JOINT REPLENISHMENT PROBLEM WITH TRUCK COST STRUCTURES

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By
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December, 2006

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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We consider inventory systems with multiple items in the presence of stochastic demand and jointly incurred order setup costs. The problem is to determine the replenishment policy that will minimize the total expected ordering, inventory holding and backorder costs; the so–called stochastic joint replenishment problem in the literature. In particular, we study the settings in which order setup costs reflect the transportation costs and have a step–wise cost structure, each step corresponding to an additional transportation vehicle. For this setting, we propose a new policy which we call the $(\mathbf{s}, Q)$ policy. Under this policy, a replenishment order of fixed size Q is triggered whenever the inventory position of one of the items drops to its reorder point s. The replenishment order is allocated to multiple items to equalize inventory positions of items to the extent possible. The policy is designed for settings in which the backorder and setup costs are high, as it allows the items to independently trigger replenishment orders and fully exploits the economies of scale by consistently ordering the same quantity. A numerical study is conducted to confirm that the policy works as designed and to compare its performance against the $(Q, \mathbf{S})$ and $(Q, \mathbf{s}, \mathbf{S})$ policies that were suggested earlier in the literature. The study shows that the proposed $(\mathbf{s}, Q)$ policy outperforms the $(Q, \mathbf{S})$ when the backorder and setup costs are high and when the vehicles are not capacitated. When the vehicles are capacitated, the new policy outperforms both other policies under the most settings considered.

Keywords: Inventory theory, stochastic joint replenishment problem, truck cost structure.

ÖZET

ARAÇ MALİYET YAPILI TOPLU SİPARİŞ PROBLEMİ

Mehmet Mustafa Tanrıkulu
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Tez Yöneticisi: Yrd. Doç. Dr. Alper Şen , Yrd. Doç. Dr. Osman Alp
Aralık, 2006

Bu tez çalışmasında rassal talep ve toplu sipariş maliyetlerini içeren çok ürünlü envanter sistemleri incelenmiştir. Özellikle ilgilenilen problem, amacı toplam beklenen sipariş verme, envanter tutma ve ardısmarlama maliyetlerini en aza indiren politikayı bulmak olan ve literatürde rassal toplu sipariş problemi adı verilen problemdir. Literatürden farklı olarak bu çalışmada her basamağı ilave taşıt kapasitesine karşılık gelen, basamaklı maliyet yapısı incelenmiştir. Böyle bir yapıda (s, Q) adı verilen yeni bir politika önerilmiştir. Bu politikada, bir ürünün envanter pozisyonu yeniden ısmarlama noktası s'e düştüğünde, sabit miktarlı sipariş tetiklenmektedir. Bu ısmarlanan miktar, ürunler arasında, envanter pozisyonları mümkün olduğunca eşitlenecek şekilde paylaştırılır. Bu yeni politika, ardısmarlama ve sipariş maliyetlerinin yüksek olduğu durumlar için tasarlanmış olup, herbir ürünün bağımsız olarak siparişi tetikleyebilmesine izin verir ve sürekli olarak aynı miktarda sipariş vererek ölçek ekonomisinden tümüyle faydalanır. Politikanın tasarlandığı şekilde işlediğini doğrulamak ve performansını daha önce literatürde önerilen $(Q, \mathbf{S})$ ve $(Q, \mathbf{S}, \mathbf{s})$ politikalarıyla karşılaştırmak için bir sayısal çalışma yapılmıştır. Bu Çalışmanın sonucunda, önerilen $(\mathbf{s}, Q)$ politikasının yüksek ardısmarlama, sipariş verme maliyetlerinin olduğu ve kapasite kısıtının olmadığı durumlarda $(Q, \mathbf{S})$ politikasından daha iyi sonuç verdiği gözlemlenmiştir. Araçlarda kapasite kısıtı olduğunda, önerilen yeni politika, incelenen diğer iki politikadan daha iyi sonuç vermiştir.

Anahtar sözcükler: Envanter teorisi, rassal toplu sipariş problemi, araç maliyet yapısı.

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kaytarıkçılar olarak yaptığımız aktivitelere katılan ve beni ben yapan bütün arkadaşlarıma teşekkürler. . . .

To my grandmother in heaven...

Cenneteki Anneanneme...

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Chapter 1

Introduction

Many companies manage inventories of multiple items. The primary challenge in managing multi–item inventory systems is the fact that some of the costs are incurred jointly. In particular, the setup costs in production, purchasing or transportation are often incurred jointly for the multiple items that are included in the production batch, purchase order or the shipment. Joint setups can be seen as an opportunity as well as a challenge, since scale economies can be exploited to reduce setup costs or reduce cycle inventories or both, by carefully coordinating the replenishment of multiple items. The joint replenishment problem (JRP) is to determine the inventory replenishment policy of multiple items that share a common setup.

A basic example of the joint replenishment problem occurs in a setting where multiple items are sourced from a common supplier. Setup costs in this setting may include the transportation costs and purchase transaction costs. Since 1980s, many manufacturing companies are reducing their supplier bases. Examples include Xerox reducing its supplier base in early 1980s from 5000 to 400 [6], Texas Instruments reducing its MRO suppliers from 5000 to 750 between 1998 and 2000 [24], Merck reducing its total global supplier base from 40,000 in 1992 to fewer than 10,000 in 1997 [15], IBM now using only 50 suppliers for the 85 % of its requirements [9] and Sun Microsystems now using only 40 suppliers for the 90 % of its requirements [8]. Among other things, reduction of the supplier base helps companies decrease their inventory holding, transportation and purchasing costs by giving them the capability of jointly replenishing multiple items from common suppliers.

Being sourced from a common supplier is not a necessity for jointly replenishing multiple items. Companies are devising numerous strategies to leverage economies of scale of combining different items into a single delivery. Among these, the milk–run strategy allows the joint procurement of multiple SKUs from different suppliers located in close physical proximity and helps companies consolidate smaller shipments to more efficient larger shipments (or move from infrequent independent shipments to more frequent joint shipments) to reduce transportation costs and cycle stocks. For example, Toyota's Kentucky plant sources 80 % of its parts from suppliers that are located within 200 miles of the plant. Milk–run vehicles serving these suppliers help Toyota receive deliveries on a JIT basis [19]. Another example is Eastman Kodak that significantly increased the frequency of inbound shipments to its plants by successfully implementing the milk–run strategy [12]. A final example of milk–run is the commercial vehicle producer MAN. In 2004, MAN's Ankara plant successfully reduced its inbound transportation costs and component inventory by consolidating its shipments from various component manufacturers located in close proximity in Northwestern Turkey: a project jointly undertaken by MAN and Industrial Engineering Department at Bilkent University. Another strategy that allows companies to exploit economies of scale in inbound transportation is cross–docking. With cross–docking, smaller shipments from multiple suppliers can be merged in a consolidation warehouse for a larger and more economical joint delivery. Cross–docking has been a successful strategy in practice including the famous Wal–Mart implementation [31].

Joint replenishment is also relevant when replenishing a single item in multiple locations. As in the case of multi–item inventory systems, companies are developing strategies that will help them exploit economies of scale of combining shipments to multiple locations under their control. For example, a milk–run vehicle can depart from a supplier or a distribution center and visit a group of production plants to replenish them jointly, reducing the transportation costs and cycle inventories. An example of this is again Eastman Kodak which implements the milk–run strategy for its shipments from its distribution center to multiple plants as well as from multiple suppliers to its distribution centers [12]. Milk–runs are widely used to replenish multiple retail store locations from retailer owned distribution centers or from suppliers directly. Aforementioned cross–docking also enables multiple facilities to consolidate their replenishment at least for a portion of the trip. Joint replenishment of multiple locations is possible when all these locations are centrally controlled or when these locations are in a coalition for joint replenishment. Under a Vendor Managed Inventory contract between a supplier and multiple retailers (or other downstream players), the supplier takes control of the management of inventories at the retail locations. Among other benefits, VMI contracts allow the joint replenishment of multiple retail location and help reduce the transportation and inventory costs for the supply chain ([10], [11]).

In accordance with its relevance and importance in practice, the joint replenishment problem has been an extensively studied research topic for almost 40 years starting with the pioneering works of Balintfy [5] and Silver [27]. Formally the problem is to determine the replenishment and inventory policies of N items (or locations) to minimize the total setup, holding and shortage costs in the presence of setup costs that are incurred jointly. In a more general setting, in addition to the setup costs that are common and incurred with each replenishment order regardless of which items are involved (major setups), item specific setup costs may be incurred for each item in the order (minor setups). Research in this area followed two separate paths depending on whether the demands are deterministic or stochastic; the latter being referred to as the stochastic joint replenishment problem (SJRP). In this thesis, SJRP is investigated.

One major gap in the existing literature on the joint replenishment problem is the fact that the setup costs (major or minor) are independent of the size of the order. This may be a reasonable assumption when the setup costs reflect the administrative costs that are related to a purchase order or the production setups that are incurred for a production batch. However, when the setup costs are due to transportation costs (perhaps the main motivation of the joint replenishment problem), such an assumption is rather restrictive. In practice the transportation is carried out with capacitated vehicles. Thus, the setup cost structure is step–wise, each step corresponding to an additional vehicle. Such cost structures are recently being investigated in the literature (see, for example, Alp et. al [2]) for the single item inventory systems. The main contribution of this thesis is the incorporation of the transportation vehicle capacities and associated cost structures for the stochastic joint replenishment problem. In particular, we develop a replenishment policy in which a replenishment order of fixed size (perhaps the capacity of the vehicle) is created whenever the inventory position for one of the N items (locations) drop to its own reorder point. We name this policy $(\mathbf{s}, Q)$

policy, where $\mathbf{s}$ is the vector of reorder points for the N items, and Q is the constant reorder quantity.

A partial motivation for this study is our experience with a beverage producer in Turkey. This beverage producer manages the inventory of its distributors under a VMI like setting and dispatches trucks for the replenishment of about 100 SKUs at each distributor. The trucks that are used for the shipments are capacitated. Since the trucks travel large distances (up to 1000 kilometers), the transportation costs are substantial (as compared to inventory holding costs) and do not depend significantly on the load of the truck, the beverage producer almost always dispatches full trucks to its distributors. The company also wants to maintain a high service level at its distributors at which the demand for the SKUs can be highly uncertain. This rules out a policy that removes the ability of each SKU individually triggering a replenishment order. One such policy is $(Q, \mathbf{S})$ policy, in which a replenishment order of size Q is triggered whenever the total demand since last order reaches Q to bring up the inventory position of the N items to $\mathbf{S}$.

In the specific setting in which we propose our policy, there are N items (or locations). The demand for each item follows an independent Poisson process. There are no minor setup costs. Unsatisfied demand is completely backlogged. Two types of backlogging costs are incurred: per backlog occasion and based on the backlog duration. Linear inventory holding costs are charged. The problem is to determine the reorder quantity Q and reorder points $\mathbf{s}$ so that the total expected ordering, inventory holding and backlogging costs are minimized. When the items are identical, the contents of the replenishment order is decided in a way that item inventory positions are equalized (to the extent that this is possible). For the case of non–identical items, we devise a rule to allocate the fixed replenishment quantity to multiple items based on the stock–out costs. The policy is general in the sense that the same set–up cost can be incurred regardless of the reorder quantity Q. To consider the case of capacitated vehicles, we introduce a capacity C and it is sufficient to consider the case of $Q \leq C$, since we have a continuous review model.

We conduct a numerical study to asses the performance of the proposed $(\mathbf{s}, Q)$ policy against two policies in the literature. One of these policies is the $(Q, \mathbf{S})$ policy which is described earlier. The second policy is the $(Q, \mathbf{S}, \mathbf{s})$ policy in which a replenishment order is triggered whenever the total demand since the last order reaches Q or the inventory position of any of the items drops to its reorder point. Our numerical study results show that, there is no dominance relationship between these policies. One policy may outperform the others in different settings. However, when vehicle capacities are assumed to be infinite, the $(\mathbf{s}, Q)$ policy tends to outperform the $(Q, \mathbf{S})$ policy while the $(Q, \mathbf{S}, \mathbf{s})$ policy outperforms the $(\mathbf{s}, Q)$ policy in most of the cases. On the other hand, when vehicle capacities are assumed to be finite, the $(\mathbf{s}, Q)$ policy outperforms the other policies in most of the cases considered.

The rest of this thesis is organized as follows. In Chapter 2, we review the literature on the stochastic joint replenishment problem. In Chapter 3, we propose our new policy $(\mathbf{s}, Q)$ along with a review of the replenishment policies $(Q, \mathbf{S})$ and $(Q, \mathbf{S}, \mathbf{s})$. In Chapter 4, we present our numerical results that compare the three policies when the vehicle capacities are both not considered and considered. Chapter 5 concludes the thesis and suggests some avenues for future research.

Chapter 2

Literature Survey

In this chapter we review the literature on the stochastic joint replenishment problem. In the first part of the chapter, we focus on the single–echelon inventory systems. These inventory systems are discussed under two categories: periodic review and continuous review. In the second part of the chapter we review multi–echelon inventory systems and vendor–managed inventory systems. While there is a large body of literature on the deterministic joint replenishment problem, we do not review this literature here. For a review of that literature see Aksoy and Erengüç [1] and Goyal and Satir [16].

Replenishment policies are vital for an efficient inventory system. When there are multiple items, cost savings can be obtained through jointly replenishing them. The savings through joint replenishment can be substantial, when efficient joint replenishment policies are used. The previous research in this area shows that finding a good solution for the joint replenishment problem is difficult. Ignall [18] studies the replenishment problem to find the optimal joint replenishment policy. The main result of this study is that the optimal policy in joint replenishment, even for a two–item case, is complicated because of the dependency of the quantity ordered on inventory levels of two items. As the number of items in the system increases, the inventory system is more difficult to control and the implementation of the joint order policies are even more challenging. Therefore, heuristic policies are sought in the literature.

Balintfy [5] is the first to study the *stochastic joint replenishment problem.*

We begin our survey with this study. Balintfy [5] develops a continuous–review joint ordering policy, which determines the range of reorder points at which several items can be ordered simultaneously. This new policy is suitable for computer–controlled inventory systems. In individual ordering, each item triggers a replenishment order whenever its inventory position drops to a certain level, referred to as the *reorder point*. The replenishment order consists of only the item that triggered the order. For joint ordering, a new quantity called the *can–order point* is defined. The area between the can–order point and the reorder point is called the *reorder range*. Items, whose inventory positions fall within this range, are also ordered when an order is triggered.

This new policy by Balintfy [5] is referred to as can–order policy and is represented as $(\mathbf{S}, \mathbf{c}, \mathbf{s})$. $\mathbf{S}$ is the vector of order–up–to levels; $\mathbf{s}$ is the vector of reorder points and $\mathbf{c}$ is the vector of new points called the can–order point. This new policy functions as follows. An order is triggered when any of the items inventory position drops to or below its reorder point s. When the order is triggered, the inventory position of the item that triggered the ordering is raised to its order–up–to level. Simultaneously, the inventory positions of the other items are also checked. If the inventory position of any of the items is at or below the can–order point, that item's inventory position is also raised to its order–up–to level. This policy seems to be simple; unfortunately, however, it is difficult to derive cost expressions analytically.

Silver [27] studies a special case of the $(\mathbf{S}, \mathbf{c}, \mathbf{s})$ policy. In this special case, the replenishment leadtime is zero, c is assumed to be $S - 1$ and $s = 0$ for each item. Demands are assumed to be Poisson and shortages are not allowed. The objective is to minimize the expected total cost per unit time, comprised of the holding cost and the ordering cost. Silver [27] proves that the can–order policy performs better than individual ordering if the fixed ordering cost does not change with joint ordering. If the fixed costs are not equal in individual ordering and joint ordering, whether the joint replenishment will reduce the cost depends on the fixed cost for the joint replenishment. If the fixed cost for the joint replenishment lies below a critical value, joint replenishment reduces costs.

Another study on the $(\mathbf{S}, \mathbf{c}, \mathbf{s})$ policy by Silver [29] who decomposes the N–item problem with unit Poisson demands into N single–item problems to approximate the solution. This single–item problem is first analyzed by Silver [28] himself and solved optimally by Zheng [34]. The same decomposition method is used for compound Poisson demand by Thompson and Silver [32] and Silver [30]. When Poisson arrival process for the special replenishment possibilities is assumed, (i.e., the reduced cost occur probabilistically according to a Poisson process with a rate μ per year, where μ is the expected number of orders triggered per year by all other items in the group), Van Eijs [33] and Schultz and Johansen [26] show that the decomposition method performs poorly.

Federgruen et al. [14] suggest a semi–Markov decision model and use a decomposition approach similar to Silver [30]. The authors focus on calculating the control parameters of the $(\mathbf{S}, \mathbf{c}, \mathbf{s})$ policy and propose a heuristic method using a policy–iteration algorithm to find the control parameters. This decomposition approach is different than Silver [30], since it is based on the fact that under general conditions, superpositions of n point processes converge to a Poisson process as $n \longrightarrow \infty$. Using this approach, the problem becomes an n independent single–item problem.

One of the most important continuous–review control policies for the joint replenishment problem in the literature is the $(Q, \mathbf{S})$ policy. This policy is first proposed by Renberg and Planche [25]. In this thesis, we compare the performance of our proposed policy and the $(Q, \mathbf{S})$ policy. Pantumsinchai [23] subsequently studies the policy, assuming Poisson demand. The policy is simple and functions as follows: when the total amount of demand since the previous order has reached Q, an order in the amount of Q is placed with the supplier to raise the inventory positions of all of the items to $\mathbf{S}$. Q is the order quantity and $\mathbf{S}$ is the order–up–to level. Pantumsinchai [23] compares the $(Q, \mathbf{S})$ policy with the $(\mathbf{S}, \mathbf{c}, \mathbf{s})$ policy, and shows that the $(Q, \mathbf{S})$ policy performs better than the $(\mathbf{S}, \mathbf{c}, \mathbf{s})$ policy if the fixed ordering cost is high and the shortage cost is low. The $(\mathbf{S}, \mathbf{c}, \mathbf{s})$ policy only performs better if the fixed ordering cost is low.

Cheung and Lee [11] also study the $(Q, \mathbf{S})$ policy, but in a setting with single warehouse and multiple retailers. The policy works similarly with in an inventory system with a single retailer multiple items. In this multi–retailer case, an order is triggered when a total of Q units are demanded in all retailers. After an order is triggered, inventory positions of the retailers are all raised up to their maximum levels $\mathbf{S}$. Cheung and Lee [11] analyze the model exactly in a setting where the warehouse uses the $(Q, \mathbf{R})$ policy for its inventory control. They also propose a new model applying the same policy in which the stocking positions of the retailers can be rebalanced while unloading the items and find a lower and an upper bound for this model.

Atkins and Iyogun [3] propose two periodic review replenishment policies and compare them against the $(\mathbf{S}, \mathbf{c}, \mathbf{s})$ policy. These policies are referred to as $(\mathbf{R}, T)$–type policies. In this class of policies, the inventory is reviewed periodically, and inventory position of each item is raised to level $\mathbf{R}$ at the end of each period of length T, by creating a joint replenishment order. The first proposed policy of this kind is called a *periodic heuristic policy* and is represented by P. In this policy, the period lengths are identical. The second type is called a *modified periodic heuristic policy* and is represented by MP. In this policy, periods are integer multiples of a base period and periods can differ for each item. Computational results illustrate that as the fixed cost increases, P and MP type policies outperform the $(\mathbf{S}, \mathbf{c}, \mathbf{s})$ policy. Atkins and Iyogun [3] conclude that simple periodic policies seem to work better than complicated can–order policies. Pantumsinchai [23] also studies the MP type policy and shows that the performance of MP is comparable to the $(Q, \mathbf{S})$ policy.

A recent study that considers periodic and continuous review policies for the stochastic joint replenishment problem is by Cachon [7]. Cachon [7] considers three policies for dispatching trucks; the first one is the *minimum quantity continuous review policy* in which the inventory is reviewed continuously. Trucks are dispatched, (i.e., a replenishment is triggered) when a total of Q units have been ordered. The order quantity here is equal to Q, which is the demand since the last shipment. The second policy is the *full service periodic review policy*. In this case, inventory is reviewed every T time units, and trucks are dispatched to replenish the shelves of the stores, regardless of how much demand has been accumulated since the last order. The third policy is the *minimum quantity periodic review policy*, which is referred to as the $(Q, \mathbf{S}|T)$ policy. In this policy, the retailer reviews its inventory at every T time units. Trucks are dispatched when one of the trucks has at least Q units, and the others are all full.

The final study that we like to discuss under a single echelon setting is a study by Nielsen and Larsen [20]. Nielsen and Larsen propose a new policy referred to as the $Q(\mathbf{S}, \mathbf{s})$ policy. This policy functions as follows: when a total amount of Q

demands are accumulated since the last review, a replenishment order is triggered. Items whose inventory positions in this review at or below s are ordered up to S. This policy becomes a $(Q, \mathbf{S})$ policy if identical demand and identical cost structures are assumed for the items. It is shown that this policy performs better than the previous policies under certain settings.

While there is a large body of literature on multi–echelon inventory systems (for a review and two recent models see Axsater [4] and Federgruen [13]), a few number of studies look at the stochastic joint replenishment problem in a multi–echelon setting. One such study is Gürbüz et al. [17]. In this study, the supply chain consists of a cross–dock location which serves multiple identical retailers. A new replenishment is triggered when a total of Q demands are observed, or when a retailer's inventory position drops to its reorder point. Whenever a replenishment order is triggered, inventory position at each retailer is raised to its order–up–to level. Gürbüz et al. [17] compare the proposed policy with the $(Q, \mathbf{S})$ policy, the periodic review order–up–to policy $(\mathbf{S}, T)$ and the special can–order policy $(\mathbf{S}, \mathbf{c}, \mathbf{s})$. The numerical results show that the proposed policy is better than the other policies under the settings considered. Also in this paper, the authors compare this policy with the others considering additional transportation penalty costs. These penalty costs are incurred, when the number of units shipped exceed the truck capacity, and the costs are based on per–unit exceeded. The proposed policy also outperforms other policies under such transportation penalty costs.

The most recent study on stochastic joint replenishment problem in literature is by Özkaya et al. [21]. In this study they propose $(Q, \mathbf{S}, T)$ policy in a single location, N-items setting. This policy functions as follows: a new replenishment is triggered and inventory positions of all of the items are increased up-to their order-up-to points, whenever a total of Q units are demanded or when T time units elapse. In this study, it is shown that the $(Q, \mathbf{S}, T)$ policy outperforms the other joint replenishment policies in most of the problem instances considered. The new joint replenishment policy is studied and its performance is compared against other policies in a two-echelon setting in Özkaya et al. [22].

A related topic in the multi–echelon setting is the Vendor Managed Inventory (VMI) systems. It is argued that one of the benefits of VMI is the manufacturer's ability to consolidate shipments to jointly replenish the retailers. This aspect of VMI systems is studied by Çetinkaya and Lee [10]. Resupply time and the quantity is decided by the supplier using this information and substantial savings are realizable by a consolidation program that combines small shipments to create larger and more economical deliveries. In this program, replenishment orders wait at the warehouse for the allocation of a specific quantity or for a specified time. The authors study the problem from the vendor's perspective and do not consider the performance of the retailers.

The main contribution of this thesis to the existing literature on the stochastic joint replenishment problem is a new policy that considers the capacity of vehicles that deliver the joint replenishment. We call this policy the $(\mathbf{s}, Q)$ policy. In this policy, an order is triggered whenever the inventory position of an item drops to its reorder point s. The order size is always Q. Since we are not maintaining a constant inventory position for each item at the replenishment epoch, the replenishment order has to be allocated to different items. In the symmetric case the replenishment order is allocated to each item such that the inventory positions are equalized (to the extent that this is possible). The policy is designed especially for situations where the replenishments are shipped using capacitated vehicles and individual items have substantial shortage penalties. The policy is compared against the $(Q, \mathbf{S})$ and the $(Q, \mathbf{S}, \mathbf{s})$ policies under a variety of settings.

Chapter 3

Model

We consider a supply chain that consists of a single warehouse, a single retailer and N items. While a single item, multi–retailer problem is equivalent to a single retailer, multi–item problem when the warehouse has ample supply, we use the latter setting throughout the chapter for consistency. The inventories of items are controlled in a continuous and coordinated fashion by the retailer. Items are shipped from the warehouse to the retailer by a fleet of trucks each having a fixed and identical size. The warehouse has an unlimited supply capacity and the fleet size is assumed to be sufficient enabling the dispatch of the items from the warehouse whenever necessary. There is a fixed transit time from the warehouse to the retailer, which corresponds to the replenishment *leadtime*, L. The notation used throughout the chapter is introduced as need arises and it is also summarized in Table 3.1.

We assume that the demand observed by the retailer for item i follows a Poisson process with a rate of λ_i and unsatisfied demands are fully backordered. The *holding cost*, h_i, is incurred at the retailer level per item per unit time. The *backordering cost*, π_i, is the cost incurred for each unit backordered. The *shortage cost*, p_i is incurred per unit backorder per unit time. The $fixed$ *ordering cost*, K, is associated with the use of trucks, i.e., for every truck of capacity C utilized for shipment, a fixed cost of K is incurred independent of the quantity loaded.

λ_i	Arrival rate of the demand for item i at the retailer
L	Replenishment lead time
K	Fixed ordering cost associated to the use of each truck
S_i	Order up to level of item i at the retailer
s_i	Reorder point of item i at the retailer
$D_i(t)$	Demand observed at the retailer for item i during a time period of t
C	Capacity of a truck
N	Number of items in the system
p_i	Unit shortage cost of item i per unit time
π_i	Backordering cost of item i per each unit backordered
h	Unit inventory holding cost per unit time for item i
τ	Random variable denoting the time between two consecutive replenishments
$f(\cdot)$	pdf of τ
$IP_i(t)$	Inventory position of item i at time t
$IL_i(t)$	Inventory level of item i at time t
Δ_i	$S_i - s_i$

Table 3.1: Summary of Notation

The retailer aims to find a joint replenishment policy for the inventory management of her N items to minimize the total holding, backordering and the fixed costs of ordering in this particular environment. In literature, there are two different heuristic policies (the $(Q, \mathbf{S})$ policy of Cachon [7] and the $(Q, \mathbf{S}, \mathbf{s})$ policy of Gürbüz et al. [17]) proposed for this problem. In this thesis, we propose a new heuristic policy which we refer as the $(\mathbf{s}, Q)$ policy. Next, we present the detailed explanation of these policies.

3.1 The $(Q, \mathbf{S})$ policy

This policy is first suggested by Renberg and Planche [25] for the general joint replenishment problem. The policy under Poisson demand is studied by Pantumsinchai [23]. The policy under a capacitated vehicle is studied by Cachon [7]. In this policy, when the total amount of demand since the previous order reaches Q units, an order in the amount of Q is placed so that the retailer raises the inventory positions of all items up to the vector $\mathbf{S} = (S_1, S_2, ..., S_N)$, where S_i is the order–up–to level of item i. That is, when the total inventory position

$IP(t) = \sum_{i=1}^{N} IP_i(t)$ drops to $S_T - Q$, where $S_T = \sum_{i=1}^{N} S_i$, an order amount of Q is placed to raise the inventory positions of all items up to their order–up–to level S_i. $S_T - Q$ can be assumed as the system reorder point. Since fixed cost of K is incurred every time a truck is utilized, delaying the shipment of a fully loaded truck will not be optimal under a $(Q, \mathbf{S})$ policy. Hence, when optimizing the policy parameters, one should search the region $[1, C]$ for the optimal value of Q for any given truck capacity, C.

Pantumsinchai [23] presents the derivation of the total expected cost function of this policy. Since all of the inventory positions are raised to their order–up–to points, S_i, whenever an order is triggered, inventory positions become a regenerative process. Therefore, the inventory positions of items reach to a steady state and their limiting probability can be computed. The cumulative demand for an item since the last order is binomially distributed for given cumulative demand for all items. If we let X_i be the random variable for the cumulative demand since the last order for item i and X_0 be the $\sum_{i=1}^{n} X_i$, $P(X_i|X_0)$ becomes binomial with parameters x_0 and θ_i, where $\theta_i = \lambda_i/\lambda_0$, where x_0 is uniformly distributed between 0 and $Q-1$. Therefore, the marginal distribution of X_i, referred to as $u(x_i)$ becomes:

$$u(x_i) = \frac{1}{Q} \sum_{x_0=x_i}^{Q-1} \binom{x_0}{x_i} \theta^{x_i} (1-\theta)^{x_0-x_i}, \qquad x_i = 0, 1, ..., Q-1,$$

as shown in Pantumsinchai [23].

It can be shown using recursive calculations that the marginal distribution of X_i is given by:

$$u(x_i) = \frac{1}{\theta Q}(1 - B(x_i; Q, \theta)), \qquad x_i = 0, 1, ..., Q-1,$$

where $B(x_i; Q, \theta)$ is the cumulative binomial probability. The expected value of X_i is $\theta(Q-1)/2$ and the variance is $\theta(1-\theta)(Q-1)/2 + \theta^2(Q^2-1)/12$.

In order to calculate different cost components, we should first calculate the stockout probabilities and the expected backorder size at any time. Therefore, we should know about the inventory positions and the net inventories of the items. It is known that the inventory position at any time t depends on the fixed lead time L. Assuming that the inventory position of an item is z at time $t-L$ and the demand for the item between $t-L$ and t is d_i, the net inventory at any time t becomes $z-d_i = S-v$ where $v = x_i + d_i$. This is because, items ordered before time $t-L$ will be on hand by time t but the items ordered after time $t-L$ will not be on hand by time t. If we let $m(v)$ be the probability distribution of v, $m(v)$ becomes:

$$m(v) = \sum_{x=0}^{min(v,Q-1)} u(x)r(v-x), \qquad v = 0, 1, 2, ...,$$

where $r(.)$ is the probability of demand during lead time.

We now can calculate the stockout probability and expected size of backorder at any time. If we let $P(S,Q)$ be the stockout probability and $B(S,Q)$ be the expected size of backorder at any time, then the equations become:

$$P(S,Q) = Pr(v \geq S) = \sum_{v=S}^{\infty} m(v),$$

and

$$B(S,Q) = \sum_{v=S+1}^{\infty} m(v).$$

For the calculation of the expected backordering cost, we need to know the average stockouts per unit time which is represented by $\lambda P(S,Q)$ and the expected number of stockouts which is represented by $\sum_i P_i(S_i, Q)$. For the calculation of the holding costs, we use the expected on hand inventory which is equal to $S - \theta(Q-1)/2 - \lambda L + B(S,Q)$. With all this information, it is straightforward to write the expected total cost equation per period. If we let $C(Q, S_1, ..., S_n)$ be the expected total cost per period, the equation becomes:

$$\begin{aligned} C(Q, S_1, ..., S_n) &= \frac{K\lambda_0}{Q} + \sum_{i=1}^{n} h_i(S_i - \theta_i(Q-1)/2 - \lambda_i L_i) \\ &+ \sum_{i=1}^{n}(p_i + h_i)B_i(S_i, Q) + \sum_{i=1}^{n} \pi_i \lambda_i P_i(S_i, Q). \end{aligned}$$

3.2 The $(Q, \mathbf{S}, \mathbf{s})$ policy

In this section, we present the details of the $(Q, \mathbf{S}, \mathbf{s})$ policy suggested by Gürbüz et al. [17]. In this policy, when the total amount of demand since the previous order reaches Q or whenever any of the item's inventory position drops to its reorder point s_i, the inventory positions of each item at the retailer are raised to their corresponding order–up–to levels. Hence, there are two different ways of replenishing the system. The first way is only related individually to the retailer's inventory position; that is, whenever any of the retailer's inventory positions drops to its reorder point, replenishment occurs. The second way is related to the total echelon inventory position. Whenever the inventory position of the total echelon drops to $\sum_{i=1}^{N} S_i - Q$, replenishment occurs. Here, if $S_i - s_i \geq Q$ for all i, then this policy works as a $(Q, \mathbf{S})$ policy.

Gürbüz et al. [17] present exact expressions of the total expected costs realized in this policy in the case of identical items. Since the inventory positions are raised to the same level, at every replenishment epoch, we have a regenerative process as in the case of $(Q, \mathbf{S})$ policy. As a result, inventory positions of the items reach a steady state. The time between two consecutive orders is called the *cycle time* and is represented by τ. In order to calculate the expected ordering cost, we first need to find the expected cycle time. Therefore, it is necessary to calculate the probability density function of τ. Let $\Delta = S - s$, then due to the policy requirements

$$\tau = min(T_{\Delta}^{1}, ..., T_{\Delta}^{N}, T_Q),$$

where $T_{\Delta}^{i} \sim Erlang(\Delta, \lambda)$ for all i and $T_Q \sim Erlang(Q, N\lambda)$.

Here, T_{Δ}^{i} is the time at which Δ^{th} demand occurs at the retailer i and T_Q is the time when a total number of Q units is demanded within the system. T_Q would be the cycle time, only if the total demanded amount within the system reaches Q before any of the retailer's inventory positions drops to s. The cycle time is greater than t, if the retailer's inventory positions did not drop to s and the total system demand did not reach Q by that time. Therefore, the cumulative distribution function of the cycle time is driven by the following equation:

$$\begin{aligned}
F_\tau(t) &= Pr(\tau \leq t) \\
&= 1 - Pr(\tau > t) \\
&= 1 - Pr(D_1(t) \leq \Delta - 1, ..., D_N(t) \leq \Delta - 1, D_0(t) \leq Q - 1) \\
&= 1 - \sum_{d_1=0}^{(\Delta-1)\Lambda(Q-1)} \sum_{d_2=0}^{(\Delta-1)\Lambda(Q-1-d_1)} ..... \sum_{d_N=0}^{(\Delta-1)\Lambda(Q-1-d_1-...-d_{N-1}} \prod_{i=1}^{N} p(d_i; \lambda t),
\end{aligned}$$

where $D_0(t) = \sum_{i=1}^{N} D_i(t)$. and $D_i(t)$ is the demand to retailer i for a period of t time units.

The probability density function of the cycle time τ is calculated by taking the derivative of the cumulative function above and represented by the following equation:

$$\begin{aligned} f(t) &= N\lambda \sum_{d_1=0}^{(\Delta-1)\Lambda(Q-1)} \sum_{d_2=0}^{(\Delta-1)\Lambda(Q-1-d_1)} \cdots \sum_{d_{N-1}=0}^{(\Delta-1)\Lambda(Q-1-d_1-\ldots-d_{N-2})} \\ &\times \prod_{i=1}^{N-1} p(d_i;\lambda t) p((\Delta-1)\Lambda(Q-1-d_1-\ldots-d_{N-1});\lambda t). \end{aligned}$$

The equation above reveals that the distribution of the cycle time depends only on Q and Δ. This distribution is used to calculate the ordering cost.

To calculate the other cost components–holding costs and backordering costs– the probability distribution of the inventory level should be known. To derive the inventory level distribution, the random demand distribution is needed. When $Q > \Delta + (N-1)(\Delta-1)$, the total system demand will never reach Q before any of the item's inventory positions drops to s. Therefore, for this case we have,

$$P(D_i \geq n) = \begin{cases} 1 & \text{if } n = 0 \\ \int_{t=0}^{\infty} \lambda p(n-1;\lambda t)[1 - P(\Delta;\lambda t)]^{(N-1)} dt & \text{if } 1 \leq n \leq \Delta - 1 \\ 0 & \text{if } n \geq \Delta \end{cases}$$

where $P(r;\mu) = \sum_{j=r}^{\infty} p(j;\mu)$ and $p(j;\mu) = e^{-\mu}\frac{\mu^j}{j!}$.

When $Q \leq \Delta + (N-1)(\Delta-1)$, an order can be triggered in either way. The probability distribution of the amount shipped to the retailer during a cycle time, represented by Z_i, is the following:

$$P(Z \geq n \mid Q, \Delta, N) = \begin{cases} P(Z \geq n \mid Q-1, \Delta, N) + \phi(Q, \Delta, N) \\ \quad \text{if } n = 0, 1, ..., Q-1 \\ \\ P(Z \geq n \mid Q-1, \Delta, N) + \phi(Q, \Delta, N) \\ \quad \text{if } n = \min(\Delta, Q) \& \min(\Delta, Q) = \min(\Delta, Q-1) \\ \\ \sum_{d_1=n-1}^{n-1} \cdots \sum_{d_N=0}^{(\Delta-1)\Lambda(Q-1-d_1-d_2-...-d_{N-1})} \frac{h(d_1,d_2,...,d_N)}{N} \\ \quad \text{if } n = \min(\Delta, Q) \& \min(\Delta, Q) > \min(\Delta, Q-1) \end{cases}$$

where

$$D_0 = \sum_{i=1}^{N} d_i, h(d_1, d_2, ..., d_N) = \frac{D_o!}{N^{D_0} \times \prod_{i=1}^{N} d_i!},$$

and

$$\begin{aligned} \phi(Q, \Delta, N) &= \lambda \begin{pmatrix} Q-1 \\ n-1 \end{pmatrix} \left(\frac{1}{N}\right)^{n-1} \left(1 - \frac{1}{N}\right)^{Q+1-n} \\ &\times \quad (E(\tau|Q+1-n, \Delta, N-1) - E(\tau|Q-n, \Delta, N-1)). \end{aligned}$$

These equations suggest calculating the probability of Z recursively.

The relationship between the inventory position and the inventory level is $IP(t-L) - D(L) = IL(t)$, where $IL(t)$ is the inventory level at time t, and $IP(t-L)$ is the inventory position at time $t-L$. To calculate the holding costs and backordering costs, we should find the inventory level distribution using the inventory position distribution. The inventory position distribution is derived using demand distributions. For $Q = 1$,

$$P(D \geq n \mid Q) = \begin{cases} 1 & \text{if} \quad n = 0 \\ 0 & \text{if} \quad n \geq 1 \end{cases},$$

and when $Q \geq 2$,

$$P(D \geq n \mid Q) = \begin{cases} 1 & \text{if} \quad n = 0 \\ P(Z \geq n \mid Q-1) & \text{if} \quad 1 \leq n \leq \min(\Delta - 1, Q-1) \\ 0 & \text{if} \quad n \geq \min(\Delta, Q) \end{cases}.$$

Since the demand distributions are known, the inventory position distribution can easily be calculated. The equation for the inventory position distribution can be found as follows:

$$\begin{aligned} Pr(IP = j) &= \sum_{n=S-j}^{(\Delta-1)\Lambda(Q-1)} P(IP = j \mid D(\tau) = n)P(D(\tau) = n) \\ &= \sum_{n=S-j}^{(\Delta-1)\Lambda(Q-1)} \frac{P(D(\tau) = n)}{n+1}, j = S - \min(\Delta - 1, Q - 1), ..., S. \end{aligned}$$

By using these distributions, the expected value of the cycle time and the expected value of the inventory level can be calculated. These expectations are used to determine the expected ordering cost, the expected holding cost and the expected backordering cost. The total cost function is:

$$\begin{aligned} CR &= \frac{K}{E[\tau]} + N \times [(h+p)E[IL^+] + p(E[D(LT)] - E[IP]) \\ &+ \pi \times \lambda(1 - \sum_{j=(\max(S-Q+1, S-\Delta+1))^+}^{S} \sum_{l=0}^{j-1} p(l, \lambda LT)Pr(IP = j))]. \end{aligned} \quad (3.1)$$

Gürbüz et al. [17] use only one type of backordering cost: no backorder costs are charged per occasion, i.e. per unit backordered. However, in Equation 3.1 above, we also incorporate the backorder costs per occasion.

3.3 The proposed $(\mathbf{s}, Q)$ policy

In the proposed $(\mathbf{s}, Q)$ policy, a joint replenishment order of size Q is triggered when the inventory position of an item falls to its reorder point s_i. Here $\mathbf{s} = (s_1, s_2, ..., s_N)$ denotes the vector of the reorder points of items. The total order size, Q is then allocated to the items so that their inventory positions are equalized to the extent that is possible. This is achieved by employing the following procedure when all items are identical: First, the inventory position of the item which triggered the ordering with the minimum inventory position is increased up to the inventory position of the next item with the lowest inventory position and their inventory positions are equated if Q is sufficient. Otherwise all Q units are allocated to the first item. Next, we begin to increase the inventory positions of these two items up to the inventory position of the next item with the lowest inventory position and equate their inventory positions (again if Q is sufficient). This process continues in the same manner until a total amount of Q is allocated. Different than the previous policies, the inventory positions at each replenishment epoch are not necessarily equal to each other and there is no fixed order–up–to point for any of the items.

Deciding how to allocate the items is a critical issue in this policy. We employ different allocation rules depending on whether the items are identical in their backordering costs or not. When items are identical, we try to equate their inventory positions as the size of Q permits as explained above. When items are not identical we allocate the total order size of Q to items, by minimizing the total expected backordering costs in the subsequent replenishment cycle. In particular, for each of the items, we calculate how much we save from expected backordering cost in the subsequent replenishment cycle if we increase the inventory position of that item by one unit. We compare the savings and allocate one unit to the item which produces maximum savings. The same procedure is employed until all Q units are allocated. The expected backordering cost of item i with an inventory position of IP_i can be calculated as:

$$EBC(IP_i) = \pi_i.E[\max\{D_i(L) - IP_i, 0\}] = \pi_i \sum_{i=I_i}^{\infty}(D_i(L) = i)(i - IP_i).$$

Specifically, for every item i, we calculate

$$EBC(IP_i + 1) - EBC(IP_i),$$

and allocate one unit to the item which will provide the highest difference (highest reduction in cost). Note that this allocation rule is merely a heuristic rule. Several different allocation rules may be employed in this setting.

There are some advantages of Q being constant. Since we are using capacitated vehicles for shipment, companies prefer attaining stable and acceptable utilization levels on the trucks that they dispatch. Moreover, Q being constant, together with the allocation policy explained above, ordering items which have higher inventory positions can be avoided. Since, the truck capacities are constant and every time a truck is used a fixed cost of K is incurred, the decision on the value of Q will be based on how much of the capacity is utilized. Note that, delaying the shipment of a fully loaded truck cannot be optimal under the $(\mathbf{s}, Q)$ policy, similar to the previous policies. Hence, one should search the region $[1, C]$ for the optimal value of Q for any given truck capacity, C. In the $(Q, \mathbf{S})$ policy, an order may include items with unnecessarily high inventory positions since there is no individual control of items. Whereas in our proposed policy, the existence of reorder points allows for such an individual control and hence prevents unnecessarily increasing of those items that already has higher inventory positions. Therefore, the total system saves from total expected holding costs and saves from total expected backordering costs by ordering from the items which have lower inventory.

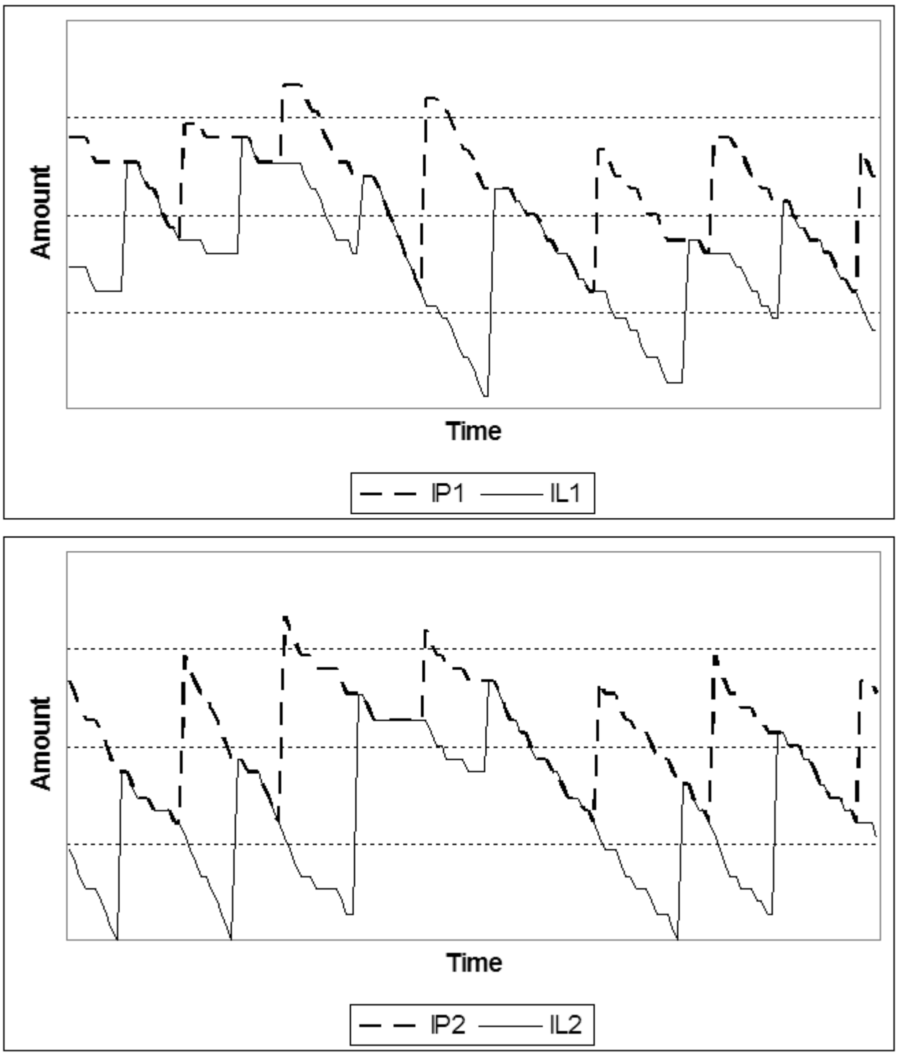
Figure 3.1 shows how the proposed $(\mathbf{s}, Q)$ policy functions. There are two identical items in the system. It can be observed from the graph that when an order is triggered the inventory positions of the items are increased in a way that their inventory positions are equalized. This reduces holding costs since, we do not order much for the item which already has a high inventory position.

The inventory positions of items under the $(\mathbf{s}, Q)$ policy can be modeled by a continuous time Markov chain. First, we explain this modeling approach for an identical items case. Due to the nature of the policy and the allocation rule employed, the inventory position of each item can take a value between $(s + 1, s + 2, ..., s + Q)$ at any given time. Let $x = (IP_1, IP_2, ..., IP_N)$ be the vector of inventory positions of all items at any given time. A continuous time Markov chain model can be constructed by defining its states by the vector x. Since the inventory position of each item can take Q different values and we have N items, this Markov chain has Q^N states. One can find the transition rates from every state to each other by considering the demand process. The system leaves its current state when one of the items observe one unit of demand. Since the demand of each item is Poisson with rate λ_i, the interarrival times of demand realizations for each item is exponentially distributed with the same rate λ_i. Hence, the time that the system stays at any given state is determined by the minimum of these interarrival times, and thus, is exponentially distributed by rate

$$\sum_{i=1}^{N} \lambda_i.$$

Moreover, the probability that the minimum is due to item i (or equivalently the

Figure 3.1: How the proposed $(\mathbf{s}, Q)$ policy functions



system leaves the current state due to item i) is

$$\frac{\lambda_i}{\sum_{i=1}^{N} \lambda_i}.$$

Thus, the rate of moving from state $(IP_1,\ IP_2,\ ...,\ IP_i,\ ...,\ IP_N)$ to $(IP_1,\ IP_2,\ ...,\ IP_i - 1,\ ...,\ IP_N)$ is

$$\frac{\lambda_i}{\sum_{i=1}^{N} \lambda_i} \times \sum_{i=1}^{N} \lambda_i = \lambda_i.$$

An example state transition rate diagram is illustrated in Figure 3.2 for an identical two–item environment with $s_1 = s_2 = 0$.

From Figure 3.2 it can be seen that, when a demand occurs for any of the items, the state of the Markov chain will change. Since there are two items, there are two possible ways out from each state as demand arrives to one of the items. Let the current state of the system be (IP_1, IP_2). When item 1 observes one unit of demand at the retailer, the state of the system moves to $(IP_1 - 1, IP_2)$ with rate λ_1 if $IP_1 - 1 \neq s_1$. If $IP_1 - 1 = s_1$, a replenishment order of size Q is initiated and this order is allocated to both items by the allocation rule described above. Hence, the state that the Markov chain jumps to depends on the allocation rule and the values of the current state variables. Note that the allocation rule for identical items suggests equating their inventory positions as far as Q permit. Therefore, when there are only two items, the Markov chain enters to a state where inventory positions are equal or to a state where the difference between inventory positions are only one after leaving the current state. For example, when a demand occurs at state $(1, 2i + 1 - Q)$ for item 1, the inventory position of item 1 falls to its reorder point ($s_1 = 0$) and it triggers the ordering. First the allocation rule increases the inventory position of item 1 up to $2i + 1 - Q$ after which only $Q - (2i + 1 - Q) = 2Q - 2i - 1$ units are left to allocate. Since the items are identical, half of the units are allocated to item 1 and the other half is allocated to item 2. As a result, inventory position of each item should be $2i + 1 - Q + \frac{2Q-2i-1}{2} = i + \frac{1}{2}$. However, since item 1 triggered the ordering and we could not have half of an item, we allocate one additional unit to item 1 by integrating the fractional parts. Hence, Markov chain enters the state $(i + 1, i)$. Note that the total of the inventory positions of the two items at the entered state is greater than Q. Otherwise, there is only two possible ways in to that state from the states $(i + 2, i)$ and $(i + 1, i + 1)$. Since the items are identical, state transition procedure is the same for item 2.

After writing all of the transition equations we calculate the steady state probability distributions. Steady state distribution of the inventory positions can be used to calculate the steady state distribution of the inventory level of the items. Inventory level probabilities are used to calculate the expected backordering and the holding costs per unit time. Let $P\{IP = y\}$ denote the steady state probability that inventory position of item i is y at any given time. Then,

$$P\{IL = x\} = \sum_{i=s+1}^{s+Q} P\{IP = i\} P\{D_i(L) = x + i\},$$

where $D_i(L)$ is Poisson distributed with rate $\lambda \times L$. Therefore, the expected backordering cost for an item becomes

$$EBC = \pi \sum_{x=-\infty}^{-1} P(IL = x),$$

and the expected holding cost for an item becomes

$$EHC = h \sum_{x=1}^{s+Q} P(IL = x).$$

The expected ordering cost for an item is

$$EOC = \frac{K\lambda_i}{Q}.$$

Therefore, since there are N identical items, the total expected cost becomes

$$ETC = N \times (EOC + EBC + EHC).$$

The equations of the steady state probabilities, when there are N identical items in the inventory, can be found in Appendix A. Next, we provide the equations when there are two identical items. Since the items are identical $\lambda_1 = \lambda_2$ and $s_1 = s_2 = s$.

The steady state probabilities can be found by solving the set of equations below. The left hand side of the equations are the outgoing rates while the right hand side is the ingoing rates for a state.

$2\Pi_{(s+Q-i)(s+Q)} = \Pi_{(s+Q-i+1)(s+Q)}$ for i = s+2,...,(s+Q-1)

$2\Pi_{(s+Q)(s+Q-i)} = \Pi_{(s+Q)(s+Q-i+1)}$ for i = s+2,...,(s+Q-1)

The first two equations are for the states when one of the items inventory position is at its maximum value Q and the difference between the inventory positions is at least 2 units.

$$2\Pi_{(s+Q)(s+Q-1)} = \Pi_{(s+Q)(s+Q)} + \Pi_{(s+1)(s+Q-1)}$$

$$2\Pi_{(s+Q-1)(s+Q)} = \Pi_{(s+Q)(s+Q)} + \Pi_{(s+Q-1)(s+1)}$$

The two equations above are for the states when the difference between inventory positions of the items is only 1.

When the inventory positions of the items are at their maximum the equation becomes:

$$2\Pi_{(s+Q)(s+Q)} = \Pi_{(s+Q)(s+1)} + \Pi_{(s+1)(s+Q)}.$$

For the rest of the states, we have a set of equations that can be expressed using the following algorithm:

for $i = s+1, ..., (s+Q-1)$

for $j = s+1, ..., (s+Q-1)$

if $|i-j| = 1$ and $2i > Q$

$$2\Pi_{(i)(j)} = \Pi_{(i+1)(j)} + \Pi_{(i)(j+1)} + \Pi_{(s+1)(i+j-Q)}$$

$$2\Pi_{(j)(i)} = \Pi_{(j+1)(i)} + \Pi_{(i)(j+1)} + \Pi_{(i+j-Q)(s+1)}$$

$$2\Pi_{(i)(i)} = \Pi_{(i+1)(i)} + \Pi_{(s+1)(2i-Q)} + \Pi_{(2i-Q)(s+1)} + \Pi_{(i)(i+1)}$$

else

$$2\Pi_{(i)(j)} = \Pi_{(i+1)(j)} + \Pi_{(i)(j+1)}$$

next j

next i

There are totally Q^2 equations above. Also we have $\sum_{i=1}^{Q}\sum_{j=1}^{Q}\Pi_{(i)(j)} = 1$. Therefore, we solve this set of equations and find all of the steady state distributions, $\Pi_{(i)(j)}$.

The modeling approach explained above can also be extended to a non-identical items setting. In this case, the allocation rule employed plays a critical role. Indeed, the state transition rates are exactly the same as in the non-identical items case, but the state reached from a boundary state (a state with having at least one of the IP_i values equal to $s_i + 1$) depends on the allocation rule employed. Note that a replenishment decision is made at a boundary state whenever an item with $IP_i = s_i + 1$ observes one unit of demand. First, one can come up with an algorithm that determines the state to be reached from a boundary state for any given allocation rule. Then, a Markov chain can easily be constructed by using the state transition rates and the output of such an algorithm; and the steady state analysis can be employed similar to the identical items case. We also point out that the modeling framework of $(\mathbf{s}, Q)$ policy differs from that of other two policies because the inventory positions of items in the $(\mathbf{s}, Q)$ policy do not form a regenerative process as there are no fixed order–up–to levels of items.

After finding the steady state probabilities, we search for the optimal s and Q values to minimize the expected total cost. Since the steady state probabilities are dependent only on Q for any s, we calculate these probabilities for a given Q only once. Then using these steady state probabilities we calculate the holding and backordering costs, therefore we search for s for given Q that minimizes the expected total cost. Our numerical studies show that the expected total cost seems to be quasi–convex in s for a given Q, which can be seen in an example in Figure 3.4. Also, the expected total cost seems to be quasi–convex in Q for a given s, which can be seen on an example in Figure 3.3. However, we were not able to show this analytically. Figure 3.5 shows the total cost as a function of Q and s for a particular problem instance.

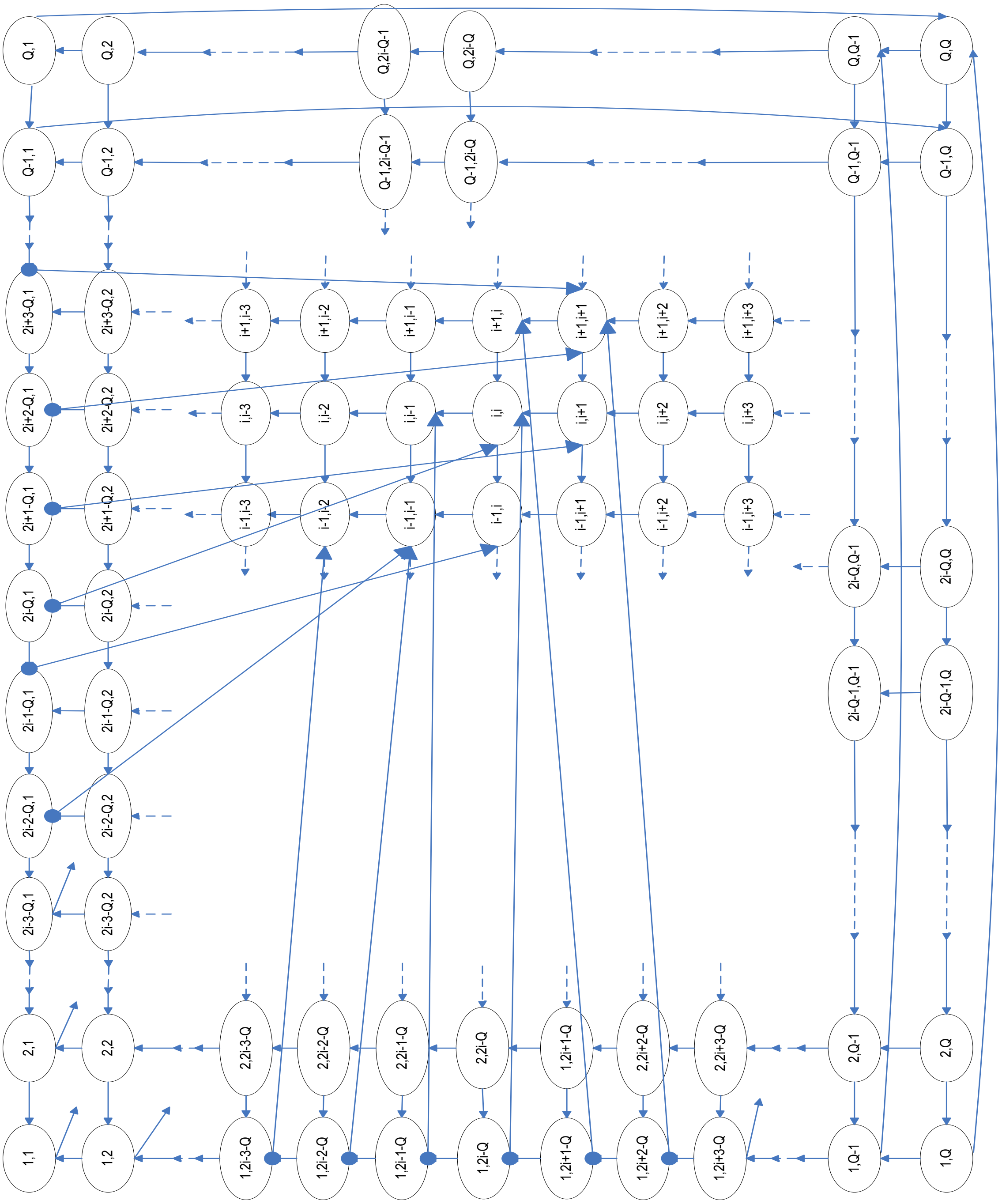


Figure 3.2: State transition rate diagram. All transition rates are equal and λ.

Figure 3.3: Total cost as a function of Q with parameters $\lambda = 5,\ h = 6,\ \pi = 200,\ K = 150,\ s = 6$

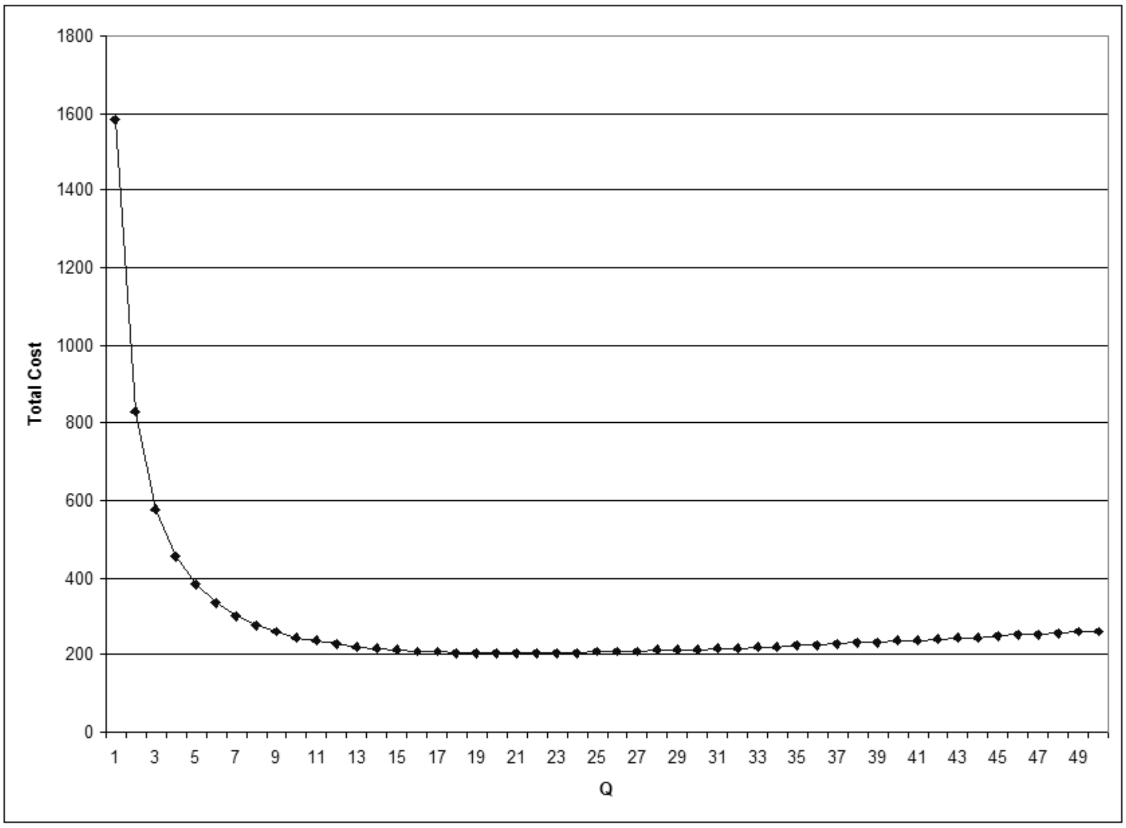


Figure 3.4: Total cost as a function of s with parameters $\lambda = 5,\ h = 6,\ \pi = 200,\ K = 150,\ Q = 17$

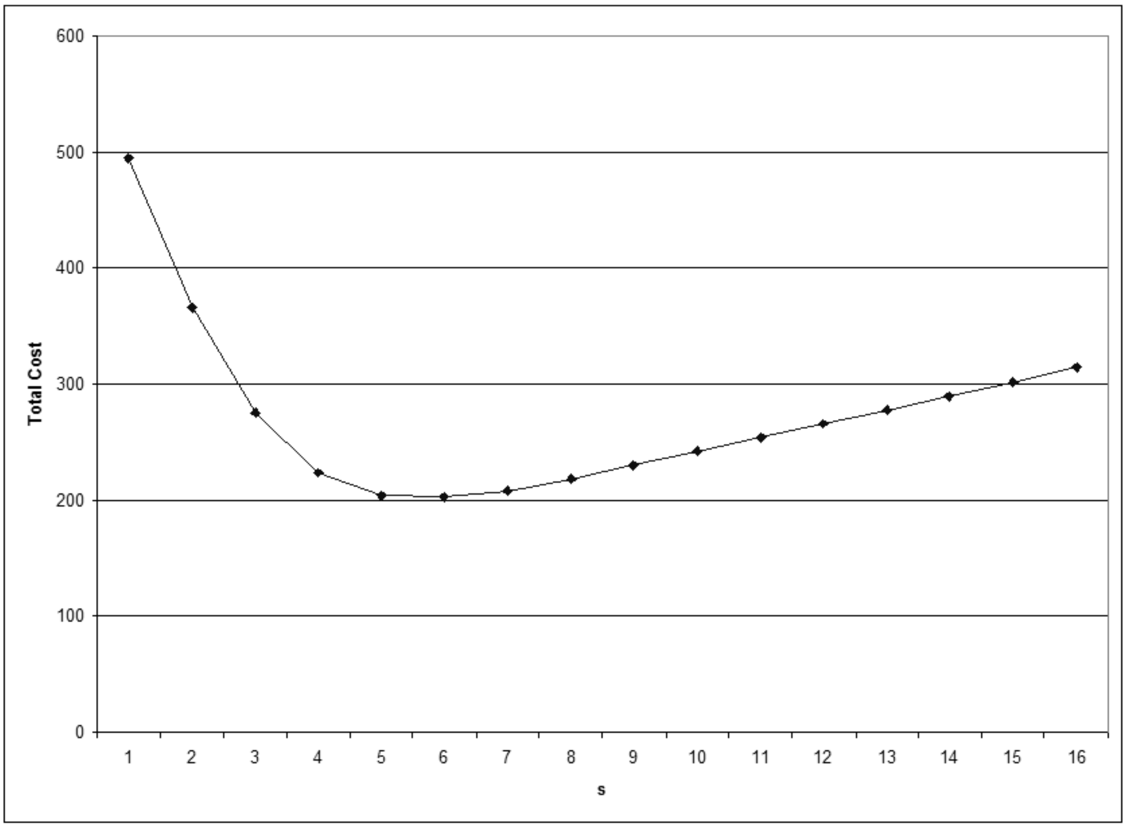
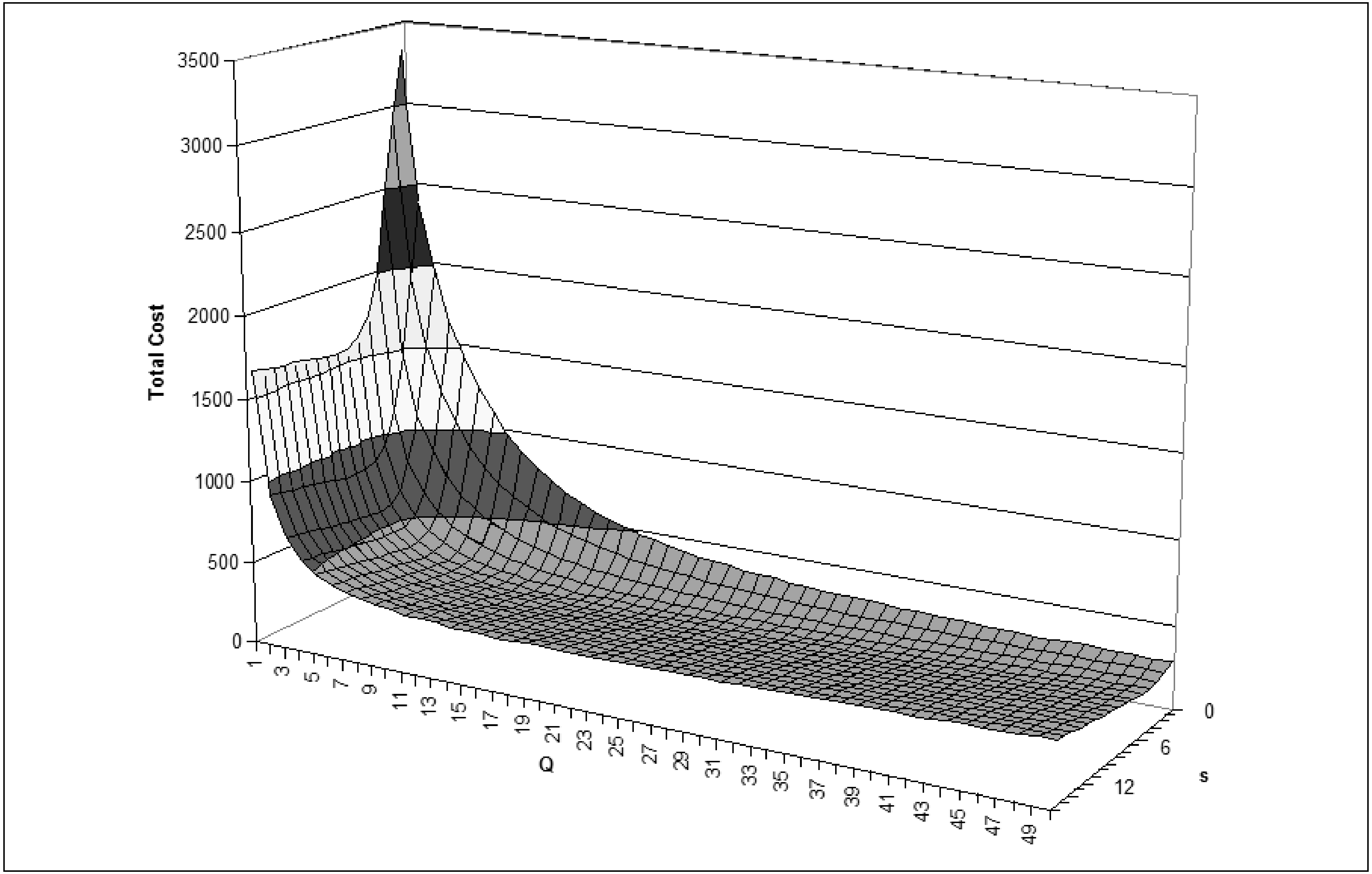


Figure 3.5: Total cost as a function of Q and s with parameters $\lambda = 5,\ h = 6,\ \pi = 200,\ K = 150$



Chapter 4

Numerical Study

In this chapter, we compare the performance of the proposed $(\mathbf{s}, Q)$ policy to that of the $(Q, \mathbf{S})$ and $(Q, \mathbf{S}, \mathbf{s})$ policies through a numerical study. The comparison is based on the optimal total cost rates of the three policies for several problem instances with different parameters including backordering costs, holding costs, fixed ordering costs, number of items, and the capacity of the trucks.

In Chapter 3, we present the total cost rate functions of each policy in terms of their corresponding policy parameters. For each of the three policies and for every problem instance considered, we find the optimal policy parameters by evaluating the total cost rate functions for a sufficiently wide range of parameter values and selecting the ones that minimize the overall cost rate. Even though we present an exact algorithm based on a Markov chain analysis to calculate the total cost rate function of the $(\mathbf{s}, Q)$ policy in Section 3.3, the numerical solutions presented in this chapter for $(\mathbf{s}, Q)$ policy are found via a simulation study. We make one replication with a run length of 100,000 time units. We verified that this run length is sufficiently long by comparing our simulation results to the exact solution. We initiate the system in a way that the inventory level and the inventory position of each item is equal to s and s+Q, respectively. Finally, note also that the optimal parameters for each policy may turn out to be different than each other in any given problem instance.

Let $TC^*_{(s,Q)}$, $TC^*_{(Q,S)}$ and $TC^*_{(Q,S,s)}$ denote the optimal cost rates of the $(\mathbf{s}, Q)$, $(Q, \mathbf{S})$ and $(Q, \mathbf{S}, \mathbf{s})$ policies, respectively. We define the following two functions to evaluate the relative performance of the $(\mathbf{s}, Q)$ policy over $(Q, \mathbf{S})$ and $(Q, \mathbf{S}, \mathbf{s})$ policies, respectively:

$$Gap(QS) = \frac{TC^*_{(Q,S)} - TC^*_{(s,Q)}}{TC^*_{(s,Q)}} \times 100 \ ,$$

and

$$Gap(QSs) = \frac{TC^*_{(Q,S,s)} - TC^*_{(s,Q)}}{TC^*_{(s,Q)}} \times 100 \ .$$

Hence, a positive value of $Gap(QS)$ for a given problem instance indicates that the $(\mathbf{s}, Q)$ policy performs better than the $(Q, \mathbf{S})$ policy. Similarly, a negative value indicates the vice versa. Similar interpretations are also true for the function $Gap(QSs)$.

We present our results in three parts. In Section 4.1, we provide the results without truck capacity constraints for cases with two and four identical items. In Section 4.2, we provide the analysis of the problems with truck capacity constraints. In Section 4.3, we assume two non-identical items and compare the results of the proposed $(\mathbf{s}, Q)$ policy to the $(Q, \mathbf{S})$ policy.

4.1 Comparison under no truck capacity constraints

In this section, we compare the performances of the three policies. Unless stated otherwise, we take $N = 2$, $\lambda = 5$ and $h = 6$ throughout this section. The remaining parameters take one of the following values: $K \in \{100, 150, 200, 500, 1000\}$, $\pi \in \{20, 40, 60, 80, 100, 120, 200, 300\}$ and $L \in \{0.25, 0.5, 1.0\}$. The truck sizes are assumed to be infinity in this section. The detailed results of each problem instance are presented in Appendix B.2. While we draw conclusions by considering average gap values, these conclusions do not always match exactly when individual cases are considered. More detailed individual comparisons are presented with figures in Appendix B.1.

Table 4.1 presents a summary of the relative performance of the $(\mathbf{s}, Q)$ policy over the $(Q, \mathbf{S})$ policy. The values presented in this table are the average

Gap(QS) values where the average is taken over all K values considered. For example, when $N = 2$, $\lambda = 5$, $h = 6$, $\pi = 300$ and $L = 1$, the $(\mathbf{s}, Q)$ policy performs 3.11% better than the $(Q, \mathbf{S})$ policy on the average for all values of $K \in \{100, 150, 200, 500, 1000\}$. Table 4.1 shows that as the unit backordering cost π increases, the $(\mathbf{s}, Q)$ policy begins to perform better than the $(Q, \mathbf{S})$ policy. The main reason for this result is that the $(Q, \mathbf{S})$ policy does not provide an individual control over the items, but controls the system in aggregate terms. Therefore, even if the inventory of a particular item is dangerously low, this policy does not replenish that item if the total amount demanded since the last order is not enough to replenish the inventory, which results in backordering. Thus, as the unit backordering cost increases, *Gap*(QS) also increases. We observe the same trend in *Gap*(QS) in terms of individual K values, for any given L value, except in one case, where $K = 100$, $l = 1$ and π increases from 200 to 300.

Table 4.1: Average percentage gap between the $(Q, \mathbf{S})$ and the $(\mathbf{s}, Q)$ policies over all K values.

	L = 1	**L = 0.5**	**L = 0.25**	**AVG** *GAP*(QS)
π = **20**	-1.49	-1.67	-1.83	-1.67
π = **40**	-0.41	0.09	0.39	0.02
π = **60**	0.38	0.95	1.68	1.00
π = **80**	0.97	1.85	2.81	1.87
π = **100**	1.44	2.42	3.84	2.56
π = **120**	1.75	2.89	4.35	3.00
π = **200**	2.59	3.97	5.51	4.02
π = **300**	3.11	4.79	6.73	4.88
AVG *GAP*(QS)	1.17	2.08	3.14	**2.15**

We also observe from Table 4.1 that the average *Gap*(QS) values increase as leadtime decreases for $\pi \geq 40$. The reason behind this result is again the individual control over the items in the $(\mathbf{s}, Q)$ policy because, whenever an item falls to its reorder point, the $(\mathbf{s}, Q)$ policy replenishes the inventory. Since replenishment occurs quickly due to short leadtime, in both of the policies system keeps less inventory and less backordering occurs. Therefore, sum of the backordering and holding costs of both of the policies reduce. However, this reduction in total backordering and holding costs of the $(\mathbf{s}, Q)$ policy is more as compared to that of the $(Q, \mathbf{S})$ policy because of the effective control at the individual level. For example, as it can be seen from Appendix B.2 for problem instance where $\pi = 120$, $K = 150$ and $l = 1.0$, sum of the backordering and holding cost is 136.07 while it is 124.54 for $l = 0.5$ for the $(\mathbf{s}, Q)$ policy. With the same parameters for the

$(Q, \mathbf{S})$ policy, this cost is 137.11 for $l = 1$ while, it is 127.83 for $l = 0.5$. Reduction is 8.47% in the $(\mathbf{s}, Q)$ policy while, it is only 6.76% in the $(Q, \mathbf{S})$ policy. Thus, for average results, we see that $Gap(QS)$ increases as leadtime decreases. For individual problem instances, in all cases for $\pi \geq 60$, this trend is same as with the average results. There are few cases violating this trend for $\pi = 40$. For $\pi = 20$ case, the trend is the opposite; $Gap(Q, S)$ decreases as leadtime decreases. In this case, backorder costs are small and the reductions in inventory holding costs outweigh the reductions in backordering costs due to the leadtime reduction.

Table 4.2: Average percentage gap between the $(Q, \mathbf{S}, \mathbf{s})$ and the $(\mathbf{s}, Q)$ policies over all K values.

	L = 1	**L = 0.5**	**L = 0.25**	**AVG** $GAP(QSs)$
π = **20**	-1.51	-1.70	-1.87	-1.70
π = **40**	-0.77	-0.59	-0.70	-0.68
π = **60**	-0.48	-0.49	-0.50	-0.49
π = **80**	-0.32	-0.33	-0.27	-0.31
π = **100**	-0.27	-0.27	-0.13	-0.22
π = **120**	-0.26	-0.22	-0.17	-0.22
π = **200**	-0.13	-0.22	-0.17	-0.17
π = **300**	-0.20	-0.10	-0.17	-0.16
AVG $GAP(QSs)$	-0.44	-0.43	-0.42	**-0.43**

Table 4.2 compares the performance of the $(\mathbf{s}, Q)$ policy to that of the $(Q, \mathbf{S}, \mathbf{s})$ policy in a similar manner. We observe that the $Gap(QSs)$ values are close to zero but, the $(Q, \mathbf{S}, \mathbf{s})$ policy outperforms the $(\mathbf{s}, Q)$ policy in 112 out of 120 individual cases. In this table, the overall averages show that as the unit backordering cost π increases, the performance of the proposed policy approaches to the performance of the $(Q, \mathbf{S}, \mathbf{s})$ policy or remains same. This is because, as the unit backordering cost increases, the ordering cost and the holding cost of the $(Q, \mathbf{S}, \mathbf{s})$ policy increases since policy begins to order more frequently to prevent stockouts. However, to balance the frequent ordering, it keeps the value of Q high, which increases the holding cost.

Although we observe in Table 4.2 that the gap seems to decrease monotonically as π increases, we see that this is not always true. When $L = 1$, the gap increases when π increases from 200 to 300. Also, when $L = 0.25$ the gap increases when π increases from 100 to 120. We suspect that these differences are because of simulation results. In individual problem instances, there is no monotonic behaviour for $Gap(QSs)$ as π increases.

It can also be observed from Table 4.2 that the effect of the leadtime on the *Gap*(QSs) between the two policies is small and non-monotonic. This is due to the fact that replenishment leadtime primarily affects backordering costs, and both policies perform similarly as they both provide individual control.

We observe that the optimal Q value of the $(\mathbf{s}, Q)$ policy is always smaller than the optimal Q value of the $(Q, \mathbf{S}, \mathbf{s})$ policy. In the $(Q, \mathbf{S}, \mathbf{s})$ policy, the optimal Q value is kept larger in order to compensate for the possible smaller orders that are triggered individually when the inventory position of an item reaches its reorder level.

Table 4.3: Average percentage gap between the $(Q, \mathbf{S})$ and the $(\mathbf{s}, Q)$ policies over all π values.

	L = 1	**L = 0.5**	**L = 0.25**	**AVG** $GAP(QS)$
K = 100	1.29	2.43	3.99	2.57
K = 150	1.26	2.31	3.67	2.41
K = 200	1.22	2.36	3.44	2.34
K = 500	1.30	2.01	2.86	2.06
K = 1000	0.77	1.27	1.76	1.27
AVG $GAP(QS)$	1.17	2.08	3.14	**2.15**

The effect of the setup cost on the relative performance of the $(\mathbf{s}, Q)$ policy over the $(Q, \mathbf{S})$ policy are given in Table 4.3. It can be seen from the table that the overall average *Gap*(QS) decreases as the setup cost, K increases. This effect is due to the fact that higher setup costs tend to produce larger ordering and inventory holding costs, and this significantly shrinks the impact of the differences in backordering costs. For example, as it can be seen from Appendix C, in a particular instance where $\pi = 120$ and $L = 0.5$ the gap between the $(\mathbf{s}, Q)$ and the $(Q, \mathbf{S})$ policy decreases as the setup cost increases. When $K = 100$, backordering cost is 7.03% of the total cost for the $(\mathbf{s}, Q)$ policy, while it is 9.53% for the $(Q, \mathbf{S})$ policy. However, when the setup cost $K = 1000$, backordering cost is only 4.59% of the total cost for the $(\mathbf{s}, Q)$ policy, while it is 6.96% for the $(Q, \mathbf{S})$ policy.

Although, for overall averages there is a monotonic behaviour that *Gap*(QS) decreases as K increases, if we look at Table 4.3 for individual leadtime values, we observe that there are two cases that violate this monotonic behaviour. One of them is when $l = 1$ and K increases from 200 to 500 and the other is when $l = 0.5$ and K increases from 150 to 200. Moreover, this non-monotonic behaviour persists in individual instances with any given π value. The non-monotonic behaviour is because of the complex dynamics between the cost components.

Table 4.4 the relative performance of the $(\mathbf{s}, Q)$ policy over the $(Q, \mathbf{S}, \mathbf{s})$ policy as a function of the setup cost. We can easily observe that there is a non-monotonic relation between $Gap(QSs)$ and K. The effect of setup costs on the relative performance of these policies is rather insignificant. This insignificant effect is due to similar behaviour of the policies in all K values. As the setup cost increases system tries to order less frequently which results in an increase in the Q value. This increase in the Q value tends to keep more inventory, and therefore, total ordering and holding costs become the major part of the total cost.

Table 4.4: Average percentage gap between the $(Q, \mathbf{S}, \mathbf{s})$ and the $(\mathbf{s}, Q)$ policies over all π values.

	L = 1	**L = 0.5**	**L = 0.25**	**AVG** *GAP(QSs)*
K = 100	-0.50	-0.51	-0.43	-0.48
K = 150	-0.51	-0.53	-0.48	-0.51
K = 200	-0.51	-0.37	-0.51	-0.46
K = 500	-0.29	-0.35	-0.35	-0.33
K = 1000	-0.37	-0.36	-0.33	-0.35
AVG *GAP(QSs)*	-0.43	-0.42	-0.42	**-0.43**

The average percentage gap between the $(\mathbf{s}, Q)$ policy and the $(Q, \mathbf{S})$ policy is 2.15%, meaning that the proposed policy performs better than the $(Q, \mathbf{S})$ policy on the average for the problem instances considered in our numerical study. For individual instances, we observe that $Gap(QS)$ value ranges between -2.22% and 7.54%. The minimum gap value is observed when $\pi = 20$, $L = 0.25$ and $K = 150$ where as the maximum value is observed when $\pi = 300$, $L = 0.25$ and $K = 200$. On the other hand, the average gap between the $(Q, \mathbf{S}, \mathbf{s})$ policy and the proposed $(\mathbf{s}, Q)$ policy is -0.43% meaning that the $(Q, \mathbf{S}, \mathbf{s})$ policy performs better than the proposed policy but, the deviation is very small. For individual instances, we observe that $Gap(QSs)$ value ranges between -2.22% and 0.60%. The minimum gap value is observed when $\pi = 20$, $L = 0.25$ and $K = 150$ where as the maximum value is observed when $\pi = 40$, $L = 0.5$ and $K = 200$. The minimum gap values are occurred at the same parameter sets since, the $(Q, \mathbf{S}, \mathbf{s})$ policy performs the same with the $(Q, \mathbf{S})$ policy for the cases where $(Q, \mathbf{S})$ policy outperforms the $(\mathbf{s}, Q)$ policy. This result is due to the policy characteristics. A detailed summary

Table 4.5: Detailed gap values for each π (total of 15×8 instances)

π	*Gap*(QS)			*Gap*(QSs)		
	min %	max %	Number of instances ≥ 0	min %	max %	Number of instances ≥ 0
20	-2.22	-1.24	0	-2.22	-1.24	0
40	-1.42	2.46	6	-1.42	0.60	1
60	-1.03	3.16	12	-1.05	-0.16	0
80	-0.04	4.30	14	-0.75	-0.12	0
100	0.73	5.16	15	-0.58	0.08	1
120	1.06	5.34	15	-0.48	0.04	1
200	2.40	6.03	15	-0.38	0.08	2
300	2.49	7.54	15	-0.56	0.02	3

Table 4.6: Detailed gap values for each K (total of 24×5 instances)

K	*Gap*(QS)			*Gap*(QSs)		
	min %	max %	Number of instances ≥ 0	min %	max %	Number of instances ≥ 0
100	-1.30%	6.87%	20	-1.30%	-0.02%	0
150	-2.22%	7.14%	20	-2.22%	-0.04%	0
200	-2.05%	7.54%	20	-2.05%	0.60%	2
500	-2.12%	6.38%	18	-2.12%	0.08%	2
1000	-2.15%	5.70%	14	-2.15%	0.08%	4

for every problem instance considered can be found in Appendix B.2. A detailed summary of the minimum and maximum gap values as well as the number of instances the $(\mathbf{s}, Q)$ policy outperforms the $(Q, \mathbf{S})$ and the $(Q, \mathbf{S}, \mathbf{s})$ policies for given π, K and L values are presented in Tables 4.5, 4.6 and 4.7.

In this section, we also investigate the case $N = 4$. We decrease the arrival rate λ from 5 to 2.5 so that the total arrival rate to the system is the same. Tables 4.8 and 4.9 depict the behaviour of each of the three policies for a limited number of problem instances. The relative performances of the $(\mathbf{s}, Q)$ policy with respect to the $(Q, \mathbf{S})$ and $(Q, \mathbf{S}, \mathbf{s})$ policies are observed to be different in $N = 4$ and $N = 2$ cases for the problem instances considered. For the case where the unit backordering cost value is small, in $N = 4$ case *Gap*(QS) and *Gap*(QSs) are less than the gap values in $N = 2$ case, which means that the $(Q, \mathbf{S}, \mathbf{s})$ and $(Q, \mathbf{S})$

Table 4.7: Detailed gap values for each L (total of 40×3 instances)

L	*Gap*(QS)			*Gap*(QSs)		
	min %	max %	Number of instances ≥ 0	min %	max %	Number of instances ≥ 0
0.25	-2.22%	7.54%	32	-2.22%	0.08%	4
0.5	-1.98%	4.95%	32	-1.98%	0.60%	3
1.0	-1.68%	3.36%	28	-1.68%	0.08%	1

Table 4.8: Comparison of the three policies for $N = 4$ and $\pi = 20$ case

K	*Gap*(QS) L = 1 %	*Gap*(QS) L = 0.5 %	*Gap*(QS) L = 0.25 %	*Gap*(QSs) L = 1 %	*Gap*(QSs) L = 0.5 %	*Gap*(QSs) L = 0.25 %
100	-3.12	-3.53	-4.04	-3.12	-3.53	-4.04
150	-2.82	-2.74	-3.34	-2.82	-2.74	-3.34
200	-1.99	-2.31	-2.42	-1.99	-2.31	-2.42

Table 4.9: Comparison of the three policies for $N = 4$ and $\pi = 120$ case

K	*Gap*(QS) L = 1 %	*Gap*(QS) L = 0.5 %	*Gap*(QS) L = 0.25 %	*Gap*(QSs) L = 1 %	*Gap*(QSs) L = 0.5 %	*Gap*(QSs) L = 0.25 %
100	1.14	2.80	4.64	-0.95	-0.45	-0.57
150	1.38	1.85	3.88	-0.70	-1.10	-0.41
200	1.13	1.89	2.98	-0.61	-1.05	-0.46
500	0.56	1.02	1.89	-0.39	-0.29	-0.33
1000	-0.54	-0.24	-0.22	-0.54	-0.24	-0.21

policies performances improve as N increases. For example, while *Gap*(QS) and *Gap*(QSs) are -1.97% when $N = 2$, they are both -2.74% when $N = 4$. Also, for large backordering cost value examined, *Gap*(QS) and *Gap*(QSs) values when $N = 4$ are again less than the corresponding values of $N = 2$ case. However, again the proposed policy performs better than the $(Q, \mathbf{S})$ policy for the large value of backordering cost unless the setup cost is not significantly large. Again, as in $N = 2$, policies perform similarly and the gaps diminish, as we increase the setup costs.

4.2 Comparison under truck capacity constraints

In this section we investigate the performances of the three policies under truck capacity constraints. Recall that, for a given truck capacity, C, one should search the region $[1, C]$ for the optimal value of Q in order to optimize policy parameters in all three policies. In order to analyze the impact of the vehicle capacity, we plot the C/Q^* versus *Gap*(QS) and *Gap*(QSs) values for several problem instances. Here, Q^* is the optimal Q value for the $(\mathbf{s}, Q)$ policy for a given problem instance when C is unlimited. Capacity of the trucks are selected from the set $C \in [1, Q^*]$ in such a way that $C/Q^* = \{0.2, 0.4, 0.6, 0.8, 1.0\}$. The other problem parameters are kept as the same. For each particular value of C, the optimal Q values of $(Q, \mathbf{S})$ and $(Q, \mathbf{S}, \mathbf{s})$ policies for each problem instance are searched over the range $[1, C]$.

In Figures 4.1, 4.2 and 4.3 we present the C/Q^* versus *Gap*(QS) values for all K values considered when $\pi = 20$ and $l = 0.25, 0.5, 1.0$, respectively. Similarly, Figures 4.4, 4.5 and 4.6 present *Gap*(QSs) values for the same problem instances. The first observation we make is the similarity of the Figures 4.1, 4.2 and 4.3 and the corresponding Figures 4.4, 4.5 and 4.6 and thus the fact that the $(Q, \mathbf{S}, \mathbf{s})$ policy behaves exactly the same as the $(Q, \mathbf{S})$ policy for small values of backordering cost. This is due to the fact that the individual control is not important when the backordering costs are small. For example, for the problem instance $C/Q^* = 0.8$, $\pi = 20$, $l = 0.5$ and $K = 150$, in both of the $(Q, \mathbf{S})$ and the $(Q, \mathbf{S}, \mathbf{s})$ policies, parameters turn out to be $Q^* = 20$ and $S^* = 12$ and s in the $(Q, \mathbf{S}, \mathbf{s})$ policy turns out to be so small, $s^* = -6$. Under these parameters, the total demand reaches to Q earlier than any of the items inventory position drops to s. Therefore, we may conclude that, if the optimal Q value is significantly greater than C, the $(Q, \mathbf{S}, \mathbf{s})$ policy works exactly the same as the $(Q, \mathbf{S})$ policy. Therefore, since the $(Q, \mathbf{S})$ policy outperforms the $(\mathbf{s}, Q)$ policy in small values of backordering cost, we expect the $(Q, \mathbf{S}, \mathbf{s})$ policy to outperform $(\mathbf{s}, Q)$ policy at this range of parameters. In any case, there are few problem instances where $(\mathbf{s}, Q)$ policy performs better than the other policies under low backordering cost. We observe such situations especially for low leadtimes and moderate capacities (like $C = 0.4$ and $C = 0.6$). Recall that, in the uncapacitated case both of the $(Q, \mathbf{S})$ and the $(Q, \mathbf{S}, \mathbf{s})$ policies outperform the $(\mathbf{s}, Q)$ policy when $\pi = 20$.

Figure 4.1: $Gap(QS)$ versus C/Q^* values when $l = 0.25$, $\pi = 20$

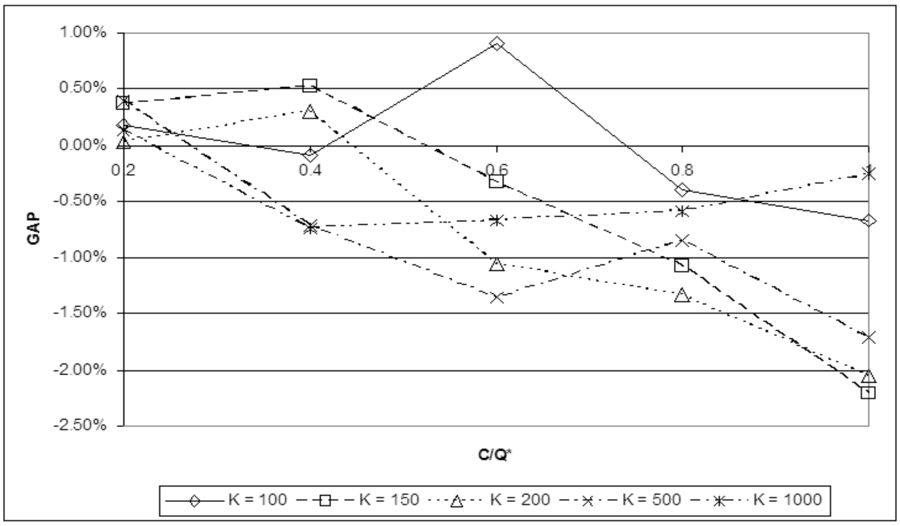


Figure 4.2: $Gap(QS)$ versus C/Q^* values when $l = 0.5$, $\pi = 20$

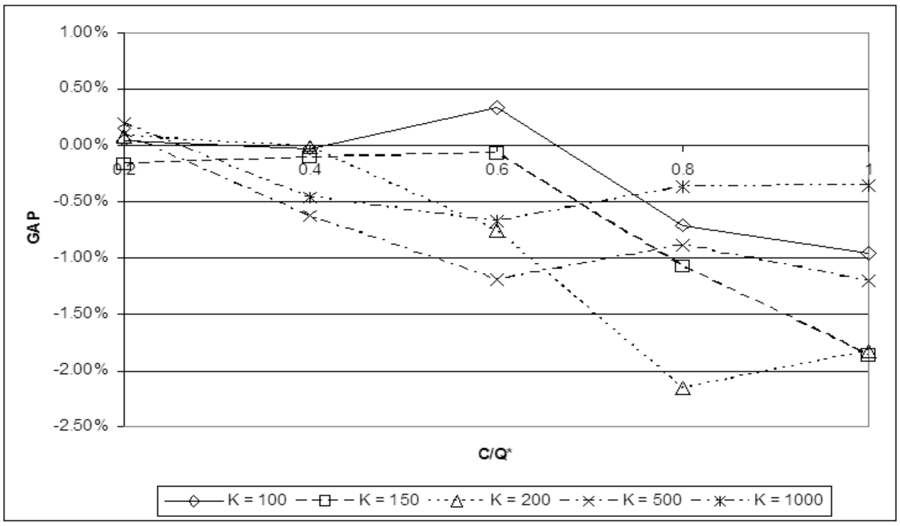


Figure 4.3: $Gap(QS)$ versus C/Q^* values when $l = 1$, $\pi = 20$

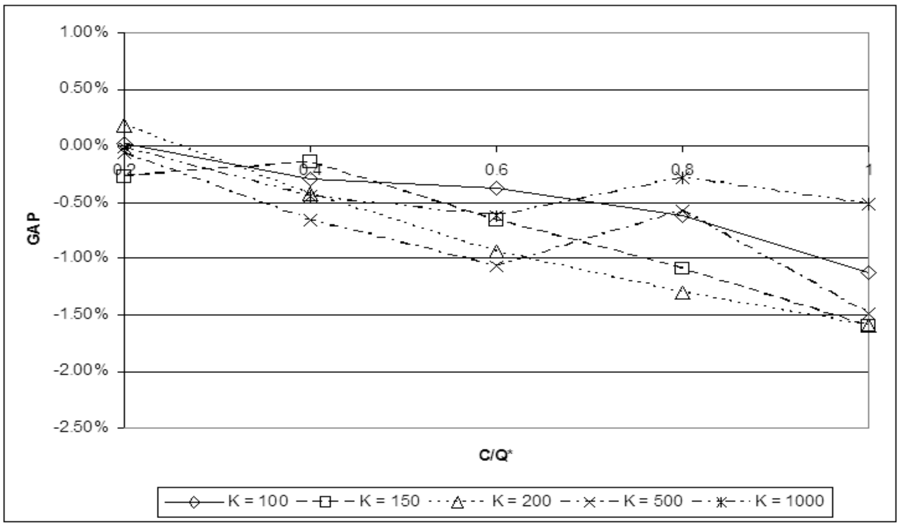


Figure 4.4: $Gap(QSs)$ versus C/Q^* values when $l = 0.25$, $\pi = 20$

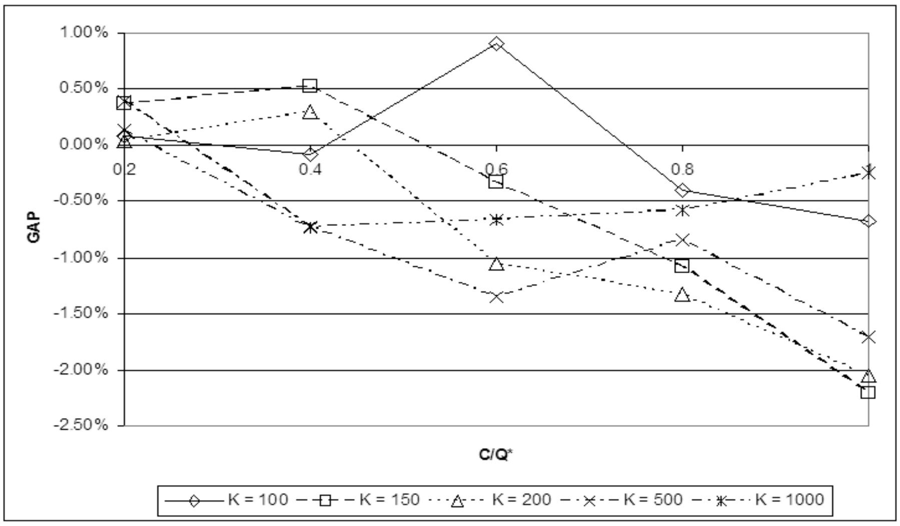


Figure 4.5: $Gap(QSs)$ versus C/Q^* values when $l = 0.5$, $\pi = 20$

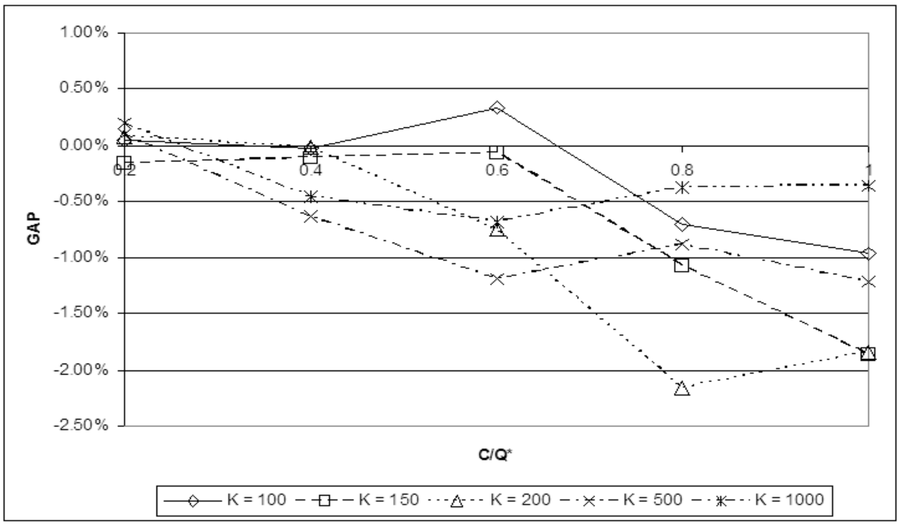
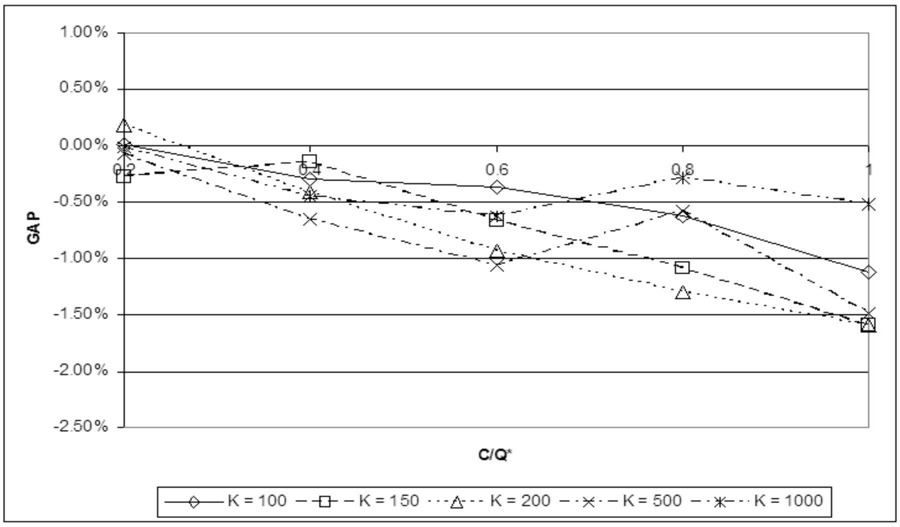


Figure 4.6: $Gap(QSs)$ versus C/Q^* values when $l = 1$, $\pi = 20$



Under small truck capacity, C, (when $C/Q^* = 0.2$) all three policies perform very similar in all of the problem instances considered and hence, the gap values are in the range of -0.40% and 0.72% and the average gap value is 0.17%. This is because, in all cases for small capacity values, optimal Q value is equal to C in all of the three policies and frequent orders are observed. Hence, the effect of other policy parameters diminish. As the available truck capacity increases, the impact of the policy characteristics begin to be effective and gap values exhibit different behaviours. Nevertheless, under relatively small truck capacity (for example when C=0.4), the optimal Q values are equal to C in both of the $(Q, \mathbf{S}, \mathbf{s})$ and $(Q, \mathbf{S})$ policies and the effect of the s parameter in the $(Q, \mathbf{S}, \mathbf{s})$ policy is not observed. Hence, both policies operate in a similar manner. However, as the capacity increases more and approaches to the optimal Q value, all of the policies begin to perform similar to their no truck capacity constraint case and policy parameters become more effective. Since both of the $(Q, \mathbf{S})$ and the $(Q, \mathbf{S}, \mathbf{s})$ policies outperform the $(\mathbf{s}, Q)$ policy for small π values in general, we observe a general decreasing behaviour as available capacity increases.

In Figures 4.7, 4.8 and 4.9 we present the C/Q^* versus $Gap(QS)$ values for all K values considered when the backordering cost, $\pi = 100$ and $l = 0.25, 0.5, 1.0$, respectively. Similarly, Figures 4.10, 4.11 and 4.12 present $Gap(QSs)$ values for the same problem instances. For small capacity values, $Gap(Q, S)$ values are close to zero, as explained above. For low and moderate setup costs, we observe that $Gap(QS)$ value increases as the available truck capacity increases. The behaviour of the $(Q, \mathbf{S}, \mathbf{s})$ policy exhibits differences characteristics as C/Q^* increases. For small capacity values, $Gap(QSs)$ is close to zero as explained above, and for large capacities ($C/Q^* = 1$), $Gap(QSs)$ values approach to the corresponding values obtained in the unconstrained case which were scattered around zero when $\pi = 100$ (see Table 4.2). For moderate capacity levels, we observe that the $(\mathbf{s}, Q)$ policy outperforms the $(Q, \mathbf{S}, \mathbf{s})$ policy. This is because, since C is less than optimal Q value, the $(Q, \mathbf{S}, \mathbf{s})$ policy chooses Q equal to C to reduce ordering cost. However, because π is moderate, in order not to increase backordering cost, policy chooses an s value such that any of the items inventory position may drop to s before a total of Q units are demanded. This results in frequent ordering in the $(Q, \mathbf{S}, \mathbf{s})$ policy and moreover, trucks are dispatched before being fully loaded. However, in the $(\mathbf{s}, Q)$ policy, trucks are always dispatched when they are fully loaded due to policy requirements. Therefore, ordering cost is higher in the $(Q, \mathbf{S}, \mathbf{s})$ policy compared to the $(\mathbf{s}, Q)$ policy. However, the sum of the holding and backordering costs in both of the policies are approximately equal. For example, for the problem instance when $C/Q^* = 0.8$, $\pi = 100$, $K = 200$ and $l = 0.25$, total ordering cost is 100.018, total holding cost is 95.461 and total backordering cost is 9.627 for the $(\mathbf{s}, Q)$ policy while, costs in the $(Q, \mathbf{S}, \mathbf{s})$ policy are 104.542, 98.631 and 5.921 respectively. The highest *Gap*(QSs) value is 2.42% when $\pi = 100$, $l = 0.25$ and $K = 200$. We still do not observe a monotonic behaviour in *Gap*(QSs) values as the setup cost increases. This is due to the complex dynamics between backordering, holding and setup costs.

In Figures 4.13, 4.14 and 4.15 we present the C/Q^* versus *Gap*(QS) values for all K values considered when the backordering cost, $\pi = 300$ and $l = 0.25, 0.5, 1.0$, respectively. Similarly, Figures 4.16, 4.17 and 4.18 present *Gap*(QSs) values for the same problem instances. From these figures we again conclude that the $(Q, \mathbf{S}, \mathbf{s})$ policy works similar to the $(Q, \mathbf{S})$ policy under low truck capacity. As explained before, for large values of backordering cost, we observe that individual control becomes important, and the proposed policy performs better for all values of the leadtime. However, as leadtime increases gap decreases. Out of 125 instances considered, in 116 cases the $(\mathbf{s}, Q)$ policy outperforms other policies when $\pi = 300$. Remember that the policy $(Q, \mathbf{S}, \mathbf{s})$ outperforms the $(\mathbf{s}, Q)$ policy for the uncapacitated case even when the backorder costs are large. The other observations on Figures 4.13 to 4.18 are similar to those made in Figures 4.7 to 4.12.

In Figures 4.1 – 4.18, we observe that in 28 of 30 cases considered, the $(\mathbf{s}, Q)$ policy outperforms other policies when $l = 0.25$ and $C/Q^* = 0.2$. This is due to effective individual control over the items in the proposed $(\mathbf{s}, Q)$ policy. In the proposed policy, because of the policy characteristics, system waits until any of the items inventory position drops to its reorder point which saves from the expected total holding cost. For example, for the problem instance $C/Q^* = 0.2$, $\pi = 300$, $l = 0.25$ and $K = 200$ holding cost in the $(\mathbf{s}, Q)$ policy is 63.665 while it is 69 in the $(Q, \mathbf{S})$ and the $(Q, \mathbf{S}, \mathbf{s})$ policies. For this problem instance, the $(\mathbf{s}, Q)$ policy gives $s = 4$ and $Q = 5$, while in the $(Q, \mathbf{S})$ and the $(Q, \mathbf{S}, \mathbf{s})$ policies $S = 8$ and $Q = 5$. Thus the system keeps at most 7 units if the $(\mathbf{s}, Q)$ policy is implemented while, the system keeps 8 units from both of the items if the other policies are implemented. Thus, in the $(Q, \mathbf{S})$ policy and in the $(Q, \mathbf{S}, \mathbf{s})$ policy, system keeps more inventory since, they do not wait inventory positions to drop to a certain level and items arrive quickly after they are ordered. Moreover, when

Figure 4.7: $Gap(QS)$ versus C/Q^* values when $l = 0.25$, $\pi = 100$

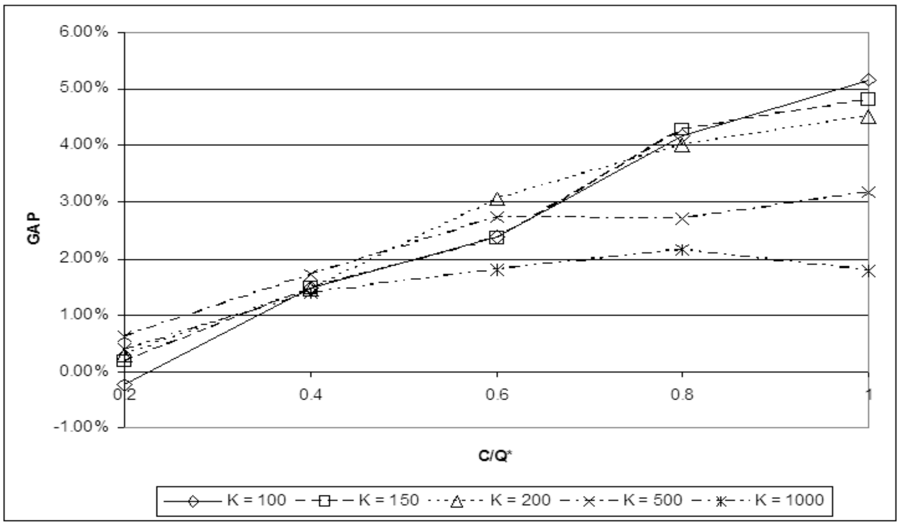


Figure 4.8: $Gap(QS)$ versus C/Q^* values when $l = 0.5$, $\pi = 100$

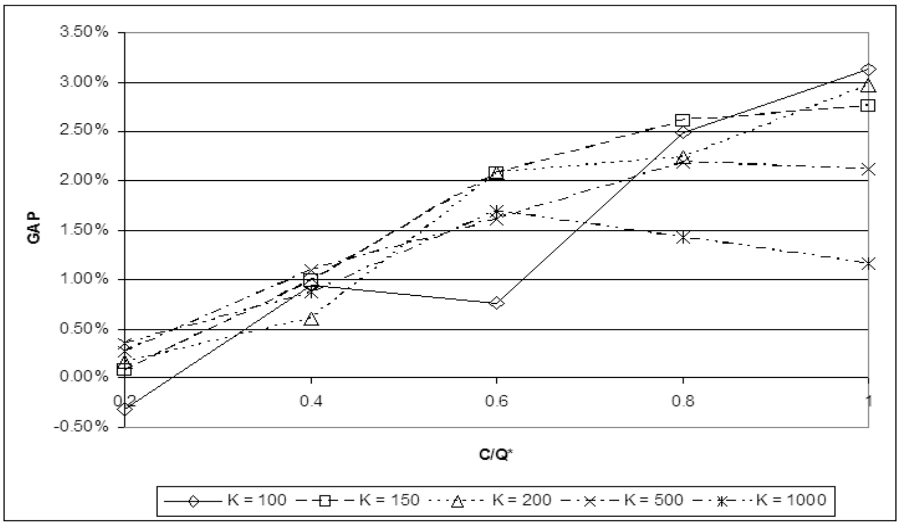


Figure 4.9: $Gap(QS)$ versus C/Q^* values when $l = 1$, $\pi = 100$

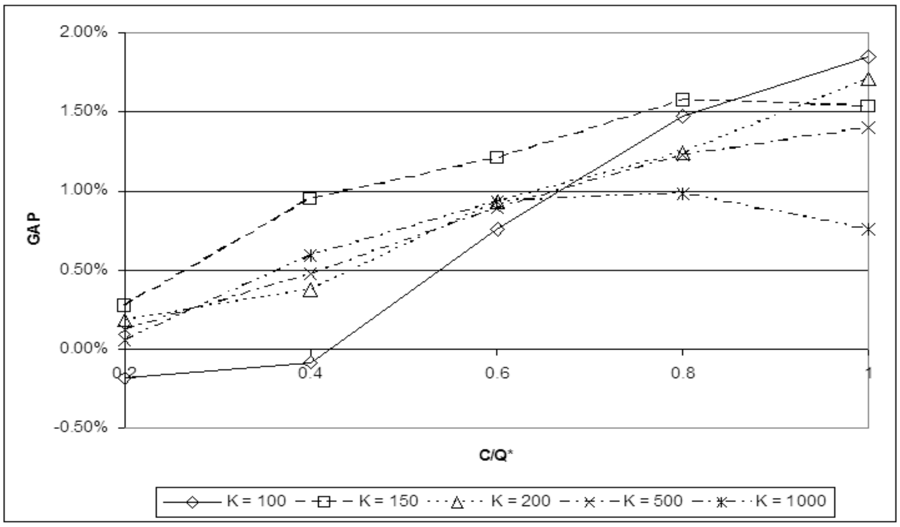


Figure 4.10: $Gap(QSs)$ versus C/Q^* values when $l = 0.25$, $\pi = 100$

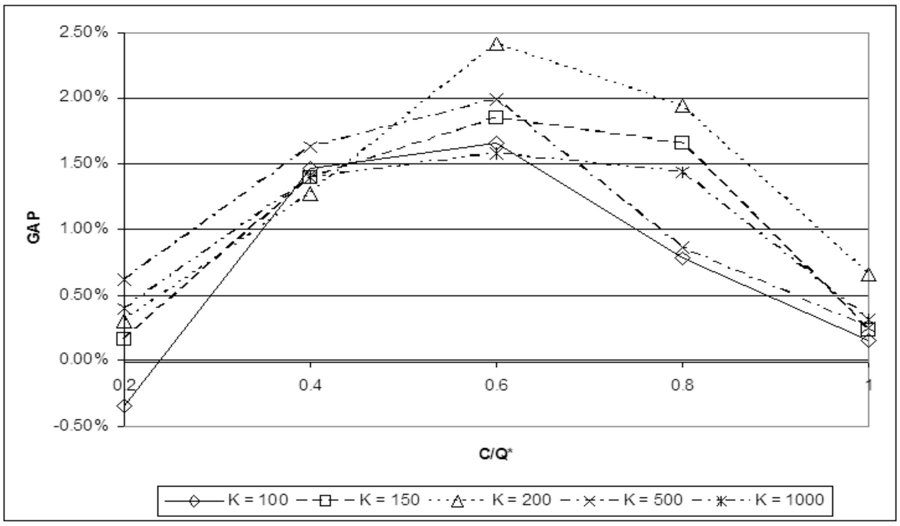


Figure 4.11: $Gap(QSs)$ versus C/Q^* values when $l = 0.5$, $\pi = 100$

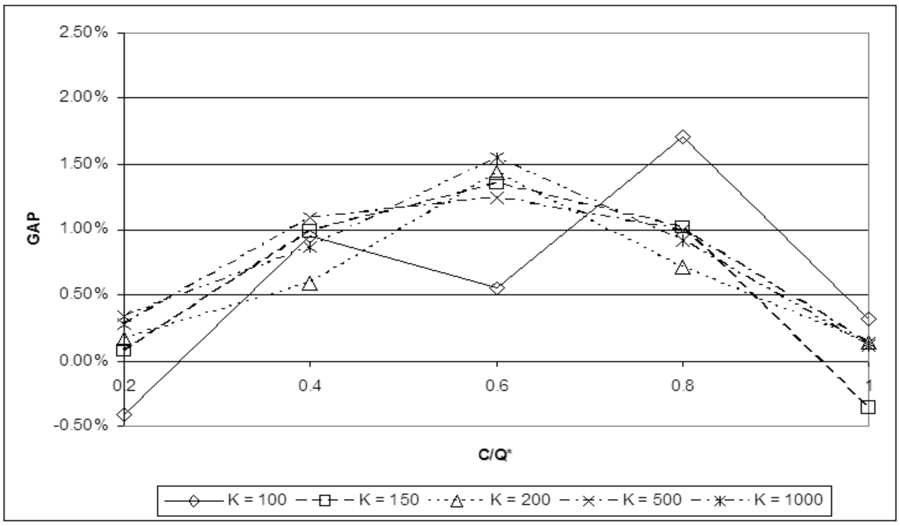


Figure 4.12: $Gap(QSs)$ versus C/Q^* values when $l = 1$, $\pi = 100$

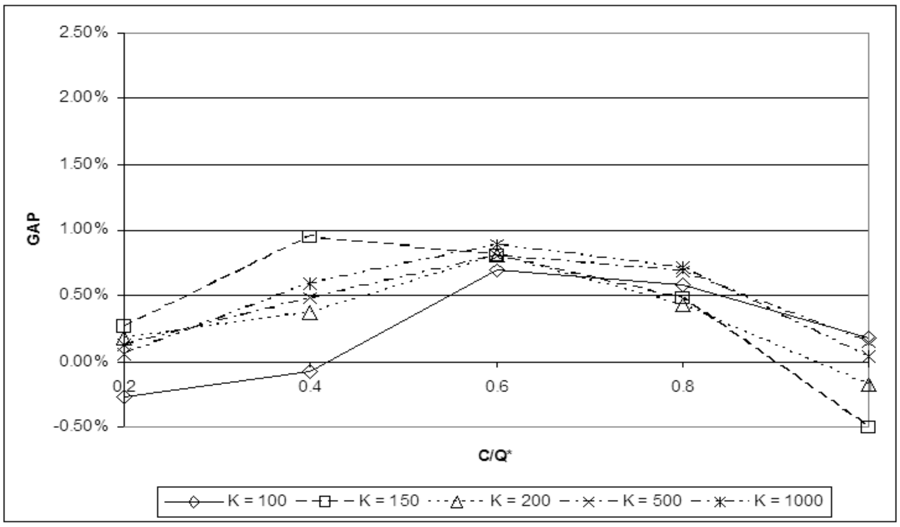


Figure 4.13: $Gap(QS)$ versus C/Q^* values when $l = 0.25$, $\pi = 300$

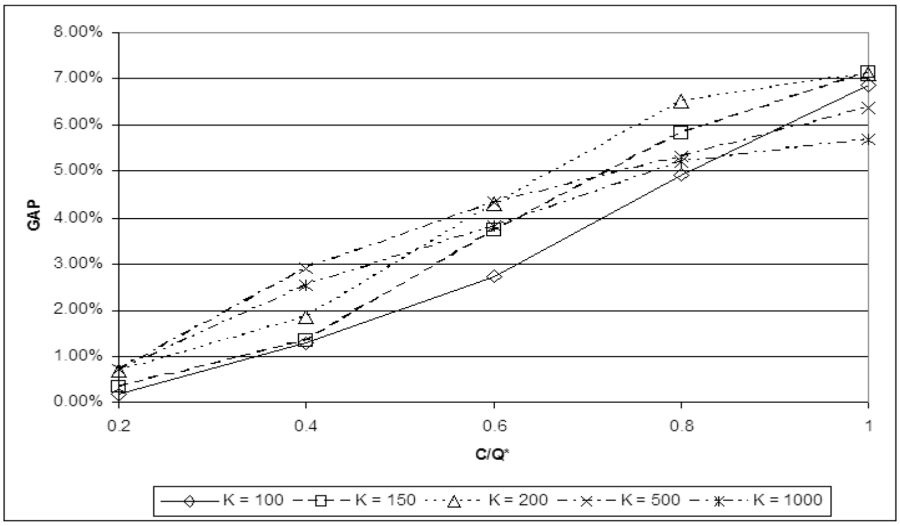


Figure 4.14: $Gap(QS)$ versus C/Q^* values when $l = 0.5$, $\pi = 300$

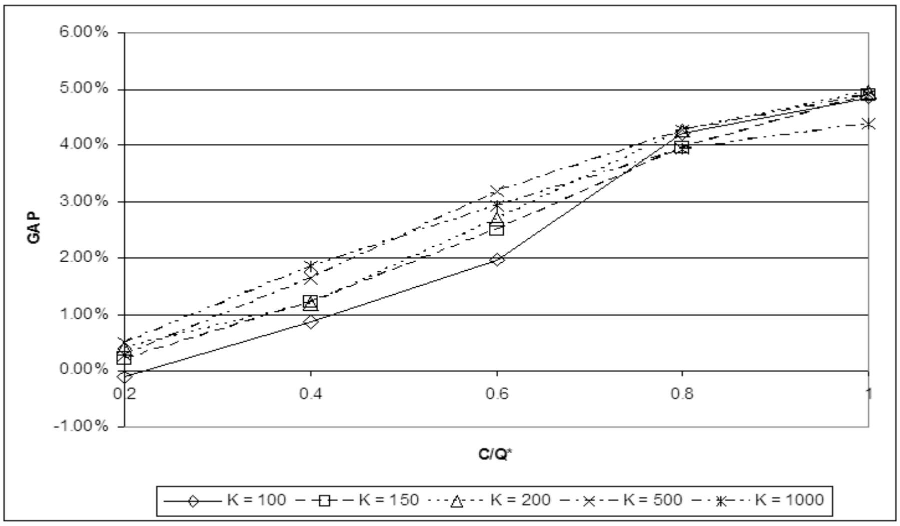


Figure 4.15: $Gap(QS)$ versus C/Q^* values when $l = 1$, $\pi = 300$

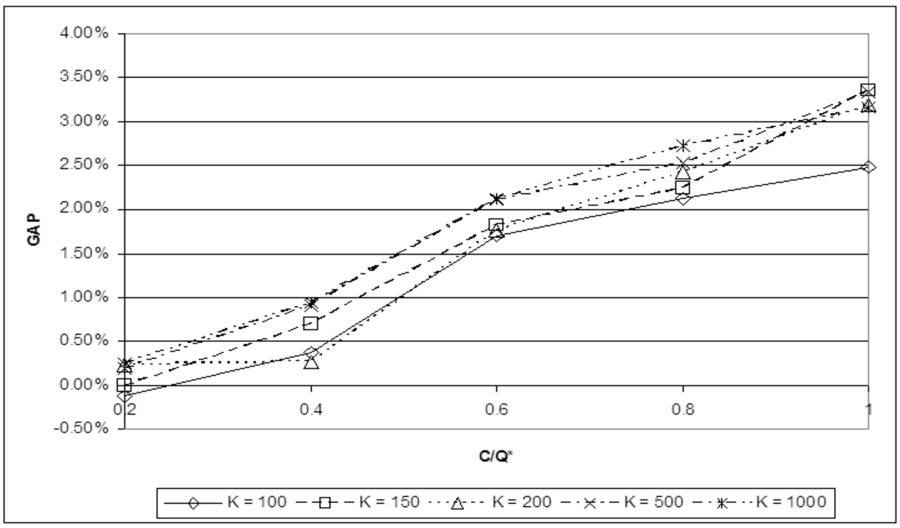


Figure 4.16: $Gap(QSs)$ versus C/Q^* values when $l = 0.25$, $\pi = 300$

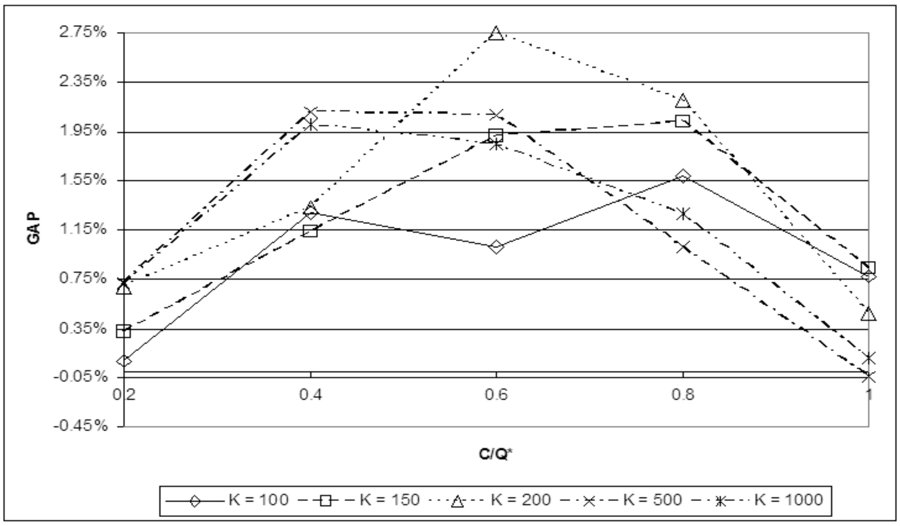


Figure 4.17: $Gap(QSs)$ versus C/Q^* values when $l = 0.5$, $\pi = 300$

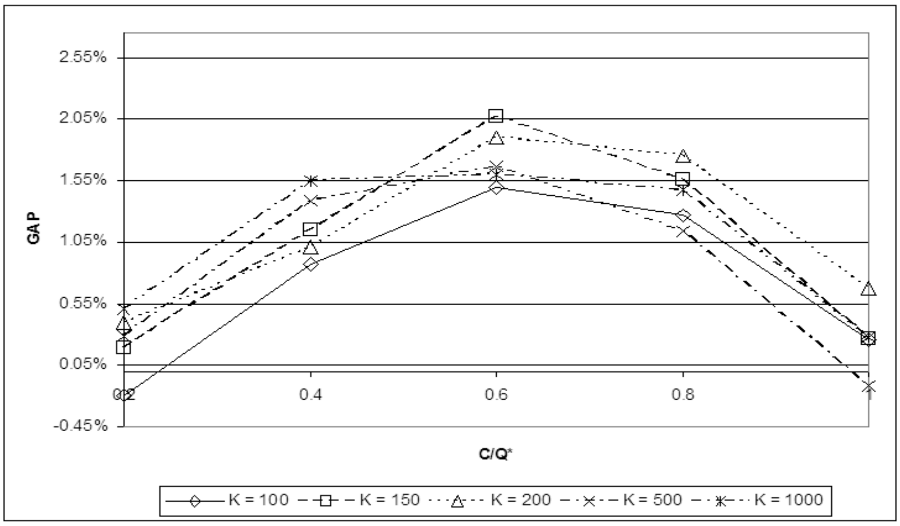
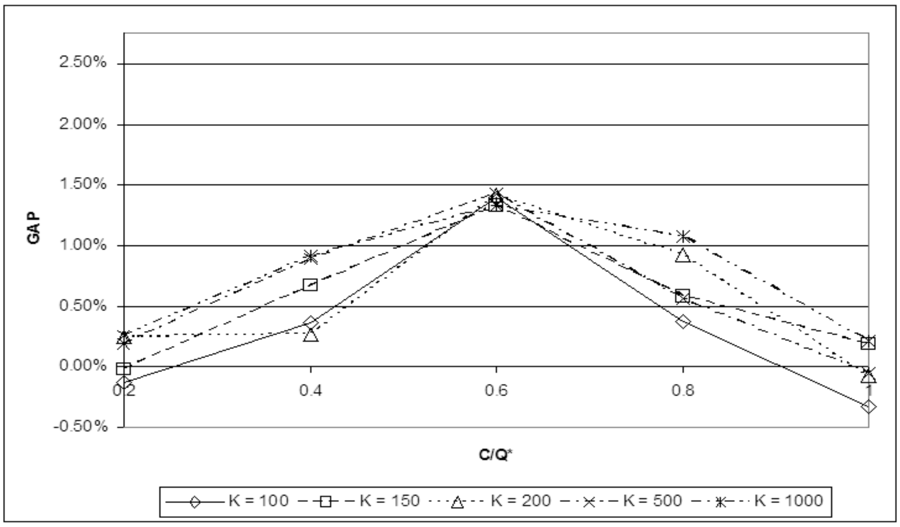


Figure 4.18: $Gap(QSs)$ versus C/Q^* values when $l = 1$, $\pi = 300$



the backordering cost is very small, individual control effect becomes unimportant for large values of leadtime since holding cost reduces more than the increase in the backordering cost. Therefore, the $(Q, \mathbf{S})$ and the $(Q, \mathbf{S}, \mathbf{s})$ policies perform better for larger values of leadtime when π is small.

The impact of the leadtime on the comparative performances of the $(Q, \mathbf{S}, \mathbf{s})$, $(Q, \mathbf{S})$ and $(\mathbf{s}, Q)$ policies remains the same, when the truck capacities are introduced. Mainly, the gap between $(\mathbf{s}, Q)$ and $(Q, \mathbf{S}, \mathbf{s})$ is not sensitive to changes in leadtime, and the gap between $(\mathbf{s}, Q)$ and $(Q, \mathbf{S})$ increases as the leadtime decreases.

4.3 Non–identical items case

In this section we assume two non-identical items with same arrival rates but different backordering costs. We compare the performances of the $(\mathbf{s}, Q)$ and the $(Q, \mathbf{S})$ policies. Items being non-identical have no impact on the implementation of the $(Q, \mathbf{S})$ policy while allocation becomes an issue in the implementation of the $(\mathbf{s}, Q)$ policy, as described in Section 3.3. Gürbüz et al. [17] pose their $(Q, \mathbf{S}, \mathbf{s})$ policy only for identical items. When the items are non-identical an allocation scheme must be adopted for this policy, too and that changes the total cost derivations of Section 3.2. Therefore, we do not include the $(Q, \mathbf{S}, \mathbf{s})$ policy in the analysis of this section. The effect of the setup cost, the effect of the leadtime and the effect of the backordering cost can be seen in Table 4.10.

As setup costs increase, both policies order less frequently and thus hold more inventory. This reduces the chances of stock–outs and diminishes the relative effectiveness of individual control for reducing backorders for the $(\mathbf{s}, Q)$ policy. In addition, when the setup costs are large, setup costs and inventory holding costs dominate the total cost and also shrinks the difference between two policies. Thus we see that the gap between two policies generally decrease as setup costs increase.

Gaps for large backordering cost values are all positive meaning that the $(\mathbf{s}, Q)$ policy outperforms the $(Q, \mathbf{S})$ policy in these cases. It may be observed from the detailed tables in the Appendix E that cost components are lower in the $(\mathbf{s}, Q)$ policy than the $(Q, \mathbf{S})$ policy due to individual control, which is even more

Table 4.10: Percentage gap between the $(\mathbf{s}, Q)$ and the $(Q, \mathbf{S})$ policies for non–identical items

K	**L**	π_1	π_2	$Gap(QS)$
100	0.5	20	80	0.98
100	0.5	80	120	3.32
100	0.5	100	200	3.54
100	0.5	100	300	3.90
150	0.5	20	80	0.76
150	0.5	80	120	3.15
150	0.5	100	200	3.48
150	0.5	100	300	3.95
200	0.5	20	80	-0.23
200	0.5	80	120	2.91
200	0.5	100	200	3.78
200	0.5	100	300	3.93
100	1	20	80	-0.14
100	1	80	120	1.88
100	1	100	200	2.20
100	1	100	300	2.36
150	1	20	80	-0.14
150	1	80	120	1.76
150	1	100	200	2.13
150	1	100	300	2.45
200	1	20	80	-0.32
200	1	80	120	1.78
200	1	100	200	2.15
200	1	100	300	2.49

effective when the leadtimes are small.

In the non-identical items case, we observe that the $(Q, \mathbf{S})$ policy performs better than the $(\mathbf{s}, Q)$ policy only in 4 cases out of 48 and only for small values of the backordering cost. This is similar to the identical item case. The problem set examined, we also observe that as backordering cost increases $Gap(QS)$ value also increases whereas, as leadtime increases $Gap(QS)$ value decreases. Also, there is no monotonic relationship between K and $Gap(QS)$ value, which is again because of the complex dynamics between backordering, holding and ordering costs.

Chapter 5

Conclusion

In this thesis, we consider a two echelon inventory system composed of a retailer and a warehouse. The particular problem we consider is referred to as the stochastic joint replenishment problem, and involves determining a replenishment policy so that the total expected cost, which is composed of holding, ordering and backordering costs, is minimized. Demands of the items are random and retailer may order the items jointly. Items are shipped from the warehouse to the retailer by capacitated vehicles. The main objective of this research is to investigate the effect of truck capacity constraint on the total cost of the system.

There are numerous policies proposed in literature for the stochastic joint replenishment problem. In this thesis, we propose a new joint replenishment policy that explicitly considers cost structures induced by transportation of the items with capacitated vehichles. We name this policy as the $(\mathbf{s}, Q)$ policy and compare it with two existing policies in the literature: the $(Q, \mathbf{S})$ and the $(Q, \mathbf{S}, \mathbf{s})$ policies.

An extensive numerical study has been conducted to assess the performances of these policies with capacitated and uncapacitated vehicles. The results with the uncapacitated vehicles show that the $(\mathbf{s}, Q)$ policy typically outperforms the $(Q, \mathbf{S})$ policy especially when the backorder penalties are high and the replenishment leadtimes are small. The gap usually diminishes when the setup costs are very large. We also see that the $(Q, \mathbf{S}, \mathbf{s})$ policy has a slightly better performance than the $(\mathbf{s}, Q)$ policy when vehicles are uncapacitated.

Results of the numerical study show that the proposed policy's performance increases as the individual control becomes important. This is because, in the proposed policy while the items are ordered jointly, there is an individual control on each item. This individual control becomes important when the backordering cost increases since, individual control results in less backordering which reduces the backordering costs. Also, as leadtime decreases items are replenished more quickly which results in keeping less inventory in the system and reduces the inventory holding costs. Leadtime reduction also prevents system from stockouts, which again results a decrease in total backordering costs.

The results with capacitated vehicles show that the policies have similar performances when the vehicle capacities are very low. When the vehicle capacities are at moderate levels, the proposed $(\mathbf{s}, Q)$ policy outperforms both policies. When the capacities are high, the results are similar to those with uncapacitated vehicles.

The numerical study shows that our policy should be an appropriate choice when the vehicles are capacitated, the backordering costs or the service levels are high, and economies of scale in transportation is important. As far as we know, there is only one other study in the literature in which vehicle capacities are explicitly considered, and this is by Cachon [7]. We propose a new policy under this setting and showed that this policy outperforms the basic policy suggested there.

The study here can be extended in multiple directions. First one can consider a case where the warehouse does not have ample supply and hence also manages inventories and operates a replenishment policy. Another extension could be the incorporation of minor setup costs in addition to the major setup costs here. Another direction could be devising a different allocation rules for the non–identical case and assessing their performances against the existing policies.

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Appendix A

MarkovChain: Equations for N identical items.

For N identical items where $\lambda_1 = \lambda_2 = ... = \lambda_N$ the equations become

$$\left.\begin{array}{ll} N\Pi_{(Q-i)(Q)...(Q)} = \Pi_{(Q-i+1)(Q)(Q)...(Q)} & \text{for } \; i = 2, ..., (Q-1) \\ N\Pi_{(Q)(Q-i)...(Q)} = \Pi_{(Q)(Q-i+1)(Q)...(Q)} & \text{for } \; i = 2, ..., (Q-1) \\ \vdots & \\ N\Pi_{(Q)(Q)...(Q-i)} = \Pi_{(Q)(Q)(Q)...(Q-i+1)} & \text{for } \; i = 2, ..., (Q-1) \end{array}\right\}$$

$$\left.\begin{array}{l} N\Pi_{(Q-1)(Q)...(Q)} = \Pi_{(Q)(Q)...(Q)} + \Pi_{(Q-1)(1)(Q)...(Q)} + \Pi_{(Q-1)(Q)(1)...(Q)} + ... + \Pi_{(Q-1)(Q)(Q)...(1)} \\ N\Pi_{(Q)(Q-1)...(Q)} = \Pi_{(Q)(Q)...(Q)} + \Pi_{(1)(Q-1)(Q)...(Q)} + \Pi_{(Q)(Q-1)(1)...(Q)} + ... + \Pi_{(Q)(Q-1)(Q)...(1)} \\ \vdots \\ N\Pi_{(Q)(Q)...(Q-1)} = \Pi_{(Q)(Q)...(Q)} + \Pi_{(1)(Q)(Q)...(Q-1)} + \Pi_{(Q)(1)(Q)...(Q-1)} + ... + \Pi_{(Q)(Q)...(1)(Q-1)} \end{array}\right\}$$

$$\left.\begin{array}{l}
\text{for } \; k \geq i,j \quad \& \quad |i-j| = 1 \quad \& \quad i+j > Q \\
N\Pi_{(i)(j)...(k)} = \Pi_{(i+1)(j)...(k)} + \Pi_{(i)(j+1)...(k)} + ... + \Pi_{(i)(j)...(k+1)} + \Pi_{(1)(i+j-Q)...(k)} \\
\\
\text{for } \; j \geq i,k \quad \& \quad |i-k| = 1 \quad \& \quad i+k > Q \\
N\Pi_{(i)(j)...(k)} = \Pi_{(i+1)(j)...(k)} + \Pi_{(i)(j+1)...(k)} + ... + \Pi_{(i)(j)...(k+1)} + \Pi_{(1)(j)...(i+k-Q)} \\
\\
\qquad\qquad\qquad\qquad \vdots \\
\text{for } \; i \geq j,k \quad \& \quad |j-k| = 1 \quad \& \quad j+k > Q \\
N\Pi_{(i)(j)...(k)} = \Pi_{(i+1)(j)...(k)} + \Pi_{(i)(j+1)...(k)} + ... + \Pi_{(i)(j)...(k+1)} + \Pi_{(i)(1)...(j+k-Q)}
\end{array}\right\}$$

$$\left.\begin{array}{l}
\text{for } \; k \geq i,j \quad \& \quad |i-j| = 1 \quad \& \quad i+j > Q \\
N\Pi_{(i)(j)...(k)} = \Pi_{(i+1)(j)...(k)} + \Pi_{(i)(j+1)...(k)} + ... + \Pi_{(i)(j)...(k+1)} + \Pi_{(i+j-Q)(1)...(k)} \\
\\
\text{for } \; j \geq i,k \quad \& \quad |i-k| = 1 \quad \& \quad i+k > Q \\
N\Pi_{(i)(j)...(k)} = \Pi_{(i+1)(j)...(k)} + \Pi_{(i)(j+1)...(k)} + ... + \Pi_{(i)(j)...(k+1)} + \Pi_{(i+k-Q)(j)...(1)} \\
\\
\qquad\qquad\qquad\qquad \vdots \\
\text{for } \; i \geq j,k \quad \& \quad |j-k| = 1 \quad \& \quad j+k > Q \\
N\Pi_{(i)(j)...(k)} = \Pi_{(i+1)(j)...(k)} + \Pi_{(i)(j+1)...(k)} + ... + \Pi_{(i)(j)...(k+1)} + \Pi_{(i)(j+k-Q)...(1)}
\end{array}\right\}$$

$$\left.\begin{array}{l}
\text{for}|i-j|=1,|i-k|=1,|j-k|=1 \\
N\Pi_{(i)(j)...(k)} = \Pi_{(i+1)(j)...(k)} + \Pi_{(i)(j+1)...(k)} + ... + \Pi_{(i)(j)...(k+1)} + \Pi_{(1)(\frac{i}{N}-\frac{Q}{N}+j)...(\frac{i}{N}-\frac{Q}{N}+k)} \\
\\
\text{for}|i-j|=1,|i-k|=1,|j-k|=1 \\
N\Pi_{(i)(j)...(k)} = \Pi_{(i+1)(j)...(k)} + \Pi_{(i)(j+1)...(k)} + ... + \Pi_{(i)(j)...(k+1)} + \Pi_{(\frac{j}{N}-\frac{Q}{N}+i)(1)...(\frac{j}{N}-\frac{Q}{N}+k)} \\
\\
\vdots \\
\text{for}|i-j|=1,|i-k|=1,|j-k|=1 \\
N\Pi_{(i)(j)...(k)} = \Pi_{(i+1)(j)...(k)} + \Pi_{(i)(j+1)...(k)} + ... + \Pi_{(i)(j)...(k+1)} + \Pi_{(\frac{k}{N}-\frac{Q}{N}+i)(\frac{k}{N}-\frac{Q}{N}+j)...(1)}
\end{array}\right\}$$

$$N\Pi_{(i)(j)...(k)} = \Pi_{(i+1)(j)...(k)} + \Pi_{(i)(j+1)...(k)} + ... + \Pi_{(i)(j)...(k+1)}$$

$$N\Pi_{(Q)(Q)...(Q)} = \Pi_{(1)(Q)...(Q)} + \Pi_{(Q)(1)...(Q)} + ... + \Pi_{(Q)(Q)...(1)}$$

Appendix B

Identical item case results:

B.1 Individual comparisons with graphs for each problem instance

Figure B.1: $Gap(QSs)$ vs π when l=1

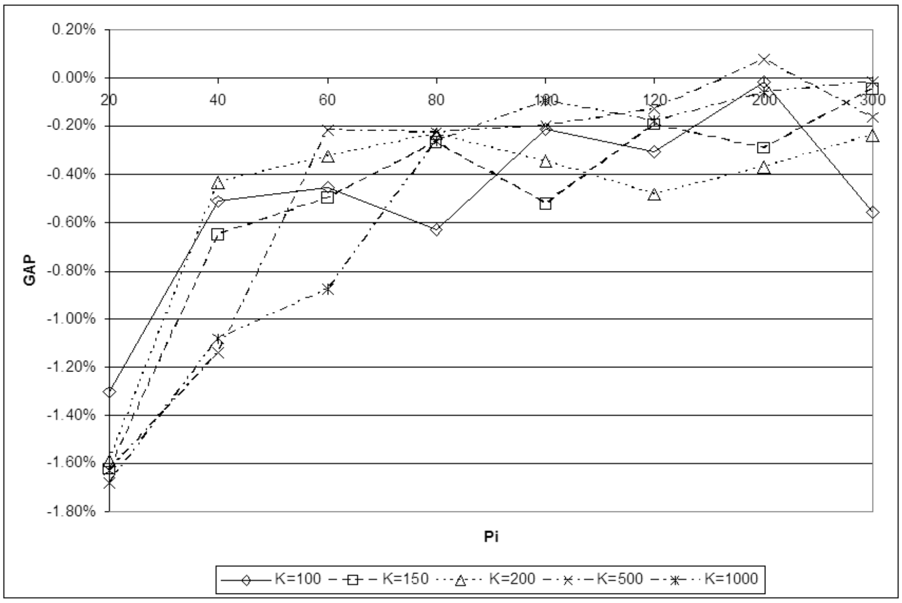


Figure B.2: $Gap(QSs)$ vs π when l=0.5

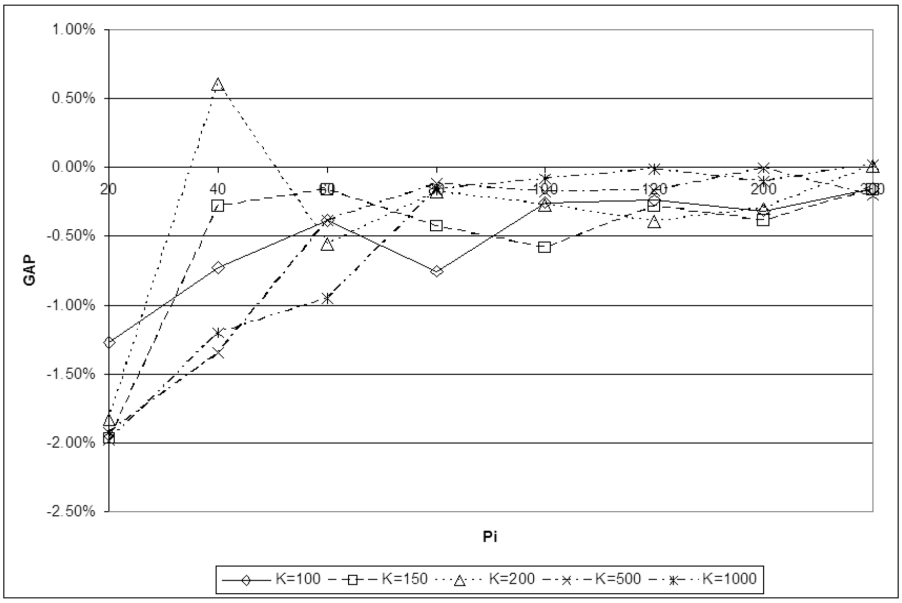


Figure B.3: $Gap(QSs)$ vs π when l=0.25

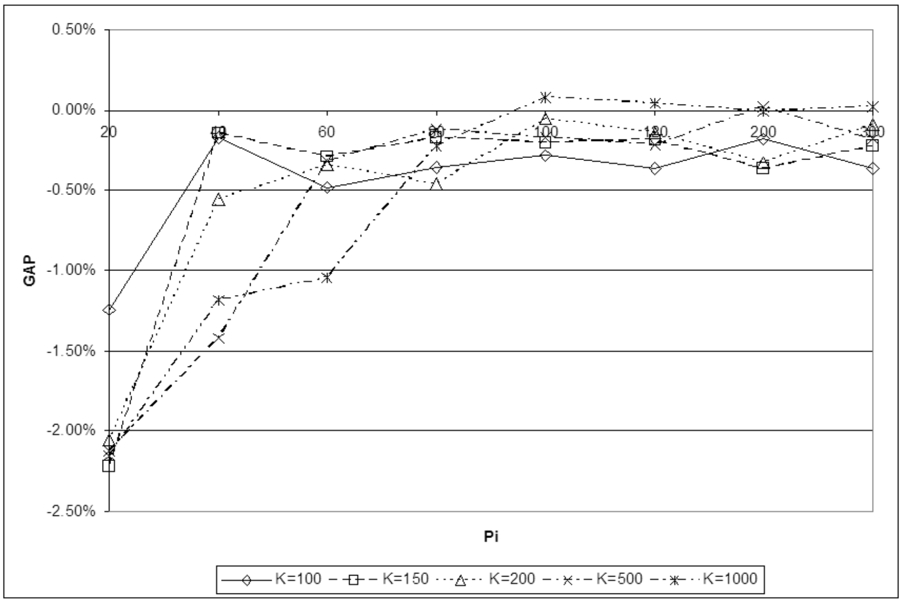


Figure B.4: $Gap(QS)$ vs π when l=1

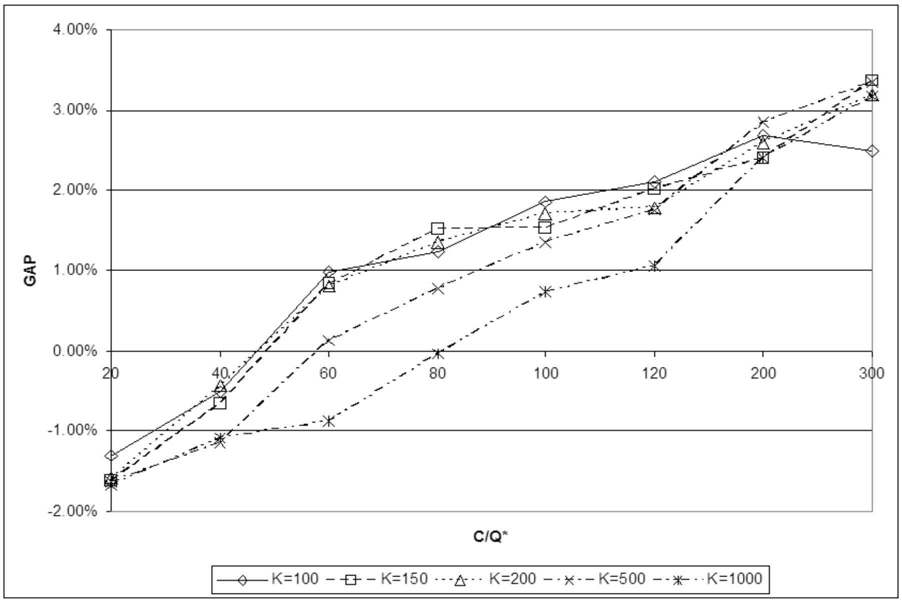


Figure B.5: $Gap(QS)$ vs π when l=0.5

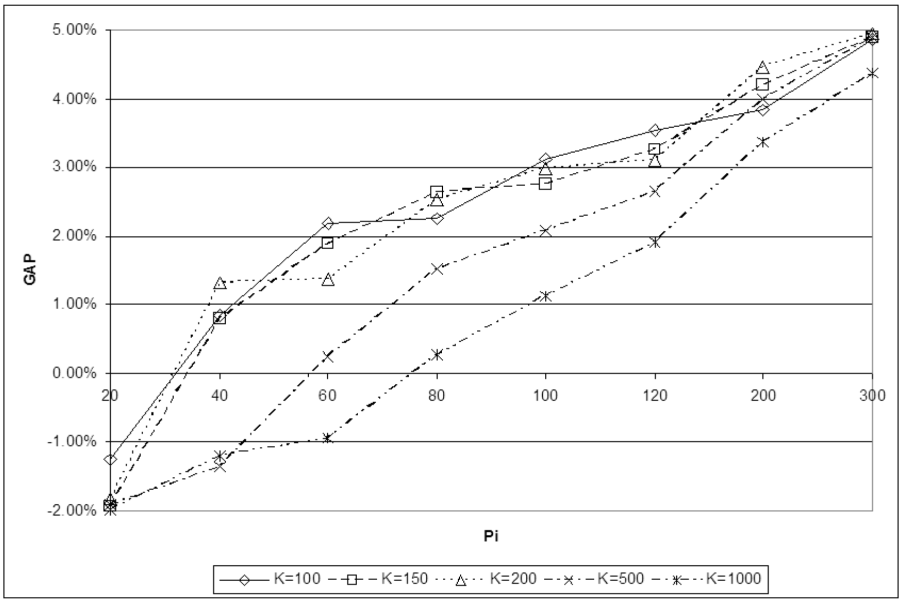
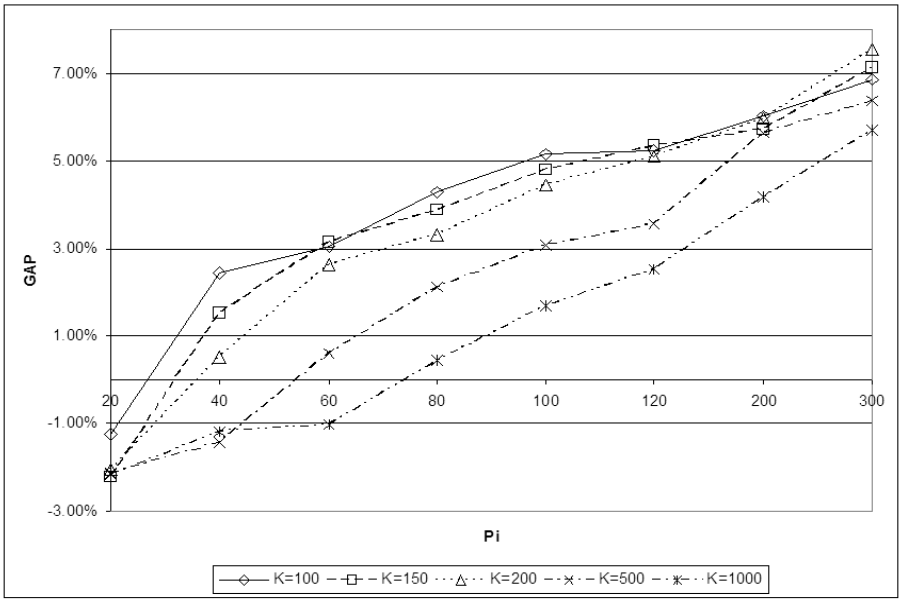


Figure B.6: $Gap(QS)$ vs π when l=0.25



B.2 Detailed results of each problem instance

(s,Q) Policy							(Q,S) Policy							(Q,S,s) Policy							
K	s	Q	OC	HC	BOC	TC	S	Q	OC	HC	BOC	TC	Gap	S	s	Q	OC	HC	BOC	TC	Gap
100	7	19	52.550	107.090	15.160	174.810	18	18	55.560	105.180	17.290	178.050	1.82 %	18	7	21	54.645	104.251	15.546	174.442	-0.21 %
150	6	26	57.690	119.620	21.100	198.420	20	23	65.217	114.292	21.952	201.460	1.51 %	20	6	25	64.237	113.165	19.991	197.393	-0.52 %
200	6	27	74.000	123.060	20.730	217.800	22	27	74.074	126.298	21.144	221.520	1.68 %	22	6	30	73.054	125.076	18.914	217.044	-0.35 %
500	5	41	121.860	158.976	23.729	304.565	29	43	116.279	162.574	29.842	308.697	1.34 %	29	5	45	118.624	164.807	20.534	303.965	-0.20 %
1000	5	59	169.310	206.931	27.147	403.388	37	61	163.934	204.936	37.470	406.341	0.73 %	37	4	62	169.641	210.486	22.888	403.015	-0.09 %
$\Pi = 100$, $\lambda = 5$, h = 6, n = 2, LT = 1.0, k = 0																					

(s,Q) Policy							(Q,S) Policy							(Q,S,s) Policy							
K	s	Q	OC	HC	BOC	TC	S	Q	OC	HC	BOC	TC	Gap	S	s	Q	OC	HC	BOC	TC	Gap
100	4	18	55.580	97.300	10.270	163.150	15	18	55.560	99.118	13.580	168.256	3.03 %	15	4	21	54.645	98.162	9.930	162.737	-0.25 %
150	3	23	65.150	103.140	19.510	187.810	17	23	65.217	108.210	19.564	192.992	2.69 %	17	3	25	64.237	107.060	15.431	186.728	-0.58 %
200	3	26	76.780	113.530	17.280	207.600	18	25	80.000	114.220	19.558	213.780	2.89 %	19	3	30	73.054	118.979	15.001	207.034	-0.27 %
500	3	40	124.885	161.253	10.934	297.072	26	42	119.048	159.404	24.812	303.266	2.04 %	26	2	45	118.624	158.722	19.210	296.556	-0.17 %
1000	2	58	172.350	209.663	15.978	397.991	34	60	166.667	201.762	34.058	402.488	1.12 %	34	2	62	173.789	208.518	15.354	397.661	-0.08 %
$\Pi = 100$, $\lambda = 5$, h = 6, n = 2, LT = 0.5, k = 0																					

(s,Q) Policy							(Q,S) Policy							(Q,S,s) Policy							
K	s	Q	OC	HC	BOC	TC	S	Q	OC	HC	BOC	TC	Gap	S	s	Q	OC	HC	BOC	TC	Gap
100	2	19	52.513	91.915	10.052	154.480	13	18	55.555	90.126	16.764	162.448	4.90 %	13	2	21	54.645	89.143	10.251	154.039	-0.29 %
150	2	22	68.136	102.503	8.643	179.282	15	22	68.181	102.154	17.572	187.911	4.59 %	15	2	25	68.275	102.124	8.521	178.919	-0.20 %
200	2	25	79.884	112.984	7.485	200.353	17	26	76.923	114.182	18.188	209.294	4.27 %	17	2	29	77.924	115.032	7.283	200.239	-0.06 %
500	1	40	124.935	152.381	14.044	291.360	24	41	121.951	153.372	24.970	300.296	2.98 %	24	1	44	123.463	154.574	12.826	290.862	-0.17 %
1000	1	57	175.230	208.766	9.885	393.881	32	58	172.414	198.592	29.498	400.504	1.65 %	32	1	61	179.395	204.826	9.960	394.180	0.08 %
$\Pi = 100$, $\lambda = 5$, h = 6, n = 2, LT = 0.25, k = 0																					

(s,Q) Policy							(Q,S) Policy							(Q,S,s) Policy							
K	s	Q	OC	HC	BOC	TC	S	Q	OC	HC	BOC	TC	Gap	S	s	Q	OC	HC	BOC	TC	Gap
100	7	20	50.000	110.590	17.140	177.730	19	20	50.000	111.202	20.270	181.470	2.06 %	19	7	22	50.093	111.236	15.858	177.187	-0.31 %
150	7	23	65.140	121.130	14.940	201.220	20	22	68.182	117.204	19.906	205.295	1.98 %	21	7	27	62.983	123.672	14.174	200.829	-0.19 %
200	7	27	73.970	134.910	12.710	221.590	22	26	76.923	129.214	19.404	225.542	1.75 %	22	6	29	73.833	125.928	20.764	220.525	-0.48 %
500	6	41	121.700	170.777	15.836	308.314	29	42	119.048	165.442	29.240	313.733	1.73 %	29	5	44	119.614	165.827	22.474	307.915	-0.13 %
1000	5	57	175.440	212.580	20.223	408.244	37	59	169.492	210.610	32.474	412.578	1.05 %	37	4	61	170.774	211.628	25.123	407.525	-0.18 %
$\Pi = 120$, $\lambda = 5$, h = 6, n = 2, LT = 1.0, k = 0																					

(s,Q) Policy							(Q,S) Policy							(Q,S,s) Policy							
K	s	Q	OC	HC	BOC	TC	S	Q	OC	HC	BOC	TC	Gap	S	s	Q	OC	HC	BOC	TC	Gap
100	4	19	52.500	101.010	11.610	165.120	15	18	55.556	99.118	16.296	170.973	3.42 %	15	4	21	54.645	98.162	11.916	164.723	-0.24 %
150	4	23	65.280	114.870	9.670	189.830	17	22	68.182	111.136	16.694	196.013	3.15 %	17	4	25	68.275	111.139	9.882	189.296	-0.28 %
200	4	25	80.010	121.950	8.640	210.610	19	26	76.923	123.152	17.090	217.167	3.02 %	19	3	29	73.833	119.833	16.123	209.790	-0.39 %
500	3	42	119.020	168.112	12.619	299.751	26	41	121.951	162.296	23.470	307.720	2.59 %	26	3	44	123.463	163.581	12.192	299.236	-0.17 %
1000	2	60	166.270	216.009	18.424	400.703	34	58	172.414	207.470	28.434	408.320	1.87 %	34	2	61	174.401	209.112	17.119	400.632	-0.02 %
$\Pi = 120$, $\lambda = 5$, h = 6, n = 2, LT = 0.5, k = 0																					

(s,Q) Policy							(Q,S) Policy							(Q,S,s) Policy							
K	s	Q	OC	HC	BOC	TC	S	Q	OC	HC	BOC	TC	Gap	S	s	Q	OC	HC	BOC	TC	Gap
100	2	19	52.540	91.910	12.048	156.498	13	17	58.823	93.070	12.804	164.700	4.98 %	13	2	20	55.175	89.668	11.085	155.928	-0.37 %
150	2	22	68.257	102.305	10.398	180.960	15	21	71.428	105.092	14.096	190.618	5.07 %	15	2	25	68.275	102.124	10.225	180.624	-0.19 %
200	2	26	76.860	116.378	8.742	201.980	17	25	80.000	117.112	15.162	212.276	4.85 %	17	2	29	77.924	115.032	8.739	201.695	-0.14 %
500	1	40	124.920	152.024	17.054	293.998	24	40	125.000	156.266	23.210	304.477	3.44 %	24	1	43	124.196	155.284	13.872	293.352	-0.22 %
1000	1	58	172.420	212.053	11.461	395.934	33	59	169.492	207.466	29.008	405.966	2.47 %	33	1	62	173.789	211.427	10.889	396.105	0.04 %
$\Pi = 120$, $\lambda = 5$, h = 6, n = 2, LT = 0.25, k = 0																					

(s,Q) Policy							(Q,S) Policy							(Q,S,s) Policy							
K	s	Q	OC	HC	BOC	TC	S	Q	OC	HC	BOC	TC	Gap	S	s	Q	OC	HC	BOC	TC	Gap
100	8	19	52.590	118.800	14.090	185.490	19	18	55.560	117.092	17.832	190.480	2.62 %	19	8	21	54.645	116.164	14.651	185.459	-0.02 %
150	8	23	65.310	132.890	11.770	209.980	21	22	68.182	129.098	17.748	215.030	2.35 %	21	7	25	64.237	125.040	20.096	209.373	-0.29 %
200	7	27	73.980	134.930	21.020	229.930	23	26	76.923	141.106	17.874	235.906	2.53 %	23	7	29	73.833	137.820	17.431	229.084	-0.37 %
500	7	41	121.890	182.367	13.204	317.461	30	41	121.951	180.190	24.362	326.504	2.77 %	30	6	44	119.614	177.705	20.388	317.707	0.08 %
1000	6	57	175.550	224.191	18.566	418.307	38	58	172.414	225.302	30.624	428.339	2.34 %	38	6	62	173.789	226.474	17.798	418.061	-0.06 %
$\Pi = 200$, $\lambda = 5$, h = 6, n = 2, LT = 1.0, k = 0																					

(s,Q) Policy							(Q,S) Policy							(Q,S,s) Policy							
K	s	Q	OC	HC	BOC	TC	S	Q	OC	HC	BOC	TC	Gap	S	s	Q	OC	HC	BOC	TC	Gap
100	5	18	55.570	109.320	7.220	172.120	16	18	55.556	111.046	12.132	178.735	3.70 %	16	4	21	50.950	106.163	14.467	171.580	-0.31 %
150	4	23	65.220	114.930	16.060	196.220	18	22	68.182	123.056	13.236	204.476	4.04 %	18	4	26	63.395	118.062	14.016	195.473	-0.38 %
200	4	26	76.900	125.520	14.090	216.520	19	24	83.333	129.062	13.774	226.170	4.27 %	19	4	28	78.379	124.488	13.012	215.879	-0.30 %
500	4	39	128.170	169.863	8.952	306.985	26	39	128.205	168.150	22.910	319.267	3.85 %	27	3	45	118.624	170.616	17.716	306.956	-0.01 %
1000	3	59	169.250	224.775	14.920	408.945	35	57	175.439	222.214	25.046	422.700	3.25 %	35	3	62	173.789	220.432	14.312	408.533	-0.10 %
$\Pi = 200$, $\lambda = 5$, h = 6, n = 2, LT = 0.5, k = 0																					

(s,Q) Policy							(Q,S) Policy							(Q,S,s) Policy							
K	s	Q	OC	HC	BOC	TC	S	Q	OC	HC	BOC	TC	Gap	S	s	Q	OC	HC	BOC	TC	Gap
100	3	18	55.557	100.290	5.746	161.593	13	16	62.500	96.037	12.795	171.333	5.68 %	14	3	21	54.645	101.110	5.553	161.308	-0.18 %
150	3	22	68.208	114.441	4.636	187.285	15	20	75.000	108.050	14.948	198.001	5.41 %	16	2	25	64.237	109.974	12.397	186.608	-0.36 %
200	2	26	76.926	116.249	14.656	207.831	17	24	83.333	120.066	16.842	220.243	5.64 %	17	2	28	78.379	115.477	13.290	207.146	-0.33 %
500	2	40	124.665	164.358	9.352	298.375	25	40	125.000	168.134	22.090	315.226	5.35 %	25	2	45	123.098	166.166	9.164	298.428	0.02 %
1000	2	57	175.140	221.065	6.678	402.883	33	56	178.571	216.198	24.908	419.679	4.00 %	33	2	61	179.395	216.786	6.658	402.838	-0.01 %
$\Pi = 200$, $\lambda = 5$, h = 6, n = 2, LT = 0.25, k = 0																					

(s,Q) Policy							(Q,S) Policy							(Q,S,s) Policy							
K	s	Q	OC	HC	BOC	TC	S	Q	OC	HC	BOC	TC	Gap	S	s	Q	OC	HC	BOC	TC	Gap
100	8	20	49.900	122.570	20.240	192.720	20	19	52.632	126.062	18.824	197.520	2.43 %	20	8	21	50.950	124.164	16.537	191.651	-0.56 %
150	8	23	65.190	132.790	17.220	215.220	22	23	65.217	138.068	19.168	222.456	3.25 %	22	8	26	63.395	136.068	15.662	215.125	-0.04 %
200	8	27	74.010	146.840	15.090	235.950	23	25	80.000	144.072	19.408	243.481	3.09 %	23	8	28	78.379	142.488	14.519	235.386	-0.24 %
500	7	42	119.065	186.014	19.800	324.879	30	39	128.205	186.098	21.444	335.749	3.24 %	31	7	45	118.624	188.604	17.118	324.346	-0.16 %
1000	7	57	175.420	236.080	14.430	425.930	39	57	175.439	240.136	23.840	439.417	3.07 %	39	7	62	173.789	238.420	13.647	425.856	-0.02 %
$\Pi = 300$, $\lambda = 5$, h = 6, n = 2, LT = 1.0, k = 0																					

(s,Q) Policy							(Q,S) Policy							(Q,S,s) Policy							
K	s	Q	OC	HC	BOC	TC	S	Q	OC	HC	BOC	TC	Gap	S	s	Q	OC	HC	BOC	TC	Gap
100	5	19	52.630	112.850	10.215	175.710	16	17	58.824	114.026	11.390	184.241	4.63 %	16	5	21	54.645	110.119	10.676	175.440	-0.15 %
150	5	22	68.110	123.510	8.930	200.560	18	21	71.428	126.032	12.906	210.369	4.66 %	18	5	25	68.275	123.108	8.863	200.246	-0.16 %
200	5	25	79.960	133.870	7.650	221.490	20	25	80.000	138.042	14.404	232.447	4.71 %	20	5	29	77.924	136.020	7.567	221.511	0.01 %
500	4	42	118.935	180.163	12.915	312.013	27	39	128.205	180.074	19.012	327.293	4.67 %	27	4	44	123.463	175.531	12.381	311.375	-0.20 %
1000	4	57	175.090	230.373	9.192	414.655	35	55	181.818	228.116	22.888	432.823	4.20 %	35	4	61	179.395	225.792	9.567	414.754	0.02 %
$\Pi = 300$, $\lambda = 5$, h = 6, n = 2, LT = 0.5, k = 0																					

(s,Q) Policy							(Q,S) Policy							(Q,S,s) Policy							
K	s	Q	OC	HC	BOC	TC	S	Q	OC	HC	BOC	TC	Gap	S	s	Q	OC	HC	BOC	TC	Gap
100	3	17	58.840	96.669	9.171	164.680	14	17	58.823	105.022	12.148	175.996	6.43 %	14	3	21	54.645	101.110	8.330	164.085	-0.36 %
150	3	21	71.385	110.896	7.452	189.733	16	21	71.428	117.032	14.818	203.280	6.66 %	16	3	25	68.275	114.096	6.931	189.302	-0.23 %
200	3	25	79.914	125.050	6.084	211.048	17	23	86.956	123.038	16.090	226.956	7.01 %	18	3	29	77.924	127.008	5.928	210.860	-0.09 %
500	2	41	121.780	167.453	14.037	303.270	25	38	131.579	174.058	16.986	322.624	6.00 %	25	2	44	123.463	166.523	12.744	302.729	-0.18 %
1000	2	58	172.260	224.454	9.387	406.101	34	56	178.571	228.108	22.560	429.241	5.39 %	33	2	60	179.727	217.092	9.348	406.167	0.02 %
$\Pi = 300$, $\lambda = 5$, h = 6, n = 2, LT = 0.25, k = 0																					

Appendix C

Summary Results: Summary of each problem instance

$n = 2,\ \lambda = 5,\ h = 6,\ L = 1,\ K = 100,\ k = 0$					
total cost	$(\mathbf{s}, Q)$ policy	$(Q, \mathbf{S})$ policy	*Gap*(QS)	$(Q, \mathbf{S}, \mathbf{s})$ policy	*Gap*(QSs)
Π=20	141.064	139.247	-1.30%	139.247	-1.30%
Π=40	157.703	158.137	-0.51%	156.899	-0.51%
Π=60	165.907	167.536	0.98%	165.157	-0.45%
Π=80	171.328	173.439	1.23%	170.253	-0.63%
Π = 100	174.810	178.046	1.85%	174.442	-0.21%
Π = 120	177.730	181.474	2.11%	177.187	-0.31%
Π = 200	185.490	190.480	2.69%	185.459	-0.02%
Π = 300	192.720	197.520	2.49%	191.651	-0.56%

$n = 2,\ \lambda = 5,\ h = 6,\ L = 0.5,\ K = 100,\ k = 0$					
total cost	$(\mathbf{s}, Q)$ policy	$(Q, \mathbf{S})$ policy	*Gap*(QS)	$(Q, \mathbf{S}, \mathbf{s})$ policy	*Gap*(QSs)
Π=20	137.477	135.750	-1.26%	135.749	-1.27%
Π=40	150.456	151.731	0.85%	149.373	-0.72%
Π=60	156.130	159.550	2.19%	155.537	-0.38%
Π=80	160.900	164.523	2.25%	159.700	-0.75%
Π = 100	163.150	168.256	3.13%	162.737	-0.25%
Π = 120	165.120	170.973	3.54%	164.723	-0.24%
Π = 200	172.120	178.735	3.84%	171.580	-0.31%
Π = 300	175.710	184.241	4.86%	175.440	-0.15%

$n = 2,\ \lambda = 5,\ h = 6,\ L = 0.25,\ K = 100,\ k = 0$					
total cost	$(\mathbf{s}, Q)$ policy	$(Q, \mathbf{S})$ policy	*Gap*(QS)	$(Q, \mathbf{S}, \mathbf{s})$ policy	*Gap*(QSs)
Π=20	135.475	133.816	-1.24%	133.815	-1.24%
Π=40	144.473	148.020	2.46%	144.220	-0.18%
Π=60	150.241	154.829	3.05%	149.518	-0.48%
Π=80	152.536	159.095	4.30%	151.989	-0.36%
Π = 100	154.480	162.448	5.16%	154.039	-0.29%
Π = 120	156.499	164.700	5.24%	155.928	-0.37%
Π = 200	161.593	171.333	6.03%	161.308	-0.18%
Π = 300	164.680	175.996	6.87%	164.085	-0.36%

$n = 2,\ \lambda = 5,\ h = 6,\ L = 1,\ K = 150,\ k = 0$					
total cost	$(\mathbf{s}, Q)$ policy	$(Q, \mathbf{S})$ policy	$Gap(QS)$	$(Q, \mathbf{S}, \mathbf{s})$ policy	$Gap(QSs)$
Π=20	160.945	158.379	-1.62%	158.378	-1.62%
Π=40	180.070	179.982	-0.65%	178.903	-0.65%
Π=60	188.660	190.244	0.84%	187.725	-0.50%
Π=80	193.730	196.679	1.52%	193.216	-0.27%
Π = 100	198.420	201.463	1.53%	197.393	-0.52%
Π = 120	201.220	205.295	2.03%	200.829	-0.19%
Π = 200	209.980	215.030	2.40%	209.373	-0.29%
Π = 300	215.220	222.456	3.36%	215.125	-0.04%

$n = 2,\ \lambda = 5,\ h = 6,\ L = 0.5,\ K = 150,\ k = 0$					
total cost	$(\mathbf{s}, Q)$ policy	$(Q, \mathbf{S})$ policy	$Gap(QS)$	$(Q, \mathbf{S}, \mathbf{s})$ policy	$Gap(QSs)$
Π=20	158.999	155.928	-1.93%	155.927	-1.97%
Π=40	173.270	174.655	0.80%	172.779	-0.28%
Π=60	179.930	183.318	1.88%	179.648	-0.16%
Π=80	184.207	189.079	2.64%	183.417	-0.43%
Π = 100	187.810	192.992	2.76%	186.728	-0.58%
Π = 120	189.830	196.013	3.26%	189.296	-0.28%
Π = 200	196.220	204.476	4.21%	195.473	-0.38%
Π = 300	200.560	210.369	4.89%	200.246	-0.16%

$n = 2,\ \lambda = 5,\ h = 6,\ L = 0.25,\ K = 150,\ k = 0$					
total cost	$(\mathbf{s}, Q)$ policy	$(Q, \mathbf{S})$ policy	$Gap(QS)$	$(Q, \mathbf{S}, \mathbf{s})$ policy	$Gap(QSs)$
Π=20	158.005	154.497	-2.22%	154.497	-2.22%
Π=40	168.993	171.586	1.53%	168.751	-0.14%
Π=60	174.059	179.554	3.16%	173.558	-0.29%
Π=80	177.524	184.396	3.87%	177.215	-0.17%
Π = 100	179.282	187.911	4.81%	178.919	-0.20%
Π = 120	180.960	190.618	5.34%	180.624	-0.19%
Π = 200	187.285	198.001	5.72%	186.608	-0.36%
Π = 300	189.733	203.280	7.14%	189.302	-0.23%

$n = 2,\ \lambda = 5,\ h = 6,\ L = 1,\ K = 200,\ k = 0$					
total cost	$(\mathbf{s}, Q)$ policy	$(Q, \mathbf{S})$ policy	$Gap(QS)$	$(Q, \mathbf{S}, \mathbf{s})$ policy	$Gap(QSs)$
Π=20	176.884	174.073	-1.59%	174.073	-1.59%
Π=40	198.599	198.439	-0.43%	197.735	-0.43%
Π=60	207.709	209.374	0.80%	207.029	-0.33%
Π=80	213.640	216.529	1.35%	213.142	-0.23%
Π = 100	217.800	221.520	1.71%	217.044	-0.35%
Π = 120	221.590	225.542	1.78%	220.525	-0.48%
Π = 200	229.930	235.906	2.60%	229.084	-0.37%
Π = 300	235.950	243.481	3.19%	235.386	-0.24%

$n = 2,\ \lambda = 5,\ h = 6,\ L = 0.5,\ K = 200,\ k = 0$					
total cost	$(\mathbf{s}, Q)$ policy	$(Q, \mathbf{S})$ policy	$Gap(QS)$	$(Q, \mathbf{S}, \mathbf{s})$ policy	$Gap(QSs)$
Π=20	175.527	172.310	-1.83%	172.310	-1.83%
Π=40	191.354	193.862	1.31%	192.512	0.60%
Π=60	200.560	203.323	1.38%	199.445	-0.56%
Π=80	204.179	209.356	2.54%	203.819	-0.18%
Π = 100	207.600	213.780	2.98%	207.034	-0.27%
Π = 120	210.610	217.167	3.11%	209.790	-0.39%
Π = 200	216.520	226.170	4.46%	215.879	-0.30%
Π = 300	221.490	232.447	4.95%	221.511	0.01%

$n = 2,\ \lambda = 5,\ h = 6,\ L = 0.25,\ K = 200,\ k = 0$					
total cost	$(\mathbf{s}, Q)$ policy	$(Q, \mathbf{S})$ policy	$Gap(QS)$	$(Q, \mathbf{S}, \mathbf{s})$ policy	$Gap(QSs)$
Π=20	174.971	171.378	-2.05%	171.377	-2.05%
Π=40	190.187	191.182	0.52%	189.131	-0.56%
Π=60	194.699	199.842	2.64%	194.038	-0.34%
Π=80	198.860	205.424	3.30%	197.942	-0.46%
Π = 100	200.353	209.294	4.46%	200.239	-0.06%
Π = 120	201.980	212.276	5.10%	201.695	-0.14%
Π = 200	207.831	220.243	5.97%	207.146	-0.33%
Π = 300	211.048	226.956	7.54%	210.860	-0.09%

$n = 2,\ \lambda = 5,\ h = 6,\ L = 1,\ K = 500,\ k = 0$					
total cost	$(\mathbf{s}, Q)$ policy	$(Q, \mathbf{S})$ policy	$Gap(QS)$	$(Q, \mathbf{S}, \mathbf{s})$ policy	$Gap(QSs)$
Π=20	234.019	230.204	-1.63%	230.204	-1.63%
Π=40	280.838	277.659	-1.14%	277.663	-1.14%
Π=60	292.579	292.971	0.13%	291.940	-0.22%
Π=80	299.849	302.173	0.78%	299.166	-0.23%
$\Pi = 100$	304.565	308.697	1.36%	303.965	-0.20%
$\Pi = 120$	308.314	313.733	1.76%	307.915	-0.13%
$\Pi = 200$	317.461	326.504	2.85%	317.707	0.08%
$\Pi = 300$	324.879	335.749	3.35%	324.346	-0.16%

$n = 2,\ \lambda = 5,\ h = 6,\ L = 0.5,\ K = 500,\ k = 0$					
total cost	$(\mathbf{s}, Q)$ policy	$(Q, \mathbf{S})$ policy	$Gap(QS)$	$(Q, \mathbf{S}, \mathbf{s})$ policy	$Gap(QSs)$
Π=20	236.709	232.164	-1.92%	232.164	-1.92%
Π=40	278.999	275.283	-1.35%	275.282	-1.35%
Π=60	288.453	289.158	0.24%	287.342	-0.39%
Π=80	293.009	297.505	1.53%	292.659	-0.12%
$\Pi = 100$	297.072	303.266	2.09%	296.556	-0.17%
$\Pi = 120$	299.751	307.720	2.66%	299.236	-0.17%
$\Pi = 200$	306.985	319.267	4.00%	306.956	-0.01%
$\Pi = 300$	312.013	327.293	4.90%	311.375	-0.20%

$n = 2,\ \lambda = 5,\ h = 6,\ L = 0.25,\ K = 500,\ k = 0$					
total cost	$(\mathbf{s}, Q)$ policy	$(Q, \mathbf{S})$ policy	$Gap(QS)$	$(Q, \mathbf{S}, \mathbf{s})$ policy	$Gap(QSs)$
Π=20	238.275	233.223	-2.12%	233.223	-2.12%
Π=40	277.837	273.950	-1.42%	273.949	-1.42%
Π=60	285.349	287.044	0.59%	284.429	-0.32%
Π=80	288.654	294.804	2.13%	288.297	-0.12%
$\Pi = 100$	291.360	300.296	3.07%	290.862	-0.17%
$\Pi = 120$	293.998	304.477	3.56%	293.352	-0.22%
$\Pi = 200$	298.375	315.226	5.65%	298.428	0.02%
$\Pi = 300$	303.270	322.624	6.38%	302.729	-0.18%

$n = 2,\ \lambda = 5,\ h = 6,\ L = 1,\ K = 1000,\ k = 0$					
total cost	$(\mathbf{s}, Q)$ policy	$(Q, \mathbf{S})$ policy	$Gap(QS)$	$(Q, \mathbf{S}, \mathbf{s})$ policy	$Gap(QSs)$
Π=20	309.909	304.702	-1.68%	304.702	-1.68%
Π=40	367.704	363.759	-1.08%	363.762	-1.08%
Π=60	389.263	385.870	-0.87%	385.877	-0.88%
Π=80	398.168	398.015	-0.04%	397.133	-0.26%
Π = 100	403.388	406.341	0.73%	403.015	-0.09%
Π = 120	408.244	412.578	1.06%	407.525	-0.18%
Π = 200	418.307	428.339	2.40%	418.061	-0.06%
Π = 300	425.930	439.417	3.17%	425.856	-0.02%

$n = 2,\ \lambda = 5,\ h = 6,\ L = 0.5,\ K = 1000,\ k = 0$					
total cost	$(\mathbf{s}, Q)$ policy	$(Q, \mathbf{S})$ policy	$Gap(QS)$	$(Q, \mathbf{S}, \mathbf{s})$ policy	$Gap(QSs)$
Π=20	316.638	310.501	-1.98%	310.501	-1.98%
Π=40	367.150	362.799	-1.20%	362.798	-1.20%
Π=60	387.161	383.516	-0.94%	383.515	-0.95%
Π=80	393.773	394.844	0.27%	393.153	-0.16%
Π = 100	397.991	402.488	1.13%	397.661	-0.08%
Π = 120	400.703	408.320	1.90%	400.632	-0.02%
Π = 200	408.945	422.700	3.36%	408.533	-0.10%
Π = 300	414.655	432.823	4.38%	414.754	0.02%

$n = 2,\ \lambda = 5,\ h = 6,\ L = 0.25,\ K = 1000,\ k = 0$					
total cost	$(\mathbf{s}, Q)$ policy	$(Q, \mathbf{S})$ policy	$Gap(QS)$	$(Q, \mathbf{S}, \mathbf{s})$ policy	$Gap(QSs)$
Π=20	320.769	313.872	-2.15%	313.872	-2.15%
Π=40	366.647	362.343	-1.19%	362.342	-1.19%
Π=60	386.345	382.348	-1.03%	382.347	-1.05%
Π=80	391.421	393.095	0.43%	390.533	-0.23%
Π = 100	393.882	400.504	1.68%	394.180	0.08%
Π = 120	395.934	405.966	2.53%	396.105	0.04%
Π = 200	402.883	419.679	4.17%	402.838	-0.01%
Π = 300	406.101	429.241	5.70%	406.167	0.02%

Appendix D

N=4 case: Detailed results.

	(s,Q) Policy							(Q,S) Policy							(Q,S,s) Policy							
K	s	Q	OC	HC	BOC	TC	S	Q	OC	HC	BOC	TC	Gap	S	s	Q	OC	HC	BOC	TC	Gap
100	3	21	47,59	143.470	29.930	220.991	10	17	58.824	132.457	32.262	223.543	1.14 %	10	3	20	57.133	130.853	30.931	218.917	-0.95 %
150	3	24	62,4525	156.325	25.663	244.441	11	21	71.429	144.499	31.943	247.871	1.38 %	11	3	26	70.887	143.852	28.014	242.754	-0.70 %
200	3	24	83,268	156.058	25.932	265.258	12	26	76.923	153.676	37.677	268.276	1.13 %	12	3	31	81.583	157.747	24.320	263.651	-0.61 %
500	2	41	121,92	201.653	31.795	355.369	16	43	116.279	199.023	42.085	357.387	0.56 %	16	2	48	121.120	203.584	29.302	354.005	-0.39 %
1000	1	59	169,45	247.848	40.829	458.127	20	62	161.290	238.796	55.577	455.663	-0.54 %	20	-11	62	161.291	238.797	55.574	455.662	-0.54 %

$\Pi = 120$, $\lambda = 2.5$, $h = 6$, $n = 4$, $LT = 1.0$, $k = 0$

	(s,Q) Policy							(Q,S) Policy							(Q,S,s) Policy							
K	s	Q	OC	HC	BOC	TC	S	Q	OC	HC	BOC	TC	Gap	S	s	Q	OC	HC	BOC	TC	Gap
100	2	16	62,406	126.827	15.599	204.832	9	19	52.632	132.312	25.793	210.736	2.80 %	9	2	25	55.493	135.029	13.390	203.912	-0.45 %
150	1	23	65,2245	133.614	33.439	232.278	9	20	75.000	129.419	32.226	236.645	1.85 %	10	1	27	62.648	141.389	25.704	229.741	-1.10 %
200	1	27	74,096	150.819	28.253	253.168	10	24	83.333	141.486	33.235	258.055	1.89 %	10	1	28	82.671	140.657	27.217	250.545	-1.05 %
500	1	38	131,595	195.355	19.477	346.427	14	41	121.951	186.893	41.169	350.013	1.02 %	14	1	47	131.551	195.139	18.726	345.416	-0.29 %
1000	0	56	178,42	241.976	31.084	451.480	18	59	169.492	229.498	51.400	450.390	-0.24 %	18	-11	59	169.493	229.498	51.400	450.390	-0.24 %

$\Pi = 120$, $\lambda = 2.5$, $h = 6$, $n = 4$, $LT = 0.5$, $k = 0$

	(s,Q) Policy							(Q,S) Policy							(Q,S,s) Policy							
K	s	Q	OC	HC	BOC	TC	S	Q	OC	HC	BOC	TC	Gap	S	s	Q	OC	HC	BOC	TC	Gap
100	1	16	62,488	117.853	13.284	193.625	7	15	66.666	111.227	25.149	203.043	4.64 %	7	1	21	67.379	111.534	13.616	192.529	-0.57 %
150	1	20	75,1065	135.554	10.517	221.178	8	19	78.947	123.304	27.860	230.111	3.88 %	9	1	29	70.564	140.268	9.443	220.275	-0.41 %
200	1	23	86,922	148.838	9.006	244.766	9	23	86.957	135.381	29.954	252.291	2.98 %	9	0	26	84.733	133.339	25.568	243.640	-0.46 %
500	-1	40	125	194.192	20.419	339.611	13	40	125.000	180.816	40.345	346.161	1.89 %	13	0	46	131.653	186.188	20.658	338.499	-0.33 %
1000	0	54	185,06	248.772	14.831	448.663	17	58	172.414	223.451	51.831	447.696	-0.22 %	17	-12	58	172.415	223.450	51.838	447.702	-0.21 %

$\Pi = 120$, $\lambda = 2.5$, $h = 6$, $n = 4$, $LT = 0.25$, $k = 0$

Appendix E

Non–identical case: Detailed Results

(s,Q) Policy											(Q,S) Policy									
K	s1	s2	Q	OC	HC1	HC2	BOC1	BOC2	TC		S1	S2	Q	OC	HC1	HC2	BOC1	BOC2	TC	Gap
100	3	6	20	49.980	36.359	46.105	10.987	13.563	157.004		15	19	21	47.619	31.488	54.144	14.546	8.990	156.793	-0.13 %
150	0	6	27	55.542	43.236	53.959	13.680	12.056	178.475		16	21	26	57.692	31.056	58.704	19.682	11.099	178.234	-0.14 %
200	0	6	28	71.411	45.059	55.587	13.127	11.633	196.819		18	23	31	64.516	35.777	63.278	19.279	13.342	196.193	-0.32 %

Π1 = 20, Π2 = 80, λ = 5, h = 6, n = 2, LT = 1.0, k = 0

(s,Q) Policy											(Q,S) Policy									
K	s1	s2	Q	OC	HC1	HC2	BOC1	BOC2	TC		S1	S2	Q	OC	HC1	HC2	BOC1	BOC2	TC	Gap
100	0	4	20	50.110	36.837	47.232	10.056	4.912	149.154		12	15	19	52.631	30.847	48.095	11.183	7.850	150.609	0.97 %
150	0	3	24	62.480	40.702	49.567	9.845	9.167	171.760		14	18	26	57.692	33.204	55.662	15.968	10.553	173.082	0.76 %
200	0	3	27	74.034	47.873	53.605	8.035	8.633	192.181		15	19	28	71.428	36.150	58.667	15.041	10.453	191.741	-0.23 %

Π1 = 20, Π2 = 80, λ = 5, h = 6, n = 2, LT = 0.5, k = 0

(s,Q) Policy											(Q,S) Policy									
K	s1	s2	Q	OC	HC1	HC2	BOC1	BOC2	TC		S1	S2	Q	OC	HC1	HC2	BOC1	BOC2	TC	Gap
100	7	7	19	52.626	52.223	54.619	6.463	8.349	174.281		18	19	19	52.632	51.143	57.069	9.259	7.448	177.553	1.84 %
150	6	7	24	62.448	57.796	60.934	8.664	7.695	197.539		20	21	24	62.500	55.703	61.605	11.386	9.813	201.007	1.73 %
200	6	6	28	71.420	61.922	64.767	8.392	10.826	217.178		21	22	26	76.923	58.704	64.607	11.099	9.702	221.036	1.75 %
$\Pi1 = 80$, $\Pi2 = 120$, $\lambda = 5$, h = 6, n = 2, LT = 1.0, k = 0																				

(s,Q) Policy											(Q,S) Policy									
K	s1	s2	Q	OC	HC1	HC2	BOC1	BOC2	TC		S1	S2	Q	OC	HC1	HC2	BOC1	BOC2	TC	Gap
100	3	4	20	49.979	47.588	50.896	8.320	5.861	162.645		14	15	18	55.556	43.641	49.559	11.141	8.148	168.047	3.21 %
150	3	4	23	65.205	52.618	56.268	7.442	5.126	186.659		16	17	22	68.182	49.651	55.568	10.798	8.347	192.547	3.06 %
200	3	4	26	76.958	58.093	61.235	6.461	4.491	207.239		18	19	26	76.923	55.662	61.576	10.553	8.545	213.261	2.82 %
$\Pi1 = 80$, $\Pi2 = 120$, $\lambda = 5$, h = 6, n = 2, LT = 0.5, k = 0																				

(s,Q) Policy										(Q,S) Policy									
K	s1	s2	Q	OC	HC1	HC2	BOC1	BOC2	TC	S1	S2	Q	OC	HC1	HC2	BOC1	BOC2	TC	Gap
100	7	8	19	52.670	54.601	58.130	7.010	7.790	180.201	18	19	18	55.556	52.599	58.546	8.645	8.916	184.263	2.20 %
150	7	8	23	65.304	61.527	65.364	5.801	6.074	204.071	20	21	22	68.182	58.602	64.549	8.294	8.874	208.504	2.13 %
200	6	7	26	76.840	61.161	64.403	10.062	11.582	224.049	22	23	27	74.074	63.149	69.077	10.572	12.089	228.964	2.15 %

Π1 = 100, Π2 = 200, λ = 5, h = 6, n = 2, LT = 1.0, k = 0

(s,Q) Policy										(Q,S) Policy									
K	s1	s2	Q	OC	HC1	HC2	BOC1	BOC2	TC	S1	S2	Q	OC	HC1	HC2	BOC1	BOC2	TC	Gap
100	4	4	19	52.290	49.210	51.513	5.203	8.860	167.376	15	16	18	55.556	49.559	55.523	6.790	6.066	173.496	3.53 %
150	3	4	23	65.205	52.617	56.267	9.303	8.944	191.937	17	18	22	68.182	55.568	61.528	6.956	6.618	198.853	3.48 %
200	3	4	27	74.056	59.695	63.271	7.720	6.936	211.677	18	19	25	80.000	57.110	63.049	9.779	10.064	220.004	3.78 %

Π1 = 100, Π2 = 200, λ = 5, h = 6, n = 2, LT = 0.5, k = 0

(s,Q) Policy											(Q,S) Policy									
K	s1	s2	Q	OC	HC1	HC2	BOC1	BOC2	TC		S1	S2	Q	OC	HC1	HC2	BOC1	BOC2	TC	Gap
100	7	9	20	49.970	56.965	65.532	6.474	4.524	183.465		18	20	19	52.632	51.143	63.031	11.574	9.412	187.794	2.31 %
150	7	8	23	65.304	60.813	66.118	6.059	8.598	206.893		20	22	23	65.217	57.146	69.034	10.976	9.584	211.960	2.39 %
200	6	8	27	74.052	65.931	69.943	8.257	8.805	226.989		21	23	25	80.000	60.147	72.036	10.756	9.704	232.645	2.43 %
Π1 = 100, Π2 = 300, λ = 5, h = 6, n = 2, LT = 1.0, k = 0																				

(s,Q) Policy											(Q,S) Policy									
K	s1	s2	Q	OC	HC1	HC2	BOC1	BOC2	TC		S1	S2	Q	OC	HC1	HC2	BOC1	BOC2	TC	Gap
100	4	5	19	52.634	51.669	55.241	4.418	5.898	169.860		14	16	17	58.824	45.090	57.013	9.855	5.695	176.479	3.75 %
150	4	5	22	68.093	57.047	60.399	3.798	4.788	194.125		16	18	21	71.429	51.100	63.016	9.791	6.453	201.791	3.80 %
200	x	x	x	77.054	60.901	65.106	6.961	4.704	214.726		18	20	25	80.000	57.110	69.020	9.779	7.202	223.113	3.76 %
Π1 = 100, Π2 = 300, λ = 5, h = 6, n = 2, LT = 0.5, k = 0																				