WARRANTY COST ANALYSIS UNDER IMPERFECT REPAIR

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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Abstract

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M.S. in Industrial Engineering

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Increasing market competition forces manufacturers to offer extensive warranties. Faced with the challenge of keeping the associated costs under control, most companies seek efficient rectification strategies. In this study, we focus on the repair strategies with the intent of minimizing the manufacturer's expected warranty cost expressed as a function of various parameters such as product reliability, structure of the cost function and the type of the warranty contract. We consider both one- and two-dimensional warranties, and use quasi renewal processes to model the product failures along with the associated repair actions. We propose static, improved and dynamic repair policies, and develop representative cost functions to evaluate the effectiveness of these alternative policies. We consider products with different reliability structures under the most commonly observed types of warranty contracts. Experimental results indicate that the dynamic policy generally outperforms both static and improved policies on highly reliable products, whereas the improved policy is the best performer for products with low reliability. Although, the increasing number of factors arising in the analysis of two-dimensional policies renders generalizations difficult, several insights can be offered for the selection of the rectification action based on empirical evidence.

Keywords: Imperfect repair, quasi renewal processes, two-dimensional warranty, warranty cost, numerical methods

Özet

NOKSAN ONARIM ALTINDA GARANTİ MALİYETİ ANALİZİ

Gülay Samatlı
Endüstri Mühendisliği Yüksek Lisans
Tez Yöneticisi: Yardımcı Doçent Mehmet Rüştü Taner
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Artmakta olan pazar rekabeti, üreticileri genişletilmiş garantiler önermeye zorlamaktadır. Garantiyle ilgili maliyetleri kontrol altında tutmakla karşı karşıya kalan çoğu firma, verimli düzeltme stratejileri aramaktadır. Bu çalışma, ürün güvenilirliği, maliyet fonksiyon yapısı ve garanti sözleşmesi gibi bir takım değişik parametrelerle açıklanan üreticinin beklenen garanti maliyetini en küçültmek amacıyla farklı onarım stratejileri üzerinde odaklanmaktadır. Ürün bozulmasıyla ilgili onarım faaliyetlerini modellerken hem bir hem de iki boyutlu garantileri göz önünde bulunduruyor ve yenilenimsi süreç yaklaşımını kullanıyoruz. Alternatif onarım politikalarını değerlendirmek için, statik, iyileştirilmiş ve dinamik onarım politikalarını öneriyor, ve hem bir hem de iki boyutlu garantiler için temsili maliyet fonksiyonları geliştiriyoruz. Farklı güvenilirlik yapılarına sahip ürünleri en yaygın olarak gözlenen garanti sözleşme çeşitleri altında ele alıyoruz. Deneysel sonuçlar yüksek güvenilirliğe sahip ürünler için dinamik politikaların genel olarak hem statik hem de iyileştirilmiş politikalara baskın geldiğini göstermektedir, iyileştirilmiş politika ise genelde düşük güvenirliliğe sahip ürünler için en iyi alternatif olarak öne çıkmaktadır. Her ne kadar iki boyutlu politikaların analizindeki artan etkenler genellemeyi zorlaştırsa da, deneysel sonuçlara dayanarak düzeltme stratejileri seçmede çeşitli bilgiler önerebilmekteyiz.

Anahtar sözcükler: Noksan onarım, yenilenimsi süreçler, iki boyutlu garantiler, garanti maliyeti, sayısal yöntemler

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Chapter 1

INTRODUCTION

A warranty is a contract made by the seller to the buyer that specifies the compensation type for a given product in the event of failure. It plays an important role to protect the consumers` interest especially for the complex products such as automobiles or electronic devices. Many consumers may be unable to evaluate the performance of these products since they do not have enough technical knowledge. Similarly, if the product related characteristics of different brands are nearly identical, consumers have difficulty deciding which one is better. So, the post-sale characteristics such as warranty, service, maintenance, and parts availability, become important in purchasing decisions. When consumers have difficulty in selecting a product, warranty is used as a signal of quality/reliability. That is, customers usually perceive a product with a longer warranty period as more reliable. Additionally, warranty reduces consumer's dissatisfaction in case of a failure through a reimbursement by the manufacturer. The type and terms of the reimbursement are specified in the warranty contract. Thus, warranty functions as a marketing tool that helps to evaluate products and differentiate among them in the competitive environments.

In addition to the protection for the consumers, warranty also provides protection for the manufacturer. It provides the guidelines for the proper use of the products by defining the usage conditions. So, it reduces excessive claims about the product and possibility of lawsuits caused by misuse of the product. In this way, it provides cost savings to the manufacturer. At the same time, it protects the manufacturer's reputation.

Warranty has also an important role as a promotional device for the manufacturer. Since longer warranty gives a message that the product performance is good, it can be a good advertising tool like price and other product characteristics. This method is very effective especially for a new product that does not exist in the market because consumers are generally uncertain about the new product performance. Although the level of uncertainty decreases when performance information about the product is spread, the dissemination of this information usually takes some time, and it may be desirable to take certain precautions to avoid low sales early on. Sales may be raised by eliminating the risk related to products, and warranty plays an important role to reduce this risk.

On the negative side, offering warranty may result in additional costs to the manufacturer over the warranty period due to such expenditures as labor cost and repair or replacement cost in case of a failure. Although, warranty increases manufacturer's total cost, it may increase sales when it is used as a marketing tool and so it may still provide an increase in profit. The magnitude of the additional cost may depend on product characteristics, warranty terms and consumers' usage patterns. The additional profit, on the other hand, depends on competitors' product characteristics such as price and performance as well as warranty terms offered for competitors' products. While assessing the benefit of the warranty, the additional cost should be compared with the expected profit. To compare the cost and profit, a detailed analysis related to cost parameters, warranty compensation and limits should be done. After the analysis, if the expected profit gained by offering warranty is larger than the additional cost, then it may be considered rational to offer warranty.

Warranty policies are defined in several ways in regards to their certain characteristics. For example, regarding the compensation types, there are two basic types of policies: the free replacement warranty (FRW) and the pro-rata warranty (PRW). In the FRW, the cost of the repair or replacement of the failed product is reimbursed by the manufacturer at no cost to the buyer, whereas in the PRW, the buyer and the manufacturer share the cost of repair or replacement. The manufacturer's responsibility in PRW is determined based on some non-increasing function of product age. FRW applies to any kind of repairable and non-repairable product, but PRW usually applies to products whose performance is affected by age, such as accumulator. In addition, hybrid warranties can be derived by combining the FRW and PRW policies.

Examples for structural characterization of warranties can be one- or two-dimensional policies. In one-dimensional warranty policies, failure models are characterized on a single scale. The scale is usually age of the product or the amount of usage. Whereas, in the two-dimensional policies, warranty is indexed on two scales: usually one representing the usage and the other age.

Another aspect of warranty analysis relates to the extent of repair after failure. There exist several repair types but the most widely used ones in the literature are perfect (as good as new), minimal and imperfect repair. In the *perfect repair* type, the failed product is brought to the same condition as a new product after the repair. On the other hand, if a repair brings the product to a working state without changing its failure rate, then it is said to be *minimal*. In contrast to the minimal repair, if the repair action changes the failure rate of the product, then it is called *imperfect repair*. An imperfect repair can lower or increase the failure rate of the product after the repair action. A repair action that lowers the failure rate is essentially an improvement that brings the product to a better than new state. In the literature, for repairable products, repair action is often modeled with perfect or minimal repair, but most of repair actions do not fall into these two categories. For instance, perfect repair may not be practical especially for expensive products. On the other hand, minimal repairs generally are appropriate for multi-component products where the product failure occurs because of a component failure, and the rectification of this component brings the product to an operational state. In many realistic situations, the repair action brings the product to an intermediate state between perfect and minimal repair. To overcome this problem, several imperfect repair models such as a combination of perfect and minimal repair and virtual age models are derived.

In this study, we examine the manufacturer's total expected warranty cost under different extents of imperfect repair for products with the different levels of reliability on the expected warranty cost. The key factor that motivates the use of imperfect repair is that it is more realistic and practical than perfect and minimal repair in most cases. Our warranty policies are one- and two-dimensional free replacement warranties. We propose a representative cost function which depends on the degree of repair. In the analysis part, firstly, we deal with one-dimensional warranty policies. In the one-dimensional analysis, we consider the static repair policies, in which the repair action is done at the same level after each failure, as well as the improved repair policy. In the improved policy, the failed product is replaced by an improved one after the first failure. In addition to these policies, we proposed the dynamic repair policies. In the dynamic policies, the repair action is determined by taking into account the time of failure. We compare the optimal static policy with the dynamic policy. Then, we generalize the one-dimensional imperfect repair concept to the two-dimensional case. In the two-dimensional case, we analyze the repair actions which are the extent of one-dimensional static and dynamic policies.

The organization of this thesis report is as follows. In Chapter 2, we give the basic concept of warranty policies and modeling issues. In Chapter 3, we present a review of literature on one- and two-dimensional warranties with various failure models. Chapter 4 presents the definition of our problem. We formulate the problem for one- and two-dimensional cases in Chapter 5. Then, we focus on the expected number of failures under different types of two-dimensional policies and propose three new types of policies. The solution approach for calculating the expected warranty cost is given in Chapter 6. Computational results are presented in Chapter 7. Finally, concluding remarks and future research directions are given in Chapter 8.

Chapter 2

WARRANTY CONCEPT AND SOME MODELING ISSUES

A warranty agreement specifies the length of warranty time, the conditions under which the warranty applies, and the compensation method in case of unsatisfactory performance within the warranty period. In this chapter, we firstly consider different types of warranty policies with respect to various criteria such as warranty coverage, rectification actions and structure; then we deal with failure modeling techniques.

Firstly, warranty policies can be grouped into two with respect to their period of coverage as renewing and non-renewing. In the renewing warranty, the warranty period, W, is not fixed. In the case of failure, the product is returned with a new warranty after rectification. The terms of this new warranty can be identical to or different from the original. In contrast to the renewing policy, the warranty period is fixed in the non-renewing warranty, usually beginning on the date of purchase. If the product fails during this period, it is replaced or repaired by the manufacturer, but this rectification action does not change the duration of the warranty. That is, if the product fails at age t, then the remaining warranty period is $W-t$ time units.

Further, these warranty policies can also be classified with respect to the type of compensation. With respect to this criterion, there are two basic types of policies. The first one is the free replacement warranty (FRW) and the second one is pro rata warranty (PRW). Under FRW, the manufacturer covers the cost of repair or replacement of the failed products within the warranty period at no cost to the buyer. This warranty type applies both to inexpensive products such as house appliances and to expensive products such as automobiles and other durable consumer goods. In contrast to FRW, the manufacturer promises to cover a fraction of the cost of repair or replacement in the PRW. The amount of compensation in PRW is determined based on some non-increasing function of the product age. This type of warranty usually applies to products whose performance is affected by age, such as car batteries. In addition, there exist policies called hybrid warranties which are combination of FRW and PRW. Under these policies, the manufacturer initially applies FRW for a certain period of time and then switches to the PRW in the remaining time within the warranty term.

Another aspect of warranty analysis relates to the extent of repair after failure. There exist several repair types in this regard but the most widely used ones in the literature are perfect (or as good as new), minimal and imperfect repair. In the *perfect repair*, the failed product is brought to the same condition as a new product after the repair. That is, the failure distribution after the repair is the same as that of a new item. If the original product's and the repaired unit's failure rates and mean times to failure are denoted as $r_i(x)$ and $E_i(x)$, i=1,2 then

$$r_2(x) = r_1(x)$$
$$E_2(x) = E_1(x)$$

for as good as new repair.

If a repair does not affect the performance of the product, then it is said to be minimal. In *minimal repair*, the failure rate after the repair is the same just before the failure occurs. Mathematically, if x_1 is the realization of the first failure time, the failure rate and the mean time to failure are;

$$r_2(x) = r_1(x_1 + x)$$
$$E_2(x) = E_1(x \mid x_1)$$

In contrast to the minimal repair, if the repair action changes the failure rate of the product, then it is called *imperfect repair*. Imperfect repair can increase (deterioration) or decrease (improvement) the failure rate of the product after the repair action. That is;

$$r_2(x) < (>) r_1(x)$$
$$E_2(x) > (<) E_1(x)$$

Reasons for deterioration may be to applying inadequate repair to the failed product or replacing the failed item with a less reliable secondhand item. To replace the failed item with an improved one is an example for the improvement resulted by imperfect repair. The deterioration or improvement can be modeled by changing the scale of failure distribution. If the time interval between the $(n\text{-}1)^{th}$ and n^{th} failure is written such that $T_n = \alpha_n X_n$, n=1,2,…, then the improvement and deterioration can be characterized by using different range of α value. For instance, if α is less than 1, it represents the deterioration of the process. Besides these approaches, the combination of the perfect and minimal repair is also called imperfect repair. For example, the failed product is replaced by a new one, if the expected repair cost is larger than a predetermined cost, otherwise it is minimally repaired. Another example for the combination of the perfect and minimal repair is that the failed product switches to the operational state with probability p or it continues in a failed state with probability 1-p after the repair.

When we consider the warranty structure, we can group policies as one-, two- and multi-dimensional. In the one-dimensional warranty policies, the warranty period is defined by an interval. This interval is specified by a single variable such as the amount of usage or time until the end of the warranty period. Whereas, in the two-dimensional policies, warranty is indexed by two scales, one representing the usage and the other age. Here, the warranty expires when the product under warranty reaches the pre-specified age or usage whichever occurs earlier. If the warranty is specified over three or more dimensions, the corresponding policy is referred to as a multi-dimensional policy. An example for multi-dimensional policies is warranty policies for aircrafts. Total time in the air, number of flights and calendar age are the three dimensions of aircraft warranty.

In practice, one- and two-dimensional warranties are frequently used. In the two-dimensional policies, based on the structure of the warranty region basically four different types have been proposed (Figure 2.2). Each of these policies tends to favor customers having different usage rates. The first policy (Contract A) is the one that the manufacturer covers the cost of repair or replacement of the product if the failure occurs up to a time limit W and usage limit U. The warranty ceases at time limit W or at usage limit U whichever occurs earlier. This policy is one of the policies that is in favor of the manufacturer. In this policy, if the customer's usage rate is low, then the warranty ceases at time W before the total usage exceeds the usage limit U. Similarly, if the rate is high, the warranty ceases at U before the time limit W is reached. This policy is very popular especially for automobiles. Under the second type of policy (Contract B) the warranty region is specified by two infinite-dimensional strips, each one of which is parallel to one axis. This policy guarantees the coverage beyond the time limit W for customers with a low usage rate, and it guarantees W units of time coverage if the total usage is larger than U. Thus, it protects the low and high usage customers. However, this policy does not favor the manufacturer. It can cause excessive warranty cost. To protect the manufacturer from excessive warranty cost under the second policy, secondary time and usage limits can be added. Under the contract B` policy, the warranty is characterized by two limited strips instead of the infinite-dimensional strips. In this type of policy, determination of W_2 and U_2 is important. If these parameters are properly selected, the warranty cost can become the same for both heavy and light users. Thus, the manufacturer provides equal coverage for both types of users. Contract C also provides a tradeoff between time and usage. It is specified by a triangle with a slope ($-U/W$). Here, the warranty expires if the total usage, x, by the failure time t satisfies the inequality $x + (U/W)t \geq U$.

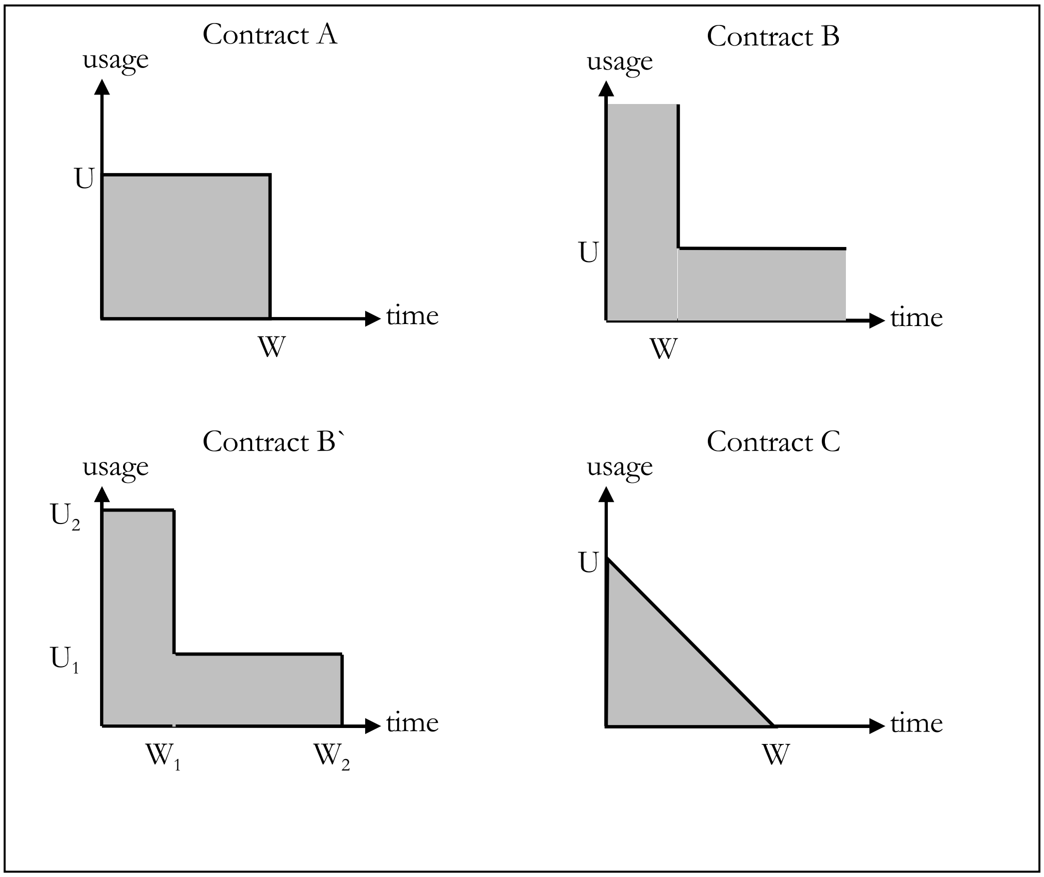

Figure 2.2: Different two-dimensional warranty policies

While modeling the products' lifetime or failure for one-dimensional policies under the rectification actions stated above, the concept of a renewal process is frequently used. Ordinary renewal processes are appropriate for as good as new repair, since after each failure the product characteristics become same as the initial product. If the repair is minimal and initial product's lifetime is exponentially distributed, then a non-homogeneous Poisson process with a cumulative failure rate of $\Lambda(x) = \int_0^x r(t)dt$ can be used since the rectification action does not change the failure rate of the product. However, there exist some cases where renewal processes are not suitable. For example, imperfect repair brings the failed product to an intermediate state between perfect and minimal repair. In the Chapter 5, we consider the imperfect repair, which changes the product's failure rate, in detail.

For two-dimensional warranty policies, the lifetime is modeled by bivariate models. These bivariate models may be grouped based on the relationship between two variables. In the first approach, the two variables, i.e. age and usage, are functionally related. This approach models product failures by using a one-dimensional point process. In this approach, one dimension is eliminated by using relation between dimensions. Instead of having functional relation, variables can be correlated. This method models failures by a bivariate distribution. If (T_n, X_n), n=1,2,… represent the time interval between the n^{th} and $(n-1)^{st}$ failure and the product usage between the two failures, then (T_n, X_n) can be modeled with a bivariate distribution function, $F_n(t, x)=P(T_n \leq t, X_n \leq x)$. Here, the structure of $F_n(t, x)$ is different for different types of rectification actions. For example, if rectification is via perfect repair, then $F_n(t, x)$'s are identical, so this case can be modeled by a two-dimensional renewal process. The two-dimensional renewal process can be analyzed by two univariate renewal processes associated with $\{T_n\}$ and $\{X_n\}$ which are the sequences of marginal distribution of $F_n(t, x)$ with respect to t and x. If

$$N^t(t) = \max\{n : S_n^t = \sum_{i=1}^{n} T_i \le t\} \quad \text{and} \quad N^x(x) = \max\{n : S_n^x = \sum_{i=1}^{n} X_i \le x\},$$

then the number of renewals in a rectangle $[0,W)\times[0,U)$ is

$$N(W,U) = \min\{N^t(W), N^x(U)\}$$

Thus,

$$P(N(W,U) = n) = F^n(W,U) - F^{n+1}(W,U)$$

where F^n is an n-fold convolution. As a result, from the above equation

$$M(W,U) = E(N(W,U)) = \sum_{n=0}^{\infty} nP(N(W,U) = n)$$

On the other hand, if imperfect repair is applied to a failed product, each $F_n(t, x)$ has a different structure and the renewal process can not cover the imperfect repairs. In Section 5.1, an alternative method is discussed to model the imperfect repair under two-dimensional warranty. Although failure models with correlated random variables may be more descriptive for product lifetime, majority of the two-dimensional warranty literature focus on the failure models in which the variables are functionally related.

Chapter 3

LITERATURE REVIEW

Warranty research dates back to 1960s. Earlier research mainly focused on identification of warranty expenses, determination of warranty reserve and usage of warranty as a marketing strategy. Issues such as determining different warranty policies with respect to repair types, warranty region and compensation characteristics, deriving models for analysis of policies and setting maintenance actions have become popular in the recent years. This chapter provides a review of the literature on one- and two-dimensional warranty policies considering different repair actions. Extensive reviews of warranty problems are provided in Blischke and Murthy (1992), Thomas and Rao (1999) and Murthy and Djamaludin(2002). Blischke and Murthy (1992-1, 2, 3) deal with consumer and manufacturer perspectives on warranty, different types of warranty policies and system characterization of warranty. In addition, they classify mathematical models for warranty cost. Thomas and Rao (1999) cover a summary of the warranty economic models and analysis methods along with the related warranty management issues. More recently, Murthy and Djamaludin (2002) review the literature over the last decade literature. The paper discusses the issues related to warranty for a new product.

In this chapter, we mainly focus on corrective maintenance actions and provide a review of these actions. Corrective maintenance refers to all type of rectification actions in case of a failure. They are usually characterized with respect to the degree of rectification. These rectification degrees can be grouped as perfect, minimal and imperfect repair. Perfect repair denotes the case where the product becomes as good as new after a repair. On the other hand, minimal repair refers the rectification action that brings the product as bad as old after repair. Perfect and minimal repair are the most common models for corrective maintenance seen in the literature, but they reflect the two extreme cases concerning to repair actions. Imperfect repairs (i.e. general repairs) have become popular in the more recent warranty/reliability literature. This type of repair may be more realistic than perfect and minimal repair since this repair returns the product to a state between as good as new and as bad as old.

3.1 Perfect Repair

Corrective maintenance is called perfect repair when failure is reimbursed by replacing the product with a new one. If product is non-repairable, there is no alternative way of rectification. In the perfect repair, the repaired product's lifetime and other characteristics become identical to that of a new product. Thus, perfect repair can be modeled as a renewal process (Blische& Murthy,1994).

Balcer& Sahin (1986) consider one-dimensional pro-rata and free replacement warranty policies in which a failure is rectified by replacement. They characterize the moments of the buyer's total cost under both policies during the product life cycle. In addition, they extend the stationary failure time distribution to the time varying failure time distribution for pro-rata warranty policy.

Murthy et al. (1995) analyze four different two-dimensional warranty policies (discussed in Chapter 2) with perfect repair. They derive the expected warranty cost per product and the expected life cycle cost for each policy by using a two-dimensional approach to model the failure distribution. In the numerical analysis, they use Beta Stacy distribution as a failure distribution. Kim and Rao (2000) also perform a similar study. They analyze the expected cost of two different two-dimensional warranty policies (Contract A and B) by using a bivariate exponential failure distribution.

Many reliability/warranty studies consider perfect repairs due to their advantage in derivation of analytical results. However, perfect repair may not be practical in certain cases. For example, for multi-component products, to replace a failed component may not return the product to an as good as new condition. Minimal repair may be an alternative modeling assumption in the cases in which perfect repair is not realistic.

3.2 Minimal Repair

Minimal repair is defined as a repair that does not affect product's failure rate. Minimal repair is generally used for products consisting of multiple-components in which the failed component does not affect the other components. The repair only brings the product to an operational state. Minimal repair is often used in combination with perfect repair to make up a repair-replace policy. Such hybrid policies are sometimes referred to as imperfect repair. Barlow and Hunter's study (1960) is the first to introduce the concept of minimal repair. They consider a policy such that product is replaced at regular intervals and it is minimally repaired if a failure occurs between replacement intervals. Boland and Proschan (1982) also consider the same policy. They determine the optimal replacement period over a finite time horizon and the total expected cost over an infinite time horizon.

Phelps (1983) compares three types of replacement policies. The first policy is to perform minimal repairs up to a certain age, then replace the failed product. The second policy sets a threshold on the number of failures. If the number of failures is less than the threshold point, product is minimally repaired; otherwise, perfect repair is applied. The third policy uses age dependent threshold point, replacement is applied only to the first failure after the threshold point and then all other failures are rectified by minimal repair. Phelps suggests using semi-Markov decision processes and concludes that the third policy is optimal for products with increasing failure rate. Jack and Murthy (2001) also study the third policy and conclude that optimality of this policy depends on the length of warranty period, replacement and repair cost. Iskandar et al. (2005) extend this policy to two-dimensional by using two rectangular regions instead of intervals. In the numerical analysis, they see that this policy is optimal when ratio of repair and replacement cost is around 0.5. If ratio approaches 1(0), then always replace (always repair) policy dominates the hybrid policy. Iskandar and Murthy (2003) also study the two-dimensional repair-replace strategies. They divide the warranty region into two non-overlapping sets and propose two policies. In the first policy, if failure occurs in the first region, it is replaced; and if it occurs in the second region, it is repaired minimally. On the other hand, in the second policy, failures in the first region are minimally repaired; and those in the second region are replaced.

Cleroux et al. (1979) and Nguyen and Murthy (1984) also discuss repair-replace policies, but they propose a different threshold type. In both papers, if the estimated repair cost is greater than a threshold point, then the failed product is replaced, otherwise the failures are reimbursed by minimal repair. Cleroux et al. take some percentage of replacement cost as a threshold point, whereas Nguyen and Murthy select a threshold point such that it minimizes the expected cost per unit time.

Jack (1991, 1992), Jack et al. (2000), Qian et al. (2003) and Sheu and Yu (2005) also discuss one-dimensional repair-replace policies. Jack (1991, 1992) considers a policy over a finite time horizon in which failures are repaired minimally before the *N*th failure and at the *N*th failure system is replaced. In addition, Jack et al. (2000) suggest replacing all failures before a specified age, then minimally repairing all other failures until the end of the warranty period.

Sandve and Aven (1999) propose different policies based on minimal repair for a system comprising of multiple-components. The first one of these policies is replacement of the system at fixed time periods. The second one is referred (T, S) policy, $T \leq S$. In this policy, a replacement is placed at time S or at the first failure after time T. In the third policy, the system is replaced at a time dependent on the condition of the system.

Another form of repair-replace policy is called (p, q) type policy. In this type of policy, each time when a failure occurs, the failed product is replaced with probability p, or it undergoes minimal repair with probability q=1- p. Block et al. (1985) discuss a policy in which the probabilities depend on the age of product at the failure time. They model failures between successive replacements by a renewal process. Makis and Jardine (1992) include the failure number while calculating (p, q) and also incorporate the alternative of scrapping and replacing the product with an additional cost if the repair is not successful. This policy is modeled as a semi-Markov decision process.

Some other examples of minimal repair for two-dimensional policies are Yun (1997) and Baik et al. (2003). Yun considers the failure-free two-dimensional warranty for repairable products. He derives the expected value and variance of the warranty cost of products. Here, the warranty period is taken as a random variable because the period is ended at the warranty age limit or the mileage limit, whichever occurs first. In the paper, failures are modeled as a non-homogeneous Poisson process. Yun derives the expected value and the variance of number of failures by conditioning on the number of repairs. Baik et al. focus on the characterization of failures under minimal repair. They extend the one-dimensional minimal repair concept to the two-dimensional and show that minimal repair over two-dimensional policies can be modeled as a non-homogeneous Poisson process. Analysis of minimal repairs provides an extension to a broader concept, the imperfect repair.

3.3 Imperfect Repair

Different ways of modeling imperfect repairs are proposed in the literature. One approach is to use a mixture of minimal and perfect repair with a threshold point based on the repair cost or the number of failures. Another approach is to use (p, q) type policy in which with probability p failed product is rectified by perfect repair and otherwise, with probability q=1-p, it is corrected with minimal repair. Examples of these two types of imperfect repairs are given in Section 3.3.

The third type of imperfect repairs changes the failure rate of the product after repair. The most widely adopted imperfect repair model is the virtual age model proposed by Kijima in1989. In this model, product can return to a state between as good as new and as bad as old after repair. Kijima constructs two models according to repair effect. The first model (Type 1), is $V_n = V_{n-1} + A_n X_n$ where V_n is the virtual age after the n^{th} failure, X_n is the inter-failure time between $(n\text{-}1)^{th}$ and n^{th} failure and A_n is the degree of the n^{th} repair. In this model, the n^{th} repair cannot remove the damages incurred before the previous repair; it reduces the additional age (X_n) by the degree of repair (Kijima, 1989). On the other hand, the second virtual age model (Type 2) is $V_n = A_n (V_{n-1} + X_n)$. In the first model, relationship between virtual age and chronological age is obvious, but in the second model it is not. In both models, if A_n is equal to 1, then it means that repair is minimal, whereas if A_n is equal to 0, rectification action is perfect repair. Kijima finds bounds for chronological age of the product with respect to two models. By numerical example, he found that difference between the expected value of the chronological age under minimal repair and under virtual age models gets larger when the degree of repair decreases.

In addition, Dagpunar (1997) defines the virtual age as a function of virtual age plus inter-failure time, i.e. $V_n = \varphi(V_{n-1} + X_n)$. This model is an extension of Kijima's Type 2 model. Dagpunar constructs integral equations for the repair density and for the joint density of repairs with respect to chronological age and virtual age. In addition, an upgraded repair strategy in which minimal repairs are applied until the product reaches a specified age is developed. In the paper, the repair density and asymptotic moments for each model are also derived.

Dimitrov et al. (2004) propose age-dependent repair model along the same way as in Kijima's Type 1 model. They analyze warranty cost for some warranty policies such as PRW, a mixture of minimal and imperfect repair and renewing and fixed warranty.

Wang and Pham (1996-1) suggest two imperfect preventive models and a cost limit repair model. In the models, preventive maintenance is applied at times kT after the k^{th} repair, where T is a non-negative constant. In their models, repair is imperfect in the sense that repair action decreases the lifetime of the product, but increases the repair time. This reduces the lifetime and increase the repair time. In the paper, repair cost also increases with each additional repair. In the first policy, repairs are imperfect between preventive maintenance periods and after preventive maintenance, product will be as good as new with probability p and as bad as old with probability 1- p; whereas, the second model assumes that after preventive maintenance the age of the product becomes x units of time younger ($0 \leq x \leq T$) and the product is replaced by a new one if it has operated for a time interval NT (Wang and Pham, 1996-1). In the third model, after k^{th} repair a failure is rectified by repair or replacement regarding its repair cost, and repair brings to as good as new state with probability p, and to as bad as old with probability 1- p. In the paper, Wang and Pham derive the long-run expected maintenance cost, asymptotic average availability and find the optimal parameters for each model. After this study, Wang and Pham (1996-2) call this repair model as a quasi-renewal process and deal with similar policies, but assume negligible repair time. In addition, Bai and Pham (2005) suggest repair-limit warranty policies such that after a failure, imperfect repair is conducted if the number of repairs is less than a threshold point. If not, the failed product is replaced. The threshold point is chosen in such a way that after this point repair becomes more costly.

3.4 Two-dimensional Warranty Examples

Most of the two-dimensional warranties consider policies with different repair types such as perfect, minimal or mixture of perfect and minimal repair. However, there are some examples that approach warranty problem in a different way. For example, Singpurwalla and Wilson (1993) derive expected utility of the manufacturer and consumers as a function of product price and warranty region. Due to competition in market, manufacturer can not freely choose a price and a warranty structure to maximize its expected utility, so Singpurwalla and Wilson handle warranty problem by the concept of two person non-zero sum game. In addition, they propose various regions for two-dimensional warranty different from the rectangular one. The rectangular warranty region has a disadvantage for the manufacturer if the product is used above the normal rate during the initial period of purchase. Other alternative regions can be constructed by shaving off some part of the rectangular region. For instance, shaving off an upper/lower triangle of the rectangular region renders a more manufacturer/consumer friendly warranty region. In order to make the warranty policy more consumer friendly, circular or parabolic warranty regions can also be adopted instead of a triangular region. The semi-infinite warranty region similar to the one suggested by Murthy et al. is not advantageous to normal users but it is so for users with an exceptionally high or low rate of usage.

Gertsbakh& Kordonsky (1998) deal with constructing individual warranties for a customer with low or high usage rate since the traditional two-dimensional warranties do not provide equal conditions for different types of customers. They construct a new time scale which is a combination of usage and mileage. Then, this time scale can be used to determine warranty region for each customer by considering his usage rate. This type of warranty may increase the number of customers and improve the manufacturer’s profit.

Singpurwalla and Wilson (1998) propose an approach for probabilistic models indexed by time and usage. They suggest three different processes to model the usage. The first one is Poisson process. It is appropriate when usage is characterized by a binary variable: down and up or the amounts of usage up to failures do not affect failure inter-arrivals. On the other hand, if using the product continuously causes wear, then the gamma process is useful for modeling the usage. Lastly, for modeling wear by continuous use with the periods of rest, the Markov additive process is suitable.

Chukova et. al. (2004) focus on the transition from the initial lifetime to the second lifetime following to the first repair and they compare different types of repairs in the case of one repair by using the distribution functions, mean time to failure and failure rate functions of the lifetime distributions. Chukova et. al. also mention the accelerated lifetime distribution functions. In the accelerated life models, the repaired item has a lifetime distribution which generates from the same family with the multiplicative scale factor to rescale the original random variable. Here, the product's reaction to failure changes according to the scale multiplier: if it is less than 1, then the product is less fond of failure than the case with the multiplier greater than 1.

3.5 Conclusion

In the literature, there are a vast number of studies which model and analyze different warranty polices. Majority of these studies deal with the one-dimensional warranty policy. Although one-dimension is enough for describing the failure process for most products, there exist some cases for which a single dimension is not sufficient to characterize the failure structure of the product. This usually occurs when the usage and age of the product affect the lifetime of the associated product such as in tires, cars etc. For such products, two-dimensional warranty policies are more suitable. However, the studies of two-dimensional warranties are limited. Thus, this concept is one of the topics that can be studied in detail. When the rectification types under the warranty policies are examined, it is seen that the rectification types are generally perfect, minimal and combination of these two. Other than considering a combination of perfect and minimal repair, imperfect repair has not received much attention. The imperfect models that change the failure rate of product are discussed only in a couple of papers. Examples of these types of imperfect repairs are limited by

Kijima's models (1989) and Wang and Pham (1996-1, 2) approaches. On the other hand, all the studies related to this type of imperfect repair consider only the univariate case. In this thesis, we focus on imperfect repairs under both one- and two-dimensional warranties. We extend the application of quasi renewal processes to model two-dimensional warranties. We then define representative cost functions and investigate the effectiveness of several repair policies under a variety of conditions.

Chapter 4

PROBLEM DEFINITION

We consider a replacement/repair warranty policy. We focus on both one- and two-dimensional cases. For one-dimensional warranties, we describe the product lifetime in terms of age. For two-dimensional cases, we characterize it in terms of age and usage, and we assume that age and usage are correlated. In the two-dimensional warranties, we investigate policies with different degrees of protection for the manufacturer and consumer. Our failure model is an imperfect repair model that is based on a quasi-renewal process. We analyze the effect of imperfect repairs on the total expected warranty cost for products of different reliability structure. While we construct and analyze the failure models, we make the following simplifying assumptions:

- Buyers have similar attitude with respect to usage when they use the same product
- All claims during the warranty period are valid
- The time to rectify a failed item is negligible

The first assumption above allows considering all the buyers simultaneously. The second assumption states that the failure does not occur as a result of improper usage. Lastly, the time to rectify a failed item can be assumed negligible, when the repair time is too small compared to the product lifetime.

With respect to corrective maintenance actions, we study imperfect repairs based on a quasi-renewal process. The quasi-renewal process is characterized by a scaling parameter that alters the random variable after each renewal. In other words, this parameter indicates the deterioration or improvement of process. For example, if the scaling parameter is between 0 and 1, it indicates deterioration; whereas if it is greater than 1, it indicates an improved policy. In our study, we refer to this parameter as extent of repair. The extent of repair also determines the amount of change in the mean of the interfailure and failure rate before and after the renewal. The quasi-renewal process allows for modeling many different extents of repair by varying the scale parameter.

To compare various policies, we use the expected total cost over the warranty period. Warranty cost includes the rectification cost in case of failure. In the literature, these costs are generally aggregated and assumed constant over the warranty period. In addition, there exist some examples in which cost depends on the product age. However, to the best of our knowledge, there has not been any attempt to model the warranty cost as a function of the repair policy adapted throughout the warranty period. In this study, we propose new cost functions that address this issue for one- and two-dimensional warranty. These functions are composed of two parts: fixed and variable components. The fixed component is paid independently of the extent of repair and represents the costs such as loss of goodwill or setup. The variable cost includes direct labor and direct material costs and it increases in parallel with the extent of repair.

The total expected warranty cost is based on the fixed and variable cost components and the expected number of failures. The fixed and variable costs are determined by the manufacturer, whereas the expected number of failures depends on the warranty length, the reliability of the product and the extent of repair. We assume that the warranty length is determined before the product is placed in the market by considering various factors such as competition in the sector, marketing strategy and product's characteristics. The reliability of the product is specified by the probability density function of the interfailure times. In this study, our aim is to find the optimal repair policy which minimizes the warranty cost.

Chapter 5

IMPERFECT REPAIR MODEL

In this chapter, we firstly introduce the univariate imperfect repair model in Section 5.1. Then in Section 5.2, we extend the concept to the multiple dimensions and focus on the bivariate case. In Section 5.3, we discuss the representative cost functions for one- and two-dimensional warranties. Then, in Section 5.4, we derive the expected number of failures under different types of two-dimensional policies. Lastly, in Section 5.5, we propose new repair strategies.

5.1 Univariate Imperfect Repair Model

In order to find the expected number of breakdowns, we try to characterize failure distribution with a model based upon the quasi-renewal process. The quasi renewal process is used as an alternative method for modeling imperfect repairs. Other approaches frequently seen in the literature are *(p, q)* models and combinations of minimal and perfect repair. In the *(p, q)* models, the product after a failure is replaced by a new one with probability p or it is repaired by minimally with probability q=1-p. Whereas, in the combination models, there is a threshold point between minimal and perfect repair. This threshold is characterized by either a maximum number of breakdowns or expected cost. That is, if the count of the last failure is larger than the maximum allowable failures, than the product is replaced instead of minimal repair. Similarly, if a predetermined limit on the expected cost of failure is greater than the threshold point, then the failure is rectified by perfect repair. Otherwise, it is corrected by minimal repair.

The quasi renewal process, on the other hand, does not restrict the possibilities of repair actions to minimal or perfect repair. Indeed, it may be considered more realistic than *(p, q)* and mixture policies since repair actions do not switch between two cases. We focus on the quasi renewal process that represents the deterioration of the product after a failure. That means probability of breakdowns increases after a failure occurs. Wang and Pham (1996-2) introduce the quasi renewal process for the univariate distribution. In this section, the concept of quasi renewal process introduced by Wang and Pham is explained. Then, we generalize the concept of quasi-renewal processes proposed by Wang and Pham (1996-2) for the univariate distribution to the case of the multivariate distribution.

Quasi-Renewal Processes:

Let $\{N(t), t > 0\}$ be a counting process and T_n be the time between the $(n\text{-}1)^{th}$ and n^{th} events of the process(n>0). The counting process $\{N(t), t > 0\}$ is said to be a quasi-renewal process with parameter α, $\alpha > 0$, if

$$T_n = \alpha^{n-1} X_n,\ n=1, 2, 3\ldots \tag{5.1}$$

where X_n's are independently and identically distributed random variables with cumulative distribution and density functions F and f , respectively and α is a constant.

The quasi-renewal process describes the case where the successive intervals $\{T_n,\ n=1, 2, 3\ldots\}$ are modeled as a fraction of the preceding interval. The implication of this process is that the distribution of the n^{th} interval is scaled by a factor, α^{n-1}, but retains the same shape. This phenomenon is depicted in the Figure 5.1 for $\alpha < 1$. As it is seen, the likelihood of successive intervals increases. So, this process can model the deterioration of a system. On the other hand, the case when $\alpha > 1$ represents the improvement of the system and may be appropriate for a reliability growth model. The case α=1 becomes the ordinary renewal process since all the intervals are distributed identically.

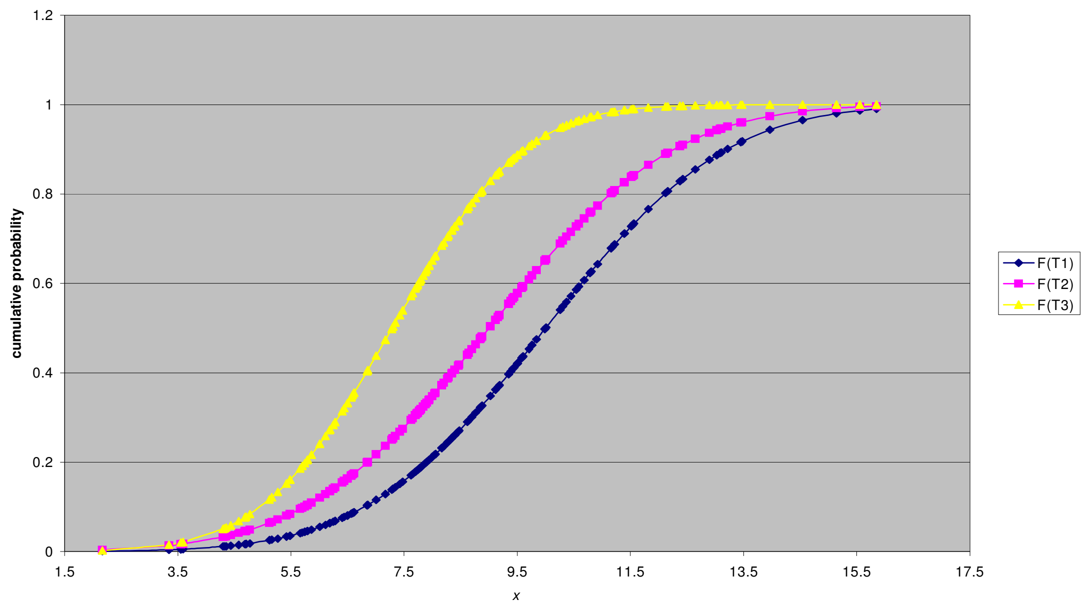

Figure 5.1: Quasi-Renewal Distribution of Successive Intervals

If F_n and f_n are respectively the cumulative and probability density function of the new system, then they are defined as follows.

$$F_n(t)=F(\alpha^{1-n}t) \quad (5.2)$$

$$f_n(t)= \alpha^{1-n} f(\alpha^{1-n}t) \quad (5.3)$$

These results are obtained by the cumulative distribution technique for functions of random variables. That is:

$$F_n(t) = P(T_n \le t) = P(\alpha^{n-1} X_n \le t) = P(X_n \le \alpha^{1-n} t) = F(\alpha^{1-n} t) \quad (5.4)$$

$$f_n(t) = \frac{\partial F_n(t)}{\partial t} = \alpha^{1-n} f(\alpha^{1-n} t) \tag{5.5}$$

Then, the probability function of $N(t)$ can be derived by using the relationship $N(t) \geq n \Leftrightarrow S_n \leq t$, where S_n is the occurrence time of the n^{th} event.

$$\begin{aligned} P(N(t) = n) &= P(S_n \leq t) - P(S_{n+1} \leq t) \\ P(N(t) = n) &= F^{(n)}(t) - F^{(n+1)}(t) \qquad n = 1,\ 2, \ldots \end{aligned} \tag{5.6}$$

where $F^{(n)}(t)$ is the convolution of the arrival times $T_1, T_2, \ldots, T_n$ and $F^{(0)}(t)=1$.

The form of the renewal function of this process is obtained in a similar way to that of the basic renewal process, but the main difference between these two renewal functions is that the intervals are not identically distributed in the quasi-renewal processes. Let the renewal function, i.e. the number of events until time t, of the quasi-renewal process be $M_q^1(t)$. Then, it can be written as:

$$M_q^1(t) = E[N(t)] = \sum_{n=0}^{\infty} nP(N(t) = n) = \sum_{n=1}^{\infty} F^{(n)}(t) \tag{5.7}$$

In order to find expected number of events up to a certain point, we firstly investigate the behavior of the convolutions. The first convolution is obviously equal to:

$$F^{(1)}(t) = \int_0^t f(x_1) dx_1$$

The second convolution is the cumulative distribution function of $T_1 + T_2$. To find this function, the joint density function of T_1 and T_2 should be found. Since X_n's are independently distributed random variables, changing the scale of these variables does not affect the independency of X_n. Thus, T_n's are also independently but not identically distributed. In the light of this information, we can write the joint density of T_1 and T_2 as the product of the marginal density functions of T_1 and T_2. That is:

$$f_{T_1,T_2}(t_1,t_2) = f_{T_1}(t_1) f_{T_2}(t_2) = f(t_1)\alpha^{1-2} f(\alpha^{1-2} t_2)$$

Then, the distribution function of $T_1 + T_2$ is

$$F^{(2)}(t) = P(T_1 + T_2 \le t) = \iint_{t_1+t_2 \le t} f_{T_1}(t_1) f_{T_2}(t_2) dt_2 dt_1 = \int_{t_1=0}^{t} \int_{t_2=0}^{t-t_1} f_{T_1}(t_1) f_{T_2}(t_2) dt_2 dt_1$$

Similarly, the third convolution is

$$F^{(3)}(t) = P(T_1 + T_2 + T_3 \le t) = \iiint_{t_1+t_2+t_3 \le t} f_{T_1}(t_1) f_{T_2}(t_2) f_{T_3}(t_3) dt_3 dt_2 dt_1$$

$$= \int_{t_1=0}^{t} \int_{t_2=0}^{t-t_1} \int_{t_3=0}^{t-t_1-t_2} f_{T_1}(t_1) f_{T_2}(t_2) f_{T_3}(t_3) dt_3 dt_2 dt_1$$

Continuing in this way, we can generalize this to the n-fold convolution as follows:

$$F^{(n)}(t) = P(T_1 + T_2 + ... + T_n \le t) = \iiint_{t_1+t_2+...+t_n \le t} ... \int f_{T_1}(t_1) f_{T_2}(t_2) ... f_{T_n}(t_n) dt_n ... dt_2 dt_1$$

$$= \int_{t_1=0}^{t} \int_{t_2=0}^{t-t_1} \int_{t_3=0}^{t-t_1-t_2} ... \int_{t_n=0}^{t-\sum_{i=1}^{n-1} t_i} f_{T_1}(t_1) f_{T_2}(t_2) f_{T_3}(t_3) ... f_{T_n}(t_n) dt_n ... dt_2 dt_1$$

$$= \int_{t_1=0}^{t} \int_{t_2=0}^{t-t_1} \int_{t_3=0}^{t-t_1-t_2} ... \int_{t_n=0}^{t-\sum_{i=1}^{n-1} t_i} f(t_1)\alpha^{-1} f(\alpha^{-1} t_2)\alpha^{-2} f(\alpha^{-2} t_3) ... \alpha^{1-n} f(\alpha^{1-n} t_n) dt_n ... dt_2 dt_1$$

Closed form of analytical expressions for $F^{(n)}$ can be secured only for a few special distributions such as the normal distribution. For this reason, a numerical method is developed to evaluate $F^{(n)}$. This method will be explained in the next chapter.

In our system, we consider the replacement and deterioration of the system. So, the quasi-renewal process, with $0< \alpha \leq 1$, is suitable for the imperfect repair type. Here, the value of α represents the extent of repair. If α is equal to 1, this means that the rectification is done by replacement of the failed product by a new one, whereas, a smaller alpha value corresponds to the case in which the product switches to an operational state inferior to that of a new one. The extent of repair that corresponds to the minimal repair is discussed in the following part.

Handling the minimal repair:

A repair action is said to be minimal if the product failure rate is the same before and after the repair action. If $F_1(t)$ is the failure distribution of the original item and t_1 is the realization of the first failure time, then the time to failure distribution after a minimal repair has the following structure.

$$F_1(t \mid t_1) = 1 - \frac{1 - F_1(t + t_1)}{1 - F_1(t_1)}$$

If the failure distribution is exponential with the failure rate of λ, then the cumulative distribution function of the item after the minimal repair is as follows.

$$\begin{aligned} F(t \mid t_1) &= 1 - \frac{1 - F(t + t_1)}{1 - F(t_1)} \\ &= 1 - \frac{1 - (1 - e^{-\lambda(t + t_1)})}{1 - (1 - e^{-\lambda t_1})} \\ &= 1 - e^{-\lambda t} \end{aligned}$$

Due to the memoryless property of the exponential distribution, the minimal repair does not change the failure distribution. In the quasi renewal concept, the extents of repair for perfect and minimal repair are equal if the failure distribution is exponential. Now, let the failure process of the item be characterized by normal distribution. Finding the extent of repair that corresponds to minimal repair is complex in this case. If T_1 is normally distributed with mean μ and standard deviation σ, then the conditional distribution of T_2 given T_1 is as follows.

$$f_1(t_2 \mid t_1) = \frac{f_1(t_2 + t_1)}{f_1(t_1)}$$

$$f_1(t_2 \mid t_1) = \frac{\frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(t_2 + t_1 - \mu)^2}{2\sigma^2})}{\frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(t_1 - \mu)^2}{2\sigma^2})}$$

$$f_1(t_2 \mid t_1) = \exp(-\frac{(t_2 + t_1 - \mu)^2}{2\sigma^2} + \frac{(t_1 - \mu)^2}{2\sigma^2})$$

$$f_1(t_2 \mid t_1) = \exp(-\frac{(t_2^2 - 2t_1t_2 - 2t_2\mu)}{2\sigma^2})$$

Then, the joint distribution function of T_1, T_2 can be found by the total probability rule. That is,

$$\begin{aligned} f_{1,2}(t_1, t_2) &= f_1(t_2 \mid t_1) f_1(t_1) \\ &= \exp(-\frac{(t_2^2 - 2t_1t_2 - 2t_2\mu)}{2\sigma^2}) \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(t_1 - \mu)^2}{2\sigma^2}) \end{aligned}$$

Lastly, the distribution function of T_2 is as follows.

$$\begin{aligned} f_{T_2}(t_2) &= \int_{-\infty}^{\infty} f_{T_1,T_2}(t_1, t_2) dt_1 \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(t_2^2 - 2t_1t_2 - 2t_2\mu)}{2\sigma^2} - \frac{(t_1 - \mu)^2}{2\sigma^2}) dt_1 \end{aligned}$$

As it is seen above, the determination of the second failure distribution is complex if the distribution does not have memoryless property. In addition, in our model after each failure, our model behaves like it starts with a new distribution with the shape but narrower scale than the previous inter failure. So, the quasi-renewal process concept does not incorporate the minimal repair.

As discussed earlier, in some cases, one dimension may not be enough to adequately model the failure characteristics of a system. For example, breakdowns of a car are generally affected by both its age and usage rate. Similarly, failure characteristic of a jet engine can be modeled by three factors such as number of flights, calendar age and total flight hours. Thus, in the following section, we will generalize the concept of quasi-renewal processes to multiple-dimensions.

5.2 Generalization of Quasi-Renewal Processes to Multiple Dimensions

For a failure defined along n dimensions, let $\mathbf{X_i}=(X_{1i}, X_{2i},\ldots, X_{ni})$, $i=1,2,3\ldots$ represent an n-dimensional random vector where X_{ki} denotes the interval of kth dimension between the $(i$-1)th and ith successive renewals with $X_{k0}=0$ for all $k=1, 2,\ldots, n$. Let $\{N(x_1, x_2,\ldots, x_n);\ x_k>0\ k=1,\ldots, n\ \}$ be a counting process such that $\mathbf{X^T=AY^T}$ where $\mathbf{A}$ is a $n\times n$ non-negative diagonal matrix and $\mathbf{Y}$ is n-dimensional independently and identically distributed random vector, then we can say $\{N(x_1, x_2,\ldots, x_n);\ x_k>0$ for $k=1,\ldots, n\ \}$ is an n-dimensional quasi-renewal process corresponding to $\mathbf{A}$. In other words,

$$\begin{bmatrix} X_{1i} & \cdots & X_{ni} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} X_{1i} \\ \vdots \\ X_{ni} \end{bmatrix} = \begin{bmatrix} \alpha_1^{i-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_n^{i-1} \end{bmatrix} \begin{bmatrix} Y_{1i} \\ \vdots \\ Y_{ni} \end{bmatrix} = \begin{bmatrix} \alpha_1^{i-1} Y_{1i} \\ \vdots \\ \alpha_n^{i-1} Y_{ni} \end{bmatrix}$$

If $\mathbf{Y_i}$=(Y_{1i}, Y_{2i},…, Y_{ni}), i=1,2,3… has a distribution function $F(y_{1i}, y_{2i},\ldots, y_{ni})$, then cumulative distribution and density function of $\mathbf{X_i}$ can be written as:

$$F_i(x_{1i},\ldots,x_{ni})=F(\alpha_1^{1-i}x_{1i},\ \ldots,\alpha_n^{1-i}x_{ni}) \tag{5.8}$$

$$f_i(x_{1i},\ldots,x_{ni})=\frac{\partial^n F_i(x_{1i},\ldots,x_{ni})}{\partial x_{1i}\ldots\partial x_{ni}}=\alpha_1^{1-i}\ldots\alpha_n^{1-i}f(\alpha_1^{1-i}x_{1i},\ldots,\ \alpha_n^{1-i}x_{ni}) \tag{5.9}$$

The results are again obtained by cumulative distribution technique for functions of multivariate random variables. That is;

$$\begin{aligned}F_i(x_{1i},\ldots,x_{ni})&=P(X_{1i}\le x_{1i},\ldots,X_{ni}\le x_{ni})\\&=P(\alpha_1^{i-1}Y_{1i}\le x_{1i},\ldots,\alpha_n^{i-1}Y_{ni}\le x_{ni})\\&=F(\alpha_1^{1-i}x_{1i},\ \ldots,\alpha_n^{1-n}x_{ni})\end{aligned} \tag{5.10}$$

The distribution function of counting process $N(x_1, x_2,\ldots, x_n)$ can be found in a similar manner as in the univariate process.

$$P(N(x_1,\ldots,\ x_n)=k)=F^{(k)}(x_1,\ldots,\ x_n)-F^{(k+1)}(x_1,\ldots,\ x_n) \qquad k=1,\ 2,\ldots \tag{5.11}$$

where $F^{(k)}$ is the k-fold convolution of F with $F^{(0)}(x_1, x_2,\ldots, x_n)=1$.

Now, we can formulate the expected number of renewals over the n-dimensional plane as follows.

$$M_q^2(x_1,\ldots,\ x_n)=E[N(x_1,\ldots,\ x_n)]=\sum_{k=0}^{\infty}kP(N(x_1,\ldots,\ x_n)=k)=\sum_{k=1}^{\infty}F^{(k)}(x_1,\ldots,\ x_n) \tag{5.12}$$

In the following section, the bivariate quasi renewal process is discussed.

5.2.1 Bivariate Quasi-Renewal Processes

Let (T_n, X_n), n=1,2,3…, be a two-dimensional random vector, where T_n represents the time interval between the n^{th} and $(n-1)^{st}$ failures and X_n represents the product usage between the same two failures with $T_n=X_n=0$. $\{N(t, x); t, x>0\}$ is a two-dimensional quasi-renewal process with parameters α_1 and α_2, $\alpha_1,\alpha_2>0$, if $T_n = \alpha_1^{n-1} Y_n$ and $X_n = \alpha_2^{n-1} Z_n$, n=1, 2, 3…, (5.13)

where the (Y_n, Z_n)'s are independently and identically distributed random variables with the cumulative distribution function $F(y, z)$ and α_1, α_2 are constants. The cumulative distribution and density functions of (T_n, X_n) become as follows.

$$F_n(t,x) = F(\alpha_1^{1-n}t,\ \alpha_2^{1-n}x) \tag{5.14}$$

$$f_n(t,x) = \frac{\partial^2 F_n(t,x)}{\partial t \partial x} = \alpha_1^{1-n}\alpha_2^{1-n} f(\alpha_1^{1-n}t,\ \alpha_2^{1-n}x) \tag{5.15}$$

It then follows that

$$P(N(t,\ x) = n) = F^{(n)}(t,\ x) - F^{(n+1)}(t,\ x) \qquad n = 1,\ 2,... \tag{5.16}$$

where $F^{(n)}(t, x)$ is the n-fold convolution of $F(t, x)$, $F^{(0)}(t, x)$=1, and the expected number of failures over $[0,t)\times[0,x)$ is expressed as

$$M_q^2(t,\ x) = E[N(t,\ x)] = \sum_{n=0}^{\infty} nP(N(t,\ x) = n) = \sum_{n=1}^{\infty} F^{(n)}(t,\ x) \tag{5.17}$$

In the above equation, the first convolution is as follows.

$$F^{(1)}(t,x) = \int_{t_1=0}^{t} \int_{x_1=0}^{x} f(t_1,\ x_1)dt_1 dx_1$$

Since the consecutive failures are independent of each other, the joint density of (T_1, X_1) and (T_2, X_2) is the product of the two marginal density functions. Thus, the convolution of (T_1, X_1) and (T_2, X_2) is equal to:

$$F^{(2)}(t,x) = P(T_1 + T_2 \le t; X_1 + X_2 \le x) = \iiint\limits_{\substack{T_1+T_2\le t\\ X_1+X_2\le x}} \cdots \int f_1(t_1,x_1) f_2(t_2,x_2) dx_2 dt_2 dx_1 dt_1$$

$$= \int\limits_{t_1=0}^{t} \int\limits_{x_1=0}^{x} \int\limits_{t_2=0}^{t-t_1} \int\limits_{x_2=0}^{x-x_1} f_1(t_1,x_1) f_2(t_2,x_2) dx_2 dt_2 dx_1 dt_1$$

The general form of n-fold convolution in the bivariate case can be written as follows.

$$F^{(n)}(t,x) = P(T_1 + ... + T_n \le t; X_1 + ... + X_n \le x)$$

$$= \iiint\limits_{\substack{T_1+...+T_n\le t\\ X_1+...+X_2\le x}} \cdots \iint f_1(t_1,x_1) ... f_n(t_n,x_n) dx_n dt_n ... dx_1 dt_1$$

$$= \int\limits_{t_1=0}^{t} \int\limits_{x_1=0}^{x} \cdots \int\limits_{t_n=0}^{t-\sum_{i=1}^{n-1} t_i} \int\limits_{x_n=0}^{x-\sum_{i=1}^{n-1} x_i} f_1(t_1,x_1) ... f_n(t_n,x_n) dx_n dt_n ... dx_1 dt_1$$

In order to calculate the n-fold convolution, we need to take $2n$ many integrals. Hence, finding the expected number of failures over a two-dimensional region is more difficult than that in the one-dimension. In the next chapter, we will develop a numerical method to serve this purpose.

5.3 Cost Function

As the repair degree improves, the repair cost increases. An appropriate cost function that displays these characteristics can be written in the following way:

$$C(W, \alpha_1, ..., \alpha_n) = \sum_{i=1}^{N(W)} (c + c_1 \alpha_i)$$

where c and c_1 are real constant corresponding to fixed and variable repair cost, respectively, α_i corresponds to the extent of i^{th} repair and $N(W)$ is the number of failures during warranty. The expected cost is then as follows:

$$EC = E(C(W, \alpha_1, ..., \alpha_n)) = \sum_{i=1}^{E(N(W))} (c + c_1 \alpha_i) \tag{5.19}$$

where $E(N(W))$ is the expected number of failures within the warranty period of length W.

For the two-dimensional case, there are two variable components. One corresponds to repair cost along the time dimension; the other along the usage dimension. Then, the total cost over warranty period (W, U) can be written in equation 5.20 and the expected total can be formulated in equation 5.21.

$$C(W, U, \alpha_{11}, ...\alpha_{1n}, \alpha_{21}, ..., \alpha_{2n}) = \sum_{i=1}^{N(W,U)} (c + c_1 \alpha_{1i} + c_2 \alpha_{2i}) \tag{5.20}$$

$$C(W, U, \alpha_{11}, ...\alpha_{1n}, \alpha_{21}, ..., \alpha_{2n}) = \sum_{i=1}^{E(N(W,U))} (c + c_1 \alpha_{1i} + c_2 \alpha_{2i}) \tag{5.21}$$

where c is a fixed component of the cost, and c_1 and c_2 are the variable components of the repair cost along the time and usage dimensions, respectively with α_{1i} and α_{2i} indicating the extent of i^{th} repair in each dimension.

5.4 Expected Number of Failures for Different Two-dimensional Warranty Policies

The expected number of failures for Contract A can be calculated based on equation 5.19. In Contract B, the warranty ceases after a failure if the time of the failure and the total usage up to the failure both exceed the warranty limits *W* and *U*, respectively. This policy is the combination of two one-dimensional policies; one is for the time dimension and the other for the usage dimension. So, the expected number of failures can be found by using the one and two-dimensional processes. The expected number of failures can be written in the following way.

$$\begin{aligned} M_q^B(W,U,\alpha_1,\alpha_2) &= M_q^2(W,\infty,\alpha_1,\alpha_2) + M_q^2(\infty,U,\alpha_1,\alpha_2) - M_q^2(W,U,\alpha_1,\alpha_2) \\ &= \sum_{n=1}^{\infty} F^{(n)}(W,\infty) + \sum_{n=1}^{\infty} F^{(n)}(\infty,U) - \sum_{n=1}^{\infty} F^{(n)}(W,U) \\ &= E(N_1(W)) + E(N_2(U)) - E(N(W,U)) \end{aligned} \tag{5.22}$$

where $N_1(W)$ and $N_2(U)$ are one-dimensional point process corresponding to the marginal distribution functions of $F(t, x)$ with respect to time and usage dimension, respectively.

In Contract C, if a failure occurs at time *t*, and the total usage of the product up to *t* is *x*, then the failure is reimbursed by the manufacturer if $K=x+mt \leq U$ where $m=U/W$. Let $K_i=X_i+mT_i$, $i \geq 1$, then $\{K_i,\ i \geq 1\}$ is a sequence of independent and identically distributed random variables with F_K such that

$$
\begin{aligned}
F_K(k) &= P(K = X_i + mT_i \le k) \\
&= \iint\limits_{x_i + mt_i \le k} f_i(t_i, x_i)\,dt_i\,dx_i \\
&= \int_{x_i=0}^{k} \int_{t_i=0}^{\frac{k-x_i}{m}} f_i(t_i, x_i)\,dt_i\,dx_i
\end{aligned}
\tag{5.23}
$$

If $N_K(U)$ denotes the expected number of failures in [0, U) corresponding to renewal process with F_K, then the expected number of failures under the warranty region is equal to

$$
M_q^C(W,\ U) = E[N(W,\ U)] = \sum_{n=1}^{\infty} F_K^{(n)}(U) \tag{5.24}
$$

In the above expectation, the first convolution is given by

$$
\begin{aligned}
F_K^{(1)}(U) &= P(K_1 = X_1 + mT_1 \le U) \\
&= \iint\limits_{X_1 + mT_1 \le U} f_1(t_1, x_1)\,dx_1\,dt_1 \\
&= \int_{t_1=0}^{W} \int_{x_1=0}^{U-mt_1} f_1(t_1, x_1)\,dx_1\,dt_1
\end{aligned}
$$

The second convolution is more complicated than the first one due to its bounds. It is explicitly

$$
\begin{aligned}
F_K^{(2)}(U) &= P((x_1 + x_2) + m(t_1 + t_2) \le U) \\
&= \iiint \int_{(x_1+x_2)+m(t_1+t_2)\le U} f_1(t_1, x_1) f_2(t_2, x_2)\,dx_2\,dt_2\,dx_1\,dt_1
\end{aligned}
$$

and the bounds of the above integrals are

$$0 \le t_1 \le W$$
$$0 \le x_1 + mt_1 \le U \Rightarrow 0 \le x_1 \le U - mt_1$$
$$0 \le t_1 + t_2 \le W$$
$$0 \le (x_1 + x_2) + m(t_1 + t_2) \le U \Rightarrow 0 \le x_2 \le U - m(t_1 + t_2) - x_1$$

Thus, 2-fold convolution is equal to

$$F_K^{(2)}(U) = P((x_1 + x_2) + m(t_1 + t_2) \le U)$$
$$= \int_{t_1=0}^{W} \int_{x_1=0}^{U-mt_1} \int_{t_2=0}^{W-t_1-\frac{x_1}{m}} \int_{x_2=0}^{U-m(t_1+t_2)-x_1} f_1(t_1, x_1) f_2(t_2, x_2) dx_2 dt_2 dx_1 dt_1$$

By continuing in this manner, the *n*-fold convolution can be written as.

$$F_K^{(n)}(U) = P(\sum_{i=1}^{n} x_i + m\sum_{i=1}^{n} t_i \le U)$$
$$= \int_{t_1=0}^{W} \int_{x_1=0}^{U-mt_1} \cdots \int_{t_n=0}^{W-\sum_{i=1}^{n-1} t_i - \sum_{i=1}^{n-1} \frac{x_i}{m}} \int_{x_n=0}^{U-m\sum_{i=1}^{n} t_i - \sum_{i=1}^{n-1} x_i} f_1(t_1, x_1) \ldots f_n(t_n, x_n) dx_n dt_n \ldots dx_1 dt_1$$

5.5 Proposed Policies

As we stated, most of the reliability literature deals with the rectification actions such as perfect repair, minimal repair and combination of these two repairs. In this section, we propose two classes of imperfect repair policies that rely on quasi-renewal processes. The first class is the ***"static repair policy"***. In the static policy, the extent of repair remains constant over the warranty period. Under the static policy, we also consider ***"improved repair policy"***. The last class of policies is called ***"dynamic***

repair policy" in which the extent of repair varies systematically over the warranty period.

5.5.1 Static Policies

In these policies, all breakdowns seen within the warranty period are rectified in the same manner. That is, α in equation $X_n = \alpha^{n-1} Y_n$ and α_1 and α_2 in equations $T_n = \alpha_1^{n-1} Y_n$ and $X_n = \alpha_2^{n-1} Z_n$ that correspond to the extent of repair are constant over the warranty period. In our case, the repair brings the product to an operational state. However, it becomes less reliable than before the failure. So, α can take values between 0 and 1. A larger α implies a better repair. When α is equal to 1, the repair action corresponds to replacement, i.e. perfect repair, of the product. Perfect repair decreases the expected number of failures more than any other imperfect repairs, but at the same time it increases unit repair cost defined as $C(W, \alpha_i) = \sum_{i=1}^{N(W)} c + c_1 \alpha_i$ for the one-dimensional and $C(W, U, \alpha_{1i}, \alpha_{2i}) = \sum_{i=1}^{N(W,U)} c + c_1 \alpha_{1i} + c_2 \alpha_{2i}$ for two-dimensional warranties. On the other hand, for small α, the unit cost is also small, but the expected number of failures gets larger. Thus, manufacturer should find a trade-off in the degree of repair that minimizes his total cost.

In the two-dimensional warranty policies, the degree of repair is represented by a two-dimensional vector (α_1, α_2). In these policies, α_1 and α_2 correspond to the degree of repair between failures along the time and usage dimensions, respectively. In this case, when α_1 is equal to α_2, the comparison between the repair degree combinations is the same as the one-dimensional case. However, α_1 does not need to be equal to α_2. For example, failed component can be replaced with a less used component at the same age. In this case, comparison between extents of repair is difficult.

5.5.2 Improved Policies

In the improved policy, the product is replaced by an improved one after the first failure. This applies usually to the high-tech products for which a newer, improved version of the product is designed and developed before the failure of the older version. Let β be the degree of improvement between these two versions of the product. For example, if the newer version has a mean time to failure which is 20% larger than that for the failed product, then β is 1.2. For one-dimensional warranties, the inter-failure times under the improved policy can be modeled as follows.

$$\begin{aligned} T_1 &= X_1 \\ T_2 &= \beta X_2 \\ T_i &= \alpha^{i-2}\beta X_i \text{ for all } i \geq 3 \end{aligned} \tag{5.25}$$

where X_i and α are defined the same in the univariate quasi-renewal process. The total warranty cost of the improved policy is defined as follows.

$$C^{(1,imp)} = (c+\beta c_1)+(c+c_1\alpha)(N(W)-1)$$

(5.26)

and the expected cost over the warranty period is

$$EC^{(1,imp)} = (c+\beta c_1)+(c+c_1\alpha)(E(N(W))-1) \tag{5.27}$$

For the two-dimensional warranties, define β_i as the degree of improvement with respect to dimension i such as time and usage. Then, the improved policy for the two-dimensional warranties can be modeled as follows.

$$\begin{aligned} T_1 &= Y_1 \\ T_2 &= \beta_1 Y_2 \\ T_i &= \alpha_1^{i-2}\beta_1 Y_i \text{ for all } i \geq 3 \end{aligned} \quad \text{and} \quad \begin{aligned} X_1 &= Z_1 \\ X_2 &= \beta_2 Z_2 \\ X_i &= \alpha_2^{i-2}\beta_2 Z_i \text{ for all } i \geq 3 \end{aligned} \tag{5.28}$$

where (Y_i, Z_i) and (α_1, α_2) are defined the same in the bivariate quasi-renewal process. The total expected warranty cost of the two-dimensional improved policy is calculated as follow.

$$EC^{(2,imp)} = (c + \beta_1 c_1 + \beta_2 c_2) + (c + \alpha_1 c_1 + \alpha_2 c_2)(E(N(W, U))\text{-}1) \tag{5.29}$$

5.5.3 Dynamic Policies

In contrast to static policies, in the dynamic policy, the extent of repair is not constant over the warranty period. In the dynamic policy, the extent of each repair changes as a decreasing function of the time of the breakdowns. As the time gets closer to the end of warranty period, the extent of repair decreases. The motivation for this policy is to decrease the expected cost by repairing the product to the extent that would carry it in an operational state until the end of the warranty period. We think that the dynamic policies may dominate the static policies since in the latter one; the failure is rectified with the same level of repair even if there is little time left until the end of warranty period. The failure time model under the dynamic policy for one-dimensional case has the following form.

$$\begin{aligned} &T_1 = X_1 \\ &T_i = \alpha(\sum_{k=1}^{i-1} T_k) X_i \quad i \geq 2 \end{aligned} \tag{5.30}$$

where X_i's are independently and identically distributed random variables with the probability density function $f(x_i)$ and $\alpha(t)$ is a non-increasing function of t which gives the degree of the repair for a failure that occurs at time t with the following general form.

$$\alpha(t) = a + bt + ct^2$$

where a, b and c are constant real numbers. It is preferable for the function $\alpha(t)$ to be concave as a good repair becomes increasingly undesirable towards the end of warranty period. Thus, the rate of decline in function alpha increases as the time approaches to the end of warranty period. An example repair degree function is given in Figure 5.2. This function alters the degree of repair between 1 and 0.75 according to the time of failure. In our computational study, we will use this particular function for the univariate and bivariate cases.

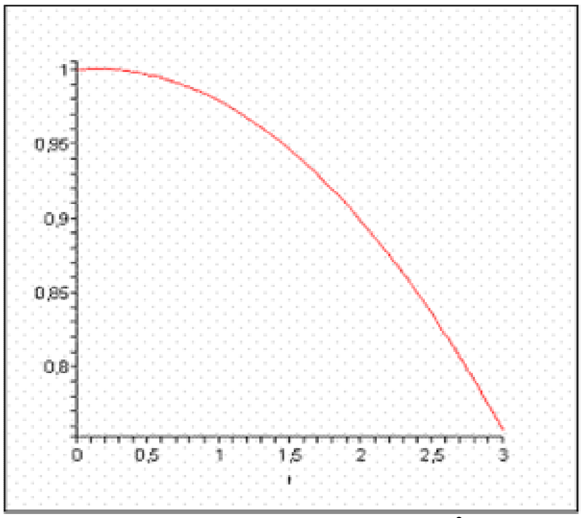

Figure 5.2: $\alpha(t)$=0.991+0.0093t-0.03t^2

The warranty cost of this new policy for one-dimensional warranty is calculated as follows:

$$C^{(1,dyn)}(W) = \sum_{i=1}^{\infty}[c + c_1 * \alpha(\sum_{k=1}^{i} T_k)] \quad (5.31)$$

and the expected cost over the warranty period is:

$$EC^{(1,dyn)} = E(C^{(1,dyn)}(W)) = \sum_{i=1}^{E(N(W))} [c + c_1 E(\alpha(\sum_{k=1}^{i} T_k))] \quad (5.32)$$

For the expected cost defined above, a bound can be found by replacing all $E(\alpha(\sum_{k=1}^{i} T_k))$, i>3 with $E(\alpha(\sum_{k=1}^{2} T_k))$. This bound provides an upper bound since $\alpha(t)$ is a monotone decreasing function with respect to t. If $T_1 = X_1$ and

$T_2 = \alpha(T_1)X_2 = \alpha(X_1)X_2$, then the formulation of $E(\alpha(T_1))$ and $E(\alpha(T_1+T_2))$ can be written in the following way for $\alpha(t) = a + bt + ct^2$

$$
\begin{aligned}
&E(\alpha(T_1)) = E(\alpha(X_1)) = E(a + bX_1 + cX_1^2) \\
&= a + bE(X_1) + cE(X_1^2) \qquad (5.33) \\
&= a + bE(X_1) + c(V(X_1) + E^2(X_1))
\end{aligned}
$$

and

$$
\begin{aligned}
&E(\alpha(T_1+T_2)) = E(a + b(T_1+T_2) + c(T_1+T_2)^2) \\
&= a + b[E(T_1) + E(T_2)] + c[V(T_1 + T_2) + E^2(T_1+T_2)] \\
&= a + b[\underbrace{E(T_1)}_{E(X_1)} + E(T_2)] + c[\underbrace{V(T_1)}_{V(X_1)} + V(T_2) + 2\operatorname{cov}(T_1,T_2) + (E(T_1)+E(T_2))^2] \\
&= a + b[E(X_1) + E(T_2)] + c[V(X_1) + V(T_2) + 2\operatorname{cov}(T_1,T_2) + E^2(T_1) + E^2(T_2) + 2E(T_1)E(T_2)]
\end{aligned}
$$

where

$$
\begin{aligned}
E(T_2) &= E(\alpha(T_1)X_2) \\
&= E((a + bX_1 + cX_1^2)X_2) \\
&= aE(X_2) + bE(X_1X_2) + cE(X_1^2X_2)
\end{aligned}
$$

Since X_1 and X_2 are independent random variables, expectation of two random variables product can be written:

$$E(X_1X_2) = E(X_1)E(X_2)$$

and so,

$$E(T_2) = aE(X_2) + b(E(X_1)E(X_2)) + c(\underbrace{E(X_1^2)}_{V(X_1)+E^2(X_1)} E(X_2)) \qquad (5.35)$$

On the other hand,

$$
\begin{aligned}
V(T_2) &= V(\alpha(T_1)X_2) \\
&= V((a+bX_1+cX_1^2)X_2) \\
&= a^2V(X_2)+b^2V(X_1X_2)+c^2V(X_1^2X_2)+2ab\operatorname{cov}(X_2,X_1X_2) \\
&\quad +2ac\operatorname{cov}(X_2,X_1^2X_2)+2bc\operatorname{cov}(X_1X_2,X_1^2X_2)
\end{aligned}
$$

$$
\begin{aligned}
&= a^2V(X_2)+b^2[E(X_1^2X_2^2)-E^2(X_1X_2)] \\
&\quad +c^2[E(X_1^4X_2^2)-E^2(X_1^2X_2)]+2ab\operatorname{cov}(X_2,X_1X_2) \\
&\quad +2ac\operatorname{cov}(X_2,X_1^2X_2)+2bc\operatorname{cov}(X_1X_2,X_1^2X_2)
\end{aligned}
\tag{5.36}
$$

$$
\begin{aligned}
&= a^2V(X_2)+b^2[E(X_1^2)E(X_2^2)-E^2(X_1)E^2(X_2)] \\
&\quad +c^2[E(X_1^4)E(X_2^2)-E^2(X_1^2)E^2(X_2)]+2ab\operatorname{cov}(X_2,X_1X_2) \\
&\quad +2ac\operatorname{cov}(X_2,X_1^2X_2)+2bc\operatorname{cov}(X_1X_2,X_1^2X_2)
\end{aligned}
$$

Since,

$$
\begin{aligned}
\operatorname{cov}(X_1,X_2) &= E((X_1-E(X_1))(X_2-E(X_2))) \\
&= E(X_1X_2)-E(X_1)E(X_2),
\end{aligned}
$$

$V(T_2)$ can be written as :

$$
\begin{aligned}
V(T_2) &= a^2V(X_2)+b^2[E(X_1^2)E(X_2^2)-E^2(X_1)E^2(X_2)] \\
&\quad +c[E(X_1^4)E(X_2^2)-E^2(X_1^2)E^2(X_2)] \\
&\quad +2ab[E(X_2X_1X_2)-E(X_2)E(X_1X_2)] \\
&\quad +2ac[E(X_2X_1^2X_2)-E(X_2)E(X_1^2X_2)] \\
&\quad +2bc[E(X_1X_2X_1^2X_2)-E(X_1X_2)E(X_1^2X_2)]
\end{aligned}
$$

$$
\begin{aligned}
V(T_2) &= a^2V(X_2)+b^2[E(X_1^2)E(X_2^2)-E^2(X_1)E^2(X_2)] \\
&\quad +c[E(X_1^4)E(X_2^2)-E^2(X_1^2)E^2(X_2)] \\
&\quad +2ab[E(X_1)E(X_2^2)-E(X_1)E^2(X_2)] \\
&\quad +2ac[E(X_1^2)E(X_2^2)-E(X_1^2)E^2(X_2)] \\
&\quad +2bc[E(X_1^3)E(X_2^2)-E(X_1)E(X_1^2)E^2(X_2)]
\end{aligned}
\tag{5.37}
$$

Lastly,

$$
\begin{aligned}
\operatorname{cov}(T_1,T_2) &= \operatorname{cov}(T_1,\alpha(T_1)X_2) \\
&= \operatorname{cov}(X_1,(a+bX_1+cX_1^2)X_2) \\
&= a\underbrace{\operatorname{cov}(X_1,X_2)}_{0} + b\operatorname{cov}(X_1,X_1X_2) \\
&\quad + c\operatorname{cov}(X_1,X_1^2X_2) \\
&= b[E(X_1X_1X_2) - E(X_1)E(X_1X_2)] \\
&\quad + c[E(X_1X_1^2X_2) - E(X_1)E(X_1^2X_2)] \\
&= b[E(X_1^2)E(X_2) - E^2(X_1)E(X_2)] \\
&\quad + c[E(X_1^3)E(X_2) - E(X_1)E(X_1^2)E(X_2)]
\end{aligned}
\tag{5.38}
$$

In this way, we can write $E(\alpha(T_1))$ and $E(\alpha(T_1+T_2))$. Some examples for expected repair degree are given in Table 5.1. In Table 5.1, zero expectation means that the product quality is so high that the probability of observing the second failure is negligible.

Table 5.1: Expected repair degree for the first and second failure

Mean interarrival times(μ_1)	Expected α for the first failure	Expected α for the second failure
0.8	0.987	1.000
1.4	0.951	0.903
2	0.891	0.538
2.6	0.809	0.000
3.2	0.703	0.000
3.8	0.575	0.000
4.4	0.424	0.000
5	0.250	0.000

In the two-dimensional policies, we first consider the use of the same the extent of repair at any given instance in time in both dimensions. Mathematical formulation of this method is:

$$T_1 = Y_1 \qquad\qquad X_1 = Z_1$$
$$T_i = \alpha(\sum_{k=1}^{i-1} T_k)Y_i \quad i \geq 2 \quad \text{and} \quad X_i = \alpha(\sum_{k=1}^{i-1} T_k)Z_i \quad i \geq 2 \qquad (5.39)$$

The total expected warranty cost is calculated

$$EC^{(2,dyn)} = E(C^{(2,dyn)}(W,U)) = \sum_{i=1}^{E(N(W,U))} [c + c_1 E(\alpha(\sum_{k=1}^{i} T_k)) + c_2 E(\alpha(\sum_{k=1}^{i} T_k))] \qquad (5.40)$$

For this method, the upper bound of the expected cost can be found in a similar manner with univariate case. For the second and subsequent failures, we use the expected repair degree of the second failure in the cost approximation. To determine the repair degree of time dimension, we use the marginal distribution of the time in equation 5.33 and 5.34.

In the second method, both dimensions are rectified with the extent of repair such that $\frac{\alpha_1(t)}{\alpha_2(t)} = \frac{\mu_1}{\mu_2}$ for all t with the condition that $\alpha_1(t),\ \alpha_2(t) \leq 1$. In this method, both dimensions are rectified with the same proportion. That is, the usage repair degree is chosen such that the ratio of the repair degree of time dimension to its first interfailure time mean is equal to the ratio of repair degree of usage to its mean. Mathematically, the second model is:

$$T_1 = Y_1 \qquad\qquad X_1 = Z_1$$
$$T_i = \alpha(\sum_{k=1}^{i-1} T_k)Y_i \quad i \geq 2 \quad \text{and} \quad X_i = \alpha_{2,i} Z_i \quad i \geq 2 \qquad (5.41)$$

where $\alpha_{2,i}$ is the extent of repair for the usage dimension such that $\frac{\alpha(\sum_{k=1}^{i-1} T_k)}{\mu_1} = \frac{\alpha_{2,i}}{\mu_2} \Rightarrow \alpha_{2,i} = \alpha(\sum_{k=1}^{i-1} T_k)\frac{\mu_2}{\mu_1}$ and (μ_1, μ_2) is vector that indicates time and usage mean until the first failure. The total expected cost is similar to the previous case. That is:

$$EC^{(2,dyn2)} = E(C^{(2,dyn2)}(W,U)) = \sum_{i=1}^{E(N(W,U))} [c + c_1 E(\alpha(\sum_{k=1}^{i} T_k)) + c_2 \frac{\mu_2}{\mu_1} E(\alpha(\sum_{k=1}^{i} T_k))]$$

(5.42)

Chapter 6

SOLUTION APPROACH

In order to make comparisons between different extents of repair, we use the following cost functions.

$$EC^{(1)} = E(C(W, \alpha_i)) = \sum_{i=1}^{E(N(W))} (c + c_1 \alpha_i)$$

for one-dimensional warranty and

$$EC^{(2)} = E(C^{(2)}(W, U, \alpha_{1i}, \alpha_{2i})) = \sum_{i=1}^{E(N(W,\,U))} (c + c_1 \alpha_{1i} + c_2 \alpha_{2i})$$

for two-dimensional warranty. To calculate these cost functions, we need to calculate the expected number of breakdowns over the warranty region. Note however that, this expectation is equal to an infinite sum of a series convolution, i.e. $M_q(W) = \sum_{n=1}^{\infty} F^{(n)}(W)$ for the one-dimensional case and $M_q^2(W,\ U) = \sum_{n=1}^{\infty} F^{(n)}(W,\ U)$ for the two-dimensional case. Due to the intractability of this expectation, we propose the use of a numerical integration method to approximate each convolution within the expectation. The numerical integration method that we choose for our calculation is the Composite Simpson's rule. It is based on Simpson's rule which approximates the integral of *f*(*x*) using a quadratic polynomial. The quadratic polynomial in Simpson's rule is chosen such that it takes the same values as *f*(*x*) at the end and midpoint of the integral. If x_i's i=1,2,…,2n are equally spaced points separated by a distance h, then the mathematical formulation of the Simpson's approximation is given as follows.

$$\int_{x_0}^{x_0+2h} f(x)dx = \frac{h}{3}\left[f(x_0) + 4f(x_0+h) + f(x_0+2h)\right]$$

Simpson's rule provides a good approximation when the interval of integration is small. If the interval is not small, the Composite Simpson's rule is more adequate. This method is an extension of Simpson's rule. In the Composite Simpson, the integration interval is divided into equally spaced M intervals. Then, for each interval the Simpson's rule is applied. Thus, the mathematical formulation of the Composite Simpson is stated as follows.

$$\int_{x_0}^{x_{2n}} f(x)dx = \frac{h}{3}\left[f(x_0) + 4\left(f(x_1) + f(x_3) + ... + f(x_{2n-1})\right) + 2\left(f(x_2) + f(x_4) + ... + f(x_{2n-2})\right) + f(x_{2n})\right]$$

$$= \frac{h}{3}\left[f(x_0) + f(x_{2n})\right] + \frac{2h}{3}\sum_{k=1}^{M-1} f(x_{2k}) + \frac{4h}{3}\sum_{k=1}^{M} f(x_{2k-1})$$

where $2M+1$ is the number of equally spaced points, h ($h=(x_{2n}-x_0)/2M$) is length between every two consecutive points and $x_k = x_0 + hk$ for k=0,1,…,2M. For a single integral, the application is rather straightforward. On the other hand, for a multidimensional integral, we need to evaluate the integrals in an iterative manner. Suppose, we have an n-dimensional integral in the following form:

$$F^{(n)}(t) = P(X_1 + X_2 + ... + X_n \le t)$$

$$= \int_{x_1=0}^{t} \int_{x_2=0}^{t-x_1} \int_{x_3=0}^{t-x_1-x_2} ... \int_{x_n=0}^{t-x_1-x_2-...x_{n-1}} f_{X_1}(x_1) f_{X_2}(x_2) f_{X_3}(x_3) ... f_{X_n}(x_n) dx_n ... dx_3 dx_2 dx_1$$

To approximate this multiple integral, we first apply the Composite Simpson's rule to the last integral, then the second last and so on through the first one at the end.

The application of the Composite Simpson rule for the n-dimensional integral in the context of our warranty analysis is summarized in the following algorithm.

***Algorithm* 1:**

W: warranty period

B: upper limit of the subsequent integral

h: interval length

n: number of integral

n' : number of integrals left

$f_i(x_i)$: density function of x_i i=1,2,…,n

1. Initialization:

   Set B=W

   Set $n' = n$

   $F^{(n)}(W) = 0$: initial value of convolution

   $Sumeven = \sum_{k=1}^{M-1} f(x_{2k}) = 0$: sum of even points' value

   $Sumodd = \sum_{k=1}^{M} f(x_{2k-1}) = 0$: sum of odd points' value

2. Evaluate the following function

   Set $M = \left\lceil \frac{B}{2h} \right\rceil$ where $\lceil a \rceil$ ($\lfloor a \rfloor$) is the smallest (largest) integer greater (less) than a and revise the interval length $h = B/2M$

   $Simpson(n', B)$ {

   2.1. If $n' > 1$

   For k=1 to M–1 Do

$Sumeven+= f_{n-n'+1}(2kh)*Simpson(n'-1, B-2kh)$

$Sumodd+= f_{n-n'+1}((2k\text{-}1)h)*Simpson(n'-1, B-(2k-1)h)$

End For

$Sumodd+= f_{n-n'+1}((2M\text{-}1)h)*Simpson(n'-1, B-(2M-1)h)$

$$F^{(n)} = \frac{h}{3}\left[f_{n-n'+1}(0)*Simpson(n'-1,B) + f_{n-n'+1}(W)*\underbrace{Simpson(n'-1,0)}_{0} \right] + \frac{2h}{3}sumeven + \frac{4h}{3}sumodd$$

Return $F^{(n)}$

2.2. Else

For k=1 to M–1 Do

$Sumeven+= f_n(2kh)$

$Sumodd+= f_n((2k-1)h)$

End For

$Sumodd+= f_n((2M-1)h)$

Return $\frac{h}{3}\left[f_n(0)+f_n(y)\right]+\frac{2h}{3}sumeven+\frac{4h}{3}sumodd$

} end $Simpson(n', B)$

3. Print $F^{(n)}$

The above algorithm is instrumental in integrating a convolution with a given number of integrals; however, the expectation is stated in the form of an infinite sum. Hence, we should truncate this infinite sum for a numerical analysis. The truncated summation gives us an approximation for the expectation. One way for truncation is to set a limit on the number of integrals at the outset. However, such an approach would not allow for a direct control over accuracy. An alternative method is to set the desired accuracy and stop when it is achieved. In this thesis, we choose truncation based on accuracy. We assume that if the probability of the n^{th} failure over the warranty region is less than 0.0001, then the probability of occurring n+1 or more failures is much smaller. Thus, to calculate further convolutions does not make much contribution to the expectation bound. Hence, if the value of the last convolution is less than 0.0001, we truncate the summation.

The above algorithm is given for one-dimensional warranties, but it can be extended for two-dimensional warranties. For two-dimensional warranties, the n-fold convolution is the following form:

$$F^{(n)}(t,x) = P(T_1 + ... + T_n \le t; X_1 + ... + X_n \le x)$$

$$= \int_{t_1=0}^{t} \int_{x_1=0}^{x} \cdots \int_{t_n=0}^{t-\sum_{i=1}^{n-1} t_i} \int_{x_n=0}^{x-\sum_{i=1}^{n-1} x_i} f_{T_1,X_1}(t_1,x_1) \ldots f_{T_n,X_n}(t_n,x_n)\, dx_n dt_n \ldots dx_1 dt_1$$

The approximation algorithm for $F^{(n)}(t,\ x)$ is given below. In this algorithm, we start applying the numerical method to t_1 and revise the upper bound of next integral with respect to t (Step 2.1). Then, for each value of t_1, we apply the method to x_1 and revise the upper bound of next integral with respect to x (Step 2.2.2). We repeat Step 2.1 and Step 2.2.2 through the last integral to the first integral. For the last integral, i.e. integral with respect to x_n, we apply Step 2.2.1. and we stop.

***Algorithm* 2:**

(W, U): warranty regions

K: upper limit of subsequent integral with respect to t

L: upper limit of subsequent integral with respect to x

h_1, h_2: interval length for t_i and x_i

n: number of integral

n' : number of integrals left

$f_i(t_i, x_i)$: joint density function of (t_i, x_i) i=1, 2, …, n

1. Initialization:

Set $K=W$

Set $L=U$

Set $n' = 2n$

$F^{(n)}(W, U) = 0$: initial value of convolution

Sumeven = 0: sum of even points' value

Sumodd = 0: sum of odd points' value

2. Evaluate the following recursive function

$Simpson(n', K, t, L)\{$

2.1. If mod(n', 2) != 1

Set $M_1 = \left\lceil \frac{K}{2h_1} \right\rceil$ and revise the new interval length $h_1 = K/\,2M_1$

For k=1 to M_1–1 Do

$Sumeven\mathrel{+}= Simpson(n'-1,\, K-2kh_1,\, 2kh_1,\, L)$

$Sumodd\mathrel{+}= Simpson(n'-1,\, K-(2k-1)h_1,\, (2k-1)h_1,\, L)$

End For

$Sumodd\mathrel{+}= Simpson(n'-1,\, K-(2M_1-1)h_1,\, (2M_1-1)h_1,\, L)$

$$F^{(n)} = \frac{h_1}{3}\left[Simpson(n'-1, K, 0, L) + \underbrace{Simpson(n'-1, 0, K, L)}_{0} \right] + \frac{2h_1}{3} sumeven + \frac{4h_1}{3} sumodd$$

Return $F^{(n)}$

2.2. If mod(n', 2) = 1

Set $M_2 = \left\lceil \frac{L}{2h_2} \right\rceil$ and revise the new interval length $h_2 = L/\ 2M_2$

2.2.1 If $n' = 1$

For k=1 to $M_2 - 1$ Do

$Sumeven$+ = $f_n(t, 2k\, h_2)$

$Sumodd$+ = $f_n(t, (2k-1)h_2)$

End For

$Sumodd$+ = $f_n(t, (2M_2 - 1)\, h_2)$

Return $\frac{h_2}{3}\left[f_n(t,0) + f_n(t,L)\right] + \frac{2h_2}{3} sumeven + \frac{4h_2}{3} sumodd$

2.2.2 Else

For k=1 to M_2–1 Do

$Sumeven$+= $f_{n-\lfloor n'/2 \rfloor}(t, 2kh_2)*Simpson(n'-1, K, t, L\text{-}2kh_2)$

$Sumodd$+= $f_{n-\lfloor n'/2 \rfloor}(t, (2k\text{-}1)h_2)*Simpson(n'-1, K, t, L\text{-}(2k\text{-}1)h_2)$

End For

$Sumodd$+= $f_{n-\lfloor n'/2 \rfloor}(t, (2M_2\text{-}1)h_2)*Simpson(n'-1, K, t, L\text{-}(2M_2\text{-}1)h_2)$

Return

$$\frac{h_2}{3} f_{n-\lfloor n'/2 \rfloor}(t,0) * Simpson(n'-1, K, t, L) + \frac{2h_2}{3} sumeven + \frac{4h_2}{3} sumodd$$

} End $Simpson(n'-1, K - 2kh_1, 2kh_1, L)$

3. Print $F^{(n)}$

As in the convolution of a univariate distribution, finding the convolution of bivariate distribution is a computationally expensive look for large n even with the numerical method. Like the one-dimensional case, we use lower bound on the expected number of breakdowns for the two-dimensional warranty analysis. The lower bound is calculated using accuracy-based truncation rule as in the one-dimensional case.

To evaluate the performance of our numerical approximation, we use the normal distribution to model failure interarrival times. If X_n's are independent and identically distributed with normal distribution, $N(\mu, \sigma^2)$, then the distribution of $T_n= \alpha^{n-1}X_n$, n=1, 2, 3… also follows a normal distribution with mean $\alpha^{n-1}\mu$ and variance $\alpha^{2(n-1)}\sigma^2$. Thus, $T_1 +\ldots+ T_n$ has a mean of $(1+\alpha+\ldots+\alpha^{n-1})\mu$ and a variance of $(1+\alpha^2+\ldots+\alpha^{2(n-1)})\sigma^2$. That is, the n-fold convolution of F, $F^{(n)}$, is distributed as $N\left(\frac{1-\alpha^n}{1-\alpha}\mu, \frac{1-\alpha^{2n}}{1-\alpha^2}\sigma^2\right)$. The expected warranty costs under different parameter sets calculated with the numerical and analytical method are showed in Table 6.1 and 6.2. In both numerical and the analytical method, we use the truncation rule with a desired accuracy of 0.0001. The results indicate that the difference between the numerical and analytical method is negligible.

Table 6.1: Errors between the numerical and analytical method

Mean time to first failure	Alpha	Expectation (Numerical Method)	Expectation (Analytical)	Error (Difference)
1	1	2.56503	2.52058	-0.04
2	1	1.05602	1.05617	0.00
3	1	0.50225	0.50234	0.00
4	1	0.15881	0.15886	0.00
5	1	0.05480	0.05484	0.00
1	0.98	2.63949	2.61832	-0.02
2	0.90	1.09533	1.09602	0.00
3	0.80	0.50615	0.50630	0.00
4	0.70	0.15951	0.15958	0.00
5	0.50	0.05538	0.05544	0.00

Table 6.2: Errors between the numerical and analytical method for a given mean time to first failure

Mean time to first failure	Alpha	Expectation (Numerical Method)	Expectation (Analytical)	Error (Difference)
2	1	1.05602	1.05617	0.00
2	0.90	1.09533	1.09602	0.00
2	0.88	1.10544	1.10649	0.00
2	0.8	1.15611	1.16238	0.01
4	1	0.15881	0.15886	0.00
4	0.90	0.15891	0.15897	0.00
4	0.70	0.15951	0.15958	0.00
4	0.50	0.16245	0.16323	0.00

Chapter 7

COMPUTATIONAL STUDY

In this chapter, we will present computational results with one- and two-dimensional warranties and discuss the behavior of the expected warranty cost under different parameter settings in different types of policies. Firstly, the experimental design for the computational study is presented. Then, the results of one- and two-dimensional warranties are discussed.

7.1 EXPERIMENTAL DESIGN

The parameters that we vary in the computational study consist of product quality in terms of the reliability structure, extent of repair for static policies and the ratio between the fixed and variable components of the repair cost. Firstly, we manipulate the reliability structure of the product by changing the mean of the interval time between the first and second failures, i.e. μ_1, for one-dimensional warranties. If this mean is large compared to the warranty limit, i.e. ratio of mean to warranty limit is larger than 1, then we say that the product is of high quality. If this ratio is less than 0.5, we say that the product is of low quality. For other values of ratio, we call the product is of medium quality. Similarly, for two-dimensional warranties, quality of the product is determined for each dimension quality in a similar manner. Secondly, the degree of repair, i.e. α, in the computational study can take non-negative real values less than or equal to 1 where α=1 corresponds to perfect repair (replacement). In our experiments, we restrict the extent of repair to take the values between 0.5 and 1 for the static policies. Lastly, as we stated in Chapter 5, we have two components of repair cost: one is fixed, c; the other is variable, c_1 and c_2. For each type of product, we examine the effect of different cost ratios (c/c_1 in the univariate case, and c/c_1, c/c_1, c_1/c_2 in the bivariate case) on the preferred cost repair policy. We assume that the warranty period is fixed for 3 years for one-dimensional warranty policies and 3 years, 30,000km (3 yr, 3 km) for two-dimensional policies. In the following, we present the computational results.

7.2 COMPUTATIONAL RESULTS

We discuss the results for one- and two-dimensional warranties in Section 7.2.1 and 7.2.2, respectively.

7.2.1 One-dimensional Warranties

In this section, we first discuss the static policies where the interfailure distribution is normal and weibull. Then, for each failure distribution the results with the improved and dynamic repair policies are compared to the results with the optimal static policy. Lastly, these three repair policies are compared simultaneously to give insights for different cost components and product reliability settings.

Case 1: Static Policies for Normal and Weibull Failure Distribution

The following expected number of failures is approximated using Algorithm 1 given in Chapter 6.

$$M_q(t) = E[N(t)] = \sum_{n=0}^{\infty} nP(N(t) = n) = \sum_{n=1}^{\infty} F^{(n)}(t)$$

where

$$F^{(n)}(t) = \int_{x_1=0}^{t} \int_{x_2=0}^{t-x_1} \int_{x_3=0}^{t-x_1-x_2} \cdots \int_{x_n=0}^{t-x_1-x_2-\ldots x_{n-1}} f_{X_1}(x_1) f_{X_2}(x_2) f_{X_3}(x_3) \ldots f_{X_n}(x_n) dx_n \ldots dx_3 dx_2 dx_1$$

If Y_i in equation 5.1 is normally distributed with $N(\mu, \sigma^2)$, then the distribution function of the interval between $(i\text{-}1)^{\text{th}}$ and i^{th} failures has the following form.

$$F_{X_i}(x_i) = P(X_i \le x_i) = P(\alpha^{i-1} Y_i \le x_i)$$
$$= \int_{-\infty}^{\alpha^{1-i} x_i} \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(y_i - \mu)^2}{2\sigma^2}) dy_i$$

and the density function is

$$f_{X_i}(x_i) = \frac{\alpha^{1-i}}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(\alpha^{1-i} x_i - \mu)^2}{2\sigma^2})$$

Similarly, if Y_i is a univariate weibull random variable with Wei(γ, φ), then the distribution function of the interval between $(i\text{-}1)^{\text{th}}$ and i^{th} has the following form.

$$F_{X_i}(x_i) = P(X_i \le x_i) = P(\alpha^{i-1} Y_i \le x_i)$$
$$= 1 - \exp\left(-(\frac{\alpha^{1-i} x_i}{\varphi})^{\gamma} \right)$$

and the density function is

$$f_{X_i}(x_i) = \gamma \left(\frac{\alpha^{1-i} x_i}{\varphi} \right)^{\gamma-1} \frac{\alpha^{1-i}}{\varphi} \exp\left(-(\frac{\alpha^{1-i} x_i}{\varphi})^{\gamma} \right)$$

where γ is the shape and φ is the scale parameter.

We select various μ_1 values between 0.5 and 5 for experimental study. For normal distribution case, for each μ_1, we assign a σ^2 so as to maintain a coefficient of variation of ¼. In this way, we force the probability of realizing a negative interfailure time to be negligible. For weibull distribution, we set the shape parameter (γ) to 2, since the shape parameter greater than 1 is suitable for representing the lifetime of a product. To get varies μ_1 values between 0.5 and 5, we change the scale parameter between 0.56 and 5.66.

Table 7.1 shows the expected number of failures of normal distribution for α=0.5, 0.6, 0.7, 0.8, 0.9, 1 with mean interarrival times of 1, 3 and 5. We observe that for a given mean (i.e. μ_1) as α gets smaller, the expected number of failures increases; whereas, for a given α, as mean gets larger, the expected number of failures decreases. This should not be surprising since increasing the extent of repair reduces the deterioration of the product and so the expected number of failures decreases and as the reliability of the product increases, the expected number of failures becomes smaller. Table 7.1 shows that if the product quality is low, the effect of repair degree is greater than the effect in the high quality product. For example, when the degree of repair is halved, the expected number of failure increases approximately by 300% if the mean of first interarrival is 1; whereas the change between the expected number of failures is about 1% when the mean of first interarrival is 5. Table 7.2 shows the relationship between the expected number of failures and the extent of repair for weibull failure distribution. The results of weibull distribution are the same as that of normal distribution.

Table 7.1: Expected number of failures with Normal failure distribution

Alpha	Expected failures with μ_1=1	Expected failures with μ_1=3	Expected failures with μ_1=5
1	2.56503	0.502253	0.054798
0.9	3.01249	0.503636	0.054814
0.8	3.84594	0.50615	0.054845
0.7	5.83678	0.510925	0.054907
0.6	>8	0.520513	0.055045
0.5	>8	0.542316	0.055375

Table 7.2: Expected number of failures with Weibull failure distribution

Alpha	Expected failures with μ_1=1	Expected failures with μ_1=3	Expected failures with μ_1=5
1	2.6034	0.575055	0.256821
0.9	3.05202	0.590264	0.259569
0.8	3.97769	0.612159	0.263441
0.7	5.38634	0.647002	0.269333
0.6	>6	0.714853	0.279043
0.5	>6	0.850689	0.299703

The behavior of the expected warranty cost as a function of the reliability structure and the repair policy is not as straightforward. The ratio of the fixed and variable components (c/c_1) of the cost function also affects this behavior. Figures 7.1-7.16 show the expected warranty cost under different scenarios.

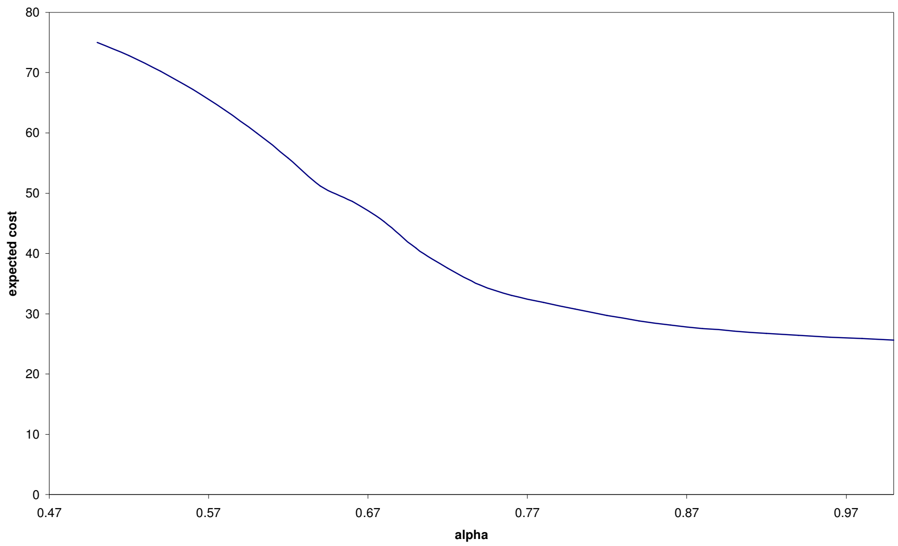

Figure 7.1: Expected cost with normal failure distribution (c/c_1=0, μ_1=1)

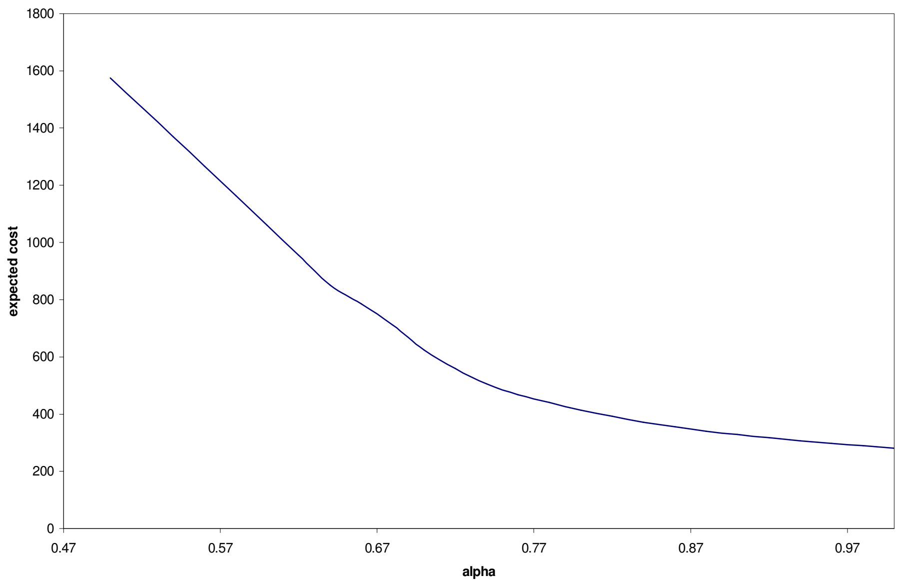

Figure 7.2: Expected cost with normal failure distribution (c/c_1=10, μ_1=1)

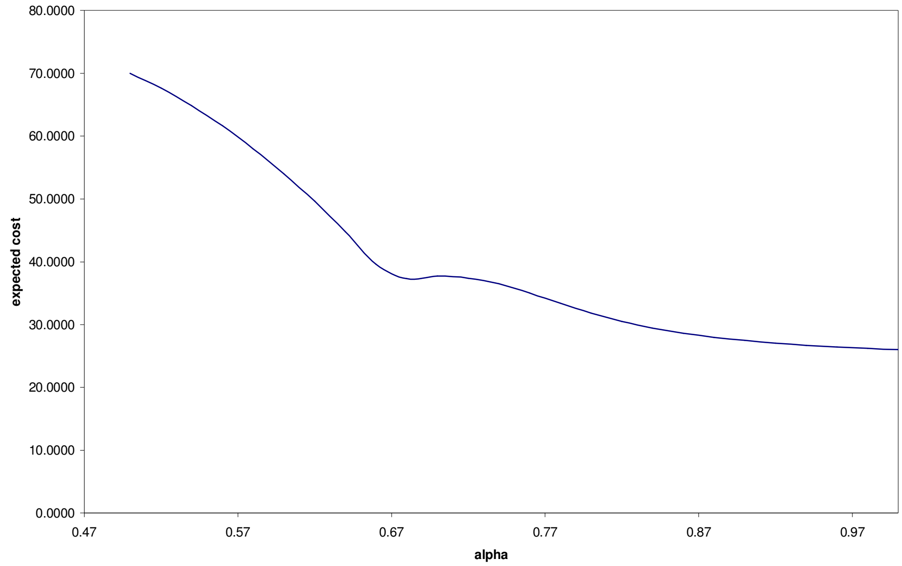

Figure 7.3: Expected cost with weibull failure distribution (c/c_1=0, μ_1=1)

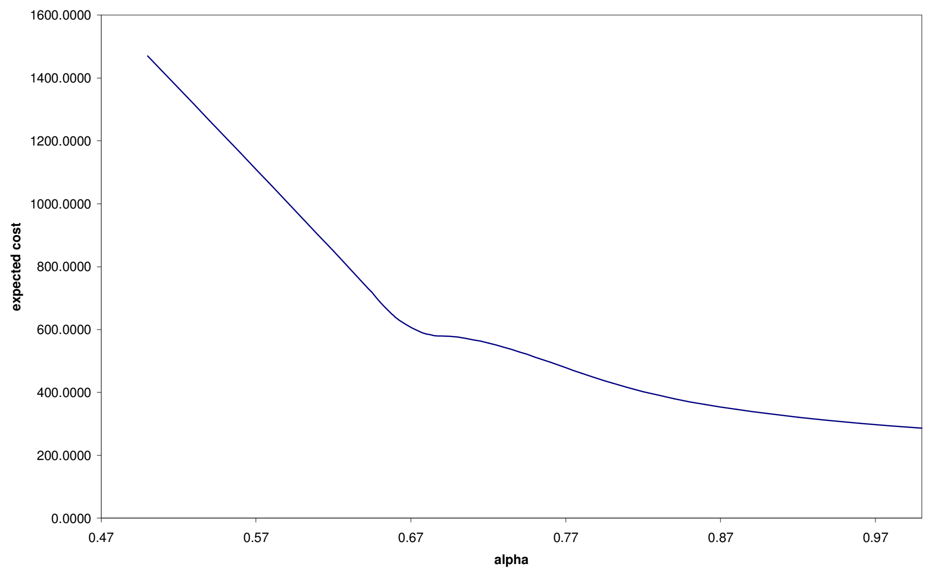

Figure 7.4: Expected cost with weibull failure distribution (c/c_1=10, μ_1=1)

Figures 7.1 and 7.2 display the expected cost as a function of extent of repair (α) for the product with normal failure distribution with μ_1=1 when c/c_1=0 and c/c_1=10, respectively. The expected cost behaves in a similar manner in both cases and it decreases as the extent of repair increases. In the Figure 7.2, the total cost is larger by $10E(N(W))$. From Figures 7.3 and 7.4, we get the same result for a low quality product with weibull failure distribution.

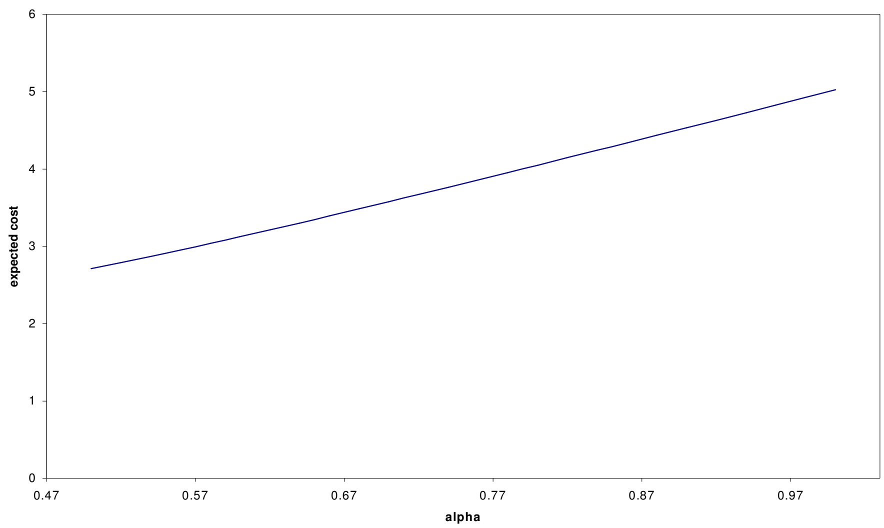

Figure 7.5: Expected cost with normal failure distribution (c/c_1=0, μ_1=3)

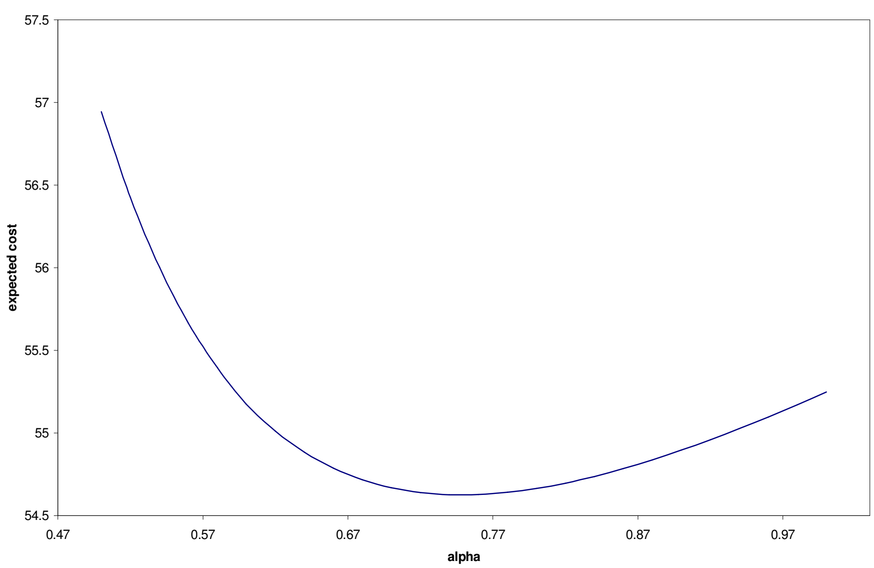

Figure 7.6: Expected cost with normal failure distribution (c/c_1=10, μ_1=3)

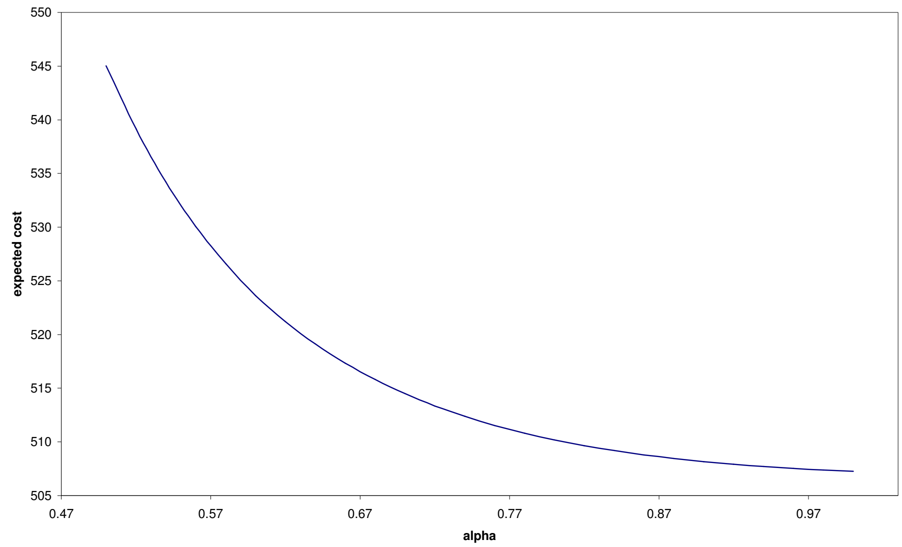

Figure 7.7: Expected cost normal failure distribution (c/c_1=100, μ_1=3)

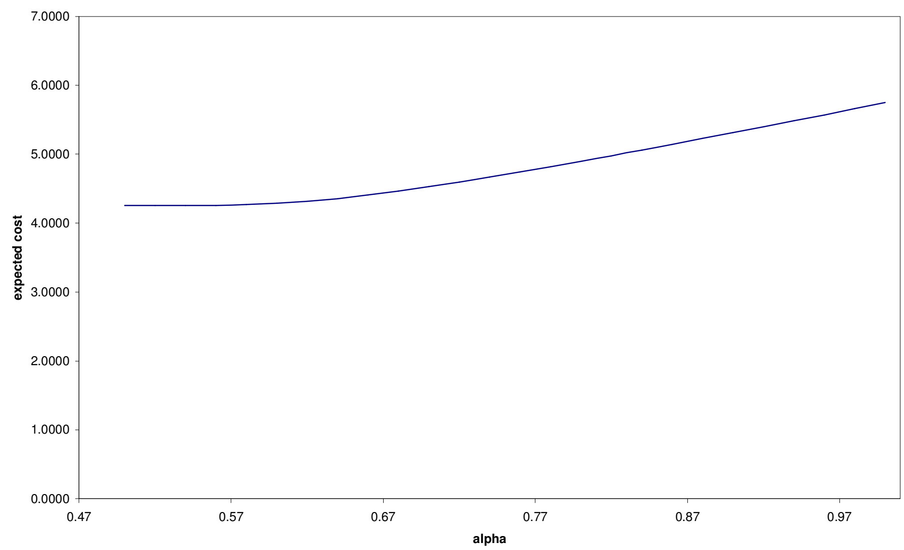

Figure 7.8: Expected cost with weibull failure distribution (c/c_1=0, μ_1=3)

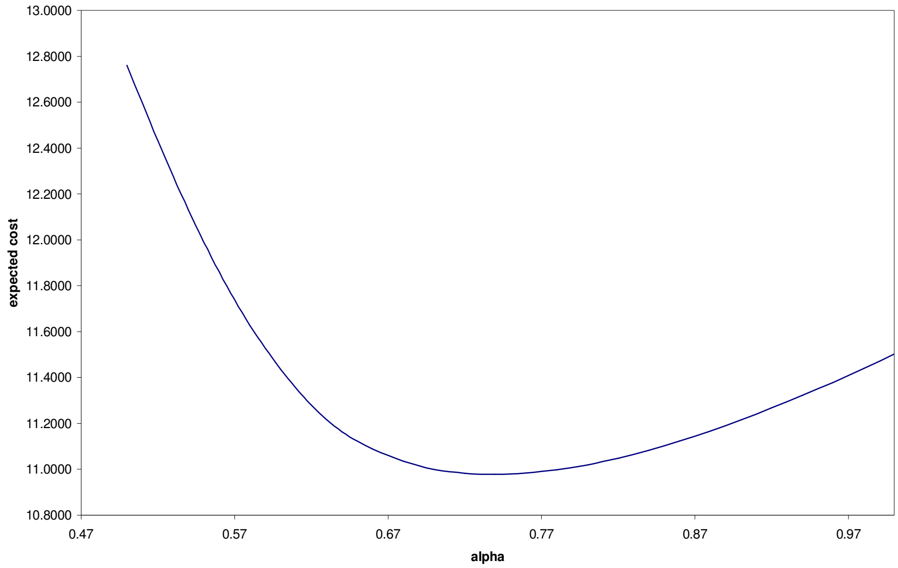

Figure 7.9: Expected cost with weibull failure distribution (c/c_1=1, μ_1=3)

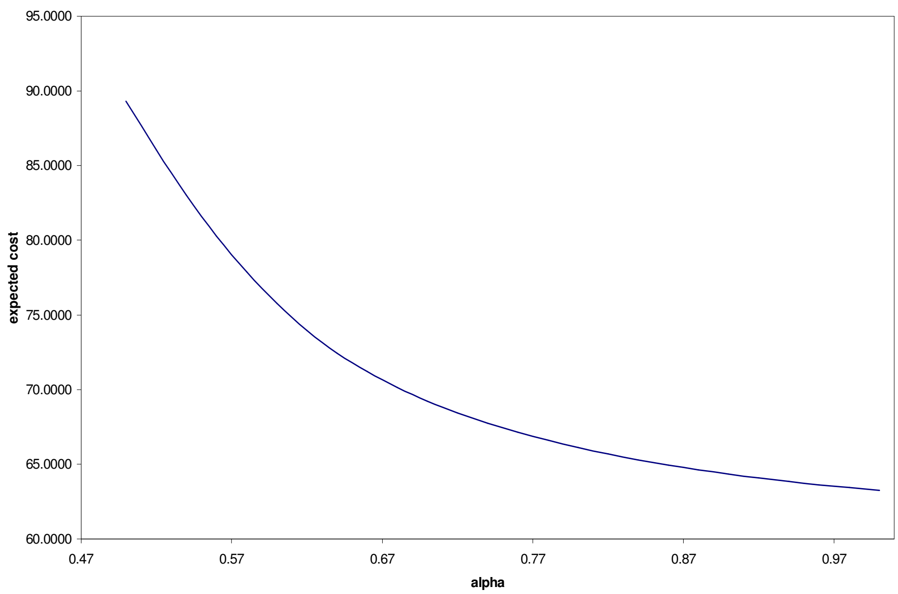

Figure 7.10: Expected cost with weibull failure distribution (c/c_1=10, μ_1=3)

Figures 7.5-7.7 show the expected cost vs. the extent of repair for the product with normal failure distribution with μ_1=3 when c/c_1=0, c/c_1=10 and c/c_1=100, respectively. When the ratio between fixed and variable component is small (e.g.

c/c_1=0), the expected cost shows an increasing linear trend although the expected number of failures decreases as α increases. For the product with average quality, the change in the expected number of failures between different α is small, so the variable component for a given extent of repair, i.e. αc_1, governs the behavior of the cost function. When the ratio between the cost components gets larger (e.g. c/c_1=10), the cost function decreases when α is between 0.5 and 0.74. However, as α is between 0.74 and 1, the cost function shows increasing trend like in the case of c/c_1=0. If the fixed component is too large compared to variable component (e.g. c/c_1=100), then the expected cost function behaves in a similar manner as the previous case (c/c_1=10) in range (0.5, 0.74). For the average quality of product, as the ratio between the fixed and variable component increases, the cost function becomes sensitive to even a small change in the expected number of failures and so the cost function shows a decreasing trend as α increases. Figures 7.8-7.10 show the behavior of the expected cost for an average quality product with weibull failure distribution when c/c_1=0, c/c_1=1 and c/c_1=10, respectively. When the ratio is small (e.g. c/c_1=0), the expected cost shows an increasing trend like in the normal distribution case, but this trend is not linear in the weibull distribution. For an average quality of product with weibull failure distribution, the impact of repair degree is more significant than for that kind of product with normal failure distribution. If the fixed component is equal to the variable component, then the cost function decreases in the repair range (0.5, 0.74) and increases in the range (0.74, 1). This behavior is the same with the case of normal failure distribution with c/c_1=10. For larger ratios (e.g. c/c_1=10), the cost function shows a decreasing trend.

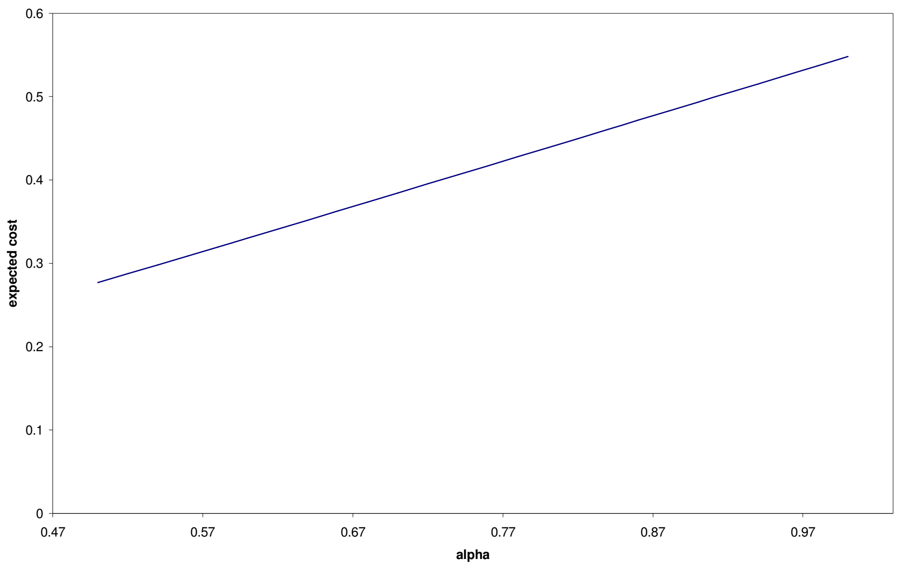

Figure 7.11: Expected cost normal failure distribution (c/c_1=0, μ_1=5)

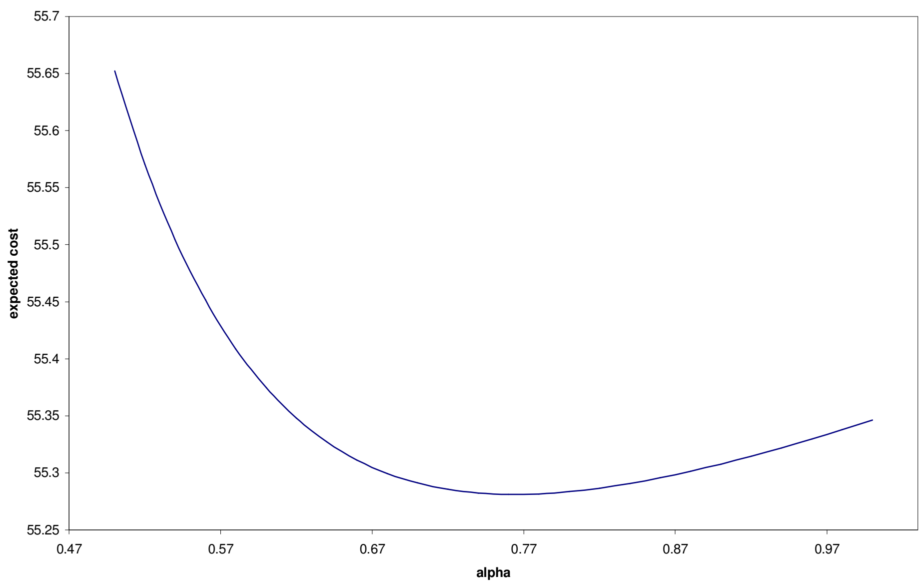

Figure 7.12: Expected cost normal failure distribution (c/c_1=100, μ_1=5)

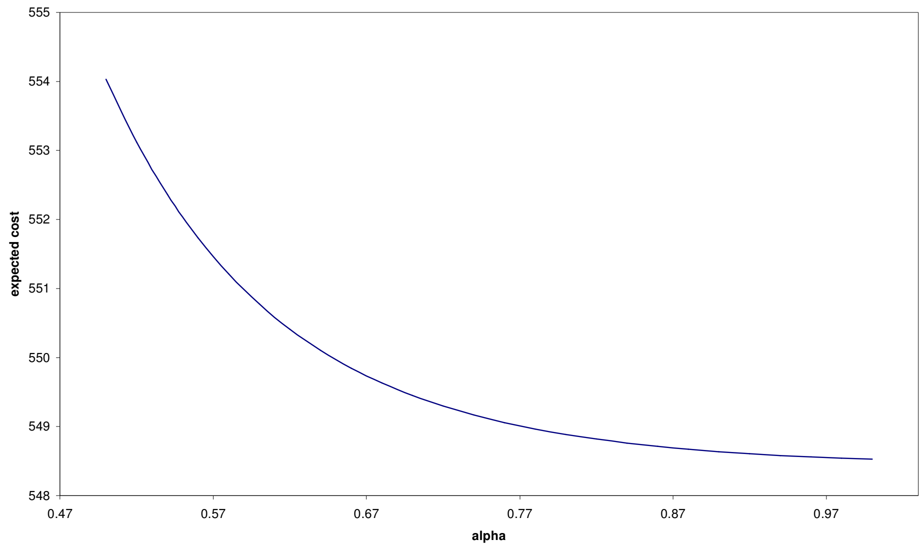

Figure 7.13: Expected cost normal failure distribution (c/c_1=1000, μ_1=5)

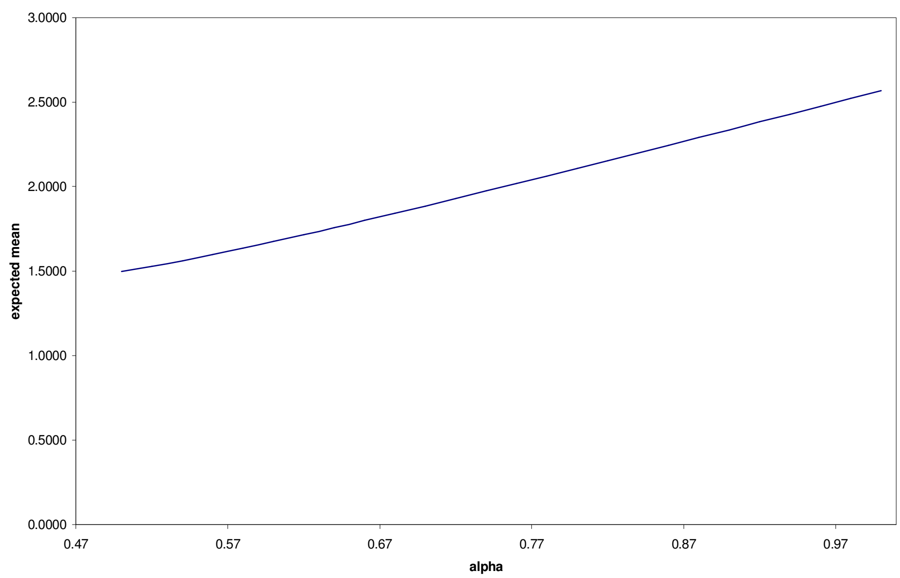

Figure 7.14: Expected cost with weibull failure distribution (c/c_1=0, μ_1=5)

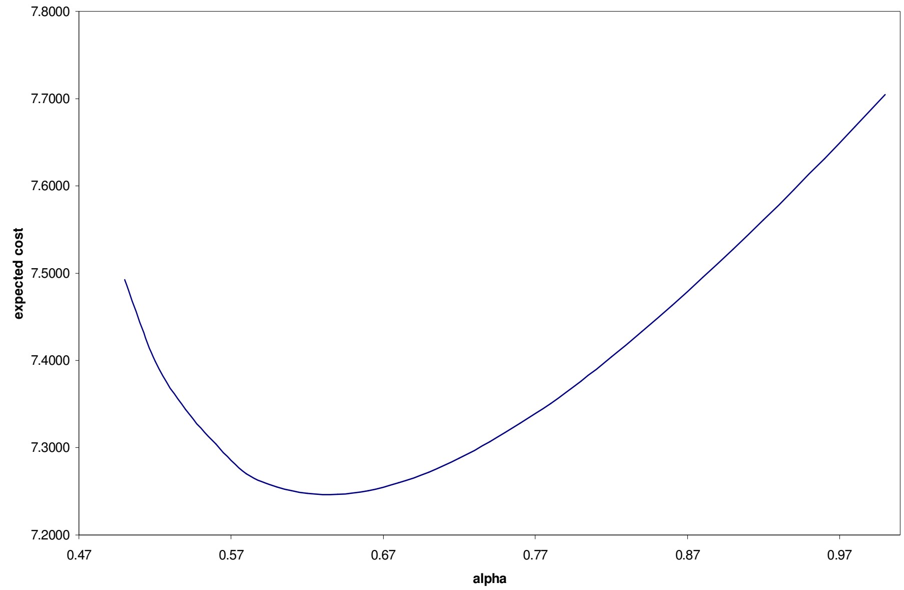

Figure 7.15: Expected cost with weibull failure distribution (c/c_1=2, μ_1=5)

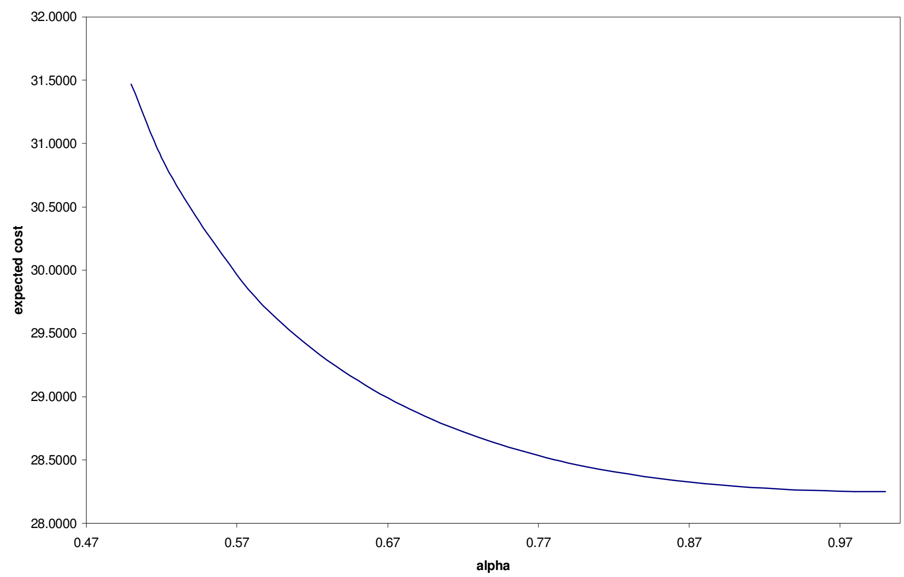

Figure 7.16: Expected cost with weibull failure distribution (c/c_1=10, μ_1=5)

Figures 7.11-7.13 shows the behavior of the expected cost function for the exceptionally reliable product with normal failure distribution, i.e. μ_1=5, when c/c_1=0, c/c_1=100 and c/c_1=1000, respectively. For these products, the cost function shows a similar behavior to the one with the average quality product, but the same trend is observed for larger cost ratios. For example, the average quality of product shows linear increasing trend for the cost ratios between 0 and 10, whereas the exceptionally reliable product shows the same trend for cost ratios less than 100. Similarly, the exceptionally reliable product has a monotone decreasing behavior for ratios greater than 1000, but this trend is observed for ratios greater than 100 for the average quality of product. Figures 7.14-7.16 show the behavior of the cost function for the exceptionally reliable product with weibull failure distribution when c/c_1=0, c/c_1=2 and c/c_1=10, respectively. The results for an exceptionally reliable product with weibull failure distribution are similar with the results for the average quality product, but in this case, the cost function is more sensitive to the increase in the fixed component of the cost than the cost of the average quality product.

In all Figures between 7.1 and 7.16, we observe that as the mean of interarrival increases and the cost ratio decreases, the expected cost function shows an increasing trend. In addition, we see that even if the cost functions show similar trends, the sensitivity of the expected cost to the change in the cost ratio varies for different failure distribution. Tables 7.3 and 7.4 show the optimum degree of repair corresponding to the minimum warranty cost for cost ratios and different mean time to failure with normal and weibull distribution, respectively. For an unreliable product with small values of mean time to first failure, perfect repair is the most suitable repair type for any cost ratio. The reason for this is the significant impact of the degree of repair on the expected number of breakdowns which more than compensates the corresponding increase in the cost. On the other hand, for a more reliable product with a large mean time to first failure, a smaller degree of repair gives the minimum cost when the fixed component of the cost is comparable with the variable component. However, if the fixed component is large, then a more extensive repair (larger α) is needed to lower the expected cost. In addition, for an average quality product, the extent of repair varies as the cost ratio changes. In particular, a more extensive repair is required as c/c_1 ratio increases for a given mean time to first failure. Eventually, when the fixed component hits a certain threshold, then perfect repair is the most preferred option for any type of product.

Table 7.3: Optimum repair degree for various normal first interarrival mean and cost ratios

Mean time to first failure	c/c_1=0	c/c_1=1	c/c_1=10	c/c_1=100	c/c_1=1000
0.5	1	1	1	1	1
0.8	1	1	1	1	1
1.1	1	1	1	1	1
1.4	0.88	1	1	1	1
1.7	0.76	1	1	1	1
2.0	0.62	0.84	1	1	1
2.3	0.50	0.68	1	1	1
2.6	0.50	0.58	0.88	1	1
2.9	0.50	0.52	0.78	1	1
3.2	0.50	0.50	0.70	1	1
3.5	0.50	0.50	0.64	0.98	1
3.8	0.50	0.50	0.60	0.92	1
4.1	0.50	0.50	0.58	0.86	1
4.4	0.50	0.50	0.54	0.82	1
4.7	0.50	0.50	0.52	0.80	1
5.0	0.50	0.50	0.50	0.76	1

Table 7.4: Optimum repair degree for various weibull first interarrival mean and cost ratios

Scale parameter(β)	mean time to first failure	c/c_1=0	c/c_1=1	c/c_1=10
0.56	0.50	1	1	1
0.86	0.76	1	1	1
1.16	1.03	1	1	1
1.46	1.29	0.98	1	1
1.76	1.56	0.88	1	1
2.06	1.83	0.80	1	1
2.36	2.09	0.74	0.98	1
2.66	2.36	0.68	0.90	1
2.96	2.62	0.64	0.84	1
3.26	2.89	0.60	0.78	1
3.56	3.15	0.52	0.74	1
3.86	3.42	0.50	0.70	1
4.16	3.69	0.50	0.66	1
4.46	3.95	0.50	0.64	1
4.76	4.22	0.50	0.62	1
5.06	4.48	0.50	0.60	1
5.66	5.01	0.50	0.56	1

Case 2: Improved Repair Policy for Normal and Weibull Failure Distribution

We now focus on the expected number of failures and expected cost under the improved repair policy. In this model, we assume that the improvement increases the interfailure time by 20%, i.e. β=1.2 in the equation 5.25 and all failures, after the first replacement with an improved one, are rectified by replacement, i.e. α=1. Tables 7.5 and 7.6 show the difference in the expected number of failures between the perfect and improved repair policy for normal and weibull failure distribution, respectively. As a baseline scenario, we consider the perfect repair, since the perfect repair provides the smallest number of failures whatever the product quality is. Table 7.5 and Table 7.6 show that the difference increases as the product quality gets worse. That is, the performance of the improved policy with respect to the expected number of failures is better for unreliable products. When the product reliability is high, then the difference between the improved and perfect repair is negligible. Tables 7.7 and 7.8 show the performance of the improved repair policy in terms of the expected cost depending on various c/c_1 ratios for normal and weibull failure distribution, respectively. These tables show that if the fixed component of the cost is relatively larger than the variable component, then the improved repair policy dominates the optimum static policy for a given mean time to first failure. As we said in the static repair policy section, when the fixed component is large compared to the variable component, more extensive repair is needed, so for large c/c_1, the improved repair policy provides more extensive repair policy than the perfect repair. On the other hand, for other cost ratios, this policy is better than the static repair policy when the product quality is low and medium. In brief, for a given mean time to first failure, the performance of the improved repair policy increases as the cost ratio increases, and for a given cost ratio, it increases as the first interarrival mean decreases.

Table 7.5: Expected number of failures under perfect and improved repair policy with normal distribution

Mean time to first failure	Expected failures (α=1)	Expected failures (improved)	Difference
0.5	6.89395	4.89210	2.00
0.8	3.37312	2.99117	0.38
1.1	2.27289	2.00615	0.27
1.4	1.68273	1.44779	0.23
1.7	1.25390	1.13163	0.12
2.0	1.05602	1.01361	0.04
2.3	0.91277	0.89911	0.01
2.6	0.73912	0.73449	0.00
2.9	0.55792	0.55619	0.00
3.2	0.40254	0.40181	0.00
3.5	0.28440	0.28407	0.00
3.8	0.20011	0.19995	0.00
4.1	0.14171	0.14163	0.00
4.4	0.10161	0.10156	0.00
4.7	0.07399	0.07397	0.00
5.0	0.05480	0.05478	0.00

Table 7.6: Expected number of failures under perfect and improved repair policy with weibull distribution

Scale parameter(β)	Mean time to first failure	Expected failures (α=1)	Expected failures (improved)	Difference
0.56	0.50	5.99783	5.09965	0.90
0.86	0.76	3.69571	3.17854	0.52
1.16	1.03	2.60340	2.26632	0.34
1.46	1.29	1.97440	1.74682	0.23
1.76	1.56	1.56824	1.41390	0.15
2.06	1.83	1.28208	1.17690	0.11
2.36	2.09	1.06780	0.99526	0.07
2.66	2.36	0.90127	0.85044	0.05
2.96	2.62	0.76897	0.73265	0.04
3.26	2.89	0.66221	0.63577	0.03
3.56	3.15	0.57506	0.55544	0.02
3.86	3.42	0.50318	0.48839	0.01
4.16	3.69	0.44337	0.43202	0.01
4.46	3.95	0.39317	0.38436	0.01
4.76	4.22	0.35074	0.34380	0.01
5.06	4.48	0.31459	0.30905	0.01
5.36	4.75	0.28359	0.27911	0.00
5.66	5.01	0.25682	0.25318	0.00

Table 7.7: Change in the expected cost under optimal static and improved repair policy with normal distribution

Mean time to first failure	c/c_1=0 (%)	c/c_1=1 (%)	c/c_1=10 (%)	c/c_1=100 (%)	c/c_1=1000 (%)
0.5	26.14	27.59	28.77	29.01	29.03
0.8	5.39	8.36	10.78	11.26	11.32
1.1	2.94	7.34	10.94	11.65	11.73
1.4	0.79	8.02	12.88	13.84	13.95
1.7	-12.29	1.78	8.30	9.59	9.74
2	-41.29	-7.26	2.29	3.83	4.00
2.3	-76.56	-17.68	-0.29	1.30	1.48
2.6	-103.55	-26.62	-1.43	0.43	0.61
2.9	-118.45	-33.52	-2.43	0.11	0.29
3.2	-126.31	-38.30	-3.22	-0.02	0.16
3.5	-130.79	-41.04	-3.83	-0.08	0.10
3.8	-133.32	-42.58	-4.30	-0.13	0.06
4.1	-134.93	-43.57	-4.68	-0.18	0.04
4.4	-136.01	-44.23	-5.00	-0.22	0.03
4.7	-136.91	-44.78	-5.30	-0.25	0.02
5	-137.43	-45.10	-5.52	-0.29	0.01

Table 7.8: Change in the expected cost under optimal static and improved repair policy with weibull distribution

Scale parameter(β)	Mean time to first failure	c/c_1=0 (%)	c/c_1=1 (%)	c/c_1=5 (%)	c/c_1=10 (%)	c/c_1=100 (%)
0.56	0.50	11.64	13.31	14.42	14.67	14.94
0.86	0.76	8.58	11.29	13.09	13.50	13.94
1.16	1.03	5.27	9.11	11.67	12.25	12.87
1.46	1.29	1.39	6.46	9.84	10.61	11.43
1.76	1.56	-4.70	3.47	7.72	8.68	9.72
2.06	1.83	-13.36	0.40	5.60	6.79	8.05
2.36	2.09	-23.66	-2.56	3.67	5.10	6.61
2.66	2.36	-31.56	-4.48	1.94	3.92	5.45
2.96	2.62	-39.64	-6.70	0.39	2.99	4.53
3.26	2.89	-47.97	-9.01	-1.04	2.25	3.80
3.56	3.15	-56.71	-11.31	-2.39	1.65	3.22
3.86	3.42	-66.03	-13.56	-3.68	1.18	2.75
4.16	3.69	-74.33	-15.71	-4.96	0.79	2.37
4.46	3.95	-81.64	-17.79	-6.29	0.46	2.05
4.76	4.22	-88.02	-19.77	-7.72	0.20	1.79
5.06	4.48	-93.55	-21.64	-9.25	-0.02	1.57
5.36	4.75	-98.35	-23.43	-10.84	-0.21	1.38
5.66	5.01	-102.74	-25.14	-12.51	-0.37	1.22

Case 2: Dynamic Repair Policy for Normal and Weibull Failure Distribution

Now, we focus on the dynamic policy. Figures 7.17 and 7.18 show the change in the expected cost as a function of the repair extent for product with normal and weibull failure distribution, respectively. When we compare the results of dynamic policy with that of optimum degree of repair in static policy for products with normal failure distribution, we see that for the comparable cost ratios (e.g.: $c/c_1<10$), the dynamic policy performs better when the product quality is low or high. However, when the fixed component becomes fairly large (e.g.: $c/c_1 \geq 10$), then the dynamic policy dominates the optimum static repair policy. On the other hand, for products with weibull failure distribution, the performance of the dynamic policy decreases as the fixed component of the cost increases opposed of the normal case. In addition, in the weibull case, the performance of the dynamic policy increases as the quality of product increases.

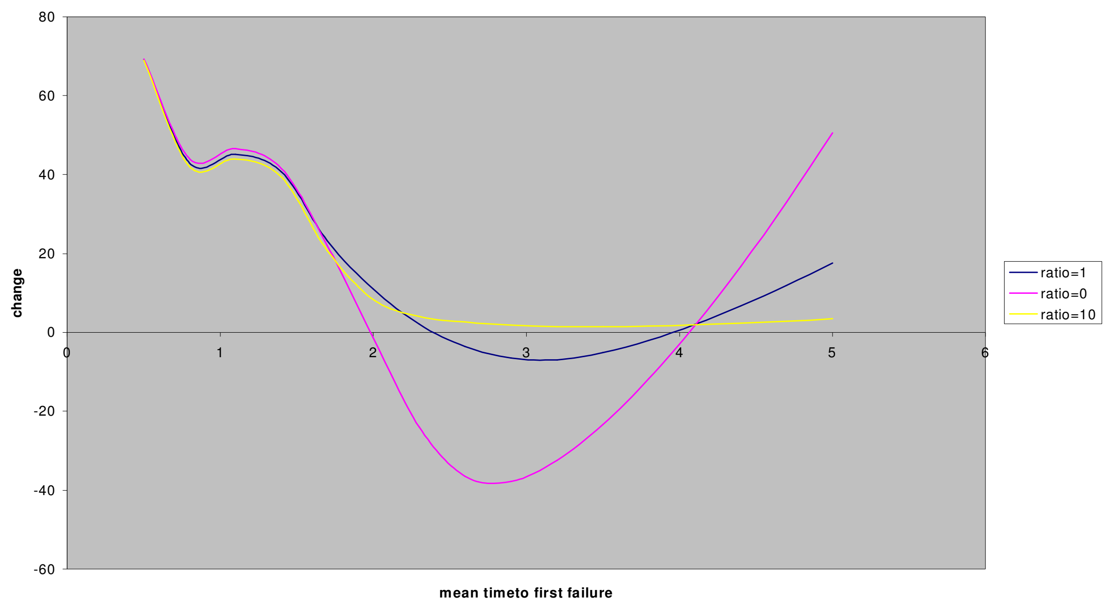

Figure 7.17: Change in the expected cost under dynamic policy with normal failure distribution

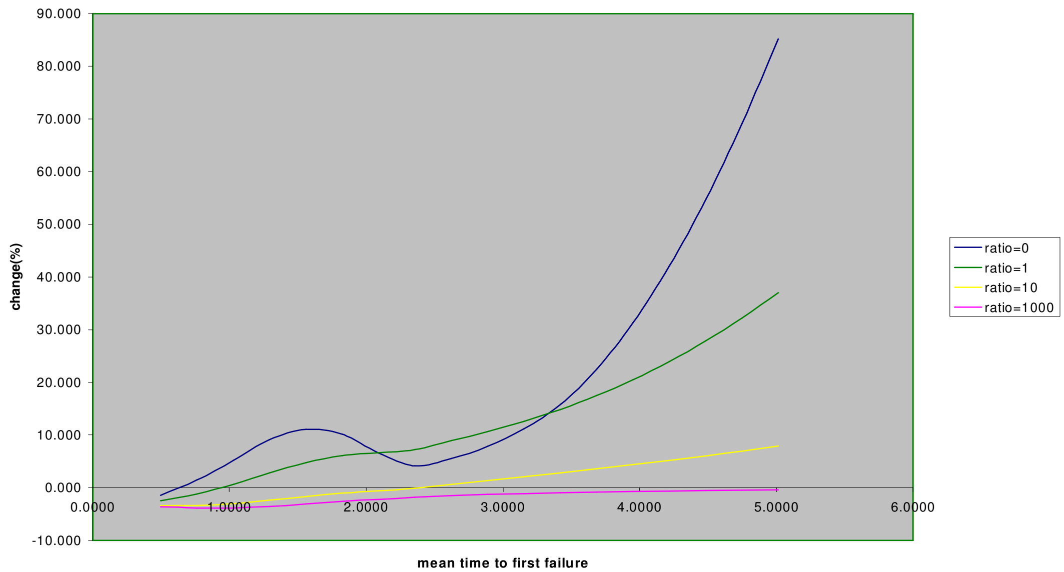

Figure 7.18: Change in the expected cost under dynamic policy with weibull failure distribution

***Comparisons the repair policies under one-dimensional warranties*:**

When we compare the performance of the optimal static, improved and dynamic policies, we see that when the failure distribution is normal (Table 7.9), the dynamic repair policy outperforms the other two policies as the fixed component of the cost function is relatively larger than the variable component. As the fixed component gets smaller, the optimal static policy is the best alternative for products with medium reliability. Whereas for products with low and high reliability, the dynamic policy outweighs the improved repair policy as cost ratio decreases. When the failure process is characterized by weibull distribution (Table 7.10), the performance of the improved policy outweighs the optimal static and dynamic as the cost ratio increases and/or the product reliability decreases. When the fixed component hits the threshold point, the improved policy becomes the best among all the repair policies.

Table 7.9: Comparisons the repair policies with normal failure distribution

Mean time to first failure	c/c_1=0 (%)	c/c_1=1 (%)	c/c_1=10 (%)	c/c_1=100 (%)
0.5	Dynamic	Dynamic	Dynamic	Dynamic
0.8	Dynamic	Dynamic	Dynamic	Dynamic
1.1	Dynamic	Dynamic	Dynamic	Dynamic
1.4	Dynamic	Dynamic	Dynamic	Dynamic
1.7	Dynamic	Dynamic	Dynamic	Dynamic
2.0	0.62	Dynamic	Dynamic	Dynamic
2.3	0.50	Dynamic	Dynamic	Dynamic
2.6	0.50	0.58	Dynamic	Dynamic
2.9	0.50	0.52	Dynamic	Dynamic
3.2	0.50	0.50	Dynamic	Dynamic
3.5	0.50	0.50	Dynamic	Dynamic
3.8	0.50	0.50	Dynamic	Dynamic
4.1	Dynamic	Dynamic	Dynamic	Dynamic
4.4	Dynamic	Dynamic	Dynamic	Dynamic
4.7	Dynamic	Dynamic	Dynamic	Dynamic
5.0	Dynamic	Dynamic	Dynamic	Dynamic

Table 7.10: Comparisons the repair policies with weibull failure distribution

Mean time to first failure	c/c_1=0 (%)	c/c_1=1 (%)	c/c_1=5 (%)	c/c_1=10 (%)
0.50	Improved	Improved	Improved	Improved
0.76	Improved	Improved	Improved	Improved
1.03	Improved	Improved	Improved	Improved
1.29	Dynamic	Improved	Improved	Improved
1.56	Dynamic	Dynamic	Improved	Improved
1.83	Dynamic	Dynamic	Improved	Improved
2.09	Dynamic	Dynamic	Improved	Improved
2.36	Dynamic	Dynamic	Improved	Improved
2.62	Dynamic	Dynamic	Improved	Improved
2.89	Dynamic	Dynamic	Improved	Improved
3.15	Dynamic	Dynamic	Dynamic	Improved
3.42	Dynamic	Dynamic	Dynamic	Improved
3.69	Dynamic	Dynamic	Dynamic	Improved
3.95	Dynamic	Dynamic	Dynamic	Improved
4.22	Dynamic	Dynamic	Dynamic	Improved
4.48	Dynamic	Dynamic	Dynamic	Improved
4.75	Dynamic	Dynamic	Dynamic	Improved
5.01	Dynamic	Dynamic	Dynamic	Improved

7.2.2 Two-dimensional Warranties

In this section, we analyze three different two-dimensional warranty policies. We start with Contract A, then B and C are considered respectively. Firstly, we discuss Contract A. Under Contract A, we start with the discussion of the static policies with bivariate normal and weibull failure distributions. Then, the results with the improved and dynamic repair policies for Contract A are evaluated. We also compare these policies with each other. Secondly, we focus on the static repair policies for Contract B and lastly, we examine Contract C.

Case 1: Static, Improved and Dynamic Repair Policies for Contract A

For Contract A, we consider both bivariate normal and weibull failure distribution for each repair policy in Case 1.1, 1.2 and 1.3, respectively.

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Case 1.1: Static Policies for Bivariate Normal and Weibull Failure Distributions

For Policy A, the mathematical formulation of the expected number of failures is as follows.

$$M_q^2(t,\ x) = \sum_{n=1}^{\infty} F^{(n)}(t,\ x)$$

If (Y_n, Z_n)'s in equation 5.13 are normally distributed two-dimensional random variables with $N\left(\begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix}, \begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{bmatrix}\right)$, then the distribution of n^{th} failure becomes

$$N\left(\begin{bmatrix}\dfrac{1-\alpha_1^n}{1-\alpha_1}\mu_1\\ \dfrac{1-\alpha_2^n}{1-\alpha_2}\mu_2\end{bmatrix}, \begin{bmatrix}\dfrac{1-\alpha_1^{2n}}{1-\alpha_1^2}\sigma_{11} & \dfrac{1-\alpha_1^n\alpha_2^n}{1-\alpha_1\alpha_2}\sigma_{12}\\ \dfrac{1-\alpha_1^n\alpha_2^n}{1-\alpha_1\alpha_2}\sigma_{12} & \dfrac{1-\alpha_2^{2n}}{1-\alpha_2^2}\sigma_{22}\end{bmatrix}\right)$$

. If the distribution of (Y_i, Z_i)'s is bivariate weibull with the following probability density function

$$f_{Y_i,Z_i}(y_i,z_i)=\frac{\gamma_1}{\theta_1}\left(\frac{y_i}{\theta_1}\right)^{\frac{\gamma_1}{\delta}-1}\frac{\gamma_2}{\theta_2}\left(\frac{z_i}{\theta_2}\right)^{\frac{\gamma_2}{\delta}-1}\left[\left(\frac{y_i}{\theta_1}\right)^{\frac{\gamma_1}{\delta}}+\left(\frac{z_i}{\theta_2}\right)^{\frac{\gamma_2}{\delta}}\right]^{\delta-2}$$
$$\left\{\left[\left(\frac{y_i}{\theta_1}\right)^{\frac{\gamma_1}{\delta}}+\left(\frac{z_i}{\theta_2}\right)^{\frac{\gamma_2}{\delta}}\right]^{\delta}+\frac{1}{\delta}-1\right\}\exp\left(-\left[\left(\frac{y_i}{\theta_1}\right)^{\frac{\gamma_1}{\delta}}+\left(\frac{z_i}{\theta_2}\right)^{\frac{\gamma_2}{\delta}}\right]^{\delta}\right)$$

where γ_1, γ_2 are shape; θ_1, θ_2 are scale parameters of time and usage dimensions, respectively and δ is a common shape parameter, then the density function of (T_i, X_i) is written as follows.

$$f_{T_i,X_i}(t_i,x_i)=\gamma_1\frac{\alpha_1^{1-i}}{\theta_1}\left(\frac{\alpha_1^{1-i}t_i}{\theta_1}\right)^{\frac{\gamma_1}{\delta}-1}\gamma_2\frac{\alpha_2^{1-i}}{\theta_2}\left(\frac{\alpha_2^{1-i}x_i}{\theta_2}\right)^{\frac{\gamma_2}{\delta}-1}\left[\left(\frac{\alpha_1^{1-i}t_i}{\theta_1}\right)^{\frac{\gamma_1}{\delta}}+\left(\frac{\alpha_2^{1-i}x_i}{\theta_2}\right)^{\frac{\gamma_2}{\delta}}\right]^{\delta-2}$$
$$\left\{\left[\left(\frac{\alpha_1^{1-i}t_i}{\theta_1}\right)^{\frac{\gamma_1}{\delta}}+\left(\frac{\alpha_2^{1-i}x_i}{\theta_2}\right)^{\frac{\gamma_2}{\delta}}\right]^{\delta}+\frac{1}{\delta}-1\right\}\exp\left(-\left[\left(\frac{\alpha_1^{1-i}t_i}{\theta_1}\right)^{\frac{\gamma_1}{\delta}}+\left(\frac{\alpha_2^{1-i}x_i}{\theta_2}\right)^{\frac{\gamma_2}{\delta}}\right]^{\delta}\right)$$

where α_1 and α_2 measure the extent of repair for time and usage, respectively. As in the normal case, the mean time and usage to the first failure are μ_1 and μ_2, respectively.

For the two-dimensional warranties, we first consider the case in which the mean time and usage to the first failure are equal (μ_1=μ_2). Then, we consider the case with unequal means. In each case, we consider mean values between 1 and 5. For the normal case, the variance of each dimension is selected as in the one-dimensional case and the covariance is chosen such that the correlation coefficient (ρ) is equal to 0.2, 0.5 and 0.9. For the weibull case, we set the shape parameters (γ_1, γ_2) equal to 2, and we assign the scale parameters to set the mean of the dimensions at the desired values. As in the normal case, the common shape parameter in the weibull case is selected so that the correlation coefficient is equal to 0.2, 0.5 and 0.9. Tables 7.11 and 7.12 show the expected number of failures for bivariate normal and weibull distributions with equal and unequal mean time and usage to the first failure, respectively. The expected numbers of failures are found under different extent of repair combinations when the correlation coefficient is 0.2. We observe that for a given mean vector, i.e. (μ_1, μ_2), the expected number of failures increases as the extent of repair decreases on at least one dimension. If the product reliability becomes low, the expected number of failures increases significantly. So, for these products, we present a lower bound. Whereas for a given extent of repair for each dimension, the expected number of failures decreases as the reliability of the product increases. In addition, Table 7.11 suggests that if the reliability along both dimensions decreases simultaneously, then the impact of the extent of repair gets more significant. For instance, when means are equal to 5, the expected numbers of failures are almost the same between the largest and smallest combination of repair degree when the failure distribution is normal, but when means are 1.5, then it changes more than 16%. The same observation is valid for the weibull distribution. For the unequal means case, Table 7.12 shows the similar results with Table 7.13. These results are in agreement with the one-dimensional case. Table 7.13 shows the effect of the correlation coefficient on the expectation. We observe that as the time and usage dimension become more dependent to each other, the expected number of failures increases. In addition, it is observed that there is no significant interaction between (α_1, α_2) and the correlation coefficient with respect to the expected number of failures.

Table 7.11: Expected number of failures with bivariate normal and weibull failure distribution for equal means under Contract A (ρ=0.2)

(α_1, α_2)	Bivariate Normal Distribution with (μ_1, μ_2)			Bivariate Weibull Distribution with (μ_1, μ_2)		
	(5, 5)	**(3, 3)**	**(1.5, 1.5)**	**(5, 5)**	**(3, 3)**	**(1.6, 1.6)**
(1.0, 1.0)	0.006120	0.282020	1.30157	0.090153	0.338829	1.15429
(1.0, 0.8)	0.006120	0.282058	1.42095	0.090466	0.343277	1.22612
(1.0, 0.5)	0.006120	0.282265	1.52067	0.091333	0.354343	1.34137
(0.8, 0.8)	0.006120	0.282148	1.61625	0.090917	0.349766	1.34158
(0.8, 0.5)	0.006120	0.282678	1.83357	0.092233	0.367364	>1.35
(0.5, 0.5)	0.006123	0.285225	>1.84	0.095038	0.416865	>1.35

Table 7.12: Expected number of failures with bivariate normal and weibull failure distribution for unequal means under Contract A (ρ=0.2)

(α_1, α_2)	Bivariate Normal Distribution with (μ_1, μ_2)			Bivariate Weibull Distribution with (μ_1, μ_2)		
	(5, 3)	**(5, 1)**	**(3, 1)**	**(5, 3)**	**(5, 1)**	**(3, 1)**
(1.0, 1.0)	0.035645	0.054791	0.502419	0.16396	0.257874	0.620392
(1.0, 0.8)	0.036123	0.054793	0.504250	0.16471	0.258058	0.62301
(1.0, 0.5)	0.036126	0.054810	0.504940	0.166356	0.258111	>0.82
(0.8, 1.0)	0.036123	0.054840	0.506542	0.165217	0.264394	0.659544
(0.8, 0.8)	0.036125	0.054847	0.506548	0.166384	0.264819	0.666086
(0.8, 0.5)	0.036134	0.054849	0.506591	0.168995	0.265148	>0.82
(0.5, 1.0)	0.03613	0.055404	0.541821	0.169103	0.292529	0.817921
(0.5, 0.8)	0.036142	0.055510	0.543897	0.17247	0.295131	>0.82
(0.5, 0.5)	0.036208	0.055614	0.545218	0.180836	0.296013	>0.82

Table 7.13: Expected number of failures with equal means for different ρ values under Contract A

(α_1, α_2)	ρ	Bivariate Normal Distribution with (μ_1, μ_2)			Bivariate Weibull Distribution with (μ_1, μ_2)		
		(5, 5)	**(3, 3)**	**(1.5, 1.5)**	**(5, 5)**	**(3, 3)**	**(1.6, 1.6)**
(1.0, 1.0)	0.2	0.006120	0.282020	1.301570	0.090153	0.338829	1.154290
	0.5	0.013815	0.333447	1.356080	0.138094	0.409353	1.243220
	0.9	0.035227	0.429388	1.463800	0.222465	0.565944	1.565110
(0.8, 0.8)	0.2	0.006120	0.282148	1.616250	0.090917	0.349766	1.341580
	0.5	0.013816	0.333965	1.673540	0.140160	0.428501	1.469650
	0.9	0.035243	0.431593	1.812550	0.228486	0.608947	1.830538
(0.5, 0.5)	0.2	0.006123	0.285225	>1.82	0.095038	0.416865	>1.84
	0.5	0.013841	0.341547	>1.82	0.151606	0.503631	>1.84
	0.9	0.035471	0.454591	>1.82	0.25765	0.696287	>1.84

Similar to the one-dimensional warranties, the expected warranty cost is a function of reliabilities and repair degrees of both dimensions in the two-dimensional case. This cost function is also affected by the different cost ratios such as c/c_1, c/c_2 and c_1/c_2. Tables 7.14-7.17 show the optimal extent of repair combination that gives the minimum expected warranty cost under different costs formulations. Tables 7.14 and 7.15 present the results when $c_1=c_2$. If $\mu_1=\mu_2$, optimal extent of repair (α_1, α_2) increases as the fixed component of the cost function increases behind the certain threshold relative to variable components. When the fixed component becomes too large, the optimal extent of repair is the replacement policy for each dimension. If the product reliability is low, the optimal extent of repair along each dimension converges more quickly to the replacement policy. When $\mu_1 \neq \mu_2$, we get similar results with the previous case. However, note that if the means are not equal, applying better repair to the high quality dimension provides smaller expected cost when the variable cost components are equal. From Tables 7.14 and 7.15, it is seen that small extent of repair is enough for both dimensions if the reliability of at least one dimension is high and the fixed cost is comparable with the variable cost components. Tables 7.16 and 7.17 present the optimal extent of repair when the fixed component is equal to one of the variable components, i.e. c=c_1. The results indicate that the extent of repair is larger for the dimension with a lower variable cost component. From Tables 7.14-7.17, we observe that the correlation coefficient does not affect the optimal repair degree of each dimension since the interaction between (α_1, α_2) and the correlation coefficient with respect to the expected number of failures is negligible.

Table 7.14: Optimal repair degree combination of equal means for bivariate normal and weibull distribution under Contract A (c_1= c_2)

$\frac{c}{c_1 = c_2}$	ρ	Bivariate Normal Distribution with (μ_1, μ_2)			Bivariate Weibull Distribution with (μ_1, μ_2)		
		(5, 5)	**(3, 3)**	**(1.5, 1.5)**	**(5, 5)**	**(3, 3)**	**(1.6, 1.6)**
0.01	0.2	(0.5, 0.5)	(0.5, 0.5)	(0.5, 1.0)	(0.5, 0.5)	(0.5, 0.5)	(0.5, 1.0)
	0.5	(0.5, 0.5)	(0.5, 0.5)	(0.5, 1.0)	(0.5, 0.5)	(0.5, 0.5)	(0.5, 1.0)
	0.9	(0.5, 0.5)	(0.5, 0.5)	(0.5, 1.0)	(0.5, 0.5)	(0.5, 0.5)	(0.5, 1.0)
1	0.2	(0.5, 0.5)	(0.5, 0.5)	(0.5, 1.0)	(0.5, 0.5)	(0.5, 0.5)	(0.5, 1.0)
	0.5	(0.5, 0.5)	(0.5, 0.5)	(0.5, 1.0)	(0.5, 0.5)	(0.5, 0.5)	(0.5, 1.0)
	0.9	(0.5, 0.5)	(0.5, 0.5)	(0.5, 1.0)	(0.5, 0.5)	(0.5, 0.5)	(0.5, 1.0)
10	0.2	(0.5, 0.5)	(0.5, 0.5)	(1.0, 1.0)	(0.5, 0.8)	(0.8, 1.0)	(1.0, 1.0)
	0.5	(0.5, 0.5)	(0.5, 0.5)	(1.0, 1.0)	(0.5, 0.8)	(0.8, 1.0)	(1.0, 1.0)
	0.9	(0.5, 0.5)	(0.8, 0.5)	(1.0, 1.0)	(0.5, 1.0)	(0.8, 1.0)	(1.0, 1.0)
200	0.2	(0.5, 0.5)	(0.8, 1.0)	(1.0, 1.0)	(1.0, 1.0)	(1.0, 1.0)	(1.0, 1.0)
	0.5	(0.8, 0.5)	(1.0, 1.0)	(1.0, 1.0)	(1.0, 1.0)	(1.0, 1.0)	(1.0, 1.0)
	0.9	(0.8, 0.5)	(1.0, 1.0)	(1.0, 1.0)	(1.0, 1.0)	(1.0, 1.0)	(1.0, 1.0)
10000	0.2	(0.8, 0.8)	(1.0, 1.0)	(1.0, 1.0)	(1.0, 1.0)	(1.0, 1.0)	(1.0, 1.0)
	0.5	(1.0, 1.0)	(1.0, 1.0)	(1.0, 1.0)	(1.0, 1.0)	(1.0, 1.0)	(1.0, 1.0)
	0.9	(1.0, 1.0)	(1.0, 1.0)	(1.0, 1.0)	(1.0, 1.0)	(1.0, 1.0)	(1.0, 1.0)

Table 7.15: Optimal repair degree combination of unequal means for bivariate normal and weibull distribution under Contract A ($c_1 = c_2$)

$\frac{c}{c_1 = c_2}$	ρ	Bivariate Normal Distribution with (μ_1, μ_2)				Bivariate Weibull Distribution with (μ_1, μ_2)			
		(5, 3)	**(5, 1)**	**(3, 1)**	**(2,1.5)**	**(5, 3)**	**(5, 1)**	**(3, 1)**	**(2,1.6)**
0.01	0.2	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(0.8,0.5)	(0.5,0.5)	(0.5,0.5)	(0.8,0.8)	(1.0,0.8)
	0.5	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(0.8,0.5)	(0.5,0.5)	(0.5,0.5)	(0.8,0.8)	(1.0,0.8)
	0.9	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(0.8,0.5)	(0.5,0.5)	(0.5,0.5)	(0.8,0.8)	(1.0,0.8)
1	0.2	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(1.0,0.5)	(0.5,0.5)	(0.5,0.5)	(0.8,0.8)	(1.0,0.8)
	0.5	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(1.0,0.5)	(0.5,0.5)	(0.5,0.5)	(0.8,0.8)	(1.0,0.8)
	0.9	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(1.0,0.5)	(0.5,0.5)	(0.5,0.5)	(1.0,0.8)	(1.0,0.8)
10	0.2	(0.5,0.5)	(0.5,0.5)	(0.8,0.5)	(1.0,0.5)	(0.8,0.5)	(1.0,0.5)	(1.0,0.8)	(1.0,1.0)
	0.5	(0.5,0.5)	(0.5,0.5)	(0.8,0.5)	(1.0,0.5)	(1.0,0.5)	(1.0,0.5)	(1.0,0.8)	(1.0,1.0)
	0.9	(0.5,0.5)	(0.5,0.5)	(0.8,0.5)	(1.0,0.5)	(1.0,0.5)	(1.0,0.5)	(1.0,0.8)	(1.0,1.0)
10000	0.2	(1.0,1.0)	(1.0,1.0)	(1.0,1.0)	(1.0,1.0)	(1.0,1.0)	(1.0,1.0)	(1.0,1.0)	(1.0,1.0)
	0.5	(1.0,1.0)	(1.0,1.0)	(1.0,1.0)	(1.0,1.0)	(1.0,1.0)	(1.0,1.0)	(1.0,1.0)	(1.0,1.0)
	0.9	(1.0,1.0)	(1.0,1.0)	(1.0,1.0)	(1.0,1.0)	(1.0,1.0)	(1.0,1.0)	(1.0,1.0)	(1.0,1.0)

Table 7.16: Optimal repair degree combination of equal means for bivariate normal and weibull distribution under Contract A ($c = c_1$)

$\frac{c = c_1}{c_2}$	ρ	Bivariate Normal Distribution with (μ_1, μ_2)			Bivariate Weibull Distribution with (μ_1, μ_2)		
		(5, 5)	**(3, 3)**	**(1.5, 1.5)**	**(5, 5)**	**(3, 3)**	**(1.6, 1.6)**
0.1	0.2	(0.5, 0.5)	(0.5, 0.5)	(1, 0.5)	(0.5, 0.5)	(1, 0.5)	(1, 0.5)
	0.5	(0.5, 0.5)	(0.5, 0.5)	(1, 0.5)	(0.8, 0.5)	(1, 0.5)	(1, 0.5)
	0.9	(0.5, 0.5)	(0.5, 0.5)	(1, 0.5)	(1, 0.5)	(1, 0.5)	(1, 0.5)
1	0.2	(0.5, 0.5)	(0.5, 0.5)	(1, 0.5)	(0.5, 0.5)	(0.5, 0.5)	(1, 0.5)
	0.5	(0.5, 0.5)	(0.5, 0.5)	(1, 0.5)	(0.5, 0.5)	(0.5, 0.5)	(1, 0.5)
	0.9	(0.5, 0.5)	(0.5, 0.5)	(1, 0.5)	(0.5, 0.5)	(0.5, 0.5)	(1, 0.5)
10	0.2	(0.5, 0.5)	(0.5, 0.5)	(0.5, 1)	(0.5, 0.8)	(0.5, 1)	(0.5, 1)
	0.5	(0.5, 0.5)	(0.5, 0.5)	(0.5, 1)	(0.5, 1)	(0.5, 1)	(0.5, 1)
	0.9	(0.5, 0.5)	(0.5, 0.8)	(0.5, 1)	(0.5, 1)	(0.5, 1)	(0.5, 1)
1000	0.2	(0.5, 0.8)	(0.5, 1)	(0.5, 1)	(0.5, 1)	(0.5, 1)	(0.5, 1)
	0.5	(0.5, 1)	(0.5, 1)	(0.5, 1)	(0.5, 1)	(0.5, 1)	(0.5, 1)
	0.9	(0.5, 1)	(0.5, 1)	(0.5, 1)	(0.5, 1)	(0.5, 1)	(0.5, 1)
10000	0.2	(0.5, 1)	(0.5, 1)	(0.5, 1)	(0.5, 1)	(0.5, 1)	(0.5, 1)
	0.5	(0.5, 1)	(0.5, 1)	(0.5, 1)	(0.5, 1)	(0.5, 1)	(0.5, 1)
	0.9	(0.5, 1)	(0.5, 1)	(0.5, 1)	(0.5, 1)	(0.5, 1)	(0.5, 1)

Table 7.17: Optimal repair degree combination of unequal means for bivariate normal and weibull distribution under Contract A ($c = c_1$)

$\frac{c=c_1}{c_2}$	ρ	Bivariate Normal Distribution with (μ_1, μ_2)				Bivariate Weibull Distribution with (μ_1, μ_2)			
		(5, 3)	**(5, 1)**	**(3, 1)**	**(2,1.5)**	**(5, 3)**	**(5, 1)**	**(3, 1)**	**(2,1.6)**
0.1	0.2	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(1.0,0.5)	(0.8,0.5)	(0.8,0.5)	(1.0,0.8)	(1.0,0.8)
	0.5	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(1.0,0.5)	(0.8,0.5)	(0.8,0.5)	(1.0,0.8)	(1.0,0.8)
	0.9	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(1.0,0.5)	(0.8,0.5)	(0.8,0.5)	(1.0,0.8)	(1.0,0.8)
1	0.2	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(1.0,0.5)	(0.5,0.5)	(0.5,0.5)	(0.8,0.8)	(1.0,0.8)
	0.5	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(1.0,0.5)	(0.5,0.5)	(0.5,0.5)	(0.8,0.8)	(1.0,0.8)
	0.9	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(1.0,0.5)	(0.5,0.5)	(0.5,0.5)	(0.8,0.8)	(1.0,0.8)
10	0.2	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(0.5,1.0)	(0.5,1.0)	(0.5,0.5)	(0.8,0.8)	(0.5,1.0)
	0.5	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(0.5,1.0)	(0.5,1.0)	(0.5,0.5)	(0.8,0.8)	(0.5,1.0)
	0.9	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(0.5,1.0)	(0.5,1.0)	(0.5,0.5)	(0.8,0.8)	(0.5,1.0)
1000	0.2	(0.5,1.0)	(0.5,1.0)	(0.5,1.0)	(0.5,1.0)	(0.5,1.0)	(0.5,1.0)	(0.8,1.0)	(0.5,1.0)
	0.5	(0.5,1.0)	(0.5,1.0)	(0.5,1.0)	(0.5,1.0)	(0.5,1.0)	(0.5,1.0)	(0.8,1.0)	(0.5,1.0)
	0.9	(0.5,1.0)	(0.5,1.0)	(0.5,1.0)	(0.5,1.0)	(0.5,1.0)	(0.5,1.0)	(0.8,1.0)	(0.5,1.0)

Case 1.2: Improved Policy for Bivariate Normal and Weibull Failure Distribution

We now focus on the expected number of failures and expected cost with the two-dimensional models under the improved repair policy. In this model, we assume that the improvement increases the time and total usage until the first failure by 20%, i.e. β_1=β_2=1.2 in equation 5.28. All failures, after the first replacement, are rectified via replacement, i.e. α_1= α_2=1. Tables 7.18 and 7.19 show the performance of the improved policy under bivariate normal and weibull failure distribution with respect to the expected number of failures, respectively. As a baseline scenario, we consider applying perfect repair to both dimensions. These tables show that failure distribution, the difference between the improved and perfect repair policy is negligible when the reliability of at least one dimension is high. For other cases, the difference increases as the reliability decreases.

Table 7.18: Expected number of failures of perfect and improved repair policy with bivariate normal failure distributions under Contract A

ρ	(μ_1, μ_2)	expected # of failures with bivariate normal distribution		difference
		(α_1,α_2)=(1.0,1.0)	Improved	
0.2	(5.0, 5.0)	0.006120	0.005591	0.00
0.5		0.013815	0.012834	0.00
0.9		0.035227	0.033249	0.00
0.2	(5.0, 3.0)	0.035645	0.034421	0.00
0.5		0.047812	0.046741	0.00
0.9		0.054762	0.054751	0.00
0.2	(5.0, 1.0)	0.054791	0.054782	0.00
0.5		0.054735	0.054719	0.00
0.9		0.058268	0.058253	0.00
0.2	(3.0, 3.0)	0.282020	0.265386	0.02
0.5		0.333447	0.316718	0.02
0.9		0.429388	0.412117	0.02
0.2	(3.0, 1.0)	0.502392	0.501069	0.00
0.5		0.502540	0.501230	0.00
0.9		0.515219	0.513896	0.00
0.2	(2.0, 1.0)	1.060700	1.015620	0.05
0.5		1.060820	1.015930	0.04
0.9		1.061460	1.016880	0.04
0.2	(1.5, 1.5)	1.301570	1.127020	0.17
0.5		1.356080	1.171620	0.18
0.9		1.463800	1.261190	0.20

Table 7.21: Expected number of failures of perfect and improved repair policy with bivariate weibull failure distribution under Contract A

ρ	(μ_1, μ_2)	Expected # of failures with bivariate weibull distribution		difference
		(α_1,α_2)=(1.0,1,0)	improved	
0.2	(5.0, 5.0)	0.090153	0.089789	0.00
0.5		0.138094	0.136997	0.00
0.9		0.222465	0.219171	0.00
0.2	(5.0, 3.0)	0.16396	0.162791	0.00
0.5		0.209592	0.207164	0.00
0.9		0.254765	0.251165	0.00
0.2	(5.0, 1.0)	0.257874	0.254006	0.00
0.5		0.258764	0.255008	0.00
0.9		0.260816	0.256939	0.00
0.2	(3.0, 3.0)	0.338829	0.333615	0.01
0.5		0.409353	0.399727	0.01
0.9		0.565944	0.544716	0.02
0.2	(3.0, 1.0)	0.620392	0.596171	0.02
0.5		0.625082	0.601337	0.02
0.9		0.640547	0.615024	0.03
0.2	(1.6, 1.6)	1.154290	1.062830	0.09
0.5		1.243220	1.139150	0.10
0.9		1.565110	1.402460	0.16

Tables 7.20 and 7.21 show % change between the costs of the optimal static policy and improved policy. The improved policy generally dominates the static policy when the fixed component of the cost function is very large compared to the variable components for any reliability of product. In other words, this policy dominates the case in which the perfect repair is optimal among the static policies and the improved policy provides significant decline in the expectation.

Table 7.20: Change(%) in the expected cost under optimal static and improved repair policy with bivariate normal and weibull distribution under Contract A (μ_1=μ_2)

$\frac{c}{c_1 = c_2}$	ρ	Bivariate Normal Distribution with (μ_1, μ_2)			Bivariate Weibull Distribution with (μ_1, μ_2)		
		(5, 5)	**(3, 3)**	**(1.5, 1.5)**	**(5, 5)**	**(3, 3)**	**(1.6, 1.6)**
1	0.2	-55.23	-58.18	0.54	-60.61	-36.05	-7.01
	0.5	-57.62	-57.64	-2.41	-53.62	-34.93	-8.55
	0.9	-59.35	-54.11	-9.11	-44.61	-32.99	-3.93
10	0.2	-2.93	-4.89	10.85	-6.83	-2.13	5.04
	0.5	-4.52	-4.53	11.14	-5.18	-0.77	5.69
	0.9	-5.66	-4.17	11.56	-5.30	0.20	7.78
10000	0.2	8.64	5.89	13.41	0.40	1.53	7.92
	0.5	7.10	5.01	13.60	0.79	2.35	8.37
	0.9	5.61	4.02	13.84	1.48	3.75	10.39

Table 7.21: Change(%) in the expected cost under optimal static and improved repair policy with bivariate normal and weibull distribution under Contract A ($\mu_1 \neq \mu_2$)

$\frac{c}{c_1 = c_2}$	ρ	Bivariate Normal Distribution with (μ_1, μ_2)				Bivariate Weibull Distribution with (μ_1, μ_2)		
		(5, 3)	**(5, 1)**	**(3, 1)**	**(2,1.5)**	**(5, 3)**	**(5, 1)**	**(3, 1)**
1	0.2	-61.61	-68.07	-56.45	-27.82	-53.04	-45.88	-17.04
	0.5	-65.15	-68.20	-56.69	-28.91	-47.45	-46.59	-17.30
	0.9	-68.11	-68.41	-56.61	-29.72	-46.03	-45.64	-17.19
10	0.2	-7.16	-11.45	-8.54	-1.34	-5.71	-6.23	-0.56
	0.5	-9.51	-11.53	-8.54	-2.25	-5.48	-6.81	-1.06
	0.9	-11.48	-11.67	-8.48	-2.98	-6.06	-6.31	-1.41
10000	0.2	3.22	-0.20	0.05	3.22	0.67	1.46	3.87
	0.5	2.03	-0.23	0.05	4.00	1.12	1.41	3.76
	0.9	-0.20	-0.24	0.04	4.24	1.37	1.45	3.95

Case 1.2: Dynamic Policy for Bivariate Normal and Weibull Failure Distribution

While analyzing the dynamic repair policies for two-dimensional warranties, we first focus on the policies with the same the extent of repair at any given instance in time in both dimensions. Tables 7.22 and 7.23 show the performance of the dynamic policy with respect to the expected number of failures for normal and weibull failure distribution, respectively. The performance of this policy decreases as the reliability of the product decreases. In this policy, for high reliability of products, the difference between the dynamic and perfect repair policy is negligible.

Table 7.22: Expected number of failures of perfect and dynamic(1) repair policy with bivariate normal failure distribution under Contract A

ρ	(μ_1, μ_2)	Expected # of failures with bivariate normal distribution		difference
		(α_1,α_2)=(1.0,1,0)	dynamic(1)	
0.2	(5.0, 5.0)	0.006120	0.005591	0.00
0.5		0.013815	0.012834	0.00
0.9		0.035227	0.033258	0.00
0.2	(5.0, 3.0)	0.035645	0.032642	0.00
0.5		0.047812	0.043265	0.00
0.9		0.054762	0.049244	0.01
0.2	(5.0, 1.0)	0.054791	0.049270	0.01
0.5		0.054713	0.049210	0.01
0.9		0.058240	0.052561	0.01
0.2	(3.0, 3.0)	0.282020	0.265427	0.02
0.5		0.333447	0.316921	0.02
0.9		0.429388	0.413178	0.02
0.2	(3.0, 1.0)	0.502392	0.470217	0.03
0.5		0.502540	0.470324	0.03
0.9		0.515219	0.480948	0.03
0.2	(2.0, 1.0)	1.060700	1.075570	-0.01
0.5		1.060820	1.075300	-0.01
0.9		1.061460	1.076130	-0.01
0.2	(1.5, 1.5)	1.301570	1.337480	-0.04
0.5		1.356080	1.388390	-0.03
0.9		1.463800	1.491390	-0.03

Table 7.23: Expected number of failures of perfect and dynamic(1) repair policy with bivariate weibull failure distribution under Contract A

ρ	(μ_1, μ_2)	Expected # of failures with bivariate weibull distribution		difference
		(α_1,α_2)=(1.0,1,0)	dynamic(1)	
0.2	(5.0, 5.0)	0.090153	0.087849	0.00
0.5		0.138094	0.135538	0.00
0.9		0.222465	0.219682	0.00
0.2	(5.0, 3.0)	0.16396	0.159191	0.00
0.5		0.209592	0.204243	0.01
0.9		0.254765	0.247695	0.01
0.2	(5.0, 1.0)	0.257874	0.250329	0.01
0.5		0.258764	0.250941	0.01
0.9		0.260816	0.252057	0.01
0.2	(3.0, 3.0)	0.338829	0.332922	0.01
0.5		0.409353	0.405011	0.00
0.9		0.565944	0.564506	0.00
0.2	(3.0, 1.0)	0.620392	0.612879	0.01
0.5		0.625082	0.616882	0.01
0.9		0.640547	0.635774	0.00
0.2	(1.6, 1.6)	1.154290	1.161230	-0.01
0.5		1.243220	1.250730	-0.01
0.9		1.565110	1.573340	-0.01

Tables 7.24 and 7.25 show the change of the expected cost between the dynamic policy with same extents of repair and the optimal static repair policy. This policy dominates the static policy when the expected number of breakdowns and the total variable cost components of the optimal static policy are both larger than the expected number of breakdowns and the total variable cost component of the dynamic policy, respectively. For a given high reliability of product, as the fixed component of the cost increases, the optimal static policy increases, but the difference between the expected number of breakdowns becomes negligible; so the performance of the dynamic policy decreases. On the other hand, for a given average quality of product, when the fixed component is small, the total variable component of the cost function

in the optimal static policy is smaller than the total variable component in the dynamic policy. So, when the fixed component is small, the static policy dominates the dynamic policy although the expected number of breakdowns in the static policy is larger than the expected number in the dynamic. However, for a given average quality of product, as the fixed component increases, the total variable cost also increases but the difference between the expected number of failures becomes negligible. So, the dynamic policy dominates the static policy.

Table 7.24: Change in the expected cost under optimal static and dynamic(1) repair policy with bivariate normal and weibull distribution under Contract A ($\mu_1=\mu_2$)

$\frac{c}{c_1=c_2}$	ρ	Bivariate Normal Distribution with (μ_1, μ_2)			Bivariate Weibull Distribution with (μ_1, μ_2)		
		(5, 5)	**(3, 3)**	**(1.5, 1.5)**	**(5, 5)**	**(3, 3)**	**(1.6, 1.6)**
1	0.2	31.55	-15.49	1.60	45.23	6.75	3.84
	0.5	30.49	-15.15	-1.24	47.02	8.00	2.12
	0.9	29.72	-12.80	-7.75	49.47	6.34	4.70
10	0.2	12.84	2.86	-1.02	14.15	6.58	1.27
	0.5	11.50	3.14	-0.57	14.53	6.40	1.48
	0.9	10.51	3.29	0.07	13.30	3.21	1.60
10000	0.2	8.66	5.89	-2.76	2.57	1.75	-0.60
	0.5	7.11	4.96	-2.38	1.87	1.07	-0.60
	0.9	5.60	3.78	-1.88	1.27	0.26	-0.52

Table 7.25: Change in the expected cost under optimal static and dynamic(1) repair policy with bivariate normal and weibull distribution under Contract A ($\mu_1 \neq \mu_2$)

$\frac{c}{c_1 = c_2}$	ρ	Bivariate Normal Distribution with (μ_1, μ_2)				Bivariate Weibull Distribution with (μ_1, μ_2)		
		(5, 3)	**(5, 1)**	**(3, 1)**	**(2,1.5)**	**(5, 3)**	**(5, 1)**	**(3, 1)**
1	0.2	32.42	33.34	-7.18	-10.21	47.84	49.89	16.25
	0.5	32.59	33.27	-7.33	-12.13	49.33	49.72	16.24
	0.9	33.33	32.96	-7.48	-13.50	49.80	50.20	15.68
10	0.2	13.95	15.12	9.67	-1.46	15.09	14.00	5.24
	0.5	14.17	15.04	8.11	-3.23	14.58	13.66	4.97
	0.9	15.11	14.64	9.90	-4.49	14.09	14.34	3.90
10000	0.2	8.44	10.09	6.41	-1.75	2.93	2.94	1.22
	0.5	9.52	10.07	6.42	-1.97	2.57	2.97	1.32
	0.9	10.09	9.77	6.66	-2.20	2.79	3.38	0.75

Tables 7.26 and 7.27 show the performance of the second type of dynamic policy under normal and weibull failure distribution, respectively. In the second policy, both dimensions are rectified so as to set $\frac{\alpha_1(t)}{\alpha_2(t)} = \frac{\mu_1}{\mu_2}$ for all t with the condition that $\alpha_1(t),\ \alpha_2(t) \leq 1$ where t is the time of failure. The performance of this method is relatively better than the previous dynamic method. Tables 7.26 and 7.27 show that the performance of the second dynamic policy is similar that of the first dynamic policy. Table 7.30 shows the change of the expected cost between the second dynamic and the optimal static policy. As in the previous dynamic policy, the second dynamic policy dominates the optimal static policy as long as the total variable cost and the expected number of failures of the dynamic policy are smaller than the total variable cost and the expected number of failures of the static policy, respectively. As the fixed cost increases, the difference between the expected number of failures of optimal static and dynamic policies decreases; so does the performance of the dynamic policy.

Table 7.26: Expected number of failures of perfect and dynamic(2) repair policy with bivariate normal failure distribution under Contract A

ρ	(μ_1, μ_2)	Expected # of failures with bivariate normal distribution		difference
		(α_1,α_2)=(1.0,1,0)	dynamic(2)	
0.2	(5.0, 3.0)	0.035645	0.032641	0.00
0.5		0.047812	0.043257	0.00
0.9		0.054762	0.049226	0.01
0.2	(5.0, 1.0)	0.493264	0.049222	0.01
0.5		0.499220	0.049173	0.01
0.9		0.502293	0.052556	0.01
0.2	(3.0, 1.0)	0.502392	0.467735	0.03
0.5		0.502540	0.468940	0.03
0.9		0.515219	0.480890	0.03
0.2	(2.0, 1.0)	1.060700	1.044020	0.02
0.5		1.060820	1.061480	0.00
0.9		1.061460	1.076540	-0.02

Table 7.27: Expected number of failures of perfect and dynamic(2) repair policy with bivariate weibull failure distribution under Contract A

ρ	(μ_1, μ_2)	Expected # of failures with bivariate weibull distribution		difference
		(α_1,α_2)=(1.0,1,0)	dynamic(2)	
0.2	(5.0, 3.0)	0.16396	0.161141	0.00
0.5		0.209592	0.206325	0.00
0.9		0.254765	0.248409	0.01
0.2	(5.0, 1.0)	0.257874	0.248006	0.01
0.5		0.258764	0.243235	0.02
0.9		0.260816	0.238590	0.02
0.2	(3.0, 1.0)	0.620392	0.611649	0.01
0.5		0.625082	0.607451	0.02
0.9		0.640547	0.622846	0.02

Table 7.28: Change(%) in the expected cost under optimal static and dynamic(2) repair policy with bivariate normal and weibull distribution under Contract A ($\mu_1 \neq \mu_2$)

$\frac{c}{c_1 = c_2}$	ρ	Bivariate Normal Distribution with (μ_1, μ_2)				Bivariate Weibull Distribution with (μ_1, μ_2)		
		(5, 3)	**(5, 1)**	**(3, 1)**	**(2,1.5)**	**(5, 3)**	**(5, 1)**	**(3, 1)**
1	0.2	36.92	42.27	14.72	2.59	48.84	53.45	32.50
	0.5	37.09	42.20	14.40	1.06	50.41	54.30	33.40
	0.9	37.79	41.90	14.31	-1.54	51.23	55.80	33.29
10	0.2	14.77	16.81	14.01	4.20	14.37	15.42	9.22
	0.5	15.00	16.72	12.33	2.33	14.02	16.92	10.17
	0.9	15.94	16.28	13.79	-0.86	14.15	19.51	9.63
10000	0.2	8.33	10.13	6.90	2.03	1.74	3.84	1.42
	0.5	9.46	10.09	6.69	8.01	1.58	6.02	2.83
	0.9	10.05	9.73	6.66	-0.88	2.53	8.54	2.77

***Comparisons of the repair policies under Contract A*:**

Table 7.29 and 7.30 show the optimal repair policy for bivariate normal and weibull distributions when the fixed components of costs are equal. When the reliability of each dimension is very high and equal to each other, the dynamic policy 1 or 2 performs better the other repair policies. For product with equal dimension reliability (Table 7.30), the improved policy generally outperforms, as the cost ratio increases and/or the product reliability decreases. When $\mu_1 \neq \mu_2$ (Table 7.30), the dynamic policy outweighs the optimal static and improved policy for most cost ratios and product reliability, but the product reliability is low and the fixed component is very large, the improved policy performs better than the dynamic policy. When one of the variable components is equal to the fixed component, then the dynamic policy outperforms when the product reliability is high.

Table 7.29: Optimal repair policy under Contract A with equal means for bivariate normal and weibull distribution (c_1=c_2)

$\frac{c}{c_1 = c_2}$	ρ	Bivariate Normal Distribution with (μ_1, μ_2)			Bivariate Weibull Distribution with (μ_1, μ_2)		
		(5, 5)	**(3, 3)**	**(1.5, 1.5)**	**(5, 5)**	**(3, 3)**	**(1.6, 1.6)**
0.01	0.2	Dyn-1,2	Dyn-1,2	(0.5, 1.0)	Dyn-1,2	Dyn-1,2	(0.5, 1.0)
	0.5	Dyn-1,2	(0.5, 0.5)	(0.5, 1.0)	Dyn-1,2	Dyn-1,2	(0.5, 1.0)
	0.9	Dyn-1,2	(0.5, 0.5)	(0.5, 1.0)	Dyn-1,2	Dyn-1,2	Dyn-1,2
1	0.2	Dyn-1,2	(0.5, 0.5)	Dyn-1,2	Dyn-1,2	Dyn-1,2	Dyn-1,2
	0.5	Dyn-1,2	(0.5, 0.5)	(0.5, 1.0)	Dyn-1,2	Dyn-1,2	Dyn-1,2
	0.9	Dyn-1,2	(0.5, 0.5)	(0.5, 1.0)	Dyn-1,2	Dyn-1,2	Dyn-1,2
10	0.2	Dyn-1,2	Dyn-1,2	Improved	Dyn-1,2	Dyn-1,2	Improved
	0.5	Dyn-1,2	Dyn-1,2	Improved	Dyn-1,2	Dyn-1,2	Improved
	0.9	Dyn-1,2	Dyn-1,2	Improved	Dyn-1,2	Improved	Improved
10000	0.2	Dyn-1,2	Improved	Improved	Dyn-1,2	Dyn-1,2	Improved
	0.5	Dyn-1,2	Improved	Improved	Dyn-1,2	Improved	Improved
	0.9	Improved	Improved	Improved	Improved	Improved	Improved

Table 7.30: Optimal repair policy under Contract A with unequal means for bivariate normal and weibull distribution (c_1=c_2)

$\frac{c}{c_1 = c_2}$	ρ	Bivariate Normal Distribution with (μ_1, μ_2)				Bivariate Weibull Distribution with (μ_1, μ_2)		
		(5, 3)	**(5, 1)**	**(3, 1)**	**(2,1.5)**	**(5, 3)**	**(5, 1)**	**(3, 1)**
0.01	0.2	Dyn-2	Dyn-2	Dyn-2	(0.8,0.5)	Dyn-2	Dyn-2	Dyn-2
	0.5	Dyn-2	Dyn-2	Dyn-2	(0.8,0.5)	Dyn-2	Dyn-2	Dyn-2
	0.9	Dyn-2	Dyn-2	Dyn-2	(0.8,0.5)	Dyn-2	Dyn-2	Dyn-2
1	0.2	Dyn-2	Dyn-2	Dyn-2	Dyn-2	Dyn-2	Dyn-2	Dyn-2
	0.5	Dyn-2	Dyn-2	Dyn-2	Dyn-2	Dyn-2	Dyn-2	Dyn-2
	0.9	Dyn-2	Dyn-2	Dyn-2	(1.0,0.5)	Dyn-2	Dyn-2	Dyn-2
10	0.2	Dyn-2	Dyn-2	Dyn-2	Dyn-2	Dyn-1	Dyn-2	Dyn-2
	0.5	Dyn-2	Dyn-2	Dyn-2	Dyn-2	Dyn-1	Dyn-2	Dyn-2
	0.9	Dyn-2	Dyn-2	Dyn-2	(1.0,0.5)	Dyn-2	Dyn-2	Dyn-2
10000	0.2	Dyn-1	Dyn-2	Dyn-2	Improved	Dyn-1	Dyn-2	improved
	0.5	Dyn-1	Dyn-2	Dyn-2	Improved	Dyn-1	Dyn-2	improved
	0.9	Dyn-1	Dyn-1	Dyn-2	Improved	Dyn-1	Dyn-2	improved

Table 7.31: Optimal repair policy under Contract A with equal means for bivariate normal and weibull distribution ($c_1=c$)

$\frac{c=c_1}{c_2}$	ρ	Bivariate Normal Distribution with (μ_1, μ_2)			Bivariate Weibull Distribution with (μ_1, μ_2)		
		(5, 5)	**(3, 3)**	**(1.5, 1.5)**	**(5, 5)**	**(3, 3)**	**(1.6, 1.6)**
0.1	0.2	Dyn-1,2	(0.5, 0.5)	(1, 0.5)	Dyn-1,2	(1, 0.5)	(1, 0.5)
	0.5	Dyn-1,2	(0.5, 0.5)	(1, 0.5)	Dyn-1,2	(1, 0.5)	(1, 0.5)
	0.9	Dyn-1,2	(0.5, 0.5)	(1, 0.5)	Dyn-1,2	(1, 0.5)	(1, 0.5)
1	0.2	Dyn-1,2	(0.5, 0.5)	Dyn-1,2	Dyn-1,2	Dyn-1,2	Dyn-1,2
	0.5	Dyn-1,2	(0.5, 0.5)	(0.5, 1.0)	Dyn-1,2	Dyn-1,2	Dyn-1,2
	0.9	Dyn-1,2	(0.5, 0.5)	(0.5, 1.0)	Dyn-1,2	Dyn-1,2	Dyn-1,2
10	0.2	Dyn-1,2	(0.5, 0.5)	(0.5, 1)	Dyn-1,2	(0.5, 1)	(0.5, 1)
	0.5	Dyn-1,2	(0.5, 0.5)	(0.5, 1)	Dyn-1,2	(0.5, 1)	(0.5, 1)
	0.9	Dyn-1,2	(0.5, 0.8)	(0.5, 1)	Dyn-1,2	(0.5, 1)	(0.5, 1)
10000	0.2	Dyn-1,2	(0.5, 1)	(0.5, 1)	Dyn-1,2	(0.5, 1)	(0.5, 1)
	0.5	Dyn-1,2	(0.5, 1)	(0.5, 1)	Dyn-1,2	(0.5, 1)	(0.5, 1)
	0.9	Dyn-1,2	(0.5, 1)	(0.5, 1)	Dyn-1,2	(0.5, 1)	(0.5, 1)

Table 7.32: Optimal repair policy under Contract A with unequal means for bivariate normal and weibull distribution ($c_1=c$)

$\frac{c=c_1}{c_2}$	ρ	Bivariate Normal Distribution with (μ_1, μ_2)				Bivariate Weibull Distribution with (μ_1, μ_2)		
		(5, 3)	**(5, 1)**	**(3, 1)**	**(2,1.5)**	**(5, 3)**	**(5, 1)**	**(3, 1)**
0.1	0.2	Dyn-2	Dyn-2	(0.5,0.5)	(1.0,0.5)	Dyn-2	Dyn-2	Dyn-2
	0.5	Dyn-2	Dyn-2	(0.5,0.5)	(1.0,0.5)	Dyn-2	Dyn-2	Dyn-2
	0.9	Dyn-2	Dyn-2	(0.5,0.5)	(1.0,0.5)	Dyn-2	Dyn-2	Dyn-2
1	0.2	Dyn-2	Dyn-2	Dyn-2	Dyn-2	Dyn-2	Dyn-2	Dyn-2
	0.5	Dyn-2	Dyn-2	Dyn-2	Dyn-2	Dyn-2	Dyn-2	Dyn-2
	0.9	Dyn-2	Dyn-2	Dyn-2	(1.0,0.5)	Dyn-2	Dyn-2	Dyn-2
10	0.2	Dyn-2	Dyn-2	Dyn-2	Dyn-2	Dyn-2	Dyn-2	Dyn-2
	0.5	Dyn-2	Dyn-2	Dyn-2	Dyn-2	Dyn-2	Dyn-2	Dyn-2
	0.9	Dyn-2	Dyn-2	Dyn-2	Dyn-2	Dyn-2	Dyn-2	Dyn-2
1000	0.2	Dyn-2	Dyn-2	Dyn-2	(0.5,1.0)	Dyn-2	Dyn-2	Dyn-2
	0.5	Dyn-2	Dyn-2	Dyn-2	(0.5,1.0)	Dyn-2	Dyn-2	Dyn-2
	0.9	Dyn-2	Dyn-2	Dyn-2	(0.5,1.0)	Dyn-2	Dyn-2	Dyn-2

Case 2: Static Repair Policies for Contract B

For Contract B, we consider normal failure distribution to calculate the following expected number of failures over the warranty region

$$M_q^B(W,U,\alpha_1,\alpha_2) = N_1(W) + N_2(U) - N(W,U)$$

Tables 7.33 and 7.34 show the expected number of failures of Contract B under bivariate normal failure distribution with a correlation coefficient of 0.2. As in Contract A, the expected number of failures increases as the extent of repair of at least one dimension decreases and/or the reliability of product decreases. Since, the warranty range of this policy is larger than that of Contract A, the expected number of breakdowns under this policy are larger than that under Contract A. However, unlike Contract A, in this case, we observe that the expected number of failures decreases as the correlation coefficient increases (Table 7.35). This phenomenon can be explained by the fact that a high correlation coefficient triggers frequent failures motivated by the forces at play along both dimensions and results in quick move outside the warranty region. On the other hand, when the correlation is low there is more likelihood to operate within the warranty region even after coverage expires along one of the two-dimensions.

Table 7.33: Expected number of failures with bivariate normal failure distribution under Contract B (ρ=0.2)

(α_1, α_2)	Bivariate Normal Distribution with (μ_1, μ_2)		
	(5, 5)	**(3, 3)**	**(1.5, 1.5)**
(1.0, 1.0)	0.103477	0.722486	1.720690
(1.0, 0.8)	0.103523	0.726345	1.949780
(1.0, 0.5)	0.104054	0.762304	4.211240
(0.8, 0.8)	0.103570	0.730152	2.198530
(0.8, 0.5)	0.104100	0.765788	4.342390
(0.5, 0.5)	0.104628	0.799407	>4.35

Table 7.34: Expected number of failures with bivariate normal failure distribution under Contract B (ρ =0.2)

(α_1, α_2)	Bivariate Normal Distribution with (μ_1, μ_2)		
	(5, 3)	**(5, 1.5)**	**(3, 1.5)**
(1.0, 1.0)	0.521407	1.511143	1.511711
(1.0, 0.8)	0.524826	1.859609	1.859777
(1.0, 0.5)	0.560988	4.316366	4.316297
(0.8, 1.0)	0.520975	1.511152	1.512785
(0.8, 0.8)	0.524870	1.859610	1.860097
(0.8, 0.5)	0.561027	4.316364	4.316151
(0.5, 1.0)	0.521498	1.511269	1.528722
(0.5, 0.8)	0.525384	1.859634	1.868255
(0.5, 0.5)	0.561484	4.316338	4.317115

Table 7.35: Expected number of failures with equal means under different ρ values for Contract B

(α_1, α_2)	ρ	Bivariate Normal Distribution with (μ_1, μ_2)		
		(5, 5)	**(3, 3)**	**(1.5, 1.5)**
(1.0, 1.0)	0.2	0.103477	0.722486	1.720690
	0.5	0.095782	0.671059	1.666180
	0.9	0.074370	0.575118	1.558460
(0.8, 0.8)	0.2	0.103570	0.730152	2.102950
	0.5	0.095874	0.678335	2.045660
	0.9	0.074446	0.580707	1.906650
(0.5, 0.5)	0.2	0.104628	0.799407	>2.11
	0.5	0.096909	0.743085	>2.05
	0.9	0.075279	0.630041	>1.91

Tables 7.36-7.39 show the optimal extent of repair combination that gives the minimum expected warranty cost under different scenarios. Under Contract B, the behavior of the expected cost function is similar to the behavior of the cost under Contract A. That is, for a given cost ratio, i.e. $c/c_1 = c_2$ or $c = c_1/c_2$, the extent of repair for any dimension increases as the reliability of the dimension decreases and for a given reliability the extent increases as the cost ratio increases. Also, the correlation

coefficient does not affect the optimal repair degree of each dimension. However, the difference between these two policies is that for a given cost structure, under Contract B the optimal extent of repair for any dimension is equal or greater than the optimal extent under Contract A, since this policy favors the consumers. In addition, in this policy if the reliabilities of dimensions are not equal, applying better extent of repair to the less reliable dimension provides smaller cost.

Table 7.36: Optimal repair degree combination of equal means for bivariate normal distribution under Contract B($c_1 = c_2$)

$\frac{c}{c_1 = c_2}$	ρ	Bivariate Normal Distribution with (μ_1, μ_2)		
		(5, 5)	**(3, 3)**	**(1.5, 1.5)**
0.01	0.2	(0.5, 0.5)	(0.5, 0.5)	(1.0, 1.0)
	0.5	(0.5, 0.5)	(0.5, 0.5)	(1.0, 1.0)
	0.9	(0.5, 0.5)	(0.5, 0.5)	(1.0, 1.0)
1	0.2	(0.5, 0.5)	(0.5, 0.5)	(1.0, 1.0)
	0.5	(0.5, 0.5)	(0.5, 0.5)	(1.0, 1.0)
	0.9	(0.5, 0.5)	(0.5, 0.5)	(1.0, 1.0)
10	0.2	(0.5, 0.5)	(0.8, 0.8)	(1.0, 1.0)
	0.5	(0.5, 0.5)	(0.8, 0.8)	(1.0, 1.0)
	0.9	(0.5, 0.5)	(0.8, 0.8)	(1.0, 1.0)
200	0.2	(0.8, 0.8)	(1.0, 1.0)	(1.0, 1.0)
	0.5	(0.8, 0.8)	(1.0, 1.0)	(1.0, 1.0)
	0.9	(0.8, 0.8)	(1.0, 1.0)	(1.0, 1.0)
500	0.2	(1.0, 1.0)	(1.0, 1.0)	(1.0, 1.0)
	0.5	(1.0, 1.0)	(1.0, 1.0)	(1.0, 1.0)
	0.9	(1.0, 1.0)	(1.0, 1.0)	(1.0, 1.0)

Table 7.37: Optimal repair degree combination of unequal means for bivariate normal distribution under Contract B ($c_1 = c_2$)

$\frac{c}{c_1 = c_2}$	ρ	Bivariate Normal Distribution with (μ_1, μ_2)			
		(5, 3)	**(5, 1.5)**	**(3, 1.5)**	**(2,1.5)**
0.01	0.2	(0.5,0.5)	(0.5,1.0)	(0.5,1.0)	(0.8,1.0)
	0.5	(0.5,0.5)	(0.5,1.0)	(0.5,1.0)	(0.8,1.0)
	0.9	(0.5,0.5)	(0.5,1.0)	(0.5,1.0)	(0.8,1.0)
1	0.2	(0.5,0.5)	(0.5,1.0)	(0.5,1.0)	(0.8,1.0)
	0.5	(0.5,0.5)	(0.5,1.0)	(0.5,1.0)	(0.8,1.0)
	0.9	(0.5,0.5)	(0.5,1.0)	(0.5,1.0)	(0.8,1.0)
10	0.2	(0.5,1.0)	(0.5,1.0)	(0.8,1.0)	(1.0,1.0)
	0.5	(0.5,1.0)	(0.5,1.0)	(0.8,1.0)	(1.0,1.0)
	0.9	(0.5,1.0)	(0.5,1.0)	(0.8,1.0)	(0.8,1.0)
1000	0.2	(0.8,1.0)	(0.5,1.0)	(1.0,1.0)	(1.0,1.0)
	0.5	(0.8,1.0)	(0.5,1.0)	(0.8,1.0)	(1.0,1.0)
	0.9	(0.8,1.0)	(0.5,1.0)	(0.8,1.0)	(0.8,1.0)
50000	0.2	(0.8,1.0)	(1.0,1.0)	(1.0,1.0)	(1.0,1.0)
	0.5	(1.0,1.0)	(0.5,1.0)	(1.0,1.0)	(1.0,1.0)
	0.9	(1.0,1.0)	(0.5,1.0)	(0.8,1.0)	(0.8,1.0)

Table 7.38: Optimal repair degree combination of equal means for bivariate normal distribution under Contract B ($c = c_1$)

$\frac{c = c_1}{c_2}$	ρ	Bivariate Normal Distribution with (μ_1, μ_2)		
		(5, 5)	**(3, 3)**	**(1.5, 1.5)**
0.1	0.2	(0.5, 0.5)	(0.5, 0.5)	(1.0, 1.0)
	0.5	(0.5, 0.5)	(0.5, 0.5)	(1.0, 1.0)
	0.9	(0.5, 0.5)	(0.5, 0.5)	(1.0, 1.0)
1	0.2	(0.5, 0.5)	(0.5, 0.5)	(1.0, 1.0)
	0.5	(0.5, 0.5)	(0.5, 0.5)	(1.0, 1.0)
	0.9	(0.5, 0.5)	(0.5, 0.5)	(1.0, 1.0)
10	0.2	(0.5, 0.5)	(0.5, 0.5)	(1.0, 1.0)
	0.5	(0.5, 0.5)	(0.5, 0.5)	(1.0, 1.0)
	0.9	(0.5, 0.5)	(0.5, 0.5)	(1.0, 1.0)
1000	0.2	(0.5, 0.5)	(0.5, 0.5)	(1.0, 1.0)
	0.5	(0.5, 0.5)	(0.5, 0.5)	(1.0, 1.0)
	0.9	(0.5, 0.5)	(0.5, 0.5)	(1.0, 1.0)

Table 7.39: Optimal repair degree combination of unequal means for bivariate normal distribution under Contract B ($c = c_1$)

$\frac{c=c_1}{c_2}$	ρ	Bivariate Normal Distribution with (μ_1, μ_2)			
		(5, 3)	**(5, 1.5)**	**(3, 1.5)**	**(2,1.5)**
0.1	0.2	(0.5,0.5)	(0.5,1.0)	(0.5,1.0)	(1.0,1.0)
	0.5	(0.5,0.5)	(0.5,1.0)	(0.5,1.0)	(1.0,1.0)
	0.9	(0.5,0.5)	(0.5,1.0)	(0.5,1.0)	(1.0,1.0)
1	0.2	(0.5,0.5)	(0.5,1.0)	(0.5,1.0)	(0.8,1.0)
	0.5	(0.5,0.5)	(0.5,1.0)	(0.5,1.0)	(0.8,1.0)
	0.9	(0.5,0.5)	(0.5,1.0)	(0.5,1.0)	(0.8,1.0)
10	0.2	(0.5,0.8)	(0.5,1.0)	(0.5,1.0)	(0.8,1.0)
	0.5	(0.5,0.8)	(0.5,1.0)	(0.5,1.0)	(0.8,1.0)
	0.9	(0.5,0.8)	(0.5,1.0)	(0.5,1.0)	(0.8,1.0)
1000	0.2	(0.5,1.0)	(0.5,1.0)	(0.5,1.0)	(0.8,1.0)
	0.5	(0.5,1.0)	(0.5,1.0)	(0.5,1.0)	(0.8,1.0)
	0.9	(0.5,1.0)	(0.5,1.0)	(0.5,1.0)	(0.8,1.0)

Case 3: Static Repair Policies for Contract C

For Contract C, we consider normal failure distribution with the following expected number of failures

$$M_q^D(W,\ U) = E[N(W,\ U)] = \sum_{n=1}^{\infty} F_K^{(n)}(U)$$

where F_K is the distribution function of $K_i=X_i+mT_i$, $i \geq 1$. In this contract, we change the warranty limits such a way that the areas of coverage in Contract A and C are equal. That is, W_c and U_c chosen so as to set $WU = \frac{W_c U_c}{2}$. Since, in Contract A, $W=U=3$, in Contract C, the limits are $W_c=U_c=4.24$.

Tables 7.40 and 7.41 show the expected number of failures for Contract C under bivariate normal failure distribution with a correlation coefficient of 0.2. As in

Contract A and B, in this policy the effect of extent of repair is almost negligible when the reliability of the product is high. Although, for highly reliable products the effect of correlation coefficient on the expected number of failures is similar to the effect in Contract A, for products with low reliability, the expected number of breakdowns decreases as the correlation between the dimensions increases (Table 6.42).

Table 7.40: Expected number of failures with bivariate normal failure distribution under Contract C (ρ=0.2)

(α_1, α_2)	Bivariate Normal Distribution with (μ_1, μ_2)		
	(5, 5)	**(3, 3)**	**(1.5, 1.5)**
(1.0, 1.0)	0.000932	0.042527	0.960079
(1.0, 0.8)	0.000932	0.042527	0.962677
(1.0, 0.5)	0.000932	0.042529	0.972267
(0.8, 0.8)	0.000932	0.042528	0.967665
(0.8, 0.5)	0.000932	0.042531	0.985222
(0.5, 0.5)	0.000932	0.042551	>0.99

Table 7.41: Expected number of failures with bivariate normal failure distribution under Contract C (ρ=0.2)

(α_1, α_2)	Bivariate Normal Distribution with (μ_1, μ_2)		
	(5, 3)	**(5, 1.5)**	**(3, 1.5)**
(1.0, 1.0)	0.005653	0.034464	0.289859
(1.0, 0.8)	0.005653	0.034465	0.289875
(1.0, 0.5)	0.005653	0.034466	0.289937
(0.8, 1.0)	0.005653	0.034465	0.289887
(0.8, 0.8)	0.005653	0.034466	0.289928
(0.8, 0.5)	0.005653	0.034470	0.290079
(0.5, 1.0)	0.005653	0.034475	0.290086
(0.5, 0.8)	0.005654	0.034484	0.290294
(0.5, 0.5)	0.005655	0.034512	0.291065

Table 7.42: Expected number of failures with equal means under different ρ values for Contract C

(α_1, α_2)	ρ	Bivariate Normal Distribution with (μ_1, μ_2)		
		(5, 5)	**(3, 3)**	**(1.5, 1.5)**
(1.0, 1.0)	0.2	0.000932	0.042527	0.960079
	0.5	0.002728	0.061763	0.943630
	0.9	0.006846	0.085624	0.923835
(0.8, 0.8)	0.2	0.000932	0.042528	0.967665
	0.5	0.002728	0.061769	0.954429
	0.9	0.006847	0.085663	0.937646
(0.5, 0.5)	0.2	0.000932	0.042551	>1.5
	0.5	0.002729	0.061887	>1.5
	0.9	0.006856	0.086086	>1.5

Tables 7.43-7.46 show optimal extents of repair for each dimension under different cost ratios. Since the effect of repair extent is negligible, minimum extents of repair for any dimension give the smallest warranty cost for most cost ratio. Only when the fixed cost becomes extremely large, the replacement policies for both dimensions provide the smallest cost for medium and low quality products. Like in Contract A, the dimension whose variable cost component is high has a smaller extent of repair than the dimension with low variable cost component.

Table 7.43: Optimal repair degree combination of equal means for bivariate normal distribution under Contract C($c_1 = c_2$)

$\frac{c}{c_1 = c_2}$	ρ	Bivariate Normal Distribution with (μ_1, μ_2)		
		(5, 5)	**(3, 3)**	**(1.5, 1.5)**
0.01	0.2	(0.5,0.5)	(0.5,0.5)	(0.5,0.8)
	0.5	(0.5,0.5)	(0.5,0.5)	(0.5,0.8)
	0.9	(0.5,0.5)	(0.5,0.5)	(0.5,1.0)
1	0.2	(0.5,0.5)	(0.5,0.5)	(0.5,0.8)
	0.5	(0.5,0.5)	(0.5,0.5)	(0.5,0.8)
	0.9	(0.5,0.5)	(0.5,0.5)	(0.5,1.0)
10	0.2	(0.5,0.5)	(0.5,0.5)	(0.5,0.8)
	0.5	(0.5,0.5)	(0.5,0.5)	(0.5,0.8)
	0.9	(0.5,0.5)	(0.5,0.5)	(0.5,1.0)
1000	0.2	(0.5,0.5)	(0.5,0.8)	(1.0,1.0)
	0.5	(0.5,0.5)	(0.5,1.0)	(1.0,1.0)
	0.9	(0.5,1.0)	(0.8,1.0)	(1.0,1.0)
500000	0.2	(0.5,0.5)	(1.0,1.0)	(1.0,1.0)
	0.5	(1.0,1.0)	(1.0,1.0)	(1.0,1.0)
	0.9	(1.0,1.0)	(1.0,1.0)	(1.0,1.0)

Table 7.44: Optimal repair degree combination of unequal means for bivariate normal distribution under Contract C ($c_1 = c_2$)

$\frac{c}{c_1 = c_2}$	ρ	Bivariate Normal Distribution with (μ_1, μ_2)			
		(5, 3)	**(5, 1.5)**	**(3, 1.5)**	**(2,1.5)**
0.01	0.2	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(0.8,0.5)
	0.5	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)
	0.9	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)
1	0.2	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)
	0.5	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)
	0.9	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)
10	0.2	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)
	0.5	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)
	0.9	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)
1000	0.2	(0.5,0.5)	(0.8,0.5)	(1.0,0.5)	(1.0,0.8)
	0.5	(0.8,0.5)	(1.0,0.5)	(1.0,0.8)	(1.0,1.0)
	0.9	(1.0,0.5)	(1.0,0.5)	(1.0,1.0)	(1.0,1.0)
500000	0.2	(1.0,1.0)	(1.0,1.0)	(1.0,1.0)	(1.0,1.0)
	0.5	(1.0,1.0)	(1.0,1.0)	(1.0,1.0)	(1.0,1.0)
	0.9	(1.0,1.0)	(1.0,1.0)	(1.0,1.0)	(1.0,1.0)

Table 7.45: Optimal repair degree combination of equal means for bivariate normal distribution under Contract C($c = c_1$)

$\frac{c=c_1}{c_2}$	ρ	Bivariate Normal Distribution with (μ_1, μ_2)		
		(5, 5)	**(3, 3)**	**(1.5, 1.5)**
0.1	0.2	(0.5,0.5)	(0.5,0.5)	(0.5,0.8)
	0.5	(0.5,0.5)	(0.5,0.5)	(0.5,0.8)
	0.9	(0.5,0.5)	(0.5,0.5)	(0.5,0.8)
1	0.2	(0.5,0.5)	(0.5,0.5)	(0.5,0.8)
	0.5	(0.5,0.5)	(0.5,0.5)	(0.5,0.8)
	0.9	(0.5,0.5)	(0.5,0.5)	(0.5,1.0)
10	0.2	(0.5,0.5)	(0.5,0.5)	(0.5,1.0)
	0.5	(0.5,0.5)	(0.5,0.5)	(0.5,1.0)
	0.9	(0.5,0.5)	(0.5,0.5)	(0.5,1.0)
10000	0.2	(0.5,0.5)	(0.5,1.0)	(0.5,1.0)
	0.5	(0.5,1.0)	(0.5,1.0)	(0.5,1.0)
	0.9	(0.5,1.0)	(0.5,1.0)	(0.5,1.0)

Table 7.46: Optimal repair degree combination of unequal means for bivariate normal distribution under Contract C ($c = c_1$)

$\frac{c=c_1}{c_2}$	ρ	Bivariate Normal Distribution with (μ_1, μ_2)			
		(5, 3)	**(5, 1.5)**	**(3, 1.5)**	**(2,1.5)**
0.1	0.2	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)
	0.5	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)
	0.9	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)
1	0.2	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)
	0.5	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)
	0.9	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)
10	0.2	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)
	0.5	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(0.5,0.8)
	0.9	(0.5,0.5)	(0.5,0.5)	(0.5,0.5)	(0.5,0.8)
1000	0.2	(0.5,1.0)	(0.5,1.0)	(0.5,1.0)	(0.5,1.0)
	0.5	(0.5,1.0)	(0.5,1.0)	(0.5,1.0)	(0.5,1.0)
	0.9	(0.5,1.0)	(0.5,1.0)	(0.5,1.0)	(0.5,1.0)

Chapter 8

CONCLUSION AND FUTURE RESEARCH DIRECTIONS

Increasing market competition forces many manufacturers to offer extensive warranties. Faced with the challenge of keeping the associated costs under control, most companies seek efficient rectification strategies. Most products are marketed with one- or two-dimensional warranties. The one-dimensional warranty is suitable for a product whose lifetime is affected only by time or usage. If the lifetime of a product is affected by both time and usage, then the two-dimensional warranty is more descriptive. In the literature, the problem of warranty cost minimization is generally considered by using one-dimensional warranties.

In this thesis, the repair strategies are investigated with the intent of minimizing the expected warranty cost expressed as a function of different parameters such as reliability of the product, relationship between cost components and structure of the warranty contract. It may be worthwhile to mention that to the best of author's knowledge, this study is the first to consider an age reduction approach to model the imperfect repairs under the two-dimensional coverage.

We first addressed one-dimensional policies and adopted a quasi renewal process to model the product failure and associated repair action. The quasi renewal process employs a scaling parameter that reflects the extent of repair. Then, we generalized quasi renewal process to multivariate scenarios, and we considered two-dimensional warranty policies using bivariate quasi renewal process with two scaling parameters factors: one for each dimension. In bivariate case, two scaling factors may not be equal.

The warranty related studies in the literature consider either a constant repair cost or some functions of the time and/or count the number of failures. Our cost function is more realistic as it incorporates both fixed and variable components. The fixed component is a constant and it is paid independently from the extent of repair, whereas, the variable component increases with the extent of repair. It may be important to note that a particular cost function in the study is adopted to provide a good representation of different scenarios and the general approach can be repeated with other cost functions if necessary.

Through computational experimentations, we investigated the effects of the repair mechanism on the expected warranty cost under different combinations of problem parameters. These parameters correspond to product reliability, failure distribution, ratio between cost components and type of warranty contract. With respect to the mean time to the first failure, we grouped products as of high, medium and poor reliability. For each group, we modeled the distribution of failures as univariate normal and weibull in the one-dimensional warranties, and bivariate normal or weibull in the two-dimensional warranties. We also examined the effect of relative magnitude of fixed and variable components under one- and two-dimensional warranties. Finally, we examined different contract types offering different degrees of protection to the manufacturer and consumer in the case of two-dimensional warranties.

Under the static repair, the expected number of failures in the one-dimensional policies increases as the extent of repair decreases. When the mean time to failure is large, the effect of the extent of repair is negligible, but as the mean gets smaller, the effect becomes more significant. Although we can easily characterize the behavior of the expected number of failures under different mean values, the behavior of the cost function depends on both the cost and reliability structure. In the one-dimensional warranties, the cost function shows an increasing trend, as the mean time to the first failure increases and the ratio between the fixed and variable components decreases. For a less reliable product with a small mean time to the first failure, perfect repair tends to be optimal for any value of the cost ratio. On the other hand, for an exceptionally reliable product, the smallest extent of repair seems to be preferred unless the fixed component is much larger than the variable component. In all of our experiments, if the fixed component is large, then a more extensive repair minimizes the cost function. When the fixed component reaches a certain threshold, then perfect repair becomes the optimal repair policy for any type of product. After the first failure if the product is replaced by an improved one and if the succeeding failures are rectified via replacement, the expected number of failures is smaller than that of the perfect repair. The performance of this policy improves as the product becomes less reliable. In our experiments with the dynamic policy, we observe that it dominates the static policy when the product is highly reliable and/or the fixed component is much larger than the variable component.

In the two-dimensional policies, the behavior of the expected number of failures follows a similar pattern in general to that in the univariate case. However, the computational experiments suggest several additional insights regarding the expected warranty cost. When the means along the two dimensions are equal with equal variable components, the optimal extent of repair increases as the fixed component increases beyond a certain threshold. On the other hand, when the means are not equal, the extent of repair in the more reliable dimension tends to be more elaborate than that of the other. As in the one-dimensional case, when the fixed component becomes much larger than the variable components, replacement appears to be the optimal policy. If the variable components are not equal, the extent of repair is larger in the dimension with a lower variable component. In the case of the improved or dynamic policy, we observe in our experiments that these policies dominate the optimal static policy when their expected number of failures and total variable cost are smaller than that of optimal static policy.

Although, this thesis focuses on one- and two-dimensional warranties, the experimentation can be extended for multi-dimensional warranties with the model given in Chapter 5. For example, the three-dimensional quasi renewal process may be used to model the warranty policy offered for the flight engines.

Several preventive maintenance methods may be integrated to this model. For instance, the original product may be repaired to bring it as god as new state after the first year of its purchase. The maintenance may also be applied after a specific number of failures or at times kT, $k = 1,2,\ldots$ where T is a prevent maintenance interval.

In addition, the negligible repair time assumption can be removed. The repair time can be assumed to be either an independently and identically distributed random variable or modeled as a quasi renewal process. If the repair time is modeled as a quasi renewal process, its corresponding scaling parameter may be greater than 1. This scaling factor indicates the increase in the repair time as the number of failures increases.

This study can be generalized to accommodate multi-component systems. In this case, each component failure process may be modeled with the quasi renewal process.

Finally, this problem can be approached with a game theoretic method. In this case, a repair policy is selected such that under the selected policy, both the manufacturer's and customer's costs are at a minimum.

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