GEODESICS OF THREE-DIMENSIONAL WALKER MANIFOLDS

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

GEODESICS OF THREE-DIMENSIONAL WALKER MANIFOLDS

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We review some basic facts of Lorentzian geometry including causality and geodesic completeness. We depict the properties of curves and planes in three-dimensional Minkowski space. We define the Walker manifolds, that is, a Lorentzian manifold which admits a parallel degenerate distribution. We calculate the Christoffel symbols and Levi-Civita connection components, Riemann curvature and Ricci curvature components for an arbitrary three-dimensional Walker manifold and strictly Walker manifold.

Finally, we derive the geodesic equations of a three-dimensional Walker manifold and investigate the geodesic curves in it, particularly the ones with a constant component. We prove that any straight line with a constant third component is a geodesic in any Walker manifold with the causality depending on its second component. We prove that the existence of a geodesic in a Walker manifold with a linear third component implies that the manifold is strict. We also show that any three-dimensional Walker manifold is geodesically complete.

Keywords: Walker manifold, Lorentzian manifold, geodesic.

ÖZET

ÜÇ BOYUTLU WALKER MANİFOLDLARDA JEODEZİKLER

Gökçen Büyükbaş Çakar
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Lorentz geometri ile ilgili bazı bilinen gerçekleri, nedensellik ve jeodeziklerce tamlık gibi, gözden geçirdik. Üç boyutlu Minkowski uzayındaki eğri ve düzlemlerin özelliklerini inceledik. Walker manifoldunu tanımladık, öyle ki, paralel dejenere bir dağılım kabul eden Lorentz manifoldlarına Walker manifoldu denir. Herhangi bir üç boyutlu Walker ve mutlak Walker manifoldunun Christoffel sembollerini, Riemann ve Ricci eğriliklerini hesapladık.

Son olarak herhangi bir üç boyutlu Walker manifoldunun jeodezik denklemlerini bulduk ve jeodezik eğrilerini araştırdık, özellikle de bir bileşeni sabit olan eğrileri inceledik. Üçüncü bileşeni sabit olan herhangi bir düz çizginin bir Walker manifoldu üzerinde her zaman bir jeodezik eğrisi olduğunu ve bu eğrinin nedenselliğinin ikinci bileşenine bağlı olduğunu kanıtladık. Bir Walker manifoldu üzerinde doğrusal bir üçüncü bileşeni olan bir jeodezik eğrisinin varlığının bu manifoldun tekdüze olmasını gerektirdiğini kanıtladık. Ayrıca her üç boyutlu Walker manifoldunun jeodeziklerce tam olduğunu göstedik.

Anahtar sözcükler: Walker manifoldu, Lorentz manifoldu, jeodezik.

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Contents

Chapter 1

Introduction

It is known that if there exists a parallel line field on a Riemannian manifold, then the manifold decomposes locally as a direct product. If the line field is non-degenerate, i.e., is generated by a non-null locally defined vector field, then this property can be extended to semi-Riemannian manifolds (see [1], [2]). However, the geometrical consequences of the case that there exists a parallel degenerate line field on the manifold are not yet well understood. A *Walker structure* is a parallel degenerate plane field on a manifold. The existence of Walker structures yields many of the fundamental differences between the Riemannian and semi-Riemannian geometries. Walker described the canonical form for a space with parallel field of null planes in [3]. In [1], Chaichi, García-Río and Vázquez-Abal investigated the curvature properties of a three-dimensional Walker manifold. They obtained a complete characterization of Walker three-manifolds being locally symmetric or locally conformally flat. Moreover they showed nice families of examples of such properties in the Lorentzian setting. Then, Calvaruso and De Leo investigated Ricci solitons on three-dimensional Walker manifold in [2]. They proved that there exist non-trivial Ricci solitons on several classes of these manifolds, depending on the defining function f. In [4], Nadjafikhah and Jafari applied the Lie symmetry group method to determine the Lie point symmetry group and provided example of solution of the system of partial differential equations that is determined general form of four-dimensional Einstein Walker manifold. In [5], Bejan and Druţă-Romaniuc establish that on a three-dimensional Walker manifold M admitting a unit spacelike vector field, any lightlike curve that is normal to this vector field is a reparametrization of a lightlike geodesic. They also obtain magnetic curves corresponding to a Killing vector field admitted by M. Moreover, some characterization of the normal magnetic trajectories associated to the Killing vector field ∂_x on M are obtained and some examples of Killing magnetic curves on such manifolds are provided.

Lorentzian geometry is the mathematical theory used in general relativity. A *Lorentzian manifold* is a smooth paracompact manifold furnished with a non-degenerate metric of signature $(1, n-1)$ (or equivalently $(n-1, 1)$). It is well known that any non-compact paracompact manifold M admits a Lorentzian structure. However, the only compact manifolds admitting a Lorentzian structure are the manifolds with Euler characteristic zero (see [6], [7]).

In the second chapter of this thesis we recall some basic definitions and properties of the Lorentzian geometry by following [6], [7], [8], [9], [10], [11]. In section 2.2, we define the Lorentzian metric space $\mathbb{E}_1^3$ and the causal character of tangent vectors and subspaces. Any nonzero vector v in the tangent space of M is said to be *timelike* (respectively *spacelike*, *lightlike*) if $g(v,v) < 0$ (respectively $g(v,v) > 0$, $g(v,v) = 0$). A Lorentzian manifold M admitting a continuous, nowhere vanishing, timelike vector field X is said to be *time oriented*. By using this vector field, the non-spacelike vectors are separated into two classes at each point. These two classes are called *past directed* and *future directed*. A Lorentzian manifold (M, g) together with a choice of time orientation is called a *space-time* (see [7], [9]). In section 2.3, we introduce the parametrized curves in Lorentzian space [12], [9] and give some examples of plane and spatial curves in this space. In section 2.4, we introduce the Frenet trihedron $\{T, N, B\}$ as an orthonormal basis for $\mathbb{E}_1^3$ and find the Frenet equations of curves depending on their causal characters. Finally in section 2.6, we introduce the connections and curvatures of a semi-Riemannian manifold. There is a unique affine connection on M compatible with the metric which is called the *Levi-Civita connection*. To describe the geometry of an n-dimensional manifold, we use curvature definition of Riemann and some other notions such as Ricci curvature and scalar curvature.

One can find the details about these notions in a number of book and papers such as [6], [7], [8], [2], [13], [14]. We define the geodesic completeness of a Lorentzian manifold in section 2.8 (see [7], [15]).

In chapter 3, we introduce three-dimensional Walker manifolds, that is, a three-dimensional Lorentzian manifold admitting a parallel degenerate line field. If the underlying line field can be generated by a null vector, then the manifold is called *strictly Walker manifold* [2], [1]. The Walker manifolds are described in terms of local coordinates $\{x, y, z\}$ and a defining function $f = f(x, y, z)$. There is a large class of Walker manifolds, depending on the defining function f. If the Walker manifold is strict, then it can be characterized by $f = f(y, z)$. In section 3.1, we define the vector product for a Walker manifold and construct an orthonormal basis [5]. In section 3.2, we calculate the connections and curvature components of an arbitrary three-dimensional Walker manifold and a strictly Walker manifold (see [2], [13]).

Finally in chapter 4, we focus on the geodesics of a three-dimensional Walker manifold (M, g_f), in particular, the ones with a constant or linear component. Suppose that $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ is a curve in (M, g_f), we prove that straight-lines on M with a constant γ_3 is a geodesic in M and its causal character depends on its second component, that is, if γ_2 is constant, then γ is a light-like geodesic; if it is not constant, the curve is a timelike geodesic (see Corollary 4.0.10). Moreover, we establish that if there exists a geodesic in M with a linear third component, then M is a strictly Walker manifold (see Corollary 4.0.11). In a strictly Walker manifold, if γ is a geodesic, then γ_3 is either constant or linear (see Corollary 4.0.12). We also show that any Walker manifold of dimension three is geodesically complete (see Theorem 4.0.15) [1], [10].

Chapter 2

Preliminaries

In this chapter we state some basic definitions and results from semi-Riemannian and Lorentzian Geometry. These definitions and results may be found in a number of books such as [6] and [7]. Our definition of curvature is in agreement with [7], but it is the negative of the definition used in [6].

2.1 Manifold Theory

In this section, we define some basic terms from general manifold theory and also introduce some notational conventions.

Throughout this work any manifold M is assumed to be connected, Hausdorff, para-compact, and smooth. The set of all tangent vectors to M at $p \in M$ is denoted by $T_p(M)$. The set of all tangent vectors to M is denoted by $T(M)$.

A smooth section X of $T(M)$ is a *vector field* in M. The set of all vector fields in M is denoted by $\mathfrak{X}(M)$. Let $\gamma\colon I \to M$ be a smooth curve, then a vector field along γ is a smooth mapping $V\colon I \to T(M)$ with $V(t) \in T_{\gamma(t)}(M)$ for all $t \in I$. Let $\mathfrak{X}(\gamma)$ denote the set of all vector fields along γ.

Let M be a smooth manifold. A *metric tensor* g in M is a $(0,2)$ tensor field in M of constant index that is symmetric and non-degenerate. The *signature* of g is denoted by (r,s) and the index of g is r where g has r negative eigenvalues and $s=n-r$ positive eigenvalues.

A *semi-Riemannian* manifold (M,g) is a smooth manifold furnished with a metric tensor g. (M,g) is called a *Riemannian* manifold if the index of g is zero, i.e, g has no negative eigenvalues. (M,g) is called a *Lorentzian* manifold if the signature of g is $(1,n-1)$ (or, equivalently, $(n-1,1)$) and the dimension of M is greater than or equal to 2. Throughout this work any manifold M is assumed to be Lorentzian.

We use $\langle\ ,\ \rangle$ as an alternative notation for $g(v,w)=\langle v,w\rangle\in\mathbb{R}$ for tangent vectors, and $g(V,W)=\langle V,W\rangle\in\mathfrak{X}(M)$ for vector fields.

2.2 The Lorentz-Minkowski Space $\mathbb{E}_1^3$

In this section we introduce the Lorentz-Minkowski space with an interest on the curves on this space. First, we state some basic definitions and then we develop the Frenet equations for curves in $\mathbb{E}_1^3$. The definitions and propositions in this section can be found in [9].

Let $\mathbb{R}^3$ be the real vector space with the canonical basis $B_u=\{E_1,E_2,E_3\}$ where,

$$E_1=(1,0,0),\quad E_2=(0,1,0),\quad E_3=(0,0,1).$$

The three-dimensional real vector space with the Euclidean metric, $(\mathbb{R}^3,\langle,\rangle_e)$, is denoted by $\mathbb{E}^3$.

Definition 2.2.1. The metric space $\mathbb{E}_1^3=(\mathbb{R}^3,\langle,\rangle)$ with the *Lorentz metric*

$$\langle u,v\rangle=u_1v_1+u_2v_2-u_3v_3,\qquad \text{where } u=(u_1,u_2,u_3),\ v=(v_1,v_2,v_3)$$

is called the *Lorentz-Minkowski space.*

In Lorentz-Minkowski space, each vector have a causal character as follows:

Definition 2.2.2. A vector $v \in \mathbb{E}_1^3$ is called

(i) *timelike* if $\langle v, v \rangle < 0$,

(ii) *null* or *lightlike* if $\langle v, v \rangle = 0$ and $v \neq 0$,

(iii) *spacelike* if $\langle v, v \rangle > 0$ or $v = 0$.

The set of all null vectors in $\mathbb{E}_1^3$ is called the *light cone*, that is,

$$\mathcal{C} = \{(x, y, z) \in \mathbb{E}_1^3 | x^2 + y^2 - z^2 = 0\} - \{(0, 0, 0)\}.$$

The set of all timelike vectors in $\mathbb{E}_1^3$ is

$$\mathcal{T} = \{(x, y, z) \in \mathbb{E}_1^3 | x^2 + y^2 - z^2 < 0\}.$$

Let $U \subset \mathbb{R}^3$ be a subspace and the induced metric $\langle u, v \rangle_U$ on U,

$$\langle u, v \rangle_U = \langle u, v \rangle, \quad u, v \in U.$$

The characterization of the causality of a subspace U is as follows:

(i) U is called spacelike if $\langle , \rangle_U$ is positive definite.

(ii) U is called timelike if the index of $\langle , \rangle_U$ is 1.

(iii) U is called null if $\langle , \rangle_U$ is degenerate.

Proposition 2.2.3. *Let $U \subset \mathbb{E}_1^3$ be a subspace. Then*

(i) $dim(U^\perp) = 3 - dim(U)$.

(ii) $(U^\perp)^\perp = U$.

(iii) If U is non-degenerate, then $U^\perp$ is non-degenerate.

(iv) U is spacelike (respectively timelike, null) if and only if $U^\perp$ is timelike (respectively, spacelike, null).

(v) If v is a timelike or spacelike vector, then $\mathbb{E}^3_1 = Span\{v\} \oplus Span\{v\}^{\perp}$.

Proposition 2.2.4. *(i) Let $u, v \in \mathbb{E}^3_1$ be lightlike vectors. $\langle u, v\rangle = 0$ if and only if u and v are linearly dependent.*

(ii) If $u, v \in \mathbb{E}^3_1$ are two non-spacelike vectors with $\langle u, v\rangle = 0$, then they are lightlike vectors.

(iii) If $u, v \in \mathbb{E}^3_1$ are timelike vectors, then $\langle u, v\rangle \neq 0$.

(iv) If U is a null subspace, $dim(U \cap U^{\perp}) = 1$.

Proposition 2.2.5. *Let $W \subset \mathbb{E}^3_1$ be a vector plane. The following are equivalent:*

(i) There exists a timelike vector in W.

(ii) There exist two linearly independent lightlike vectors in W.

(iii) W is a timelike subspace.

Proposition 2.2.6. *Let $U \subset \mathbb{E}^3_1$ be a subspace. The following are equivalent:*

(i) U is a null subspace.

(ii) There exists a lightlike vector in U but there does not exist a timelike vector.

(iii) $U \cap \mathcal{C} = L - \{0\}$, $dim(L) = 1$.

Proposition 2.2.7. *Let $W \subset \mathbb{E}^3_1$ be a vector plane given by an orthogonal vector $\vec{n}_e$. Then W is a spacelike (respectively timelike, lightlike) plane if and only if $\vec{n}_e$ is a timelike (respectively spacelike, lightlike) vector.*

We define the *norm* of a vector $u \in \mathbb{E}^3_1$ as

$$|u| = \sqrt{|\langle u, u\rangle|}.$$

The vector u is said to be *unitary* if $|u| = 1$.

Definition 2.2.8. Let $u \in \mathbb{E}^3_1$ be a timelike vector. The *timelike cone* of u is the set:

$$C(u) = \{v \in \mathcal{T} | \langle u, v\rangle < 0\}.$$

Note that $C(u) \neq \emptyset$ since $u \in C(u)$. Furthermore, if there exists another timelike vector $v \in \mathbb{E}^3_1$, $\langle u, v\rangle < 0$ or $\langle u, v\rangle > 0$ (by Proposition 2.2.4) and $\mathcal{T}$ is the disjoint union of $C(u)$ and $C(-u)$.

Proposition 2.2.9. *Let u and v be two timelike vectors.*

(i) $v \in C(u)$ if and only if $C(v) = C(u)$.

(ii) u and v are in the same timelike cone if and only if $\langle u, v\rangle < 0$.

(iii) The timelike cones are convex sets.

Let $B = \{e_1, e_2, e_3\}$ be an ordered basis for $\mathbb{E}^3_1$. B is called a *null basis* (or *null frame*) if e_1 is a unit spacelike vector and $e_2, e_3 \in Span\{e_1\}^\perp$ are null vectors lying in the same component of $\mathcal{C}$, that is, $\langle e_2, e_3\rangle = -1$.

A difference between $\mathbb{E}^3$ and $\mathbb{E}^3_1$ is the Cauchy-Schwarz inequality. Recall that the Cauchy-Schwarz inequality for $u, v \in \mathbb{E}^3$ is

$$|\langle u, v\rangle| \leq |u||v|$$

and the equality holds if and only if u and v are linearly dependent.

In Lorentz-Minkowski space, there exists a *backwards Cauchy-Schwarz inequality* for the timelike vectors which is the reverse of the inequality in $\mathbb{E}^3$.

Theorem 2.2.10. *Let $u, v \in \mathbb{E}^3_1$ be timelike vectors. Then*

$$|\langle u, v\rangle| \geq |u||v|$$

and the equality holds if and only if u and v are linearly dependent. If u and v lie in the same timelike cone, then

$$\langle u, v\rangle = -|u||v| \cosh \alpha \tag{2.2.1}$$

for some unique number $\alpha \geq 0$ which is called the hyperbolic angle between u and v.

Now, to define the angle between any two vectors $u, v \in \mathbb{E}^3_1$, we first assume that u and v are linearly independent and they are not null. Let W be the plane generated by u and v. The induced metric on W can be Riemannian, Lorentzian or degenerate.

In the Riemannian case, the angle between two spacelike vectors is defined same as the usual definition in Euclidean space.

In the Lorentzian case, the plane is isometric to $\mathbb{E}^2_1$, the Lorentz-Minkowski plane, and an isometry preserves the angle. The angle between two timelike vectors lying in the same timelike cone is defined in Definition 2.2.1.

$\mathbb{U}^2_1$, the set of unit vectors in $\mathbb{E}^2_1$, has four components,

$$\begin{aligned}
\mathbb{H}^1_- &= \{(x, y) \in \mathbb{E}^2_1 | x^2 - y^2 = -1, y < 0\}, \\
\mathbb{H}^1_+ &= \{(x, y) \in \mathbb{E}^2_1 | x^2 - y^2 = -1, y > 0\}, \\
\mathbb{S}^{1-}_1 &= \{(x, y) \in \mathbb{E}^2_1 | x^2 - y^2 = 1, x < 0\}, \\
\mathbb{S}^{1+}_1 &= \{(x, y) \in \mathbb{E}^2_1 | x^2 - y^2 = 1, x > 0\}.
\end{aligned}$$

$\mathbb{H}^1_- \cup \mathbb{H}^1_+$ consists of the timelike vectors and $\mathbb{S}^{1-}_1 \cup \mathbb{S}^{1+}_1$ consists of the spacelike vectors.

Definition 2.2.11. Assume that $u, v \in \mathbb{E}^2_1$ are non-zero spacelike vectors such that $\frac{u}{|u|}$ and $\frac{v}{|v|}$ are in the same component of $\mathbb{U}^2_1$. The angle α between u and v is uniquely defined by

$$\cosh \alpha = \frac{\langle u, v \rangle}{|u||v|}. \tag{2.2.2}$$

Note that we do not define the angle between two vectors if they do not lie in the same component of $\mathbb{U}^2_1$.

Finally, in the case that the plane is degenerate, again, we do not define the angle between the vectors.

Now, we define the *time orientation* for the Lorentz-Minkowski space. Let $\mathcal{B}$ denote the set of all ordered orthonormal bases such that $B = \{e_1, e_2, e_3\} \in \mathcal{B}$ and e_3 is timelike. Then the equivalence relation $\sim$ between two basis $B = \{e_1, e_2, e_3\}$ and $B' = \{e'_1, e'_2, e'_3\}$ is defined as

$$B \sim B', \text{ if } e_3 \text{ and } e'_3 \text{ are in the same timelike cone, i.e., } \langle e_3, e'_3 \rangle < 0.$$

There are two equivalence classes determined by this equivalence relation. these equivalence classes are called the *timelike orientations*. An ordered pair $(\mathbb{E}^3_1, [B])$ for some $B \in \mathcal{B}$ is said to be *timelike oriented*.

Definition 2.2.12. Let $B_u = \{E_1, E_2, E_3\}$ with $E_3 = (0, 0, 1)$. A timelike vector $v = (v_1, v_2, v_3)$ is called *future directed* (respectively *past directed*) if $v \in C(E_3)$, that is, $v_3 > 0$ (respectively $v \in C(-E_3)$, that is, $v_3 < 0$).

Conversely, if we fix the timelike cone $C(E_3)$, then an orthonormal basis $B = (e_1, e_2, e_3)$ is said to be future directed if $e_3 \in C(E_3)$ or equivalently, if e_3 is future directed.

Now we define the *Lorentzian vector product*.

Definition 2.2.13. The *Lorentzian vector product* of any two vectors $u, v \in \mathbb{E}^3_1$ is the vector which is unique and denoted by $u \times v$ that satisfies

$$\langle u \times v, w \rangle = \det(u, v, w), \tag{2.2.3}$$

where $w \in B_u$ and $\det(u, v, w)$ is the determinant of the matrix consists of the vectors u, v and w as its rows.

The Lorentzian vector product $u \times v$ can be obtained in coordinates with respect to the basis B_u as follows:

$$u \times v = \begin{vmatrix} i & j & -k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}. \tag{2.2.4}$$

Remark 2.2.14. Let u and v be non-degenerate vectors. $B = \{u, v, u \times v\}$ forms a basis for $\mathbb{E}_1^3$ and the time orientation of B is determined by the causal characters of u and v, that is, if u and v are both spacelike, then $u \times v$ is timelike and B is negatively oriented. If u and v have different causalities, then B is positively oriented.

2.3 Curves in Minkowski Space

In this section we define the curves in Minkowski space and their causalities. Then we construct the theory of the Frenet trihedron for curves in $\mathbb{E}_1^3$. We followed the book [12] and the paper [9] for the basic definitions.

A (smooth) curve γ is a smooth map $\gamma\colon I \to \mathbb{E}_1^3$ where $I \subset \mathbb{R}$ is an open interval.

A curve is called *regular* if $\gamma'(t) \neq 0$ for all $t \in I$. Any timelike or lightlike curve is regular.
Throughout this thesis any curve γ is assumed to be smooth and regular.

Definition 2.3.1. Let γ be a regular curve in M. γ is said to be

(i) spacelike at t if $\langle \gamma'(t), \gamma'(t) \rangle > 0$,

(ii) timelike at t if $\langle \gamma'(t), \gamma'(t) \rangle < 0$,

(iii) lightlike at t if $\langle \gamma'(t), \gamma'(t) \rangle = 0$.

The curve γ is spacelike (respectively timelike, lightlike) if $\gamma'(t)$ is spacelike (respectively timelike, lightlike) for all $t \in I$.

Note that a curve may not be one of these types. For example, let $\gamma\colon \mathbb{R} \to \mathbb{E}_1^3$ be defined as:

$$\gamma(t) = (\sin t, \cos t, \frac{t^2}{2}).$$

$\gamma'(t) = (\cos t, -\sin t, t)$, so γ is a regular curve. As $\langle \gamma', \gamma' \rangle = 1 - t^2$, the curve γ is timelike in $(-\infty, -1) \cup (1, \infty)$, spacelike in $(-1, 1)$ and lightlike in $\{-1, 1\}$.

Example 2.3.2. Let γ be a plane curve, i.e., it lies in an affine plane of $\mathbb{R}^3$. We determine its causal character.

(1) If γ is a straight-line, that is, $\gamma(t) = p + tv$, $p, v \in \mathbb{R}^3$ with $v \neq 0$, then its causal character is the same as the vector v.

(2) If γ is a circle of radius r and in the form $\gamma(t) = r(\cos t, \sin t, 0)$, then it is a spacelike curve lying in the spacelike plane define by $z = 0$.

(3) If γ is a hyperbola in the form $\gamma(t) = r(0, \sinh t, \cosh t)$, then it is a spacelike curve lying in the timelike plane define by $x = 0$.

(4) If γ is a hyperbola in the form $\gamma(t) = r(0, \cosh t, \sinh t)$, then it is a timelike curve lying in the timelike plane define by $x = 0$.

(5) If γ is a parabola in the form $\gamma(t) = (t, t^2, t^2)$, then it is a spacelike curve lying in the lightlike plane define by $y - z = 0$.

Example 2.3.3. Let γ be a spatial curve and $r > 0$, $h \neq 0$.

(1) The helix $\gamma(t) = (r \cos t, r \sin t, ht)$ of pitch $2\pi h$. This curve lies in the cylinder defined by $x^2 + y^2 = r^2$. γ is a timelike (respectively spacelike, lightlike) curve if $r^2 < h^2$ (respectively $r^2 > h^2$, $r^2 = h^2$).

(2) The curve $\gamma = (ht, r \sinh t, r \cosh t)$ is a spacelike and lies in the hyperbolic cylinder defined by $y^2 - z^2 = -r^2$.

(3) The curve $\gamma = (ht, r \cosh t, r \sinh t)$ lies in the hyperbolic cylinder defined by $y^2 - z^2 = r^2$. γ is timelike (respectively spacelike, lightlike) if $h^2 - r^2 < 0$ (respectively $h^2 - r^2 > 0$, $h^2 - r^2 = 0$).

Proposition 2.3.4. *Let $\gamma \colon I \to \mathbb{E}^3_1$ be a non-spacelike curve. There exist smooth functions $f, g \colon J \subset \mathbb{R} \to \mathbb{R}$ such that $t = \phi(s)$ and*

$$\psi(s) = \gamma(\phi(s)) = (f(s), g(s), s).$$

Let $\gamma\colon \mathbb{R} \to \mathbb{E}_1^3$ be a closed curve. There exists a minimum value $T > 0$ such that $\gamma(T+t) = \gamma(t)$, i.e., γ is periodic. Particularly, the trace of γ is compact.

Theorem 2.3.5. *Let γ be a closed curve. If γ is a spacelike curve lying on a plane $W \subset \mathbb{E}_1^3$, then the plane W is spacelike.*

The theory of closed spacelike plane curves is same as the theory in an Euclidean plane since a closed spacelike plane curve is isometric to a closed curve in an Euclidean plane. However there is no such theory of closed non-spacelike curves.

Theorem 2.3.6. *There do not exist closed timelike or lightlike curves in $\mathbb{E}_1^3$.*

Proposition 2.3.7. *For any timelike or spacelike curve $\gamma\colon I \to \mathbb{E}_1^3$ with given $t_0 \in I$, there exists a reparametrization $\psi = \gamma \circ \phi$ satisfying $|\psi'(s)| = 1$ for all $s \in (-\epsilon, \epsilon)$ for some $\epsilon, \delta > 0$ and a diffeomorphism $\phi\colon (-\epsilon, \epsilon) \to (t_0 - \delta, t_0 + \delta)$.*

Lemma 2.3.8. *If the curve γ is lightlike and its trace is different than a straight-line, then we can reparametrize γ as $\psi = \gamma \circ \phi$ such that $|\psi''(s)| = 1$. This parametrization is called pseudo-parametrization by arc length.*

Remark 2.3.9. Let $\psi = \gamma \circ \phi$ be a reparametrization of a curve γ. Then γ and ψ have the same causal character.

2.4 Curvature and Torsion of Curves in Minkowski Space

We want to find a basis for $\mathbb{E}_1^3$ and describe the geometry of a curve γ with the variation of the basis at each point of the curve. We will use the Frenet trihedron $\{T(s), N(s), B(s)\}$ to do this. The Frenet frame is an orthonormal basis of Euclidean space with $B = T \times N$ and this basis is positively oriented. See [9] for details.

Let the curve $\gamma \in \mathbb{E}^3_1$ be parametrized by the arc length or pseudo arc length. Recall that $T(s)$ is the velocity vector of γ, that is, $\gamma'(s) = T(s)$. There are some problems appearing in the Minkowski space.

(1) Let γ be a lightlike curve. Then T is a lightlike vector and therefore, $\{T, N, B\}$ is not an orthonormal basis. In this case, we will use the concept of null frame.

(2) Let $\{T, N, B\}$ be an orthonormal basis for $\mathbb{E}^3_1$, where we define $B = T \times N$. Then the basis $\{T, N, B\}$ may not be positively oriented. As an example, if T and N are spacelike, then the basis is negatively oriented.

(3) The orthonormal basis $\{T, N, B\}$ is not necessarily future directed.

Consider a straight-line parametrized by $\gamma(s) = p + sv$. Then $\gamma''(s) = 0$. In this case, the curvature of γ is said to be 0.

Conversely, let γ be a regular curve satisfying $\gamma''(s) = 0$ for all s. Then, by an integration, γ is found as $\gamma(s) = p + sv$, a straight-line, for some $p, v \in \mathbb{E}^3_1$, $v \neq 0$. Observe that there are other parametrizations of a straight-line, such as $\gamma(s) = (s^3 + s^2, 0, 0)$ where $\gamma'' \neq 0$.

Consider a curve $\gamma \colon I \to \mathbb{E}^3_1$ parametrized by the arc length or pseudo arc length. The vector

$$T(s) = \gamma'(s)$$

is called the *tangent vector* at s. Then $\langle T(s), T(s) \rangle$ is constant, in fact, equals $-1, 0$ or 1. After differentiating the inner product with respect to s, we obtain $\langle T(s), T'(s) \rangle = 0$, and this means that $T'(s)$ is orthogonal to $T(s)$. To avoid the straight-lines, we assume that $T'(s) \neq 0$ for all $s \in I$ and $T'(s) \neq \lambda T(s)$ for all $\lambda \in \mathbb{R}$ and each s.

Depending on the causality of $T(s)$, we find the Frenet equations. In the following discussion we assume that the curve γ is parametrized by the arc length or pseudo arc length.

(1) If the curve γ is timelike, then $T(s) = \gamma'(s)$ is a timelike vector and $T'(s)$ is spacelike by Proposition 2.2.3. Since $T'(s)$ is spacelike, it is different than zero and $T(s)$ and $T'(s)$ are linearly independent. The curvature of γ at s is defined as

$$\kappa(s) = |T'(s)|.$$

$N(s)$, the normal vector at s, is defined by

$$N(s) = \frac{T'(s)}{\kappa(s)}.$$

This gives $T'(s) = \kappa(s)N(s)$. Furthermore $\kappa(s) = \langle T'(s), N(s)\rangle$. The binormal vector $B(s)$ is defined by

$$B(s) = T(s) \times N(s).$$

This gives a unitary spacelike binormal vector. $\{T(s), N(s), B(s)\}$ forms an orthonormal basis for $\mathbb{E}_1^3$ for each s and is called the *Frenet trihedron* of γ at s. This basis is positively oriented since

$$\det(T, N, B) = \langle T \times N, B\rangle = \langle B, B\rangle = 1 > 0.$$

The *torsion* τ of γ at s is defined by

$$\tau(s) = \langle N'(s), B(s)\rangle.$$

We differentiate the vector functions T, N, B and write them in coordinates by using the Frenet basis. Finally, we have the Frenet equations (Frenet formula), that is,

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}. \tag{2.4.1}$$

(2) If the curve γ is spacelike, then $T(s) = \gamma'(s)$ is a spacelike vector and $T'(s)$ may be timelike, spacelike or lightlike by Proposition 2.2.3. We analyze each case.

(i) If $T'(s)$ is timelike, the curvature of γ is

$$\kappa(s) = |T'(s)| = \sqrt{-\langle T'(s), T'(s)\rangle}.$$

The normal vector is

$$N(s) = \frac{T'(s)}{\kappa(s)},$$

and the binormal vector is

$$B(s) = T(s) \times N(s).$$

Note that $B(s)$ is a spacelike vector and so, the Frenet basis is positively oriented.

The torsion of γ is $\tau(s) = \langle N'(s), B(s)\rangle$. The Frenet equations are obtained as

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}. \tag{2.4.2}$$

(ii) If $T'(s)$ is spacelike, we find the curvature of γ as

$$\kappa(s) = |T'(s)|,$$

the normal vector as

$$N(s) = \frac{T'(s)}{\kappa(s)},$$

and the binormal vector as

$$B(s) = T(s) \times N(s).$$

In this case $B(s)$ is a timelike vector and therefore, the Frenet basis is negatively oriented since

$$\det(T, N, B) = \langle T \times N, B\rangle = \langle B, B\rangle = -1 < 0.$$

We obtain the Frenet equations as

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}. \tag{2.4.3}$$

The torsion of γ is

$$\tau(s) = -\langle N'(s), B(s)\rangle.$$

(iii) If the vector $T'(s)$ is lightlike, the curvature of γ is not defined. There is the definition of *pseudo-torsion* τ of γ which is obtained by

$$\tau = -\langle N', B\rangle$$

where the normal vector is defined as $N(s) = T'(s)$ and the binormal vector is defined as the unique lightlike vector $B(s)$ satisfying $\langle N(s), B(s)\rangle = -1$. The Frenet equations are

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \tau & 0 \\ 1 & 0 & -\tau \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}. \tag{2.4.4}$$

Note that N and T are linearly independent and B and T are orthogonal to each other. However, $\{T(s), N(s), B(s)\}$ does not form a basis for $\mathbb{E}^3_1$ because $N(s)$ and $B(s)$ are lightlike, it is a *null frame* and we do not know its time-orientation.

(3) Let the curve γ be lightlike. Then the tangent vector is $T(s) = \gamma'(s)$ and the normal vector is defined as $N(s) = T'(s)$ which is spacelike. The binormal vector is defined as the unique lightlike vector orthogonal to $N(s)$ and satisfying $\langle T(s), B(s)\rangle = -1$. The vectors T and B are null, therefore $\{T, N, B\}$ is a null frame of $\mathbb{E}^3_1$. The *pseudo-torsion* of γ is defined as

$$\tau = -\langle N', B\rangle.$$

However, the curvature of the curve is not defined in this case such as the spacelike curves with lightlike normal vector.
The Frenet equations for this curve are

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \tau & 0 & 1 \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}. \tag{2.4.5}$$

Remark 2.4.1. A curve may not have the same causal character in the whole interval I, that is, $T'(s)$ may have changing causal characters in I. In the above discussion, it is assumed that the causal character of $\gamma(s)$, so the causal character of $T'(s)$ is the same in whole I.

Example 2.4.2. [9]

(1) Consider the timelike curve

$$\gamma(s) = \left(\frac{hs}{\sqrt{r^2-h^2}}, r\cosh\left(\frac{s}{\sqrt{r^2-h^2}}\right), r\sinh\left(\frac{s}{\sqrt{r^2-h^2}}\right)\right)$$

with $r^2 - h^2 > 0$. Then

$$T(s) = \gamma'(s) = \frac{1}{\sqrt{r^2-h^2}}\left(h, r\sinh\left(\frac{s}{\sqrt{r^2-h^2}}\right), r\cosh\left(\frac{s}{\sqrt{r^2-h^2}}\right)\right).$$

$T(s)$ is a timelike and future directed vector. Then we have

$$T'(s) = \frac{r}{r^2-h^2}\left(0, \cosh\left(\frac{s}{\sqrt{r^2-h^2}}\right), \sinh\left(\frac{s}{\sqrt{r^2-h^2}}\right)\right)$$

and

$$\kappa = \frac{r}{r^2-h^2}.$$

Hence

$$N(s) = \frac{T'(s)}{\kappa} = \left(0, \cosh\left(\frac{s}{\sqrt{r^2-h^2}}\right), \sinh\left(\frac{s}{\sqrt{r^2-h^2}}\right)\right).$$

It follows that

$$B(s) = \frac{1}{\sqrt{r^2-h^2}}\left(-r, -h\sinh\left(\frac{s}{\sqrt{r^2-h^2}}\right), -h\cosh\left(\frac{s}{\sqrt{r^2-h^2}}\right)\right).$$

We obtain

$$\tau = \langle N', B\rangle = \frac{h}{r^2-h^2}.$$

(2) Let $\gamma(s) = r(0, \cosh(s/r), \sinh(s/r))$. Then we have

$$T(s) = \left(0, \sinh\left(\frac{s}{r}\right), \cosh\left(\frac{s}{r}\right)\right) \text{ and}$$

$$T'(s) = \frac{1}{r}\left(0, \cosh\left(\frac{s}{r}\right), \sinh\left(\frac{s}{r}\right)\right)$$

We find $\kappa = 1/r$. Furthermore,

$$N(s) = \left(0, \cosh\left(\frac{s}{r}\right), \sinh\left(\frac{s}{r}\right)\right)$$

and $B(s) = (-1, 0, 0)$. Hence, $\tau = 0$.

(3) Let $\gamma(s) = \dfrac{1}{r^2}(\cosh(rs), rs, \sinh(rs))$. We find the vectors T, N, B as follows:

$$\begin{aligned} T(s) &= \frac{1}{r}(\sinh(rs), 1, \cosh(rs)), \\ N(s) = T'(s) &= (\cosh(rs), 0, \sinh(rs)), \\ B(s) &= \frac{r}{2}(\sinh(rs), -1, \cosh(rs)). \end{aligned}$$

It follows that the pseudo-torsion is $\tau = -\dfrac{r^2}{2}$.

2.5 Exponential Map and Operators

In this section, we briefly define the *exponential map* and recall the definitions of some basic operators. See [6] for details.

Definition 2.5.1. Let M be a semi-Riemannian manifold, $p \in M$ and $\mathcal{D}_p$ be the set of vectors $v \in T_p(M)$. There exists a unique *maximal* (i.e., *inextendible*) geodesic $\gamma_v \colon I \to M$ defined at least on $[0,1]$ such that $\gamma_v(0) = p$ and $\gamma_v'(0) = v$. The map

$$\exp_p \colon \mathcal{D}_p \to M,$$

defined as $\exp_p(v) = \gamma_v(1)$ for all v, is called the *exponential map* [6].

Note that $\mathcal{D}_p$ is the largest subset of $T_p(M)$ such that $\exp_p$ can be defined on it. $\mathcal{D}_p = T_p(M)$ for all $p \in M$ if M is complete.

Let $v \in \mathcal{D}_p$ and $t \in \mathbb{R}$. Then

$$\exp_p(tv) = \gamma_{tv}(1) = \gamma_v(t).$$

This means that the lines passing through the origin of $T_p(M)$ are carried to geodesics of M passing through p by the exponential map $\exp_p$.

Proposition 2.5.2. *Let $\{e_1, \ldots, e_n\}$ be any basis of $T_p(M)$. For $(x^1, \ldots, x^n)$ in a neighborhood of the origin, the map*

$$x^1 e_1 + \cdots + x^n e_n \longrightarrow \exp_p(x^1 e_1 + \cdots + x^n e_n)$$

is a diffeomorphism from $\mathcal{D}_0$ onto a neighborhood $U \subset M$ of p.

The point $(x^1, \dots, x^n)$ and $\exp_p(x^1 e_1 + \cdots + x^n e_n)$ in U together gives us a coordinate chart for M, called *normal coordinates based at p* for U. If for any two points in U there exists a unique geodesic segment of (M, g) between these points contained entirely in U, then the set U is called a *convex neighborhood* of p. One can prove that for each point p in a semi-Riemannian (hence Lorentzian) manifold, there exist arbitrarily small convex neighborhoods of p [16]. The set U is called a *convex normal neighborhood* if for each $q \in U$, there are normal coordinates based at q containing U [17].

Definition 2.5.3. For an ordered pair $X, Y \in \mathfrak{X}(M)$, the *Lie bracket* is a vector field $[X, Y] \in \mathfrak{X}(M)$ such that for a smooth function f

$$[X, Y](f) = X(Y(f)) - Y(X(f)).$$

Specifically, for $X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}$ and $Y = \sum_{i=1}^n Y^i \frac{\partial}{\partial x^i}$,

$$[X, Y] = \sum_{i=1}^n \sum_{j=1}^n \big(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j}\big) \frac{\partial}{\partial x^i}.$$

The metric $g \in \mathfrak{T}_2^0(M)$ is represented in the local coordinates $(U, (x^1, \dots, x^n))$ in M by

$$g|_U = \sum_{i,j=1}^n g_{ij}(x)\, dx^i \otimes dx^j \qquad \text{or} \qquad \big(g|_U\big) = \big(g_{ij}(x)\big)_{n \times n}.$$

Assume that $f \colon M \to \mathbb{R}$ is a smooth function. The *gradient* of f, denoted by $grad(f) \in \mathfrak{T}_1^0(M)$, is defined as

$$Y(f) = df(Y) = g(grad(f), Y)$$

where $Y \in \mathfrak{X}(M)$ is any vector field and $df \in \mathfrak{T}_1^0(M)$. In local coordinates $(U, (x^1, \dots, x^n))$,

$$grad(f) = \sum_{i,j=1}^n g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}.$$

2.6 Connections and Curvature

In differential geometry, it is too difficult to describe the infinitesimal geometry of a manifold of dimension at least three by a single number at a given point. Riemann found an abstract and rigorous way, now known as the curvature tensor. Similar notions have found applications everywhere in differential geometry. In this section, we define these notions.

Definition 2.6.1. Let M be a differentiable manifold.
A map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ is called an *affine connection* in M if

(i) $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$;

(ii) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$;

(iii) $\nabla_X(fY) = X(f)Y + f\nabla_X Y$

for all $X, Y, Z \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M, \mathbb{R})$ (note that $\nabla_X Y := \nabla(X, Y)$).

The vector field $\nabla_X Y$ is also known as the *covariant derivative* of Y along X. [8]

Theorem 2.6.2. *On a semi-Riemannian manifold (M, g) there is a unique connection ∇ which is symmetric and compatible with g such that*

(i) $[X, Y] = \nabla_X Y - \nabla_Y X$ *and*

(ii) $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$,

for all $X, Y, Z \in \mathfrak{X}(M)$. ∇ is called the Levi-Civita connection of M and satisfies the Koszul formula

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle =& X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ & - \langle X, [Y, Z]\rangle + \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle. \end{aligned}$$

Definition 2.6.3. In local coordinates $(x^1, \ldots, x^n)$, the Christoffel symbols for the Levi-Civita connection are the real-valued functions Γ^i_{jk} on a neighborhood $U \subset M$ such that

$$\nabla_{\partial_i}\partial_j = \sum_k \Gamma^k_{ij}\partial_k \quad (1 \leq i, j \leq n) \tag{2.6.1}$$

where

$$\Gamma^i_{jk} = \frac{1}{2}\sum_{l=1}^{n} g^{il}\left(\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right) \tag{2.6.2}$$

and $\partial_i, \partial_j, \partial_k$ are the coordinate vector fields $\partial/\partial x^i, \partial/\partial x^j, \partial/\partial x^k$, respectively and $g^{ij} = (g_{ij})^{-1}$ [8].

On a semi-Riemannian manifold (M, g) with the Levi-Civita connection, the function $R : \mathfrak{X}(M)^3 \longrightarrow \mathfrak{X}(M)$ defined as

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \tag{2.6.3}$$

is a $(1,3)$ tensor field in M called the *Riemannian curvature tensor* of M. There is an alternative notation for $R(X,Y)Z$ as $R_{XY}Z$.

If $X, Y \in T_p(M)$, the $f-$linear operator

$$R(X,Y) : T_p(M) \longrightarrow T_p(M)$$

sending each Z to $R(X,Y)Z$ is called a *curvature operator.*
The Riemannian curvature tensor has the following *symmetries:*

Proposition 2.6.4. *[6] For $X, Y, Z, W \in T_p(M)$,*

1. $R(X,Y) = -R(Y,X)$,
2. $\langle R(X,Y)Z, W\rangle = -\langle R(X,Y)W, Z\rangle$,
3. $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ *(First Bianchi identity),*
4. $\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle$ *(Symmetry by pairs).*

Lemma 2.6.5. *[6] On the coordinate neighborhood of a coordinate system $x^1, \dots, x^n$,*

$$R(\partial_i, \partial_j)\partial_k = \sum_l R^l_{kij}\partial_l,$$

where the components of R are given by

$$R^l_{kij} = \frac{\partial}{\partial_j}\Gamma^l_{ik} - \frac{\partial}{\partial_i}\Gamma^l_{jk} + \sum_m \Gamma^l_{jm}\Gamma^m_{ik} - \sum_m \Gamma^l_{im}\Gamma^m_{jk}.$$

Definition 2.6.6. A semi-Riemannian manifold M is said to be *flat* if its curvature tensor R is zero at every point $p \in M$.

The trace of the Riemann curvature tensor is the *Ricci curvature*, a symmetric $(0, 2)$ tensor. The components of the Ricci curvature are

$$R_{ij} = \sum_l R^l_{ilj}. \tag{2.6.4}$$

Let $\{E_1, \dots, E_n\}$ be an *orthonormal frame field.* Then the Ricci curvature can be defined as

$$Ric(X, Y) = \sum_{m=1}^{n} \varepsilon_m \langle R(E_m, Y)X, E_m\rangle \tag{2.6.5}$$

where $X, Y \in \mathfrak{X}(M)$ and $\varepsilon_m = \langle E_m, E_m\rangle$.

The *Ricci tensor* ρ is the $(1, 1)$ tensor field that corresponds to the Ricci curvature. The components of the Ricci tensor can be obtained as follows:

$$R^i_j = \sum_{m=1}^{n} g^{mi}R_{mj} = \sum_{m=1}^{n} g^{mi}R_{jm}. \tag{2.6.6}$$

Ricci curvature plays a key role in general relativity since it is the one which enters into the Einstein field equations rather than the full Riemann curvature [18]. A manifold (M, g) is said to be an *Einstein manifold* if its Ricci tensor is proportional to the metric tensor g at every point $p \in M$, i.e, $\rho = \lambda g$. In particular, M is called *Ricci flat* if its Ricci tensor is identically zero. A flat manifold is certanly Ricci flat, but the converse does not hold [6].

The trace of the Ricci tensor is the *scalar curvature* τ.

$$\tau = \sum_{i=1}^{n} R_i^i = \sum_{i,j=1}^{n} g^{ij} R_{ij}. \tag{2.6.7}$$

2.7 Causality

In this section, we briefly state some basic definitions about the causality of Lorentzian manifolds.

Definition 2.7.1. The *causal character* of a nonzero vector $x \in T_pM$ is

- *timelike* if $g(x,x) < 0$,
- *null* or *lightlike* if $g(x,x) = 0$,
- *spacelike* if $g(x,x) > 0$.

The set of all null vectors in T_pM is called the *light cone* at p that is:

$$\mathcal{C} = \{X \in \mathbb{E}_1^3 | g(x,x) = 0\} - \{(0,0,0)\}.$$

If M has a timelike vector field X, then we say M is time-oriented by X. In this case, a non-spacelike tangent vector $Y_p \in T_p(M)$ is called *future directed* (respectively, *past directed*) if $g(X_p, Y_p) < 0$ (respectively, $g(X_p, Y_p) > 0$). A *space-time* (M, g) is a time-oriented *Lorentzian manifold.*

Let $p, q \in M$. If there exists a smooth future directed timelike curve from p to q, we write $p \ll q$. If $p = q$ or if there exists a smooth future directed non-spacelike curve from p to q, we write $p \leqslant q$. The *chronological future* of p is defined as

$$I^+(p) = \{q \in M | p \ll q\}$$

and the *chronological past* is defined as

$$I^-(p) = \{q \in M | q \ll p\}.$$

The *causal future* of p is defined as

$$J^+(p) = \{q \in M | p \leqslant q\}$$

and the *causal past* is

$$J^-(p) = \{q \in M | q \leqslant p\}.$$

The sets $I^+(p)$ and $I^-(p)$ are always open in any space-time, while $J^+(p)$ and $J^-(p)$ are neither open nor closed in general.

A space-time (M, g) is called *chronological* if $p \notin I^+(p)$ for all $p \in M$, i.e., there does not exist any closed timelike curves in (M, g). (M, g) is called *causal* if there exists no pair of distinct points $p, q \in M$ satisfying $p \leqslant q \leqslant p$. This is equivalent to that (M, g) contains no closed non-spacelike curves. Note that if M is compact, then (M, g) contains a closed timelike curve. Therefore in general relativity, the space-times are assumed to be non-compact.

Let U be an open set in a space-time. U is called *causally convex* if it does not intersect with a non-spacelike curve in a disconnected set. A space-time (M, g) is called *strongly causal* at $p \in M$ if all neighborhoods of p are causally convex. If a space-time is strongly causal at each point, it is said to be a *strongly causal space-time*.

Let g_1 and g_2 be strongly causal Lorentzian metrics in M. The future and past sets are the same at all points for g_1 and g_2 if and only if $g_1 = \omega g_2$ for some smooth function $\omega : M \longrightarrow (0, \infty)$, i.e., g_1 and g_2 are globally conformal. Let $C(M, g)$ denote the set of all Lorentzian metrics globally conformal to g. Thus, the basic properties of the causality theory is independent of the choice of Lorentzian metric representing $C(M, g)$.

Definition 2.7.2. Let (M, g) be a Lorentzian manifold and $p, q \in M$, with $p \leqslant q$. If $\gamma \colon [0, 1] \to M$ is a future directed piecewise smooth non-spacelike curve from p to q differentiable except at $0 = t_1 < t_2 < \cdots < t_k = 1$, the *Lorentzian arc length* of γ, $L_g(\gamma)$, is defined as

$$L_g(\gamma) = \sum_{i=1}^{k-1} \int_{t_i}^{t_{i+1}} \sqrt{-g(\gamma'(t), \gamma'(t))}dt.$$

Finally, the *Lorentzian distance* $d : M \times M \to \mathbb{R} \cup \{\infty\}$ is defined as:

$$d(p,q) = \begin{cases} 0 & \text{if } q \notin J^+(p), \\ \sup\{L_g(\gamma) | \gamma \in \Omega_{p,q}\} & \text{if } q \in J^+(p) \end{cases}$$

where $p, q \in M$ and $\Omega_{p,q}$ is the set of all future directed piecewise smooth non-spacelike curves.

In Lorentzian manifolds, if $p \leqslant q \leqslant r$ then

$$d(p,q) + d(q,r) \leq d(p,r)$$

which is known as the *reverse triangle inequality.*

2.8 Completeness

In this section, we briefly state some basic terms about *completeness* of Lorentzian manifolds.

A (smooth) *curve* on a manifold M is a smooth map $\gamma\colon I \longrightarrow M$ where $I \subset \mathbb{R}$ is an open interval. The curve γ is called *regular* if $\gamma'(t) \neq 0$ for all $t \in I$.

Throughout the manuscript, any curve γ is assumed to be regular.

Definition 2.8.1. Let M be a Lorentzian manifold. A smooth curve $\gamma\colon I \longrightarrow M$ is called *geodesic* if its vector field $\gamma'(t)$ is parallel, i.e., its acceleration is zero: $\gamma''(t) = 0$ for all $t \in I$ where $\gamma''(t) = \nabla_{\gamma'}\gamma'(t)$. Also note that if γ is a geodesic, then $g(\gamma'(t), \gamma'(t))$ is constant for all $t \in I$.

A smooth curve $\gamma\colon I \to M$ in a semi-Riemannian manifold is called a *pre-geodesic* if there exists a repametriziation of it which is a geodesic. A parameter s is called an affine parameter for a pre-geodesic γ if $\gamma''(s) = 0$.

A smooth curve $\gamma\colon (a,b) \to M$ in a semi-Riemannian manifold is said to be *inextendible* to $t = a$ (respectively to $t = b$) if the $\lim_{t \to a^+} \gamma(t)$ (respectively $\lim_{t \to b^-} \gamma(t)$) does not exist.

Recall that if γ is a geodesic, then its tangent vector field $\gamma'(t)$ has the same causal character for all $t \in I$. Thus, a geodesic is called spacelike (respectively timelike, null) depending on the causal character of its tangent vector $\gamma'(t)$ for any $t \in I$.

The Hopf-Rinow Theorem ([7]) states that in a Riemannian manifold, the metric completeness and geodesic completeness are equivalent.

Definition 2.8.2. Let (M, g) be a Lorentzian manifold and $\gamma(t)$ be a geodesic on it. γ is called *complete* if it can be defined on $\mathbb{R}$.

The Lorentzian manifold (M, g) is spacelike (respectively timelike, null) complete if all spacelike (respectively timelike, null) inextendible geodesics are complete. A space-time is called a *geodesically singular space-time* if it is non-spacelike incomplete. Note that spacelike completeness, timelike completeness and null completeness are independent ([7]).

A Lorentzian manifold (M, g) is called *geodesically complete* if every geodesic curve γ in M can be defined on $\mathbb{R}$.

Chapter 3

Three-Dimensional Lorentzian Walker Manifolds

In this chapter, we give the definition of a Walker manifold and a strict Walker manifold. Then we investigate their connection components and curvature tensors.

Let M be a semi-Riemannian manifold and $T(M) = \mathcal{V}_1 \oplus \mathcal{V}_2$, where $\mathcal{V}_1$ and $\mathcal{V}_2$ are smooth subbundles, called *distributions*. This defines two complementary projection π_1 and π_2 of $T(M)$ onto $\mathcal{V}_1$ and $\mathcal{V}_2$. $\mathcal{V}_1$ is said to be a *parallel distribution* if $\nabla\pi_1 = 0$. Equivalently this means that if X_1 is any smooth vector field taking values in $\mathcal{V}_1$, then ∇X_1 also takes values in $\mathcal{V}_1$. In the Riemannian setting, we can take $\mathcal{V}_2 = \mathcal{V}_1^{\perp}$ to be the orthogonal complement of $\mathcal{V}_1$ and then $\mathcal{V}_2$ is again parallel. If M is semi-Riemannian, then $\mathcal{V}_1 \cap \mathcal{V}_2$ is not necessarily trivial. $\mathcal{V}_1$ is said to be a null parallel distribution if $\mathcal{V}_1$ is parallel and if the induced metric on $\mathcal{V}_1$ vanishes identically. A manifold is called a *Walker manifold* if it admits a null parallel distribution and it is called a *strictly Walker manifold* if this distribution is spanned by a null vector [4], [2].

Definition 3.0.3. A three-dimensional *Walker manifold* (M, g_f) is a three-dimensional Lorentzian manifold admitting a parallel degenerate line field with the local coordinates (x, y, z) where the metric tensor is expressed as

$$g_f = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & f \end{pmatrix}, \text{ with the inverse } \quad g_f^{-1} = \begin{pmatrix} -f & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{3.0.1}$$

for some smooth function $f(x, y, z)$ defined on the manifold M and $\varepsilon = \pm 1$. Throughout this work we will take $\varepsilon = -1$. Note that the Walker manifold (M, g_f) has signature $(2, 1)$ if $\varepsilon = 1$ and $(1, 2)$ if $\varepsilon = -1$ and is Lorentzian in both cases.

Throughout this manuscript M is assumed to be a three-dimensional Walker manifold.

3.1 Vector Product

We defined the Lorentzian vector product (see Definition 2.2.13). Now we will define the vector product for the special case, that is, the metric is a Walker metric g_f.

Let $u, v \in M$. The vector product of u and v, $u \times v$, is the unique vector satisfying

$$\langle u \times v, w \rangle = \det(u, v, w) = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{3.1.1}$$

where $w \in \{E_1, E_2, E_3\}$. We obtain the product vector as

$$u \times v = (u_1 v_2 - u_2 v_1 - (u_2 v_3 - u_3 v_2) f, u_1 v_3 - u_3 v_1, u_2 v_3 - u_3 v_2). \tag{3.1.2}$$

Now we can construct an orthonormal frame on (M, g_f). Let $u = (0, 1, 0)$ and $v = (0, 0, \frac{1}{\sqrt{|f|}})$. Then

$$u \times v = (-\sqrt{f}, 0, \frac{1}{\sqrt{f}}).$$

Note that $|u| = |v| = |u \times v| = 1$. Here we assume that $f > 0$, since one can study the case $f < 0$ similarly. We obtain an orthonormal frame

$$e_1 = -\sqrt{f}\partial_x + \frac{1}{\sqrt{f}}\partial_z, \quad e_2 = \partial_y, \quad e_3 = \frac{1}{\sqrt{f}}\partial_z. \tag{3.1.3}$$

Moreover,

$$\langle e_1, e_1 \rangle = -1, \quad \langle e_2, e_2 \rangle = -1, \quad \langle e_3, e_3 \rangle = 1.$$

$\{e_1, e_2, e_3\}$ forms an orthonormal basis for $T(M)$.

3.2 Connections and Curvature

We begin our investigation by calculating the Christoffel symbols and the Levi-Civita connection.

Proposition 3.2.1. *We can calculate the Christoffel symbols by using the formula in 2.6.2*

$$\Gamma^i_{jk} = \frac{1}{2}\sum_{l=1}^{n} g^{il}\left(\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right)$$

and we find the possible nonzero Christoffel symbols of a Walker metric g_f as follows,

$$\begin{aligned}
\Gamma^1_{13} = \Gamma^1_{31} &= \frac{1}{2}f_x, \\
\Gamma^1_{23} = \Gamma^1_{32} &= \frac{1}{2}f_y, \\
\Gamma^1_{33} &= \frac{1}{2}\left(f_z + f f_x\right), \\
\Gamma^2_{33} &= \frac{1}{2}f_y, \\
\Gamma^3_{33} &= -\frac{1}{2}f_x.
\end{aligned}$$

Proof. By using 2.6.2, we establish the the Christoffel symbols as follows.

$$\begin{aligned}
\Gamma^1_{11} &= \frac{1}{2}\sum_{l=1}^{n} g^{1l}\left(\frac{\partial g_{1l}}{\partial x} + \frac{\partial g_{1l}}{\partial x} - \frac{\partial g_{11}}{\partial x^l}\right) \\
&= \frac{1}{2}g^{11}\left(\frac{\partial g_{11}}{\partial x} + \frac{\partial g_{11}}{\partial x} - \frac{\partial g_{11}}{\partial x}\right) + \frac{1}{2}g^{12}\left(\frac{\partial g_{12}}{\partial x} + \frac{\partial g_{12}}{\partial x} - \frac{\partial g_{11}}{\partial y}\right) \\
&\quad - \frac{1}{2}g^{13}\left(\frac{\partial g_{13}}{\partial x} + \frac{\partial g_{13}}{\partial x} - \frac{\partial g_{11}}{\partial z}\right) \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\Gamma^1_{13} &= \frac{1}{2}\sum_{l=1}^{n} g^{1l}\left(\frac{\partial g_{3l}}{\partial x} + \frac{\partial g_{1l}}{\partial z} - \frac{\partial g_{13}}{\partial x^l}\right) \\
&= \frac{1}{2}g^{11}\left(\frac{\partial g_{31}}{\partial x} + \frac{\partial g_{11}}{\partial z} - \frac{\partial g_{13}}{\partial x}\right) + \frac{1}{2}g^{12}\left(\frac{\partial g_{32}}{\partial x} + \frac{\partial g_{12}}{\partial z} - \frac{\partial g_{13}}{\partial y}\right) \\
&\quad - \frac{1}{2}g^{13}\left(\frac{\partial g_{33}}{\partial x} + \frac{\partial g_{13}}{\partial z} - \frac{\partial g_{13}}{\partial z}\right) \\
&= \frac{1}{2}f_x.
\end{aligned}$$

Similarly, we can calculate all of the components Γ^i_{jk} of the Christoffel symbols and find as given above. □

Now, as we know the Christoffel symbols, we can calculate the components of the Levi-Civita connection.

Theorem 3.2.2. *The possible nonzero components of the Levi-Civita connection of any metric g_f are*

$$\begin{aligned}
\nabla_{\partial_x}\partial_z &= \frac{1}{2}f_x\partial_x \\
\nabla_{\partial_y}\partial_z &= \frac{1}{2}f_y\partial_x \\
\nabla_{\partial_z}\partial_z &= \frac{1}{2}(ff_x + f_z)\partial_x + \frac{1}{2}f_y\partial_y - \frac{1}{2}f_x\partial_z.
\end{aligned}$$

Proof. After a straightforward calculation by using (2.6.1) and Proposition 3.2.1,

$$\begin{aligned}
\nabla_{\partial_x}\partial_x &= \sum_k \Gamma^k_{11}\partial_k \\
&= \Gamma^1_{11}\partial_x + \Gamma^2_{11}\partial_y + \Gamma^3_{11}\partial_z \\
&= 0,
\end{aligned}$$

$$
\begin{aligned}
\nabla_{\partial_x}\partial_z &= \sum_k \Gamma_{13}^k \partial_k \\
&= \Gamma_{13}^1\partial_x + \Gamma_{13}^2\partial_y + \Gamma_{13}^3\partial_z \\
&= \frac{1}{2} f_x \partial_x.
\end{aligned}
$$

Similarly, we can find all components $\nabla_{\partial_i}\partial_j$ of the Levi-Civita connection of g_f as given above. $\square$

Theorem 3.2.3. *As given in (2.6.3),*

$$
R(\partial_i, \partial_j)\partial_k = \nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i}\partial_k - \nabla_{[\partial_i, \partial_j]}\partial_k, \tag{3.2.1}
$$

where $\partial_i, \partial_j, \partial_k$ is the coordinate basis, the Riemann curvature of the manifold M is determined by the following possibly non-zero components,

$$
\begin{aligned}
R(\partial_x, \partial_z)\partial_x &= \frac{1}{2} f_{xx}\partial_x, \\
R(\partial_x, \partial_z)\partial_y &= \frac{1}{2} f_{xy}\partial_x, \\
R(\partial_y, \partial_z)\partial_x &= \frac{1}{2} f_{xy}\partial_x, \\
R(\partial_y, \partial_z)\partial_y &= \frac{1}{2} f_{yy}\partial_x, \\
R(\partial_x, \partial_z)\partial_z &= \frac{1}{2} f f_{xx}\partial_x + \frac{1}{2} f_{xy}\partial_y - \frac{1}{2} f_{xy}\partial_z, \\
R(\partial_y, \partial_z)\partial_z &= \frac{1}{2} f f_{xy}\partial_x + \frac{1}{2} f_{yy}\partial_y - \frac{1}{2} f_{xy}\partial_z.
\end{aligned}
$$

Proof. By using (2.6.3),

$$
\begin{aligned}
R(\partial_x, \partial_y)\partial_x &= \nabla_{\partial_x}\nabla_{\partial_y}\partial_x - \nabla_{\partial_y}\nabla_{\partial_x}\partial_x - \nabla_{[\partial_x, \partial_y]}\partial_x \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
R(\partial_x, \partial_z)\partial_x &= \nabla_{\partial_x}\nabla_{\partial_z}\partial_x - \nabla_{\partial_z}\nabla_{\partial_x}\partial_x - \nabla_{[\partial_x, \partial_z]}\partial_x \\
&= \nabla_{\partial_x}(\frac{1}{2} f_x \partial_x) \\
&= \frac{1}{2} f_{xx}\partial_x.
\end{aligned}
$$

Similarly, all components $R(\partial_i, \partial_j)\partial_k$ of the Riemann curvature can be calculated as above. $\square$

Theorem 3.2.4. *The Ricci tensor of a three-dimensional Walker manifold (M, g_f) is*

$$\rho = \begin{pmatrix} \frac{1}{2}f_{xx} & \frac{1}{2}f_{xy} & \frac{1}{2}f_{yy} \\ 0 & 0 & -\frac{1}{2}f_{xy} \\ 0 & 0 & \frac{1}{2}f_{xx} \end{pmatrix}.$$

ρ has eigenvalues $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = \frac{1}{2}f_{xx}$.

Proof. To calculate the Ricci curvature, we use the formula given in (2.6.5), that is,

$$Ric(X, Y) = \sum_{m=1}^{n} \varepsilon_m \langle R(e_m, Y)X, e_m \rangle$$

and the orthonormal frame in (3.1.3). After a straightforward calculation by using these and Theorem 3.2.3, (2.6.6), we find the Ricci tensor as above. □

Corollary 3.2.5. *A three-dimensional Walker manifold (M, g_f) is flat (Definition 2.6.6) if and only if*

$$f(x, y, z) = x\alpha(z) + y\beta(z) + \zeta(z). \tag{3.2.2}$$

Theorem 3.2.6. *(M, g_f) is flat if and only if it is Ricci flat.*

Proof. It is clear that if M is flat, it is Ricci flat. Conversely, if M is Ricci flat, then we have

$$f_{xx} = f_{xy} = f_{yy} = 0,$$

which implies that the Riemann curvature tensor is zero and so M is flat. □

The trace of ρ is the scalar curvature τ of (M, g_f) and is

$$\tau = f_{xx}.$$

The Lorentzian three-manifolds admitting a parallel null vector field which is spanned by ∂_x are called *strictly Walker three-manifolds* . A strictly Walker three-manifold can be characterized by $f = f(y, z)$, i.e., f is independent of the parameter x (see [1]).

Corollary 3.2.7. *Let (M, g_f) be a strictly Walker three-manifold.*

1. *The non-zero components of the Christoffel symbols of M are*

$$\Gamma^1_{23} = \Gamma^1_{32} = \frac{1}{2} f_y$$
$$\Gamma^1_{33} = \frac{1}{2} f_z$$
$$\Gamma^2_{33} = \frac{1}{2} f_y.$$

2. *The non-zero components of the Riemannian curvature tensor of M are*

$$R(\partial_y, \partial_z)\partial_y = \frac{1}{2} f_{yy} \partial_x$$
$$R(\partial_y, \partial_z)\partial_z = \frac{1}{2} f_{yy} \partial_y.$$

3. *The Ricci tensor ρ of M is*

$$\rho = \begin{pmatrix} 0 & 0 & \frac{1}{2} f_{yy} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

   and the eigenvalues of ρ are equal to zero.

4. *The scalar curvature of M is zero.*

Chapter 4

Curves on a Walker Manifold

In this chapter, we discuss the geodesic curves of a three-dimensional Walker manifold, with a focus on the curves with a constant or linear component.

Recall that a curve γ in M is said to be timelike (respectively spacelike, lightlike) at t if $\gamma'(t)$ is a timelike (respectively spacelike, lightlike) vector. The curve γ is timelike (respectively spacelike, lightlike) if $\gamma'(t)$ is timelike (respectively spacelike, lightlike) for all $t \in I$. [9]

Example 4.0.8. Consider the plane curves in (M, g_f).

(1) Let the curve be the straight-line $\gamma(t) = p + tv$ with $p, v \in \mathbb{R}^3$ and $v \neq 0$. The causal character of this curve is same as the causal character of the vector v.

(2) The hyperbola $\gamma(t) = r(\cosh t, \sinh t, 0)$ is a timelike curve lying in the lightlike plane defined by $z = 0$.

Proof. (1) $\gamma'(t) = v$, and so $\langle \gamma', \gamma' \rangle = \langle v, v \rangle$. Thus, γ has the same causal character as the vector v.

(2) $\gamma'(t) = r(\sinh t, \cosh t, 0)$. Then

$$\langle \gamma', \gamma' \rangle = -\cosh^2 t$$

which is negative for all $t \in \mathbb{R}$. Therefore, the hyperbola γ is timelike.

The induced metric on the plane $z = 0$ is $g_f|_{z=0} = -y^2$, so it is degenerate. Hence, the plane represented by $z = 0$ is a lightlike plane and the curve γ lies on it.

□

The Euclidean geometry has no curvature and so a straight-line is the shortest path between two points in the Euclidean geometry. However, in General Relativity, the space is curved by the mass. A geodesic is the notion of a straight line to the curved space, so that a geodesic is (locally) the shortest path between points in the space.

Recall that a curve $\gamma\colon I \longrightarrow M$ on a semi-Riemannian manifold M is called a *geodesic* if $\gamma''(t) = \nabla_{\gamma'}\gamma'(t) = 0$.

Corollary 4.0.9. *Let $\gamma(t)$ be a curve defined on $U \subset M$ with the coordinate functions $\gamma_1(t), \gamma_2(t), \gamma_3(t)$. γ is a geodesic of M if and only if its coordinate functions γ_k satisfy*

$$\frac{\mathrm{d}^2 \gamma_k}{\mathrm{d}\, t^2} + \sum_{i,j} \Gamma_{ij}^k(\gamma) \frac{\mathrm{d}\,\gamma_i}{\mathrm{d}\, t} \frac{\mathrm{d}\,\gamma_j}{\mathrm{d}\, t} = 0$$

for $k = 1, 2, 3$.

It follows that the geodesic equations of a three-dimensional Walker manifold (M, g_f) are

1. $\dfrac{\mathrm{d}^2 \gamma_1}{\mathrm{d}\, t^2} + f_x \dfrac{\mathrm{d}\,\gamma_1}{\mathrm{d}\, t} \dfrac{\mathrm{d}\,\gamma_3}{\mathrm{d}\, t} + f_y \dfrac{\mathrm{d}\,\gamma_2}{\mathrm{d}\, t} \dfrac{\mathrm{d}\,\gamma_3}{\mathrm{d}\, t} + \dfrac{1}{2}(f f_x + f_z) \left(\dfrac{\mathrm{d}\,\gamma_3}{\mathrm{d}\, t}\right)^2 = 0,$

2. $\dfrac{\mathrm{d}^2 \gamma_2}{\mathrm{d}\, t^2} + \dfrac{1}{2} f_y \left(\dfrac{\mathrm{d}\,\gamma_3}{\mathrm{d}\, t}\right)^2 = 0,$

3. $\dfrac{\mathrm{d}^2 \gamma_3}{\mathrm{d}\, t^2} - \dfrac{1}{2} f_x \left(\dfrac{\mathrm{d}\,\gamma_3}{\mathrm{d}\, t}\right)^2 = 0,$

4. $\langle \gamma', \gamma' \rangle = \begin{cases} < 0, & \gamma \text{ is timelike} \\ 0, & \gamma \text{ is lightlike} \\ > 0, & \gamma \text{ is spacelike} \end{cases}.$

For an arbitrary f,

(i) If we have a curve with constant γ_1, the curve $\gamma(t) = (c, \gamma_2, \gamma_3)$ satisfying the following equations is a geodesic in M.

1. $f_y \dfrac{\mathrm{d}\,\gamma_2}{\mathrm{d}\,t}\dfrac{\mathrm{d}\,\gamma_3}{\mathrm{d}\,t} + \dfrac{1}{2}(f f_x + f_z)\left(\dfrac{\mathrm{d}\,\gamma_3}{\mathrm{d}\,t}\right)^2 = 0,$

2. $\dfrac{\mathrm{d}^2\,\gamma_2}{\mathrm{d}\,t^2} + \dfrac{1}{2} f_y \left(\dfrac{\mathrm{d}\,\gamma_3}{\mathrm{d}\,t}\right)^2 = 0,$

3. $\dfrac{\mathrm{d}^2\,\gamma_3}{\mathrm{d}\,t^2} - \dfrac{1}{2} f_x \left(\dfrac{\mathrm{d}\,\gamma_3}{\mathrm{d}\,t}\right)^2 = 0,$

4. $\langle \gamma', \gamma' \rangle = -\left(\dfrac{\mathrm{d}\,\gamma_2}{\mathrm{d}\,t}\right)^2 + f\left(\dfrac{\mathrm{d}\,\gamma_3}{\mathrm{d}\,t}\right)^2 = \begin{cases} < 0, & \gamma \text{ is timelike} \\ 0, & \gamma \text{ is lightlike} \\ > 0, & \gamma \text{ is spacelike} \end{cases}.$

(ii) If we have a curve with constant γ_2, the curve $\gamma(t) = (\gamma_1, c, \gamma_3)$ satisfying the following equations is a geodesic in M.

1. $\dfrac{\mathrm{d}^2\,\gamma_1}{\mathrm{d}\,t^2} + f_x \dfrac{\mathrm{d}\,\gamma_1}{\mathrm{d}\,t}\dfrac{\mathrm{d}\,\gamma_3}{\mathrm{d}\,t} + \dfrac{1}{2}(f f_x + f_z)\left(\dfrac{\mathrm{d}\,\gamma_3}{\mathrm{d}\,t}\right)^2 = 0,$

2. $\dfrac{1}{2} f_y \left(\dfrac{\mathrm{d}\,\gamma_3}{\mathrm{d}\,t}\right)^2 = 0,$

3. $\dfrac{\mathrm{d}^2\,\gamma_3}{\mathrm{d}\,t^2} - \dfrac{1}{2} f_x \left(\dfrac{\mathrm{d}\,\gamma_3}{\mathrm{d}\,t}\right)^2 = 0,$

4. $\langle \gamma', \gamma' \rangle = 2\dfrac{\mathrm{d}\,\gamma_1}{\mathrm{d}\,t}\dfrac{\mathrm{d}\,\gamma_3}{\mathrm{d}\,t} + f\left(\dfrac{\mathrm{d}\,\gamma_3}{\mathrm{d}\,t}\right)^2 = \begin{cases} < 0, & \gamma \text{ is timelike} \\ 0, & \gamma \text{ is lightlike} \\ > 0, & \gamma \text{ is spacelike} \end{cases}.$

Notice that, if $\frac{\mathrm{d}\gamma_3}{\mathrm{d}t} \neq 0$, then it follows from (2) that $f_y = 0$ and so $f = f(x, z)$.

(iii) If we have a curve with constant γ_3, the curve $\gamma(t)$ satisfying the following equations is a geodesic in M.

1. $\dfrac{\mathrm{d}^2\gamma_1}{\mathrm{d}t^2} = 0,$

2. $\dfrac{\mathrm{d}^2\gamma_2}{\mathrm{d}t^2} = 0,$

3. $\dfrac{\mathrm{d}^2\gamma_3}{\mathrm{d}t^2} = 0.$

   The curve is in the form $\gamma = (a_1 t + a_2, b_1 t + b_2, c)$.

4. $\langle \gamma', \gamma' \rangle = -b_1^2 = \begin{cases} < 0, & \gamma \text{ is timelike} \\ 0, & \gamma \text{ is lightlike} \end{cases}.$

$\gamma(t) = (a_1 t + a_2, b_1 t + b_2, c)$ is a timelike geodesic if $b_1 \neq 0$. If $b_1 = 0$, $\gamma(t) = (a_1 t + a_2, b_2, c)$ is a lightlike geodesic in M.

Corollary 4.0.10. *Any straight-line $\gamma(t) = p + tv$ with $p, v \in M$ and $v = (v_1, v_2, 0)$ is a geodesic in the manifold (M, g_f) with an arbitrary f and its causal character is the same as v.*

Proof. Let $\gamma(t) = p + tv$ and $v = (v_1, v_2, 0)$.

$$\gamma' = v = (v_1, v_2, 0).$$

Hence γ has the same causal character as v. Moreover, $\gamma(t)$ satisfies the geodesic equations for any f. □

Corollary 4.0.11. *Let (M, g_f) be a Walker manifold. If there exists a geodesic curve in the form $\gamma(t) = (\gamma_1(t), \gamma_2(t), c_1 t + c_2)$ in M where $c_1 \neq 0$, then M is a strictly Walker manifold. Moreover if γ_2 is also linear with respect to t, then $f = f(z)$.*

Proof. Let $\gamma(t) = (\gamma_1(t), \gamma_2(t), c_1 t + c_2)$ be a geodesic in M, where $c_1 \neq 0$. Then γ satisfies the geodesic equations

1. $\dfrac{\mathrm{d}^2 \gamma_1}{\mathrm{d}\, t^2} + c_1 f_x \dfrac{\mathrm{d}\,\gamma_1}{\mathrm{d}\, t} + c_1 f_y \dfrac{\mathrm{d}\,\gamma_2}{\mathrm{d}\, t} + \dfrac{1}{2} c_1^2 \left(f f_x + f_z \right) = 0,$

2. $\dfrac{\mathrm{d}^2 \gamma_2}{\mathrm{d}\, t^2} + \dfrac{1}{2} c_1^2 f_y = 0,$

3. $-\dfrac{1}{2} c_1^2 f_x = 0.$

Since $c_1 \neq 0$, $f_x = 0$ followed from (3). Thus $f = f(y, z)$, that is, M is a strictly Walker manifold.

For the second part, let $\gamma(t) = (\gamma_1(t), b_1 t + b_2, c_1 t + c_2)$. Then $\gamma' = (\gamma_1'(t), b_1, c_1)$ and the geodesic equations become

1. $\dfrac{\mathrm{d}^2 \gamma_1}{\mathrm{d}\, t^2} + c_1 f_x \dfrac{\gamma_1}{\mathrm{d}\, t} + \dfrac{1}{2} c_1^2 \left(f f_x + f_z \right) = 0,$

2. $\dfrac{1}{2} c_1^2 f_y = 0,$

3. $-\dfrac{1}{2} c_1^2 f_x = 0.$

It follows from (2) and (3) that; $f_x = f_y = 0$. Therefore $f = f(z)$. □

Corollary 4.0.12. *Let M be a strictly Walker three-manifold, admitting a parallel null vector field spanned by $\{\partial_x\}$. M can be characterized by $f = f(y, z)$ so that f is independent of the parameter x. In this case, $f_x = 0$ and the geodesic curves in this manifold satisfy the following equations:*

1. $\dfrac{\mathrm{d}^2 \gamma_1}{\mathrm{d}\, t^2} + f_y \dfrac{\mathrm{d}\,\gamma_2}{\mathrm{d}\, t} \dfrac{\mathrm{d}\,\gamma_3}{\mathrm{d}\, t} + \dfrac{1}{2} f_z \left(\dfrac{\mathrm{d}\,\gamma_3}{\mathrm{d}\, t} \right)^2 = 0,$

2. $\dfrac{d^2 \gamma_2}{d t^2} + \dfrac{1}{2} f_y \left(\dfrac{d \gamma_3}{d t} \right)^2 = 0,$

3. $\dfrac{d^2 \gamma_3}{d t^2} = 0,$

4. $\langle \gamma', \gamma' \rangle = \begin{cases} < 0, & \gamma \textit{ is timelike} \\ 0, & \gamma \textit{ is lightlike} \\ > 0, & \gamma \textit{ is spacelike} \end{cases} .$

This means that $\dfrac{d \gamma_3}{d t}$ is a constant c_1 so that any curve in the form

$$\gamma(t) = (\gamma_1(t), \gamma_2(t), c_1 t + c_2)$$

with γ_1, γ_2 satisfying the equations above are geodesics in M for any $f(y, z)$.

Moreover, if $c_1 = 0$, then γ is a straight-line:

$$\gamma(t) = (a_1 t + a_2, b_1 t + b_2, c_2).$$

The causality of γ depends on b_1, that is,

- if $b_1 = 0$, then γ is a lightlike geodesic,
- if $b_1 \neq 0$, then γ is a timelike geodesic.

Example 4.0.13. Let M be a Walker manifold with the defining function $f(x, y, z) = x/z$ where $z \neq 0$ and x, z are linear in t with unit speed.

(i) If $\gamma(t) = (a' + at, b' + bt, c)$ then $\gamma(t)$ is a timelike geodesic in M.

(ii) If $\gamma(t) = (a' + at, b, c)$, then it is a lightlike geodesic in M.

Example 4.0.14. Let (M, g_f) be a strictly Walker manifold with $f = y + z$ where $y = t$, $z = t^2$. A unit-speed timelike geodesic on this manifold is either $\gamma(t) = (c_1 t + c_2, -t + c_3, c_4)$ or $\gamma(t) = (c_1 t + c_2, t + c_3, c_4)$. Moreover, $\gamma(t) = (c_1 t + c_2, c_3, c_4)$ is a lightlike geodesic on this manifold for any $c_1, c_2, c_3, c_4 \in \mathbb{R}$.

A semi-Riemannian manifold M is called *geodesically complete* if every geodesic curve γ in M is defined on $\mathbb{R}$.

Theorem 4.0.15. *A Walker three-manifold M with the metric g_f as given in (3.0.1) is geodesically complete.*

Proof. To prove this, we use a criterion from [10] for geodesic completeness that "a semi-Riemannian metric defined globally on $\mathbb{R}^n$ whose Christoffel symbols satisfy

$$\Gamma^i_{jk} = 0$$

for all $j, k < i$ is geodesically complete."It is seen from Proposition 3.2.1 that all of the Christoffel symbols Γ^i_{jk} with $j, k < i$ are equal to zero. Therefore, (M, g_f) is geodesically complete [1]. $\square$

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