ROBUST DECENTRALIZED INVESTMENT GAMES

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By

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Robust Decentralized Investment Games
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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

ROBUST DECENTRALIZED INVESTMENT GAMES

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In the first part of the thesis, assuming a one-period economy with an investor and two portfolio managers who are experts in investing each in a risky asset (or an index) with first and second moment information available to all parties, we consider the problem of the principal in distributing her wealth optimally among the two managers as well as setting optimally the fees to the portfolio managers under the condition that the principal wants to safeguard against uncertainty in the expert forecasts of the managers regarding the mean return of assets. In the second part, simple games are devised to ensure a fair allocation of contracts between the two managers under the conditions assumed in the first part. Furthermore, the game concept is extended in which three or more managers are involved.

Keywords: Robust optimization, decentralized investment, game theory.

ÖZET

MERKEZİ OLMAYAN GÜRBÜZ YATIRIM OYUNLARI

Burak Çelik
Endstri Mühendisliği, Yüksek Lisans
Tez Danışmanı: Mustafa Ç. Pınar
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Tezin ilk bölümünde tek periyotlu bir ekonomi olduğunu, ilk ve ikinci moment bilgilerinin herkesçe bilindiğini ve tek bir yatırımcı ile riskli varlıklara yatırım yapmada uzman iki portföy yöneticisi olduğunu varsaydık. Yatırımcının bu iki portföy yöneticisinin ortalama varlık kazançları üzerindeki bilirkişi tahminlerini göze alarak kendisini piyasadaki belirsizliklere karşı güvence altına almak istediği, bunun sonucu olarak da sermayesini bu iki portföy yöneticisine en iyi nasıl dağıtabileceği ve sonrasında bunların sözleşme ücretlerini en iyi nasıl ayarlayabileceği problemini değerlendirdik. Tezin ikinci bölümde ise ilk bölümdeki varsayımları kabul edip ortaya çıkan sonuçları değerlendirdikten sonra sözleşme ücretlerinin daha adil bir şekilde belirlenmesini sağlayacak basit bir oyun kurguladık. Daha sonra, bu oyun konseptini üç veya daha fazla portföy yöneticisinin de olabileceği yeni bir varyasyonunu geliştirdik.

Anahtar sözcükler: Gürbüz optimizasyon, merkezi olmayan yatırım, oyun teorisi.

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Chapter 1

Introduction

The first mathematical model for portfolio selection was introduced by Markowitz [1]. In his model, the return of a portfolio is measured by the overall expected return and the risk of a portfolio is estimated by the overall variance of assets in that portfolio. Although it is simple, this model provides investors of all types with enough flexibility and it is a powerful tool to construct their own strategies.

Later, as the savings of individuals and institutions increased, the need of an intermediary party arouse for managing these portfolios in place of asset holders. Therefore, delegated portfolio management concept was introduced, and numerous studies and implications were developed in the literature. The seminal study on this subject is by Bhattacharya and Pfleiderer [2]. We will give a detailed review in the next chapter.

The use of robust optimization techniques in portfolio selection is rather new and it was first introduced by El Ghaoui and Lebret [3], and by Ben-Tal and Nemirovski [4]. We use their combined work [5] to help our calculations for finding optimal solutions to our model. Extended from Bhattacharya and Pfleiderer [2] and developed by Fabretti and Herzel [6], we continue their work by using robust optimization techniques in portfolio optimization.

Apart from portfolio selection and robust optimization, we also use the concepts of fair allocation and envy-free division. This topic is vast and it can be as simple as cake cutting (i.e. the procedure of "I cut, you choose") and as complicated as political turmoil (i.e. determining borders in a territorial dispute). We also use very basic tools of game theory since the second part of this thesis includes simple games between agents who are involved in the mathematical model.

The organization of remainder chapters is as follows. We provide literature review for all main aspects in *Chapter 2*. These are delegated portfolio management, robust optimization and fair allocation. In *Chapter 3*, we introduce our mathematical model on robust decentralized portfolio management. In the model, we consider a one-period economy, a single risk neutral investor who has a capital W_0 and needs to allocate it between two managers having exponential utility parameter β_i, in which the coefficient of risk aversion is known to all parties. Two cases are considered. In the first one, each manager invests into one asset. The aim of the principal is to maximize her utility by allocating her wealth between managers. She also sets premia for the managers and considers the worst case scenario for the rate of return from risky assets and the managers' expertise. In the second one, we investigate the case in which both managers invest into both assets. This time, the principal wants to guard herself against the forecast error of the managers. We also provide some numerical examples by manipulating the model's parameter. In *Chapter 4*, inspiring from the results of our model in *Chapter 3*, we devise simple games that includes managers involved in wealth allocation to ensure that contracts rewarded to managers will be envy-free. We divide this chapter into three cases, and in each one we use the direct results of our model in *Chapter 3* for calculating managers' contract values. In case I, managers announce their desired contract levels after we calculate their optimal contract values. In the second one, this time they announce it after they know their calculated contract values. Later, we extend this idea when there are more than two managers involved in the wealth allocation process. We also provide numerical examples for this case. Finally, we end the thesis with conclusions in *Chapter 5*.

Chapter 2

Literature Review

The modern portfolio management techniques were established by Markowitz [1]. He introduced mean-variance analysis and optimization as a basis on this topic. Since then, his ideas were adopted by many colleagues, and there have been tremendous contributions arising from it. The idea behind mean-variance analysis is very simple yet quite efficient. It analyses both the expected performance (mean) and the risk factor (variance) of the investment, and the investor tries to find the best parameters according to her investment profile.

Another aspect that the mean-variance framework contributed is the concept of diversification. Since allocating risk factors into different elements of the market was the common motive, Markowitz developed this concept of diversification as the notion of covariance between market assets and the combined differences within a specific portfolio. This can be explained as a simple strategy that you should not invest all your wealth in similar asset groups. If one of the assets performs badly, there is a considerable chance that the remaining asset group will also perform poorly due to the correlation between them.

As a sub-implementation of portfolio management, delegated portfolio management is probably one of the most important aspects on financial markets. A big portion of investments is not directly controlled by asset holders, but rather by intermediary agents. These agents (or portfolio managers) are delegated and contracted by the investor (or the principal), hence they manage principal's wealth and therefore receive payment for their efforts.

The seminal study of delegated portfolio management was done by Bhattacharya and Pfleiderer [2]. They developed a model to give a reasoning why an experienced manager should be adequately contracted so that he would be interested in performing his best based on the information he has by monitoring the rate of return on a risky financial asset. In their model, agents receive signals both at the precontracting stage and after the contracting is concluded. Bhattacharya and Pfleiderer found that by penalizing the deviations between these two, managers are required to reveal their truthful information at the precontracting stage. Their premise and results were very interesting, and for this reason there have been tremendous contributions on the literature stemming from this study.

Later, Stoughton [7] found out that as opposed to Bhattacharya and Pfleiderer, linear contracting may not be optimal for the manager. Therefore, he considered the implementation of nonlinear contracting for better motivation for portfolio managers to reveal information. He overcame this problem by using quadratic contracts.

Further review on this topic can also be found in Stracca [8].

A more recent study by Fabretti and Herzel [6] considered a restricted case of the problem. They investigated how to determine manager's compensation when setting their contracts if they are restricted to limited amount of investment set such as green assets. When manager's investment set is restricted, this leads to a loss in expected earnings, and therefore the principal must give compensation to the manager based on the realized return in order to attract his interest. Fabretti and Herzel calculated the optimal bonus by considering risk aversion and expertise of the manager as well as the restriction on the portfolio. Under the assumption that managers have the same skill, they showed that as the manager's expertise in the green market increases, his incentive to accept a contract with a smaller bonus also increases compared to a manager with less expertise in the that area. We use the same problem setup to find the optimum wealth allocation and contracts given to managers. Our findings, in a away, will also be similar. As the manager's expertise on the portfolio assets increases, he will be less risk averse to obtain better expected returns. We will also find that a manager who offers to the principal a more risky portfolio is rewarded by a more lucrative contract, which is an obvious result. Expanding this to results from Fabretti and Herzel [6], expertise in the green set provides manager better earnings which leads to less compensation bonus at the end.

In this thesis, we use robust optimization methods to find the optimum values for our delegation problem. Robust optimization in financial applications is rather new to the literature, and its main aim is to deal with the uncertain data that come from market's unpredictability. We are also using ellipsoidal uncertainty sets for our problem setup. This topic was thoroughly investigated in recent decade and we will present the most relative ones in this section of literature review.

The first modern robust optimization technique was implemented by El Ghaoui and Lebret [3]. They minimized the worst-case residual error by using convex programming. Later, Ben-Tal and Nemirovski [9] showed that the robust counterpart of Linear Programming with ellipsoidal uncertainty set can be solved in polynomial time. We mainly use the combined work of Ben-Tal, El Ghaoui and Nemirovski [5] for calculating optimal solutions for our robust model.

Costa and Paiva [10] considered the problem of robust optimal portfolio selection given the mean return of both risk-free and risky assets, and the covariance matrix of risky assets that constitutes a convex polytope. They showed that the problems when finding a portfolio that has minimum worst case volatility of tracking error with guaranteed both fixed minimum target-expected performance and fixed maximum volatility are computationally equivalent to solving Linear Mixed Integer optimization problems.

Goldfarb and Iyengar [11] introduced "uncertainty structures" for the market parameters. In their model, the mean return vector μ, the factor loading matrix V, the covariance matrices of the factor return vector f, and the residual error vector ϵ can be represented by well defined uncertainty sets. They also showed that the robust portfolio problems derived from these uncertainty structures can benefit from second-order cone programs, in which the computational requirements are similar to that from convex quadratic programs.

In this area, one important study was presented by Bertsimas and Sim [12]. They dealt with complex and practically efficient methods in discrete robust optimization under ellipsoidal uncertainty sets. They first showed that robust counterpart of the linear program can be NP-hard even if the corresponding problem is polynomially solvable. Then, with distinct linear objectives, they showed that the robust problem can be solved as a collection of nominal problems. Finally, they proposed a generalization of the robust optimization which allowed increased flexibility and less conservatism, but keeping the complexity of the nominal problem at the same time.

Tütüncü and Koenig [13] investigated the problem of finding an optimal of funds over different asset groups when expected returns are uncertain. Regarding the previous approaches, their novelty was to treat described input estimates as the form of uncertainty sets. This approach reveals assets that have the best worst-case behavior by preserving conservatism. They proposed an algorithm that provides robust portfolios by using an interior-point method for saddle point problems, and then discussed its implementations.

There were also more detailed implementations of these techniques in recent years. One of them was a joint work of Zhu and Fukushima [14]. They considered the worst-case Conditional Value-at-Risk when only partial information on the underlying probability distribution is available. They investigated the minimization of the worst-case CVaR under ellipsoidal uncertainty and applied it to robust portfolio optimization.

The second part of the thesis will revolve around how the principal can provide managers with fair contracts so that they will not have envy for other managers' contracts. As our results will show, optimal allocation does not necessarily result in fair allocation. Concepts such as fairness, envy-free allocation, and bargaining are all parts of our lives. There are numerous applications of these concepts both theoretically and practically. We mainly used three studies to better understand the concept and to find the best possible way to solve our problem. We researched for both effective and practical ways of dealing with the issue at hand.

The first study we looked at is Brams and Taylor [15]. In their book, fair allocation is the main subject. From cutting a cake to determining the borders in a major international disagreement, they analyzed lots of real life situations. They analyzed simple solutions such as "I cut, you choose" from cutting a cake problem to a complex real life applications such as allocation of properties in an estate to the owners. Starting with the simplest form of dividing between $n = 2$ agents, they extended ideas to further $n > 2$. Although we will not specifically use any of these procedures from this book, it inspired us to come up with new ideas while analyzing our own problem.

In the fair allocation problems, preferences are private information, and in our problem, managers' opinions about their desired contracts (in monetary values) are also private information. As the authors suggest, people generally are not willing to reveal their preferences unless they are benefiting from it. In our case by inspiring from this, we tried to design a procedure that managers will be willing to announce this specific information since it provides them enough benefit at the end. Brams and Taylor also gave procedures when $n = 3$ and $n = 4$ for envy-free division and later extended this to arbitrary n. The main focus was cake cutting in their studies but it was enough for us to extend our idea further when three or more managers are involved in determining optimal and envy-free contracts.

Korth [16] made a good analysis on bargaining theory, game theory and fairness in his book. He emphasizes the importance of economic bargaining theory by reasoning that most of the human interactions are bargaining situations. In order to better understand and formalize these situations, one can utilize the tools of mathematical theory of games. His contribution to the literature is mainly in form of behavioral economics and he emphasizes how important the mechanism of a game is and how it affects the behavior of the players. As we can see, both Brams and Taylor, and Korth consider this scheme as a core of a game to ensure fairness of it. Knowing this significant factor, we will always consider the behavioral results of our proposed games in this thesis. In every step of the proposed game, we will check whether specific mechanism leads to desired envy-free results or not. Another factor that Korth highlights is that all parties involved in a game should be able to affect the results so that fairness of the game will be preserved. Our proposed game will consider this aspect in a way that managers are also a part of determining their contracts not only indirectly (by setting asset weights in their portfolio) but also directly when they are making declarations as a phase of the game.

For more examples and applications of fair allocation on different areas, we recommend Young [17].

Chapter 3

Robust Decentralized Investment

Let us consider the problem of a Centralized Investor (e.g. the Head of a Pension Fund) who has to allocate a capital W_0 between two managers. The managers have an exponential utility with parameter β_i, and are rewarded with a contract based on the wealth obtained from their trading activity. The managers' knowledge of the market is modeled by a private signal.

In the basic problem, the market consists of two risky assets with rate of return X that is bi-variate normal $N(\bar{X}, \Sigma)$ and one riskless asset with rate of return R. Let us denote the variance of the return of asset i by σ_i^2.

The principal allocates a portion $0 < \alpha < 1$ of her wealth to the first manager and $1 - \alpha$ to the second manager. In case 1, we investigate that each manager invest in only one of the two risky assets, that is manager 1 is restricted to invest in asset 1 and manager 2 in asset 2. In case 2, both managers invest in both risky assets.

Manager i receives a private signal

$$S_i = X_i + \epsilon^i,$$

where ϵ^i is normally distributed with zero mean and standard deviation $\sigma_{\epsilon,i}$. Since

ϵ^i is the noise of the signal, $\sigma_{\epsilon,i}$ represents the manager's expertise.

3.1 Case I: Each manager invests into one asset

Let $\omega_i(S_i)$ be the allocation of manager i, that produces a return W_i. Manager i receives the compensation $AR + b_i W_i$, $i = 1, 2$, where the parameter A is a fixed amount received at the beginning of the period. Hence manager i solves the problem:

$$\max_{\omega_i} -E[\exp(-\beta_i b_i W_i)|S].$$

The principal wants to maximize her utility by allocating α and setting the premia b_1 and b_2, but also considering the worst case scenario for S_i:

$$\max_{\alpha, b_1, b_2} \min_{S \in U_S} E[W_1 + W_2 - (b_1 W_1 + b_2 W_2)] - 2AR$$

subject to

$$b_1 \alpha \geq d_1$$

$$b_2(1 - \alpha) \geq d_2$$

where d_1 and d_2 are suitable constants obtained by evaluating the reservation utilities of the two respective managers, and W_1 and W_2 represent the "optimal" return produced by the two respective managers after observing the signal in S_1 and S_2.

Solving the problem of the manager i by simple algebra, we obtain the following optimal investment rule into asset i:

$$\omega_i^* = \frac{1}{\beta_i b_i} \left[\frac{\sigma_{\epsilon,i}^2 (\bar{X}_i - R) + \sigma_2^2 (S_i - R)}{\sigma_i^2 \sigma_{\epsilon,i}^2} \right]. \tag{3.1}$$

Substituting the above expressions into the objective function of the principal we obtain the following expression for the principal's expected return

$$\begin{aligned}\phi(S_1,S_2,\alpha,b_1,b_2) &= (1-b_1)\left[\alpha W_0R+\frac{1}{\beta_1 b_1}\left(\frac{\sigma^2_{\epsilon,1}(\bar{X}_1-R)+\sigma_1^2(\bar{X}_1-R)(S_1-R)}{\sigma_1^2\sigma^2_{\epsilon,1}}\right)\right]\\ &\quad +(1-b_2)\left[(1-\alpha)W_0R+\frac{1}{\beta_2 b_2}\left(\frac{\sigma^2_{\epsilon,2}(\bar{X}_2-R)+\sigma_2^2(\bar{X}_2-R)(S_2-R)}{\sigma_2^2\sigma^2_{\epsilon,2}}\right)\right].\end{aligned}$$

Now, the principal accepts an ellipsoidal uncertainty set $\mathcal{U}_S$ of the type

$$\mathcal{U}_S=\{S|S=\bar{S}+\Xi^{1/2}u:\|u\|_2\le\varepsilon\}$$

for some 2×2 positive-definite symmetric matrix Ξ, and positive constant ε, and is interested in solving the problem

$$\max_{\alpha,b_1,b_2}\min_{S\in U_S}\phi(S_1,S_2,\alpha,b_1,b_2)-2AR$$

subject to

$$b_1\alpha\ge d_1$$

$$b_2(1-\alpha)\ge d_2.$$

Evaluating the inner minimization over S_1,S_2 we obtain the objective function

$$\begin{aligned}\psi(\alpha,b_1,b_2) &= (1-b_1)\left[\alpha W_0R+\frac{1}{\beta_1 b_1}\left(\frac{(\bar{X}_1-R)^2}{\sigma_1^2}+\frac{(\bar{X}_1-R)(\bar{S}_1-R)}{\sigma^2_{\epsilon,1}}\right)\right]\\ &\quad +(1-b_2)\left[(1-\alpha)W_0R+\frac{1}{\beta_2 b_2}\left(\frac{(\bar{X}_2-R)^2}{\sigma_2^2}+\frac{(\bar{X}_2-R)(\bar{S}_2-R)}{\sigma^2_{\epsilon,2}}\right)\right]\\ &\quad -\varepsilon\left\|\Xi^{1/2}\begin{pmatrix}\frac{(1-b_1)(\bar{X}_1-R)}{\beta_1 b_1\sigma^2_{\epsilon,1}}\\ \frac{(1-b_2)(\bar{X}_2-R)}{\beta_2 b_2\sigma^2_{\epsilon,2}}\end{pmatrix}\right\|_2.\end{aligned}$$

Hence, we have posed the problem (PrP) of the principal who wants to safeguard herself against the forecast errors of the managers by taking a worst-case approach:

$$\max_{\alpha,b_1,b_2}\psi(\alpha,b_1,b_2)-2AR$$

subject to

$$b_1\alpha \geq d_1$$

$$b_2(1-\alpha) \geq d_2,$$

where $0 \leq \alpha \leq 1$, and $0 \leq b_1, b_2 \leq 1$. Let us rewrite the objective function into an easier to read form:

$$\begin{aligned} \psi(\alpha, b_1, b_2) &= W_0R(1-\alpha b_1 + \alpha b_2 - b_2) + \frac{C_1}{b_1} + \frac{C_2}{b_2} \\ &\quad -\varepsilon \left\| \Xi^{1/2} \begin{pmatrix} \frac{(1-b_1)}{b_1} C_3 \\ \frac{(1-b_2)}{b_2} C_4 \end{pmatrix} \right\|_2 - 2AR - C_1 - C_2. \end{aligned}$$

where

$$C_1 = \frac{1}{\beta_1}\frac{(\bar{X}_1 - R)^2}{\sigma_1^2} + \frac{(\bar{X}_1 - R)(\bar{S}_1 - R)}{\sigma_{\epsilon,1}^2},$$

$$C_2 = \frac{1}{\beta_2}\frac{(\bar{X}_2 - R)^2}{\sigma_2^2} + \frac{(\bar{X}_2 - R)(\bar{S}_2 - R)}{\sigma_{\epsilon,2}^2},$$

$$C_3 = \frac{(\bar{X}_1 - R)}{\sigma_{\epsilon,1}^2},$$

and

$$C_4 = \frac{(\bar{X}_2 - R)}{\sigma_{\epsilon,2}^2}.$$

Ignoring momentarily the bounds on the variables and the constant terms, we form the Lagrange function

$$\begin{aligned} L(\alpha, b_1, b_2, \lambda_1, \lambda_2) &= W_0R(1-\alpha b_1 + \alpha b_2 - b_2) + \frac{C_1}{b_1} + \frac{C_2}{b_2} \\ &\quad -\varepsilon \left\| \Xi^{1/2} \begin{pmatrix} \frac{(1-b_1)}{b_1} C_3 \\ \frac{(1-b_2)}{b_2} C_4 \end{pmatrix} \right\|_2 + \lambda_1(\alpha b_1 - d_1) + \lambda_2(b_2(1-\alpha) - d_2) \end{aligned}$$

where λ_1 and λ_2 are non-negative multipliers. Assuming that both constraints are binding at a candidate solution α^*, b_1^*, b_2^* where $0 < \alpha^*, b_1^*, b_2^* < 1$ we obtain the following result.

Proposition 1 *If α^*, b_1^*, b_2^* where $0 < \alpha^*, b_1^*, b_2^* < 1$ solve (PrP) with both reservation utility constraints binding, then there exist non-negative multipliers λ_1^*, λ_2^* such that the following system of equations hold at $(\alpha^*, b_1^*, b_2^*, \lambda_1^*, \lambda_2^*)$:*

$$(\lambda_1 - W_0R)\alpha - \frac{C_1}{d_1^2}\alpha^2 - \frac{2\varepsilon\Xi_{11}C_3^2(d_1\alpha^2 - \alpha^3)}{d_1^3\mathcal{A}} + \frac{2\varepsilon\Xi_{12}C_3C_4(\alpha^2 - \alpha^3 - \frac{d_2\alpha^2}{1-\alpha} + \frac{d_2\alpha^3}{1-\alpha})}{d_1^2d_2\mathcal{A}} = 0, \tag{3.2}$$

$$(W_0R - \lambda_2)\alpha - W_0R - \frac{C_2}{d_2^2}(1-\alpha)^2 - \frac{2\varepsilon\Xi_{12}(\alpha - d_1)C_3C_4(1-\alpha)^2}{d_1d_2^2\mathcal{A}} - \frac{2\varepsilon\Xi_{22}C_4^2(\frac{d_2}{1-\alpha} - 1)}{(\frac{d_2}{1-\alpha})^3\mathcal{A}} = 0 \tag{3.3}$$

$$\lambda_1 = \frac{W_0R(b_1 - b_2) + \lambda_2 b_2}{b_1} \tag{3.4}$$

$$\lambda_2 = \frac{W_0R(b_2 - b_1) - \lambda_1 b_1}{b_2} \tag{3.5}$$

$$\frac{d_1}{b_1} + \frac{d_2}{b_2} = 1, \tag{3.6}$$

where

$$\mathcal{A} = \sqrt{\Xi_{11}C_3^2(\frac{\alpha - d_1}{d_1})^2 + 2\Xi_{12}C_3C_4(\frac{\alpha - d_1}{d_1})(\frac{1-\alpha-d_2}{d_2}) + \Xi_{22}C_4^2(\frac{1-\alpha-d_2}{d_2})^2}.$$

Proof. The proof is obtained by manipulating the first-order conditions, i.e., differentiating the Lagrange function with respect to α, b_1, b_2 and applying the First-Order Necessary Conditions Theorem, e.g., Theorem 9.1.1 of [18] after observing that the gradient vectors of the two utility reservation constraints are always linearly independent at α^*, b_1^*, b_2^* with the assumed properties, and hence the Kuhn-Tucker constraint qualification is satisfied. □

While it does not seem possible to solve the equations in *Proposition 1* in closed form, one can solve the optimization problem (PrP) numerically using *Proposition 1.* We shall base our investigation on the numerical solution of the optimization problem.

In *Figure 1*, we report the results of a simple numerical experiment in MATLAB where we used $\bar{X}_1 = 0.2380$ $\bar{X}_2 = 0.1870$, $\sigma_1^2 = 0.9350$, $\sigma_2^2 = 0.3650$, i.e., the first asset commanded by Manager 1 has a higher expected return with a higher variability. We assumed identical managers with $\beta_1 = \beta_2 = 0.5$, $\sigma^2_{\epsilon,1} = \sigma^2_{\epsilon,2} = 0.01$, $R = 0.064$, $W_0 = 100$, $A = 0.5$, and $d_1 = d_2 = 0.1$. Taking $\bar{S}_1 = 0.24$ and $\bar{S}_2 = 0.19$ and

$$\Xi = \begin{pmatrix} 0.94 & -0.39 \\ -0.39 & 0.36 \end{pmatrix}$$

we plot the optimal solution components α, b_1, b_2 as a function of increasing disbelief by the principal in the signal returns of the managers, i.e., increasing ε. With her increasing disbelief, the principal allocates less and less to the manager commanding the more profitable but riskier asset, but rewards an increasingly lucrative contract to that manager, while an increasingly less attractive contract to the manager commanding the less profitable but less risky asset.

For the experiments of *Figure 2* and *3*, we keep everything constant except the risk aversion levels of the managers. In *Figure 2*, we take the first manager to be more risk averse with $\beta_1 = 2$ while $\beta_2 = 0.5$. With the principal's increasing disbelief, she allocates less and less to the more risk averse manager commanding the more profitable but riskier asset, but rewards a decreasingly less lucrative contract to that manager, while an increasingly but slightly more attractive contract to the less risk averse manager commanding the less profitable but less risky asset.

In *Figure 3*, we take the second manager to be more risk averse with $\beta_1 = 0.5$ while $\beta_2 = 5$. With the principal's increasing disbelief, she allocates less and less to the less risk averse manager commanding the more profitable but riskier asset, but rewards an increasingly more lucrative contract to that manager, while an increasingly less attractive contract to the more risk averse manager commanding the less profitable but less risky asset.

Two important observations can be made from these three figures. The first one is consistent with the literature, that is, as the manager becomes less risk averse, his expected return increases provided that he has the expertise, and consequently he is rewarded with more lucrative contract compared to the more risk averse manager. The second observation is related to α value. If the principal's disbelief is low (e.g. $\varepsilon < 0.1$), she becomes less conservative about her wealth allocation and invests most of her wealth into the manager who commands riskier asset but having a higher expected return. However, as the principal becomes more concerned about the market (e.g. $\varepsilon > 0.3$), she becomes more conservative about the wealth allocation and begins to allocate in a balanced way between the managers (e.g. α converges to near 0.5).

3.2 Case II: Both managers invest into both assets

Now, we distinguish the skills of the two managers by the matrices $\Sigma_{\epsilon,i}$, $i = 1, 2$. Let $S^i \in \mathbb{R}^2$ denote the forecast of the manager $i = 1, 2$. In this case, the optimal allocation by manager i is given by [6]:

$$\omega_i^* = \frac{1}{\beta_i b_i} V_i^{-1}(M_i(S) - R\mathbf{1})$$

where $\mathbf{1}$ is the two-dimensional vector of ones, $M_i(S) = \bar{\mathbf{X}} + \Sigma\Sigma_{S,i}^{-1}(S^i - \bar{\mathbf{X}})$ $\Sigma_{S,i} = \Sigma + \Sigma_{\epsilon,i}$ and $V_i = \Sigma - \Sigma\Sigma_{S,i}^{-1}\Sigma$. The portfolio returns for the two managers are expressed as

$$W_1 = \mathbf{X}^T\omega_1^* + (\alpha W_0 - \mathbf{1}^T\omega_1^*)R,$$

$$W_2 = \mathbf{X}^T\omega_2^* + ((1-\alpha)W_0 - \mathbf{1}^T\omega_2^*)R.$$

The principal who wants to protect herself against the forecast errors of the managers wishes to decide the values of α, b_1 and b_2 that will maximize

$$\min_{S \in \mathcal{U}_S} \mathbb{E}[(1-b_1)W_1 + (1-b_2)W_2 - 2AR],$$

where $S = \begin{pmatrix} S^1 \\ S^2 \end{pmatrix}$ is a four-dimensional vector and $\mathcal{U}_S$ is a suitable uncertainty set, subject to the constraints

$$b_1\alpha \geq d_1$$

$$b_2(1-\alpha) \geq d_2,$$

where $0 \leq \alpha \leq 1$, and $0 \leq b_1, b_2 \leq 1$ as in the previous section. Let

$$\mathcal{U}_S = \{S | S = \bar{S} + P^{1/2}u : \|u\|_2 \leq \varepsilon\}$$

where $\bar{S} = \begin{pmatrix} \bar{S}^1 \\ \bar{S}^2 \end{pmatrix}$ is a four-dimensional vector with the two dimensional block components $\bar{S}^1$ and $\bar{S}^2$, and P is 4×4 symmetric and positive-definite matrix. After some calculations we can pose the problem of the principal as

$$\max_{\alpha, b_1, b_2} \sum_{i=1}^{2} \frac{(1-b_i)}{\beta_i b_i}\kappa_i + (1-b_1)\alpha W_0 R + (1-b_2)(1-\alpha)W_0 R - 2AR - \varepsilon \|P \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}\|_2$$

where γ_1 and γ_2 are two-dimensional vectors given by

$$\gamma_1 = \frac{1-b_1}{\beta_1 b_1} V_1^{-1} \Sigma \Sigma_{S,1}^{-1} (\bar{\mathbf{X}} - R\mathbf{1})$$

$$\gamma_2 = \frac{1-b_2}{\beta_2 b_2} V_2^{-1} \Sigma \Sigma_{S,2}^{-1} (\bar{\mathbf{X}} - R\mathbf{1}),$$

and the constants κ_i, $i = 1, 2$ are expressed as

$$\begin{aligned} \kappa_i \;\; = \;\; & \bar{\mathbf{X}}^T V_i^{-1} \bar{\mathbf{X}} + \bar{\mathbf{X}}^T V_1^{-1} \Sigma \Sigma_{S,1}^{-1} \bar{S}^i - \bar{\mathbf{X}}^T V_1^{-1} \Sigma \Sigma_{S,1}^{-1} \bar{\mathbf{X}} - 2R\mathbf{1}^T V_i^{-1} \bar{\mathbf{X}} - R\mathbf{1}^T V_1^{-1} \Sigma \Sigma_{S,1}^{-1} \bar{S}^i \\ & + R\mathbf{1}^T V_1^{-1} \Sigma \Sigma_{S,1}^{-1} \bar{\mathbf{X}} + R^2 \mathbf{1}^T V_i^{-1} \mathbf{1}, \end{aligned}$$

subject to the restrictions

$$b_1 \alpha \geq d_1$$

$$b_2(1-\alpha) \geq d_2,$$

where $0 \leq \alpha \leq 1$, and $0 \leq b_1, b_2 \leq 1$.

Now, we investigate numerically the previous model. Our base data are as follows. We have $\bar{\mathbf{X}}_1 = 0.2380$, $\bar{\mathbf{X}}_2 = 0.1870$, $\Sigma = \begin{pmatrix} 0.9359 & -0.392 \\ -0.392 & 0.363 \end{pmatrix}$, $\Sigma_{\epsilon,1} = \begin{pmatrix} 0.1 & 0.01 \\ 0.01 & 0.1 \end{pmatrix}$, $\Sigma_{\epsilon,2} = \begin{pmatrix} 0.2 & 0.01 \\ 0.01 & 0.2 \end{pmatrix}$. Due to the larger diagonal entries of $\Sigma_{\epsilon,2}$

the second manager is assumed to be less credible in the quality in his forecasts. We take $\mathbf{S}^1 = \mathbf{S}^2 = (0.24\ 0.20)^T$, and

$$P = \begin{pmatrix} \Sigma_{S,1} & -0.1E \\ -0.1E & \Sigma_{S,2} \end{pmatrix}$$

where E is a 2×2 matrix of ones.

In *Figure 4*, we plot the behavior of optimal values of α, b_1 and b_2 for increasing values of ε when $\beta_1 = \beta_2 = 1$. With the principal's increasing disbelief, she begins to allocate slightly less to the first manager commanding the more profitable but riskier asset but who is more reliable in forecasts, but rewards an increasingly more lucrative contract to that manager, while a less attractive contract to the second manager commanding the less profitable but less risky asset but with a less reliable forecast history.

In *Figure 5*, we plot the behavior of optimal values of α, b_1 and b_2 for increasing values of ε when $\beta_1 = 1$, $\beta_2 = 10$. With the principal's increasing disbelief, she begins to allocate much less to the first manager commanding the more profitable but riskier asset but who is more reliable in forecasts, but rewards slightly more lucrative contract to that manager, while a bigger chunk of the budget but a less attractive contract to the second manager commanding the less profitable but less risky asset.

From these two figures, we can see that they are a bit different than the previous ones in terms of their pattern. First, we do not see any convergence to $\alpha = 0.5$, actually it is quite the opposite. In *Figure 4*, α value is always greater than 0.85, and often close to 1. In *Figure 5*, α value starts close to 1 and stays there for a while then shifts rapidly towards 0 in relatively small ε range. We can argue that in *Figure 5* we choose relatively distinct risk aversion levels, however, even though the managers have equal degree of risk aversion in *Figure 4*, the principal mostly sticks with manager 1 throughout different ε levels in which this contradicts the concept of diversification of the risk. We will investigate this problem in the next chapter and offer an alternative way to resolve the issue and improve the condition of the managers, i.e., more fair contracts rewarded to them.

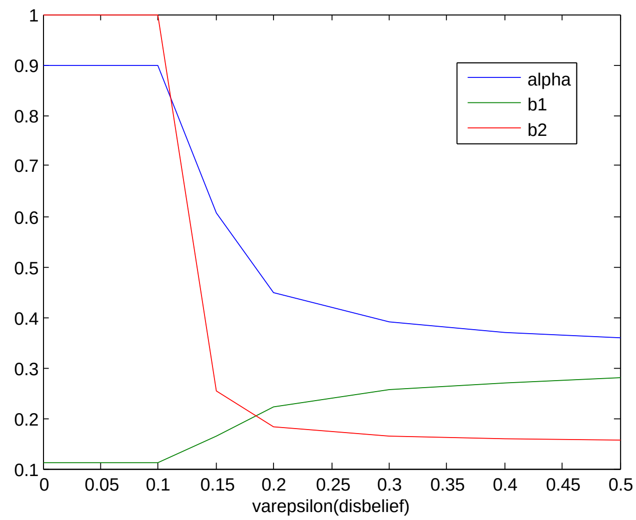

Figure 3.1: Two identical managers
Assuming two identical managers, with increasing disbelief, the principal allocates less and less to the manager commanding the more profitable but riskier asset, but rewards an increasingly lucrative contract to that manager, while an increasingly less attractive contract to the manager commanding the less profitable but less risky asset.

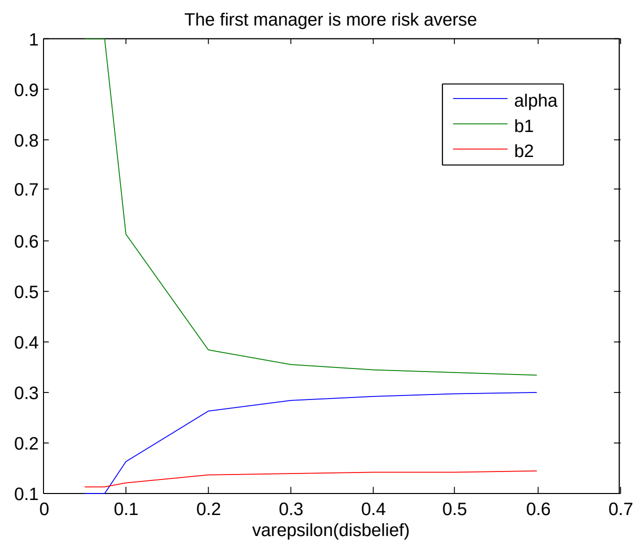

Figure 3.2: The first manager is more risk averse
Assuming the first manager to be more risk averse, with increasing disbelief, the principal allocates less and less to the more risk averse manager commanding the more profitable but riskier asset, but rewards a decreasingly less lucrative contract to that manager, while an increasingly but slightly more attractive contract to the less risk averse manager commanding the less profitable but less risky asset.

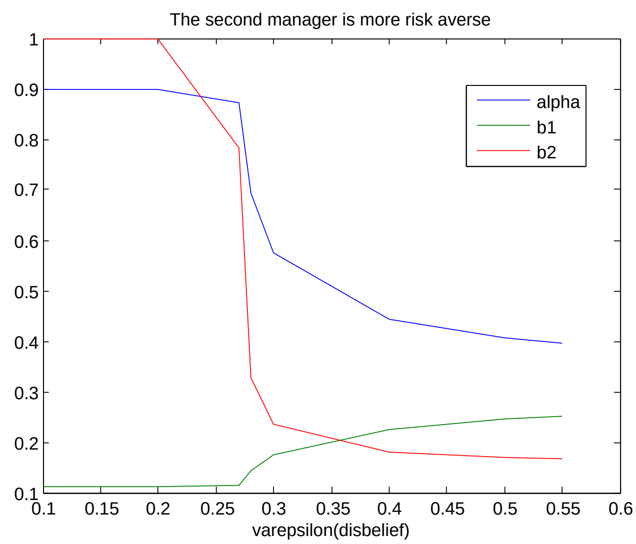

Figure 3.3: The second manager is more risk averse
Assuming the first manager to be less risk averse, with increasing disbelief, the principal allocates less and less to the less risk averse manager commanding the more profitable but riskier asset, but rewards an increasingly more lucrative contract to that manager, while an increasingly less attractive contract to the more risk averse manager commanding the less profitable but less risky asset.

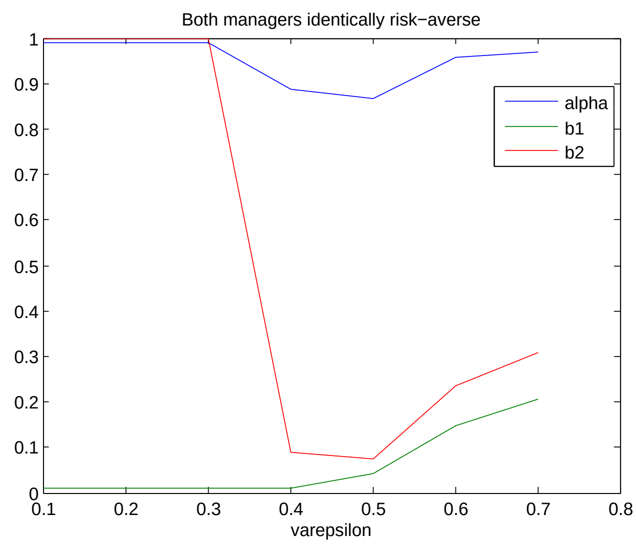

Figure 3.4: Both managers are identically risk averse
Assuming both managers to have equal degree of risk aversion. With increasing disbelief, the principal begins to allocate slightly less to the first manager commanding the more profitable but riskier asset but who is more reliable in forecasts, but rewards an increasingly more lucrative contract to that manager, while a less attractive contract to the second manager commanding the less profitable but less risky asset but with a less reliable forecast history.

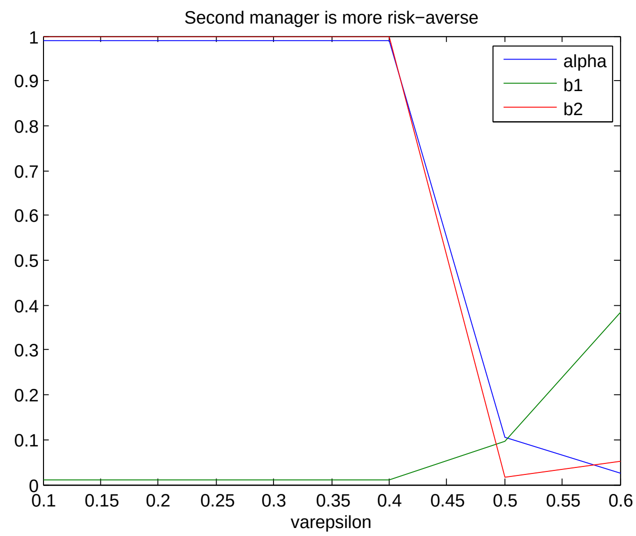

Figure 3.5: The second manager is more risk averse
Assuming the second manager (with less reliable forecasts) to have larger degree of risk aversion ($\beta_2 = 10$, $\beta_1 = 1$). With increasing disbelief, the principal begins to allocate much less to the first manager commanding the more profitable but riskier asset but who is more reliable in forecasts, but rewards slightly more lucrative contract to that manager, while a bigger chunk of the budget but a less attractive contract to the second manager commanding the less profitable but less risky asset.

Chapter 4

Games for the Design of Fair Contracts

In the previous section, our results have revealed that there may occur two major problems regarding results of our mathematical model. One of them is pertaining to allocation of the principal's wealth to the managers in which depending on the parameters one manager may obtain huge portion of the principal's wealth, i.e., close to 1 as in the *Figure 3.4 and 3.5*, and therefore the other manager is left with almost non-existent capital to perform his task. This also contradicts the idea of risk reduction through diversification and negates the reason that the principal needs to have two managers to begin with. The second problem is the unfairness of contracts rewarded to managers again due to the heavy shifting of the results given specific parameters, i.e., from almost all of our figures this pattern can be observed.

These problems arise since the managers' knowledge of the market is a private information and therefore there is no direct way to know their true risk aversion levels. If we do not assume that they will reveal their true risk aversion levels, then one manager can exploit this by being less risk averse (i.e. he increases the weight of riskier asset in his portfolio) and hence obtaining more lucrative contract. The situation gets worse if the other manager also decides to lower his risk aversion level which leads to a race between managers and greatly distorts the outcomes and results in a way that contradicts the principal's aim of adopting a worst-case *max-min* approach. There is still a problem even if we assume that the managers announce their true risk aversion levels, e.g, if one manager is more risk averse and the other manager is less risk averse, then the first manager will be rewarded by a much less attractive contract, on the other hand the second manager will be rewarded by a more lucrative contract that may not reflect their true efforts. This results in unfair contracts rewarded to the managers and violates an envy-free division.

To remedy these undesired outcomes, we introduce two new approaches to determine their contracts. In the first one, we assume the original problem's setup. Given the risk aversion levels, by introducing a mini game between managers, we can induce more balanced capital allocation between managers and at the same time the contracts rewarded to managers will be envy-free. In the second one, again by introducing another mini game between managers, but this time, played before they reveal their risk aversion levels without assuming that they will reveal the true one. This way, we can ensure a fair environment for managers with envy-freeness of their contracts as well as a balanced capital allocation between them which also benefits the principal. Later, we will also extend this game concept in which there are more than two managers involved in the process.

4.1 Case I: Both managers announce after knowing their contracts

In the original problem, managers' contracts might be unfairly distributed, e.g., if one manager is more risk averse but the other manager is less risk averse then they will be rewarded with unfair contracts that might not reflect their true efforts. The principal would not want to create a non-envy-free environment for managers since the principal's utility is dependent of the manager's success on decision making. Among other things, for example, with the principal's increasing disbelief in the expertise announced by the managers, she begins to allocate much less to a manager commanding the more profitable but riskier asset but who is more reliable in forecasts, but rewards slightly more lucrative contract to that manager, while a bigger chunk of the budget but a less attractive contract to the second manager commanding the less profitable but less risky asset. This behavior is not an intuitive one in the sense of risk management for a principal with an increased disbelief, since allocating a very big portion of the capital into a single manager who is more risk averse and hence probably less skillful than the other manager as well as giving him a less attractive contract while trusting him that much does not seem to be rational.

In this case, to be able to prevent this distortion, we introduce the following procedure. Each manager announce their desired contracts in monetary value, after the initial problem setup has been established, that is, each manager knows about the original contracts before announcement is made. After this announcement, we calculate their new contracts given updated risk aversion levels. Before revealing the mechanics of the game, we need to make some assumptions:

Assumption 1. Managers reveal their true risk aversion levels as in the original problem, thus they do not try to make their contracts more attractive by being the less risk averse manager.

Assumption 2. Managers do not cooperate after the initial problem setup has been established, e.g., they try to maximize their own contracts not the summation of them.

Assumption 3. Managers' skill levels are private information. When managers try to estimate their desired contracts before announcing it, they make a biased estimation according to their skill levels. For example, if one manager thinks that he should be rewarded with a more lucrative contract then his incentive is to be a less risk averse manager so that he may get a better contract.

4.1.1 Algorithm of the game

Initial phase Managers announce their risk aversion levels and we solve the problem numerically as we did in *Chapter 3*. Let us assume manager 1 gets the contract c_1 and manager 2 gets the contract c_2 in monetary values and for simplicity let us also assume $c_1 < c_2$.

Announcement phase: Two managers now announce their desired contracts which is a monetary value within a range that the managers are still willing to do their jobs and the manager who announces the lower contract value wins the game.

∘ Manager 1 announces ξ_1. Best response for manager 1 is between c_1 and c_1'

$$\xi_1 \in (c_1, c_1'],$$

where c_1' is the highest contract level that he is content with even if he loses the game.

∘ Manager 2 announces ξ_2. Best response for manager 2 is between c_2' and c_2

$$\xi_2 \in [c_2', c_2),$$

where c_2' is the lowest contract level that he is content with doing this job.

Update phase: After their announcements, contracts are updated as follows:

∘ If $\xi_1 < \xi_2$, then manager 1 wins

Manager 1's contract is updated to ξ_1 and his risk aversion is adjusted to β_1' to satisfy given β_2 and ξ_1. The reason why we adjust manager 1's risk aversion level is that, since he thinks that he deserves more lucrative contract, it is more natural for him to be less risk averse rather than manager 2 being more risk averse in a forced way.

Manager 2's contract is updated to c_2'', given fixed β_1', β_2. We do not allow c_2'' to be lower than c_1 and higher than c_2 to prevent the same issue with reversed positions. Hence

$$c_2'' = \begin{cases} c_1 & \text{if } c_2'' \leq c_1 \\ c_2'' & \text{if } c_1 \leq c_2'' \leq c_2 \\ c_2 & \text{if } c_2 \leq c_2''. \end{cases}$$

∘ If $\xi_1 \geq \xi_2$, then manager 2 wins

Manager 2's contract is updated to ξ_2 and his risk aversion is adjusted to β_2' to satisfy given fixed ξ_2 and β_1. The reason why we adjust manager 2's risk aversion level follows from the above intuition. Since he thinks that he is still content with reduced contract value, it is more natural for him to be more risk averse rather than manager 1 being less risk averse in a forced way.

Manager 1's contract is updated to c_1'' to satisfy given β_1 and β_2'. We do not allow c_1'' to be lower than c_1 and higher than ξ_2 to prevent the same issue with reversed positions. Hence

$$c_1'' = \begin{cases} c_1 & \text{if } c_1'' \leq c_1 \\ c_1'' & \text{if } c_1 \leq c_2'' \leq \xi_2 \\ \xi_2 & \text{if } \xi_2 \leq c_1''. \end{cases}$$

Proposition 2 *Under the assumption that $c_1 < c_2$, manager 1 announces ξ_1 where c_1' is the highest he would be willing to bet and manager 2 announces ξ_2 where c_2' is the lowest he would be willing to bet. Then*

$$\xi_1 \in (c_1, c_1']$$

$$\xi_2 \in [c_2', c_2),$$

and

$$c_1', c_2' \in (c_1, c_2).$$

Proof. Since the game mechanics do not allow managers to get a contract outside of the interval $[c_1, c_2]$, all announcements made by the managers from the interval $[0, c_1)$ is weakly dominated by

$$\xi_1 = \xi_2 = c_1.$$

Furthermore, for manager 1, announcing $\xi_1 = c_1$ is weakly dominated by

$$\xi_1 = c_1 + \epsilon,\ \ \epsilon > 0.$$

Knowing this information, manager 2 will not announce $\xi_2 = c_1$. For manager 2, all announcements from the interval (c_2, ∞) is weakly dominated by

$$\xi_2 = c_2$$

because of the game mechanics. Knowing this, manager 1 cannot get a contract from the interval $[c_2, \infty)$, e.g, even if he announces $\xi_1 = c_2$, he cannot win since from $\xi_1 = \xi_2 = c_2$ manager 2 wins by right. Finally knowing this, for manager 2

$$\xi_2 = c_2 - \epsilon, \epsilon > 0$$

weakly dominates $\xi_2 = c_2$. $\square$

Proposition 3 *Assuming that Proposition 2 holds, the mechanics of the game guarantee to reduce the monetary difference between contracts rewarded to managers.*

Proof. This one is trivial. Since we showed in *Proposition 2* that

$$\xi_1, \xi_2 \in (c_1, c_2),$$

if $\xi_1 < \xi_2$, manager 1 wins

$$c_2 - \xi_1 < c_2 - c_1,$$

and if $\xi_2 < \xi_1$, manager 2 wins

$$\xi_2 - c_1 < c_2 - c_1$$

even before adjusting the other manager's contract. $\square$

4.2 Case II: Both managers announce while not knowing their contracts

In the first case after the initial phase, managers have an insight of what situation they are in. This knowledge has a great impact on what they are going to declare at the announcement phase. In order to remove this bias or provide managers a less strict situation where they can estimate their desired contract values more accurately, in this case managers announce their contract and risk-aversion levels at the same time. Notation is the same as in *Case I*.

c_i: Obtained contract for manager i through declaring risk-aversions, and it is calculated numerically as in *Chapter 3* given β_1 and β_2.

ξ_i: Desired contract value for manager i by his biased estimation based on his skill, satisfaction etc., and it is a private information before declaring it.

4.2.1 Algorithm of the game

Announcement phase: From β_1 and β_2, we find c_1 for manager 1. He also announced ξ_1 at the same time. From β_1 and β_2, we find c_2 for manager 2. He also announced ξ_2 at the same time.

After the announcement phase, we determine both the winning contract and the winning agent. The winning contract is

$$\min\{\xi_1, c_1, \xi_2, c_2\},$$

and the winning agent is

$$\begin{cases} \text{manager 1} & \text{if } \xi_1 \vee c_1 \in \min\{\xi_1, c_1, \xi_2, c_2\} \\ \text{manager 2} & \text{otherwise.} \end{cases}$$

For simplicity, let us assume manager 1 wins the game. All possible outcomes are as follows:

Manager 1 wins through c_1

$$\begin{aligned} &\textbf{(1a)}\ c_1 < c_2 \ll \xi_1 < \xi_2 \ \ \textbf{or} \ \ c_1 < c_2 \ll \xi_2 < \xi_1 \\ &\textbf{(2a)}\ c_1 < \xi_1 \ll c_2 < \xi_2 \ \ \textbf{or} \ \ c_1 < \xi_1 \ll \xi_2 < c_2 \\ &\textbf{(3a)}\ c_1 < \xi_2 < c_2 < \xi_1 \ \ \textbf{or} \ \ c_1 < \xi_2 < \xi_1 < c_2, \end{aligned}$$

and through ξ_1

$$\begin{aligned} &\textbf{(1b)}\ \xi_1 < \xi_2 \ll c_1 < c_2 \ \ \textbf{or} \ \ \xi_1 < \xi_2 \ll c_2 < c_1 \\ &\textbf{(2b)}\ \xi_1 < c_1 \ll \xi_2 < c_2 \ \ \textbf{or} \ \ \xi_1 < c_1 \ll c_2 < \xi_2 \\ &\textbf{(3b)}\ \xi_1 < c_2 < \xi_2 < c_1 \ \ \textbf{or} \ \ \xi_1 < c_2 < c_1 < \xi_2. \end{aligned}$$

Update phase: After the announcements, contracts are updated as follows:

Outcomes (1a)(1b)

These kinds of results may arise when one manager thinks that he is more risk averse than his counterpart or vice versa. For example, if he thinks that he is more risk averse, then he is most likely to underestimate his desired contract value when his counterpart is not actually less risk averse. Furthermore, if the other manager also thinks this way that his counterpart is less risk averse, then he will also underestimate his desired contract value. As a result, both managers will underestimate their contract values and outcome **1a** will be realized. The realization of outcome **1b** follows from the same intuition but in reversed situation for both managers.

∘ In **1a**, managers overestimate their desired contracts. Contract of manager 1 is updated to

$$c_2,$$

and contract of manager 2 is updated to

$$c_2 \cdot (\frac{c_2}{c_1}).$$

Conclusion 1. Managers will obtain new contracts with increased monetary amounts compared to what they would obtain by numerical calculation given initial risk aversion levels.

Conclusion 2. Principle needs to pay more to the managers in total

$$c_1 + c_2 < c_2 \cdot (1 + \frac{c_2}{c_1}) \text{ since } c_2 > c_1.$$

∘ In **1b**, managers underestimate their desired contracts. Contract of manager 1 is updated to

$$\xi_2,$$

and contract of manager 2 is updated to

$$\xi_2 \cdot (\frac{c_2}{c_1}).$$

Conclusion 1. Managers will obtain new contracts with decreased monetary amounts compared to what they would obtain by numerical calculation given initial risk aversion levels.

Conclusion 2. Principle needs to pay less to the managers in total

$$c_1 + c_2 > \xi_2 \cdot (1 + \frac{c_2}{c_1}) \text{ since } c_1 > \xi_2.$$

Remark. Multiplication of the contracts with $\frac{c_2}{c_1}$ is for keeping the ratio of the manager's contracts in monetary values with respect to the original problem's setup so that we do not reward any manager more compared to the other relatively.

Remark. In these outcomes, capital W_0 will be allocated in a more balanced way relative to other outcomes since managers' risk aversion levels are close to each other.

Proposition 4 *Under the assumption that managers' overestimation and underestimation are normally distributed with mean c_n for $n = 1, 2$, the principal's gain and loss as an expected value on contracts rewarded to managers will neutralize each other.*

Proof. In **1a**, the difference is

$$c_2 \cdot (1 + \frac{c_2}{c_1}) - (c_2 + c_1) = \frac{{c_2}^2 - {c_1}^2}{c_1},$$

and in the **1b**, the difference is

$$\xi_2 \cdot (1 + \frac{c_2}{c_1}) - (c_2 + c_1) = (c_2 - \xi_2) + \frac{\xi_2 c_2 - {c_1}^2}{c_1}.$$

Hence $1a = 1b$, since

$$c_2 = \mathbb{E}(\xi_2) \text{ and,}$$

$$\frac{{c_2}^2 - {c_1}^2}{c_1} = \frac{\mathbb{E}(\xi_2) c_2 - {c_1}^2}{c_1}$$

under the normality of overestimations and underestimations. □

As a result of *Proposition 4*, updates for this outcome can be considered "a hedge" or "an insurance" for agents when they overestimate or underestimate their contract values.

Outcomes (2a)(2b)

These outcomes are the most problematic ones. Manager 1 has a considerably greater risk aversion than manager 2, which results in higher capital allocation to manager 1 where he is getting the worst contract. This problem is identical to problem in *Case I*, therefore we can simply apply the idea in *Case I* and adjust capital allocation and rewarded contracts in a balanced way.

○ In **2a**, we should note that $\xi_1 \in (c_1, c_2)$ (the same condition for manager 1 as in *Case* I). Hence, manager 1's contract is updated to ξ_1 and his risk aversion is adjusted to β_1' to satisfy given ξ_1 with β_2.

Manager 2's contract is updated to c_2'', given fixed β_1', β_2. We do not allow c_2'' to be lower than c_1 and higher than c_2. Hence

$$c_2'' = \begin{cases} c_1 & \text{if } c_2'' \leq c_1 \\ c_2'' & \text{if } c_1 \leq c_2'' \leq c_2 \\ c_2 & \text{if } c_2 \leq c_2''. \end{cases}$$

Remark. As shown in *Case I*, this update phase guarantees to reduce the monetary gap between managers' contracts.

○ In **2b**, both manager 1's and manager 2's contracts will be the same as c_1 and c_2 respectively.

Remark. Manager 1's desired contract level is even lower than his theoretical contract level. Since he will be content with even better contract, there is no need to mess up with the risk aversion levels and force the issue. Therefore, we do not make any updates in this case and contracts will stay the same.

Outcomes (3a)(3b)

Among all, these outcomes are the most desired ones since managers' declarations do not refer to any unintended situations that are mentioned before. There is no need to reallocate the capital by manipulating the risk aversion levels, and it can be argued that c_i's are fine as it is. However, for the sake of the game mechanics and to force managers to make their best efforts to estimate their desired contract values and try not to exploit the game by manipulating their declared parameters, we need to adjust the contracts as follows:

∘ In **3a**, manager 1's contract is updated to

$$\frac{c_1 + \min\{\xi_1, c_2\}}{2},$$

and manager 2's contract is updated to

$$\frac{\xi_2 + \min\{\xi_1, c_2\}}{2}.$$

∘ In **3b**, manager 1's contract is updated to

$$\frac{\xi_1 + \min\{c_1, \xi_2\}}{2},$$

and manager 2's contract is updated to

$$\frac{c_2 + \min\{c_1, \xi_2\}}{2}.$$

Proposition 5 *Under the assumption that all of these four outcomes in **3a** and **3b** are realized equally likely and managers' overestimations and underestimations are normally distributed with mean c_n for $n = 1, 2$, in expected value the principal will be indifferent between the original contracts and the updated ones in terms of total monetary value.*

Proof. Following the assumption that all of these four outcomes in **3a** and **3b**

realize equally likely, the principal has to reward the managers in expected value

$$\frac{c_1 + c_2 + \mathbb{E}(\xi_1) + \mathbb{E}(\xi_2)}{2},$$

and also following the assumption that managers' overestimations and underestimations are normal

$$\frac{c_1 + c_2 + \mathbb{E}(\xi_1) + \mathbb{E}(\xi_2)}{2} = c_1 + c_2. \qquad \square$$

Thus, as a result of *Proposition 5*, the principal's position on this case is neutral compared to the theoretical one from the results of *Chapter 3*.

4.3 Case III: More than two managers are involved

For this case, we are going to investigate what might happen when three or more managers are involved. As more and more managers are involved, it is much harder for the principal to calculate how to distribute her wealth to the managers optimally and set their contracts fairly. Whether or not it is optimal and fair, the principal nonetheless has to determine a way to do that, and at the end there is a chance that contracts provided to the manager might be unfair.

Our procedure to determine managers' contracts before the game is performed for this case is to utilize the model established in *Chapter 3* such a way that it covers all the cases where more than two managers are involved. Assuming there are n managers, first, we choose one of the managers and calculate his contract value and wealth allocation α head-to-head against every other manager. Afterwards, we choose the second manager and continue this process until all managers have their contract values and wealth allocations calculated with every other manager. At the end, we simply have to make $\binom{n}{2}$ calculations. After these calculations, we take average of $n-1$ contracts that a specific manager is involved as his final contract and take the summed weight allocation for a specific manager divided by the summed weight allocations over all managers as his final wealth allocation. As an example, let us assume there are $n = 4$ managers. We start with choosing manager 1 and manager 2, then calculate their contracts and wealth allocation between them. After that, we choose manager 1 and manager 3, then calculate their contracts and wealth allocation between them, and also repeat this process between managers 1 and 4. We continue with manager 2 now and calculate these values against managers 3 and 4. Finally, we conclude this phase by calculating these numerical values between manager 3 and 4. Let c_{ij} be the monetary contract value for manager i that is calculated between managers i and j, hence c_{ji} will be the contract value for manager j that is calculated between managers j and i. Furthermore, let α_{ij} be the portion of wealth allocated to manager i which is the result from calculations between managers i and j, hence α_{ji} will be the portion of wealth allocated to manager j, that is

$$\alpha_{ji} = 1 - \alpha_{ij}.$$

The finalized contract value c_i for manager i will be

$$c_i = \frac{c_{ij}}{\sum_{i=1}^{n} \sum_{j>i}^{n} c_{ij}},$$

and the finalized portion of wealth allocation α_{ij} for manager i will be

$$\alpha_i = \frac{\sum_{j \neq i}^{n} \alpha_{ij}}{\sum_{i=1}^{n} \sum_{j \neq i}^{n} \alpha_{ij}}.$$

This method of determining contracts and portion of wealth allocations seems simplistic, however it is quite effective and overall reliable. If we want a method that tries to find exact optimum values for each cases, e.g. a method for $n = 7$ or another method for $n = 10$, it will get complicated, and the necessary calculations become tedious or even impossible. Therefore, we adopt this procedure so that we have a method that covers all instances with a minimal computational effort.

Even though the principal follows our method or any other procedure for determining the contracts, as the number of managers and assets increase, there is an elevated chance to encounter an outlier case in which one or two managers may obtain unfair contracts that may not reflect their incentive efforts. Managers, on the other hand, may still feel some discomfort or discontent about their contracts even if there is no obvious reason, for example one will always have an urge that he should have obtained a better contract compared to others. On the other side, it is the principal's duty to provide the best environment for them while having their full content. In order to achieve this, by devising a simple game between the managers, we also include them in the process of determining the contracts and give them the responsibility for their actions so that they will obtain more fair contracts in terms of both objective and subjective values compared to predetermined ones.

4.3.1 Algorithm of the game

We assume that there are $n \geq 3$ managers.

Announcement and Initialization Phase: At the beginning of the game, they determine their desired contract levels ξ_n that they are content with, and consider that it is the fair amount for their efforts. They also assign assets' weights in their portfolios at the same time. Hence, in this way, the principal can allocate her wealth to the managers and calculate their contracts c_n numerically by using either our procedure that we have just explained above or her own predetermined method after observing each of these values.

After the announcement and calculation phase, we have at the hand all the numerical values (in monetary terms). We rank these values numerically from minimum to maximum (i.e. $\xi_2 < \xi_3 < c_1 < \xi_1 < c_4 < ...$), which we call as 'contact array' and denote it by C. We also denote the location in the array as $i = 1, 2, ..., 2n$.

Update Phase I: Let K_n be the current contract value for manager n. If $\xi_n \leq c_n$ for manager n, then we set

$$\tilde{\xi_n} = \xi_n$$

to prevent any confusion and set K_n as

$$K_n = \min\{\xi_n, c_n\} \ , \forall n \in C.$$

These values may change later on according to dynamics of the game. We will investigate at the end what happens when there is no update at all, which is the case of $\xi_n > c_n$ for all managers. Next, we pick the minimum of $\tilde{\xi_n}$ value and denote it by Z since it is a fixed value and we will use it again a couple of times. We then remove the right side from the contract array C and call it C'. That is, the maximum value of C' is Z. We also separately remove the left side of Z from the contract array C and call it C''. That is, the minimum value of C'' is Z. Next, we pick the minimum ratio of $\frac{\tilde{\xi_n}}{c_n}$ among the managers who have lower value ξ_n than c_n and denote it as Y since it is a fixed value, that is

$$Y = \min\{\frac{\tilde{\xi_n}}{c_n}\} \ , n|\{\xi_n \leq c_n\}.$$

If $\xi_n > c_n$ for $c_n \in C''$, then we update K_n as

$$K_n = \max\{Y \cdot c_n, c_n - (\xi_n - c_n)\}$$

for manager n who has $\xi_n > c_n \in C''$. At first glance, it may seem unnecessary to update contracts for managers who have $\xi_n > c_n$, but it is a very important aspect of the game as well as to preserve its reliability. By having this step, we motivate managers to estimate desired contracts more accurately. We can also avoid a situation in which a manager who knows that he should be obtaining a much better contract would just announce a very high random value to ensure his contract. This kind of behavior puts the principal in a though situation by forcing her to deal with larger prediction errors. It would also be disrespectful to the other managers who try their best to comply with the concept. Hence, it is better to stay safe here than be sorry.

Proposition 6 *For update phase I, managers who think they have $c_n \in C''$ do not try to overestimate or underestimate their contract levels ξ_n, that is, with perfect information over c_n's, their best response would be $\xi_n = c_n$.*

Proof. First, a manager would not be interested in overestimating his contract value. If he tries to overestimate, for example, he declares $\xi_n = c_n + \epsilon$, then his updated contract K_n will be

$$K_n = \max\{Y \cdot c_n, c_n - ((c_n + \epsilon) - c_n)\} = \max\{Y \cdot c_n, c_n - \epsilon\}.$$

From above, the maximum value he can obtain is c_n since $Y \leq 1$, which is guaranteed when $\epsilon = 0$. Hence, he will not try to overestimate and his best response will be from the interval $[0, c_n]$.

Second, he would not be interested in underestimating his contract value. If he tries to underestimate, for example he declares $\xi_n = c_n - \epsilon$, then his updated contract K_n will be

$$K_n = \min\{\xi_n, c_n\}$$

since $\xi_n \leq c_n$. As a result, the maximum value he can obtain is c_n, which is guaranteed when $\epsilon = 0$. Therefore, he will not try to underestimate and his best response will be from the interval $[c_n, \infty]$. Combining these two results and with perfect information over c_n's, his best response when declaring ξ_n is c_n. $\square$

Remark. In update phase I, one important observation is that the managers do not know other managers' asset weights in their portfolios. Therefore, the best strategy for a manager would be trying to guess c_n as if it is his envy-free contract level. As a result, what managers are really trying not to overestimate or underestimate is their envy-free contract levels based on the data that they have.

Update Phase II: We first calculate the contribution values P_n for all n from $c_n \in C''$ as follows

$$P_n = c_n - K_n,$$

then we calculate the update values U_n for all $n|\{c'_n, \xi'_n\} \in C'$ as follows

$$U_n = \frac{Z - c_n}{\sum_{i \in C'}(Z - c_i)}\left(\sum_{i \in C''} P_i\right),$$

where $\sum_{i \in C''} P_i$ is simply the sum of excess monetary value after we update manager n's contract from c_n to $\tilde{\xi_n}$ since $\xi_n \leq c_n$. It should be noted that we only calculated U_n's for the managers who deserve c_n and announced ξ_n which is less than or equal to Z. This is for the reason that managers should try to guess their contract accurately without attempting to overestimate. It is also not reasonable for a manager who announces a larger value to get an update than a manager who actually would have obtained a better contact if we did not perform this game. Next, we update contract values K_n for all n from $c_n \in C'$ as follows

$$K_n = \min\{Z, \xi_n, c_n + U_n\}.$$

Update Phase III(if necessary): For this phase to be active, there has to be still left money in the pool $\sum_{i \in C''} P_i$. Let $\tilde{P}$ be the total summed value, that is

$$\tilde{P} = \sum_{i \in C''} P_i - \sum_{i \in C'}(K_i - c_i).$$

We would want this excess money to return to the managers in a some way so as to comply with our fairness policy. To do that, we first update contract values for managers who have $\{\xi_i > Z\}_{i \in C'}$ as follows

$$K_n = \max\{\min\{K_i\}_{i|\{c_n \leq K_i \leq Z\}}, Z - (\xi_n - Z)\}.$$

By this last update, we again encourage the managers to estimate their contract values as accurate as possible. After the latest update, if $\tilde{P}$ is still positive, we continue with calculating the update values U''_n in the same manner as in update phase II for all $n|\{c_n \in C''\}$ as follows

$$U''_n = \frac{P_n}{\sum_{i \in C''} P_i} \cdot \tilde{P},$$

then we finalize the contract values K_n by updating K_n where $n \in C''$ as follows

$$K_n = K_n + U''_n.$$

Proposition 7 *Under the algorithm of update phase III, managers who think they have $c_n \in C'$ will try to overestimate their contract, and their best response will be Z with perfect information over c_n's and ξ_n's.*

Proof. Assuming there are enough money in the pool $\tilde{P}$, their contracts will be updated to either

$$K_n = \min\{Z, \xi_n, c_n + U_n\},$$

or

$$K_n = \max\{\min\{K_i\}_{i|\{c_n \le K_i \le Z\}}, Z - (\xi_n - Z)\}.$$

Hence, the best they can obtain is Z as is their best response. □

Remark. Although this result is trivial, implications are very important. Coinciding with the aim of our game, there is nothing wrong for those managers who are getting relatively worse contracts to try declaring higher ξ_n than c_n. The question for them is that they neither know their c_n value nor the Z value before the announcement phase. Therefore, their best strategy would be to try to estimate their ideal contract level as accurately as possible in order not to be discontent after contracts have been finalized.

Special Case: We will now investigate what the algorithm would be when the case of $\xi_n > c_n$ happens for all managers. Since there is no update at all, K_n stays as c_n after the initialization. Although it should be rare, there is a small chance that managers who obtain the lowest and the highest contracts may have very high prediction errors about their contracts, which may lead to an unfair condition (i.e., one or more managers who obtain the highest contracts will have relatively much better contracts and one or more managers who obtain the lowest contracts will have much worse contracts that in both cases will not reflect what they deserve). First of all, in the case of

$$\max\{c_n\} < \min\{\xi_n\},$$

for example, $c_2 < c_3 < c_1 < c_4 < \cdots < c_8 < \xi_3 < \xi_1 < \xi_4 < \xi_2 < \cdots < \xi_7$, we do not make any update and we should not indeed since we cannot trust managers' declarations of ξ_n's. The principal may have the option of either repeating the game or accepting the results as it is. For all other cases, we first find the $Z = \min\{\xi_n\}$ of C. Second, we determine the upper part array C^u, which includes the right side after Z(i.e. the minimum value for C^u would be Z) and the lower part array C^l, which includes the left side before Z(i.e. the maximum value for C^l would be Z). Then we calculate the following values for the upper part

$$A = \sum_{i \in C^u} (c_i - Z),$$

and for the lower part

$$B = \sum_{i \in C^l} (Z - c_i).$$

If $A < B$, then for manager n from $c_n \in C^u$

$$K_n = Z,$$

and for manager n from $c_n \in C^l$

$$K_n = \frac{Z - c_n}{B} \cdot A + c_n.$$

And if $A > B$, then for manager n from $c_n \in C^l$

$$K_n = Z,$$

and for manager n from $c_n \in C^u$

$$K_n = \frac{c_n - Z}{A} \cdot (A - B) + Z.$$

For the trivial case of $A = B$, all managers get the same contracts, that is

$$K_n = Z \ , \ n \in C.$$

Proposition 8 *The principal's stand for this game is monetarily neutral. In other words, she is neither gaining nor losing any financial value after managers' contracts are finalized after the algorithm of our game is applied.*

Proof. Let us investigate this for every possible scenario. Let n be the total number of managers who are involved in the game and k be the number of contracts c_n that has lower value than Z, and hence $n - k$ will be number of contracts c_n that has greater than or equal value to Z.

First of all, in the trivial case of $\max\{c_n\} < \min\{\xi_n\}$, we do not make any update due to the reason that is mentioned in special case section. Since we do not make any update, there will not be any change at all.

If special case is applied, then there will be three sub-cases. Before update phase, we had total contract value of

$$\sum_i c_i = \sum_{i \in C^u} c_i + \sum_{i \in C^l} c_i = A + (n-k)Z + kZ - B = A + nZ - B.$$

When $A = B$, then

$$\sum_i c_i = nZ,$$

which is what we would obtain as total contract value after updates if we set $K_n = Z$ for all n. When $A < B$, then after updates we will have total contract value of

$$\begin{aligned}
&(n-k)Z + \big(\sum_{i\in C^l} \frac{Z-c_i}{B}\big)\cdot A + \sum_{i\in C^l} c_i \\
=&\big(\sum_{i\in C^u} c_i\big) - A + A + \sum_{i\in C^l} c_i \\
=&\sum_{i\in C} c_i,
\end{aligned}$$

and when $A > B$

$$\begin{aligned}
&kZ + \Big(\sum_{i\in C^l} \frac{c_i - Z}{A}\cdot (A-B)\Big) + (n-k)Z \\
=&kZ + A - B + nZ - kZ = A + nZ - B \\
=&\sum_{i\in C} c_i.
\end{aligned}$$

For all other scenarios, first we should note that we only use money from $\tilde{P}$ which comes from the managers. Therefore, the principal does not give away any monetary value from her wealth. We just need to check whether or not $\tilde{P}$ is emptied at the end. Second, we should also note that, leftover contribution value P_n after update phase II cannot be larger than update value U''_n for all n from $c_n \in C''$ since

$$U''_n = \frac{P_n}{\sum_{i\in C''} P_i}\cdot \tilde{P}$$

can be at most P_n when there is no update in update phase II (i.e. $Z = c_n$, $\forall n \in C'$). Hence, the principal does not add any value to her wealth from these scenarios as well. $\square$

4.3.2 Numerical examples

Example 1 There are $n = 5$ managers. They announced $\xi_n = \{90, 110, 105, 180, 125\}$ for $n = 1, \ldots, 5$, and they would obtain $c_n = \{60, 80, 120, 150, 160\}$ for $n = 1, \ldots, 5$ by our proposed method or the principal's predetermined procedure. Therefore, we have the contract array C as

$$C : c_1 = 60 < c_2 = 80 < \xi_1 = 90 < \xi_3 = 105 < \xi_2 = 110 < c_3 = 120 < \xi_5 = 125$$
$$< c_4 = 150 < c_5 = 160 < \xi_4 = 180.$$

For the update phase I, we pick the ξ_n's that are $\xi_n \leq c_n$. These are $\xi_3 = 105 < c_3 = 120$ and $\xi_5 = 125 < c_5 = 160$. Next we set $\tilde{\xi_3} = \xi_3 = 105$, $\tilde{\xi_5} = \xi_5 = 125$ and initialize $K_n = \min\{\xi_n, c_n\}$

$$K_n = \{60, 80, 105, 150, 125\}.$$

We next choose the minimum one from $\tilde{\xi_n}$'s, that is

$$Z = \min\{\tilde{\xi_n}\} = \min\{\tilde{\xi_3} = 105, \tilde{\xi_5} = 125\} = \tilde{\xi_3} = 105.$$

To find the reduced array C', we remove the part after $\tilde{\xi_3}$ and get

$$C' : c_1 = 60 < c_2 = 80 < \xi_1 = 90 < \xi_3 = 105,$$

and for C'', we remove the part before $\tilde{\xi_3}$ and get

$$C'' : \xi_3 = 105 < \xi_2 = 110 < c_3 = 120 < \xi_5 = 125 < c_4 = 150 < c_5 = 160 < \xi_4 = 180.$$

We now find the minimum ratio of $\frac{\tilde{\xi_n}}{c_n}$, that is

$$Y = \min\{\frac{\tilde{\xi_n}}{c_n}\} = \min\{\frac{105}{120}, \frac{125}{160}\} = 0.78125.$$

Since manager 4 has $\xi_4 = 180 > c_4 = 150$, we again update K_n for

$$K_4 = \max\{0.78125 \cdot 150 = 117.1875, 150 - (180 - 150) = 120\} = 120.$$

For the update phase II, we first calculate the contribution values P_n for manager 3,4 and 5 since $c_3, c_4, c_5 \in C''$

$$P_{3,4,5} = \{120, 150, 160\} - \{105, 120, 125\} = \{15, 30, 35\},$$

then we calculate the update values U_n for $c_1 \in C'$ only since $\xi_2 = 110 > \tilde{\xi_3} = 105$.

$$U_1 = \frac{105-60}{105-60}(15+30+35) = 80.$$

Then we update contract values $K_n \in C'$

$$K_1 = \min\{105, \xi_1 = 90, c_1 + U_1 = 60 + 80 = 140\} = 90,$$

and

$$K_2 = \min\{105, \xi_2 = 110, c_2 + U_2 = 80 + 0 = 80\} = 80.$$

Basically, we did no need to calculate for manager 2 as his contract value will not be updated since he announced $\xi_2 = 110 > \min\{\tilde{\xi_3}\} = 105$. We still have excess money $\tilde{P} = 80 - (90 - 60) = 50$, hence we can continue with update phase III. We will distribute this excess money back to the manager 2 who have $\{\xi_2 = 110 > Z = 105\}$ and then to their owners (manager 3, 4 and 5) according to their contribution ratios. For manager 2

$$K_2 = \max\{\min\{K_1 = 90\} = 90, 105 - (110 - 105) = 100\} = 100.$$

We still have positive $\tilde{P} = 80 - (90 - 60) - (100 - 80) = 30$, hence we continue with calculating update values U''_n for $n = 3, 4, 5$ where $c_n \in C''$

$$U''_3 = \frac{120-105}{80} \cdot 30 = 5.625$$

$$U''_4 = \frac{150-120}{80} \cdot 30 = 11.25$$

$$U''_5 = \frac{160-125}{80} \cdot 30 = 13.125.$$

We then finalize the contract values for $K_n \in C''$:

$$\begin{aligned} K_{3,4,5} &= \{K_3 + U''_3 = 105 + 5.625, K_4 + U''_4 = 120 + 11.25, K_5 + U''_5 = 125 + 13.125\} \\ &= \{110.625, 132.25, 138.125\}. \end{aligned}$$

Hence, the finalized values are

$$K_n = \{90, 100, 110.625, 132.25, 138.125\}\ , \ n = 1, \ldots, 5.$$

Example 2 There are $n = 5$ managers. They announced $\xi_n = \{100, 85, 110, 120, 130\}$ for $n = 1, \ldots, 5$, and they would obtain $c_n = \{90, 75, 80, 115, 105\}$ for $n = 1, \ldots, 5$ by our proposed method or the principal's predetermined procedure. Therefore, we have the contract array C as

$$\begin{aligned} C : c_2 = 75 < c_3 = 80 < \xi_2 = 85 < c_1 = 90 < \xi_1 = 100 < c_5 = 105 < \xi_3 = 110 \\ < c_4 = 115 < \xi_4 = 120 < \xi_5 = 130. \end{aligned}$$

This is a case in which $\xi_n > c_n$ for all $n = 1, \ldots, 5$, hence we are going to implement special case. We first find the $Z = \min\{\xi_n\} = \xi_2 = 85$ of C. Second, we determine the upper part array C^u

$$\begin{aligned} C^u : \xi_2 = 85 < c_1 = 90 < \xi_1 = 100 < c_5 = 105 < \xi_3 = 110 < c_4 = 115 < \xi_4 = 120 \\ < \xi_5 = 130. \end{aligned}$$

and the lower part array C^l

$$C^l : c_2 = 75 < c_3 = 80 < \xi_2 = 85.$$

For the upper part

$$A = (90 - 85) + (115 - 85) + (105 - 85) = 55,$$

and for the lower part

$$B = (85 - 75) + (85 - 80) = 15.$$

Since $A > B$, we set K_n for manager 2 and 3 as $Z = 85$, that is

$$K_{2,3} = \{85\},$$

for manager 1

$$K_1 = \frac{90 - 85}{55}(55 - 15) + 85 = 88.64,$$

for manager 4

$$K_4 = \frac{115 - 85}{55}(55 - 15) + 85 = 106.82,$$

and for manager 5

$$K_5 = \frac{105 - 85}{55}(55 - 15) + 85 = 99.54.$$

Hence, the finalized values are

$$K_n = \{88.64, 85, 85, 106.82, 99.54\}\ ,\ n = 1, \ldots, 5.$$

Chapter 5

Conclusion

In this thesis, we investigated a model in which we considered a one-period economy, two managers having exponential utility, and a risk neutral investor who has to allocate her wealth between two managers and setting their fees optimally in a way that the principle safeguards herself against forecast errors of the managers considering the mean return of assets. The managers' knowledge of the market was modeled as a private signal and they are rewarded through a contract which is calculated based on the performance of their trading activity. In the second part, we devised simple games between managers to create more fair contracts for them regarding the results that was found in part one.

In the first section of *Chapter 3*, we investigated the case where each manager is restricted to invest into one asset apart from the risk-free asset. We solved the principle's utility maximization problem who accepts an ellipsoidal uncertainty set $\mathcal{U}_S$ by taking a worst-case approach. After setting up the Lagrangian function, we obtained optimal equations for our problem. Although the equations are not in closed form, we can solve them numerically to find the optimum allocation values as well as contract levels that are rewarded to managers. In the second section, we distinguished the skills of the managers according to their forecasts. This time we solved the problem of the principle who wants to protect herself against the forecast errors. We also provided a MATLAB code where we can solve these equations numerically.

Chapter 4 consists of our efforts to create games that ensure envy-free contracts that the managers obtain as their fee. Since the optimal results from *Chapter 3* do not guarantee the assumption of fair contracts, we constructed an algorithm based on a game between managers and presented our findings. In the first game format, we assumed the results of *Chapter 3*, and let the managers announce their desired contract levels. After the updates, our algorithm guarantees to reduce the monetary difference between contracts. In the second game format, we let the managers announce their desired contract levels before they know their actual contracts which will be calculated by our model in *Chapter 3*. In this way, the managers have an opportunity to estimate their desired contracts without any bias. In all possible outcomes, we increased the envy-freeness of their contracts through the implementation of game mechanism. In the last game format, we extended our ideas to include more than two managers participating in the game and provide them to obtain more fair contracts. Our algorithm is applicable for any number of managers that are involved in the game. We also provided some numerical examples for this format to make it more understandable.

It is possible that our design and algorithm of the games can be improved in some ways. For the game between more than two managers, we do not update wealth allocations for managers as we do in two managers game formats. As more and more managers are involved in the game, the number of possible distinct contract arrays grows factorially. Therefore, it would not be practical to go in that direction at the moment. However, it can be implemented by a heuristic that divides the contract array in known patterns and allows us to work on them. This can be a main motivation for further studies.

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