DISTANCE BETWEEN A MAXIMUM MODULUS POINT AND THE ZERO SET OF AN ENTIRE FUNCTION

A DISSERTATION SUBMITTED TO

THE DEPARTMENT OF MATHEMATICS

AND THE INSTITUTE OF ENGINEERING AND SCIENCE

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

By

Adem Ersin Üreyen

November, 2006

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Assoc. Prof. Dr. H. Turgay Kaptanoğlu (Supervisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Asst. Prof. Dr. Seçil Gergün

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Assoc. Prof. Dr. Aurelian Gheondea

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Assoc. Prof. Dr. Tuğrul Hakioğlu

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Assoc. Prof. Dr. A. Sinan Sertöz

Approved for the Institute of Engineering and Science:

Prof. Dr. Mehmet B. Baray
Director of the Institute

ABSTRACT

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Adem Ersin Üreyen
Ph.D. in Mathematics
Supervisor: Assoc. Prof. Dr. H. Turgay Kaptanoğlu
November, 2006

We obtain asymptotical bounds from below for the distance between a maximum modulus point and the zero set of an entire function. Known bounds (Macintyre, 1938) are more precise, but they are valid only for some maximum modulus points. Our bounds are valid for all maximum modulus points and moreover, up to a constant factor, they are unimprovable.

We consider entire functions of regular growth and obtain better bounds for these functions. We separately study the functions which have very slow growth. We show that the growth of these functions can not be very regular and obtain precise bounds for their growth irregularity.

Our bounds are expressed in terms of some smooth majorants of the growth function. These majorants are defined by using orders, types, (strong) proximate orders of entire functions.

Keywords: Entire function, Maximum modulus point, Zero set, Order, Type, Proximate order, Regular growth.

ÖZET

BİR TÜM FONKSİYONUN BİR MAKSİMUM MODÜL NOKTASI İLE SIFIR KÜMESİ ARASINDAKİ UZAKLIK

Adem Ersin Üreyen
Matematik, Doktora
Tez Yöneticisi: Doç. Dr. H. Turgay Kaptanoğlu
Kasım, 2006

Bir tüm fonksiyonun maksimum modül noktası ile sıfır kümesi arasındaki uzaklık için aşağıdan asimptotik sınır buluyoruz. Bilinen sınırlar (Macintyre, 1938) daha kesin, ama sadece bazı maksimum modül noktaları için geçerli. Bulduğumuz sonuçlar tüm maksimum modül noktaları için geçerli ve bir sabit çarpan haricinde iyileştirilemez.

Ek olarak, düzenli büyüyen tüm fonksiyonları inceliyoruz ve bu tip fonksiyonlar için daha iyi sınırlar buluyoruz. Çok yavaş büyüyen tüm fonksiyonları ayrıca inceliyor, bu fonksiyonların çok düzenli büyüyemeyeceğini gösteriyor ve büyüme düzensizlikleri hakkında kesin sınırlar buluyoruz.

Bulduğumuz sonuçlar büyüme fonksiyonunun bazı düzgün üst sınırları cinsinden ifade edilmiştir. Bu üst sınırlar tüm fonksiyonların mertebeleri, tipleri ve (kuvvetli) yaklaşık mertebeleri kullanılarak tanımlanır.

Anahtar sözcükler: Tüm fonksiyon, Maksimum modül noktası, Sıfır kümesi, Mertebe, Tip, Yaklaşık mertebe, Kuvvetli yaklaşık mertebe, Düzenli büyüme.

Acknowledgement

The true supervisor of this thesis is Iossif V. Ostrovskii. He proposed the problem, and the thesis is completed wholly under his supervision. For administrative reasons, the supervisor had to be present at the defense. I. V. Ostrovskii's duties were performed by H. T. Kaptanoğlu during the defense as he was unable to attend it and H. T. Kaptanoğlu's name was written as supervisor on the final copy of the thesis.

I would like to express my deep gratitude to my supervisor Prof. I. V. Ostrovskii for his excellent guidance, valuable suggestions, encouragements, and patience.

I would like to thank H. T. Kaptanoğlu and Seçil Gergün for careful reading of the text and valuable remarks.

I would like to thank the jury members for serving on the thesis committee.

Finally, I would like to thank my family and friends for their encouragements and supports.

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Chapter 1

Introduction

A function $f : \mathbb{C} \to \mathbb{C}$ which is analytic in the whole complex plane is called an entire function. It can be represented by an everywhere convergent power series,

$$f(z) = \sum_{k=0}^{\infty} c_k z^k, \qquad z \in \mathbb{C}.$$

In the above series, if only finitely many of the coefficients c_n are nonzero, then f is called a polynomial. Otherwise it is called transcendental.

To characterize the asymptotic behavior of an entire function f, we introduce the growth function

$$M(r, f) := \max_{|z|=r} |f(z)|. \tag{1.1}$$

It follows from the maximum modulus principle that $M(r, f)$ is a nondecreasing function of $r \in \mathbb{R}_+$. If f is not constant, then $M(r, f)$ strictly increases and tends to ∞ as $r \to \infty$.

Let f be a polynomial of degree n,

$$f(z) = \sum_{k=0}^{n} c_k z^k, \qquad c_n \neq 0.$$

It can be easily shown that

$$\lim_{r\to\infty} \frac{\log M(r, f)}{\log r} = n.$$

Furthermore, f has exactly n zeros in $\mathbb{C}$. This shows that there is a close connection between the asymptotic behavior and the set of zeros of a polynomial. The main subject matter of the entire function theory is to establish relations between the growth of an entire function and the distribution of its zeros (see, e.g., [7], [10], [11]). The aim of this work is to obtain such a relation: We investigate the distance between the zero set of an entire function and points where the function is "large" in the sense we will describe below.

For each $r > 0$, there are points on the circle $\{z : |z| = r\}$ where the maximum in (1.1) is attained. We will denote such a point by w and call it a *maximum modulus point*. Equivalently, a point w is a maximum modulus point if

$$|f(w)| = M(|w|, f).$$

We denote by Z_f the zero set of the entire function f, i.e., $Z_f = \{z : f(z) = 0\}$. For each maximum modulus point w, we denote by $R(w, f)$ the distance between w and Z_f,

$$R(w, f) := \inf\{|w - z| : f(z) = 0\}.$$

The aim of this work is to obtain asymptotic (as $|w| \to \infty$) bounds for $R(w, f)$ from below. The first results in this direction were obtained by A.J. Macintyre.

Theorem A ([12]) (i) *The following inequality holds*

$$\limsup_{|w|\to\infty} \frac{1}{|w|} R(w, f)(\log M(|w|, f))^{1/2} > 0. \tag{1.2}$$

(ii) *For each $\epsilon > 0$ the following inequality holds*

$$\liminf_{\substack{|w|\to\infty \\ |w|\notin A_\epsilon}} \frac{1}{|w|} R(w, f)(\log M(|w|, f))^{\frac{1}{2}+\epsilon} > 0, \tag{1.3}$$

where $A_\epsilon \subset \mathbb{R}_+$ is such that

$$\int_{A_\epsilon} \frac{dt}{t} < \infty. \tag{1.4}$$

The inequality (1.2) gives an asymptotic bound for $R(w, f)$ from below only on *a sequence* of values of $|w| \to \infty$. The inequality (1.3) gives a little less precise bound that is valid outside of a “small” set. The following problem arises: To obtain bounds for $R(w, f)$ from below that are valid *for all sufficiently large* values of $|w|$. Main results of this work (see Theorem 1, Theorem 2 below) give such bounds.

Note that the bounds in (1.2)-(1.3) are inversely related to the growth of f: the slower the growth of f is, the better the bounds are. One significance of Theorem A is that its results are directly in terms of $M(r, f)$. Our results, on the other hand, are in terms of some “smooth majorants” of $M(r, f)$. We will explain the meaning of “smooth majorant” in Chapter 2.

The results of this dissertation have been published in [15], [16], [18], and will be published in [13], [14].

Chapter 2

Main Definitions

2.1 Order and type of an entire function

To measure the growth of an entire function f, we consider a class of "simple" and "smooth" functions and compare $M(r,f)$ with the elements of this class. In this and the following sections we will describe some special classes of comparison functions that are commonly used in the entire function theory.

It is easy to see that if an entire function f satisfies

$$\liminf_{r\to\infty} \frac{M(r,f)}{r^n} < \infty,$$

for some positive integer n, then f is a polynomial of degree at most n. Therefore, to measure the growth of transcendental (non-polynomial) entire functions it is necessary to use comparison functions that grow faster than powers of r. In entire function theory most commonly used comparison functions are of the form

$$e^{r^k}, \quad k > 0.$$

An entire function f is said to be of *finite order* if there exists a positive constant k such that the inequality

$$M(r,f) < e^{r^k} \tag{2.1}$$

holds *asymptotically*, i.e., for all sufficiently large values of r. The *order* (or the order of growth) of an entire function f is the greatest lower bound of those values of k for which inequality (2.1) is asymptotically valid. We denote the order of an entire function by $\rho = \rho_f$. Hence

$$\rho = \inf\{k : M(r, f) < e^{r^k} \quad \text{for } r > r_k\}.$$

It follows from the above definition that if f is an entire function of order ρ, and if ϵ is an arbitrary positive number, then

$$e^{r^{\rho-\epsilon}} < M(r, f) < e^{r^{\rho+\epsilon}}, \tag{2.2}$$

where the inequality on the left is satisfied for some sequence r_n tending to infinity and the inequality on the right is satisfied for all sufficiently large values of r. By taking logarithms twice, we obtain from (2.2) that

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r}. \tag{2.3}$$

Examples.

1. Let $f(z) = e^{z^n}$, $n \in \mathbb{N}$. Then $M(r, f) = e^{r^n}$, and using (2.3) we see that $\rho_f = n$.

2. Let
$$f(z) = \sin z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}.$$
Then $M(r, f) = (e^r - e^{-r})/2$ and $\rho_f = 1$.

3. Let
$$f(z) = \frac{\sin\sqrt{z}}{\sqrt{z}} = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{(2k+1)!}.$$
Then $M(r, f) = (e^{\sqrt{r}} - e^{-\sqrt{r}})/(2\sqrt{r})$ and $\rho_f = \frac{1}{2}$.

4. Let
$$f(z) = \prod_{k=1}^{\infty} \left(1 + \left(\frac{z}{e^{k^4}}\right)^{k^3}\right).$$

It can be shown that (see the proof of Theorem 3 below)

$$\log M(r, f) = \frac{1}{2}\log^2 r + O(\log^{3/2} r).$$

Then, by (2.3), $\rho_f = 0$.

5. Let $f(z) = e^{e^z}$. Then $M(r, f) = e^{e^r}$ and $\rho_f = \infty$.

Note that among the functions that have the same order, there are functions that grow in different ways. For example, it is possible to construct entire functions f_1, f_2, f_3 such that

$$M(r, f_1) \sim e^{r/\log r}, \quad M(r, f_2) \sim e^r, \quad M(r, f_3) \sim e^{r\log r}.$$

Although each of these functions has order 1, their asymptotical growth is apparently different. To distinguish functions that have the same order, we use another characteristic, the *type*.

An entire function f of order ρ is said be of *finite type* if there exists a positive constant A such that the inequality

$$M(r, f) < e^{Ar^\rho} \tag{2.4}$$

holds asymptotically. The greatest lower bound of those values of A for which the inequality (2.4) is asymptotically fulfilled is called the *type of* f (with respect to order ρ). We denote the type of an entire function f by $\sigma = \sigma_f$. Thus

$$\sigma = \inf\{A : M(r, f) < e^{Ar^\rho} \quad \text{for } r > r_A\}.$$

It follows that if ϵ is an arbitrary positive number, then

$$e^{(\sigma-\epsilon)r^\rho} < M(r, f) < e^{(\sigma+\epsilon)r^\rho}, \tag{2.5}$$

where the inequality on the left is satisfied for some sequence $r_n \to \infty$ and the inequality on the right is satisfied for all sufficiently large values of r. After taking a logarithm, we obtain from (2.5) that

$$\sigma = \limsup_{r\to\infty} \frac{\log M(r, f)}{r^\rho}.$$

If $\sigma_f = 0$, the function f is said to be of *minimal* type, if $0 < \sigma_f < \infty$, of *normal* type, and if $\sigma_f = \infty$, of *maximal* type.

Examples.

1. Let $f(z) = e^{\sigma z^n},\ n \in \mathbb{N},\ 0 < \sigma < \infty$. Then $\rho_f = n$ and $\sigma_f = \sigma$.
2. Let $M(r, f_1) \sim e^{r/\log r}$. Then $\rho_{f_1} = 1$ and $\sigma_{f_1} = 0$: f_1 is of minimal type.
3. Let $M(r, f_2) \sim e^r$. Then $\rho_{f_2} = 1$ and $\sigma_{f_2} = 1$: f_2 is of normal type.
4. Let $M(r, f_3) \sim e^{r \log r}$. Then $\rho_{f_3} = 1$ and $\sigma_{f_3} = \infty$: f_3 is of maximal type.

2.2 Proximate orders

Order and type are the simplest and the most common notions used for measuring the growth of entire functions. But they are rather coarse. That is, there are entire functions which have the same order and type but grow in substantially different ways. It follows from a theorem of Clunie and Kövari that (see Theorem C, p. 18), there exists entire functions f_1, f_2, f_3 such that

$$\begin{array}{lll} \log M(r, f_1) = r^\rho + O(1),\ r \to \infty; & \rho_{f_1} = \rho; & \sigma_{f_1} = 1 \\ \log M(r, f_2) = r^\rho \log r + O(1),\ r \to \infty; & \rho_{f_2} = \rho; & \sigma_{f_2} = \infty \\ \log M(r, f_3) = r^\rho \log^2 r + O(1),\ r \to \infty; & \rho_{f_3} = \rho; & \sigma_{f_3} = \infty. \end{array}$$

Observe that $\log M(r, f_2)/\log M(r, f_1) \sim \log r$ and these functions have different types. On the other hand, it is also true that $\log M(r, f_3)/\log M(r, f_2) \sim \log r$. However, these functions have the same order and type, i.e., it is not possible to distinguish them by using the usual order and type. Likewise, there exists entire functions g_1, g_2, g_3 such that

$$\begin{array}{lll} \log M(r, g_1) = r^\rho + O(1),\ r \to \infty; & \rho_{g_1} = \rho; & \sigma_{g_1} = 1 \\ \log M(r, g_2) = r^\rho/\log r + O(1),\ r \to \infty; & \rho_{g_2} = \rho; & \sigma_{g_2} = 0 \\ \log M(r, g_3) = r^\rho/\log^2 r + O(1),\ r \to \infty; & \rho_{g_3} = \rho; & \sigma_{g_3} = 0. \end{array}$$

Here, again, $\log M(r, g_2)/\log M(r, g_3) \sim \log r$, but these functions have the same order and type. It is easy to see that the problem is related to functions that have either minimal or maximal type. To avoid this, it is necessary to use larger class of comparison functions than functions of the form $e^{\sigma r^\rho}$ and make all functions of normal type. This can be done by using *proximate orders* introduced by Valiron at the beginning of the 20^{th} century.

We will define proximate orders separately for each of the following three cases: $0 < \rho < \infty$, $\rho = 0$, and $\rho = \infty$.

Definition (Valiron). (Proximate order when $0 < \rho < \infty$) A continuously differentiable positive function $\rho(r)$ on $\mathbb{R}_+$ is called a *proximate order* if it satisfies the conditions

$$\lim_{r\to\infty} \rho(r) = \rho, \qquad 0 < \rho < \infty; \tag{2.6}$$

$$\lim_{r\to\infty} r\rho'(r)\log r = 0. \tag{2.7}$$

If the inequalities

$$0 < \sigma_f := \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho(r)}} < \infty, \tag{2.8}$$

hold, then $\rho(r)$ is called a *proximate order of* f and σ_f is called the *type of* f *with respect to the proximate order* $\rho(r)$.

We call $r^{\rho(r)}$ a *smooth majorant* of $\log M(r,f)$ if (2.8) is satisfied.

Roughly speaking, by using proximate orders we can consider any function as of normal type. Following examples make the point more clear. Note that $r^{(\log c/\log r)} = c$. We assume $0 < \rho < \infty$.

Examples.

1. Let $\rho_1(r) \equiv \rho$. Evidently, ρ_1 satisfies (2.6)-(2.7) and therefore is a proximate order. If f_1 is of order ρ and of normal type σ in the usual sense, then $\rho_1(r)$ is a proximate order of f_1 and σ is the corresponding type.

2. Let $\rho_2(r) = \rho + \log\log r/\log r$. It is easy to see that ρ_2 satisfies (2.6)-(2.7), and $r^{\rho_2(r)} = r^\rho \log r$. So, if $\log M(r,f_2) \sim r^\rho \log r$, then f_2 is of maximal type in the usual sense, but it is of normal type with respect to $\rho_2(r)$.

3. Let $\rho_3(r) = \rho + 2\log\log r/\log r$. Then $r^{\rho_3(r)} = r^\rho \log^2 r$. Hence, if $\log M(r,f_3) \sim r^\rho \log^2 r$, then f_3 is of maximal type in the usual sense,

but it is of normal type with respect to $\rho_3(r)$.
Note that $\rho_2(r)$ is not a proximate order of f_3 and $\rho_3(r)$ is not a proximate order of f_2.

4. Let $\rho_4(r) = \rho - m \log\log\log r / \log r$. Then $r^{\rho_4(r)} = r^\rho / \log^m(\log r)$. If $\log M(r, f_4) \sim r^\rho / \log^m(\log r)$, then f_4 is of minimal type in usual sense, but it is of normal type with respect to $\rho_4(r)$.

Remark. Proximate order of an entire function f is not uniquely determined. If $\rho(r)$ is a proximate order of f and σ is the corresponding type, then $\tilde{\rho}(r) = \rho(r) + \log c / \log r$ is also a proximate order of f with corresponding type σ/c.

Valiron's theorem (see [10, p. 35]) shows that proximate orders form a scale of growth of entire functions of finite and positive order in the following sense: For each entire function f of order ρ, $0 < \rho < \infty$, there exists a proximate order $\rho(r) \to \rho$ such that (2.8) holds.

The definition of proximate order for functions of order ρ, $0 < \rho < \infty$, is well known and generally accepted. The situation is different for functions of zero or infinite order. For our purposes the following definitions are suitable.

In the case $\rho = 0$ the definition below is close to that of Levin [10, Ch. 1] but contains more restrictions.

Definition (Proximate order when $\rho = 0$.) We call a function $\rho(r), r \in \mathbb{R}_+$, a *zero proximate order* if it is representable in the form

$$\rho(r) = \frac{\vartheta(\log r)}{\log r}, \qquad r \geq r_0 > 1,$$

where $\vartheta(x) \in C^1(\mathbb{R}_+)$ is a positive concave function such that

$$\lim_{x\to\infty} \frac{e^{\vartheta(x)}}{x} = \infty \qquad \text{and} \qquad \lim_{x\to\infty} \frac{\vartheta(x)}{x} = 0. \tag{2.9}$$

The class of zero proximate orders is sufficient for measuring the growth of all transcendental entire functions of order zero. That is, for every entire function f of order zero, there exists a zero proximate order such that (2.8) holds (see [10, p. 35].)

Example. Let $\rho(r) = n \log\log r / \log r,\ n > 1$. Then $\rho(r)$ is a zero proximate order with $\vartheta(x) = n \log x$. If f is an entire function such that $\log M(r, f) \sim (\log r)^n$, then f is of order zero and of maximal type in the usual sense, but it is of normal type with respect to $\rho(r)$.

For functions of infinite order we will use the following definition which is based on the results of Earl and Hayman [4].

Definition (Proximate order when $\rho = \infty$.) We call a function $\rho(r), r \in \mathbb{R}_+$, an *infinite proximate order* if it is representable in the form

$$\rho(r) = \frac{\vartheta(\log r)}{\log r}, \qquad r \geq r_0 > 1,$$

where $\vartheta(x) \in C^2(\mathbb{R}_+)$ is a positive convex function such that

$$\lim_{x\to\infty} \vartheta'(x) = \infty, \qquad \text{and} \qquad \lim_{x\to\infty} \frac{\vartheta''(x)}{\vartheta'^2(x)} = 0. \tag{2.10}$$

It is proved in [4] that for every entire function f of infinite order there exists an infinite proximate order such that (2.8) holds. That is, the class of infinite proximate orders is sufficient for measuring the growth of entire functions of infinite order.

Example. Let $\rho(r) = r/\log r$. Then $\rho(r)$ is an infinite proximate order with $\vartheta(x) = e^x$. If f is an entire function such that $\log M(r, f) \sim e^r$, then $\rho(r)$ is a proximate order of f.

Using proximate orders is not the only way of measuring the growth of entire functions. For some applications, it is necessary to use functions that have smoother behavior than required by equations (2.6)-(2.7), (2.9), and (2.10).

In the remaining sections of this chapter we will describe some subclasses of proximate orders that we will need for our purposes.

2.3 Strong proximate orders

A *strong proximate order*, introduced by Levin in the $1950's$ of 20^{th} century (see [10, Ch. 1], is a *twice* continuously differentiable proximate order that satisfies the additional conditions stated below. Strong proximate orders form a proper subclass of proximate orders; nevertheless, they too are sufficient for complete characterization of the growth of entire functions.

As before we will define strong proximate orders separately for each of the following cases: $0 < \rho < \infty$, $\rho = 0$, $\rho = \infty$.

Definition (Levin) (Strong proximate order when $0 < \rho < \infty$.) A *strong proximate order* is a function $\rho(r) \in C^2(\mathbb{R}_+)$ representable in the form

$$\rho(r) = \rho + \frac{\vartheta_1(\log r) - \vartheta_2(\log r)}{\log r}, \quad r \geq r_0 > 1, \tag{2.11}$$

where $0 < \rho < \infty$, and $\vartheta_j, j = 1, 2$, is a concave function of $C^2(\mathbb{R}_+)$ satisfying the conditions

$$\lim_{x\to\infty} \vartheta_j(x) = \infty, \quad \lim_{x\to\infty} \frac{\vartheta_j(x)}{x} = 0, \quad \lim_{x\to\infty} \frac{\vartheta_j''(x)}{\vartheta_j'(x)} = 0. \tag{2.12}$$

It is easy to check that any strong proximate order is a proximate order, i.e., satisfies (2.6)-(2.7).

For any strong proximate order $\rho(r)$, if the inequality (2.8) is satisfied, then we say that $\rho(r)$ is a strong proximate order of f and σ_f is the corresponding type. Given any entire function f of order ρ, $0 < \rho < \infty$, there exists a strong proximate order $\rho(r)$ such that (2.8) holds. This is proved by Levin ([10, pp. 39–41].)

We note that if an entire function f has maximal type with respect to the usual order ρ (i.e., if $\sigma = \limsup_{r\to\infty} \log M(r, f)/r^\rho = \infty$), then we can choose $\vartheta_2 \equiv 0$ in (2.11). Likewise, if f has minimal type in the usual sense, then we can choose $\vartheta_1 \equiv 0$. The first assertion immediately follows from the construction of Levin. We will prove the second assertion later (see Lemma 4.2.)

Definition (Strong proximate order when $\rho = 0$.) We call a zero proximate order $\rho(r) = \vartheta(\log r)/\log r$, a *zero strong proximate order*, if the following additional condition is satisfied:

$$\vartheta''(x) + \vartheta'^2(x) > 0, \qquad x \geq x_0 > 0. \tag{2.13}$$

Instead of (2.13), Levin uses the slightly weaker condition $\vartheta''(x)/\vartheta'(x) \to 0$, $x \to \infty$. We will need (2.13) to guarantee that the function $r^{\rho(r)} = e^{\vartheta(\log r)}$ is convex with respect to $\log r$.

We will prove later (see Lemma 4.3) that any entire function of order zero has a zero strong proximate order.

When $\rho = \infty$, we will call an infinite proximate order also as an *infinite strong proximate order*.

2.4 Admissible and strongly admissible proximate orders

Later it will be necessary for us to consider separately the class of entire functions that satisfy

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{\log^2 r} = \infty. \tag{2.14}$$

To measure the growth of such entire functions in addition to to proximate and strong proximate orders, we will also use the following subclass of strong proximate orders.

Definition. For $0 < \rho \leq \infty$ (including infinity) we will call any strong proximate order *admissible*. For $\rho = 0$, we will call a zero strong proximate order $\rho(r) = \vartheta(\log r)/\log r$ *admissible* if it satisfies

$$2\vartheta''(x) + \vartheta'^2(x) > 0 \qquad \text{and} \qquad \limsup_{x\to\infty} \frac{e^{\vartheta(x)}}{x^2} = \infty. \tag{2.15}$$

Admissible proximate orders form a sufficient class to characterize the growth of entire functions that satisfy (2.14). For functions of positive order (including ∞), this follows from the sufficiency of strong proximate orders. For $\rho = 0$ this is a consequence of Lemma 4.4 below.

We will also need a subclass of admissible proximate orders that consists of *three times* continuously differentiable functions:

When $0 < \rho < \infty$, we will call an admissible proximate order $\rho(r)$ in the form (2.11) *strongly admissible* if the functions ϑ_j, in addition to (2.12), satisfy

$$\lim_{x\to\infty} \vartheta_j'''(x) = 0, \quad j = 1, 2. \tag{2.16}$$

When $\rho = \infty$, we will call an admissible proximate order $\rho(r) = \vartheta(\log r)/\log r$ *strongly admissible* if $\vartheta(x)$, in addition to (2.10), satisfies

$$\lim_{x\to\infty} \frac{\vartheta'''(x)}{\vartheta'^3(x)} = 0. \tag{2.17}$$

When $\rho = 0$, we will call an admissible proximate order $\rho(r) = \vartheta(\log r)/\log r$ *strongly admissible* if $\vartheta(x)$ additionally satisfies

$$\frac{\vartheta'''(x)}{\vartheta'^3(x)} = O(1), \quad x \to \infty. \tag{2.18}$$

Strongly admissible proximate orders also form a sufficient class for complete characterization of the growth of entire functions that satisfy (2.14). This can be shown by applying a suitable smoothing procedure to the functions used for showing the completeness of admissible proximate orders.

Chapter 3

Statement of Results

In Section 3.1 we will state our main theorem (Theorem 1) related to the asymptotic behavior of $R(w, f)$ for arbitrary entire functions *without any restriction.* For the characterization of the growth of $M(r, f)$ we will use proximate orders.

In Section 3.2 we will consider entire functions of *regular growth*, i.e., functions for which the following limit exists

$$\sigma = \lim_{r\to\infty} \frac{\log M(r, f)}{r^{\rho(r)}}.$$

We will obtain a better estimate of $R(w, f)$ for this class of functions (see Theorem 2). For the characterization of the growth of $M(r, f)$, we will need smoother functions and make use of strong proximate orders. We will then show that it is possible to put Theorem 2 in a simple form if we restrict ourselves to functions of not very slow growth (i.e., functions that satisfy (2.14)) and if we use admissible proximate orders (see Corollary 2.) We believe that Corollary 2 is valid even for entire functions of arbitrarily slow growth. The reason for this is the fact that the growth of slowly growing entire functions can not be "very regular" (see Theorem D.)

We will show in Theorem 3 that Corollary 2 is sharp for the subclass of admissible proximate orders that we called strongly admissible in Chapter 2.

In Section 3.3 we will study more deeply the growth irregularity of very slowly growing entire functions that satisfy $\log M(r,f) = o(\log^2 r)$.

3.1 Distance between a maximum modulus point and the zero set of an entire function without assumption of regular growth

Let $\rho(r) \to \rho$, $0 \le \rho \le \infty$, be a proximate order. Further, we will write

$$V(r) = r^{\rho(r)}. \tag{3.1}$$

We remind that if the inequalities

$$0 < \sigma = \limsup_{r\to\infty} \frac{\log M(r,f)}{V(r)} < \infty \tag{3.2}$$

hold, then we say that $\rho(r)$ is a proximate order of f and σ is the corresponding type. Let us denote by $[\rho(r),\sigma]$, $0 < \sigma < \infty$, the class of all entire functions for which $\rho(r)$ is a proximate order and σ is corresponding type. That is, $f \in [\rho(r),\sigma]$ if and only if (3.2) holds.

Our main theorem of this section is the following:

Theorem 1 *Let $\rho(r)$ be a proximate order and let V be defined by (3.1).*

(i) *If $f \in [\rho(r),\sigma]$, then*

$$\liminf_{|w|\to\infty} R(w,f)V'(|w|) \ge \frac{1}{e^2\sigma}. \tag{3.3}$$

(ii) *There exists $f \in [\rho(r),\sigma]$ such that*

$$\liminf_{|w|\to\infty} R(w,f)V'(|w|) \le \frac{\pi}{\sigma}. \tag{3.4}$$

Examples.

1. Suppose f is of order ρ ($0 < \rho < \infty$) and type σ ($0 < \sigma < \infty$) in the usual sense. That is, suppose that
$$\limsup_{r\to\infty} \frac{\log M(r,f)}{r^\rho} = \sigma, \quad 0 < \rho < \infty, \quad 0 < \sigma < \infty.$$
Since $\rho(r) \equiv \rho$ is a proximate order of f, (3.3) implies (with $V(r) = r^\rho$),
$$\liminf_{|w|\to\infty} R(w,f)|w|^{\rho-1} \geq \frac{1}{e^2\sigma\rho}.$$

2. Let $f_1(z) = \sin z$ (see Example 2, page 5.) Then (3.3) implies (with $\rho(r) \equiv 1$, $\sigma = 1$, $V(r) = r$)
$$\liminf_{|w|\to\infty} R(w,f_1) \geq \frac{1}{e^2}.$$
Note that the maximum modulus points of f_1 are the whole imaginary axis and the zero set of f_1 is $Z_{f_1} = \{z = n\pi,\ n \in \mathbb{Z}\}$. Therefore $R(w,f_1) = |w|$.

3. Let $V(r) = r^\rho(\log r)^m$, $0 < \rho < \infty$, $m \in \mathbb{R}$. Then (3.3) implies
$$\liminf_{|w|\to\infty} \frac{R(w,f)}{|w|}|w|^\rho(\log|w|)^m \geq \frac{1}{\rho e^2\sigma}.$$

4. Let $V(r) = (\log r)^m$, $m > 1$. Then (3.3) implies
$$\liminf_{|w|\to\infty} \frac{R(w,f)}{|w|}(\log|w|)^{m-1} \geq \frac{1}{me^2\sigma}.$$

5. Let $V(r) = e^{r^m}$, $m > 0$. Then (3.3) implies
$$\liminf_{|w|\to\infty} \frac{R(w,f)}{|w|}e^{|w|^m}|w|^m \geq \frac{1}{me^2\sigma}.$$

We do not know whether the constant $(e^2\sigma)^{-1}$ on the right hand side of (3.3) is the best possible. Nevertheless, (3.4) shows that the best possible constant is not greater than π/σ.

Let us compare Theorem 1 with Macintyre's Theorem A. If $f \in [\rho(r),\sigma]$, $0 < \rho < \infty$, then with some positive constant C, Theorem A implies

(i') For some sequence of w tending to ∞,
$$R(w,f) > C\frac{|w|}{(V(|w|))^{1/2}}. \tag{3.5}$$

(ii') For $w \notin A_\epsilon$, where $A_\epsilon \subset \mathbb{R}_+$ satisfies (1.4),

$$R(w, f) > C \frac{|w|}{(V(|w|))^{1/2+\epsilon}}. \tag{3.6}$$

Note that for $0 < \rho < \infty$, by (2.6)-(2.7),

$$rV'(r) = (\rho + o(1))V(r). \tag{3.7}$$

Therefore, part (i) of Theorem 1 implies

$$R(w, f) > C \frac{|w|}{V(|w|)}.$$

This estimate is less precise than (3.5) and (3.6), but it is valid for *all* w. Moreover, part (ii) of Theorem 1 shows that Macintyre's estimates can not be valid for all w.

3.2 Distance between a maximum modulus point and the zero set of an entire function with assumption of regular growth

Part (ii) of Theorem 1 shows that, up to a constant factor, the bound in (3.3) cannot in general be improvable. Later, when we prove Theorem 3.1, we will construct a function that satisfies the properties stated in part (ii). That function has an *irregular growth* in the following sense:

$$\limsup_{r\to\infty} \frac{\log M(r, f)}{V(r)} = \sigma, \quad 0 < \sigma < \infty, \qquad \liminf_{r\to\infty} \frac{\log M(r, f)}{V(r)} = 0.$$

This suggests the following question: Can we obtain a better bound for $R(w, f)$ if we assume that f has *regular growth*, i.e., if we assume that the following limit exists

$$\lim_{r\to\infty} \frac{\log M(r, f)}{V(r)} = \sigma\,?$$

We will answer this question when $\rho(r)$ is a strong proximate order. We will assume that f has regular growth in the following sense:

Definition. Let $\rho(r)$ be a strong proximate order and V be defined by (3.1). We say that an entire function $f \in [\rho(r), \sigma]$ is a function of (V, θ)-*regular growth* if

$$\log M(r, f) = \sigma V(r) + O(\theta(r)), \quad r \to \infty, \tag{3.8}$$

where θ is a positive non-decreasing function on $\mathbb{R}_+$ satisfying the conditions

$$\text{(i)} \quad \theta(r) = o(V(r)), \qquad r \to \infty, \tag{3.9}$$

$$\text{(ii)} \quad \theta\Big(r \exp\Big\{\frac{V(r)}{(rV'(r))}\Big\}\Big) = O(\theta(r)), \qquad r \to \infty. \tag{3.10}$$

It is first necessary to answer the following question: Given V and θ, is there any function of (V, θ)-regular growth? For $\log r = O(\theta(r))$, $r \to \infty$, this question is answered by the following theorem of Clunie and Kövari.

Theorem B ([3, p. 13]) *Let φ be an increasing function convex in $\log r$ such that*

$$\lim_{r\to\infty} \frac{\varphi(r)}{\log r} = \infty.$$

Then there exists an entire function f such that

$$\log M(r, f) = \varphi(r) + O(\log r).$$

It can be easily shown that if $\rho(r)$ is a strong proximate order, then the conditions above are satisfied by $\varphi(r) = V(r)$.

Another result of [3] is the following:

Theorem C ([3, p. 19]) *For any function φ representable in the form*

$$\varphi(r) = \int_1^r \psi(t)\, d\log t, \qquad r \geq r_0 > 1, \tag{3.11}$$

where ψ is a positive increasing function satisfying the condition

$$\psi(cr) - \psi(r) \geq 1 \quad \textit{for some } c > 1 \textit{ and for all } r \geq r_0 > 1, \tag{3.12}$$

there exists an entire function f such that

$$\log M(r, f) = \varphi(r) + O(1), \qquad r \to \infty. \tag{3.13}$$

It is straightforward to show that if $\rho(r) \to \rho > 0$ is a strong proximate order, then conditions (3.11)-(3.12) are satisfied by $\varphi(r) = V(r)$. Therefore, when $0 < \rho \leq \infty$, for each strong proximate order $\rho(r)$ and for each θ satisfying (3.9)-(3.10), there exists functions of (V, θ)-regular growth.

Our main result for functions of regular growth is the following:

Theorem 2 *Let $\rho(r)$ be a strong proximate order and let V be defined by (3.1). If f is of (V, θ)-regular growth, then for all sufficiently large values of $|w|$, the inequality*

$$\frac{R(w,f)}{|w|} \geq 1 - \exp\left\{ -\frac{C}{|w|V'(|w|)} \sqrt{\frac{V(|w|)}{\theta(|w|)}} \right\} \tag{3.14}$$

holds, where C is a positive constant.

The following corollary of Theorem 2 is immediate.

Corollary 1 *If conditions of Theorem 2 are satisfied and, moreover,*

$$\liminf_{r \to \infty} rV'(r) \sqrt{\frac{\theta(r)}{V(r)}} > 0, \tag{3.15}$$

then

$$\liminf_{|w| \to \infty} R(w,f) V'(|w|) \sqrt{\frac{\theta(|w|)}{V(|w|)}} > 0. \tag{3.16}$$

Examples.

1. Let $f_1(z) = \sin z$. Since $\log M(r, f_1) = r + O(1)$ (see Example 2, page 5), (3.16) implies (with $V(r) = r,\ \theta(r) = 1$)

   $$\liminf_{|w| \to \infty} \frac{R(w, f_1)}{\sqrt{|w|}} > 0.$$

   One can compare this with Example 2, page 16.

2. Let $V(r) = r^{\rho} (\log r)^m$, $0 < \rho < \infty$, $m \in \mathbb{R}$. Assume $\log M(r, f) = V(r) + O(1)$. Then (3.16) implies (with $\theta(r) = 1$)

   $$\liminf_{|w| \to \infty} \frac{R(w,f)}{|w|} \sqrt{|w|^{\rho} (\log |w|)^m} > 0.$$

3. Let $V(r) = (\log r)^m$, $m > 1$, and $\theta(r) = \log r$. Then (3.16) implies

$$\liminf_{|w|\to\infty} \frac{R(w,f)}{|w|}\sqrt{(\log|w|)^{m-1}} > 0.$$

4. Let $V(r) = e^{r^m}$, $m > 0$, and $\theta(r) = 1$. Then (3.16) implies

$$\liminf_{|w|\to\infty} \frac{R(w,f)}{|w|}\sqrt{e^{|w|^m}}|w|^m > 0.$$

Evidently, (3.16) gives a better estimate than (3.3). Moreover, the bound (3.16) depends on θ and the smaller θ is, the better the bound is.

When $0 < \rho < \infty$, using (3.7) we can put (3.16) into the form

$$\liminf_{|w|\to\infty} \frac{R(w,f)}{|w|}\sqrt{\theta(|w|)V(|w|)} > 0. \tag{3.17}$$

If $\theta(r) = O(1)$, $r \to \infty$, then the bound (3.17) is just Macintyre's bound in Theorem A (i) with *limsup* replaced by *liminf* and $\log M(|w|, f)$ replaced by $V(|w|)$. So, generally speaking, for functions of "very regular growth" Macintyre's bound is valid without any exceptional set.

We note that if $\rho(r)$ is an admissible proximate order, then (3.15) holds for $\theta \equiv 1$ and hence for any non-decreasing positive θ. This is obvious when $\rho = \infty$. When $0 < \rho < \infty$, it follows immediately from (3.7). When $\rho = 0$, condition (2.15) implies that $e^{\vartheta/2}$ is convex and $e^{\vartheta(x_n)/2}/x_n \to \infty$ for some sequence $x_n \to \infty$. This shows that $(e^{\vartheta(x)/2})' \to \infty$ as $x \to \infty$. Since $V(r) = e^{\vartheta(\log r)}$, it follows that

$$\frac{rV'(r)}{\sqrt{V(r)}} = 2r\left(e^{\vartheta(\log r)/2}\right)' \to \infty, \quad \text{as } r \to \infty.$$

Therefore Corollary 1 implies the following:

Corollary 2 *Assume that f satisfies (2.14) and $\rho(r)$ is an admissible proximate order of f. If f is of (V,θ)-regular growth, then (3.16) holds.*

We conjecture that (3.16) remains valid even for entire functions that do not satisfy (2.14). The reason is that entire functions of very slow growth can not be of "very regular growth" as the following theorem shows.

Theorem D *Let $\rho(r)$ be a strong proximate order and let f be an entire function satisfying*

$$\log M(r,f) = o(\log^2 r), \quad r \to \infty. \tag{3.18}$$

If f is of (V,θ)-regular growth, then

$$\limsup_{r\to\infty} rV'(r)\sqrt{\frac{\theta(r)}{V(r)}} > 0. \tag{3.19}$$

This theorem shows that the function θ in (3.8) has growth restrictions from below. For example, if $V(r) = \log^\beta r,\ 1 < \beta < 2$, then there is no entire function f of (V,θ)-regular growth with $\theta(r) = o(\log^{2-\beta} r),\ r \to \infty$. In the next section we will study this situation more deeply. We will not prove Theorem D since it can be proved in much the same way as Theorem 4 below.

To consider the question whether the bound (3.16) is improvable or not, we need examples of entire functions f for which

(a) $|\log M(r,f) - \sigma V(r)|$ is relatively small,

(b) maximum modulus points of f are extremely close to its zero set.

For condition (a), we can use results of Clunie and Kövari [3] mentioned above. Unfortunately, the method of these authors does not permit one to locate positions of zeros required for (b).

Nevertheless, we can prove that (3.16) is sharp if $\theta(r)$ is not of very slow growth and has some special form, and if $\rho(r)$ belongs to the class of strongly admissible proximate orders.

Theorem 3 *Let $\rho(r)$ be a strongly admissible proximate order and let V be defined by (3.1). Given $\frac{1}{3} \le \alpha < 1$, put*

$$\theta(r) = V(r)(rV'(r))^{\alpha-1}.$$

There exists an entire function f of (V,θ)-regular growth such that

$$\liminf_{|w|\to\infty} R(w,f)V'(|w|)\sqrt{\frac{\theta(|w|)}{V(|w|)}} \le \pi. \tag{3.20}$$

3.3 Growth irregularity of slowly growing entire functions

Let $\mathcal{A}$ be the set of all increasing functions φ defined for $r > 0$, convex in $\log r$, and satisfying

$$\lim_{r\to\infty} \frac{\varphi(r)}{\log r} = \infty. \tag{3.21}$$

If f is a transcendental entire function, then the maximum modulus principle and the Hadamard three circles theorem imply that $\log M(r, f) \in \mathcal{A}$. It is well known that $\mathcal{A}$ is wider than the class of all functions of the form $\log M(r, f)$. The following specific property of the latter can be mentioned: $\log M(r, f)$ must be piecewise analytic (see, e.g., [19], p. 14, or [7], p. 11). The problem of the asymptotic (at ∞) approximation of a function $\varphi \in \mathcal{A}$ by functions of the form $\log M(r, f)$ can be viewed as the problem of existence of an entire function with prescribed growth. From this point of view the problem has been studied by Edrei and Fuchs [5], Clunie [2], and Clunie and Kövari [3]. Most complete results are contained in [3] (see Theorem B and Theorem C, page 18 above.)

Recall that if the function $\psi(r) := (d/d\log r)\varphi(r)$ satisfies for some $c > 1$ the condition

$$\psi(cr) - \psi(r) \geq 1, \quad r \geq 1, \tag{3.22}$$

then, by Theorem C, there exists an entire function f such that

$$\log M(r, f) - \varphi(r) = O(1), \quad r \to \infty. \tag{3.23}$$

The restriction (3.22) implies that

$$\liminf_{r\to\infty} \varphi(r)(\log r)^{-2} > 0.$$

Therefore Theorem C is not applicable to functions $\varphi \in \mathcal{A}$ such that

$$\varphi(r) = o(\log^2 r), \quad r \to \infty. \tag{3.24}$$

Our aim in this section is to study this case. Our result concerns functions that belong to a subset of $\mathcal{A}$, which we describe now.

Let us change the scale by setting $\log r = x$. If f is a transcendental entire function satisfying

$$\log M(r,f) = o(\log^2 r), \quad r \to \infty, \tag{3.25}$$

then $\log M(e^x, f)$ has growth (as a function of x) not less than of order 1 and maximal type and not greater than of order 2 and minimal type. By Levin's theorem (see Section 2.3.), there exists a strong proximate order $\lambda(x)$ of the form

$$\lambda(x) = \lambda + \frac{\vartheta_1(\log x) - \vartheta_2(\log x)}{\log x},$$

where $1 \le \lambda \le 2$, $\vartheta_j \in C^2(\mathbb{R}_+), j = 1, 2$, is a concave function satisfying (2.12), and

$$\limsup_{x\to\infty} \frac{\log M(e^x, f)}{x^{\lambda(x)}} = 1.$$

Moreover, as we have noted in Section 2.3, if f has maximal (minimal) type, then one has $\vartheta_2 \equiv 0\,(\vartheta_1 \equiv 0)$.

Definition. We denote by $\mathcal{B}$ the set of all functions φ representable in the form

$$\varphi(e^x) = w(x),$$

where w is defined by

$$w(x) := x^\lambda e^{\vartheta_1(\log x) - \vartheta_2(\log x)}, \tag{3.26}$$

$1 \le \lambda \le 2$, ϑ_1 and ϑ_2 have properties (2.12), and, moreover, if $\lambda = 1$, then $\vartheta_2 \equiv 0$, if $\lambda = 2$, then $\vartheta_1 \equiv 0$.

The simplest examples of $\varphi \in \mathcal{B}$ are functions defined for sufficiently large r in the form

$$\varphi(r) = (\log r)^{p_1} (\log_2 r)^{p_2} \dots (\log_m r)^{p_m},$$

where $\log_k$ denotes the k^{th} iteration of log, and $p_1, \dots, p_m \in \mathbb{R}$ are chosen in such a way that (3.21) and (3.24) are satisfied.

Our main result of this section is the following:

Theorem 4 *Let $\varphi \in \mathcal{B}$ and $w(x) = \varphi(e^x)$.*

(i) *There does not exist an entire function f such that*

$$\log M(e^x, f) - w(x) = o\left(\min\left(\frac{1}{w''(x)}, x\right)\right), \quad x \to \infty. \tag{3.27}$$

(ii) *The previous assertion ceases to be true if one replaces "o" by "O".*

Examples.

1. Set $w(x) = x^p, 1 < p < 2$. There is no entire function f such that

$$\log M(r, f) - \log^p r = o(\log^{2-p} r), \quad 1 < p < 2.$$

2. Set $w(x) = x^2(\log x)^{-1}$. There is no entire function f such that

$$\log M(r, f) - (\log r)^2(\log\log r)^{-1} = o\left((\log\log r)\right).$$

3. Set $w(x) = x(\log x)^{1/2}$. There is no entire function f such that

$$\log M(r, f) - (\log r)(\log\log r)^{1/2} = o\left((\log r)\right).$$

Moreover, the above assertions ceases to be true if "o" is replaced by "O".

Note that in Examples 1 and 2, $\min(1/w''(x), x) = 1/w''(x)$, whereas in Example 3, $\min(1/w''(x), x) = x$.

Since $\lim_{x\to\infty} w''(x) = 0$ (see Lemma 9.1), part (i) of Theorem 4 implies that (3.23) is impossible for any $\varphi \in \mathcal{B}$. Moreover, the *slower* the growth of φ is, the *worse* the rate of its asymptotic approximation by functions of the form $\log M(r, f)$ is. In other words, if f satisfies (3.25), then the slower the growth of the transcendental entire function f is, the more irregular it is.

Note that, in the theory entire functions, there are many facts of opposite character: when the growth of $\log M(r, f)$ decreases, the asymptotic behavior of f at ∞ becomes more similar to that of a polynomial. Let us mention, e.g., Wiman's theorem on functions of order less than 1/2, theorems on functions of order zero ([19], Sec.9, 15, 16, 26; [9]). Therefore, one might expect that, if the growth of $\log M(r, f)$ diminishes, then its regularity increases. Theorem 4 shows that this is not the case for entire functions satisfying (3.25).

Chapter 4

Preliminaries

Let f be an entire function of order ρ, $0 < \rho < \infty$. It is proved by Levin [10, pp. 39–41] that f has its own strong proximate order. That is, there exists a strong proximate order $\rho(r)$ of the form (2.11) for which (2.8) holds. We have stated in Section 2.3 that if the function f has maximal type with respect to the order ρ, then one can choose $\vartheta_2 \equiv 0$. Correspondingly, if f has minimal type, then one can choose $\vartheta_1 \equiv 0$. The first assertion easily follows from Levin's construction and we omit its proof. We will deduce the second assertion from the following lemma.

Lemma 4.1 *Let $\varphi(x)$ be a non-positive continuous function on $\mathbb{R}_+$ such that*

(i) $\lim_{x\to\infty} \varphi(x) = -\infty$,

(ii) $\limsup_{x\to\infty} \varphi(x)/x = 0$.

Then there exists a decreasing convex function $\psi \in C^2(\mathbb{R}_+)$ such that

(a) $\psi(x) \geq \varphi(x), \quad \forall x > 0$,

(b) $\psi(x) = \varphi(x)$ *on an unbounded set,*

(c) $\lim_{x\to\infty} \psi''(x)/\psi'(x) = 0$.

Proof. Let l be the graph of φ :

$$l = \{(x, y) : y = \varphi(x), \quad 0 < x < \infty\}.$$

We will construct ψ step by step.

Step 1. Consider the family of functions $\{l_1(t) : t > 0\}$, where

$$l_1(t) = \{(x, y) : y = y_1(x, t) = \frac{1}{x+1} - t,\ x > 0\}.$$

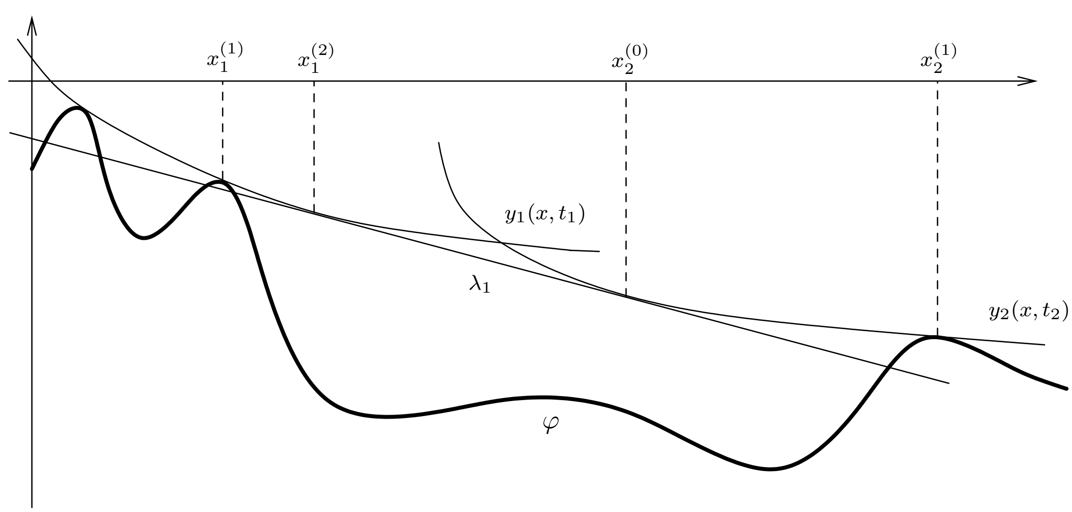
There exists a $t_1 > 0$ such that the curve $l_1(t_1)$ touches the curve l from above. (This can be shown by using the arguments we apply in the proof of Lemma 4.3.) The set of touching points is closed by continuity, and bounded by (i). Let $x_1^{(1)}$ be the abscissa of the last touching point. Take $\epsilon_1 > 0$ and choose $x_1^{(2)} > x_1^{(1)} + 1$ so large that

$$\left| \frac{y_1''(x_1^{(2)}, t_1)}{y_1'(x_1^{(2)}, t_1)} \right| < \epsilon_1.$$

Consider the line tangent to $l_1(t_1)$ at the point $x_1^{(2)}$:

$$\lambda_1 := \{(x, y) : y = \tilde{y}_1(x) = y_1'(x_1^{(2)}, t_1)(x - x_1^{(2)}) + y_1(x_1^{(2)}, t_1), \quad x \geq x_1^{(2)}\}.$$

Note that by condition (ii), λ_1 must intersect l.



Step 2. Consider the family $\{l_2(t),\ t \geq 0\}$, where

$$l_2(t) = \{(x, y) : y = y_2(x, t) = y_1(x - t, t_1) + y_1'(x_1^{(2)}, t_1)t, \quad x \geq t\}.$$

Observe that $l_2(0) = l_1(t_1)$ and $l_2(t)$ is a shift of $l_1(t_1)$ along the line λ_1.

As before, there exists $t_2 > 0$ such that $l_2(t_2)$ lies above l and touches it along a compact set. Let $x_2^{(0)}$ be the touching point of $l_2(t_2)$ and λ_1 and let $x_2^{(1)}$ be the abscissa of the last touching point of $l_2(t_2)$ and l. Take $\epsilon_2,\ 0 < \epsilon_2 < \epsilon_1$, and choose $x_2^{(2)} > x_2^{(1)} + 1$ so large that

$$\left| \frac{y_2''(x_2^{(2)}, t_2)}{y_2'(x_2^{(2)}, t_2)} \right| < \epsilon_2.$$

Denote the line tangent to $l_2(t_2)$ at the point $x_2^{(2)}$ by λ_2:

$$\lambda_2 := \{(x, y) : y = \tilde{y}_2(x) = y_2'(x_2^{(2)}, t_2)(x - x_2^{(2)}) + y_2(x_2^{(2)}, t_2), \quad x \geq x_2^{(2)}\}.$$

We repeat this process indefinitely where the numbers $\epsilon_1 > \epsilon_2 > \epsilon_3 > \ldots$ are chosen in such a way that $\epsilon_n \downarrow 0$.

With $x_1^{(0)} = 0$, we set

$$\psi(x) = \begin{cases} y_j(x, t_j), & x_j^{(0)} \leq x < x_j^{(2)}, \\ \tilde{y}_j(x), & x_j^{(2)} \leq x < x_{j+1}^{(0)}. \end{cases}$$

Then

$$\psi(x) \geq \varphi(x) \quad \text{and} \quad \psi(x_j^{(1)}) = \varphi(x_j^{(1)}).$$

Also

$$\begin{aligned} \left| \frac{\psi''(x)}{\psi'(x)} \right| &< \epsilon_j, & x_j^{(0)} < x < x_j^{(2)}; \\ \frac{\psi''(x)}{\psi'(x)} &= 0, & x_j^{(2)} < x < x_{j+1}^{(0)}. \end{aligned}$$

Note that ψ'' does not exist at the points $x_j^{(0)}$ and $x_j^{(2)}$ but we can smooth ψ at these points in such a way that all the properties (a)-(c) are preserved. $\square$

Lemma 4.2 *Let f be an entire function of order ρ, $0 < \rho < \infty$, and of minimal type. Then f has a strong proximate order $\rho(r)$ of the form (2.11) with $\vartheta_1 \equiv 0$.*

Proof. Let

$$\theta(r) := \frac{\log M(r,f)}{r^{\rho}}.$$

Since f is of order ρ and minimal type, we have

$$\limsup_{r\to\infty} \frac{\log\theta(r)}{\log r} = 0 \tag{4.1}$$

and

$$\lim_{r\to\infty} \theta(r) = 0. \tag{4.2}$$

Let $\varphi(x) := \log\theta(e^x)$. Then, because of (4.1) and (4.2), φ satisfies the hypothesis of Lemma 4.1. Thus there exists a ψ satisfying (a)-(c) of Lemma 4.1. We set $\vartheta_2 \equiv -\psi$. Using (a)-(b) we obtain

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho}e^{-\vartheta_2(\log r)}} = 1.$$

Therefore, $\rho(r) = \rho - \vartheta_2(\log r)/\log r$ is a strong proximate order of f. □

If the order of an entire function is greater than 0, it is proved in [10] for $0 < \rho < \infty$ and [4] for $\rho = \infty$ that f has its own strong proximate order. It remains to show that any entire function f of order zero has a zero strong proximate order.

We remind that a zero strong proximate order $\rho(r)$ is a function representable in the form $\rho(r) = \vartheta(\log r)/\log r$, where ϑ is a positive concave function satisfying

$$\lim_{x\to\infty} \frac{e^{\vartheta(x)}}{x} = \infty; \quad \lim_{x\to\infty} \frac{\vartheta(x)}{x} = 0; \quad \vartheta''(x) + \vartheta'^2(x) > 0, \quad \text{for} \quad x \geq x_0 > 0. \tag{4.3}$$

Lemma 4.3 *Every transcendental entire function f of order zero has a zero strong proximate order.*

Proof. We follow the idea of Levin's proof [10, p. 39]. We write $x = \log r$ and $y = \varphi(x)$, where $\varphi(x) = \log\log M(e^x, f)$. Then φ is continuous and $\varphi(x) \to \infty$ as $x \to \infty$.

Since f is of order zero, we have

$$\limsup_{x\to\infty} \frac{\varphi(x)}{x} = 0.$$

Therefore, for arbitrary $\epsilon > 0$, the curve $y = \varphi(x)$ lies below the line $y = \epsilon x$ for sufficiently large values of x. Consider the smallest convex domain containing all the points of the curve $y = \varphi(x)$ and the positive x-axis. Denote by $y = \psi(x)$ the boundary of this domain. Existence of ψ can be justified by using Zorn's Lemma. We will use a similar argument in the proof of Theorem 1 (ii) (see page 44), and we will not repeat it here.

The function ψ satisfies the following properties:

(a) ψ is concave.

(b) $\lim_{x\to\infty} \psi(x)/x = 0$.

(c) $\varphi(x) \leq \psi(x)$.

(d) $\lim_{x\to\infty} e^{\psi(x)}/x = \infty$. This is because f is transcendental.

(e) It is easy to see that if $\psi(x) \neq \varphi(x)$, then ψ is linear in some neighborhood of x. We call a point x as an extreme point if x has no neighborhood in which ψ is linear. Then, at extreme points $\varphi(x) = \psi(x)$.

(f) There exists a sequence of extreme points tending to infinity. Otherwise, by part (e), there exists a c, $0 \leq c < \infty$, and an x_0 such that $\psi(x) = c(x-x_0)+\psi(x_0)$ for $x \geq x_0$. But then c must be 0 by part (b) and therefore ψ must be bounded. This contradicts (d).

We now construct ϑ piece by piece by joining together some smooth majorants of ψ.

Step 1. Let (l_0) be a line of support of ψ. On the line (l_0) take a point (x_0, y_0) and consider the curve

$$(l_1): \; y = c_0^{(1)} + c_1^{(1)}(x - x_0) + \log\left(x - x_0 + c_2^{(1)}\right), \qquad x \geq x_0, \tag{4.4}$$

that is tangent to the line (l_0) at the point (x_0, y_0). Here, $c_0^{(1)}, c_1^{(1)}$ and $c_2^{(1)}$ are positive parameters with $c_1^{(1)}$ initially set to be $y_0'/2$, where y_0' is the slope of the line (l_0). Then the coefficients $c_0^{(1)}$ and $c_2^{(1)}$ are determined by the condition that

(l_0) and (l_1) are tangent at the point (x_0, y_0):

$$c_2^{(1)} = \frac{1}{y_0' - c_1^{(1)}}, \qquad c_0^{(1)} = y_0 - \log \frac{1}{y_0' - c_1^{(1)}}. \tag{4.5}$$

On the curve (l_1) we have

$$y' = c_1^{(1)} + \frac{1}{x - x_0 + c_2^{(1)}}, \qquad y'' = -\frac{1}{\left(x - x_0 + c_2^{(1)}\right)^2}, \tag{4.6}$$

so that $y'' + y'^2 > 0$, when $x > x_0$.

If the abscissa x_0 is sufficiently large, then that part of the curve (l_1) lying to the right of x_0 is above the curve $y = \psi(x)$. Choosing (l_1) in this manner and then decreasing $c_1^{(1)}$ while keeping the point (x_0, y_0) fixed, we can find a $c_1^{(1)} > 0$ such that this curve touches the curve $y = \psi(x)$ from above. To see this, consider the continuous function

$$g(c_1^{(1)}, x) := c_0^{(1)} + c_1^{(1)}(x - x_0) + \log\left(x - x_0 + c_2^{(1)}\right) - \psi(x),$$

$$0 \le c_1^{(1)} \le y_0'/2, \quad x \ge x_0,$$

where $c_0^{(1)}$ and $c_2^{(1)}$ satisfy (4.5). If $c_1^{(1)} > 0$, then $g(c_1^{(1)}, x) \to \infty$ as $x \to \infty$, so that we can define

$$m(c_1^{(1)}) := \min_{x \ge x_0} g(c_1^{(1)}, x), \qquad 0 < c_1^{(1)} \le y_0'/2.$$

Clearly m is continuous and $m(y_0'/2) > 0$. Also, because of (d), for arbitrary large values of M, we have

$$\psi(x) \ge M + \log x, \qquad \text{when } x \text{ is large enough}. \tag{4.7}$$

Therefore $\lim_{x\to\infty} g(0, x) = -\infty$ and, by continuity of g, $\lim_{c_1^{(1)}\to 0} m(c_1^{(1)}) = -\infty$. Thus there exists a $c_1^{(1)}$, $0 < c_1^{(1)} < y_0'/2$, such that $m(c_1^{(1)}) = 0$, and therefore the curve (l_1) touches ψ at some point $(\tilde{x}_0, \tilde{y}_0)$. Since (l_1) contains no line segments, the touching point $(\tilde{x}_0, \tilde{y}_0)$ must be an extreme point.

This finishes the first step of the construction.

Step 2. For the second step we initially set $c_1^{(2)} = c_1^{(1)}/2$ and choose a point (x_1, y_1), $x_1 > \tilde{x}_0$, $x_1 > x_0 + 1$, on (l_1) far enough so that that part of the curve

$$(l_2): \; y = c_0^{(2)} + c_1^{(2)}(x - x_1) + \log\left(x - x_1 + c_2^{(2)}\right), \qquad x \ge x_1,$$

(this curve is tangent to the curve (l_1) at the point (x_1, y_1)), lying to the right of this point, lies above the curve $y = \psi(x)$. Then without changing the point (x_1, y_1) we decrease $c_1^{(2)}$ so that (l_2) touches the curve ψ. As in the first step we have $c_1^{(2)} > 0$.

Next we set $c_1^{(3)} = c_1^{(2)}/2$ and take a point $(x_2, y_2),\ x_2 > x_1 + 1$, on (l_2), and form a curve (l_3) etc.

Now we form ϑ from the segments of the curves $(l_0),\ (l_1), \ldots$ taken between the points of contact, i.e. $\vartheta(x) = l_j(x)$ when $x_{j-1} \le x < x_j,\ j \ge 1$ and $\vartheta(x) = l_0(x)$ when $0 < x < x_0$. Clearly, ϑ satisfies first two conditions of (4.3). Also $\vartheta(x) \ge \psi(x)$, with equality holding for a sequence of extreme points of ψ tending to infinity. Using (c) and (e) we deduce that

$$\limsup_{r\to\infty} \frac{\log M(r, f)}{e^{\vartheta(\log r)}} = 1.$$

Evidently, $\vartheta(x)$ is twice continuously differentiable and satisfies the third condition of (4.3) except at the contact points x_j. It is not difficult to see that we can smooth ϑ in such a way that all the properties mentioned above are preserved. This completes the proof of the lemma. □

We noted in Section 2.4 that admissible proximate orders form a sufficient class for characterization of the growth of entire functions that satisfy

$$\limsup_{r\to\infty} \frac{\log M(r, f)}{\log^2 r} = \infty. \tag{4.8}$$

This is evident if the order of f is greater than zero, since for $\rho > 0$, any strong proximate order is admissible. The following lemma shows that if f is of order zero and satisfies (4.8), then there exists an admissible proximate order $\rho(r) = \vartheta(\log r)/\log r$ such that (2.15) and (2.8) holds.

Lemma 4.4 *Every entire function f of order zero satisfying (4.8) has an admissible proximate order.*

Proof. The proof is similar to the proof of Lemma 4.3 with a few modifications listed below. We change property (d) mentioned in the proof of Lemma 4.3 with (d') $\limsup_{x\to\infty} e^{\psi(x)}/x^2 = \infty$.

We change (4.4) to

$$(l_1):\ \ y = c_0^{(1)} + c_1^{(1)}(x - x_0) + 2\log\left(x - x_0 + c_2^{(1)}\right), \qquad x \geq x_0,$$

so that $2\vartheta''(x) + \vartheta'^2(x) > 0$. We change (4.7) to

$$\vartheta(x_n) \geq M + 2\log x_n, \qquad \text{for some sequence } x_n.$$

$\square$

Chapter 5

Auxiliary Results

In this chapter we will state and prove some auxiliary results that we will need in the sequel.

5.1 Properties of proximate orders

Let $\rho(r)$ be a proximate order. We will write

$$V(r) = r^{\rho(r)}. \tag{5.1}$$

It follows from the definitions in Section 2.2 that if $\rho = 0$ or ∞, then

$$V(r) = e^{\vartheta(\log r)}, \tag{5.2}$$

where, ϑ is a concave function that satisfies (2.9) if $\rho = 0$, and ϑ is a convex function that satisfies (2.10) if $\rho = \infty$.

It is straightforward to check that in the case $0 < \rho < \infty$,

$$rV'(r) = (\rho + o(1))V(r). \tag{5.3}$$

Lemma 5.1 ([10], Ch.1) *Let $\rho(r)$ be a proximate order such that $\rho(r) \to \rho$, $0 \le \rho < \infty$. Let V be defined by (5.1). Then*

$$\lim_{r\to\infty} \frac{V(kr)}{V(r)} = k^{\rho}, \tag{5.4}$$

uniformly an each interval $0 < a \le k \le b < \infty$.

Proof. We have

$$\log \frac{V(kr)}{V(r)} = [\rho(kr) - \rho(r)] \log r + \rho(kr) \log k.$$

Let us assume that $0 < a \le k \le 1$. By Lagrange's theorem, there exists some c between kr and r such that

$$|\rho(kr) - \rho(r)| = (r - kr)|\rho'(c)|.$$

It is easy to check that (2.7) remains valid in the case $\rho = 0$. Therefore, for arbitrary $\epsilon > 0$ and for all sufficiently large values of r, we have

$$|\rho(kr) - \rho(r)| \log r < \epsilon \frac{r - kr}{kr \log kr} \log r \le \epsilon \left(\frac{1}{a} - 1 \right) \frac{\log r}{\log ar}.$$

Hence

$$\lim_{r \to \infty} \log \frac{V(kr)}{V(r)} = \rho \log k$$

uniformly for $0 < a \le k \le 1$.

The case $k > 1$ can be treated in a similar way. □

We will now define a function ξ that we will use frequently. Let

$$\xi(r) := r \exp \left\{ \frac{V(r)}{rV'(r)} \right\}, \qquad 0 \le \rho \le \infty. \tag{5.5}$$

Then, by (5.2) and (5.3),

$$\log \frac{\xi(r)}{r} = \frac{1}{\vartheta'(\log r)}, \qquad \rho = 0 \text{ or } \rho = \infty; \tag{5.6}$$

$$\log \frac{\xi(r)}{r} = \frac{1}{\rho} + o(1), \qquad 0 < \rho < \infty. \tag{5.7}$$

Evidently $\xi(r)$ is greater than r. But it is not too much greater and we can compare $V(\xi(r))$ and $V(r)$:

Lemma 5.2 *Let ξ be defined by (5.5). Then*

$$\limsup_{r \to \infty} \frac{V(\xi(r))}{V(r)} \le e, \qquad 0 \le \rho \le \infty. \tag{5.8}$$

Proof. We first deal with the case $0 < \rho < \infty$. Since the convergence is uniform in Lemma 5.1, it follows from (5.7) and (5.4) that the limit in (5.8) exists and equals e.

When $\rho = 0$, using that ϑ in (5.2) is concave, we deduce

$$\vartheta(\log \xi(r)) - \vartheta(\log r) = \vartheta\left(\log r + \frac{1}{\vartheta'(\log r)}\right) - \vartheta(\log r) \leq 1.$$

The proof for the case $\rho = \infty$ is longer. We first show that

$$\lim_{r\to\infty} \frac{\vartheta'(\log \xi(r))}{\vartheta'(\log r)} = 1, \qquad (\rho = \infty). \tag{5.9}$$

To see this, observe that by the second condition in (2.10), for each $\epsilon > 0$, there exists an x_ϵ such that

$$\frac{1}{y-x}\left(\frac{1}{\vartheta'(x)} - \frac{1}{\vartheta'(y)}\right) = \frac{1}{y-x}\int_x^y \frac{\vartheta''(t)}{\vartheta'^2(t)}dt < \epsilon, \qquad y > x > x_\epsilon.$$

Letting $y = \log \xi(r)$, $x = \log r$, we obtain

$$0 < \frac{1}{\log(\xi(r)/r)} \frac{\vartheta'(\log \xi(r)) - \vartheta'(\log r)}{\vartheta'(\log \xi(r))\vartheta'(\log r)} \overset{(5.6)}{=} \frac{\vartheta'(\log \xi(r)) - \vartheta'(\log r)}{\vartheta'(\log \xi(r))} < \epsilon, \quad r > r_\epsilon.$$

This implies (5.9).

To see (5.8), note that since ϑ is convex, we have

$$\vartheta(\log \xi(r)) - \vartheta(\log r) \leq \log(\xi(r)/r)\vartheta'(\log \xi(r)) = \frac{\vartheta'(\log \xi(r))}{\vartheta'(\log r)}.$$

Now (5.8) follows from this and (5.9). □

The following corollary is immediate.

Corollary 5.3 *Let ξ be defined by (5.5). Then*

$$\limsup_{r\to\infty} \frac{V(\xi(r))}{rV'(r)\log(\xi(r)/r)} \leq e. \tag{5.10}$$

Lemma 5.4 *Let $C > 0$ be a constant. Then*

$$\limsup_{r\to\infty} \frac{V'\left(r + \frac{1}{CV'(r)}\right)}{V'(r)} \leq 1. \tag{5.11}$$

Proof. For simplicity, let us write

$$R := r + \frac{1}{CV'(r)}. \tag{5.12}$$

Consider first the case $\rho = 0$. We have

$$\frac{V'(R)}{V'(r)} = \frac{\vartheta'(\log R)}{\vartheta'(\log r)} \cdot \frac{V(R)}{V(r)} \cdot \frac{r}{R} =: P_1 \cdot P_2 \cdot P_3. \tag{5.13}$$

Evidently $P_3 \leq 1$. Also, since ϑ is concave, $P_1 \leq 1$. To deal with P_2, note that

$$\begin{aligned} 0 \leq \vartheta(\log R) - \vartheta(\log r) &\leq \vartheta'(\log r)(\log R - \log r) = \vartheta'(\log r) \log\left(1 + \frac{1}{CrV'(r)}\right) \\ &\leq \frac{\vartheta'(\log r)}{CrV'(r)} = \frac{\vartheta'(\log r)}{C\vartheta'(\log r)V(r)} = \frac{1}{CV(r)} \to 0. \end{aligned}$$

Therefore,

$$P_2 = \frac{V(R)}{V(r)} = e^{\vartheta(\log R) - \vartheta(\log r)} \to 1.$$

In the case $0 < \rho < \infty$, note that $rV'(r) \to \infty$ by (5.3). Then, it follows from (5.3) and Lemma 5.1 that the limit exists in (5.11) and equals 1.

We will now deal with the case $\rho = \infty$. Let $R, P_1 - P_3$ be as in (5.12) and (5.13). It is clear that $rV'(r) \to \infty$. Therefore

$$P_3 = \frac{r}{R} \to 1.$$

Since $V(r) \to \infty$,

$$\xi(r) - r = r\left(\exp\left\{\frac{V(r)}{rV'(r)}\right\} - 1\right) \geq r\frac{V(r)}{rV'(r)} = \frac{V(r)}{V'(r)} > \frac{C}{V'(r)}$$

for sufficiently large values of r. It follows that $R < \xi(r)$ when r is large enough. Hence, using (5.9) and monotonicity of ϑ', we obtain

$$P_1 = \frac{\vartheta'(\log R)}{\vartheta'(\log r)} \to 1. \tag{5.14}$$

Finally, by the convexity of V and (5.14),

$$\frac{V(R) - V(r)}{V(R)} \leq \frac{V'(R)(R-r)}{V(R)} = \frac{\vartheta'(\log R)V(R)}{RV(R)CV'(r)} = \frac{1}{C}\frac{\vartheta'(\log R)}{\vartheta'(\log r)}\frac{r}{R}\frac{1}{V(r)} \to 0.$$

It follows that $P_2 \to 1$ and the proof of the lemma is completed. $\square$

5.2 Properties of strong proximate orders

In this section $\rho(r)$ will denote a strong proximate order. We continue to write

$$V(r) = r^{\rho(r)}. \tag{5.15}$$

In case $\rho = 0$ or $\rho = \infty$, we have

$$V(r) = e^{\vartheta(\log r)}, \tag{5.16}$$

where, ϑ is a concave function that satisfies (2.9) and (2.13) if $\rho = 0$ and ϑ is a convex function satisfying (2.10) if $\rho = \infty$. When $0 < \rho < \infty$, let us write

$$\vartheta(x) = \vartheta_1(x) - \vartheta_2(x) + \rho x, \qquad (0 < \rho < \infty),$$

where ϑ_1 and ϑ_2 are as in (2.11). Now (5.16) holds for all $0 \leq \rho \leq \infty$. Note that, by (2.12),

$$\vartheta'(x) = \rho + o(1), \qquad (0 < \rho < \infty), \tag{5.17}$$

$$\vartheta''(x) = o(1), \qquad (0 < \rho < \infty). \tag{5.18}$$

We remind that a strong proximate order is also a proximate order. Therefore, all the results proved in the previous section remain valid.

The following lemma will be useful for us.

Lemma 5.5 *Let $\rho(r)$ be a strong proximate order and V be defined by (5.15). There exists a positive constant C not depending on r such that if*

$$\left|\log \frac{R}{r}\right| \leq \frac{1}{2}\frac{V(r)}{rV'(r)}, \tag{5.19}$$

then

$$|V(R) - V(r) - \log(R/r)rV'(r)| \leq C\log^2(R/r)\frac{\left(rV'(r)\right)^2}{V(r)}. \tag{5.20}$$

Proof. Step 1. We first note that it suffices to show that there exist constants C_1, C_2 and C_3 such that

$$|\vartheta(\log R) - \vartheta(\log r) - \log(R/r)\vartheta'(\log r)| \leq C_1\log^2(R/r)\vartheta'^2(\log r), \tag{5.21}$$

$$|\vartheta(\log R) - \vartheta(\log r)| \leq C_2, \tag{5.22}$$

$$|\vartheta(\log R) - \vartheta(\log r)| \leq C_3\left|\log(R/r)\right|\vartheta'(\log r). \tag{5.23}$$

Indeed, if (5.22) and (5.23) holds, then for some constant C_4,

$$\begin{aligned}\left|e^{\vartheta(\log R)-\vartheta(\log r)}-1-(\vartheta(\log R)-\vartheta(\log r))\right| &\leq C_4\left(\vartheta(\log R)-\vartheta(\log r)\right)^2\\ &\leq C_4C_3^2\log^2(R/r)\vartheta'^2(\log r).\end{aligned}$$

Combining this with (5.21) we obtain

$$\left|e^{\vartheta(\log R)-\vartheta(\log r)}-1-\log(R/r)\vartheta'(\log r)\right| \leq (C_4C_3^2+C_1)\log^2(R/r)\vartheta'^2(\log r).$$

This is equivalent to (5.20) since $V(r)=e^{\vartheta(\log r)}$.

Step 2. We start with the case $\rho=\infty$. Assume first that $r<R$. To see that (5.21) holds, observe that $R<\xi(r)$ where ξ is defined in (5.5). Then, using (2.10), (5.9) and monotonicity of ϑ' we obtain

$$\begin{aligned}|\vartheta(\log R)-\vartheta(\log r)-\log(R/r)\vartheta'(\log r)| &= \frac{1}{2}\log^2(R/r)\vartheta''(c)\\ &\leq \log^2(R/r)\vartheta'^2(\log R)o(1)\\ &\leq \log^2(R/r)\vartheta'^2(\log r)o(1)\end{aligned}$$

for some c between r and R. Further, it follows from the convexity of ϑ, (5.9) and (5.19) that

$$\begin{aligned}|\vartheta(\log R)-\vartheta(\log r)| &\leq \log(R/r)\vartheta'(\log R)=\log(R/r)\vartheta'(\log r)(1+o(1))\\ &\leq \frac{1}{2}+o(1).\end{aligned}$$

This shows (5.22) and (5.23).

Next we assume that $r>R$. By (2.10) and monotonicity of ϑ',

$$|\vartheta(\log R)-\vartheta(\log r)-\log(R/r)\vartheta'(\log r)|\leq\log^2(R/r)\vartheta'^2(\log r)o(1).$$

Further,

$$|\vartheta(\log R)-\vartheta(\log r)|\leq\log(r/R)\vartheta'(\log r)\overset{(5.19)}{\leq}\frac{1}{2}.$$

Step 3. In the case $0<\rho<\infty$, reasoning as above and using (5.3), (5.17) and (5.18), we easily obtain (5.21)-(5.23).

Step 4. We now deal with the case $\rho = 0$. Assume first that $r < R$. Note that (2.13) implies

$$|\vartheta''(x)| < \vartheta'^2 \tag{5.24}$$

when x is large enough. Therefore (for some c, $r < c < R$),

$$\begin{aligned} |\vartheta(\log R) - \vartheta(\log r) - \log(R/r)\vartheta'(\log r)| &= \frac{1}{2}\log^2(R/r)|\vartheta''(c)| \\ &\leq \frac{1}{2}\log^2(R/r)\vartheta'^2(\log r). \end{aligned}$$

Further, it follows from (5.19) and concavity of ϑ that

$$\vartheta(\log R) - \vartheta(\log r) \leq \log(R/r)\vartheta'(\log r) \leq \frac{1}{2}.$$

Hence (5.21)-(5.23) holds when $r < R$.

Assume now $r > R$. By (2.13), when x is sufficiently large, we have

$$0 \leq \frac{-\vartheta''(x)}{\vartheta'^2(x)} < 1.$$

Hence

$$\frac{1}{\vartheta'(y)} - \frac{1}{\vartheta'(x)} = \int_x^y \frac{-\vartheta''(t)}{\vartheta'^2(t)}dt \leq (y - x), \quad y > x > x_0.$$

Letting $x = \log R$, $y = \log r$ and using (5.19), we obtain

$$\frac{1}{\vartheta'(\log r)} - \frac{1}{\vartheta'(\log R)} \leq \log\left(\frac{r}{R}\right) \leq \frac{1}{2}\,\frac{1}{\vartheta'(\log r)}.$$

This implies

$$\vartheta'(\log R) \leq 2\vartheta'(\log r). \tag{5.25}$$

Using this and (5.24), we get

$$\begin{aligned} |\vartheta(\log R) - \vartheta(\log r) - \log(R/r)\vartheta'(\log r)| &\leq \frac{1}{2}\log^2(R/r)\vartheta'^2(\log R) \\ &\leq 2\log^2(R/r)\vartheta'^2(\log r). \end{aligned}$$

Further, by the concavity of ϑ, (5.25) and (5.19),

$$|\vartheta(\log R) - \vartheta(\log r)| \leq \log(r/R)\vartheta'(\log R) \leq 2\log(r/R)\vartheta'(\log r) \leq 1.$$

This shows (5.22) and (5.23) and the proof of the lemma is completed. □

Chapter 6

Proof of Theorem 1

By a well known theorem of Hadamard, $\log M(r,f)$ is a convex function of $\log r$. We define

$$K(r,f) := \frac{d}{d\log r}\log M(r,f) = r\,\frac{d}{dr}\log M(r,f). \tag{6.1}$$

For definiteness, we take the right derivative in (6.1). Then $K(r,f)$ is well defined for all $r > 0$.

Note that, being the derivative of a convex function, $K(r,f)$ is non-decreasing. Note also that, since $M(r,f)$ is strictly increasing, $K(r,f)$ is non-negative. Estimation of $K(r,f)$ will play a fundamental role in the proofs of Theorems 1 and 2 (see Lemma 6.1 and 7.1).

Further, without loss of generality, we shall assume that $f(0) = 1$.

Lemma 6.1 *Suppose that $f \in [\rho(r), \sigma]$. Then the following inequality holds:*

$$\limsup_{r\to\infty} \frac{K(r,f)}{rV'(r)} \le e\sigma. \tag{6.2}$$

Proof. Since $K(r,f)$ is non-negative and non-decreasing, for any $R > r$, we have

$$\begin{aligned}\log M(R,f) &= \int_0^R \frac{d}{dt}\log M(t,f)\,dt = \int_0^R \frac{K(t,f)}{t}\,dt \ge \int_r^R \frac{K(t,f)}{t}\,dt \\ &\ge K(r,f)\int_r^R \frac{dt}{t} = K(r,f)\log(R/r).\end{aligned}$$

We set $R = \xi(r)$, where ξ is defined in (5.5). Then

$$\limsup_{r\to\infty} \frac{K(r,f)}{rV'(r)} \le \limsup_{r\to\infty} \frac{\log M(R,f)}{V(R)} \frac{V(R)}{rV'(r)\log(R/r)}.$$

The desired result now follows from Corollary 5.3. □

We will need the following corollary.

Corollary 6.2 *We have*

$$\lim_{r\to\infty} rV'(r) = \infty.$$

Proof. Since f is transcendental, one has

$$\lim_{r\to\infty} K(r,f) = \lim_{r\to\infty} \frac{d\log M(r,f)}{d\log r} = \infty.$$

Therefore, the result follows from Lemma 6.1. □

We will now prove part (i) of Theorem 1. Let w be a maximum modulus point of f. We define

$$\Phi_w(z) := \frac{f(w+z)}{f(w)},$$

and

$$Q(h,w) := \max_{|z|\le h} |\Phi_w(z)|.$$

Since $|f(w+z)| \le M(|w|+|z|, f)$, we have

$$Q(h,w) \le \frac{M(|w|+h,f)}{M(|w|,f)}.$$

Therefore,

$$\begin{aligned}\log Q(h,w) &\le \log M(|w|+h,f) - \log M(|w|,f) = \int_{|w|}^{|w|+h} \frac{d}{dt}\log M(t,f)dt \\ &= \int_{|w|}^{|w|+h} \frac{K(t,f)}{t}dt \le K(|w|+h,f)\int_{|w|}^{|w|+h}\frac{dt}{t} \le K(|w|+h,f)\,\frac{h}{|w|}.\end{aligned}$$

Using Lemma 6.1, we deduce that for each $\epsilon > 0$, there exists r_ϵ such that

$$\log Q(h,w) \le (e\sigma+\epsilon)(|w|+h)V'(|w|+h)\frac{h}{|w|}, \quad |w| > r_\epsilon. \tag{6.3}$$

We will now obtain a lower bound for $R(w,f)$ in terms of $Q(h,w)$. For this, we define (see [12])

$$\eta_w(z) := \frac{Q(\Phi_w(z)-1)}{Q^2-\Phi_w(z)},$$

where $Q = Q(h,w)$. Evidently, $\eta_w(0)=0$ and $|\eta_w(z)| \le 1$ when $|z| \le h$. Applying Schwarz's Lemma we obtain

$$|\eta_w(z)| \le \frac{|z|}{h}, \qquad |z| \le h.$$

Hence,

$$Q|\Phi_w(z)-1| \le \frac{|z|}{h}|Q^2-\Phi_w(z)| \le \frac{|z|}{h}(|Q^2-1|+|\Phi_w(z)-1|).$$

This implies

$$|\Phi_w(z)-1| \le \frac{(|z|/h)(Q^2-1)}{Q-|z|/h}, \qquad |z| \le h.$$

Since the right hand side is less than 1 when $|z| < h/Q$, it follows that $\Phi_w(z) \neq 0$ and therefore $f(w+z) \neq 0$ for $|z| < h/Q$. This implies

$$R(w,f) \ge \frac{h}{Q}. \tag{6.4}$$

Combining (6.3) with (6.4) we deduce

$$R(w,f) \ge h\exp\{-(e\sigma+\epsilon)(|w|+h)V'(|w|+h)\frac{h}{|w|}\}, \quad |w| > r_\epsilon.$$

Setting

$$h = \frac{1}{e\sigma V'(|w|)},$$

we obtain

$$R(w,f)V'(|w|) \ge \frac{1}{e\sigma}\exp\left\{-\frac{(e\sigma+\epsilon)}{e\sigma}\,\frac{|w|+h}{|w|}\,\frac{V'(|w|+h)}{V'(|w|)}\right\}, \ |w| > r_\epsilon.$$

Note that by Corollary 6.2,

$$\frac{|w|+h}{|w|} \to 1 \qquad \text{as } |w| \to \infty.$$

It follows, therefore, from Lemma 5.4 that

$$\liminf_{|w|\to\infty} R(w,f)V'(|w|) \ge \frac{1}{e\sigma}\exp\left\{-\left(1+\frac{\epsilon}{e\sigma}\right)\right\}.$$

Since this is true for each $\epsilon > 0$, the proof of Theorem 1.i is completed.

We proceed to prove part (ii) of Theorem 1. We denote by $W(x)$ the greatest convex minorant of $V(e^x)$. Existence of W can be deduced from Zorn's Lemma. Details are as follows:

Denote by $\mathcal{F}$ the set of all convex minorants of $V(e^x)$ on $\mathbb{R}_+$. $\mathcal{F}$ is not empty, since if $g(x) = 0, \forall x \in \mathbb{R}_+$, then $g \in \mathcal{F}$. Let $g_1 \leq g_2$ mean

$$g_1(x) \leq g_2(x), \quad \forall x \in \mathbb{R}_+.$$

Then $\leq$ defines a partial ordering on $\mathcal{F}$. Pick any chain $\mathcal{C} \subset \mathcal{F}$. Let

$$h(x) := \sup_{g \in \mathcal{C}} g(x), \quad x \in \mathbb{R}_+.$$

The function h exists since at each point $x \in \mathbb{R}_+$, $g(x)$ is bounded by $V(e^x)$. Further, since each $g \in \mathcal{C}$ is convex, h is convex. Hence, $h \in \mathcal{F}$ and is an upper bound of the chain $\mathcal{C}$. Now, since each chain $\mathcal{C} \subset \mathcal{F}$ has an upper bound, Zorn's Lemma asserts that the set $\mathcal{F}$ has a maximal element W. Next, let us show that W is unique. If W_1 and W_2 are maximal elements of $\mathcal{F}$, set $W(x) := \sup\{W_1(x), W_2(x)\},\ x \in \mathbb{R}_+$. Then $W_i \leq W,\ (i = 1, 2)$. But, by the maximality of $W_i,\ W \leq W_i,\ (i = 1, 2)$. Therefore, $W = W_1 = W_2$.

Let us denote by H, the set of all touching points of $W(x)$ and $V(e^x)$, i.e., let

$$H = \{x \in \mathbb{R}_+ : W(x) = V(e^x)\}.$$

Note that in the case $\rho = \infty$, $V(e^x)$ is convex, $W(x) = V(e^x)$ and $H = \mathbb{R}_+$.

One important property of H we will need is that it is unbounded. To see this, first note that

$$\frac{V(e^x)}{x} \to \infty \qquad \text{as } x \to \infty. \tag{6.5}$$

In the case $0 < \rho \leq \infty$, (6.5) clearly holds. When $\rho = 0$, it is the first of the assumptions in (2.9). If a point x_0 is not an element of H, i.e., if $W(x_0) < V(e^{x_0})$, then there exists a neighborhood of x_0 in which W is linear. This can be easily shown by using the fact that $W(x)$ is the *greatest* convex minorant of $V(e^x)$. Now, suppose that H is bounded. Then, by the argument above, there exists x_1 and $c,\ 0 \leq c < \infty$, such that for $x \geq x_1$,

$$W(x) = c(x - x_1) + W(x_1), \qquad x \geq x_1.$$

But, because of (6.5), this is impossible.

Evidently, $W(x)$ is differentiable on H and $W'(x) = [V(e^x)]'(x)$ for $x \in H$. Moreover, $W'(x) \to \infty$ as $x \to \infty$, $x \in H$.

We construct a sequence $\{p_n\} \to \infty$ in the following way: We set $p_1 = 1$ and choose $p_n,\ n \geq 2$, in such a way that the following conditions hold (recall the definition of $\xi(x)$ in (5.5)):

$$\text{(i) } \log p_n \in H \tag{6.6}$$

$$\text{(ii) } p_n > e^{n-1} p_{n-1} \tag{6.7}$$

$$\text{(iii) } p_n > \xi(p_{n-1}) \tag{6.8}$$

$$\text{(iv) } p_n V'(p_n) > n\, p_{n-1} V'(p_{n-1}) \tag{6.9}$$

$$\text{(v) } \frac{\log r}{V(r)} < \frac{1}{n\, p_{n-1} V'(p_{n-1})}, \qquad \text{when } r > p_n. \tag{6.10}$$

It is possible to find such a sequence since H is unbounded, $rV'(r) \to \infty$ (by Corollary 6.2) and $\log r / V(r) \to 0$ as $r \to \infty$ (by 6.5).

We set

$$f(z) = \prod_{k=1}^{\infty} \left(1 + \left(\frac{z}{p_k}\right)^{[ap_k V'(p_k)]}\right),$$

where a is some positive constant to be determined later.

Let us first show that the following asymptotic equality holds:

$$\log M(r, f) = [ap_n V'(p_n)] \log \frac{r}{p_n} + o(V(r)), \quad \text{as } r \to \infty, \quad p_n \leq r < p_{n+1}. \tag{6.11}$$

We have

$$\begin{aligned}\log M(r,f) &= \sum_{k=1}^{\infty}\log\left(1+\left(\frac{r}{p_k}\right)^{[ap_kV'(p_k)]}\right)\\ &= [ap_nV'(p_n)]\log\frac{r}{p_n}+\sum_{k=1}^{n-1}[ap_kV'(p_k)]\log\frac{r}{p_k}+\\ &\quad \sum_{k=1}^{n}\log\left(1+\left(\frac{p_k}{r}\right)^{[ap_kV'(p_k)]}\right)+\sum_{k=n+1}^{\infty}\log\left(1+\left(\frac{r}{p_k}\right)^{[ap_kV'(p_k)]}\right)\\ &=: [ap_nV'(p_n)]\log\frac{r}{p_n}+S_1+S_2+S_3.\end{aligned}$$

We will show that

$$S_1=o(V(r)),\ S_2=O(1),\ S_3=O(1),\quad r\to\infty,\ p_n\le r<p_{n+1}.$$

We start with S_3. Since $p_k/p_{n+1}\ge e^{k-1}$ for $k\ge n+2$ (by 6.7) and $p_kV'(p_k)\to\infty$ (by Corollary 6.2), we find

$$S_3=\sum_{k=n+1}^{\infty}\log\left(1+\left(\frac{r}{p_k}\right)^{[ap_kV'(p_k)]}\right)\le 1+\sum_{k=n+2}^{\infty}\left(\frac{p_{n+1}}{p_k}\right)^{[ap_kV'(p_k)]}=O(1).$$

Similarly, since $p_n/p_k\ge e^k$ for $k\le n-1$, we obtain

$$S_2=\sum_{k=1}^{n}\log\left(1+\left(\frac{p_k}{r}\right)^{[ap_kV'(p_k)]}\right)\le 1+\sum_{k=1}^{n-1}\left(\frac{p_k}{p_n}\right)^{[ap_kV'(p_k)]}=O(1).$$

Finally,

$$S_1=\sum_{k=1}^{n-1}[ap_kV'(p_k)]\log\frac{r}{p_k}\le a(\log r)p_{n-1}V'(p_{n-1})\sum_{k=1}^{n-1}\frac{p_kV'(p_k)}{p_{n-1}V'(p_{n-1})}.$$

By condition (6.9), for $k\le n-2$ we have

$$\frac{p_kV'(p_k)}{p_{n-1}V'(p_{n-1})}<\frac{1}{n-1}.$$

Therefore, using (6.10) we obtain

$$\frac{S_1}{V(r)}\le 2a\,\frac{\log r}{V(r)}\,p_{n-1}V'(p_{n-1})<\frac{2a}{n}.$$

Now, we will show that with an appropriate choice of a, f has type σ with respect to $\rho(r)$, i.e.,

$$\sigma=\limsup_{r\to\infty}\frac{\log M(r,f)}{V(r)}.$$

By (6.11) we have

$$\begin{aligned}
\limsup_{r\to\infty} \frac{\log M(r,f)}{V(r)} &= \limsup_{\substack{r\to\infty \\ r\in[p_n,p_{n+1})}} \frac{[ap_nV'(p_n)]\log(r/p_n)+o(V(r))}{V(r)} \\
&= a \limsup_{\substack{r\to\infty \\ r\in[p_n,p_{n+1}]}} \frac{p_nV'(p_n)\log(r/p_n)}{V(r)} \\
&= a\limsup_{n\to\infty} \max_{r\in[p_n,p_{n+1})} \frac{p_nV'(p_n)\log(r/p_n)}{V(r)} \\
&=: a\tilde{\sigma}.
\end{aligned}$$

Since, by condition (6.8), $\xi(p_n)\in[p_n,p_{n+1})$, it follows from Corollary 5.3 that

$$\tilde{\sigma} \geq \limsup_{n\to\infty} \frac{p_nV'(p_n)\log(\xi(p_n)/p_n)}{V(\xi(p_n))} \geq e^{-1}.$$

We proceed to find an upper bound for $\tilde{\sigma}$. We write $x=\log r,\ x_n=\log p_n$. Then

$$\tilde{\sigma} = \limsup_{n\to\infty} \max_{x\in[x_n,x_{n+1})} \frac{[V(e^x)]'(x_n)\cdot(x-x_n)}{V(e^x)}.$$

Since $x_n=\log p_n$ belongs to H, we have $[V(e^x)]'(x_n)=W'(x_n)$. Further, by the definition of W, $W(x)\leq V(e^x)$. Hence,

$$\tilde{\sigma} \leq \limsup_{n\to\infty} \max_{x\in[x_n,x_{n+1})} \frac{W'(x_n)(x-x_n)}{W(x)}.$$

It now follows from the convexity of W that $\tilde{\sigma}\leq 1$.

Now, setting $a=\sigma/\tilde{\sigma}$, we obtain that $f\in[\rho(r),\sigma]$.

To complete the proof, it remains to show that f satisfies (3.4). Observe that since f has non-negative Taylor coefficients, each $p_k\in\mathbb{R}_+$ is a maximum modulus point of f. Further, zeros of f are located at

$$p_k\exp\left(i\pi\frac{1+2j}{[ap_kV'(p_k)]}\right), \quad j=0,1,\ldots,[ap_kV'(p_k)]-1.$$

Evidently,

$$R(p_k,f)\leq \left|p_k-p_k\exp\left(\frac{i\pi}{[ap_kV'(p_k)]}\right)\right| = 2p_k\sin\frac{\pi}{2[ap_kV'(p_k)]}.$$

Using Corollary 6.2, we deduce

$$\liminf_{k\to\infty} R(p_k,f)V'(p_k) \leq \frac{\pi}{a} = \pi\frac{\tilde{\sigma}}{\sigma} \leq \frac{\pi}{\sigma}.$$

This completes the proof of Theorem 1.

Chapter 7

Proof of Theorem 2

We start by proving the following amplification of Lemma 6.1 for functions of (V, θ)-regular growth.

Lemma 7.1 *If f is of (V, θ)-regular growth, then*

$$K(r, f) = \sigma r V'(r) + O\left(r V'(r) \sqrt{\frac{\theta(r)}{V(r)}} \right).$$

Proof. We set

$$\log \frac{R}{r} := \frac{V(r)}{rV'(r)} \sqrt{\frac{\theta(r)}{V(r)}}.$$

By conditions (3.8)-(3.10), there exists a constant D such that

$$\Big| [\log M(R, f) - \log M(r, f)] - [\sigma V(R) - \sigma V(r)] \Big| \leq D\theta(r).$$

Further,

$$\log M(R, f) - \log M(r, f) = \int_r^R \frac{K(t, f)}{t} dt \geq K(r, f) \log \frac{R}{r}.$$

We deduce

$$K(r, f) \leq \frac{1}{\log(R/r)} [\sigma V(R) - \sigma V(r) + D\theta(r)].$$

Note that because of (3.9), condition (5.19) is satisfied for sufficiently large values of r. Application of Lemma 5.5 yields

$$\begin{aligned} K(r,f) &\leq \frac{1}{\log(R/r)}\left(\sigma\log(R/r)rV'(r) + C\log^2(R/r)\frac{[rV'(r)]^2}{V(r)} + D\theta(r)\right) \\ &= \sigma rV'(r) + CrV'(r)\sqrt{\frac{\theta(r)}{V(r)}} + DrV'(r)\sqrt{\frac{\theta(r)}{V(r)}}. \end{aligned}$$

For the reverse inequality we set

$$\log\frac{r}{s} := \frac{V(r)}{rV'(r)}\sqrt{\frac{\theta(r)}{V(r)}}.$$

It follows from (3.8) and monotonicity of θ that, there exists a constant E such that

$$\Big|[\log M(r,f) - \log M(s,f)] - [\sigma V(r) - \sigma V(s)]\Big| \leq E\theta(r).$$

Further,

$$\log M(r,f) - \log M(s,f) = \int_s^r \frac{K(t,f)}{t}dt \leq K(r,f)\log\frac{r}{s}.$$

Hence,

$$K(r,f) \geq \frac{1}{\log(r/s)}\left(\sigma V(r) - \sigma V(s) - E\theta(r)\right).$$

Applying Lemma 5.5, we deduce that

$$\begin{aligned} K(r,f) &\geq \frac{1}{\log(r/s)}\left(\sigma\log(r/s)rV'(r) - C\log^2(r/s)\frac{[rV'(r)]^2}{V(r)} - E\theta(r)\right) \\ &= \sigma rV'(r) - CrV'(r)\sqrt{\frac{\theta(r)}{V(r)}} - ErV'(r)\sqrt{\frac{\theta(r)}{V(r)}}. \end{aligned}$$

This completes the proof of the lemma. $\square$

We will now prove Theorem 2. Let w be a maximum modulus point of f. We define (see [12])

$$\Omega_w(z) := \frac{f(we^z)}{f(w)}e^{-K(|w|,f)z}.$$

Let

$$P(h,w) := \max_{|z|\leq h}|\Omega_w(z)|.$$

Setting $|w| = r$ and $\operatorname{Re} z = t$, we obtain

$$\log P(h, w) \leq \max_{-h \leq t \leq h} \left(\log M(re^t, f) - \log M(r, f) - tK(r, f) \right).$$

We have

$$\log M(re^t, f) - \log M(r, f) - tK(r, f) = \int_r^{re^t} [K(u, f) - K(r, f)] \frac{du}{u}$$
$$= \int_r^{re^t} [(K(u, f) - \sigma u V'(u)) - (K(r, f) - \sigma r V'(r))] \frac{du}{u} +$$
$$+ \sigma \int_r^{re^t} [uV'(u) - rV'(r)] \frac{du}{u}.$$

Application of Lemma 7.1 yields (for some constant C_1)

$$\begin{aligned}
\log P(h, w) &\leq \max_{-h \leq t \leq h} \Big| \int_r^{re^t} C_1 u V'(u) \sqrt{\frac{\theta(u)}{V(u)}} \frac{du}{u} \Big| \\
&+ \max_{-h \leq t \leq h} \Big| \int_r^{re^t} C_1 r V'(r) \sqrt{\frac{\theta(r)}{V(r)}} \frac{du}{u} \Big| \\
&+ \sigma \max_{-h \leq t \leq h} \Big| V(re^t) - V(r) - trV'(r) \Big| \\
&=: S_1 + S_2 + \sigma S_3.
\end{aligned}$$

We set

$$h = h_r = \frac{V(r)}{rV'(r)} \frac{1}{\sqrt{\theta(r) V(r)}}. \tag{7.1}$$

Let us show that $S_i = O(1),\ i = 1, 2, 3.$

By the monotonicity of θ and by condition (3.10), we have (for some constant C_2),

$$\max_{re^{-h} \leq u \leq re^h} \theta(u) \leq C_2 \theta(r).$$

Therefore, (letting $C_3 = C_1 \cdot C_2,\ C_4 = 2\, C_3$)

$$\begin{aligned}
S_1 &\leq C_3 \sqrt{\theta(r)} \max_{-h \leq t \leq h} \left| \int_r^{re^t} \frac{V'(u)}{\sqrt{V(u)}} du \right| = C_4 \sqrt{\theta(r)} \max_{-h \leq t \leq h} |\sqrt{V(re^t)} - \sqrt{V(r)}| \\
&= C_4 \sqrt{\theta(r)} \max_{-h \leq t \leq h} \left| \frac{V(re^t) - V(r)}{\sqrt{V(re^t)} + \sqrt{V(r)}} \right| \leq C_4 \sqrt{\frac{\theta(r)}{V(r)}} \max_{-h \leq t \leq h} |V(re^t) - V(r)|
\end{aligned}$$

We apply Lemma 5.5 and deduce that (for some constant C_5)

$$\begin{aligned} S_1 &\leq C_4\sqrt{\frac{\theta(r)}{V(r)}} \max_{-h\leq t\leq h}\left(|t|rV'(r)+C_5t^2\frac{(rV'(r))^2}{V(r)}\right) \\ &\leq C_4\sqrt{\frac{\theta(r)}{V(r)}}\left(h\,rV'(r)+C_5h^2\frac{(rV'(r))^2}{V(r)}\right). \end{aligned}$$

Inserting (7.1) into the above inequality we obtain

$$S_1 \leq C_4 + \frac{C_5}{\sqrt{\theta(r)V(r)}}.$$

Thus, $S_1 = O(1)$.

Clearly,

$$S_2 = C_1 hrV'(r)\sqrt{\frac{\theta(r)}{V(r)}} = C_1.$$

Finally, application of Lemma 5.5 yields

$$S_3 \leq C_6h^2\frac{(rV'(r))^2}{V(r)} = \frac{C_6}{\theta(r)}.$$

This implies $S_3 = O(1)$, since θ is nondecreasing.

We conclude that when h is as in (7.1),

$$P(h,w) = O(1),\ r = |w| \to \infty. \tag{7.2}$$

To prove the desired bound for $R(w.f)$, we argue as we did in the proof of Theorem 1. We write $P := P(h,w)$ and define

$$\xi_w(z) = \frac{P(\Omega_w(z)-1)}{P^2-\Omega_w(z)}.$$

Application of Schwarz's Lemma shows that $|\xi_w(z)| \leq |z|/h$ for $|z| \leq h$. This implies

$$P|\Omega_w(z)-1| \leq \frac{|z|}{h}|P^2-\Omega_w(z)| \leq \frac{|z|}{h}(P^2-1+|\Omega_w(z)-1|), \qquad |z|\leq h.$$

We deduce

$$|\Omega_w(z)-1| \leq \frac{|z|/h(P^2-1)}{P-|z|/h}, \qquad |z|\leq h. \tag{7.3}$$

Since, for $|z| < h/P$, right hand side of (7.3) is less than 1, we obtain

$$\Omega_w(z) \neq 0 \quad \text{for } |z| < \frac{h}{P}.$$

Therefore,

$$f(we^z) \neq 0 \quad \text{for } |z| < \frac{h}{P}.$$

We conclude that

$$R(w,f) \geq \min_{|z|=\frac{h}{P}} |w - we^z| = |w| \min_{|z|=\frac{h}{P}} |1 - e^z| = |w| \left(1 - \exp\{-\frac{h}{P(h,w)}\}\right).$$

Inserting the value of h in (7.1) to the above inequality and using (7.2), we obtain (3.14).

Chapter 8

Proof of Theorem 3

We will prove Theorem 3 for $0 < \rho < \infty$. The cases $\rho = 0$ and $\rho = \infty$ can be dealt in a similar way, therefore we will only outline their corresponding proofs.

We remind that for $0 < \rho < \infty$, a strongly admissible proximate order is a function $\rho(r) \in C^3(\mathbb{R}_+)$ representable in the form

$$\rho(r) = \rho + \frac{\vartheta_1(\log r) - \vartheta_2(\log r)}{\log r}, \qquad r \geq r_0 > 1,$$

where ϑ_j $(j = 1, 2)$, is a concave function of $C^3(\mathbb{R}_+)$ satisfying

$$\text{(a)} \quad \lim_{x\to\infty} \vartheta_j(x) = \infty, \tag{8.1}$$

$$\text{(b)} \quad \lim_{x\to\infty} \frac{\vartheta_j(x)}{x} = 0, \tag{8.2}$$

$$\text{(c)} \quad \lim_{x\to\infty} \frac{\vartheta_j''(x)}{\vartheta_j'(x)} = 0, \tag{8.3}$$

$$\text{(d)} \quad \lim_{x\to\infty} \vartheta_j'''(x) = 0. \tag{8.4}$$

With $V(r) = r^{\rho(r)}$, a straightforward calculation shows that

$$rV'(r) = (\rho + o(1))V(r), \quad r \to \infty, \tag{8.5}$$

$$r^2V''(r) = (\rho^2 - \rho + o(1))V(r), \quad r \to \infty. \tag{8.6}$$

Given $\frac{1}{3} \leq \alpha < 1$, we set

$$\beta := \frac{1+\alpha}{1-\alpha}. \tag{8.7}$$

Note that $\beta \geq 2$. We define

$$\nu^{\beta+1}(r) := rV'(r). \tag{8.8}$$

It is easy to see that $\nu(r)$ is an increasing function when r is large enough. By changing $V(r)$ on a finite interval, we may assume that $\nu(r)$ increases for all $r > 0$. Let

$$\lambda := \nu^{-1} \tag{8.9}$$

be the inverse function of ν. We set

$$f(z) := \prod_{k=1}^{\infty} \left(1 + \left(\frac{z}{\lambda(k)}\right)^{[k^\beta]}\right). \tag{8.10}$$

(i) We first show that f satisfies (3.20). This is rather simple. Indeed, since each point of the positive ray is a maximum modulus point, we have

$$R(\lambda(k), f) \leq 2\lambda(k) \sin \frac{\pi}{2[k^\beta]}.$$

Therefore,

$$\liminf_{k\to\infty} k^\beta \frac{R(\lambda(k), f)}{\lambda(k)} \leq \pi.$$

Noting that

$$k^{\beta+1} = (\nu(\lambda(k)))^{\beta+1} = \lambda(k)V'(\lambda(k)), \tag{8.11}$$

we obtain the desired result.

(ii) Now we will prove that f is of (V, θ)-regular growth. This is much more cumbersome.

Since f has nonnegative Taylor coefficients, we have

$$\begin{aligned}
\log M(r, f) &= \sum_{k=1}^{\infty} \log \left(1 + \left(\frac{r}{\lambda(k)}\right)^{[k^\beta]}\right) \\
&= \sum_{k=1}^{n} \log \left(\frac{r}{\lambda(k)}\right)^{[k^\beta]} + \sum_{k=1}^{n} \log \left(1 + \left(\frac{\lambda(k)}{r}\right)^{[k^\beta]}\right) \\
&\quad + \sum_{k=n+1}^{\infty} \log \left(1 + \left(\frac{r}{\lambda(k)}\right)^{[k^\beta]}\right) \\
&=: S_1 + S_2 + S_3, \qquad \lambda(n) \leq r < \lambda(n+1).
\end{aligned} \tag{8.12}$$

We will first show that $S_2 = O(1)$, $S_3 = O(1)$, $r \to \infty$. We begin with finding an upper bound for $(\lambda(k)/\lambda(k+1))^k$.

Using (8.5) and (8.6) we obtain

$$\nu'(r) = \frac{\nu(r)}{r}\left(\frac{\rho}{\beta+1} + o(1)\right). \tag{8.13}$$

Since ν and λ are monotonic, we deduce

$$1 = \int_{\lambda(k)}^{\lambda(k+1)} \nu'(t)dt \leq [\lambda(k+1) - \lambda(k)]\frac{\nu(\lambda(k+1))}{\lambda(k)}\left(\frac{\rho}{\beta+1} + o(1)\right).$$

This implies,

$$\frac{\lambda(k+1)}{\lambda(k)} \geq 1 + \frac{(\beta+1)/\rho + o(1)}{(k+1)}.$$

Hence,

$$\liminf_{k\to\infty} k \log \frac{\lambda(k+1)}{\lambda(k)} \geq \frac{\beta+1}{\rho}. \tag{8.14}$$

We choose C_1 and C_2 such that

$$C_1 = C_2^2 \quad \text{and} \quad e^{-\frac{\beta+1}{\rho}} < C_1 < C_2 < 1.$$

Because of (8.14), there exists k_0 such that

$$\left(\frac{\lambda(k)}{\lambda(k+1)}\right)^k < C_1 \quad \text{when } k > k_0. \tag{8.15}$$

Let $n > k_0$. We have (recall that $\beta \geq 2$)

$$S_2 = \sum_{k=1}^{n} \log\left(1 + \left(\frac{\lambda(k)}{r}\right)^{[k^\beta]}\right) \leq \sum_{k=1}^{n}\left(\frac{\lambda(k)}{\lambda(n)}\right)^k = o(1) + \sum_{k=k_0+1}^{n}\left(\frac{\lambda(k)}{\lambda(n)}\right)^k.$$

We divide the interval $[k_0+1, n]$ into two parts: Since $\lambda(k)$ increases as k increases, there exists n' (depending on n) such that

$$\lambda(k)/\lambda(n) \leq C_2, \qquad k_0 + 1 \leq k \leq n'$$

and

$$\lambda(k)/\lambda(n) \geq C_2, \qquad n' < k \leq n. \tag{8.16}$$

We have

$$S_2 \le o(1) + \sum_{k=k_0+1}^{n'} \left(\frac{\lambda(k)}{\lambda(n)}\right)^k + \sum_{k=n'+1}^{n} \left(\frac{\lambda(k)}{\lambda(n)}\right)^k =: o(1) + T_1 + T_2.$$

Evidently,

$$T_1 \le \sum_{k=k_0+1}^{n'} C_2^k \le \sum_{k=0}^{\infty} C_2^k.$$

We use backward induction to show that the following inequality holds:

$$\left(\frac{\lambda(k)}{\lambda(n)}\right)^k \le C_2^{n-k}, \qquad n'+1 \le k \le n-1.$$

The base of induction ($k = n-1$) holds because of (8.15). We assume

$$\left(\frac{\lambda(n-j)}{\lambda(n)}\right)^{n-j} \le C_2^j, \qquad j < n-(n'+1).$$

Since $n-j > n'+1$ we have, by (8.16), $\lambda(n-j)/\lambda(n) \ge C_2$. Using (8.15) we obtain

$$\begin{aligned}\left(\frac{\lambda(n-j-1)}{\lambda(n)}\right)^{n-j-1} &= \left(\frac{\lambda(n-j-1)}{\lambda(n-j)}\right)^{n-j-1} \left(\frac{\lambda(n-j)}{\lambda(n)}\right)^{n-j} \frac{\lambda(n)}{\lambda(n-j)} \\ &\le C_1 . C_2^j . \frac{1}{C_2} = C_2^{j+1}.\end{aligned}$$

Therefore,

$$T_2 = \sum_{k=n'+1}^{n} \left(\frac{\lambda(k)}{\lambda(n)}\right)^k \le \sum_{k=0}^{\infty} C_2^k.$$

Thus $S_2 = O(1)$.

Using similar arguments it is easy to show that, when $n > k_0$,

$$\left(\frac{\lambda(n+1)}{\lambda(k)}\right)^k \le C_1^{k-n-1}, \qquad k \ge n+2.$$

Hence,

$$S_3 = \sum_{k=n+1}^{\infty} \log\left(1 + \left(\frac{r}{\lambda(k)}\right)^{[k^\beta]}\right) \le \sum_{k=n+1}^{\infty} \left(\frac{\lambda(n+1)}{\lambda(k)}\right)^k \le \sum_{k=0}^{\infty} C_1^k.$$

It remains to calculate the term S_1 in (8.12). Our aim is to show that

$$S_1 = \frac{V(r)}{\beta+1} + O\left(V^\alpha(r)\right), \qquad r \to \infty, \quad \lambda(n) \le r < \lambda(n+1). \tag{8.17}$$

We write

$$\begin{aligned} S_1 &= \sum_{k=1}^{n}[k^\beta]\log\frac{r}{\lambda(k)} = \sum_{k=1}^{n}k^\beta\log\frac{r}{\lambda(k)} - \sum_{k=1}^{n}(k^\beta-[k^\beta])\log\frac{r}{\lambda(k)} \\ &=: U_1 - U_2. \end{aligned}$$

We first deal with U_1. Define

$$u_1(t) := t^\beta \log\frac{r}{\lambda(t)}.$$

By Euler-Maclaurin sum formula, we have

$$\begin{aligned} U_1 = \sum_{k=1}^{n}u_1(k) &= \int_1^n u_1(t)dt + \frac{1}{2}\left(u_1(n)+u_1(1)\right) + \frac{1}{12}\left(u_1'(n)-u_1'(1)\right) \\ &\quad - \int_1^n u_1''(t)\frac{B_2(t-[t])}{2}dt, \end{aligned} \tag{8.18}$$

where $B_2(t)$ is 2^{nd} Bernoulli polynomial. We first evaluate $\int_1^n u_1(t)dt$. We have

$$\int_1^n u_1(t)dt = \frac{n^{\beta+1}}{\beta+1}\log r - \int_1^n t^\beta \log\lambda(t)dt + O(\log r). \tag{8.19}$$

Further, writing $t=\nu(s)$ and noting that ν and λ are inverse functions, we obtain

$$\begin{aligned} \int_1^n t^\beta\log\lambda(t)dt &= \int_{\lambda(1)}^{\lambda(n)} \nu^\beta(s)\log s\ d\nu(s) \\ &= \frac{\nu^{\beta+1}(s)\log s}{\beta+1}\Big|_{\lambda(1)}^{\lambda(n)} - \frac{1}{\beta+1}\int_{\lambda(1)}^{\lambda(n)}\frac{\nu^{\beta+1}(s)}{s}ds \\ &\overset{(8.8)}{=} \frac{n^{\beta+1}\log\lambda(n)}{\beta+1} - \frac{V(\lambda(n))}{\beta+1} + O(1). \end{aligned} \tag{8.20}$$

Combining (8.19) and (8.20) we find

$$\begin{aligned} \int_1^n u_1(t)dt &= \frac{n^{\beta+1}}{\beta+1}\log\frac{r}{\lambda(n)} + \frac{V(r)}{\beta+1} - \frac{1}{\beta+1}\Big(V(r)-V(\lambda(n))\Big) \\ &\quad + O(\log r). \end{aligned} \tag{8.21}$$

We will need the following two properties (recall that $\lambda(n)\le r<\lambda(n+1)$):

$$\frac{r}{\lambda(n)} - 1 = O(\frac{1}{n}), \qquad n\to\infty, \tag{8.22}$$

and

$$\frac{V(r)}{n^{\beta+1}} \to \frac{1}{\rho} \qquad n\to\infty. \tag{8.23}$$

To see that (8.22) holds, note first that $\lambda(n) \le r < \lambda(n+1)$ implies $n = [\nu(r)]$. Also, by (8.13),

$$\frac{\lambda'(r)}{\lambda(r)} = \frac{1}{r}\left(\frac{\beta+1}{\rho} + o(1)\right). \tag{8.24}$$

Therefore,

$$\begin{aligned} \frac{r-\lambda(n)}{r} &= \frac{1}{r}\int_{[\nu(r)]}^{\nu(r)} \lambda'(t)dt = \frac{1}{r}\int_{[\nu(r)]}^{\nu(r)} \left(\frac{\beta+1}{\rho} + o(1)\right)\frac{\lambda(t)}{t}\,dt \\ &\le \frac{((\beta+1)/\rho + o(1))\,r}{nr} = O(\frac{1}{n}). \end{aligned}$$

We have (8.23) since

$$n^{\beta+1} = \nu^{\beta+1}(\lambda(n)) \le \nu^{\beta+1}(r) = V(r)(\rho+o(1)) \le \nu^{\beta+1}(\lambda(n+1)) = (n+1)^{\beta+1}.$$

To estimate the term $V(r) - V(\lambda(n))$ that appeares in (8.21), note that by (8.6), we have (for some c between $\lambda(n)$ and r),

$$\begin{aligned} V(r) - V(\lambda(n)) - (r-\lambda(n))V'(\lambda(n)) &= \frac{1}{2}\left(\frac{r}{\lambda(n)} - 1\right)^2 \lambda^2(n)V''(c) \\ &\overset{(8.22)}{=} O(\frac{1}{n^2})O(V(r)) \\ &\overset{(8.23)}{=} O\left(V^{\alpha}(r)\right). \end{aligned} \tag{8.25}$$

Finally, combining (8.11), (8.21) and (8.25) we obtain

$$\begin{aligned} \int_1^n u_1(t)dt &= \frac{n^{\beta+1}}{\beta+1}\Big[\left(\frac{r}{\lambda(n)} - 1\right) + O\Big(\left(\frac{r}{\lambda(n)} - 1\right)^2\Big)\Big] + \frac{V(r)}{\beta+1} \\ &\quad - \frac{1}{\beta+1}\Big[n^{\beta+1}\left(\frac{r}{\lambda(n)} - 1\right) + O\left(V^{\alpha}(r)\right)\Big] \\ &= \frac{V(r)}{\beta+1} + O\left(V^{\alpha}(r)\right). \end{aligned} \tag{8.26}$$

We now estimate the remaining terms that appear on the right hand side of (8.18). It is easy to see that (8.2)-(8.4) implies

$$(\log\lambda(t))^{(j)} = O\left(\frac{1}{t^j}\right), \qquad j = 1, 2. \tag{8.27}$$

Therefore, for $j = 1, 2$, we have

$$\begin{aligned} u_1^{(j)}(t) &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-j+1)} t^{\beta-j}\log\frac{r}{\lambda(t)} - \sum_{l=0}^{j-1}\binom{j}{l}(t^{\beta})^{(l)}(\log\lambda(t))^{(j-l)} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-j+1)} t^{\beta-j}\log\frac{r}{\lambda(t)} + O(t^{\beta-j}). \end{aligned} \tag{8.28}$$

Hence, using (8.22) we find

$$u_1^{(j)}(n) = O(n^{\beta-j-1}) + O(n^{\beta-j}) = O(n^{\beta-j}), \qquad j = 1, 2. \tag{8.29}$$

Also,

$$u_1(n) = n^\beta \log \frac{r}{\lambda(n)} = O(n^{\beta-1}). \tag{8.30}$$

Finally, using (8.28) we find

$$\begin{aligned}
\left| \int_1^n u_1''(t) \frac{B_2(t-[t])}{2} dt \right| &\leq D_1 \int_1^n t^{\beta-2} \log \frac{r}{\lambda(t)} dt + D_2 \int_1^n t^{\beta-2} dt \\
&= D_1 \frac{t^{\beta-1}}{\beta-1} \log \frac{r}{\lambda(t)} \Big|_1^n + \frac{D_1}{\beta-1} \int_1^n t^{\beta-1} \frac{\lambda'(t)}{\lambda(t)} dt \\
&\quad + O(n^{\beta-1}) \\
&\overset{(8.24)}{=} O(n^{\beta-1})
\end{aligned} \tag{8.31}$$

Thus, using (8.18), (8.26), (8.29), (8.30) and (8.31) we obtain

$$U_1 = \frac{V(r)}{\beta+1} + O\left(V^\alpha(r)\right).$$

Now we are going to estimate U_2. Let

$$u_2(t) := \log \frac{r}{\lambda(t)}.$$

Then

$$U_2 \leq \sum_{k=1}^n u_2(k) = \int_1^n u_2(t)dt + \frac{1}{2}\left(u_2(n) + u_2(1)\right) + \int_1^n u_2'(t)\gamma(t)dt, \tag{8.32}$$

where $\gamma(t) = t - [t] - 1/2$. We have

$$\begin{aligned}
\int_1^n u_2(t)dt &= t \log \frac{r}{\lambda(t)} \Big|_1^n + \int_1^n \frac{t\lambda'(t)}{\lambda(t)} dt \\
&\overset{(8.24)}{=} n \log \frac{r}{\lambda(n)} + \int_1^n \left(\frac{\beta+1}{\rho} + o(1) \right) dt + O(\log r) \\
&\overset{(8.22)}{=} O(n).
\end{aligned} \tag{8.33}$$

Thus, using (8.22), (8.27), (8.32) and (8.33) we obtain (recall that $\beta \geq 2$)

$$U_2 = O(n) = O(n^{\beta-1}) = O\left(V^\alpha(r)\right).$$

This shows (8.17) and the proof of Theorem 3, for the case $0 < \rho < \infty$, is completed.

When $\rho = 0$ or $\rho = \infty$ we define ν, λ and f as in (8.8)-(8.10). In the case $\rho = \infty$ to prove that f is of (V, θ)-regular growth we first write a formula similar to (8.12) and then show that

$$S_1 = \frac{V(r)}{\beta+1} + O\Big(\frac{V^\alpha(r)}{(\vartheta'(\log r))^{1-\alpha}}\Big), \quad S_2 = O\Big(\frac{V^\alpha(r)}{(\vartheta'(\log r))^{1-\alpha}}\Big), \quad S_3 = O(1).$$

In the estimation of S_1 we again use Euler-Maclaurin sum formula (8.18), but now it is necessary to change (8.27) with

$$(\log \lambda(t))^{(j)} = O\left(\frac{1}{t^j\, \vartheta'(\log \lambda(t))}\right), \qquad j = 1, 2, \tag{8.34}$$

which follows from (2.10) and (2.17).

In the case $\rho = 0$ estimate (8.34) still holds because of (2.15) and (2.18). Using this one can show that

$$S_1 = \frac{V(r)}{\beta+1} + O\Big(\frac{V^\alpha(r)}{(\vartheta'(\log r))^{1-\alpha}}\Big), \quad S_2 = O(1), \quad S_3 = O(1).$$

Chapter 9

Proof of Theorem 4

The proof of Theorem 4 is based on a phenomenon discovered by Hayman [8] in 1966: for any transcendental entire function f, one has

$$\limsup_{r\to\infty}\left(\frac{d}{d\log r}\right)^2 \log M(r,f) \geq C_0,$$

where $C_0 > 0$ is an absolute constant. In 1974, Boichuk and Goldberg [1] proved that, for functions f with non-negative Taylor coefficients, the best possible value of C_0 is $\frac{1}{4}$. I.Ostrovskii found another proof based on a formula due to Rosenbloom [17]. This proof has been included in [1]. We will use the idea of that proof for getting a description of the set where

$$\left(\frac{d}{d\log r}\right)^2 \log M(r,f) \geq \epsilon^2, \quad 0 < \epsilon < \frac{1}{2}.$$

The corresponding Lemma 9.5 below plays a crucial role in the proof of Theorem 4.

We begin by proving some lemmas.

Lemma 9.1 *If $\varphi \in \mathcal{B}$ and $w(x) = \varphi(e^x)$, then*

$$\lim_{x\to\infty} w''(x) = 0.$$

Proof. Since w has the form (3.26), we have

$$w''(x) = \frac{w(x)}{x^2}\Big(\lambda^2 - \lambda + (2\lambda - 1)(\vartheta_1 - \vartheta_2)'(\log x) + (\vartheta_1 - \vartheta_2)'^2(\log x) + (\vartheta_1 - \vartheta_2)''(\log x)\Big). \tag{9.1}$$

Concavity of each ϑ_j, $(j = 1, 2)$ and second condition of (2.12) imply that

$$\vartheta_j'(x) \to 0, \quad x \to \infty, \quad (j = 1, 2). \tag{9.2}$$

Using this and third condition of (2.12) we find

$$w''(x) = x^{\lambda-2} e^{(\vartheta_1-\vartheta_2)(\log x)}\Big(\lambda^2 - \lambda + (2\lambda - 1)(\vartheta_1 - \vartheta_2)'(\log x) + o\left((\vartheta_1 + \vartheta_2)'(\log x)\right)\Big). \tag{9.3}$$

By the second condition of (2.12), for each $\alpha > 0$, $(\vartheta_j(x) - \alpha x) \to -\infty$ as $x \to \infty$. This implies

$$\frac{e^{\vartheta_j(\log x)}}{x^\alpha} \to 0, \quad x \to \infty \quad (\alpha > 0). \tag{9.4}$$

Now, for $1 \le \lambda < 2$, the desired result follows from (9.2)-(9.4). For $\lambda = 2$, we take into account that $\vartheta_1 \equiv 0$ and ϑ_2 satisfies the first condition of (2.12). $\square$

Lemma 9.2 *If $\varphi \in \mathcal{B}$ and $w(x) = \varphi(e^x)$, then*

$$\lim_{x\to\infty} \frac{w''(kx)}{w''(x)} = k^{\lambda-2},$$

uniformly on each interval $0 < a \le k \le b < \infty$.

Proof. By (9.3) we have

$$\begin{aligned} \frac{w''(kx)}{w''(x)} &= k^{\lambda-2} \exp\left\{(\vartheta_1 - \vartheta_2)(\log kx) - (\vartheta_1 - \vartheta_2)(\log x)\right\} \\ &\quad \times \frac{\lambda^2 - \lambda + (2\lambda - 1)(\vartheta_1 - \vartheta_2)'(\log kx) + o\left((\vartheta_1 + \vartheta_2)'(\log kx)\right)}{\lambda^2 - \lambda + (2\lambda - 1)(\vartheta_1 - \vartheta_2)'(\log x) + o\left((\vartheta_1 + \vartheta_2)'(\log x)\right)} \end{aligned}$$

Using (2.12) (and, for $\lambda = 1$, that $\vartheta_2 \equiv 0$) we get the assertion of the lemma. $\square$

Lemma 9.3 *If $\varphi \in \mathcal{B}$ and $w(x) = \varphi(e^x)$, then*

$$\lim_{x\to\infty} \left(x - \frac{w(x)}{w'(x)}\right) = \infty.$$

Proof. From (3.26) it follows

$$x - \frac{w(x)}{w'(x)} = \frac{(\lambda-1)x + x(\vartheta_1 - \vartheta_2)'(\log x)}{\lambda + (\vartheta_1 - \vartheta_2)'(\log x)}.$$

For $\lambda > 1$, assertion of the lemma follows from (9.2). For $\lambda = 1$, we take into account that $\vartheta_2 \equiv 0$ and note that it suffices to show that $\lim_{x\to\infty} x\,\vartheta_1'(\log x) = \infty$ or, equivalently,

$$\lim_{t\to\infty} e^t \vartheta_1'(t) = \infty. \tag{9.5}$$

By the third condition of (2.12), $e^t\vartheta_1'(t)$ is non-decreasing for sufficiently large values of t. Therefore, if (9.5) does not hold, then $\vartheta_1'(t) = O(e^{-t}),\ t\to\infty$. This contradicts the first condition of (2.12). □

Lemma 9.4 *It suffices to prove Theorem 4 under the additional assumption that f has nonnegative Taylor coefficients.*

Proof. We derive the lemma from the following theorem of Erdös and Kövari [6]:

Theorem. ([6]) *Let f be an entire function. There exists an entire function $\hat{f}$ with nonnegative Taylor coefficients such that*

$$\frac{1}{6} \le \frac{M(r,f)}{M(r,\hat{f})} \le 3, \quad r \ge 0.$$

By this theorem, we have $\log M(r,f) = \log M(r,\hat{f}) + O(1)$. Therefore f satisfies (3.27) (with "o" or "O") if and only if so does $\hat{f}$. □

Let f be an entire transcendental function with nonnegative Taylor coefficients,

$$f(z) = \sum_{k=0}^{\infty} d_k z^k, \quad d_k \ge 0, \ \ k = 0,1,2,\ldots \tag{9.6}$$

Then $M(r,f) = f(r)$. We set

$$\phi(x) = f(e^x), \quad \alpha(x) = \left(\log\phi(x)\right)', \quad \beta(x) = \left(\log\phi(x)\right)'',$$

The function $\alpha(x)$ is nonnegative, increasing and tends to $+\infty$ as $x \to +\infty$. Pick $\epsilon \in (0, 1/2)$. The equations

$$\alpha(a_k) = k + \epsilon, \ \ \alpha(b_k) = k + 1 - \epsilon, \quad k = k_0, k_0+1, k_0+2, \ldots, \tag{9.7}$$

where $k_0 = [\alpha(0)] + 1$, uniquely determine positive numbers

$$a_{k_0} < b_{k_0} < a_{k_0+1} < b_{k_0+1} < \cdots < a_k < b_k <\to \infty \quad \text{as} \quad k \to \infty.$$

Lemma 9.5 *Let f be a transcendental entire function with nonnegative Taylor coefficients. In the notations introduced above we have*

$$\beta(x) \geq \epsilon^2 \quad \textit{for} \;\; x \in A_\epsilon := \bigcup_{k=k_0}^{\infty} [a_k, b_k].$$

Proof. We have

$$\phi(x) = f(e^x) = \sum_{k=0}^{\infty} d_k e^{kx}, \quad d_k \geq 0, \;\; k = 0, 1, 2, \ldots.$$

The following formula is due to Rosenbloom [17]:

$$\beta(x) = \sum_{k=0}^{\infty} (k - \alpha(x))^2 \frac{d_k e^{kx}}{\phi(x)}. \tag{9.8}$$

Its proof is a direct calculation:

$$\begin{aligned}
\alpha(x) &= \frac{\phi'(x)}{\phi(x)} = \sum_{k=0}^{\infty} \frac{k d_k e^{kx}}{\phi(x)}; \\
\beta(x) &= \frac{\phi''(x)}{\phi(x)} - \left(\frac{\phi'(x)}{\phi(x)}\right)^2 = \sum_{k=0}^{\infty} \frac{k^2 d_k e^{kx}}{\phi(x)} - \alpha(x) \sum_{k=0}^{\infty} \frac{k d_k e^{kx}}{\phi(x)} \\
&= \sum_{k=0}^{\infty} (k - \alpha(x))^2 \frac{d_k e^{kx}}{\phi(x)}.
\end{aligned}$$

Since

$$\min_{k=0,1,2,\ldots} |k - \alpha(x)| \geq \varepsilon,$$

for $x \in A_\epsilon$, the assertion of the lemma follows. □

Lemma 9.6 *Let $\varphi \in \mathcal{B}$ and $w(x) = \varphi(e^x)$. In the notations above, there does not exist an entire function f with non-negative Taylor coefficients such that*

$$\alpha(x) - w'(x) = o(1), \quad x \to \infty. \tag{9.9}$$

Proof. Assume there exists such an entire function f. By (9.7) and (9.9) we have

$$1-2\epsilon = \alpha(b_k)-\alpha(a_k) = w'(b_k)-w'(a_k)+o(1) = w''(c_k)(b_k-a_k)+o(1),$$

for some $c_k \in (a_k, b_k)$. This implies (by Lemma 9.1),

$$(b_k-a_k)\to\infty \quad \text{as } k\to\infty.$$

On the other hand, by Lemma 9.1, there exists k_1 such that

$$(\alpha(x)-w'(x))' = \beta(x)-w''(x) \geq \frac{\epsilon^2}{2}, \quad x\in(a_k,b_k), \quad k\geq k_1. \tag{9.10}$$

This shows that the derivative of $(\alpha(x)-w'(x))$ has a positive lower bound on intervals of arbitrarily large length. This contradicts (9.9). □

We will now prove part (i) of Theorem 4. In what follows, we agree to denote by C positive constants.

We assume that there exists an entire function f with nonnegative Taylor coefficients which satisfies (3.27). We will show that

$$\alpha(x)-w'(x) = o(1), \quad x\to\infty,$$

which contradicts Lemma 9.6.

We write

$$\log\phi(x)-w(x) =: \tau(x),$$

where

$$|\tau(x)| \leq \epsilon(x)\min\left(\frac{1}{w''(x)}, x\right), \qquad \epsilon(x)\downarrow 0.$$

By Hadamard's three circles theorem, $\log\phi$ is convex. Therefore,

$$\alpha(x)(z-x)-[\log\varphi(z)-\log\varphi(x)] \leq 0, \qquad z>0,\ x>0.$$

Subtracting from both sides $w'(x)(z-x)-[w(z)-w(x)]$, we obtain

$$[\alpha(x)-w'(x)](z-x)-[(\log\varphi(z)-w(z))-(\log\varphi(x)-w(x))] \leq$$

$$\leq w(z)-w(x)-w'(x)(z-x) = \frac{1}{2}w''(c)(z-x)^2,$$

with some c between x and z. We restrict ourselves to z lying in the interval $|z-x| \le x/2$. Then by Lemma 9.2 there exists a positive constant C such that

$$\begin{aligned}[\alpha(x) - w'(x)](z-x) &\le 2 \max_{|t-x|\le x/2} |\tau(t)| + Cw''(x)(z-x)^2 \\ &\le C\epsilon(\frac{x}{2}) \min\left(\frac{1}{w''(x)}, x\right) + Cw''(x)(z-x)^2. \quad (9.11)\end{aligned}$$

If $x < z \le 3x/2$, then it follows that

$$\alpha(x) - w'(x) \le \frac{C\epsilon(\frac{x}{2}) \min\left(\frac{1}{w''(x)}, x\right)}{z-x} + Cw''(x)(z-x).$$

Setting

$$z - x = \sqrt{\epsilon(\frac{x}{2})} \min\left(\frac{1}{w''(x)}, x\right) \le \frac{x}{2}, \qquad x > x_0,$$

we obtain

$$\begin{aligned}\alpha(x) - w'(x) &\le C\sqrt{\epsilon(\frac{x}{2})} + Cw''(x)\sqrt{\epsilon(\frac{x}{2})} \min\left(\frac{1}{w''(x)}, x\right) \\ &= C\sqrt{\epsilon(\frac{x}{2})} + C\sqrt{\epsilon(\frac{x}{2})} \min(1, xw''(x)) \\ &\le C\sqrt{\epsilon(\frac{x}{2})} + C\sqrt{\epsilon(\frac{x}{2})} = o(1).\end{aligned}$$

If $x/2 \le z < x$, then it follows from (9.11) that

$$\alpha(x) - w'(x) \ge \frac{C\epsilon(\frac{x}{2}) \min\left(\frac{1}{w''(x)}, x\right)}{z-x} + Cw''(x)(z-x).$$

By setting

$$x - z = \sqrt{\epsilon(\frac{x}{2})} \min\left(\frac{1}{w''(x)}, x\right),$$

we obtain

$$\alpha(x) - w'(x) \ge -\left(C\sqrt{\epsilon(\frac{x}{2})} + C\sqrt{\epsilon(\frac{x}{2})}\right) = o(1).$$

This completes the proof of part (i) of Theorem 4.

We will now prove part (ii) of Theorem 4. We begin with showing that there exists x_0 such that $w'(x)$ strictly increases to ∞ for $x > x_0$. Firstly, it follows from (9.3) that, $w''(x) > 0$, for sufficiently large values of x (for $\lambda = 1$, we take into account that $\vartheta_2 \equiv 0$). Further, direct calculation shows

$$w'(x) = x^{\lambda-1} e^{(\vartheta_1 - \vartheta_2)(\log x)} \{\lambda + (\vartheta_1 - \vartheta_2)'(\log x)\}.$$

Therefore, for $1 < \lambda \leq 2$, it follows from (9.2) and (9.4) that $w'(x) \to \infty$. For $\lambda = 1$, taking into account that $\vartheta_2 \equiv 0$, first condition of (2.12) yields $w'(x) \to \infty$. We conclude that there exists unique $x_n \to \infty$ such that

$$w'(x_n) = n, \qquad n = n_0, n_0 + 1, \dots \tag{9.12}$$

We define positive numbers $c_n,\ n \geq n_0$, by the equation

$$\log c_n + n x_n = w(x_n). \tag{9.13}$$

Since $w(x)$ is strictly convex for $x > x_{n_0}$, each of the lines

$$y_n(x) := \log c_n + nx, \qquad n \geq n_0,$$

lies below the curve $y = w(x)$ and is tangent to it at the point x_n.

We set

$$f(z) = \sum_{n=n_0}^{\infty} c_n z^n.$$

Since

$$\frac{\log c_n}{n} = -x_n + \frac{w(x_n)}{w'(x_n)},$$

it follows from Lemma 9.3 that $\lim_{n\to\infty} |c_n|^{1/n} = 0$ and hence f is an entire function.

We denote by $\mu(r, f)$ the *maximum term* of f. That is,

$$\log \mu(e^x, f) = \sup_{n_0 \leq n < \infty} (\log c_n + nx).$$

The graph of $\log \mu(e^x, f)$ is a polygonal line lying below the graph of $w(x)$ and touching the latter only for $x = x_n$. Thus, there exists a $\sigma_n,\ x_n < \sigma_n < x_{n+1},\ n \geq (n_0 + 1)$, such that

$$\log \mu(e^x, f) = \log c_n + nx, \qquad \sigma_{n-1} \leq x \leq \sigma_n. \tag{9.14}$$

We shall prove the following:

$$w(x) - \log \mu(e^x, f) = O\left(\min\left(\frac{1}{w''(x)}, x\right)\right), \quad x \to \infty, \tag{9.15}$$

$$\log M(e^x, f) - \log \mu(e^x, f) = O(1), \quad x \to \infty, \tag{9.16}$$

from which Theorem 4 (ii) readily follows.

We will first show (9.15). Observe that $\log \mu(e^x, f)$ is differentiable on $\mathbb{R}_+$, except at the points σ_n, $n \geq n_0$. For $x > x_{n_0}$, we have

$$\left|\frac{d}{dx}(w(x) - \log \mu(e^x, f))\right| \leq 1, \quad x \neq \sigma_n, \ n \geq n_0.$$

The integration from x_{n_0} to x gives

$$0 \leq w(x) - \log \mu(e^x, f) \leq x. \tag{9.17}$$

Further, by Lemma 9.2, there exists a constant $C_* > 1$ such that

$$\frac{1}{C_*} \leq \frac{w''(x_1)}{w''(x_2)} \leq C_* \qquad \text{for} \quad \frac{1}{e} \leq \frac{x_1}{x_2} \leq e, \quad x_1, x_2 \geq x_{n_0}. \tag{9.18}$$

Fix a constant D satisfying

$$D > \max\left(2C_* e, \ \frac{1}{2}C_*^3\right). \tag{9.19}$$

Assume now that (9.15) is not correct. Then, since (9.17) holds, there exists $y > ex_{n_0}$ such that

$$w(y) - \log \mu(e^y, f) \geq \frac{D}{w''(y)}. \tag{9.20}$$

Using (9.17), we obtain $w''(y) \geq D/y$. Then

$$w''(t) \geq \frac{w''(y)}{C_*} \geq \frac{D}{C_* y} \geq \frac{D}{C_* et}. \tag{9.21}$$

for all t, $y/e \leq t \leq ey$.

Let n be such that $x_n \leq y < x_{n+1}$. It follows from (9.21) and (9.19) that

$$w'(ey) - w'(y) = \int_y^{ey} w''(t)dt \geq \frac{D}{C_* e} > 2.$$

This implies $ey \geq x_{n+2}$. Similarly, we find that $w'(y) - w'(y/e) > 2$ and therefore $y/e \leq x_{n-1}$. Hence we have

$$\frac{y}{e} < \sigma_{n-1} < x_n < \sigma_n < x_{n+1} < \sigma_{n+1} < ey.$$

There are two cases: Either $x_n \leq y < \sigma_n$ or $\sigma_n \leq y < x_{n+1}$. If $x_n \leq y < \sigma_n$, then by (9.13), (9.14) and (9.18),

$$\begin{aligned} w(y) - \log \mu(e^y, f) &= w(y) - \log c_n - ny \\ &= w(y) - w(x_n) - n(y - x_n) \\ &= \frac{1}{2} w''(c)(y - x_n)^2 \leq \frac{C_*}{2} w''(y)(y - x_n)^2, \end{aligned} \tag{9.22}$$

where $c \in (x_n, \sigma_n)$. Moreover,

$$1 \geq w'(y) - w'(x_n) = \int_{x_n}^{y} w''(t)dt \geq \int_{x_n}^{y} \frac{w''(y)}{C_*} dt = (y - x_n) \frac{w''(y)}{C_*}. \tag{9.23}$$

Combining (9.22) and (9.23), we find that

$$w(y) - \log \mu(e^y, f) \leq \frac{C_*^3}{2w''(y)}. \tag{9.24}$$

Similar reasoning shows that (9.24) also holds in the case $\sigma_n \leq y < x_{n+1}$. But this contradicts (9.19) and (9.20).

It remains to show that (9.16) holds. Noting that $w''(x) \to 0$ (by Lemma 9.1) and $w'(x_{n+1}) - w'(x_n) = 1$ (by 9.12), we conclude that $(x_{n+1} - x_n) \to \infty$ as $n \to \infty$. Therefore, there exists an $n_1 \geq n_0$ such that

$$x_{n+1} - x_n \geq 1, \qquad n \geq n_1.$$

Take any $x \geq x_{n_1}$. Let n be such that $x_n \leq x < x_{n+1}$. Then, by (9.13), for $n_1 \leq k \leq n - 1$,

$$\begin{aligned} \log \frac{c_k e^{kx}}{c_n e^{nx}} &= w(x_k) - w(x_n) - k(x_k - x_n) + (n - k)(x_n - x) \\ &\leq -\int_{x_k}^{x_n} (w'(t) - k)dt \leq -\int_{x_{k+1}}^{x_n} (w'(t) - k)dt \leq -\int_{x_{k+1}}^{x_n} 1\, dt \\ &\leq -(n - k - 1). \end{aligned}$$

Likewise, for $k \geq n+2$,

$$\begin{aligned}
\log \frac{c_k e^{kx}}{c_{n+1}e^{(n+1)x}} &\leq \int_{x_{n+1}}^{x_k} (w'(t)-k)dt \leq \int_{x_{n+1}}^{x_{k-1}} (w'(t)-k)dt \leq \int_{x_{n+1}}^{x_{k-1}} (-1)\, dt \\
&\leq -(k-n-2).
\end{aligned}$$

Thus, for $x_n \leq x < x_{n+1}$,

$$\begin{aligned}
M(e^x, f) &= \sum_{k=n_0}^{\infty} c_k e^{kx} \\
&= \sum_{k=n_0}^{n_1} c_k e^{kx} + \sum_{k=n_1+1}^{n-1} c_k e^{kx} + c_n e^{nx} + c_{n+1}e^{(n+1)x} + \sum_{k=n+2}^{\infty} c_k e^{kx} \\
&\leq \mu(e^x, f) \left(\sum_{k=n_0}^{n_1} \frac{c_k e^{kx}}{c_n e^{nx}} + \sum_{k=n_1+1}^{n-1} \frac{c_k e^{kx}}{c_n e^{nx}} + 2 + \sum_{k=n+2}^{\infty} \frac{c_k e^{kx}}{c_{n+1}e^{(n+1)x}} \right) \\
&\leq \mu(e^x, f) \left(O(1) + \sum_{k=n_1+1}^{n-1} e^{-(n-k-1)} + \sum_{k=n+2}^{\infty} e^{-(k-n-2)} \right) \\
&\leq C\mu(e^x, f).
\end{aligned}$$

Hence, (9.16) holds and the proof of Theorem 4 is completed.

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