HOMOTOPY COLIMITS AND DECOMPOSITIONS OF FUNCTION COMPLEXES

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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Given a functor $F : C \longrightarrow \mathbf{GSp}$, the homotopy colimit $hocolim_C F$ is defined as the diagonal space of simplicial replacement of F. Let G be a finite group and $\mathcal{F}$ be a family of subgroups of G, the classifying space $E_{\mathcal{F}}G$ can be taken as the homotopy colimit $hocolim_{\mathcal{O}_{\mathcal{F}}G}(G/H)$ over the orbit category $\mathcal{O}_{\mathcal{F}}G$. For G-spaces X and Y, let $map_G(X, Y)$ be the space formed by G-simplicial maps from X to Y. Given a functor $F : C \longrightarrow \mathbf{GSp}$ and a G-space Y, there is an isomorphism $map_G(hocolim_C F, Y) \cong holim_C(map_G(F, Y))$ [1]. We give a proof for this isomorphism by writing explicit simplicial maps in both directions. As an application we show that the generalized homotopy fixed points set $Y^{h_{\mathcal{F}}G} := map_G(E_{\mathcal{F}}G, Y)$ of a G-space Y can be calculated as the homotopy limit $holim_{H \in \mathcal{O}_{\mathcal{F}}G} Y^H$. Topological version of this is recently proved by D. A. Ramras in [2]. We also give some other applications of this isomorphism.

Keywords: Homotopy colimit, classifying space, simplicial set, homotopy limit, function complexes.

ÖZET

HOMOTOPİ EŞLİMİTLER VE FONKSİYONLAR KOMPLEKSLERİNİN AYRIŞIMLARI

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Verilen bir izleç $F : C \longrightarrow \mathbf{GSp}$ için, homotopi eşlimit $hocolim_C F$, F izleçinin simpleksel yer değiştirmesinin diyagonalı olarak tanımlanır. G bir grup olsun ve $\mathcal{F}$ G grubunun bir altgrup ailesi olsun, sınıflandırma uzayı $E_{\mathcal{F}}G$ yörünge kategorisi $\mathcal{O}_{\mathcal{F}}G$ üzerine homotopi eşlimit $hocolim_{\mathcal{O}_{\mathcal{F}}G}(G/H)$ olarak alınabilir. G-uzayları X ve Y için, $map_G(X,Y)$, X'ten Y'ye G-simpleksel fonksiyonların oluşturduğu simpleksel küme olsun. Verilen $F : C \longrightarrow \mathbf{GSp}$ bir izleç ve bir G-uzayı Y için bir izomorfizma vardır ki $map_G(hocolim_C F, Y) \cong holim_C(map_G(F,Y))$ [1]. Bu izomorfizma için iki yöndeki simpleksel fonksiyonları yazarak detaylı bir ispat veriyoruz. Bir uygulama olarak, bir G-uzayı Y'nin genelleştirilmiş homotopi sabit nokta kümesi $Y^{h_{\mathcal{F}}G} := map_G(E_{\mathcal{F}}G, Y)$'nin homotopi limit $holim_{H \in \mathcal{O}_{\mathcal{F}}G} Y^H$ olarak hesaplanabileceğini gösteriyoruz. Bunun topolojik versiyonu yakın zamanda D.A. Ramras tarafından [2]'de kanıtlanmıştır. Ayrıca bu teoremin bazı diğer uygulamalarını da veriyoruz.

Anahtar sözcükler: Homotopi eşlimit, sınıflandırma uzayı, homotopi limit, simpleksel küme, fonksiyonlar kompleksleri.

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Contents

Chapter 1

Introduction

An ordered simplicial complex is a pair $K = (V_K, S_K)$, where V_K is an ordered set called set of vertices and S_K is a set of non-empty subsets of V_K called set of simplicies of K satisfying some axioms. Simplicial sets are generalizations of the ordered simplicial complexes. A simplicial set is a functor $X : \mathbf{\Delta}^{op} \longrightarrow \mathbf{Sets}$ where $\mathbf{\Delta}$ is the category whose objects are the ordered sets $\mathbf{n} := \{0, 1, \ldots, n\}$ for each $n \in \mathbb{N}$ and morphisms are order preserving functions $\eta : \mathbf{n} \longrightarrow \mathbf{m}$. A simplicial set can be considered as a family of sets $\{X_n\}_{n\in\mathbb{N}}$ with face and degeneracy maps satisfying some axioms. For a simplicial set X and $n \in \mathbb{N}$, we have $n+1$ face maps $d_i : X_n \longrightarrow X_{n-1}$ for $0 \leq i \leq n$, and $n+1$ degeneracy maps $s_i : X_n \longrightarrow X_{n+1}$ for $0 \leq i \leq n$.

Following the convention in [3] we call a simplicial set, a space. Also, any time we attribute a topological property to a space, we mean this property is attributed to the geometric realization of that space. As an example when we say two spaces are homotopy equivalent we mean their geometric realizations are homotopy equivalent.

For a simplicial space X, i.e., a functor $X : \mathbf{\Delta}^{op} \longrightarrow \mathbf{Sp}$ where $\mathbf{Sp}$ denotes the category of spaces, we give the definition of the diagonal space $diag(X)$ and the simplicial realization $|X|$ of X and we give a proof of the known fact

$$diag(X) \cong |X|$$

by writing explicit simplicial maps from $diag(X)$ to $|X|$ and from $|X|$ to $diag(X)$.

Nerve of a category C is the space $N(C)$ where an element σ of $N(C)_n$ is of the form $\sigma = (\sigma(0) \xrightarrow{\alpha_1} \sigma(1) \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} \sigma(n))$ where $\sigma(i)$ are objects and α_i are morphisms in C with some face and degeneracy maps. For a functor $F : C \longrightarrow \mathbf{Sp}$, the simplicial replacement $\coprod_* F$ is defined as the simplicial space

$$(\coprod\nolimits_* F)_n = \coprod_{\sigma \in N(C)_n} F(\sigma(0)).$$

The homotopy colimit written by $hocolim_C F$ of a functor $F : C \longrightarrow \mathbf{Sp}$ is defined as $diag(\coprod_* F)$.

The main aim of this thesis is to give an explicit proof to a theorem on function complexes from a homotopy colimit to another space. To state this theorem we first introduce function complexes and homotopy limits.

For a category C, we define the contravariant functor

$$\begin{aligned} \underline{N} : C &\longrightarrow \mathbf{Sp} \\ c &\mapsto N(c \downarrow C) \end{aligned}$$

where $c \downarrow C$ denotes the under-category.

Also for two contravariant functors $F, F' : C \longrightarrow \mathbf{Sp}$, the natural space $Nat(F, F')$ from F to F' is defined as a subspace of

$$\prod_{c \in obj(C)} map(F(c), F'(c))$$

such that $(f_c)_{c \in C} \in Nat(F, F')$ if and only if for every morphism $\alpha \in Mor_C(c, e)$ the following diagram commutes:

$$\begin{tikzcd}
F(e) \times \Delta[m] \arrow[r, "f_e"] \arrow[d, "F(\alpha) \times id"'] & F'(e) \arrow[d, "F'(\alpha)"] \\
F(c) \times \Delta[m] \arrow[r, "f_c"] & F'(c).
\end{tikzcd}$$

We define the homotopy limit $holim_C F$ of a contravariant functor $F : C \longrightarrow \mathbf{Sp}$ as the space $Nat(\underline{N}, F)$.

For two spaces X and Y, the function complex $map(X,Y)$ from X to Y is the space where

$$map(X,Y)_n := Mor_{\mathbf{Sp}}(X \times \Delta[n], Y).$$

Then we have the following theorem:

Theorem 1.0.1. *For a functor $F : C \longrightarrow \mathbf{Sp}$ and a space X, there is an isomorphism*

$$map(\underset{C}{hocolim} F, X) \cong \underset{C}{holim}(map(F,X)).$$

This is a known isomorphism in homotopy theory. (See [1, Proposition XII.4.1]). We introduce a direct proof for this isomorphism by writing explicit simplicial maps and demonstrate some applications of this theorem. First we define

$$\psi : map(\underset{C}{hocolim} F, X) \longrightarrow \underset{C}{holim}(map(F,X))$$

and

$$\phi : \underset{C}{holim}(map(F,X)) \longrightarrow map(\underset{C}{hocolim} F, X).$$

Then, we show ψ is a simplicial map and ψ and ϕ are inverses.

Moreover, this theorem can be extended to the equivariant case as follows:

Theorem 1.0.2. *For a functor $F : C \longrightarrow \mathbf{GSp}$ and a G-space X,*

$$map_G(\underset{C}{hocolim} F, X) \cong \underset{C}{holim}(map_G(F,X)).$$

Here for two G-spaces X and Y, $map_G(X,Y)$ denotes the subspace of $map(X,Y)$ consisting of G-maps $f : X \times \Delta[n] \longrightarrow Y$ where $\Delta[n]$ is considered as a trivial G-space. The proof of the equivariant case follows similarly.

Let G be a finite group, a family of subgroups $\mathcal{F}$ is a set of subgroups of G such that if $H \in \mathcal{F}$ then for all $g \in G$, $g^{-1}Hg \in \mathcal{F}$ and for all $K \leq H$, $K \in \mathcal{F}$. A classifying space of a group G for a family of subgroups $\mathcal{F}$ is the topological space $E_{\mathcal{F}}G$ such that H-fixed points $(E_{\mathcal{F}}G)^H$ is contractible if $H \in \mathcal{F}$ and if $H \notin \mathcal{F}$ then $(E_{\mathcal{F}}G)^H$ is empty. Moreover, for a group G and a family of subgroups $\mathcal{F}$, a simplicial classifying space is the space $E_{\mathcal{F}}G$ whose geometric realization is a classifying space of G for $\mathcal{F}$.

For a group G and a family of subgroups $\mathcal{F}$, the orbit category $\mathcal{O}_{\mathcal{F}}G$ is the category of transitive G-sets G/H where $H \in \mathcal{F}$. Then, we introduce the functor

$$\begin{aligned} \mathcal{I} : \mathcal{O}_{\mathcal{F}}G &\longrightarrow \mathbf{GSp} \\ G/H &\mapsto G/H. \end{aligned}$$

Here, G/H is considered as a discrete G-space with isotropy H and morphisms are mapped to underlying functions. Then, we recall a known model $\underset{\mathcal{O}_{\mathcal{F}}G}{hocolim}\mathcal{I}$ for the simplicial classifying space using [2, Proposition 2.9] and 3.3.10 which is a special case of Thomason's Theorem [4].

Furthermore, for a right G-space X, we introduce the generalized homotopy orbit space $X_{h_{\mathcal{F}}G}$ as the space $X \times_G E_{\mathcal{F}}G$ and give a proof for the isomorphism

$$X_{h_{\mathcal{F}}G} \cong \underset{H \in \mathcal{O}_{\mathcal{F}}G}{hocolim}(X/H).$$

The case when $\mathcal{F}$ is the trivial family of subgroups is a well-known isomorphism and the statement can be found in many sources such as [3]. Even though we think this general form is also known, we could not find the statement of this isomorphism in any source.

For a G-space X, the generalized homotopy fixed points set $X^{h_{\mathcal{F}}G}$ is defined as $map_G(E_{\mathcal{F}}G, X)$. Using the fact that $E_{\mathcal{F}}G$ can be constructed as a homotopy colimit, we apply Theorem 1.0.2 and obtain

$$X^{h_{\mathcal{F}}G} = map_G(E_{\mathcal{F}}G, X) \cong \underset{\mathcal{O}_{\mathcal{F}}G}{holim}(map_G(\mathcal{I}, X)).$$

Also, showing

$$map_G(G/H, X) \cong X^H$$

we obtain

$$X^{h_{\mathcal{F}}G} \cong \underset{H \in \mathcal{O}_{\mathcal{F}}G}{holim} X^H.$$

A topological version of this theorem is proved by D.A. Ramras in [2].

As an application we observe the following corollary:

Corollary 1.0.3. *If X and Y are two G-spaces and*

$$X \cong \underset{c \in C}{hocolim} F$$

where $F : C \longrightarrow \mathcal{O}_{\mathcal{F}}G$ such that $F(c) = G/H_c$, then we have the following decomposition for the function complex

$$map_G(X, Y) \cong \underset{c \in C}{holim} Y^{H_c}.$$

The corollary above shows that if a G-space X is simplicially isomorphic to $\underset{c \in C}{hocolim}(G/H_c)$, then function complex $map_G(X, Y)$ has a homotopy limit decomposition. The question when a G-space X is simplicially isomorphic to $\underset{c \in C}{hocolim}(G/H_c)$ for such a functor is left as an open problem.

We conclude this thesis with two examples where the space X is simplicially isomorphic to homotopy colimit of a functor.

The main outline of this thesis is as follows:

In Chapter 2, we introduce the background information we use in thesis on categories, simplicial sets and group actions.

In Chapter 3, for a simplicial space X, we define diagonal space $diag(X)$ and simplicial realization $|X|$ and show that $diag(X)$ is simplicially isomorphic to $|X|$.

Also, we construct homotopy colimit and simplicial classifying space of a group G for a family of subgroups $\mathcal{F}$. Then we give the construction of generalized homotopy orbit space $X_{h_{\mathcal{F}}G}$ and give a proof for the isomorphism

$$X_{h_{\mathcal{F}}G} \cong \underset{H \in \mathcal{O}_{\mathcal{F}}G}{hocolim}(X/H).$$

In Chapter 4, we prove Theorem 1.0.1.

In Chapter 5, we prove Theorem 1.0.2 and we give some applications of these theorems.

Chapter 2

Preliminaries

In this chapter, we introduce some definitions and theorems that are the essential background for this thesis. We mainly introduce three concepts namely categories, simplicial sets and group actions.

We start the first section with some category theory. Then in the second section we define simplicial sets, the main objects of study in this thesis, and give some basic theorems about simplicial sets.

In the last section of this chapter, we introduce the concept of a group acting on a simplicial set. We start with the definition of a group action on a set and then extend to the simplicial sets.

2.1 Categories

We start by introducing categories and some theorems from this branch of mathematics used frequently in this thesis. The reader may look at [5] for more detailed information on the theory of categories.

Definition 2.1.1. A *category* C consists of a class of objects denoted by $obj(C)$ with sets of morphisms $Mor_C(c, e)$ for every $c, e \in obj(C)$ equipped with a rule of composition, which can be seen as a well-defined function for every $c_1,\ c_2,\ c_3 \in obj(C)$ from $Mor_C(c_2, c_3) \times Mor_C(c_1, c_2)$ to $Mor_C(c_1, c_3)$, where the composition of morphisms $\alpha \in Mor_C(c_2, c_3)$ and $\beta \in Mor_C(c_1, c_2)$ is denoted $\alpha \circ \beta$ and the following axioms hold:

(i) For all morphisms α, α', α'' in C where $\alpha \in Mor_C(c_1, c_2)$, $\alpha' \in Mor_C(c_2, c_3)$ and $\alpha'' \in Mor_C(c_3, c_4)$,

$$(\alpha \circ \alpha') \circ \alpha'' = \alpha \circ (\alpha' \circ \alpha'').$$

(ii) For every $c \in obj(C)$ there exists a morphism, denoted id_c, in $Mor_C(c, c)$ such that for every morphism $\alpha \in Mor_C(c, e)$ and for every morphism $\beta \in Mor_C(e', c)$, $id_c \circ \beta = \beta$ and $\alpha \circ id_c = \alpha$. This morphism id_c is called the identity morphism of the element $c \in obj(C)$.

We often omit $\circ$ in the notation of composition most of the time, i.e., $\alpha \circ \beta$ is written $\alpha\beta$.

Example 2.1.2. Sets with functions as morphisms with the usual composition rule and identity maps form a category. We denote this category by **Sets**.

Example 2.1.3. Groups also form a category with homomorphisms as morphisms with the usual composition rule and identity maps denoted by **Grps**.

A morphism $\alpha \in Mor_C(c, e)$ is called an *isomorphism* if there exists a morphism $\beta \in Mor_C(e, c)$ such that

$$\alpha\beta = id_e \text{ and } \beta\alpha = id_c.$$

If there is an isomorphism between two objects c and e, we say c and e are *isomorphic* and write $c \cong_C e$ or, if there is no ambiguity about the category, simply $c \cong e$.

Given a category C, we can define some related categories. These constructions are commonly used in category theory and related subjects and play a major role in this thesis.

Definition 2.1.4. For a category C, a *subcategory* E is a category such that $obj(E) \subseteq obj(C)$ and $Mor_E(c, e) \subseteq Mor_C(c, e)$ for all $c, e \in obj(E)$ with the same composition rule of morphisms and identity morphisms.

For C a category and $S \subseteq obj(C)$, the *full subcategory generated by* S is the subcategory E of C such that $obj(E) = S$ and

$$Mor_E(c, e) = Mor_C(c, e) \text{ for all } c, e \in obj(E).$$

Definition 2.1.5. The *opposite category* of a category C, denoted C^{op}, is the category with the objects $obj(C^{op}) = obj(C)$ and $Mor_{C^{op}}(c, e) = Mor_C(e, c)$. The composition of two morphisms α and β in C^{op} is defined by the composition $\beta\alpha$ in the category C. It can be seen that C^{op} is C with the arrows reversed.

When we have two categories, we can relate these two categories using functors.

Definition 2.1.6. A *covariant functor* F between two categories C and E is a set of maps each denoted F, $F : obj(C) \longrightarrow obj(E)$ and for every c, $c' \in obj(C)$, $F : Mor_C(c, c') \longrightarrow Mor_E(F(c), F(c'))$ satisfying the following conditions.

(i) $F(id_c) = id_{F(c)}$ for each $c \in obj(C)$, and

(ii) $F(\alpha\beta) = F(\alpha)F(\beta)$ for each morphisms α and β in C.

The category whose objects are categories and whose morphisms are functors is itself a category which we denote **Cats**. We also recall that a contravariant functor is defined similarly to a covariant functor as follows.

Definition 2.1.7. A contravariant functor F between two categories C and E consists of a map from $obj(C)$ to $obj(E)$ and a map for every c, $c' \in obj(C)$, from $Mor_C(c, c')$ to $Mor_E(F(c'), F(c))$ each denoted F, satisfying the following conditions.

(i) $F(id_c) = id_{F(c)}$ for each $c \in obj(C)$, and

(ii) $F(\alpha\beta) = F(\beta)F(\alpha)$ for each morphisms α and β in C.

Example 2.1.8. The assignment from **Grps** to **Sets** that maps any group to its underlying set of elements and a group homomorphism to itself viewed as a function is a functor.

Definition 2.1.9. For two functors $F,\ F' : C \longrightarrow E$, a *natural transformation* $\tau : F \Rightarrow F'$ is a collection of maps $\tau_c \in Mor_E(F(c), F'(c))$ for each $c \in obj(C)$ such that for each morphism $\alpha : c \to e$ in C the following diagram commutes:

$$\begin{tikzcd}
F(c) \arrow[r, "\tau_c"] \arrow[d, "F(\alpha)"'] & F'(c) \arrow[d, "F'(\alpha)"] \\
F(e) \arrow[r, "\tau_e"] & F'(e)
\end{tikzcd}.$$

For two categories C and E, we can form a category with functors from C to D as the objects and natural transformations as its morphisms. We denote this category $\mathbf{Func}(C, E)$. If two functors F and F' are isomorphic in this category we say that F and F' are *naturally isomorphic*. The natural concept of isomorphism of categories in **Cats** is too strong for many purposes, so we work instead with the following concept:

Definition 2.1.10. Two categories C and E are called equivalent if there exist functors $F : C \longrightarrow E$ and $F' : E \longrightarrow C$ such that

$$FF' \cong id_C \text{ and } F'F \cong id_E$$

in functor categories.

2.2 Simplicial Sets

In this section, before we define simplicial sets, we define some preliminary notions. Then we give the definition of a simplicial set investigating the structure of a simplicial set and we state the morphisms between simplicial sets.

Also in this section, we explain some notations on simplicial sets we use. Moreover, we note that Theorem 2.2.12, proved in this section, is also used frequently in this thesis to prove that two simplicial sets are simplicially isomorphic.

Lastly in this section, we give a method to construct a topological space using a simplicial set which yields an important relationship between simplicial sets and topological spaces.

Definition 2.2.1. An *ordered simplicial complex* K consists of a partially ordered set V_K and a set S_K of non-empty subsets of V_K such that

(i) For each $v \in V_K$, $\{v\} \in S_K$,

(ii) If $A \in S_K$ and $B \subset A$ then $B \in S_K$, and

(iii) Partial order on V_K induces total order on any $s \in S_K$.

For an ordered simplicial complex $K = (V_K, S_K)$, we call the elements of V_K *vertices* and the elements of S_K *simplicies*.

We note that ordered simplicial complexes form a category denoted by **OSC** with order-preserving functions, which maps simplicies to simplicies [3].

Example 2.2.2. An important example of an ordered simplicial complex, which we denote by $\mathbf{n}$, consists of $V_{\mathbf{n}} = \{0, 1, 2, 3, \ldots, n\}$ with usual ordering as vertices and $S_{\mathbf{n}} = P(V_{\mathbf{n}}) \setminus \{\varnothing\}$ as simplicies where $P(V)$ is the power set of V.

We emphasize this particular example because it is used frequently later in the thesis to construct other important objects.

Definition 2.2.3. $\boldsymbol{\Delta}$ is the full subcategory of **OSC** generated by the objects $\mathbf{n}$ for each $n \in \mathbb{N}$.

There are certain morphisms in $\boldsymbol{\Delta}$ for which we introduce some notation for future reference.

Let d_n^i be the morphism from $\mathbf{n-1}$ to $\mathbf{n}$ given by

$$d_n^i(j) = \begin{cases} j & j < i \\ j+1 & j \geq i \end{cases}$$

and s_n^i from $\mathbf{n+1}$ to $\mathbf{n}$ given by

$$s_n^i(j) = \begin{cases} j & j \leq i \\ j-1 & j > i. \end{cases}$$

We note that for each n there are $n+1$ distinct d_n^i and s_n^i by this definition. We write d^i for d_n^i and s^i for s_n^i for each n. We assume these functions are defined for whichever $\mathbf{n}$ is suitable in the context.

Remark 2.2.4. All morphisms in $\mathbf{\Delta}$ can be written as a unique composition of d^i and s^i. Moreover, these morphisms satisfy the following relations [3]:

$$\begin{aligned}
&d^j \circ d^i = d^i \circ d^{j-1} \text{ and } s^j \circ d^i = d^i \circ s^{j-1} \text{ if } i < j, \\
&s^j \circ s^i = s^i \circ s^{j+1} \text{ if } i \leq j, \\
&s^j \circ d^j = id = s^j \circ d^{j+1}, \text{ and} \\
&s^j \circ d^i = d^{i-1} \circ s^j \text{ if } i-1 > j.
\end{aligned}$$

These relations can be shown by direct calculation.

Definition 2.2.5. A simplicial set X is a functor

$$X : \mathbf{\Delta}^{op} \longrightarrow \mathbf{Sets}.$$

We refer to a simplicial set as a *space* from time to time, following the convention of [3].

A simplicial set can be seen as a family of sets $\{X_n\}_{n \in N}$ with for all n, $n+1$ morphisms $X(d^i) : X_n \longrightarrow X_{n-1}$ denoted by d_i and $n+1$ morphisms

$X(s^i) : X_n \longrightarrow X_{n+1}$ denoted by s_i, satisfying the following relations:

$$\begin{aligned}
&d_i \circ d_j = d_{j-1} \circ d_i \text{ and } d_i \circ s_j = s_{j-1} \circ d_i \text{ if } i < j, \\
&s_i \circ s_j = s_{j+1} \circ s_i \text{ if } i \leq j, \\
&d_j \circ s_j = \mathrm{id} = d_{j+1} \circ s_j, \mathit{and} \\
&d_i \circ s_j = s_j \circ d_{i-1} \text{ if } j < i-1.
\end{aligned}$$

These relations follow directly from the relations in 2.2.4.

We note that we call d_i a *face* map and s_i a *degeneracy* map. We sometimes use this second interpretation of the definition of a simplicial set. Hence, to define a simplicial set X, we define the sets X_n for each n and define $X(\eta)$ for each morphism $\eta \in \mathbf{\Delta}$ by either giving the actual definition of $X(\eta)$ or giving the definition of $X(d^i)$ and $X(s^i)$ for each face and degeneracy map. Since each morphism $\eta \in \mathbf{\Delta}$ has a unique decomposition into face and degeneracy maps 2.2.4, the definition of $X(\eta)$ follows directly from these definitions.

Remark 2.2.6. For a simplicial set X, we often denote $X(\eta)$ by η^*.

We note that, for a given simplicial set X, we define a *subsimplicial set* Y as the simplicial set where $Y_m \subseteq X_m$ with restricted face and degeneracy maps of X. We also call subsimplicial sets *subspaces*.

Example 2.2.7. A set X can be considered as a space with $X_n = X$ for all n where all face and degeneracy maps are taken to be identity. We call this a *discrete space*.

Definition 2.2.8. For two spaces X and Y:

(i) The space $X \times Y$ is defined as the space where for all $0 \leq n$, $(X \times Y)_n = X_n \times Y_n$ and for a morphism η in $\mathbf{\Delta}$

$$\eta^*(x, y) = (\eta^* x, \eta^* y).$$

(ii) The space $X \coprod Y$ is defined as the space $(X \coprod Y)_n = X_n \coprod Y_n$ and for a morphism η in $\mathbf{\Delta}$

$$\eta^*(x) = X(\eta)(x) \text{ for } x \in X \text{ and } \eta^*(y) = Y(\eta)(y) \text{ for } y \in Y.$$

Definition 2.2.9. Let X be a space and assume there is an equivalence relation for each X_n. Also assume for a morphism $\eta \in \mathbf{\Delta}$, if $x \sim y$ then $\eta^* x \sim \eta^* y$. Then, $X/\sim$ is the space $(X/\sim)_n = X_n/\sim$ and

$$\eta^*[x] = [\eta^* x].$$

Example 2.2.10. An important example of a space is $\Delta[m]$ for $m \in \mathbb{N}$, which is defined as

$$\Delta[m]_n = Mor_{\mathbf{\Delta}}(\mathbf{n}, \mathbf{m})$$

and for $\alpha : \mathbf{k} \longrightarrow \mathbf{n}$,

$$\alpha^* : Mor_{\mathbf{\Delta}}(\mathbf{n}, \mathbf{m}) \longrightarrow Mor_{\mathbf{\Delta}}(\mathbf{k}, \mathbf{m})$$
$$\eta \mapsto \eta\alpha.$$

Since a simplicial set is a functor, morphisms between simplicial sets are defined to be natural transformations.

Definition 2.2.11. For two spaces X and Y a *simplicial map* f from X to Y is a set of morphisms

$$f_n : X_n \longrightarrow Y_n$$

such that for each morphism $\eta : \mathbf{m} \to \mathbf{n}$ in $\mathbf{\Delta}$ the following diagram commutes:

$$\begin{tikzcd}
X_n \arrow[r, "f_n"] \arrow[d, "\eta^*"'] & Y_n \arrow[d, "\eta^*"] \\
X_m \arrow[r, "f_m"] & Y_m.
\end{tikzcd}$$

For a simplicial map f, we often write f for each f_n. Simplicial sets and simplicial maps form a category which we denote by **Sp**. We say two spaces are simplicially isomorphic when these spaces are isomorphic in **Sp**.

Theorem 2.2.12. *For two spaces X, Y, if $f : X \longrightarrow Y$ is a simplicial map, and if f induces a bijection on each X_n then X is simplicially isomorphic to Y.*

Proof. Let X, Y spaces and $f : X \longrightarrow Y$ be a simplicial map such that each f_n is a bijection. Then we define the simplicial map

$$g : Y \longrightarrow X$$

by $g_n := f_n^{-1}$. Then fg and gf are clearly the identity. The only remaining thing to show is that g is in fact a simplicial map. Let $y \in Y_n$ and $\eta : \mathbf{m} \longrightarrow \mathbf{n}$ be a morphism in $\mathbf{\Delta}$. Then,

$$\eta^* g_n(y) = g_m f_m \eta^* g_n(y) = g_m \eta^* f_n g_n(y) = g_m \eta^*(y).$$

Therefore, g is a simplicial map and X and Y are simplicially isomorphic. $\square$

We recall how an ordered simplicial complex gives a topological space.

Definition 2.2.13. For an ordered simplicial complex $K = (V_K, S_K)$, we define the *geometric realization* of K as in [3], denoted $|K|$ as topological space constructed as follows:

(i) For each $s \in S$ we define the space $\{(t_0, t_1 \ldots, t_{|s|}) | \sum_{i=0}^{|s|} t_i = 1, 0 \leq t_i \leq 1\}$ as a subspace of $R^{|s|+1}$.

(ii) We take the disjoint union of these spaces with the finest topology agrees on the intersections.

We write Δ^n for $|\mathbf{n}|$. A morphism $\eta : \mathbf{n} \longrightarrow \mathbf{m}$ induces a continuous function $|\eta| : |\mathbf{n}| \longrightarrow |\mathbf{m}|$ (See [3]).

Remark 2.2.14. For any $\eta : \mathbf{n} \to \mathbf{m}$, there is a simplicial map

$$\eta_* : \Delta[n] \longrightarrow \Delta[m]$$
$$\mu \mapsto \eta\mu.$$

Definition 2.2.15. For a simplicial set X, the *geometric realization* of X is the quotient topological space

$$\Big(\coprod_{n \in \mathbb{N}} X_n \times \Delta^n\Big)/_\sim$$

with relations $(\eta_* x, p) \sim (x, |\eta| p)$ for each η in $\mathbf{\Delta}$.

Definition 2.2.16. A simplicial object in category C is a functor

$$X : \mathbf{\Delta}^{op} \longrightarrow C.$$

We alternate the name of the simplicial object according to the category C. For example, if the category is **Grps** then we refer to a simplicial object $X : \Delta^{op} \longrightarrow \mathbf{Grps}$ as a simplicial group.

2.3 Group Actions

Definition 2.3.1. For a group G, a left G-set X is a set with a function

$$\phi : G \times X \longrightarrow X$$
$$(g, x) \mapsto gx$$

satisfying the following conditions:

(i) For each $x \in X$, $1x = x$ where $1 \in G$ the identity, and

(ii) For each $g, h \in G$ and $x \in X$ $g(hx) = (gh)x$.

We note that a right G-set X is also defined similarly.

Definition 2.3.2. Letting G be a group, a right G-set is defined as a set X with a function

$$\phi : X \times G \longrightarrow X$$
$$(x, g) \mapsto xg$$

which satisfies

(i) For each $x \in X$, $x1 = x$, and

(ii) For each $g, h \in G$ and $x \in X$ $(xg)h = x(gh)$.

Remark 2.3.3. We note that in this thesis each G-action is assumed to act from the left unless stated otherwise.

For a group G and a set X, the trivial G-action on X is defined by $gx = x$ for each $g \in G$.

Remark 2.3.4. For a group G and a subgroup $H \leq G$, the set of cosets

$$G/H = \{gH \mid g \in G\}.$$

This is a G-set with the action $g(g'H) = (gg')H$.

We recall some definitions and results on the subject of group actions. For further details one may check [6].

Definition 2.3.5. For a G-set X and an element $x \in X$, the *orbit* of x is defined as the set

$$Gx := \{gx \mid g \in G\}.$$

For any G-set X and $x \in X$, Gx is a G-set with the action $g(g'x) = (gg')x$.

For any G-set X, being in the same G-orbit gives an equivalence relation. Thus, any G-set X can be written as the disjoint union of the orbits of some elements.

Definition 2.3.6. For a G-set X, the *orbit set* X/G of X is defined to be the set

$$X/G = \{Gx \mid x \in X\}.$$

If a G-set contains only one orbit i.e., for any $x, y \in X$ there exists a $g \in G$ such that $gx = y$, then X is called a *transitive* G-set.

Definition 2.3.7. For an element x in a G-set X, the subgroup

$$G_x := \{g \in G \mid gx = x\}$$

is called the *stabilizer* of x.

Definition 2.3.8. For two G-sets X and Y, a *G-map* is a function $f : X \longrightarrow Y$ such that for each $g \in G$ and $x \in X$, $f(gx) = gf(x)$.

G-sets with G-maps form a category **GSets**. We denote the full subcategory of **GSets** generated by the transitive G-sets by $\mathcal{O}_G$.

Theorem 2.3.9. *Let X, Y be G-sets and let $f : X \longrightarrow Y$ be a G-map. If f is a bijection then X and Y are isomorphic as G-sets.*

Proof. If a G-map $f : X \longrightarrow Y$ is a bijection it has an inverse f^{-1} in **Sets**. Hence, it is enough to show that f^{-1} is a G-map. For $g \in G$, since f is a G-map,

$$f^{-1}(gx) = f^{-1}(gff^{-1}(x)) = f^{-1}(fgf^{-1}(x)) = gf^{-1}(x).$$

Thus, f^{-1} is a G-map, i.e., X and Y are isomorphic as G-sets. □

Now we state the theorem which gives a decomposition for every G-set and a classification for transitive G-sets.

Theorem 2.3.10. *For a group G and an element x in a G-set X*

$$Gx \cong G/G_x$$

as G-sets.

Proof. We define

$$\phi : Gx \longrightarrow G/G_x$$
$$gx \mapsto gG_x.$$

Note that this is a well-defined map because if $g_1x = g_2x$, then $g_1^{-1}g_2 \in G_x$ so, $g_1G = g_2G$. This is a G-map because

$$\phi(g(g'x)) = \phi(gg'x) = gg'G_x = g\phi(g'x).$$

This map is clearly onto since for any $gG_x \in G/G_x$, $\phi(gx) = gG_x$.

Let $\phi(gx) = \phi(g'x)$, then $gG_x = g'G_x$, which implies $g^{-1}g' \in G_x$ i.e., $g^{-1}g'x = x$. Then

$$gx = gg^{-1}g'x = g'x.$$

This shows ϕ is one-to-one and finishes the proof by 2.3.9. $\square$

A direct corollary of this theorem is the following.

Corollary 2.3.11. *Any G-set X can be decomposed as follows.*

$$X \cong \coprod_{Gx \in X/G} Gx \cong \coprod_{Gx \in X/G} G/G_x.$$

In particular if X is a transitive G-set, then $X \cong G/H$ where $H = G_x$ for any $x \in X$.

By this corollary the objects of $\mathcal{O}_G$ can be seen as G/H for each H subgroup of G. We now describe the morphisms of $\mathcal{O}_G$.

Theorem 2.3.12. *For G-sets G/H and G/K and an element $\gamma \in G$ satisfying $\gamma^{-1}H\gamma \leq K$, the map*

$$\bar{\gamma} : G/H \longrightarrow G/K$$
$$gH \mapsto g\gamma K$$

is a G-map. Moreover, all G-maps from G/H to G/K are of this form.

Proof. Let $gH = g'H$. Then, $g^{-1}g' \in H$. Therefore

$$\bar{\gamma}(g'H) = g'\gamma K = (g\gamma)(g\gamma)^{-1}g'\gamma K = g\gamma\gamma^{-1}g^{-1}g'\gamma K = g\gamma\gamma^{-1}h\gamma K$$

for some $h \in H$. Since $\gamma^{-1}H\gamma \leq K$,

$$g\gamma\gamma^{-1}h\gamma K = g\gamma K = \bar{\gamma}(g'H).$$

Hence, $\bar{\gamma}$ is well-defined.

Moreover, for $g, g' \in G$,

$$\bar{\gamma}(g(g'H)) = \bar{\gamma}(gg'H) = gg'\gamma K = g(\bar{\gamma}(g'H)).$$

Hence, $\bar{\gamma}$ is a G-map.

Now, let f be a G-map from G/H to G/K. Then $f(H) = \gamma K$ for some $\gamma \in G$. We claim that $f = \bar{\gamma}$. Let $gH \in G/H$, then

$$f(gH) = gf(H) = g\gamma K = \bar{\gamma}(gH).$$

Moreover, for an element $h \in H$,

$$f(h^{-1}H) = f(H) = h^{-1}\gamma K = \gamma K.$$

Therefore $h^{-1}\gamma K = \gamma K$, i.e., $(h^{-1}\gamma)^{-1}\gamma = \gamma^{-1}h\gamma \in K$. Hence, $\gamma^{-1}H\gamma \leq K$. $\square$

The concept of groups acting on a set can be extended to other objects. One example is groups acting on spaces.

Definition 2.3.13. Let G be a group. A G-*space* is a space X, i.e., simplicial set, where each X_n is a G-set such that for any morphism η in $\mathbf{\Delta}$,

$$\eta^*(gx) = g\eta^*(x).$$

Definition 2.3.14. For two G-spaces X and Y, a G-simplicial map from X to Y is a simplicial map $f : X \longrightarrow Y$ such that

$$f(gx) = gf(x).$$

For a G-space X, X/G is defined as the space $(X/G)_n = X_n/G$ and for a morphism η in $\mathbf{\Delta}$,

$$\eta^*[x] = [\eta^* x].$$

We note that left G-spaces form a category with G-simplicial maps which we denote by **GSp**.

Chapter 3

Homotopy Colimits and Classifying Spaces

In this chapter, we focus on the definition of homotopy colimits and some interpretations of this definition. Defining homotopy colimits requires some constructions. In the first section, we give the necessary background. We define the simplicial realization and the diagonal of a simplicial space. Then, we prove these two spaces are simplicially isomorphic.

In the second section, we introduce the simplicial replacement of a functor. This is necessary to define the homotopy colimit.

In the last section of this chapter, we introduce the definition of a simplicial classifying space of a group G for a family of subgroups $\mathcal{F}$ which we denote by $E_{\mathcal{F}}G$. For a group G and a family of subgroups $\mathcal{F}$ we give the construction of a particular simplicial classifying space following [2]. Also in this section, we prove a special case of Thomason's theorem [4]. Finally, we define $X \times_G E_{\mathcal{F}}G$ and the functor X_{-} for a right G-space X and prove Theorem 3.3.13 which is the main result of this chapter and states

$$\underset{\mathcal{O}_{\mathcal{F}}G}{hocolim} X_{-} \cong X \times_G E_{\mathcal{F}}G.$$

3.1 Simplicial Realization and the Diagonal of Simplicial Spaces

Recall that a simplicial space is a simplicial object in the category of spaces. Therefore for a simplicial space, i.e., a functor $X : \mathbf{\Delta}^{op} \longrightarrow \mathbf{Sp}$ and a morphism $\eta : \mathbf{m} \to \mathbf{n}$, $X(\eta)$ is a simplicial map

$$X(\eta) : X_n \longrightarrow X_m.$$

We denote this map $X(\eta)$ by η_h^*. Also, for $k \in \mathbb{N}$, X_k is a simplicial set, so we have

$$X_k(\eta) : X_{kn} \longrightarrow X_{km}$$

which we denote by η_v^*. Hence, a simplicial space can be considered as an array of sets X_{kn} with commuting horizontal and vertical face and degeneracy maps.

$$\begin{tikzcd}
\vdots \arrow[d, shift right, "d_i^v"'] & & \\
X_{01} \arrow[u, shift right, "s_i^v"'] \arrow[r, shift left, "s_i^h"] \arrow[d, shift right, "d_i^v"'] & X_{11} \arrow[l, shift left, "d_i^h"] \arrow[d, shift right, "d_i^v"'] & \\
X_{00} \arrow[u, shift right, "s_i^v"'] \arrow[r, shift left, "s_i^h"] & X_{10} \arrow[u, shift right, "s_i^v"'] \arrow[l, shift left, "d_i^h"] \arrow[r, shift left, "s_i^h"] & \cdots \arrow[l, shift left, "d_i^h"]
\end{tikzcd}$$

Definition 3.1.1. For a simplicial space X, the *simplicial realization* of X is the quotient space

$$|X| := \left(\coprod_{n \in \mathbb{N}} X_n \times \Delta[n] \right) \Big/ \sim$$

with the equivalence relation $(\mu_h^* x, \eta) \sim (x, \mu_* \eta)$ for any morphism μ in $\mathbf{\Delta}$.

This construction can be seen as a functor from $\mathbf{sSp}$, the category of simplicial spaces, to the category $\mathbf{Sp}$. This functor maps each simplicial space to its realization and a morphism $\alpha : X \longrightarrow Y$ of simplicial spaces to the morphism

$$\begin{aligned} |\alpha| : |X| &\longrightarrow |Y| \\ [x, \eta] &\mapsto [\alpha(x), \eta]. \end{aligned}$$

Definition 3.1.2. For a simplicial space X, the diagonal of X is the space $diag(X)_n = X_{nn}$ and for a morphism η in $\mathbf{\Delta}$,

$$\eta^* = \eta_h^* \eta_v^*.$$

We also consider the diagonal construction as a functor from **sSp** to **Sp** that takes a simplicial space to its diagonal and a morphism between two simplicial spaces to the restriction map to the diagonal.

Theorem 3.1.3. *If X is a simplicial space, then $diag(X)$ is simplicially isomorphic to $|X|$. Moreover, the isomorphism between $diag(X)$ to $|X|$ gives a natural transformation between the diagonal and simplicial realization functors.*

Proof. For the proof of this theorem we define a simplicial map from $diag(X)$ to $|X|$ and then we show this map induces a bijection on each $diag(X)_m$.

Let

$$\begin{aligned} \psi_m : diag(X)_m &\longrightarrow |X|_m \\ x &\mapsto [x, id_{\mathbf{m}}]. \end{aligned}$$

This map is well-defined since for $x \in X_{mm}$ and $id_{\mathbf{m}} \in \Delta[m]_m$.

Now let $x \in diag(X)_m$ and $\mu : \mathbf{n} \to \mathbf{m}$ be a morphism in $\mathbf{\Delta}$. We have

$$\psi\mu^*(x) = \psi\mu_h^*\mu_v^*(x) = [\mu_h^*\mu_v^*(x), id_{\mathbf{n}}].$$

Then from the equivalence relations

$$[\mu_h^*\mu_v^*(x), id_{\mathbf{n}}] = [\mu_v^*(x), \mu] = \mu^*[x, id_{\mathbf{m}}] = \mu^*\psi(x).$$

Therefore ψ is a simplicial map.

To show ψ induces a bijection on each $diag(X)_m$, we define an inverse map $\phi : |X| \longrightarrow diag(X)$. Set

$$\widetilde{\phi_m} : \left(\coprod_{n\in\mathbb{N}} X_n \times \Delta[n]\right)_m \longrightarrow diag(X)_m$$
$$[x,\eta] \mapsto X(\eta)x.$$

For $\eta : \mathbf{m} \to \mathbf{n}$, $X(\eta)$ is a map from X_{nm} to X_{mm}, so the image of $\widetilde{\phi_m}$ is an element of the $diag(X)$. Moreover,

$$\phi_m([\mu_h^* x, \eta]) = X(\eta)(\mu_h^* x) = X(\eta)X(\mu)x = X(\mu\eta)x = \phi_m([x, \mu_*\eta].$$

Hence, $\widetilde{\phi_m}$ induces a well-defined map

$$\phi_m : \big(\coprod_{n\in\mathbb{N}} X_n \times \Delta[n]\big)_m \Big/ \sim \longrightarrow diag(X)_m.$$

Moreover,

$$\phi_m\psi_m(x) = \phi_m[x, id_{\mathbf{m}}] = x$$

and

$$\psi_m\phi_m([x,\eta]) = \psi(X(\eta)x) = [X(\eta)x, id_{\mathbf{m}}] = [x,\eta].$$

Therefore $\phi_m\psi_m$ and $\psi_m\phi_m$ are both identity maps for each m. This shows that

$$diag(X) \cong |X|$$

by 2.2.12.

Lastly, since for a morphism $f : X \longrightarrow Y$,

$$\psi \circ diag(f)(x) = [f(x), id_{\mathbf{m}}] = |f|([x, id_{\mathbf{m}}] = |f| \circ \psi(x),$$

ψ gives a natural transformation from the diagonal functor to the simplicial realization functor. Hence, this finishes the proof. $\square$

3.2 Definition of Homotopy Colimit

The main goal of this section is to give the definition of homotopy colimits. We first introduce the nerve of a category and then construct the simplicial replacement of a functor. This allows us to define the homotopy colimits. After we give the definition of homotopy colimits we briefly explain the motivation to define homotopy colimits and end this section with an example.

Definition 3.2.1. The *nerve* of a category C is the space $N(C)$, where $N(C)_n$ is the set of n morphisms that can be consecutively composed in C. Therefore an element $\sigma \in N(C)_n$ is of the form $\sigma = (\sigma(0) \xrightarrow{\alpha_1} \sigma(1) \xrightarrow{\alpha_2} \sigma(2) \dots \xrightarrow{\alpha_n} \sigma(n))$. The face maps of the nerve space are defined as

$$\begin{aligned} d_i : N(C)_n &\longrightarrow N(C)_{n-1} \\ (\sigma(0) \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} \sigma(n)) &\mapsto (\sigma(0) \dots \sigma(i-1) \xrightarrow{\alpha_{i+1}\alpha_i} \sigma(i+1) \dots \sigma(n)) \end{aligned}$$

for $i \neq 0$ or $i \neq n$. d_0 and d_n are defined by omitting α_1 or α_n respectively. The degeneracy maps are defined by

$$\begin{aligned} s_i : N(C)_n &\longrightarrow N(C)_{n-1} \\ (\sigma(0) \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} \sigma(n)) &\mapsto (\sigma(0) \dots \sigma(i) \xrightarrow{id} \sigma(i) \dots \sigma(n)). \end{aligned}$$

Definition 3.2.2. Let $F : C \longrightarrow \mathbf{Sp}$ be a functor. The *simplicial replacement* of F is the simplicial space

$$\Big(\coprod_* F\Big)_n := \coprod_{\sigma \in N(C)_n} F(\sigma(0)).$$

Hence, an element of $(\coprod_* F)_n$ can be represented by (x, σ) where $\sigma \in N(C)_n$ and $x \in F(\sigma(0))$. The horizontal face and degeneracy maps of $\coprod_* F$ are given by

$$\begin{aligned} d_i : \Big(\coprod_* F\Big)_n &\longrightarrow \Big(\coprod_* F\Big)_{n-1} \\ (x, \sigma) &\mapsto \begin{cases} (F(\alpha_1)x, d_0\sigma) & \text{for } i = 0, \\ (x, d_i\sigma) & \text{for } i \neq 0 \end{cases} \end{aligned}$$

and $s_i(x, \sigma) = (x, s_i\sigma)$ for each i.

Now we give the definition of homotopy colimits.

Definition 3.2.3. Let $F : C \longrightarrow \mathbf{Sp}$ be a functor. The *homotopy colimit* of F is the space

$$\underset{C}{hocolim} F := diag\Big(\coprod_* F\Big).$$

We note that for a functor $F : C \longrightarrow \mathbf{Sp}$, an element of $(hocolim_C F)_n$ is of the form (x, σ) where $\sigma \in N(C)_n$ and $x \in F(\sigma(0))_n$. Moreover, if F is a functor from C to $\mathbf{GSp}$, then $hocolim_C F$ is a left G-space with the action defined by $g(x, \sigma) = (gx, \sigma)$.

Now, we briefly talk about the motivation of this construction. Thus, lets consider the following example.

Example 3.2.4. For an ordered simplicial complex $\mathbf{m}$, $\partial\mathbf{m}$ is the ordered simplicial complex with the same vertices of $\mathbf{m}$ and all simplices of $\mathbf{m}$ except $\{0, 1, 2, \ldots, m\}$. We define the space $\partial\Delta[m]$ as

$$(\partial\Delta[m])_n = Mor_{\mathbf{OSC}}(\mathbf{n}, \partial\mathbf{m})$$

and η^* is defined as in 2.2.10 for a morphism η in $\mathbf{\Delta}$.

Consider the category

$$C : \begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & b \\ c & \end{tikzcd}$$

and the following two functors:

$$F : \begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & b \\ c & \end{tikzcd} \mapsto \begin{tikzcd} \partial\Delta[m] \arrow[r, "i"] \arrow[d, "i"] & \Delta[m] \\ \Delta[m] & \end{tikzcd}$$

where both morphisms are mapped to the inclusions,

$$F' : \begin{tikzcd} a \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & b \\ c & \end{tikzcd} \mapsto \begin{tikzcd} \partial\Delta[m] \arrow[r] \arrow[d] & \Delta[0] \\ \Delta[0] & \end{tikzcd}$$

and where both morphisms are mapped to the trivial morphism. By the definition of a push-out, the colimit of the functor F is the push-out (See [5]). Hence, even though $|F(a)| \simeq |F'(a)|$, $|F(b)| \simeq |F'(b)|$, $|F(c)| \simeq |F'(c)|$ naturally, the geometric realization of the colimit of the functor F is not weakly equivalent to the geometric realization of the colimit of the functor F', because the colimit of the functor F is $\partial\Delta[m+1]$, while the colimit of the functor F' is $\Delta[0]$. Therefore the usual colimit is not homotopy invariant. However, homotopy colimits do not have this problem.

Theorem 3.2.5. *Let $F, F' : C \longrightarrow \mathbf{Sp}$ be two functors. If for each $c \in C$, $|F(c)| \simeq |F'(c)|$ naturally, then $|\underset{C}{hocolim} F| \simeq |\underset{C}{hocolim} F'|$.*

Proof. See [3, Remark 4.14]. $\square$

Example 3.2.6. For a space X, if we consider the constant functor

$$F : C \longrightarrow \mathbf{Sp}$$
$$c \mapsto X$$

such that $F(\alpha) = id$ for each morphism α in C, then

$$\underset{C}{hocolim} F \cong X \times N(C).$$

To show this we define a simplicial map

$$\psi_n : (\underset{C}{hocolim} F)_n \longrightarrow X_n \times N(C)_n$$
$$(x, \sigma) \mapsto (x, \sigma)$$

for $x \in F(\sigma(0))_n = X_n$ and $\sigma \in N(C)_n$. This map is a simplicial map since for each morphism α in C, $F(\alpha) = id$ and moreover, this simplicial map is clearly one-to-one and onto for each degree n. Hence, by 2.2.12, $\underset{C}{hocolim} F \cong X \times N(C)$.

3.3 Classifying Spaces

In this section we first state the definition of a classifying space of a group G for a family of subgroups $\mathcal{F}$ and present a particular model for this classifying space. Then, using a special case of Thomason's Theorem, we show that this space can be seen as the homotopy colimit of a certain functor (See [4]).

We then introduce the construction $X \times_G E_{\mathcal{F}}G$ for a right G-space X. The main theorem of this section states that $X \times_G E_{\mathcal{F}}G$ can also be considered as the homotopy colimit of a functor.

We finish this section with an application of this theorem.

Definition 3.3.1. For a group G, a *family of subgroups* $\mathcal{F}$ is a set of subgroups of G satisfying

(i) If $H \in \mathcal{F}$ and $g \in G$, then $g^{-1}Hg \in \mathcal{F}$, and

(ii) if $H \in \mathcal{F}$ and $K \leq H$, then $K \in \mathcal{F}$.

Definition 3.3.2. Let G be a group and $\mathcal{F}$ be a collection of subgroups of G. A *classifying space* for the family $\mathcal{F}$ is a G-topological space $E_{\mathcal{F}}G$ such that $(E_{\mathcal{F}}G)^H$ is contractible for $H \in \mathcal{F}$ and $(E_{\mathcal{F}}G)^H$ is empty if $H \notin \mathcal{F}$.

The concept of a classifying space can be extended to the category of simplicial sets.

Definition 3.3.3. For a group G and a collection of subgroups $\mathcal{F}$, a simplicial classifying space for $\mathcal{F}$ is a G-space of which geometric realization is a classifying space for $\mathcal{F}$. We also denote simplicial classifying space of a group G for a family of subgroups $\mathcal{F}$ by $E_{\mathcal{F}}G$.

We now give a category construction of a simplicial classifying space for a collection of subgroups $\mathcal{F}$ of a group G.

Definition 3.3.4. Let C be a category and $F : C \longrightarrow \mathbf{Sets}$ be a functor. The *Grothendieck construction* of F over C is the category whose objects are pairs (c, x) with $c \in obj(C)$ and $x \in F(c)$ and whose morphisms from (c, x) to (e, y) are morphisms $\alpha : c \to e$ in C such that

$$F(\alpha)(x) = y.$$

We denote this category by $\int_C F$.

Remark 3.3.5. For a functor $F : C \longrightarrow \mathbf{GSets}$, the $\int_C F$ has a G-action on the objects defined by $g(c, x) = (c, gx)$. Therefore, $N(\int_C F)$ is a left G-space with the action defined by

$$g[(c_0, x_0) \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} (c_n, x_n)] = [(c_0, gx_0) \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} (c_n, gx_n)].$$

We note that α_i is a morphism from (c_{i-1}, gx_{i-1}) to (c_i, gx_i) since

$$F(\alpha_i)(gx_{i-1}) = gF(\alpha_i)(x_{i-1}) = gx_i.$$

Definition 3.3.6. For a finite group G and a family of subgroups $\mathcal{F}$ of G, the *orbit category* $\mathcal{O}_{\mathcal{F}}G$ of G for $\mathcal{F}$ is the full subcategory of $\mathcal{O}_G$ generated by the transitive G-sets G/H for $H \in \mathcal{F}$.

Remark 3.3.7. We define the functor

$$\mathcal{I} : \mathcal{O}_{\mathcal{F}}G \longrightarrow \mathbf{GSets}$$
$$G/H \mapsto G/H$$

where the image of $\mathcal{I}$ is considered as a discrete G-space. Similarly for a morphism $\mathcal{I}(f) = f$ is considered as a simplicial map.

Theorem 3.3.8. *Let G be a group and $\mathcal{F}$ be a family of subgroups of G. Then G-space $N(\int_{\mathcal{O}_{\mathcal{F}}G} \mathcal{I})$ (see 3.3.5) is a simplicial classifying space of G for $\mathcal{F}$.*

We note that this is a known model for the classifying space for a family of subgroups $\mathcal{F}$. For a proof reader may see [2, Proposition 2.9].

Remark 3.3.9. We take $N(\int_{\mathcal{O}_{\mathcal{F}}G} \mathcal{I})$ as model for $E_{\mathcal{F}}G$.

For a functor $F : C \longrightarrow \mathbf{GSp}$, $hocolim_C F$ is a G-space with the action $g(x, \sigma) = (gx, \sigma)$.

Also, note that **GSets** can be considered as a subcategory of **GSp** regarding each G-set as a discrete space.

Theorem 3.3.10. *If $F : C \longrightarrow$ **GSets** is a functor, then $hocolim_C F$ is simplicially isomorphic to $N(\int_C F)$. Moreover, these two left G-spaces are isomorphic in the category* **GSp** *(See [4, Theorem 1.2]).*

Proof. We prove this theorem by defining a G-simplicial map and its inverse from $hocolim_C F$ to $N(\int_C F)$.

An element of $(hocolim_C F)_m$ is represented by (x, σ) where $\sigma \in N(C)_m$ and $x \in F(\sigma(0))_m$.

An element τ of $N(\int_C F)_m$ is of the form

$$(c_0, x) \xrightarrow{\alpha_1} (c_1, F(\alpha_1)x) \xrightarrow{\alpha_2} (c_2, F(\alpha_2\alpha_1)x) \xrightarrow{\alpha_3} \dots \xrightarrow{\alpha_m} (c_m, F(\alpha_m \dots \alpha_2\alpha_1)x)$$

where the c_i are objects of C and α_i are morphisms in C with $x \in F(c_0)$. With this setup we define

$$\begin{aligned}\psi_m : (hocolim_C F)_m &\longrightarrow N(\int_C F)_m \\ (x, \sigma) &\mapsto (\sigma(0), x) \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_m} (\sigma(m), F(\alpha_m \dots \alpha_2\alpha_1)x)\end{aligned}$$

where $\sigma \in N(C)_m$ is the nerve element $\sigma(0) \xrightarrow{\alpha_1} \sigma(1) \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_m} \sigma(m)$.

Let $(x, \sigma) \in hocolim_C F$. We show that ψ is a simplicial map using face and degeneracy maps. Since $F : C \longrightarrow \mathbf{GSets}$, $F(\sigma(0))$ is a discrete space for each σ, i.e., vertical face and degeneracy maps of $\coprod_* F$ are all identity. Now, for d_0

$$\begin{aligned}\psi_{m-1} d_0(x, \sigma) &= \psi_{m-1}(F(\alpha_1)x, d_0\sigma) \\ &= (\sigma(1), F(\alpha_1)x) \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_m} ((\sigma(m), F(\alpha_m \dots \alpha_1)x)).\end{aligned}$$

On the other hand,

$$\begin{aligned}d_0(\psi_m(x, \sigma)) &= d_0((\sigma(0), x) \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_m} (\sigma(m), F(\alpha_m \dots \alpha_1)x)) \\ &= (\sigma(1), F(\alpha_1)x) \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_m} ((\sigma(m), F(\alpha_m \dots \alpha_1)x)).\end{aligned}$$

For d_m,

$$\begin{aligned}\psi_{m-1} d_m(x, \sigma) &= \psi_{m-1}(x, d_m\sigma) \\ &= (\sigma(0), x) \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_{m-1}} (\sigma(m-1), F(\alpha_{m-1} \dots \alpha_1)x)\end{aligned}$$

and

$$\begin{aligned}d_m(\psi_m(x, \sigma)) &= d_m((\sigma(0), x) \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_m} (\sigma(m), F(\alpha_m \dots \alpha_1)x)) \\ &= (\sigma(0), x) \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_{m-1}} (\sigma(m-1), F(\alpha_{m-1} \dots \alpha_1)x).\end{aligned}$$

Now let $0 < i < m$,

$$\psi_{m-1}d_i(x,\sigma) = \psi_{m-1}(x, d_i\sigma) = (\sigma(0), x) \xrightarrow{\alpha_1} \dots$$
$$\dots(\sigma(i-1), F(\alpha_{i-1}\dots\alpha_1)x) \xrightarrow{\alpha_{i+1}\alpha_i} (\sigma(i+1), F(\alpha_{i+1}\dots\alpha_1)x)\dots$$
$$\dots \xrightarrow{\alpha_m} (\sigma(m), F(\alpha_m\dots\alpha_1)x).$$

Also,

$$d_i(\psi_m(x,\sigma)) = d_i((\sigma(0), x) \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_m} (\sigma(m), F(\alpha_m\dots\alpha_2\alpha_1)x))$$

and this is equal to

$$(\sigma(0), x) \xrightarrow{\alpha_1} \dots(\sigma(i-1), F(\alpha_{i-1}\dots\alpha_1)x) \xrightarrow{\alpha_{i+1}\alpha_i} (\sigma(i+1), F(\alpha_{i+1}\dots\alpha_1)x)\dots$$
$$\dots \xrightarrow{\alpha_m} (\sigma(m), F(\alpha_m\dots\alpha_1)x).$$

Now, for any i,

$$\psi_{m+1}s_i(x,\sigma) = \psi_{m+1}(x, s_i\sigma) = (\sigma(0), x) \xrightarrow{\alpha_1} \dots$$
$$\dots(\sigma(i), F(\alpha_i\dots\alpha_1)x) \xrightarrow{id} (\sigma(i), F(\alpha_i\dots\alpha_1)x)\dots$$
$$\dots \xrightarrow{\alpha_m} (\sigma(m), F(\alpha_m\dots\alpha_1)x).$$

Also,

$$s_i(\psi_m(x,\sigma)) = s_i((\sigma(0), x) \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_m} (\sigma(m), F(\alpha_m\dots\alpha_2\alpha_1)x))$$

and this is equal to

$$(\sigma(0), x) \xrightarrow{\alpha_1} \dots(\sigma(i), F(\alpha_i\dots\alpha_1)x) \xrightarrow{id} (\sigma(i), F(\alpha_i\dots\alpha_1)x)\dots$$
$$\dots \xrightarrow{\alpha_m} (\sigma(m), F(\alpha_m\dots\alpha_1)x).$$

Therefore ψ is a simplicial map.

Now, again for $(x,\sigma) \in (hocolim_C F)_m$

$$\psi(g(x,\sigma)) = \psi(gx,\sigma) = (\sigma(0), gx) \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_m} (\sigma(n), F(\alpha_m...\alpha_2\alpha_1)gx).$$

We recall that F is a functor to **GSets** so each $F(\alpha_i)$ is a G-simplicial map, i.e.,

$$\psi(g(x,\sigma)) = (\sigma(0), gx) \xrightarrow{\alpha_1} (\sigma(1), gF(\alpha_1)x) \xrightarrow{f_3} \dots \xrightarrow{\alpha_m} (\sigma(n), gF(\alpha_m...\alpha_2\alpha_1)x)$$
$$= g\psi(x,\sigma).$$

Thus, ψ is a G-simplicial map.

Now we define the inverse map of ψ_m for each m.

$$\phi_m : N(\int_C F)_m \longrightarrow (\underset{C}{hocolim} F)_m$$
$$(c_0, x) \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_m} (c_m, F(\alpha_m \dots \alpha_1)x) \mapsto (x, c_0 \xrightarrow{\alpha_1} c_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_m} \alpha_m).$$

Let $(x, \sigma) \in (hocolim_C F)_m$ with $\sigma := [c_0 \xrightarrow{\alpha_1} c_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_m} c_m]$.

$$\phi\psi(x, \sigma) = \phi(x, (\sigma(0), x) \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_m} (\sigma(n), F(\alpha_m \dots \alpha_1)x)) = (x, \sigma).$$

Now let $\tau := [(c_0, x) \xrightarrow{\alpha_1} (c_1, F(\alpha_1)x) \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_m} (c_m, F(\alpha_m \dots \alpha_1)x)]$ be an element of $N(\int_C F)_m$.

$$\begin{aligned} \psi\phi(\tau) &= \psi(x, c_0 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_m} c_m) \\ &= (c_0, x) \xrightarrow{\alpha_1} (c_1, F(\alpha_1)x) \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_m} (c_m, F(\alpha_m \dots \alpha_1)x) = \tau. \end{aligned}$$

Hence, ϕ_m is the inverse of ψ_m for each m. By 2.2.12, to finish the proof we only need to show that ϕ_m are G-maps. Instead of showing that directly we use the fact that ϕ_m is the inverse of ψ_m and ψ_m is a G-map. Hence, for an element τ of $N(\int_C F)_m$

$$g\phi_m(\tau) = \phi_m\psi_m(g\phi_m(\tau)) = \phi_m(g\psi_m(\phi_m(\tau))) = \phi_m(g\tau).$$

$\square$

Corollary 3.3.11. *For a group G and family of subgroups $\mathcal{F}$, the space $\underset{\mathcal{O}_{\mathcal{F}}G}{hocolim}\mathcal{I}$ is a model of classifying space for $\mathcal{F}$.*

Proof. By 3.3.10, $E_{\mathcal{F}}G = N(\int_{\mathcal{O}_{\mathcal{F}}G} \mathcal{I}) \cong \underset{\mathcal{O}_{\mathcal{F}}G}{hocolim}\mathcal{I}$. $\square$

For a right G-space X and a left G-space Y, $X \times_G Y$ is defined as $(X \times Y)/G$ where G acts diagonally on $X \times Y$:

$$g(x, y) = (xg^{-1}, gy).$$

Also, for a right G-space X, we define the functor

$$X_{-} : \mathcal{O}_{\mathcal{F}}G \longrightarrow \mathbf{GSp}$$

$$\begin{tikzcd}
G/H \arrow[r, mapsto] \arrow[d, "\bar{\gamma}"'] & X/H \arrow[d, "xH \mapsto x\gamma K"] \\
G/K \arrow[r, mapsto] & X/K
\end{tikzcd}$$

where $\bar{\gamma}$ is the G-map with $\bar{\gamma}(H) = \gamma K$ We note that $X_{-}(\bar{\gamma})$ is well-defined because if $xH = yH$ there is an element $h \in H$ such that $xh = y$. Also, we recall that $\gamma^{-1}h\gamma \in K$ by 2.3.12. Then, $x\gamma K = y\gamma K$ in X/K since

$$x\gamma K = x\gamma(\gamma^{-1}h\gamma)K = y\gamma K.$$

Definition 3.3.12. For a group G and a right G-space X, we define generalized homotopy orbit space denoted by $X_{h_{\mathcal{F}}G}$ as

$$X_{h_{\mathcal{F}}G} := X \times_G E_{\mathcal{F}}G.$$

Theorem 3.3.13. *For a right G-space X and a family of subgroups $\mathcal{F}$,*

$$\underset{\mathcal{O}_{\mathcal{F}}G}{hocolim} X_{-} \cong X \times_G E_{\mathcal{F}}G = X_{h_{\mathcal{F}}G}.$$

Proof. We define the simplicial map

$$\begin{aligned}
\psi : \underset{\mathcal{O}_{\mathcal{F}}G}{hocolim} X_{-} &\longrightarrow X \times_G E_{\mathcal{F}}G \\
(xH_0, \sigma) &\mapsto [x, (G/H_0, H_0) \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} (G/H_n, \alpha_n \dots \alpha_1(H_0))]
\end{aligned}$$

where $\sigma = [G/H_0 \xrightarrow{\alpha_1} G/H_1, \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} G/H_n]$ and $xH_0 \in X/H_0$.

To show this map is well-defined, let $xH_0 = yH_0$. Then, $xh = y$ for some $h \in H_0$.

$$\begin{aligned}
\psi(xH_0, \sigma) &= [x, (G/H_0, H_0) \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} (G/H_n, \alpha_n \dots \alpha_1(H_0))] \\
&= [x, (G/H_0, h^{-1}H_0) \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} (G/H_n, h^{-1}\alpha_n \dots \alpha_1(H_0))] \\
&= [xh, (G/H_0, H_0) \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} (G/H_n, \alpha_n \dots \alpha_1(H_0))] \\
&= [y, (G/H_0, H_0) \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} (G/H_n, \alpha_n \dots \alpha_1(H_0))] = \psi(yH_0, \sigma).
\end{aligned}$$

by the quotient relation on $X \times_G E_{\mathcal{F}}G$.

Now, let

$$[x, (G/H_0, g_0H_0) \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} (G/H_n, g_nH_n)] \in (X \times_G E_{\mathcal{F}}G)_n.$$

Then,

$$\begin{aligned}\psi(xg_0H_0, G/H_0 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} G/H_n) &= [xg_0, (G/H_0, H_0) \xrightarrow{\alpha_1} \dots (G/H_n, g_0^{-1}g_nH_n)] \\ &= [x, (G/H_0, g_0H_0) \xrightarrow{\alpha_1} \dots (G/H_n, g_nH_n)].\end{aligned}$$

since $\alpha_i \dots \alpha_1(H_0) = g_0^{-1}g_iH_i$. Hence, ψ is onto.

Also, if $\psi(xH_0, \sigma) = \psi(yH_0, \sigma')$, then

$$\begin{aligned}(xg^{-1}, (G/H_0, gH_0) \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} (G/H_n, g\alpha_n \dots \alpha_1(H_0))) = (y, (G/H_0', H_0') \xrightarrow{\alpha_1'} \dots \\ \dots \xrightarrow{\alpha_n'} (G/H_n', \alpha_n' \dots \alpha_1'(H_0'))).\end{aligned}$$

This implies

$$\begin{aligned}(G/H_0, gH_0) \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} (G/H_n, g\alpha_n \dots \alpha_1(H_0)) = (G/H_0', H_0') \xrightarrow{\alpha_1'} \dots \\ \dots \xrightarrow{\alpha_n'} (G/H_n', \alpha_n' \dots \alpha_1'(H_0')).\end{aligned}$$

i.e., for each i, $H_i = H_i'$, $\alpha_i = \alpha_i'$ and $g \in H_0$. Since $g \in H_0$ and $xg^{-1} = y$, $xH_0 = yH_0$. Therefore ψ is one-to-one.

Now, we show ψ is a simplicial map by considering face and degeneracy maps. Letting $\sigma := [G/H_0 \xrightarrow{\alpha_1} G/H_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} G/H_n] \in N(\mathcal{O}_{\mathcal{F}}G)$ and $\alpha_1(H_0) = \gamma H_1$

$$\begin{aligned}\psi d_0(xH_0, \sigma) =& \psi(d_0(x)\gamma H_1, G/H_1 \xrightarrow{\alpha_2} G/H_2 \xrightarrow{\alpha_3} \dots \xrightarrow{\alpha_n} G/H_n) \\ =& [d_0(x)\gamma, (G/H_1, H_1) \xrightarrow{\alpha_2} (G/H_2, \alpha_2(H_1)) \xrightarrow{\alpha_3} \dots \\ & \dots \xrightarrow{\alpha_n} (G/H_n, \alpha_n \dots \alpha_2(H_1))] \\ =& [d_0(x), (G/H_1, \gamma H_1) \xrightarrow{\alpha_2} (G/H_2, \alpha_2(\gamma H_1)) \xrightarrow{\alpha_3} \dots \\ & \dots \xrightarrow{\alpha_n} (G/H_n, \alpha_n \dots \alpha_2(\gamma H_1))].\end{aligned}$$

On the other hand,

$$\begin{aligned}d_0\psi(xH_0, \sigma) =& d_0[x, (G/H_0, H_0) \xrightarrow{\alpha_1} (G/H_1, \alpha_1(H_0)) \xrightarrow{\alpha_2} \dots \\ & \dots \xrightarrow{\alpha_n} (G/H_n, \alpha_n \dots \alpha_1(H_0))] \\ =& [d_0(x), (G/H_1, \gamma H_1) \xrightarrow{\alpha_2} (G/H_2, \alpha_2(\gamma H_1) \xrightarrow{\alpha_3} \dots \\ & \dots \xrightarrow{\alpha_n} (G/H_n, \alpha_n \dots \alpha_2(\gamma H_1))].\end{aligned}$$

Also, for $0 < i < n$,

$$\begin{aligned}\psi d_i(xH_0,\sigma) =&\psi(d_i(x)H_0, G/H_0 \xrightarrow{\alpha_1} \dots G/H_{i-1} \xrightarrow{\alpha_{i+1}\alpha_i} G/H_{i+1} \dots \xrightarrow{\alpha_n} G/H_n)\\ =&[d_i(x),(G/H_0,H_0)\dots(G/H_{i-1},\alpha_{i-1}\dots\alpha_1(H_0)) \xrightarrow{\alpha_{i+1}\alpha_i}\\ &(G/H_{i+1},\alpha_{i+1}\dots\alpha_1(H_0))\dots(G/H_n,\alpha_n\dots\alpha_1(H_0))]\end{aligned}$$

and

$$\begin{aligned}d_i\psi(xH_0,\sigma) =&d_i[x,(G/H_0,H_0) \xrightarrow{\alpha_1} (G/H_1,\alpha_1(H_0)) \xrightarrow{\alpha_2} \dots\\ &\dots \xrightarrow{\alpha_n} (G/H_n,\alpha_n\dots\alpha_1(H_0))]\\ =&[d_i(x),(G/H_0,H_0)\dots(G/H_{i-1},\alpha_{i-1}\dots\alpha_1(H_0)) \xrightarrow{\alpha_{i+1}\alpha_i}\\ &(G/H_{i+1},\alpha_{i+1}\dots\alpha_1(H_0))\dots(G/H_n,\alpha_n\dots\alpha_1(H_0))].\end{aligned}$$

Finally, consider d_n:

$$\begin{aligned}\psi d_n(xH_0,\sigma) =&\psi(d_n(x)H_0, G/H_0 \xrightarrow{\alpha_1} G/H_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} G/H_{n-1})\\ =&[d_n(x),(G/H_0,H_0) \xrightarrow{\alpha_1} (G/H_1,\alpha_1(H_0)) \xrightarrow{\alpha_2} \dots\\ &\dots \xrightarrow{\alpha_{n-1}} (G/H_{n-1},\alpha_{n-1}\dots\alpha_1(H_0))]\end{aligned}$$

and

$$\begin{aligned}d_n\psi(xH_0,\sigma) =&d_n[x,(G/H_0,H_0) \xrightarrow{\alpha_1} (G/H_1,\alpha_1(H_0)) \xrightarrow{\alpha_2} \dots\\ &\dots \xrightarrow{\alpha_n} (G/H_n,\alpha_n\dots\alpha_1(H_0))]\\ =&[d_n(x),(G/H_0,H_0) \xrightarrow{\alpha_1} (G/H_1,\alpha_1(H_0)) \xrightarrow{\alpha_2} \dots\\ &\dots \xrightarrow{\alpha_{n-1}} (G/H_{n-1},\alpha_{n-1}\dots\alpha_1(H_0))].\end{aligned}$$

We now consider degeneracy maps. For any $0 \le i \le n$,

$$\begin{aligned}\psi s_i(xH_0,\sigma) =&\psi(s_i(x)H_0, G/H_0 \xrightarrow{\alpha_1} \dots G/H_i \xrightarrow{id} G/H_i \dots \xrightarrow{\alpha_n} G/H_n)\\ =&[s_i(x),(G/H_0,H_0)\dots(G/H_i,\alpha_i\dots\alpha_1(H_0)) \xrightarrow{id}\\ &(G/H_i,\alpha_i\dots\alpha_1(H_0))\dots(G/H_n,\alpha_n\dots\alpha_1(H_0))]\end{aligned}$$

where

$$\begin{aligned}s_i\psi(xH_0,\sigma) =&s_i[x,(G/H_0,H_0) \xrightarrow{\alpha_1} (G/H_1,\alpha_1(H_0)) \xrightarrow{\alpha_2} \dots\\ &\dots \xrightarrow{\alpha_n} (G/H_n,\alpha_n\dots\alpha_1(H_0))]\\ =&[s_i(x),(G/H_0,H_0)\dots(G/H_i,\alpha_i\dots\alpha_1(H_0)) \xrightarrow{id}\\ &(G/H_i,\alpha_i\dots\alpha_1(H_0))\dots(G/H_n,\alpha_n\dots\alpha_1(H_0))].\end{aligned}$$

Hence, ψ is a simplicial map and by 2.2.12

$$\underset{\mathcal{O}_{\mathcal{F}}G}{hocolim} X_{-} \cong X \times_G E_{\mathcal{F}}G = X_{h_{\mathcal{F}}G}.$$

$\square$

For a group G, EG denotes $E_{\{1\}}G$ where $\{1\}$ denotes the family of subgroups which contains only the trivial subgroup.

The G can be considered as a category, written $\mathbf{G}$ with only one object $*$ and

$$Mor_{\mathbf{G}}(*, *) = \{g \mid g \in G\}.$$

Moreover, a right G-space X can be considered as a functor,

$$\begin{aligned} \mathbf{X} : \mathbf{G}^{op} &\longrightarrow Sp \\ * &\mapsto X \end{aligned}$$

and for a morphism $g \in \mathbf{G}^{op}$,

$$\begin{aligned} \mathbf{X}(g) : X &\longrightarrow X \\ x &\mapsto xg. \end{aligned}$$

Corollary 3.3.14. *For a right G-space X,*

$$\underset{\mathbf{G}^{op}}{hocolim} \mathbf{X} \cong X \times_G EG.$$

Proof. We note that $\mathcal{O}_{\{1\}}G \cong \mathbf{G}^{op}$. Also, X_{-} from $\mathcal{O}_{\{1\}}G$ is equal to $\mathbf{X}$. Now, for $\mathcal{F} = \{1\}$, by 3.3.13 we obtain

$$hocolim_{\mathbf{G}} \mathbf{X} \cong X \times_G EG.$$

$\square$

Chapter 4

A Theorem on Function Complexes

For two spaces X and Y, we can introduce a simplicial set structure to the simplicial maps from X to Y. We denote this function complex by $map(X, Y)$. In the first section, we give the definition of the function complex from X to Y. We also introduce homotopy limit $holim_C F$ of a contravariant functor $F : C \longrightarrow \mathbf{Sp}$ in this section.

In the second section, we state that a function complex from homotopy colimit of a covariant functor F to a space X can be decomposed as the homotopy limit of a particular functor. Giving a direct proof to Theorem 4.2.1 is the main goal of this chapter.

4.1 Function Complexes and Homotopy Limits

Definition 4.1.1. For two simplicial sets X and Y we define the *function complex* $map(X, Y)$ as the simplicial set

$$map(X, Y)_m = Mor_{\mathbf{Sp}}(X \times \Delta[m], Y)$$

and for $\nu : \mathbf{n} \to \mathbf{m}$ in $\mathbf{\Delta}$,

$$\begin{aligned} \nu^* : map(X,Y)_m &\longrightarrow map(X,Y)_n \\ f &\mapsto (\nu^* f)(x,\eta) = f(x,\nu\eta). \end{aligned}$$

Note that for a simplicial map $f : X \times \Delta[m] \longrightarrow Y$, $\nu^* f : X \times \Delta[n] \longrightarrow Y$ is a simplicial map.

For further information on function complexes, the reader may check [7].

Definition 4.1.2. Let $F, F' : C \longrightarrow \mathbf{Sp}$ be two contravariant functors. The *natural space* from F to F', denoted by $Nat(F,F')$, is defined as the subspace

$$Nat(F,F') \subseteq \prod_{c \in obj(C)} map(F(c), F'(c))$$

of elements $(f_c)_{c\in C} \in \prod_{c\in C} map(F(c), F'(c))$ such that $(f_c)_{c\in C} \in Nat(F,F')$ if and only if for every morphism $\alpha \in Mor_C(c,e)$ the following diagram commutes:

$$\begin{tikzcd} F(e) \times \Delta[m] \arrow[r, "f_e"] \arrow[d, "F(\alpha)\times id"'] & F'(e) \arrow[d, "F'(\alpha)"] \\ F(c) \times \Delta[m] \arrow[r, "f_c"] & F'(c). \end{tikzcd}$$

We note that $Nat(F,F')$ for covariant functors can be defined similarly.

We write (f_c) for an element $(f_c)_{c\in C} \in \prod_{c\in C} map(F(c), F'(c))$. Also, we refer an element $(f_c) \in Nat(F,F')$ as *natural*.

Now, we should show that $Nat(F,F')$ is actually a space. For that, we show for any element $(f_c) \in Nat(F,F')$ and for any morphism ν in $\mathbf{\Delta}$, $\nu^*(f_c)$ is again natural.

For any morphism $\nu \in \mathbf{\Delta}$, we consider $F'(\alpha)(\nu^* f_e)(x,\eta)$ where $\alpha : c \to e$ is a morphism in C, $x \in F(e)$ and $\eta \in \Delta[m]$. Then, since (f_c) is natural

$$\begin{aligned} F'(\alpha)(\nu^* f_e)(x,\eta) &= F'(\alpha) f_e(x, \nu\eta) \\ &= f_c(F(\alpha)\times id)(x,\nu\eta) = f_c(F(\alpha)x, \nu\eta) \\ &= (\nu^* f_c)(F(\alpha)\times id)(x,\eta). \end{aligned}$$

Thus, $\nu^*(f_c)$ is natural for $(f_c) \in Nat(F, F')$.

Definition 4.1.3. For a category C and an object $c \in obj(C)$, the *under-category* $c \downarrow C$ of c is the category whose objects are the morphisms $\alpha : c \to e$ in C and morphisms from $\alpha : c \to e$ to $\alpha' : c \to e'$ are the morphism $\beta : e \to e'$ in C such that the following diagram commutes.

$$\begin{tikzcd} e \arrow[rr, "\beta"] & & e' \\ & c \arrow[ul, "\alpha"] \arrow[ur, "\alpha'"'] & \end{tikzcd}$$

Definition 4.1.4. For a category C we define contravariant functor

$$\begin{aligned} \underline{N} : C &\longrightarrow Sp \\ c &\mapsto N(c \downarrow C). \end{aligned}$$

For an element

$$\tau = \left[\begin{tikzcd} c_0 \arrow[r, "\alpha_1"] & c_1 \arrow[r, "\alpha_2"] & c_2 \arrow[r, "\alpha_3"] & \dots \arrow[r, "\alpha_n"] & c_n \\ c \arrow[u, "\gamma_0"] \arrow[r, phantom, "="] & c \arrow[u, "\gamma_1"] \arrow[r, phantom, "="] & c \arrow[u, "\gamma_2"] \arrow[r, phantom, "="] & \dots \arrow[r, phantom, "="] & c \arrow[u, "\gamma_n"'] \end{tikzcd} \right] \in N(c \downarrow C)$$

and a morphism $\alpha : e \to c$, $\underline{N}(\alpha) : N(c \downarrow C) \longrightarrow N(e \downarrow C)$ is defined by

$$\underline{N}(\alpha)(\tau) = \left[\begin{tikzcd} c_0 \arrow[r, "\alpha_1"] & c_1 \arrow[r, "\alpha_2"] & c_2 \arrow[r, "\alpha_3"] & \dots \arrow[r, "\alpha_n"] & c_n \\ e \arrow[u, "\gamma_0\alpha"] \arrow[r, phantom, "="] & e \arrow[u, "\gamma_1\alpha"] \arrow[r, phantom, "="] & e \arrow[u, "\gamma_2\alpha"] \arrow[r, phantom, "="] & \dots \arrow[r, phantom, "="] & e \arrow[u, "\gamma_n\alpha"'] \end{tikzcd} \right] \in N(e \downarrow C).$$

For simplicity of notation we write an element $\tau \in N(c \downarrow C)$ as

$$\tau = \left[\begin{tikzcd} c_0 \arrow[r, "\alpha_1"] & c_1 \arrow[r, "\alpha_2"] & \dots \arrow[r, "\alpha_n"] & c_n \\ & c \arrow[ul, "\gamma_0"] \arrow[u, "\gamma_1"'] \arrow[urr, "\gamma_n"'] & & \end{tikzcd} \right].$$

Definition 4.1.5. For a contravariant functor $F : C \longrightarrow \mathbf{Sp}$, the *homotopy limit* of F is defined as the space

$$\underset{C}{holim} F := Nat(\underline{N}, F).$$

For a covariant functor $F : C \longrightarrow \mathbf{Sp}$ and a space X, the functor denoted by $map(F, X)$ is defined as

$$\begin{tikzcd} map(F,X): & C \arrow[r] & \mathbf{Sp} \\ & c \arrow[r] \arrow[d, "\alpha"'] & map(F(c), X) \\ & e \arrow[r] & map(F(e), X). \arrow[u, "f \mapsto f(F(\alpha)x, \eta)"'] \end{tikzcd}$$

4.2 A Proof of the Theorem on Function Complexes

The main goal of this section is to prove Theorem 4.2.1. To prove this theorem we define a simplicial map ψ from $map(hocolim_C F, X)$ to $holim_C(map(F, X))$. Then, we show that for each $m \in \mathbb{N}$, ψ_m is one-to-one and onto by introducing an inverse map ϕ_m for each ψ_m.

Theorem 4.2.1. *For a covariant functor $F : C \longrightarrow \mathbf{Sp}$ and a space X,*

$$map(\underset{C}{hocolim} F, X) \cong \underset{C}{holim}(map(F, X))$$

as simplicial sets.

This isomorphism is known in homotopy theory and for a statement reader may see [1, Theorem XII.4.1]. In this thesis, we introduce a proof for this isomorphism explicitly writing simplicial maps between these two spaces.

To prove this theorem, we introduce some notations and prove an auxiliary Lemma 4.2.4.

For an element

$$\tau = \left[\begin{tikzcd} c_0 \arrow[r, "\alpha_1"] & c_1 \arrow[r, "\alpha_2"] & \dots \arrow[r, "\alpha_n"] & c_n \\ & c \arrow[ul, "\gamma_0"] \arrow[u, "\gamma_1"'] \arrow[urr, "\gamma_n"'] & & \end{tikzcd}\right] \in N(c \downarrow C),$$

$\widetilde{\tau}$ denotes the element

$$\widetilde{\tau} = [c_0 \xrightarrow{\alpha_1} c_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} c_n] \in N(C).$$

Also, for $\sigma = [c_0 \xrightarrow{\alpha_1} c_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} c_n] \in N(C)$, let

$$\bar{\sigma} = \left[\begin{tikzcd} c_0 \arrow[r, "\alpha_1"] & c_1 \arrow[r, "\alpha_2"] & \dots \arrow[r, "\alpha_n"] & c_n \\ & c_0 \arrow[ul, "id"] \arrow[u, "\alpha_1"'] \arrow[urr, "\alpha_n \dots \alpha_1"'] & & \end{tikzcd}\right] \in N(c_0 \downarrow C).$$

Remark 4.2.2. The following properties can be shown by direct calculation. For $\sigma = c_0 \xrightarrow{\alpha_1} c_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} c_n \in N(C)$, $\tilde{\bar{\sigma}} = \sigma$ and if $i \neq 0$ then $d_i(\bar{\sigma}) = \overline{d_i(\sigma)}$. Also, $d_0(\bar{\sigma}) = \underline{N}(\alpha_1)(\overline{d_0\sigma})$ and for all i, $s_i(\bar{\sigma}) = \overline{s_i(\sigma)}$.

Moreover, for

$$\tau = \left[\begin{tikzcd} c_0 \arrow[r, "\alpha_1"] & c_1 \arrow[r, "\alpha_2"] & \dots \arrow[r, "\alpha_n"] & c_n \\ & c \arrow[ul, "\gamma_0"] \arrow[u, "\gamma_1"'] \arrow[urr, "\gamma_n"'] & & \end{tikzcd}\right] \in N(c \downarrow C)$$

$\underline{N}(\gamma_0)(\bar{\tilde{\tau}}) = \tau$ and for all i, $d_i(\tilde{\tau}) = \widetilde{d_i(\tau)}$. Similarly $s_i(\tilde{\tau}) = \widetilde{s_i(\tau)}$.

Let $F : C \longrightarrow \mathbf{Sp}$ be a functor. For $\sigma = [c_0 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} c_n] \in N(C)$ and η a morphism in $\mathbf{\Delta}$, we define

$${}_\sigma(\eta^*) : F(\sigma(0)) \longrightarrow F(\eta^*\sigma(0))$$

using d^i and s^i by

$$\begin{aligned} {}_\sigma(s^{i^*})x &= x \text{ for each } 0 \leq i \leq n, \\ {}_\sigma(d^{i^*})x &= x \text{ if } i \neq 0, \text{ and} \\ {}_\sigma(d^{0^*})x &= F(\alpha_1)x. \end{aligned}$$

Remark 4.2.3. For a functor $F : C \longrightarrow \mathbf{Sp}$ and a space X, we recall that if $(f_c) \in Nat(\underline{N}, map(F, X))$, then for $\sigma \in N(C)_n$ and η a morphism in $\mathbf{\Delta}$,

$$(f_{\sigma(0)})_n(\bar{\sigma}, \eta) : F(\sigma(0)) \times \Delta[n] \longrightarrow X$$

is a simplicial map.

Lemma 4.2.4. *If $F : C \longrightarrow \mathbf{Sp}$ is a functor and X is a space, then for $\sigma \in N(C)_n$ and $x \in F(\sigma(0))$,*

$$(f_{\mu^*\sigma(0)})_n(\overline{\mu^*\sigma}, \eta)_k({}_\sigma(\mu^*)x, \mu) = (f_{\sigma(0)})_n(\mu^*\overline{\sigma}, \eta)_k(x, \mu).$$

Proof. We prove this lemma by considering two different cases. In the first case, μ does not contain any d^0 in its decomposition into d^i and s^i. Then,

$$(f_{\mu^*\sigma(0)})_n(\overline{\mu^*\sigma}, \eta)_k({}_\sigma(\mu^*)x, \mu) = (f_{\sigma(0)})_n(\mu^*\overline{\sigma}, \eta)_k(x, \mu).$$

Now, let $\mu = d^0$ as the second case. Then,

$$(f_{\mu^*\sigma(0)})_n(\overline{\mu^*\sigma}, \eta)_k({}_\sigma(\mu^*)x, \mu) = (f_{\sigma(1)})_n(\overline{d_0(\sigma)}, \eta)_k(F(\alpha_1)x, \mu)$$

and by naturality of (f_c)

$$\begin{aligned}(f_{\sigma(1)})_n(\overline{d_0(\sigma)}, \eta)_k(F(\alpha_1)x, \mu) &= (f_{\sigma(0)})_n(\underline{N}(\alpha_1)\overline{d_0(\sigma)}, \eta)_k(x, \mu) \\ &= (f_{\sigma(0)})_n(d_0\bar{\sigma}, \eta)_k(x, \mu).\end{aligned}$$

Since each morphism in $\mathbf{\Delta}$ decomposes in terms of d^i and s^i this completes the proof. $\square$

Proof of Theorem 4.2.1. Recall that $hocolim_C F = diag(\coprod_* F) \cong |\coprod_* F|$ by 3.1.3. Therefore, the homotopy colimit is the space

$$(\underset{C}{hocolim} F)_k = \left(\coprod_{n\geq 0} \left(\coprod_{\sigma\in N(C)_n} F(\sigma(0))_k \right) \times \Delta[n]_k \right) \Big/ \sim$$

with the quotient relations we defined in 3.1.1. Thus, an element of $hocolim_C F$ can be represented by $[(x, \sigma), \eta]$, where for some $n, k \in \mathbb{N}$, $\sigma \in N(C)_n$, $x \in F(\sigma(0))_k$ and $\eta \in \Delta[n]_k$.

Now we define

$$\psi : map(\underset{C}{hocolim} F, X) \longrightarrow \underset{C}{holim}(map(F, X))$$
$$f \mapsto (\psi f)_c$$

where $(\psi f)_c$ is the map

$$(\psi f)_c : N(c \downarrow C) \times \Delta[m] \longrightarrow map(F(c), X)$$

such that for

$$\tau = \left[\begin{tikzcd} c_0 \arrow[r, "\alpha_1"] & c_1 \arrow[r, "\alpha_2"] & \dots \arrow[r, "\alpha_n"] & c_n \\ & c \arrow[ul, "\gamma_0"] \arrow[u, "\gamma_1"'] \arrow[urr, "\gamma_n"'] & & \end{tikzcd} \right] \in N(c \downarrow C)$$

and $\eta \in \Delta[m]_n$, $(\psi f_c)_n(\tau, \eta)$ is defined as the map

$$(\psi f_c)_n(\tau, \eta)_k : F(c)_k \times \Delta[n]_k \longrightarrow X_k$$
$$(x, \mu) \mapsto f([F(\gamma_0)x, \tilde{\tau}, \mu], \eta\mu).$$

We first show that this map is well-defined and for that we show that for any $f \in map(hocolim_C F, X)$, we have $\psi f_c \in Nat(\underline{N}, map(F, X))$. Let f be an element of $map(hocolim_C F, X)$ and let ν be a morphism in $\mathbf{\Delta}$.

$$\begin{aligned} \psi f_c(\nu^*(\tau, \eta))(x, \mu) &= \psi f_c(\nu^*\tau, \eta\nu)(x, \mu) \\ &= f([(F(\gamma'_0)x, \widetilde{\nu^*\tau}), \mu], \eta\nu\mu) \end{aligned}$$

where

$$\nu^*\tau = \left[\begin{tikzcd} c'_0 \arrow[r, "\alpha'_1"] & c'_1 \arrow[r, "\alpha'_2"] & \dots \arrow[r, "\alpha'_n"] & c'_{n'} \\ & c \arrow[ul, "\gamma'_0"] \arrow[u, "\gamma'_1"'] \arrow[urr, "\gamma'_{n'}"'] & & \end{tikzcd} \right].$$

In addition:

$$\begin{aligned} \nu^*\psi f_c(\tau, \eta)(x, \mu) &= \psi f_c(\tau, \eta)(x, \nu\mu) \\ &= f([(F(\gamma_0)x, \tilde{\tau}), \nu\mu], \eta\nu\mu) \\ &= f([\nu^*(F(\gamma_0)x, \tilde{\tau}), \mu], \eta\nu\mu) \\ &= f([(F(\gamma'_0)x, \widetilde{\nu^*\tau}), \mu], \eta\nu\mu). \end{aligned}$$

Hence, $\psi f_c : N(c \downarrow C) \times \Delta[m] \longrightarrow map(F(c), X)$ is a simplicial map. Moreover, $\psi f_c(\tau, \eta) : F(c) \times \Delta[n] \longrightarrow X$ is a simplicial map for each $(\tau, \eta) \in N(c \downarrow C) \times \Delta[m]$ since

$$\begin{aligned}(\psi f_c)_n(\tau,\eta)_k(\nu^*(x,\mu)) &= \psi f_c(\tau,\eta)(\nu^* x, \mu\nu)\\ &= f([(F(\gamma_0)\nu^* x, \tau), \mu\nu], \eta\mu\nu)\\ &= f\nu^*([(F(\gamma_0)x, \tau), \mu], \eta\mu)\\ &= \nu^* f([(F(\gamma_0)x, \tau), \mu], \eta\mu)\end{aligned}$$

and

$$\nu^* f([(F(\gamma_0)x, \tau), \mu], \eta\mu) = \nu^*(\psi f_c)_n(\tau,\eta)_k(x,\mu).$$

Now we show ψf_c is natural. Let $\alpha : c \to e$ in C. We have the following diagram:

$$\begin{tikzcd} N(e \downarrow C) \times \Delta[m] \arrow[r, "\psi f_e"] \arrow[d, "\underline{N}(\alpha) \times id"'] & map(F(e), X) \arrow[d, "\bar{\alpha}"] \\ N(c \downarrow C) \times \Delta[m] \arrow[r, "\psi f_c"] & map(F(c), X) \end{tikzcd} .$$

Here $\bar{\alpha}$ denotes $map(F, X)(\alpha)$ and

$$\Big(\bar{\alpha}(\psi f_e)_n(\tau,\eta)\Big)_k(x,\mu) = (\psi f_e)_n(\tau,\eta)_k(F(\alpha)x, \mu).$$

Also,

$$(\psi f_e)_n(\tau,\eta)_k(F(\alpha)x,\mu) = f([F(\gamma_0\alpha)x, \tilde{\tau}, \mu], \eta\mu),$$

and

$$\begin{aligned}(\psi f_c)_n((\underline{N}(\alpha) \times id)(\tau,\eta))_k(x,\mu) &= (\psi f_c)_n(\underline{N}(\alpha)\tau, \eta)_k(x,\mu)\\ &= (\psi f_c)_n(\underline{N}(\alpha)\tau, \eta)_k(x,\mu)\\ &= f([F(\gamma_0\alpha)x, \tilde{\tau}, \mu], \eta\mu).\end{aligned}$$

This shows that $\psi f_c \in Nat(\underline{N}, map(F, X))$. Hence, we have a well-defined map $\psi : map(hocolim_C F, X) \longrightarrow holim_C(map(F, X))$. Now we show the map ψ is a simplicial map. This is a consequence of the fact that pre-composition commutes with post-composition.

Let f be an element of $map(hocolim_C F, X)$ and $\nu : \mathbf{m} \to \mathbf{l}$ in $\mathbf{\Delta}$. Then,

$$\psi(\nu^* f)_c(\tau, \eta)(x, \mu) = \nu^* f([F(\gamma_0)x, \tau, \mu], \eta\mu) = f([F(\gamma_0)x, \tilde{\tau}, \mu], \nu\eta\mu).$$

Also, $\nu^*\psi(f)_c(\tau, \eta)(x, \mu) = \psi(f)_c(\tau, \nu\eta)(x, \mu) = f(F(\gamma_0)x, \tilde{\tau}, \mu], \nu\eta\mu)$. This shows that ψ is a simplicial map.

Now, we define the map

$$\phi_m : \underset{C}{holim}(map(F, X)) \longrightarrow map(\underset{C}{hocolim} F, X)$$

where

$$\phi_m(f_c) : \underset{C}{hocolim} F \times \Delta[m] \longrightarrow X$$

is a simplicial map such that

$$\phi_m(f_c)_k([(x, \sigma), \eta], \mu) = (f_{\eta^*\sigma(0)})_k(\overline{\eta^*\sigma}, \mu)_k({}_\sigma(\eta^*)x, id).$$

Note that we omit m from the notation of ϕ_m after this point. We show that $\phi(f_c)$ is well-defined, i.e., $\phi(f_c) \in map(hocolim_C F, X)$.

For a morphism, ν in $\mathbf{\Delta}$

$$\begin{aligned}
\phi(f_c)_k([(x, \sigma), \nu_*\eta], \mu) &= \phi(f_c)_k([(x, \sigma), \nu\eta], \mu) \\
&= (f_{(\nu\eta)^*\sigma(0)})_k(\overline{(\nu\eta)^*\sigma}, \mu)_k({}_\sigma(\nu\eta)^*x, id) \\
&= (f_{\eta^*\nu^*\sigma(0)})_k(\overline{\eta^*\nu^*\sigma}, \mu)_k({}_\sigma(\eta^*)_\sigma(\nu^*)x, id) \\
&= \phi(f_c)_k([\nu_h^*(x, \sigma), \eta], \mu).
\end{aligned}$$

Here, ν_h^* denotes the horizontal map of the simplicial replacement of F. Hence, this shows that if $[(x, \sigma), \eta] \sim [(x', \sigma'), \eta']$ then

$$\phi(f_c)([(x, \sigma), \eta], \mu) = \phi(f_c)([(x', \sigma'), \eta'], \mu).$$

Also, if we let $(f_c) \in Nat(\underline{N}, map(F, X))$, then $\phi(f_c)$ is a simplicial map: For a morphism ν in $\mathbf{\Delta}$, since f_c is a simplicial map for all c and by 4.2.4,

$$\begin{aligned}
\phi(f_c)(\nu^*([(x, \sigma), \eta], \mu)) &= \phi(f_c)([(\nu^*x, \sigma), \eta\nu], \mu\nu) \\
&= (f_{\nu^*\eta^*\sigma(0)})_k(\overline{\nu^*\eta^*\sigma}, \mu\nu)_k({}_\sigma(\nu^*)_\sigma(\eta^*)\nu^*x, id) \\
&= (f_{\eta^*\sigma(0)})_k(\nu^*\overline{\eta^*\sigma}, \mu\nu)_k({}_\sigma(\nu^*)_\sigma(\eta^*)\nu^*x, id) \\
&= (f_{\eta^*\sigma(0)})_k(\overline{\eta^*\sigma}, \mu)_k(\nu_\sigma^*(\eta^*)x, \nu).
\end{aligned}$$

Also, since $(f_{\eta^*\sigma(0)})_k(\overline{\eta^*\sigma},\mu)$ is a simplicial map,

$$\begin{aligned}(f_{\eta^*\sigma(0)})_k(\overline{\eta^*\sigma},\mu)_k({}_\sigma(\eta^*)\nu^*x,\nu) &= \nu^*(f_{\eta^*\sigma(0)})_k(\overline{\eta^*\sigma},\mu)_k({}_\sigma(\eta^*)x,id)\\ &= \nu^*\phi(f_c)([(x,\sigma),\eta],\mu).\end{aligned}$$

At this point the only remaining thing to show is that ϕ_m and ψ_m are inverse of each other. Then, by 2.2.12, the proof will be completed. Let $f \in map(hocolim_C F, X)_m$.

$$\begin{aligned}\phi\psi f([(x,\sigma),\eta],\mu) &= (\psi(f)_{\eta^*\sigma(0)})_k(\overline{\eta^*\sigma},\mu)_k({}_\sigma(\eta^*)(x),id)\\ &= f([({}_\sigma(\eta^*)(x),\widetilde{\overline{\eta^*\sigma}}),id],\mu)\\ &= f([\eta_h^*(x,\sigma),id],\mu)\\ &= f([(x,\sigma),\eta],\mu).\end{aligned}$$

Now, let $(f_c) \in holim_C(map(F,X))$.

$$\begin{aligned}(\psi\phi f_c)_n(\tau,\eta)_k(x,\mu) =&\phi f([(F(\gamma_0)x,\tilde{\tau}),\mu],\eta\mu)\\ =&(f_{\mu^*\tilde{\tau}(0)})_k(\overline{\mu^*\tilde{\tau}},\eta\mu)_k({}_{\tilde{\tau}}(\mu^*)F(\gamma_0)x,id).\end{aligned}$$

By 4.2.4

$$(f_{\mu^*\tilde{\tau}(0)})_k(\overline{\mu^*\tilde{\tau}},\eta\mu)_k({}_{\tilde{\tau}}(\mu^*)F(\gamma_0)x,id) = (f_{\tilde{\tau}(0)})_k(\mu^*\overline{\tilde{\tau}},\eta\mu)_k(F(\gamma_0)x,id).$$

Then since $\tilde{\tau} = c_0$ from the previous notation and f_{c_0} is a simplicial map we have

$$(f_{c_0})_k(\mu^*\overline{\tilde{\sigma}},\eta\mu)_k(F(\gamma_0)x,id) = (f_{c_0})_n(\bar{\tilde{\sigma}},\eta)_k(F(\gamma_0)x,\mu).$$

Lastly, since (f_c) is natural

$$(f_{c_0})_n(\bar{\tilde{\sigma}},\eta)_k(F(\gamma_0)x,id) = (f_c)_n(\underline{N}(\gamma_0)\bar{\tilde{\sigma}},\eta)_k(x,\mu) = (f_c)_n(\sigma,\eta)_k(x,\mu)$$

and this completes the proof. $\square$

Chapter 5

Applications

It is proved by D.A. Ramras [2] that for a topological space X there is a homeomorphism

$$map_G(E_{\mathcal{F}}G, X) \cong \underset{\mathcal{O}_{\mathcal{F}}G}{holim} X^{-}.$$

In the first section, we give a proof for Theorem 5.1.6 which states that $map_G(E_{\mathcal{F}}G, X)$ and $\underset{\mathcal{O}_{\mathcal{F}}G}{holim} X^{-}$ are simplicially isomorphic as simplicial sets. In spite of the fact that one may consider Theorem 5.1.6 to be a simplicial set version of [2, Theorem 3.2], we note that our theorem does not claim to be an alternative proof of the theorem [2, Theorem 3.2]. The proof of this theorem is an application of 4.2.1.

In the second section, we observe some other applications of Theorem 4.2.1.

5.1 Decomposition of Generalized Homotopy Fixed Points Space

We begin by giving the background definitions and some auxiliary results that give a relation between fixed points functor and mapping space functor.

Definition 5.1.1. For a G-space X and a subgroup $H \leq G$, we define H*-fixed* *points space* X^H of X as the subspace of X formed by elements fixed by H, i.e.,

$$X^H = \{x \in X \mid hx = x \text{ for all } h \in H\}.$$

Here, we note that for a G-space X and $H \leq G$, if $x \in X_n^H$ and η is a morphism in $\mathbf{\Delta}$, $\eta^* x \in X^H$ since for $h \in H$

$$h\eta^* x = \eta^* hx = \eta^* x.$$

Hence, X^H is a subspace.

Definition 5.1.2. For a group G, a family $\mathcal{F}$ of subgroups of G and a G-space X, the *fixed point* functor is defined as

$$\begin{aligned} X^- : \mathcal{O}_{\mathcal{F}}G &\longrightarrow \mathbf{Sp} \\ G/H &\mapsto X^H. \end{aligned}$$

On morphisms, given a G-map $f : G/H \longrightarrow G/K$ determined by $f(H) = \gamma K$ for $\gamma \in G$, we have

$$\begin{aligned} X^-(f) : X^K &\longrightarrow X^H \\ x &\mapsto \gamma x. \end{aligned}$$

We need to show that $X^-(f)$ is well-defined. By 2.3.12 we know that $\gamma^{-1}H\gamma \leq K$. For $x \in X^K$, $\gamma x \in X^H$ since for $h \in H$,

$$h\gamma x = \gamma\gamma^{-1}h\gamma x = \gamma k x$$

for some $k \in K$, so $h\gamma x = \gamma x$.

Here we want to extend the previous definition of function complexes to the case where X and Y are G-spaces.

Definition 5.1.3. For two G-spaces X and Y, $map_G(X, Y)$ is defined as the subspace of the $map(X, Y)$ such that $f \in map_G(X, Y)$ if and only if $f(gx, \eta) = gf(x, \eta)$ for each $g \in G$.

Lemma 5.1.4. *If X, X' and Y are G-spaces with $X \cong X'$, then*

$$map_G(X,Y) \cong map_G(X',Y).$$

Proof. If $X \cong X'$, then there are G-simplicial maps $\psi : X \longrightarrow X'$ and $\phi : X' \longrightarrow X$ such that $\psi\phi$ and $\phi\psi$ are identity. We define

$$\begin{aligned}\tilde{\psi} : map_G(X',Y) &\longrightarrow map_G(X,Y)\\ f &\mapsto f(\psi(x),\eta)\end{aligned}$$

and

$$\begin{aligned}\tilde{\phi} : map_G(X,Y) &\longrightarrow map_G(X',Y)\\ f &\mapsto f(\phi(x),\eta).\end{aligned}$$

These maps are well-defined since ψ and ϕ are G-simplicial maps. Let η be a morphism in $\mathbf{\Delta}$.

$$\eta^*\tilde{\psi}(f(x,\mu)) = f(\psi(x),\eta\mu) = \tilde{\psi}\eta^*(f(x,\mu)).$$

Similarly,

$$\eta^*\tilde{\phi}(f(x,\mu)) = f(\phi(x),\eta\mu) = \tilde{\phi}\eta^*(f(x,\mu)).$$

Therefore, both $\tilde{\psi}$ and $\tilde{\phi}$ are simplicial maps. Furthermore,

$$\tilde{\psi}\tilde{\phi}f(x,\eta) = f(\phi\psi(x),\eta) = f(x,\eta)$$

and similarly,

$$\tilde{\phi}\tilde{\psi}f(x,\eta) = f(\psi\phi(x),\eta) = f(x,\eta).$$

Hence,

$$map_G(X,Y) \cong map_G(X',Y).$$

□

Definition 5.1.5. For a G-space X, the *generalized homotopy fixed points space* $X^{h_{\mathcal{F}}G}$ is defined by

$$X^{h_{\mathcal{F}}G} := map_G(E_{\mathcal{F}}G, X).$$

Theorem 5.1.6. *For a group G, a family of subgroups $\mathcal{F}$ of G and a G-space X, there is an isomorphism of spaces*

$$X^{h_{\mathcal{F}}G} = map_G(E_{\mathcal{F}}G, X) \cong \underset{\mathcal{O}_{\mathcal{F}}G}{holim} X^-.$$

In [2], D.A. Ramras introduces the definitions of the topological spaces $X^{h_{\mathcal{F}}G} = map_G(E_{\mathcal{F}}G, X)$ and $\underset{\mathcal{O}_{\mathcal{F}}G}{holim} X^-$ for a topological space X. Then, he shows that these two topological spaces are homeomorphic. Theorem 5.1.6 is motivated by this homeomorphism.

The rest of this section is devoted to proving Theorem 5.1.6. First, for a functor $F : C \longrightarrow \mathbf{GSp}$, we define the functor

$$\begin{aligned} map_G(F, X) : C &\longrightarrow \mathbf{Sp} \\ c &\mapsto map_G(F(c), X) \end{aligned}$$

and for $\alpha : e \to c$ a morphism in C,

$$\begin{aligned} map_G(F, X)(\alpha) : map_G(F(c), X) &\longrightarrow map_G(F(e), X) \\ f &\mapsto f(F(\alpha)x, \eta) \end{aligned}$$

for $f \in map_G(F(c), X)$.

Theorem 5.1.7. *For a G-space X, the two functors X^- and $map_G(\mathcal{I}, X)$ are naturally isomorphic. (See 3.3.7).*

Proof. We first define the natural transformation $\tau : X^- \Rightarrow map_G(\mathcal{I}, X)$. We note that $\tau_{G/H} \in Mor_{\mathbf{Sp}}(X^H, map_G(G/H, X))$. We define τ by

$$\begin{aligned} (\tau_{G/H})_m(x) : G/H \times \Delta[m] &\longrightarrow X \\ (gH, \eta) &\mapsto \eta^*(gx). \end{aligned}$$

$(\tau_{G/H})_m(x)$ is well-defined because if $gH = g'H$, then $g^{-1}g' \in H$, so for $x \in X^H$,

$$(\tau_{G/H})_m(x)(gH, \eta) = \eta^*(gg^{-1}g'x) = (\tau_{G/H})_m(x)(g'H, \eta).$$

To see that $(\tau_{G/H})_m(x)$ is a G-map, fix $g' \in G$. Then

$$(\tau_{G/H})_m(x)(g'(gH), \eta) = \eta^*(g'gx).$$

We recall that X is a G-space, so η^* is a G-map of G-sets and

$$\eta^*(g'gx) = g'\eta^*(gx) = g'(\tau_{G/H})_m(x)(gH, \eta).$$

Finally, $(\tau_{G/H})_m$ defines a simplicial map, since for $\nu : \mathbf{n} \to \mathbf{m}$

$$(\tau_{G/H})_n(\nu^* x)(gH, \eta) = \eta^*(g\nu^* x) = \eta^*\nu^*(gx) = (\nu\eta)^*(gx)$$

and

$$\nu^*(\tau_{G/H})_m(x)(gH, \eta) = (\tau_{G/H})_m(x)(gH, \nu\eta) = (\nu\eta)^*(gx).$$

To show that τ is natural, we consider the following diagram for a G-map $f : G/H \to G/K$ that maps H to γK:

$$\begin{tikzcd}
X^K \arrow[r, "\tau_{G/K}"] \arrow[d, "X^-(f)"'] & map_G(G/K, X) \arrow[d, "\bar{f}"] \\
X^H \arrow[r, "\tau_{G/H}"] & map_G(G/H, X)
\end{tikzcd}.$$

Here, $\bar{f}$ denotes $map_G(\mathcal{I}, X)(f)$ and this diagram commutes because

$$\begin{aligned}
\bar{f}\tau_{G/K}(x)(gH, \eta) &= (\tau_{G/K}(x))(\mathcal{I}(f)(gH), \eta) = (\tau_{G/K}(x))(f(gH), \eta) \\
&= (\tau_{G/K}(x))(gf(H), \eta) = (\tau_{G/K}(x))(g\gamma K, \eta) \\
&= \eta^*(g\gamma x)
\end{aligned}$$

and

$$\tau_{G/H} X^-(f)(x)(gH, \eta) = \tau_{G/H}(\gamma x)(gH, \eta) = \eta^*(g\gamma x).$$

Thus, τ is a natural transformation.

We construct an inverse to τ. Let ρ be the natural transformation

$$\begin{aligned}
(\rho_{G/H})_m : map_G(G/H, X)_m &\longrightarrow X_m^H \\
f &\mapsto f(H, id_{\mathbf{m}}).
\end{aligned}$$

Here, $f(H, id_{\mathbf{m}})$ is an element of X_m^H because for an element $h \in H$,

$$hf(H, id_{\mathbf{m}}) = f(hH, id_{\mathbf{m}}) = f(H, id_{\mathbf{m}}).$$

Moreover, $(\rho_{G/H})$ defines a simplicial map because for a morphism $\eta : \mathbf{n} \to \mathbf{m}$,

$$\begin{aligned}(\rho_{G/H})_n \eta^*(f) = \eta^* f(H, id_{\mathbf{n}}) &= f(H, \eta) \\ &= \eta^* f(H, id_{\mathbf{m}}) \\ &= \eta^* (\rho_{G/H})_m (f).\end{aligned}$$

To show ρ is natural we consider the following diagram.

$$\begin{tikzcd} map_G(G/K, X) \arrow[r, "\rho_{G/K}"] \arrow[d, "\bar{f}"'] & X^K \arrow[d, "X^-(f)"] \\ map_G(G/H, X) \arrow[r, "\rho_{G/H}"] & X^H \end{tikzcd}$$

for a G-map $f : G/H \longrightarrow G/K$ sending H to γK. Let $\xi : G/K \times \Delta[m] \longrightarrow X$ be a G-map, so

$$\begin{aligned}X^-(f)\rho_{G/K}(\xi) &= X^-(f)\xi(K, id_{\mathbf{m}}) \\ &= \gamma\xi(K, id_{\mathbf{m}}) \\ &= \xi(\gamma K, id_{\mathbf{m}}).\end{aligned}$$

In addition, we have

$$\rho_{G/H}\bar{f}(\xi) = \bar{f}(\xi)(H, id_{\mathbf{m}}) = \xi(f(H), id_{\mathbf{m}}) = \xi(\gamma K, id_{\mathbf{m}}).$$

Hence, the diagram commutes and ρ is a natural transformation.

Now we show τ and ρ are inverses. Let G/H be an object of $\mathcal{O}_{\mathcal{F}}G$ and $x \in X_m^H$, then

$$\rho_{G/H}\tau_{G/H}(x) = \tau_{G/H}(x)(H, id_{\mathbf{m}}) = id^*(x) = x$$

by the definition of τ.

Now we consider $\tau_{G/H}\rho_{G/H}$ acting on object $f \in map_G(G/H, X)_m$.

$$\tau_{G/H}\rho_{G/H}(f)(gH, \eta) = \eta^*(g\rho_{G/H}(f)) = \eta^*(gf(H, id)) = f(gH, \eta).$$

Therefore, natural transformations τ and ρ are inverses of each other, and X^- and $map_G(\mathcal{I}, X)$ are naturally isomorphic. □

Now, we investigate if this relation extends to a relationship between homotopy limits of these functors.

Theorem 5.1.8. *Let $F, F_1', F_2' : C \longrightarrow \mathbf{Sp}$ be functors such that F_1' and F_2' are naturally isomorphic. Then, $Nat(F, F_1')$ is simplicially isomorphic to $Nat(F, F_2')$.*

Proof. Fix natural isomorphisms $\tau : F_1' \Rightarrow F_2'$ and $\rho : F_2' \Rightarrow F_1'$ such that τ and ρ are inverses to each other.

First we define $\psi : Nat(F, F_1') \longrightarrow Nat(F, F_2')$ such that for $(f_c) \in Nat(F, F_1')_m$

$$\begin{aligned}(\psi(f_c))_{c'} : F(c') \times \Delta[m] &\longrightarrow F_2'(c')\\ (x, \eta) &\mapsto \tau_{c'} f_{c'}(x, \eta).\end{aligned}$$

Note that $\psi(f_c)$ is in $Nat(F, F_2')$ since for a morphism $\alpha : e \to c'$ the following diagram is commutative.

$$\begin{tikzcd} F(c') \times \Delta[m] \arrow[r, "f_{c'}"] \arrow[d, "F(\alpha) \times id"'] & F_1'(c') \arrow[r, "\tau_{c'}"] \arrow[d, "F_1'(\alpha)"] & F_2'(c') \arrow[d, "F_2'(\alpha)"] \\ F(e) \times \Delta[m] \arrow[r, "f_e"] & F_1'(e) \arrow[r, "\tau_e"] & F_2'(e) \end{tikzcd}$$

Commutativity of the first square follows from (f_c) being in $Nat(F, F_1')$ and commutativity of the second square follows from τ being a natural transformation.

To show ψ is a simplicial map, we consider $\mu : \mathbf{m} \to \mathbf{n}$ in $\mathbf{\Delta}$. Then,

$$(\mu^* \psi(f_c))_{c'}(x, \eta) = \psi(f_c)_{c'}(x, \mu\eta) = \tau_{c'} f_{c'}(x, \mu\eta).$$

Also,

$$\mu^*(\psi(f_c)_{c'}(x, \eta)) = \mu^*(\tau_{c'} f_{c'}(x, \eta)) = \tau_{c'} f_{c'}(x, \mu\eta).$$

Hence, ψ is a simplicial map.

Now we define the inverse simplicial map of ψ for $(f_c) \in Nat(F, F_2')_m$ by

$$\begin{aligned}(\phi(f_c))_{c'} : F(c') \times \Delta[m] &\longrightarrow F_1'(c')\\ (x, \eta) &\mapsto \rho_{c'} f_{c'}(x, \eta).\end{aligned}$$

To show $\phi(f_c)$ is natural we consider the following diagram for a morphism $\alpha : e \to c'$ in C:

$$\begin{tikzcd}
F(c') \times \Delta[m] \arrow[r, "f_{c'}"] \arrow[d, "F(\alpha) \times id"'] & F_2'(c') \arrow[r, "\rho_{c'}"] \arrow[d, "F_2'(\alpha)"] & F_1'(c') \arrow[d, "F_1'(\alpha)"] \\
F(e) \times \Delta[m] \arrow[r, "f_e"] & F_2'(e) \arrow[r, "\rho_e"] & F_1'(e)
\end{tikzcd}$$

Similar to the argument before, this diagram commutes because (f_c) is an element of $Nat(F, F_2')$ and ρ is a natural transformation.

The last thing we need to show is that ϕ and ψ are inverses of each other. This directly follows from the fact that τ is the inverse of ρ. For an element f of $Nat(F, F_1')_m$,

$$(\phi\psi(f_c))_{c'}(x, \eta) = \rho_{c'}\tau_{c'}f_{c'}(x, \eta) = f_{c'}(x, \eta).$$

Similarly, for $f \in Nat(F, F_2')$,

$$(\psi\phi(f_c))_{c'}(x, \eta) = \tau_{c'}\rho_{c'}f_{c'}(x, \eta) = f_{c'}(x, \eta).$$

□

Corollary 5.1.9. *For the functors X^- and $map_G(\mathcal{I}, X)$ defined as before,*

$$\underset{\mathcal{O}_{\mathcal{F}}G}{holim} X^- \cong \underset{\mathcal{O}_{\mathcal{F}}G}{holim}(map_G(\mathcal{I}, X)).$$

Proof. Follows from 5.1.7 and 5.1.8. □

The next theorem gives a G-equivariant version of Theorem 4.2.1.

Theorem 5.1.10. *For a functor $F : C \longrightarrow$ **GSp** and a G-space X, there is a simplicial isomorphism of simplicial sets*

$$map_G(\underset{C}{hocolim} F, X)) \cong \underset{C}{holim}(map_G(F, X)).$$

Proof. First, we recall that the simplicial set $map_G(hocolim_C F, X)$ is a subspace of $map(hocolim_C F, X)$ and that $holim_C(map_G(F, X))$ is a subspace of $holim_C(map(F, X))$. Define

$$\tilde{\psi} : map_G(\underset{C}{hocolim} F, X) \longrightarrow \underset{C}{holim}(map_G(F, X))$$
$$f \mapsto \psi(f)$$

and

$$\tilde{\phi} : \underset{C}{holim}(map_G(F, X)) \longrightarrow map_G(\underset{C}{hocolim} F, X)$$
$$(f_c) \mapsto \phi(f_c).$$

We use the notation from the proof of 4.2.1.

Now we show if we pick $f \in map_G(hocolim_C F, X)_m$, i.e., f is a G-simplicial map

$$f : \underset{C}{hocolim} F \times \Delta[m] \longrightarrow X$$

then $(\psi f_c) \in Nat(\underline{N}, map_G(F, X))$:

$$(\psi f_c)_n(\sigma, \eta)_k(gx, \mu) = f([(F(\gamma_0)gx, \tilde{\sigma}), \mu], \mu\eta).$$

Since $F(\gamma_0)$ is a G-map,

$$f([(F(\gamma_0)gx, \tilde{\sigma}), \mu], \mu\eta) = f([g(F(\gamma_0)x, \tilde{\sigma}), \mu], \mu\eta).$$

Also, since f is a G-map

$$\begin{aligned} f([g(F(\gamma_0)x, \tilde{\sigma}), \mu], \mu\eta) = f(g[(F(\gamma_0)x, \tilde{\sigma}), \mu], \mu\eta) &= gf([(F(\gamma_0)x, \tilde{\sigma}), \mu], \mu\eta) \\ &= g(\psi f_c)_n(\sigma, \eta)_k(x, \mu). \end{aligned}$$

Therefore (ψf_c) is a map from $N(c \downarrow C) \times \Delta[m]$ to $map_G(F(c), X)$. Thus, $\tilde{\psi}$ is well defined.

To show $\phi(f_c) \in map_G(hocolim_C F, X)_m$ for $(f_c) \in holim_C(map_G(F, X))_m$ all we need to show is that $\phi(f_c)$ is a G-simplicial map at this point.

Let $(f_c) \in holim_C(map_G(F, X))$, so

$$\phi(f_c)([(gx, \sigma), \eta], \mu) = (f_{\eta^*\sigma(0)})_k(\overline{\eta^*\sigma}, \mu)_k({}_\sigma(\eta^*)gx, id).$$

By the definition of ${}_{\sigma}(\eta^*)$ stated in proof of 4.2.1, this last expression is equal to

$$(f_{\eta^*\sigma(0)})_k(\overline{\eta^*\sigma},\mu)_k(gx,id) \text{ or } (f_{\eta^*\sigma(0)})_k(\overline{\eta^*\sigma},\mu)_k(F(\alpha_1)gx,id)$$

and in any case

$$(f_{\eta^*\sigma(0)})_k(\overline{\eta^*\sigma},\mu)_k({}_{\sigma}(\eta^*)gx,id) = (f_{\eta^*\sigma(0)})_k(\overline{\eta^*\sigma},\mu)_k(g({}_{\sigma}(\eta^*)x),id)$$

since $F(\alpha_1)$ is a G-simplicial map. Since $(f_c) \in holim_C(map_G(F,X))$, $(f_{\eta^*\sigma(0)})_k(\overline{\eta^*\sigma},\mu)_k$ is a G-map. Therefore

$$\begin{aligned}(f_{\eta^*\sigma(0)})_k(\overline{\eta^*\sigma},\mu)_k(g_{\sigma}(\eta^*)x,id) &= g(f_{\eta^*\sigma(0)})_k(\overline{\eta^*\sigma},\mu)_k({}_{\sigma}(\eta^*)x,id)\\ &= g\phi(f_c)([(x,\sigma),\eta],\mu).\end{aligned}$$

Hence, $\phi(f_c) \in map_G(hocolim_C F, X)$.

By Proof of 4.2.1, $\tilde{\psi}$ and $\tilde{\phi}$ are inverse to each other. Moreover, since ψ is a simplicial map, $\tilde{\psi}$ is also a simplicial map. Hence, by 2.2.12

$$map_G(\underset{C}{hocolim} F, X)) \cong \underset{C}{holim}(map_G(F,X)).$$

□

Now we prove the main theorem of this section.

Proof of 5.1.6. Let $\mathcal{I}$ be the functor defined as in Chapter 3. By 3.3.10,

$$\underset{\mathcal{O}_{\mathcal{F}}G}{hocolim}\mathcal{I} \cong N(\int_{\mathcal{O}_{\mathcal{F}}G}\mathcal{I})$$

and $N(\int_{\mathcal{O}_{\mathcal{F}}G}\mathcal{I}) = E_{\mathcal{F}}G$. Hence we conclude

$$map_G(E_{\mathcal{F}}G, X) \cong map_G(\underset{\mathcal{O}_{\mathcal{F}}G}{hocolim}\mathcal{I}, X).$$

Moreover, by 5.1.9 and 5.1.10

$$map_G(\underset{\mathcal{O}_{\mathcal{F}}G}{hocolim}\mathcal{I}, X) \cong \underset{\mathcal{O}_{\mathcal{F}}G}{holim}(map_G(\mathcal{I},X)) \cong \underset{\mathcal{O}_{\mathcal{F}}G}{holim} X^-$$

so we conclude that

$$map_G(E_{\mathcal{F}}G, X) \cong \underset{\mathcal{O}_{\mathcal{F}}G}{holim} X^-$$

and this finishes the proof. □

We note that $\mathcal{O}_{\mathcal{F}}G$ can be considered as a subcategory of **GSets** considering G/H as a discrete G-space. Then, a corollary of Theorem 5.1.10 is as follows.

Corollary 5.1.11. *Let X and Y be G-spaces and for a functor F defined by*

$$\begin{aligned} F:\ C &\longrightarrow \mathcal{O}_{\mathcal{F}}G \\ c &\longrightarrow G/H_c \end{aligned}$$

suppose that $X \cong hocolim_C F$. Then,

$$map_G(X,Y) \cong \underset{c\in C}{holim}\, Y^{H_c}.$$

Proof. Since, $X \cong hocolim_C F$, by 5.1.4, we have

$$map_G(X,Y) \cong map_G(\underset{C}{hocolim} F, Y)$$

and by 5.1.10

$$map_G(\underset{C}{hocolim} F, Y) \cong \underset{c\in C}{holim}(map_G(G/H_c, Y)).$$

Furthermore, by 5.1.9

$$\underset{c\in C}{holim}(map_G(G/H_c, Y)) \cong \underset{c\in C}{holim}\, Y^{H_c}.$$

Hence, this completes the proof. □

5.2 Examples

In the Section 5.1, to obtain Result 5.1.6 we use the fact

$$E_{\mathcal{F}}G \cong \underset{\mathcal{O}_{\mathcal{F}}G}{hocolim}\, \mathcal{I}.$$

If a G-space X can be constructed as the homotopy colimit of a functor $F : C \longrightarrow$ **GSp**, then Theorem 5.1.10 can be applied to the function complex from the space X to give a similar decomposition. Now, we consider such an example.

For the poset category $\mathbf{1}$ with $obj(\mathbf{1}) = \{0,1\}$ and one non-identity morphism $\alpha : 0 \to 1$ and for a group G, we consider the

$$\begin{tikzcd} \mathcal{U}: & \mathbf{1} \arrow[r] & \mathbf{GSets} \\ & 0 \arrow[r] \arrow[d, "\alpha"'] & G/1 \arrow[d, "g\{1\}\mapsto 1G"] \\ & 1 \arrow[r] & G/G. \end{tikzcd}$$

Since $map_G(G/H, Y) \cong Y^H$, $map(\mathcal{U}, Y)$ is naturally isomorphic to the functor

$$\begin{tikzcd} \widetilde{\mathcal{U}}: & \mathbf{1} \arrow[r] & \mathbf{GSets} \\ & 0 \arrow[r] \arrow[d, "\alpha"'] & Y \\ & 1 \arrow[r] & Y^G. \arrow[u, hook, "inclusion"'] \end{tikzcd}$$

By 5.1.10 and 5.1.8, we have

$$map_G(\underset{\mathbf{1}}{hocolim}\mathcal{U}, Y) \cong \underset{\mathbf{1}}{holim}(map_G(\mathcal{U}, Y)) \cong \underset{\mathbf{1}}{holim}\widetilde{\mathcal{U}}.$$

Hence, $T := \underset{\mathbf{1}}{holim}(map_G(\mathcal{U}, Y))$ is a subspace of the function complex

$$map(N(0 \downarrow \mathbf{1}), Y) \times map(N(1 \downarrow \mathbf{1}), Y^G)$$

where $\{f_0, f_1\} \in T_m$ if and only if the following diagram commutes:

$$\begin{tikzcd} N(1 \downarrow \mathbf{1}) \times \Delta[m] \arrow[r, "f_1"] \arrow[d, "\underline{N}(\alpha)"'] & Y^G \arrow[d, hook, "inclusion"] \\ N(0 \downarrow \mathbf{1}) \times \Delta[m] \arrow[r, "f_0"] & Y. \end{tikzcd}$$

Moreover, $N(0 \downarrow \mathbf{1}) \cong \Delta[1]$ and $N(1 \downarrow \mathbf{1}) \cong \Delta[0]$. Hence, we conclude that T is the subspace of $map_G(\Delta[1], Y)$ such that $f \in T$ if and only if $f(1, \eta) \in Y^G$ for each η.

Let $X := |\underset{\mathbf{1}}{hocolim}\mathcal{U}|$. For a fibrant G-space Y (see [8, Definition 8.2.7])

$$map_G(X, |Y|) = map_G(|\underset{\mathbf{1}}{hocolim}F|, |Y|) \simeq |map_G(\underset{\mathbf{1}}{hocolim}F, Y)| \cong |T|$$

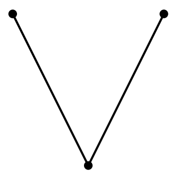
by [9, Theorem A.3.2]. Also, if we define the space

$$\mathbf{T} := \{f \in map_G(I, |Y|) \mid f(1) \in |Y|^G\}$$

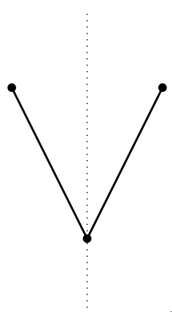
where I denotes the unit interval, then $|T| \simeq \mathbf{T}$ by [9, Theorem A.3.2]. Furthermore, $\mathbf{T} \cong |Y|^G$ by [10, Theorem 2.2.7]. Hence, we conclude that

$$map_G(X, |Y|) \simeq |Y|^G.$$

As an example of this let $G = C_2$ and let $X_{C_2} := |\underset{\mathbf{1}}{hocolim} F_{C_2}|$. We observe that

$X_{C_2} =$ 

where the action of the non-trivial element $g \in C_2$ on X_{C_2} is defined as the flip over the vertical axis:



For a fibrant space Y,

$$map_G(X_{C_2}, |Y|) \simeq Y^{C_2}.$$

We now consider another example of the applications of Theorem 4.2.1.

Definition 5.2.1. For a left G-space X, the *homotopy fixed points* of X written by X^{hG} is defined as

$$X^{hG} = map_G(EG, X).$$

Definition 5.2.2. If X is a G-space, then the *homotopy orbit space* of X written X_{hG} is $X \times_G EG$.

For X a right G-space and Y a space, $map(X, Y)$ is a left G-space with the action defined by

$$(gf)(x) = f(xg).$$

Theorem 5.2.3. *For a right G-space X,*

$$map(X_{hG}, Y) \cong (map(X, Y))^{hG}.$$

Proof. By definition, $X_{hG} = X \times_G EG$ and by 3.3.14

$$X \times_G EG \cong \underset{\mathbf{G}^{op}}{hocolim} \mathbf{X}.$$

By 4.2.1,

$$map(\underset{\mathbf{G}^{op}}{hocolim} \mathbf{X}, Y) \cong \underset{\mathbf{G}^{op}}{holim}(map(\mathbf{X}, Y)).$$

Hence, by the definition of homotopy limit, $\underset{\mathbf{G}^{op}}{holim}(map(\mathbf{X}, Y))$ is the natural subspace of $map(N(* \downarrow \mathbf{G}^{op}), map(X, Y))$. Finally, since $N(* \downarrow \mathbf{G}^{op}) \cong EG$ and the condition of being natural corresponds to the condition to be a G-map with G-actions on EG and $map(X, Y)$ this natural subspace is equal to

$$map_G(EG, map(X, Y)) = (map(X, Y))^{hG}.$$

□

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