surface in our case), which can be described as the group of algebraic equivalence classes of divisors in X. This group is closely related to the Picard group of the algebraic variety which can be seen as subgroup of Div X under linear equivalence. Here Div X is the group of all divisors on X, which can be regarded as the free abelian group on the prime divisors on X, and we say two divisors are linearly equivalent if the difference of the divisors is a principal divisor. We will also denote the subgroup of Pic X that consists of divisor classes algebraically equivalent to zero by $\text{Pic}^0\ X$ so that Pic $X/\text{Pic}^0\ X$ is a finitely generated abelian group which is the Néron-Severi group, $NS(X)$. The fact that the Picard rank $\rho(X)$ of the surface (i.e. rank of the $NS(X)$) is finite comes from the Néron-Severi theorem[5]. Hence we can observe this relation between Pic X and $NS(X)$ in the form of an exact sequence

$$1 \to \text{Pic}^0\ X \to \text{Pic}\ X \to NS(X) \to 0.$$

Moreover, when we reduce the case to the field of complex numbers, $\mathbb{C}$, the exponential function exp yields an exact sequence of abelian groups

$$0 \to \mathbb{Z} \to \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \to 0$$

where one considers $\mathbb{C}$ as additive group and $\mathbb{C}^*$ as multiplicative group. From this sequence we derive the exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$$

where $\mathbb{Z}$ is the constant sheaf, $\mathcal{O}_X$ is the sheaf of holomorphic functions on X(i.e. structure sheaf), and $\mathcal{O}_X^*$ is the subsheaf consisting of the non-vanishing holomorphic functions(i.e. invertible elements of $\mathcal{O}_X$ under multiplication). This sequence gives rise to an exact sequence of cohomology groups

$$0 \to H^1(X_h, \mathbb{Z}) \to H^1(X_h, \mathcal{O}_{X_h}) \to H^1(X_h, \mathcal{O}^*_{X_h}) \to H^2(X_h, \mathbb{Z}) \to \dots \qquad (2.1)$$

where X_h is a complex analytic space associated with X. We recall the fact that

$$H^1(X_h, \mathcal{O}^*_{X_h}) \cong \text{Pic } X_h$$

(for details cf. [5]). Now Serre's theorem on coherent sheaves implies

$$\text{Pic } X_h \cong H^1(X_h, \mathcal{O}^*_{X_h}) \cong H^1(X, \mathcal{O}^*_X) \cong \text{Pic } X.$$

Hence the exact sequence 2.1 can be revised into

$$0 \to H^1(X_h, \mathbb{Z}) \to H^1(X_h, \mathcal{O}_{X_h}) \xrightarrow{c_1} \text{Pic } X \to H^2(X_h, \mathbb{Z}) \to \dots . \qquad (2.2)$$

Notice that X_h are compact complex manifolds which indicates that $H^i(X_h, \mathbb{Z})$ are finitely generated abelian groups. Moreover, one can see that Im c_1 is isomorphic to Pic X/Pic^0 $X = NS(X)$. Thus $NS(X)$ is a subgroup of $H^2(X_h, \mathbb{Z})$ which makes it a finitely generated group.

2.3 Smith Normal Form

In order to understand the structures of modules that we obtain in our work, we will use Smith normal forms of the relation matrices for the corresponding epimorphisms.

Definition 2.3.1. Let R be a principal ideal domain and let A be a $k \times l$ matrix with entries in R. We say that A is in Smith normal form if A is a diagonal matrix with only nonzero elements $a_1, \dots, a_m \in R$ lying in the first m entries of the diagonal and satisfy $a_i | a_{i+1}$ for each $i < m$. That is, A is of the form

$$A = \begin{bmatrix} a_1 & 0 & 0 & \dots & & 0 \\ 0 & a_2 & 0 & \dots & & 0 \\ 0 & 0 & \ddots & & & 0 \\ \vdots & & & a_m & & \vdots \\ & & & & \ddots & \\ 0 & & & \dots & & 0 \end{bmatrix}.$$

It is of immediate concern whether one has a corresponding Smith normal form of any given matrix or not. We give an existence theorem that fits our needs.

Theorem 2.3.2 ([6], Theorem 2.1, p.7)**.** *If M is a matrix with entries in a principal ideal domain R, then there are invertible matrices P and Q over R such that PMQ is in Smith normal form.*

The proof of this theorem is a well-known algorithm which calculates unit matrices P and Q by elementary row and column operations while relying on Bézout's identity.

At this point, we also define the so called determinantal divisors which will be useful later.

Definition 2.3.3. The i-th determinantal divisor of a matrix $A \in M_{k\times l}$, $d_i(A)$, is defined as the gcd of all the i-th minors of A.

Here gcd is considered among the non-zero minors only, if all i-th minors are zero than $d_i(A)$ is considered zero. It is also worth mentioning that these divisors have this following property.

Corollary 2.3.4. *For given $1 \leqslant i \leqslant \min(k,l)$, $d_i(A) = d_i(B)$ if $A = PBQ$ i.e. if A and B are equivalent matrices.*

Indeed, each $k \times k$ minor of A can be written as a linear combination of $k \times k$ minors of B and vice versa [7], giving us the equivalence of greatest common divisors.

If A is the Smith normal form of $M \in M_{k\times l}$, by Theorem 2.3.2, A is related to M, i.e. $A = PMQ$, hence Corollary 2.3.4 suggests that we can calculate entries of A by using $d_i(M)$. Let us fix $d_0(M) = 1$, then we can define a_i with the relation $d_i(M) = a_i d_{i-1}(M)$ for $i \geqslant 1$. Notice that, this also shows that there exists a unique matrix in Smith normal form, equivalent to a given matrix M.

Remark 2.3.5. Note that this way of calculation is not efficient when working with big matrices, however it is efficient enough for small matrices such as the ones that we work on. For bigger matrices most computer algortihms employ the algorithm used in proving Theorem 2.3.2.

Smith normal form is useful for our work since one can easily determine the structure of a given module with its help. This connection comes with the relation matrices which correspond to the kernel of the R-module homomorphism $\alpha : R^n \to M$ (cf. [6]).

Proposition 2.3.6. *Let M be an R-module. Suppose that A is a relation matrix for M. If there exist unit matrices P and Q with PAQ a diagonal matrix with entries a_i, then $M \cong \bigoplus R/(a_i)$.*

Proof. Notice that PAQ is also a relation matrix for the same module, being eqivalent to A. Let $\{m_i\}_{i=1}^n$ be a generating set for M. We have the corresponding homomorphism $\alpha : R^n \to M$ defined as $(r_1, \ldots, r_n) \mapsto \sum_{i=1}^n r_i m_i$. Notice that the relation submodule is the kernel of α, giving us $M \cong R/\text{Ker}\ \alpha$. Moreover, the relation submodule for PAQ, being diagonal, also corresponds to the kernel of the surjection $R^n \to \bigoplus R/(a_i)$ defined as $(r_1, \ldots, r_n) \mapsto (r_1 + (a_n), \ldots, r_n + (a_n))$. Hence $R/\text{Ker}\alpha \cong \bigoplus R/(a_i)$, i.e. $M \cong \bigoplus R/(a_i)$. □

Now, considering Theorem 2.3.2 along with Proposition 2.3.6 we get the interesting result stated below.

Corollary 2.3.7 ([6], Corollary 2.2, p.8)**.** *If M is a finitely generated module over a principal ideal domain R, then there are elements $a_1, \ldots, a_m \in R$ such that $a_i | a_{i+1}$ for each $i = 1, \ldots, m-1$, and an integer $t \geq 0$ such that $M \cong R/(a_1) \oplus \ldots \oplus R/(a_m) \oplus R^t$.*

Chapter 3

Delsarte Surfaces

In this chapter we give information about our main study objects, Delsarte surfaces.

3.1 Definitions

Definition 3.1.1. A Delsarte surface is a zero set $X_A \subset \mathbb{P}^3$ of a homogeneous polynomial of the form

$$\sum_{i=0}^{3} \prod_{j=0}^{3} z_j^{a_{ij}}. \tag{3.1}$$

We represent this polynomial with $A := [a_{ij}]$, the exponent matrix which satisfies the following conditions:

1. each entry a_{ij} , $0 \leqslant i, j \leqslant 3$, is a non-negative integer;
2. each column of A has at least one zero;
3. $(1, 1, 1, 1)^t$ is an eigenvector of A, i.e.,$\sum_{j=0}^{3} a_{ij} = \lambda = const(i)$;
4. A is non-degenerate, i.e., $det A \neq 0$.

Notice that a Delsarte surface cannot contain any of the coordinate planes, seen by (2). The polynomial defining the surface is homogeneous, as desired, (3) also confirms that the degree of this polynomial is λ, the eigenvalue of the representing matrix A corresponding to $(1,1,1,1)^t$.

Shioda defines the cofactor matrix corresponding to a given Delsarte surface X_A in [1] as $A^* := (detA)A^{-1}$, and

$$d := gcd(a^*_{ij}), \quad m := |detA| \quad B := mA^{-1} = \pm d^{-1}A^* \tag{3.2}$$

by using these, one can define the projection mappings

$$\pi_B : (z_i) \mapsto \left(\prod_{j=0}^{3} z_j^{b_{ij}}\right) \quad \& \quad \pi_A : (z_i) \mapsto \left(\prod_{j=0}^{3} z_j^{a_{ij}}\right) \tag{3.3}$$

of the desired diagram

$$X_m \xrightarrow{\pi_B} X_A \xrightarrow{\pi_A} X := X_1. \tag{3.4}$$

The covering mappings, π_A and $\pi_B \circ \pi_A : (z_i) \mapsto (z_i^m)$, both are ramified coverings, and the ramification points lies in coordinate planes. These points gives us the ramification locus $R := R_0 + R_1 + R_2 + R_3 \subset X$, where $R_i := X \cap \{z_i = 0\}$. Hence $X \setminus R$ is unramified.

Moreover, the fundamental group $\pi_1(X \setminus R)$ is known to be abelian which means it is isomorphic to $H_1(X \setminus R)$ by Hurewicz theorem. Hence, using the Poincaré-Lefschetz duality we get $H_1(X \setminus R) = H^3(X, D)$. This means one can obtain the following [2];

$$\pi_1(X \setminus R) = H^2(R)/H^2(X) = \mathbb{G}$$

where $\mathbb{G} \cong \mathbb{Z}^3$ is a multiplicative abelian group generated by four generators, say, t_0, t_1, t_2, t_3 with the relation $t_0t_1t_2t_3 = 1$. Additionally each generator $t_i \in \mathbb{G}$ can be assessed on fundamental classes of R_j as δ_{ij}, the Kronecker delta. We see that the topological covering π_A is unramified outside any neighbourhood of the ramification locus and is determined uniquely by a subgroup of $\mathbb{G}$ of finite index. We can identify this subgroup with kernel of an epimorphism $\alpha : \mathbb{G} \twoheadrightarrow G$.

Homogeneity property of Delsarte surfaces allows us to consider mappings related to A and B (3.3) as endomorphisms of $\mathbb{G}$, inducing endomorphisms $A_m, B_m : \mathbb{G}/m\mathbb{G} \to \mathbb{G}/m\mathbb{G}$. Specifically, a generator $t_i \in \mathbb{G}/m\mathbb{G}$ acts on X_m by multiplication of the i-th coordinate with a fixed m-th root of unity. Hence we deduce;

$$m\mathbb{G} \subset \text{Ker } \alpha, \quad \Gamma := \text{Ker } \alpha/m\mathbb{G} = \text{Ker } B_m = \text{Im } A_m, \quad \text{Im } B_m = \text{Ker } A_m$$

these imply that X_A is birationally isomorphic to X_m/Γ.

At this point we can give another definition for Delsarte surfaces, relying only on α [2];

Definition 3.1.2. For a given $\alpha : \mathbb{G} \twoheadrightarrow G$ with finite $|G|$, the corresponding Delsarte surface X_α can be defined as (any) smooth analytic compactification of the (unramified) covering of the complement $X \smallsetminus R$.

3.2 Divisors

Initially, we consider the Fermat surface X_m as a covering, via choosing an m that satisfies $m\mathbb{G} \subset \text{Ker } \alpha$. Now considering the straight lines in X_m as components of pull-backs of $L_i := X \cap \{z_0 + z_i = 0\}$, we obtain m^2 lines for each i. Namely;

$$\begin{aligned} L_1(\zeta, \eta) :&(r : \omega\zeta r : s : \omega\eta s) \\ L_2(\zeta, \eta) :&(r : s : \omega\zeta r : \omega\eta s) \\ L_3(\zeta, \eta) :&(r : s : \omega\eta s : \omega\zeta r) \end{aligned}$$

where both ζ and η are m-th roots of unity, $\omega := \exp(\pi i/m)$, and $(r : s) \in \mathbb{P}^1$.

In order to introduce V_α, we first fix an epimorphism $\alpha : \mathbb{G} \to G$, along with the covering projection mapping $\pi : X_\alpha \to X$. Then we consider the ramification locus $R_\Sigma := R_0 + R_1 + R_2 + R_3$ where $R_i := X \cap \{z_i = 0\}$. Recalling that the $3m^2$ straight lines in Fermat surface (which is itself a Delsarte surface, but also a covering space of X_α for our purposes) comes from the pre-image of the lines

L_i for $i = 1, 2, 3$, we also define $L_\Sigma := L_1 + L_2 + L_3$ so that we can define the divisors on X_α;

$$R_\varsigma[\alpha] := \pi^* R_\varsigma, \quad L_\varsigma[\alpha] := \pi^* L_\varsigma, \quad V_\alpha := R_\Sigma[\alpha] + L_\Sigma[\alpha]$$

where ς can be any one of the indexes $\{0, 1, 2, 3, \Sigma\}$ except for the undefined L_0.

Hence we see that the divisor V_α consists of the components of ramification locus of the above covering, images of straight lines in the covering Fermat surface X_m, and the exceptional divisors (coming from the resolution of singularities)[2].

Chapter 4

Main Results

We are interested in the epimorphism $B_m : \mathbb{G}/m\mathbb{G} \twoheadrightarrow G \subset \mathbb{G}/m\mathbb{G}$ where $\mathbb{G}/m\mathbb{G} := \mathbb{G}_m$ is generated by $< t_0, \ldots, t_3 | t_i^m = 1 >$. On this setting, we can replace each t_i with an m-th root of unity, via fixing a primitive root ζ and taking $t_i = \zeta^{k_i}$. Hence we have;

$$\begin{array}{ccc} z_i & \xrightarrow{\quad t_i \quad} & \zeta^{k_i} z_i \\ \Big\downarrow{\scriptstyle \pi_B} & & \Big\downarrow{\scriptstyle \pi_B} \\ \prod_{j=0}^{3} z_j^{b_{ij}} & \longrightarrow & \prod_{j=0}^{3} \left(\zeta^{k_j} z_j\right)^{b_{ij}} \end{array} .$$

Following this diagram, we can explicitly write the kernel of the epimorphism as

$$\Gamma = \{[\zeta^{k_0} : \zeta^{k_1} : \zeta^{k_2} : \zeta^{k_3}] \in \mathbb{G}_m | \prod_{j=0}^{3} \zeta^{k_j b_{ij}} = \text{const}(i)\}$$

where $\text{const}(i)$ is a constant w.r.t i.

4.1 The Relation Matrix

We want to translate $\mathbb{G}_m$ into a module in order to be able to employ Smith normal form method in our work. In order to do so, we use the identification of $[\zeta^{k_0} : \zeta^{k_1} : \zeta^{k_2} : \zeta^{k_3}]$ with (k_0, k_1, k_2, k_3), following Shioda [1]. This enables us to see $\mathbb{G}_m$ as a module

$$M_m := \{(k_0, k_1, k_2, k_3) \in (\mathbb{Z}_m)^4 | \sum_i k_i \equiv 0 \mod m\}$$

and the image of the endomorphism $B_m : M_m \to M_m$ becomes

$$L_A := \{(k_0, k_1, k_2, k_3)B | (k_0, k_1, k_2, k_3) \in M_m\}.$$

Now we can consider calculating Smith normal forms of the corresponding relation matrices to determine the structure of these modules.

Lemma 4.1.1. *We have $L_A = M_m/\Gamma$, where Γ is generated by $(1, 1, 1, 1)$ and the rows of A.*

Proof. We need to show that

$$\Gamma = \text{Ker } B_m \supset \{(k_0, k_1, k_2, k_3)A | (k_0, k_1, k_2, k_3) \in M_m\}.$$

Using the relation $AB = BA = m1_4$ where 1_4 is the 4×4 identity matrix, we see that any element $\alpha \in M_m$ of the form $(k_0, k_1, k_2, k_3)A$ with $a_{ij} \in A$ gives us

$$(k_0, k_1, k_2, k_3)AB = (k_o, k_1, k_2, k_3)m \equiv (0, 0, 0, 0) \text{ mod } m.$$

Hence elements of the form $(k_0, k_1, k_2, k_3)A$ are in Γ. Now, take an element from Γ, say $k := (k_0, k_1, k_2, k_3)$, so that $kB \equiv \delta(1, 1, 1, 1) \text{ mod } m$ for some $\delta \in \mathbb{Z}$. So we get;

$$kB = (\delta + m\beta_0, \delta + m\beta_1, \delta + m\beta_2, \delta + m\beta_3)$$
$$kB = (\delta, \delta, \delta, \delta) + (\beta_0, \beta_1, \beta_2, \beta_3)AB.$$

Since A and B are both non-degenerate, there exists a unique k s.t. $kB = (1,1,1,1)\delta + (\beta_0, \beta_1, \beta_2, \beta_3)AB$. If we consider each row of A, $r_i^A = (a_{i0}, a_{i1}, a_{i2}, a_{i3})$, we see that $r_0^A B = (m, 0, 0, 0)$ and similarly for other i's. Hence we can write an element of Γ as $(1,1,1,1)B^{-1}\delta + \beta_0 r_0^A + \beta_1 r_1^A + \beta_2 r_2^A + \beta_3 r_3^A$. □

Thus one can build a relation matrix corresponding to B_m using (generators of) Γ. For each B we define;

$$Rel_B = \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{13} & a_{23} & a_{33} \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{4.1}$$

the relation matrix corresponding to L_A. This completes the proof Theorem 1.1.2.

4.2 An Example

We give an example, in order to illustrate how we use Smith normal form on the relation matrix 4.1 to determine if a Delsarte surface is cyclic or not.

Example 4.2.1. Take the singular Delsarte surface defined as the zero set of

$$z_0^{n-2} z_1 z_3 + z_1^{n-2} z_2 z_3 + z_2^n + z_2 z_3^{n-1} = 0.$$

This is the Delsarte surface with index 9 in the Appendix of [3]. It has the corresponding matrices

$$A = \begin{bmatrix} n-2 & 1 & 0 & 1 \\ 0 & n-2 & 1 & 1 \\ 0 & 0 & n & 0 \\ 0 & 0 & 1 & n-1 \end{bmatrix}$$

$$B = \begin{bmatrix} n(n-1)(n-2) & -n(n-1) & 2(n-2) & -n(n-3) \\ 0 & n(n-1)(n-2) & -(n-2)^2 & -n(n-2) \\ 0 & 0 & (n-1)(n-2)^2 & 0 \\ 0 & 0 & -(n-2)^2 & n(n-2)^2 \end{bmatrix}$$

with $m = n(n-1)(n-2)^2$. So we see that the module L_A consists of elements of the form;

$$\begin{aligned}
&(n(n-1)(n-2)k_0, \ldots \\
&-n(n-1)k_0 + n(n-1)(n-2)k_1, \ldots \\
&2(n-2)k_0 - (n-2)^2k_1 + (n-1)(n-2)^2k_2 - (n-2)^2k_3, \ldots \\
&-n(n-3)k_0 - n(n-2)k_1 + n(n-2)^2k_3).
\end{aligned}$$

We see that these elements (hence L_A) are generated by

$$\alpha = (n(n-1)(n-2), -n(n-1)^2, 2(n-2), n)$$

which has order $(n-1)(n-2)^2$, giving us $L_A \cong \mathbb{Z}_{(n-1)(n-2)^2}$.

However, if we build the relation matrix for L_A as in 4.1, calculating its Smith normal form gives us the same information but in a more efficient way;

$$Rel_B = \begin{bmatrix} n-2 & 1 & 0 & 1 \\ 0 & n-2 & 1 & 1 \\ 0 & 0 & n & 0 \\ 0 & 0 & 1 & n-1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & (n-1)(n-2)^2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

here $\cong$ used to imply that the right-hand side matrix is the Smith normal form of left-hand side matrix. This matrix implies, by Corollary 2.3.7, that $L_A \cong \mathbb{Z}_{(n-1)(n-2)^2}$.

Calculating all the Smith normal forms corresponding to each Delsarte surface with only ADE singularities gives us:

51 out of 83 resulted in a Smith normal form of the form;

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & f \end{bmatrix}$$

where $f|m, f < m$, these are the cyclic Delsarte surfaces with $L_A \cong \mathbb{Z}_f$.

16 out of 83 resulted in a Smith normal form of the form;

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & m \end{bmatrix}$$

these are cyclic with maximum possible order m, i.e $L_A \cong \mathbb{Z}_m$.

15 out of 83 resulted in a Smith normal form of the form;

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & q \end{bmatrix}$$

where $p|m$ and $q|m$ but possibly $q = m$, these are the cases with $L_A \cong \mathbb{Z}_p \oplus \mathbb{Z}_q$.

last case results in a Smith normal form of the form;

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{bmatrix}$$

which is the Fermat surface of degree m, with $L_A = M_m \cong \mathbb{Z}_m^3$. Again, explicit results are in the Appendix A.

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Appendix A

Data

In the below table, we give explicit results for calculated Delsarte surfaces. All surfaces in this list are taken from [3]. To the left is just an index, as in [3] with the slight change that the ones marked with * are the cyclic Delsarte surfaces, in the middle are the polynomials that generate the surface and to the right are the corresponding modules L_A. Notice that 83-th case (Fermat case) has a mark too, because it is also one of cases mentioned in [2].

1*	$z_0^{n-2}z_1z_2 + z_1^{n-2}z_2z_3 + z_2^n + z_0z_3^{n-1}$	$\mathbb{Z}_{((n-1)(n-2)^2+1)}$
2*	$z_0^{n-2}z_1z_2 + z_1^{n-2}z_2z_3 + z_2^n + z_1z_3^{n-1}$	$\mathbb{Z}_{((n-2)(n^2-3n+1))}$
3*	$z_0^{n-2}z_1z_2 + z_0z_1^{n-2}z_2 + z_2^n + z_0z_3^{n-1}$	$\mathbb{Z}_{((n-1)(n^2-4n+3))}$
4*	$z_0^{n-2}z_1z_2 + z_0z_1^{n-2}z_3 + z_2^n + z_0z_3^{n-1}$	$\mathbb{Z}_{((n-1)(n^2-4n+3)+1)}$
5*	$z_0^{n-2}z_1z_2 + z_0z_1^{n-2}z_3 + z_2^n + z_1z_3^{n-1}$	$\mathbb{Z}_{((n-1)(n^2-4n+3)-(n-2))}$
6*	$z_0^{n-2}z_1z_2 + z_0z_1^{n-2}z_3 + z_2^n + z_2z_3^{n-1}$	$\mathbb{Z}_{((n-1)(n^2-4n+3))}$
7*	$z_0^{n-2}z_1z_3 + z_1^{n-2}z_2z_3 + z_2^n + z_0z_3^{n-1}$	$\mathbb{Z}_{((n-1)(n-2)^2-(n-3))}$
8*	$z_0^{n-2}z_1z_3 + z_1^{n-2}z_2z_3 + z_2^n + z_1z_3^{n-1}$	$\mathbb{Z}_{((n-2)(n^2-3n+1))}$
9*	$z_0^{n-2}z_1z_3 + z_1^{n-2}z_2z_3 + z_2^n + z_2z_3^{n-1}$	$\mathbb{Z}_{((n-1)(n-2)^2)}$
10*	$z_0^{n-2}z_1z_3 + z_0z_1^{n-2}z_3 + z_2^n + z_0z_3^{n-1}$	$\mathbb{Z}_{((n-1)(n^2-4n+3)-(n-3))}$
11	$z_0^{n-2}z_1z_3 + z_0z_1^{n-2}z_3 + z_2^n + z_2z_3^{n-1}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_{((n-1)(n^2-4n+3)/2)}$
12*	$z_0^{n-1}z_1 + z_1^{n-1}z_2 + z_2^n + z_0z_1z_3^{n-2}$	$\mathbb{Z}_{((n-1)^2(n-2))}$
13*	$z_0^{n-1}z_1 + z_1^{n-1}z_2 + z_2^n + z_0z_2z_3^{n-2}$	$\mathbb{Z}_{((n-1)^2(n-2))}$
14*	$z_0^{n-1}z_1 + z_1^{n-1}z_2 + z_2^n + z_1z_2z_3^{n-2}$	$\mathbb{Z}_{((n-1)^2(n-2))}$
15*	$z_0^{n-1}z_1 + z_1^{n-1}z_3 + z_2^n + z_0z_1z_3^{n-2}$	$\mathbb{Z}_{((n-1)(n^2-3n+1)+1)}$
16*	$z_0^{n-1}z_1 + z_1^{n-1}z_3 + z_2^n + z_0z_2z_3^{n-2}$	$\mathbb{Z}_{((n-1)^2(n-2)+1)}$
17*	$z_0^{n-1}z_1 + z_1^{n-1}z_3 + z_2^n + z_1z_2z_3^{n-2}$	$\mathbb{Z}_{((n-1)(n^2-3n+1))}$
18	$z_0^{n-1}z_1 + z_0z_1^{n-1} + z_2^n + z_0z_1z_3^{n-2}$	$\mathbb{Z}_{(n-2)} \oplus \mathbb{Z}_{(n(n-2))}$
19*	$z_0^{n-1}z_1 + z_0z_1^{n-1} + z_2^n + z_0z_2z_3^{n-2}$	$\mathbb{Z}_{((n(n-2)^2)}$
20	$z_0^{n-1}z_2 + z_1^{n-1}z_2 + z_2^n + z_0z_1z_3^{n-2}$	$\mathbb{Z}_{(n-1)} \oplus \mathbb{Z}_{((n-1)(n-2))}$
21*	$z_0^{n-1}z_2 + z_1^{n-1}z_3 + z_2^n + z_0z_1z_3^{n-2}$	$\mathbb{Z}_{((n-1)(n^2-3n+1))}$
22*	$z_0^{n-1}z_2 + z_1^{n-1}z_3 + z_2^n + z_0z_2z_3^{n-2}$	$\mathbb{Z}_{((n-1)^2(n-2))}$
23*	$z_0^{n-1}z_2 + z_1^{n-1}z_3 + z_2^n + z_1z_2z_3^{n-2}$	$\mathbb{Z}_{((n-1)(n^2-3n+1))}$
24*	$z_0^{n-1}z_3 + z_1^{n-1}z_3 + z_2^n + z_0z_1z_3^{n-2}$	$\mathbb{Z}_{(n(n-1)(n-3))}$
25*	$z_0^{n-1}z_3 + z_1^{n-1}z_3 + z_2^n + z_0z_2z_3^{n-2}$	$\mathbb{Z}_{((n-1)(n^2-3n+1))}$
26*	$z_0^n + z_1^n + z_2^{n-1}z_3 + z_0z_1z_3^{n-2}$	$\mathbb{Z}_{(n(n-1)(n-2))}$
27*	$z_0^n + z_1^n + z_2^{n-1}z_3 + z_0z_2z_3^{n-2}$	$\mathbb{Z}_{(n(n^2-3n+1))}$
28*	$z_0^n + z_1^n + z_0z_2^{n-1} + z_0z_1z_3^{n-2}$	$\mathbb{Z}_{(n(n-1)(n-2))}$
29*	$z_0^n + z_1^n + z_0z_2^{n-1} + z_0z_2z_3^{n-2}$	$\mathbb{Z}_{(n(n-1)(n-2))}$
30*	$z_0^n + z_1^n + z_0z_2^{n-1} + z_1z_2z_3^{n-2}$	$\mathbb{Z}_{(n(n-1)(n-2))}$
31*	$z_0^{n-1}z_1 + z_1^{n-1}z_2 + z_0z_2^{n-2}z_3 + z_0z_1z_3^{n-2}$	$\mathbb{Z}_{((n-2)(n^2-4n+5))}$
32*	$z_0^{n-1}z_1 + z_1^{n-1}z_2 + z_0z_2^{n-2}z_3 + z_0z_2z_3^{n-2}$	$\mathbb{Z}_{((n-2)^3-1)}$
33*	$z_0^{n-1}z_1 + z_1^{n-1}z_2 + z_0z_2^{n-2}z_3 + z_1z_2z_3^{n-2}$	$\mathbb{Z}_{((n-2)^3)}$
34*	$z_0^{n-1}z_1 + z_1^{n-1}z_2 + z_1z_2^{n-2}z_3 + z_0z_2z_3^{n-2}$	$\mathbb{Z}_{((n-2)^3-(n-1))}$
35*	$z_0^{n-1}z_1 + z_1^{n-1}z_2 + z_0z_1z_2^{n-2} + z_0z_2z_3^{n-2}$	$\mathbb{Z}_{((n-2)^3)}$
36*	$z_0^{n-1}z_1 + z_0z_1^{n-1} + z_0z_2^{n-2}z_3 + z_1z_2z_3^{n-2}$	$\mathbb{Z}_{((n-1)(n-2)(n-3))}$
37*	$z_0^{n-1}z_2 + z_1^{n-1}z_2 + z_0z_2^{n-2}z_3 + z_0z_1z_3^{n-2}$	$\mathbb{Z}_{((n-1)(n^2-5n+7))}$

38*	$z_0^{n-1}z_2 + z_1^{n-1}z_3 + z_0z_2^{n-2}z_3 + z_0z_1z_3^{n-2}$	$\mathbb{Z}_{((n-2)(n^2-4n+3)+1)}$
39*	$z_0^{n-1}z_2 + z_1^{n-1}z_3 + z_0z_2^{n-2}z_3 + z_0z_2z_3^{n-2}$	$\mathbb{Z}_{((n-2)(n^2-4n+3))}$
40*	$z_0^{n-1}z_2 + z_1^{n-1}z_3 + z_0z_2^{n-2}z_3 + z_1z_2z_3^{n-2}$	$\mathbb{Z}_{((n-2)(n^2-4n+2))}$
41*	$z_0^{n-1}z_2 + z_1^{n-1}z_3 + z_1z_2^{n-2}z_3 + z_0z_1z_3^{n-2}$	$\mathbb{Z}_{((n-2)^3)}$
42	$z_0^{n-1}z_2 + z_1^{n-1}z_3 + z_1z_2^{n-2}z_3 + z_0z_2z_3^{n-2}$	$\mathbb{Z}_{(n-2)} \oplus \mathbb{Z}_{((n-2)^2)}$
43*	$z_0^n + z_1^n + z_0z_2^{n-2}z_3 + z_0z_1z_3^{n-2}$	$\mathbb{Z}_{(n(n-2)^2)}$
44*	$z_0^n + z_1^n + z_0z_2^{n-2}z_3 + z_0z_2z_3^{n-2}$	$\mathbb{Z}_{(n(n^2-4n+3))}$
45*	$z_0^n + z_1^n + z_0z_2^{n-2}z_3 + z_1z_2z_3^{n-2}$	$\mathbb{Z}_{(n(n^2-4n+3))}$
46*	$z_0^n + z_1^n + z_2^{n-1}z_3 + z_0z_3^{n-1}$	$\mathbb{Z}_{(n(n-1)^2)}$
47	$z_0^n + z_1^n + z_2^{n-1}z_3 + z_2z_3^{n-1}$	$\mathbb{Z}_n \oplus \mathbb{Z}_{(n(n-2))}$
48	$z_0^n + z_1^n + z_0z_2^{n-1} + z_0z_3^{n-1}$	$\mathbb{Z}_{(n-1)} \oplus \mathbb{Z}_{(n(n-1))}$
49	$z_0^n + z_1^n + z_0z_2^{n-1} + z_1z_3^{n-1}$	$\mathbb{Z}_{(n-1)} \oplus \mathbb{Z}_{(n(n-1))}$
50*	$z_0^{n-2}z_1z_2 + z_1^{n-2}z_2z_3 + z_0z_2^{n-2}z_3 + z_0z_3^{n-1}$	$\mathbb{Z}_{((n-2)(n^2-5n+7))}$
51*	$z_0^{n-2}z_1z_2 + z_1^{n-2}z_2z_3 + z_0z_2^{n-2}z_3 + z_1z_3^{n-1}$	$\mathbb{Z}_{((n-2)^3-(n-2)^2+1))}$
52*	$z_0^{n-2}z_1z_2 + z_1^{n-2}z_2z_3 + z_1z_2^{n-2}z_3 + z_0z_3^{n-1}$	$\mathbb{Z}_{((n-2)^3-(n^2-5n+7))}$
53*	$z_0^{n-2}z_1z_2 + z_1^{n-2}z_2z_3 + z_0z_1z_2^{n-2} + z_0z_3^{n-1}$	$\mathbb{Z}_{((n-2)^2(n-3))}$
54*	$z_0^{n-2}z_1z_2 + z_1^{n-2}z_2z_3 + z_0z_2^{n-2}z_3 + z_3^n$	$\mathbb{Z}_{((n-2)^3-(n-3))}$
55*	$z_0^{n-2}z_1z_2 + z_1^{n-2}z_2z_3 + z_1z_2^{n-2}z_3 + z_3^n$	$\mathbb{Z}_{((n-2)(n^2-4n+3))}$
56*	$z_0^{n-2}z_1z_2 + z_1^{n-2}z_2z_3 + z_0z_1z_2^{n-2} + z_3^n$	$\mathbb{Z}_{((n-2)^3-2n+5))}$
57	$z_0^{n-2}z_1z_2 + z_0z_1^{n-2}z_2 + z_0z_1z_2^{n-2} + z_3^n$	$\mathbb{Z}_{(n-3)} \oplus \mathbb{Z}_{(n(n-3))}$
58*	$z_0^{n-1}z_1 + z_1^{n-1}z_2 + z_2^{n-1}z_3 + z_0z_1z_3^{n-2}$	$\mathbb{Z}_{((n-2)(n^2-3n+3))}$
59*	$z_0^{n-1}z_1 + z_1^{n-1}z_2 + z_2^{n-1}z_3 + z_0z_2z_3^{n-2}$	$\mathbb{Z}_{((n-2)^3+(n^2-4n+3))}$
60*	$z_0^{n-1}z_1 + z_1^{n-1}z_2 + z_2^{n-1}z_3 + z_1z_2z_3^{n-2}$	$\mathbb{Z}_{((n-1)(n-2)^2)}$
61*	$z_0^{n-1}z_1 + z_1^{n-1}z_2 + z_0z_2^{n-1} + z_1z_2z_3^{n-2}$	$\mathbb{Z}_{((n-2)(n^2-3n+3))}$
62*	$z_0^{n-1}z_1 + z_1^{n-1}z_2 + z_1z_2^{n-1} + z_0z_2z_3^{n-2}$	$\mathbb{Z}_{((n-1)(n-2)^2)}$
63*	$z_0^{n-1}z_1 + z_1^{n-1}z_3 + z_2^{n-1}z_3 + z_0z_1z_3^{n-2}$	$\mathbb{Z}_{((n-1)(n-2)^2)}$
64*	$z_0^{n-1}z_1 + z_1^{n-1}z_3 + z_2^{n-1}z_3 + z_0z_2z_3^{n-2}$	$\mathbb{Z}_{((n-1)(n-2)^2)}$
65*	$z_0^{n-1}z_1 + z_1^{n-1}z_3 + z_2^{n-1}z_3 + z_1z_2z_3^{n-2}$	$\mathbb{Z}_{((n-1)^2(n-3))}$
66*	$z_0^{n-1}z_1 + z_1^{n-1}z_3 + z_1z_2^{n-1} + z_0z_2z_3^{n-2}$	$\mathbb{Z}_{((n-1)(n^2-4n+5))}$
67	$z_0^{n-1}z_1 + z_0z_1^{n-1} + z_2^{n-1}z_3 + z_0z_1z_3^{n-2}$	$\mathbb{Z}_{(n-2)} \oplus \mathbb{Z}_{((n-1)(n-2))}$
68*	$z_0^{n-1}z_1 + z_0z_1^{n-1} + z_2^{n-1}z_3 + z_0z_2z_3^{n-2}$	$\mathbb{Z}_{((n-2)(n^2-3n+1))}$
69*	$z_0^{n-1}z_1 + z_1^{n-1}z_2 + z_2^{n-1}z_3 + z_3^n$	$\mathbb{Z}_{((n-1)^3)}$
70*	$z_0^{n-1}z_1 + z_1^{n-1}z_2 + z_0z_2^{n-1} + z_3^n$	$\mathbb{Z}_{(n(n^2-3n+3))}$
71*	$z_0^{n-1}z_1 + z_1^{n-1}z_2 + z_1z_2^{n-1} + z_3^n$	$\mathbb{Z}_{(n(n-1)(n-2))}$
72	$z_0^{n-1}z_1 + z_1^{n-1}z_3 + z_2^{n-1}z_3 + z_3^n$	$\mathbb{Z}_{(n-1)} \oplus \mathbb{Z}_{((n-1)^2)}$
73	$z_0^{n-1}z_1 + z_1^{n-1}z_3 + z_1z_2^{n-1} + z_3^n$	$\mathbb{Z}_{(n-1)} \oplus \mathbb{Z}_{((n-1)^2)}$
74*	$z_0^{n-1}z_1 + z_0z_1^{n-1} + z_2^{n-1}z_3 + z_3^n$	$\mathbb{Z}_{(n(n-1)(n-2))}$

75	$z_0^n + z_1^n + z_2^n + z_0 z_1 z_3^{n-2}$	$\mathbb{Z}_n \oplus \mathbb{Z}_{(n(n-2))}$
76	$z_0^n + z_1^n + z_2^n + z_0 z_3^{n-1}$	$\mathbb{Z}_n \oplus \mathbb{Z}_{(n(n-1))}$
77*	$z_0^{n-2} z_1 z_2 + z_1^{n-2} z_2 z_3 + z_0 z_2^{n-2} z_3 + z_0 z_1 z_3^{n-2}$	$\mathbb{Z}_{((n-3)^3+(n^2-5n+7))}$
78*	$z_0^{n-1} z_1 + z_1^{n-1} z_2 + z_2^{n-1} z_3 + z_0 z_3^{n-1}$	$\mathbb{Z}_{((n^3-4n^2+6n-4))}$
79*	$z_0^{n-1} z_1 + z_1^{n-1} z_2 + z_2^{n-1} z_3 + z_1 z_3^{n-1}$	$\mathbb{Z}_{((n-1)(n^2-3n+3))}$
80*	$z_0^{n-1} z_1 + z_1^{n-1} z_2 + z_2^{n-1} z_3 + z_2 z_3^{n-1}$	$\mathbb{Z}_{((n-1)^2(n-2))}$
81	$z_0^{n-1} z_1 + z_1^{n-1} z_2 + z_1 z_2^{n-1} + z_2 z_3^{n-1}$	$\mathbb{Z}_{(n-1)} \oplus \mathbb{Z}_{((n-1)(n-2))}$
82	$z_0^{n-1} z_1 + z_0 z_1^{n-1} + z_2^{n-1} z_3 + z_2 z_3^{n-1}$	$\mathbb{Z}_{(n-2)} \oplus \mathbb{Z}_{(n(n-2))}$
83*	$z_0^n + z_1^n + z_2^n + z_3^n$	$\mathbb{Z}_n \oplus \mathbb{Z}_n \oplus \mathbb{Z}_n$