

OPTIMAL BUNDLE FORMATION AND PRICING OF TWO PRODUCTS WITH LIMITED STOCK

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By

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ABSTRACT

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In this study, we consider the stochastic modeling of a retail firm that sells two types of perishable products in a single period not only as independent items but also as a bundle. Our emphasis is on understanding the bundling practices on the inventory and pricing decisions of the firm. One of the issues we address is to decide on the number of bundles to be formed from the initial product inventory levels and the price of the bundle to maximize the expected profit. Product demands follow a Poisson Process with a price dependent rate. Customer reservation prices are assumed to have a joint distribution. We study the impact of reservation price distributions, initial inventory levels, product prices, demand arrival rates and cost of bundling. We observe that the expected profit decreases as the correlation coefficient increases. With negative correlation, bundling cost has a significant impact on the number of bundles formed. When the product prices are low, the retailer sells individual products as well as the bundle (mixed bundling), when they are high, the retailer sells only bundles (pure bundling). The expected profit and the number of bundles offered decrease as the variance of the reservation price distribution increases. For high starting inventory levels, the retailer reduces bundle price and offers more bundles. The number of bundle sales decreases and the number of individual product sales increases when the arrival rate increases since the need for bundling decreases. Impacts of substitutability and complementarity of products are also investigated. The retailer forms more bundles, or charges higher prices for the bundle or both as the products become more complementary and less substitutable.

Keywords: Bundling, Pricing, Stochastic Demand, Revenue Management.

ÖZET

KISITLI STOĞA SAHİP İKİ ÜRÜN DEN OPTİMAL PAKET OLUŞTURULMASI VE FİYATLANDIRILMASI

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Bu çalışmada, iki tür ürünün, sadece bağımsız ürünler olarak değil karma paket olarak da satışını yapan bir firmanın envanter ve fiyatlandırma kararlarında paketleme uygulamalarının rassal modellemesi incelenmiştir. Beklenen kârı en çoklayacak şekilde başlangıç ürün envanter seviyesinin ne kadarından paket oluşturulması ve paketin fiyatının belirlenmesi kararları üzerinde durulmuştur. Ürün talepleri, oranı fiyata bağlı Poisson sürecini takip etmektedir. Müşteri rezervasyon fiyatlarının birleşik dağılıma sahip olduğu varsayılmıştır. Rezervasyon fiyat dağılımlarının, başlangıç envanter seviyelerinin, ürün fiyatlarının, talep geliş oranlarının ve paketleme maliyetlerinin beklenen kâra, paketlenen ürün sayılarına ve paket fiyatlarına olan etkileri sayısal bir çalışmayla araştırılmıştır. Bağımlılık katsayısı arttıkça beklenen kârın azaldığı gözlemlenmiştir. Negatif bağımlılıkta paketleme maliyeti, oluşturulan paket sayısında anlamlı bir etkiye sahiptir. Ürünlerin yalnız başına olan fiyatları düşük iken, firma tek tek ürünlerle birlikte paket de satmaktadır (karışık paketleme); ürün fiyatları yüksek iken, firma sadece paket satmaktadır (saf paketleme). Rezervasyon fiyat dağılımının değişkenliği arttıkça beklenen kâr ve sunulan paket sayısı azalır. Yüksek başlangıç envanteri için, firma paket fiyatını düşürür ve daha çok paket sunar. Paketlemeye olan ihtiyaç azaldığı için geliş oranı arttıkça paket satış miktarı azalır ve tek tek ürün satışı artar. Bütün bunlara ilave olarak, ürünlerin yerine geçme ve tamamlayıcılıklarının etkisi de incelenmiştir. Ürünler daha çok tamamlayıcı daha az yerine geçen olduğunda, firma daha çok paket oluşturur ya da paketi daha yüksek fiyatlandırır ya da her ikisini birden yapar.

Anahtar sözcükler: Paketleme, Fiyatlandırma, Rassal Talep, Gelir Yönetimi.

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Contents

| | | |
|----------|---|-----------|
| 1 | INTRODUCTION AND LITERATURE REVIEW | 1 |
| 1.1 | Introduction | 1 |
| 1.2 | Related Literature | 7 |
| 1.3 | Scope of Our Study | 13 |
| 1.4 | Summary of Our Results | 14 |
| 2 | MODEL and THE ANALYSIS | 17 |
| 2.1 | Problem Definition | 18 |
| 2.2 | Problem Formulation | 20 |
| 2.2.1 | Purchasing Probabilities | 20 |
| 2.2.2 | Switching Probabilities | 22 |
| 2.3 | Sales Probabilities and the Objective Function | 28 |
| 2.4 | Superadditivity and Subadditivity of Reservation Prices | 39 |
| 2.4.1 | Purchasing Probabilities | 40 |
| 2.4.2 | Switching Probabilities | 41 |

| | |
|--------------------------|-----------|
| <i>CONTENTS</i> | vii |
| 3 NUMERICAL STUDY | 48 |
| 4 CONCLUSION | 62 |
| A TABLES | 65 |

List of Figures

| | | |
|------|--|----|
| 3.1 | Base case - Profit vs. correlation | 51 |
| 3.2 | Profit vs. product price for $\rho = -0.9$ | 53 |
| 3.3 | Profit vs. product price for $\rho = 0$ | 53 |
| 3.4 | Profit vs. product price for $\rho = 0.9$ | 54 |
| 3.5 | Profit vs. standard deviation for $\rho = -0.9$ | 55 |
| 3.6 | Profit vs. standard deviation for $\rho = 0$ | 56 |
| 3.7 | Profit vs. standard deviation for $\rho = 0.9$ | 56 |
| 3.8 | Profit vs. arrival rate for $\rho = -0.9$ | 57 |
| 3.9 | Profit vs. arrival rate for $\rho = 0$ | 58 |
| 3.10 | Profit vs. arrival rate for $\rho = 0.9$ | 58 |
| 3.11 | Profit vs. degree of contingency for $\rho = -0.9$ | 60 |
| 3.12 | Profit vs. degree of contingency for $\rho = 0$ | 60 |
| 3.13 | Profit vs. degree of contingency for $\rho = 0.9$ | 61 |

List of Tables

| | | |
|-----|---|----|
| 2.1 | Cases of realizations to derive the expected profit | 29 |
| 3.1 | Base case | 51 |
| A.1 | The impact of product prices on the expected profit | 66 |
| A.2 | The impact of initial inventory levels on the expected profit . . . | 67 |
| A.3 | The impact of variance of reservation price distributions on the expected profit | 68 |
| A.4 | The impact of arrival rate on the expected profit | 69 |
| A.5 | The impact of degree of contingency on the expected profit for $\rho = -0.9$ | 70 |
| A.6 | The impact of degree of contingency on the expected profit for $\rho = 0$ | 70 |
| A.7 | The impact of degree of contingency on the expected profit for $\rho = 0.9$ | 71 |

Chapter 1

INTRODUCTION AND LITERATURE REVIEW

1.1 Introduction

One of the main objectives of a firm is to maximize its profits. Profits can be increased through decreasing costs or increasing revenues or both. While most of the operational activities of a firm and academic literature in operations research focus on reduction of costs, significant profit improvements are also possible by managing demand and increasing revenue. Recently, several industries and many researchers are turning their attention to the demand side of the equation under the concept of Yield Management.

Yield Management, also known as Revenue Management, is described as “the process of managing the sales of perishable assets by controlling price and inventory so as to maximize profit” (Swann [26]). In this study we incorporate product bundling to the practice of Yield Management. We consider the optimal bundle pricing of two products that face stochastic demand during a finite selling horizon.

We consider the bundle formation and pricing problem of a retailer that sells

two products facing random demand, under inventory constraints over a finite selling horizon. Before providing a more detailed description of our model and findings we would like to introduce and discuss main concepts related to revenue management and pricing.

The revenue management literature is also referred to as Perishable Asset Revenue Management (PARM) by Weatherford and Bodily [29]. Some examples of perishable goods are airline seats, hotel rooms, seasonal and fashion goods, high technology goods, seats for the theater and sporting events and traffic on network lines. The main characteristics of Revenue Management given by Swann [26] are listed as:

- *Limited Capacity*: The capacity of the firm is considered limited since extra capacity has a high cost and replenishment takes long time. It is also referred to as “fixed number of product units”.
- *Market Segmentation*: A company can differentiate the product’s markets via timing of purchase, timing of delivery, etc.
- *Perishable Inventory*: Products considered in revenue management perish after a specific date.
- *Product Sold in Advance*: Time of purchase is often used to differentiate products and as a result, in the airline case, the firm should decide the number of seats reserved for low prices and the date of offering only high price seats.
- *Variable Demand*: Demand varies seasonally, weekly or even daily for products. Revenue management is used to smooth the demand variability. As an example, increasing the price during periods of high demand and decreasing the price during periods of low demand can smooth the demand.

By the help of fast and cheap electronic communication and internet, dynamic pricing is now easily applicable. Companies that successfully implement dynamic pricing to match their and demand include fashion retailer Marks and

Spencer that offers markdowns for its swimsuits before the end of summer season. Delta Airlines offers last-minute cheap tickets to capture the customers with low reservation prices. They categorize fares as five groups (First, Business, Flexible, Discounted, Deeply-discounted). Low fares have more restrictions. (You cannot change your flight date or you can with high penalties compared to others). As the airplane's seat capacity is fixed, when to offer those tickets and with what price is a valuable research area of dynamic pricing.

Now we review the general pricing concepts.

Noble and Gruca [20] classify the existing theoretical pricing research into a two level framework for industrial goods pricing. The first level consists of four pricing situations: New Product, Competitive, Product Line and Cost-based and the second level contains the pricing strategies appropriate for those situations. The pricing environments and the corresponding strategies are briefly explained below:

1) New Product pricing applies to the early life of a product. Strategies of new product pricing are listed as:

a) Price Skimming: The initial price is set high and systematically reduced over time. Customers expect prices to eventually fall. The main objective is to attract customers who are insensitive to the initial high price. After this segment gets saturated, the price is lowered to increase the demand of the product.

b) Penetration Pricing: In this strategy, the price of the product is set low. The aim is to accelerate the product adoption among customers.

c) Experience Curve Pricing: The initial price is set low again to increase the volume of production and reduce costs through accumulated experience. As a result the unit product cost will be down.

2) Competitive pricing is used when pricing decision is given considering the pricing decisions of competitors. The employed strategies are:

a) Leader Pricing: The price leader initiates price changes and expects that

others in the industry will apply price adjustments in the similar direction. In general, the price of an identical product is higher if the leader company sells it due to the brand advantage.

b) Parity Pricing: The firm matches the price set by the overall market or the price leader firm.

c) Low Price Supplier: The firm sets the price lower than its competitors and it aims to have higher demand than the others.

3) Product Line pricing is appropriate when the price of the core product is influenced by the other related products or services of the same company. The pricing strategies are:

a) Complementary Product Pricing: The price of the core product is set low and the complementary items such as accessories, supplies, spare parts, services etc. have prices with a higher premium.

b) Price Bundling: The product is offered as a part of a bundle that contains several products, usually at a total price that gives customers an attractive savings over the sum of individual products.

c) Customer Value Pricing: One version of the product is offered at a very competitive price level; however the product contains fewer features than the main version.

4) Cost-based pricing focuses on the internal costs of the firm including fixed and variable costs and contribution margins.

As mentioned before, bundling is an effective tool for pricing. Bundling does not have a consistent and universally accepted definition. Adams and Yellen [1] define bundling as “selling goods in package” and Gultinan [13] defines it as “the practice of marketing two or more products and/or services in a single package for a special price”. The definition by Stremersch and Tellis [25] is the “sale of two or more separate products in one package”. They consider separate products as products with separate markets exist and products include both

goods and services. A well-known example of bundling is encountered at fast food restaurants. Burger King's Whopper Menu contains a big Whopper, fried potatoes and a bottle of Coke. Offering a hotel room and a breakfast is another bundling practice applied in the service industry. Two-way tickets of airline companies are another form of bundling.

Nalebuff [19] provides a good review of bundling concepts and the motivations behind bundling. The author discusses two main reasons for bundling: efficiency reasons and strategic reasons. It is noted that since bundling improves efficiency, even a monopolist firm would have an incentive to offer bundle. The efficiency reasons are listed as:

a) Economies of scale: This is the most apparent reason. For example delivering a suite of software on a single CD is cheaper than sending each one on a separate CD.

b) Improving quality: Through the same example, a suite of software can offer integration and improved functionality.

c) Reducing pricing inefficiencies: The majority of economics literature has developed the theory of how bundling can be used as a price discrimination tool by a multi-good monopoly. (Stigler [24], Adams and Yellen [1], McAfee, McMillan and Whinston [18]). Bundling leads to more homogenous valuations among customers, thus, the monopolist can capture more consumer surplus. They set the individual prices inefficiently high. Customers evaluate those products with total price, they buy both or neither. Bundling reduces the total price and increases the total profit.

On the other hand, strategic reasons of bundling are given as:

a) Entry deterrence and mitigation of competition: Bundling complementary products prevents the rivals to enter the market. Colgate Motion battery-powered toothbrush and Colgate Total toothpaste are owned by Colgate-Palmolive and they make a promotion bundle to increase the battery-powered toothbrush market share. It is hard for a company, producing toothbrushes, to price as high as

Colgate toothbrushes, since customers buy the complementary toothpaste from Colgate.

b) Gain competitive advantage: Nestle introduced its new product Nescafe Ice through bundling it with its well known product Nescafe. It uses the market power of Nescafe to gain a competitive advantage.

c) Hiding prices: Bundling can hide the individual product prices. It is not easy for a customer to calculate the products' prices individually.

Bundling is categorized as Product bundling and Price bundling. Stremersch and Tellis [25] refer to *price bundling* as “the sale of two or more separate products in a package at a discount, without any integration of products” and to *product bundling* as “the integration and sale of two or more separate products or services at any price”. Price bundling is a pricing and promotional tool whereas product bundling is more of a strategic tool used to create added value. The integration in product bundling provides added value, such as compactness (multimedia PC), seamless interaction (cars), nonduplicating coverage (all-in-one insurance), reduced risk (mutual fund) or interconnectivity (Bluetooth phone and headset).

Following Adams and Yellen [1], bundling strategies can be classified as pure bundling, mixed bundling and unbundling. *Unbundling* is “a strategy in which a firm sells the products separately only, not as a bundle”. *Pure Bundling* is “a strategy in which a firm sells the products only as a bundle but not separately”. Pure bundling is sometimes called *tying* in the economics and law literature. *Mixed Bundling* is “a strategy in which a firm sells the products both as a bundle and separately”.

Reservation price is defined as “the maximum price a customer is willing to pay for the product” by Stremersch and Tellis [25]. Purchasing decision is determined by customer surplus value, which is the total gain of the customer through that purchase (the difference between his reservation price and the product price). If the reservation price is greater than or equal to the product price then he purchases the product, otherwise he does not. In literature, reservation price

behavior of a customer group is modeled as a random variable having mostly a continuous distribution.

Another factor in bundling is the degree of contingency (degree of substitutability and complementarity) of the products in the bundle. If the products are substitutable, customers are inclined to buy only one of them at a time. Then, a customer's reservation price for the bundle would be subadditive (less than the sum of the reservation prices). As an example, different brands of toothpastes Signal White and Colgate Total are substitutes of each other. An increase at the sales amount of one can cause a decrease for the other. When the products are complements, a customer's reservation price for the bundle is superadditive (more than the sum of the reservation prices). Nestle's cream, Coffeemate is a complement of Nestle's coffee, Nescafe. An increase at the sales amount of one product can also cause an increase at the sales of other. As noted in Venkatesh and Kamakura [28], "correlation in reservation prices and the degree of contingency are two distinct notions. While the degree of contingency parameter, θ , captures perceived value enhancement or reduction within each consumer, the correlation in reservation prices of two products, ρ , shows how stand-alone reservation prices relate to each other across consumers".

Next we want to discuss the pricing and bundling literature related to our study.

1.2 Related Literature

In the literature there are only a few studies combining bundling and pricing issues and mostly these issues are considered separately. Hence, we also divide the literature section into two main parts. The first part focuses on the pricing issues of perishable products and the second part contains bundling and bundle pricing literature.

Pricing Literature

Gallego and van Ryzin [12] is one of the first studies related to dynamic pricing of perishable inventories when demand is price sensitive and the firm's objective function is to maximize the expected revenues. The authors formulate the problem using intensity control and obtain structural monotonicity results for optimal intensity as a function of the stock level and the length of the selling season. For a particular exponential family of demand functions, they find the optimal pricing policy in closed form. For general demand functions, they find an upper bound on the expected revenue based on analyzing the deterministic version of the problem. They extend their results to the case where demand is compound Poisson; only a finite number of prices is allowed; the demand rate is time dependent; holding cost is incurred and the initial stock level is the decision variable. Feng and Xiao [11] comment on this paper as the model cannot work for short remaining lives and small inventory amounts.

Rajan, Rakesh and Steinberg [21] study the relationship between pricing and ordering decisions of a monopolist retailer facing a known demand function where the products may exhibit physical decay or decrease in market value. They investigate the linear and nonlinear demand cases and provide results on the optimal price changes and optimal cycle length.

Yıldırım, Gürler and Berk [30] study the dynamic pricing of perishable products with random lifetimes and a general distribution. Demand comes from a Poisson Process with a price dependent rate. There is a fixed cost of a price change. They maximize the expected profit with the optimal pricing policy and optimal initial inventory level. They conclude that single price policy results in significantly lower profits through their model.

Feng and Gallego [10] consider the optimal timing of the price change from the initial price to a lower or higher price. They conclude that switching to a lower price as soon as time-to-go falls below a threshold level that depends on the number of stocks at hand. They develop an algorithm to compute the optimal value function and the optimal pricing policy.

Bitran and Caldentey [3] examine the studies on dynamic pricing policies and their relation to Revenue Management. The survey paper is based on a generic Revenue Management problem in which perishable and non-renewable set of resources satisfy stochastic price-sensitive demand processes over a finite period of time.

Bitran and Mondschein [5] study intertemporal pricing policies when selling seasonal products in a retail store. They present a continuous time model where a seller faces a stochastic arrival of customers with different valuation of the product. In this model, they characterize the optimal pricing policies as functions of time and inventory and that model is used as a benchmark for the periodic pricing model. Their model proves the procedures obtained in practice such as retail stores successively discount the product during the season and promote a liquidation sale at the end of the planning horizon. They give some economical insights as uncertainty in the demand of new products leads to higher prices, larger discounts and more unsold inventory.

Bitran, Caldentay and Mondschein [4] propose a methodology to set prices of perishable items in the context of a retail chain with coordinated prices among its stores. The authors formulate a stochastic dynamic programming problem and develop heuristic solutions that approximate optimal solutions.

Federgruen and Heching [9] consider the pricing decision with the inventory decision to maximize profit under a single item, periodic review model. Price dependent stochastic demand function is utilized. They study both finite and infinite horizon models under the assumption that prices can be adjusted arbitrarily or that they can only be decreased. Through numerical studies, the authors characterize various qualitative properties of the optimal strategies and corresponding optimal profit values.

Elmegraby and Keskinocak [7] provide a survey of the literature and current practices about dynamic pricing. Although dynamic pricing is applicable to most markets, their focus is on the dynamic pricing in the presence of inventory considerations.

After giving related pricing literature, now we present a brief discussion on bundle pricing.

Bundling and Bundle Pricing Literature

Stigler [24] is the first study that provides a clear recognition of bundling as a price discrimination tool and is based on stylized examples with a discrete number of customers. In these examples, the reservation values for the components of the bundle are negatively correlated.

Similarly, Adams and Yellen [1] investigate the use of bundling as a price discrimination device. Their study is an important and widely cited essay. They consider a monopolist producing two products with constant unit cost and facing independent demands of customers with different tastes. Customers can buy only one unit of product. The two goods are independent in demand for all customers. Hence customer's reservation price for one product is independent of the market price of the other. Therefore the maximum amount a customer is willing to pay for a bundle is the sum of the two reservation prices (strict additivity of reservation prices). The reservation prices of the products are negatively correlated. They consider three different sales strategies (unbundling, mixed bundling and pure bundling). They discuss the effects of switching between those strategies. They observe that one strategy generates more profit than others depending on the prevailing level of costs and on the distribution of customers' reservation prices. They argue that mixed bundling at least dominates pure bundling when the customer's reservation prices are negatively correlated since the customers with negatively correlated reservation prices prefer the individual products while the others prefer the bundle.

Schamelensee [23] relaxes the negative correlation assumption of Adams and Yellen [1] and adds an additional assumption that customers' reservation price pairs follow a bivariate normal distribution. Profitability of bundling is examined as a function of production costs, the mean and variance values of customer reservation prices for each product and the correlation between the reservation prices of those products. One of the findings of the study is that even though the profit function is not globally concave, a unique profit-maximizing price always exists,

and it depends on unit cost and the mean of the reservation price distribution. In addition, the effects of changes in standard deviation of reservation price are explored. The author compares mixed bundling case with pure bundling and unbundling cases. It is shown explicitly that pure bundling operates by reducing the effective dispersion of customers' tastes. As long as the reservation prices are not perfectly correlated, the standard deviation of reservation prices for the bundle is less than the sum of the standard deviations for the two component goods. The increase in profit caused by pure bundling is apparently greater than the fall in consumer surplus, as a result pure bundling increases net welfare. The author mentions that mixed bundling combines the advantages of pure bundling and unbundled sales. Hence mixed bundling is a powerful price discrimination device. A surprising result of analysis is that bundling can be profitable even when demands are uncorrelated or even positively correlated. Two comments are written to this study by Long [17] and Jeuland [16].

Jeuland [16] states that profitability ranking of the bundling strategies (pure bundling, mixed bundling and unbundling) depends on the distribution of the customers' reservation prices. Long [17] takes a continuous reservation price distribution without restricting it to any particular form for his comments and he states that profitable bundling has a necessary condition of the heterogeneity in customer tastes. If an increase in products' prices results in individual product sales, bundling will increase the profit. Another comment of the author is that if the reservation prices of individual products are not positively correlated, bundling increases the profit. The author states that bundling is more powerful as a price discrimination device when the individual products are substitutes of each other.

Salinger [22] provides a graphical analysis of bundling with two products. Additive reservation prices are assumed and main finding of the study is that if bundling does not cause a decrease in cost, then it tends to be profitable when reservation prices are negatively correlated. However if it lowers cost and costs are higher than reservation prices, bundling is profitable when demands for the components are highly positively correlated and component costs are high.

Telser [27] stresses the importance of complementarity between products as a source for bundling. Complementarity between components yields a valuation for the bundle which is super-additive, and this clearly enhances the chances that bundling is profitable.

These early studies do not focus on optimizing bundle prices. Hanson and Martin [14] broaden the bundling studies to find an optimal bundle price to maximize the profit. They investigate how a single monopolist firm, facing a segmented customer demand and product specific cost, can determine optimal product line selection and pricing. They relax the strict additivity of reservation prices. Their formulated problem is a deterministic mixed integer linear programming model. Deciding optimal number of components included in a bundle is exponentially growing problem as the number of possible components in a product line increases. Through their computational studies, they give an algorithm for deciding the optimal contents of bundles and their prices.

Stremersch and Tellis [25] combine the bundling concepts referred in marketing, economics, law and engineering literature into the same terminology. Therefore they remove the inconsistencies and ambiguity in the use of terms. The article clearly and consistently defines the bundling terms and enables a comprehensive classification of bundling strategies. They give twelve propositions that suggest which bundling strategy is optimal in various contexts.

Ernst and Kouvelis [8] study a newsboy type modeling framework, the common practice of retail firms to sell products not only as independent items but also as part of a bundle. They study a problem similar to ours, in which their focus is the inventory decisions, taking the price set as a given parameter. They also examine the switching between the individual products and the packaged good in the stock out situations. Through the numerical studies, they provide some economical insights. Positive correlation of original demands favors increased stocking levels of a multi-product package, while negative correlation tends to have opposite effect. Demand correlation of individual items and bundle, either positive or negative, results in higher profitability of the inventory system as compared to the uncorrelated case. The higher the negative correlation of the

original demands, the more profitable the inventory system. The stronger the substitution pattern, the higher the optimal stocking level of bundle.

Bulut, Gurler and Sen [6] study the single period pricing of two perishable products which are sold individually and as a bundle. Their customer demand has a Poisson distribution with a price dependent arrival rate. Assuming a general reservation prices distribution, they determine the optimal product prices that maximize the expected revenue. They also compare the performance of three bundling strategies under different conditions such as different reservation price distributions, demand arrival rates and starting inventory levels. Their numerical study demonstrates that, when individual product prices are fixed to high values, the expected revenue is a decreasing function of the correlation coefficient, while for low prices the expected revenue is an increasing function of the correlation coefficient. They indicate that, bundling is least effective in case of limited supply and their numerical studies show that the mixed bundling strategy outperforms the others, especially when the customer reservation prices are negatively correlated.

In the next section we will give the scope and the motivation of our study.

1.3 Scope of Our Study

Most of the studies done in the bundling literature search for the conditions and strategies where bundling is profitable. These studies usually assume that the products are available in unlimited amounts and the prices are fixed. However, in real life, supply may be constrained and retailers have control over prices. This gap in the literature has motivated our study.

Specifically, we consider a retail firm that sells two types of perishable products over a finite selling season. The starting inventory levels of these two products are fixed and at the beginning of the season, the retailer uses all or a portion of these initial stocks to form product bundles. The retailer then sells the bundles as well as the individual products (mixed bundling) by charging fixed prices over

the selling season. We determine the optimal number of product bundles that the retailer should form and the optimal individual and bundle prices that the retailer should charge so as to maximize his expected profit over the selling season. No replenishments are allowed during the selling season.

As previously explained nearly all bundling papers investigate the performance of pure and mixed bundling strategies. We also give similar strategy comparisons and give some managerial insights about inventory and pricing decisions under bundling.

In our model we consider product bundling and assume that bundles are formed at the beginning of the selling horizon and separation of bundles into individual items is not allowed. We also investigate the effect of cost of forming bundles, as these costs could be non negligible in some industries. For example, combining separate PC components into a PC requires technicians to work on, which adds a labor cost to bundling (See Ansari et al. [2] for a model to determine the number of bundles to be formed).

Customer arrivals to the store follow a Poisson Process with a constant arrival rate and their choice of individual products or the product bundle is governed by their reservation prices. “Posted” product prices are used which means prices are known by the customers, however they do not know the available inventory before they actually arrive at the store. When they arrive, if their preferred product is not available, they may switch to another product or not make a purchase. These switching probabilities are related to the reservation prices.

1.4 Summary of Our Results

In our numerical study we investigate the effects of various factors on our model, such as the correlation between the reservation price distributions, the variance of the reservation price distributions, initial inventory levels, the unit bundle formation cost, and the intensity of the customer arrivals. We also consider the effects of the degree of product complementarity and substitutability, also known

as the degree of contingency.

We observe from the results of our numerical study that the expected profit decreases as the correlation coefficient increases and increase in bundling cost lowers the profit. With negative correlation, bundling cost has a significant impact on the number of bundles formed. However with positive correlation, this effect is negligible.

When the product prices are below the mean value of the reservation price distribution, the optimal bundle price is the maximum of the possible bundle prices, the retailer sells individual products as well as the bundle (mixed bundling). When the individual products are high, the retailer charges a bundle price such that only bundles are sold (pure bundling).

The expected profit and the optimal number of bundles formed decreases as the variance of the reservation price distribution increases. The optimal bundle price has different behavior with respect to the correlation coefficient. For negative correlation, optimal bundle price decreases as the standard deviation increases. However for positive correlation, the optimal bundle price is an increasing function of the standard deviation. For the uncorrelated case, the optimal bundle price is a decreasing function of the standard deviation for small bundling cost values and it is an increasing function for large bundling cost values.

When the initial inventory levels are high, the retailer reduces bundle price and also offers more bundles for sale as the customer arrival rate is constant. The expected profit is an increasing function of the arrival rate. The number of bundle sales decreases and the number of individual product sales increases when the arrival rate increases since the need for bundling decreases; the retailer can easily sell its products individually.

Finally we perform analysis to investigate the impact of product substitutability and complementarity. As stated before the products are substitutable when the degree of contingency, $\theta < 0$ (bundle reservation price is subadditive) and complementary when the degree of contingency, $\theta > 0$ (bundle reservation price is superadditive). For all correlation values and bundle formation costs, we see

that the retailer is forming more bundles, or charging higher prices for the bundle or both as the degree of contingency increases. As a result, expected profit is an increasing function of degree of contingency. The impact of the degree of contingency is more pronounced when the product prices are uncorrelated and positively correlated.

The rest of the thesis is organized as follows. Chapter 2 formulates the problem investigated and explains the stochastic model used in our study. In Chapter 3 we give results of our numerical studies and in the final chapter we conclude with the discussion of our major findings and the avenues for future research.

Chapter 2

MODEL and THE ANALYSIS

As listed in the literature section, most of the studies related to bundle pricing are focused on the determination of optimal inventory levels under a given price set. Our motivation in this study is to combine the pricing decision with the inventory level decision, which is more applicable to the real world business problems. We aim to maximize the expected profit under a mixed bundling strategy for two types of products by determining the optimal number of bundles to be formed and the optimal price set. Our retailer has a finite selling season and a given initial inventory for Product 1 and Product 2. Effect of the bundling cost is also considered in the model.

Before explaining further details of the problem, we mention some of the main assumptions used in our model. The purchasing behaviors of the customers are governed by the reservation prices. Customers decide to buy products by comparing their reservation price and the product price. If their reservation price is higher than the product price they decide to buy the product. Since we do not know the customers' reservation prices exactly we consider a reservation price distribution to define the customers' purchasing behavior. We refer the reader to Jedidi and Zhang [15] for further details about estimating the customer reservation prices.

Our model is structured on the main assumption that reservation price for

the bundle is the sum of the reservation prices of individual products that form the bundle. This is one of the common assumptions used in the bundling literature such as Adams and Yellen [1], Schmalensee [23] and McAfee [18]. Gultinan [13] called this assumption as “the assumption of strict additivity”. However Venkatesh and Kamatura [28] added the product substitutability and complementarity to this assumption. If the products that form the bundle are substitute products then the reservation price for the bundle is less than the sum of the reservation prices of the individual products as the customers’ willingness to buy both is less. However if the products are complements, customers’ willingness to buy both products at the same time will be higher. At the second part of the study we also investigate the effects of substitution and complementarity among products.

Considering the stock-out situations, we assume the following. When one product incurs shortage, the customer either switches to one of the other products or leaves the store without any purchase. For these cases we assume that switching behavior will follow a Multinomial distribution since there are three possible choices. If two types of products incur shortage, the customer has the alternatives to switch to the available product or to leave without any purchase; therefore binomial distribution is used to calculate the switching probabilities.

2.1 Problem Definition

We consider a retailer with two main product types (Product 1 and Product 2) and a bundle (Product B), formed with the two products, which is sold as a separate product. The aim is to maximize the profit over a single selling season. Stock amounts, Q_1 and Q_2 of Product 1 and 2 are given and the retailer decides the number of bundles (n_b) to be formed with this initial inventory with a unit bundling cost of c . Once the number of bundles to be formed at the beginning of the season, no new bundles are formed and none of them are unbundled to offer individual products during the season.

A customer is allowed to buy only one type of product which means he or she can buy one unit of Product 1, Product 2 or a Bundle but not any combination of products. As in real life situations, a customer can also leave the store without buying any product.

To maximize the profit at the selling season, the prices of Product 1, Product 2 and Bundle (p_1 , p_2 and p_b) are optimally decided, where it is assumed that $p_b \leq p_1 + p_2$. Product prices are set only at the beginning of the selling season and they are fixed during the period.

Customers' arrival to the store has a Poisson distribution with a fixed arrival rate, λ customers per season. The decision to buy a product is determined by the comparison of the customer's reservation price with the product prices. The reservation price for a product is the maximum price that a customer wants to pay for that product. A customer buys a product if its price is less than the customer's reservation price. Purchase probabilities for Products 1, 2 and the bundle are denoted as $m_1(p_1, p_2, p_b)$, $m_2(p_1, p_2, p_b)$ and $m_b(p_1, p_2, p_b)$ respectively. For brevity, we express $m_i(p_1, p_2, p_b)$ as m_i for $i = 1, 2, b$. Then $m_0 = 1 - m_1 - m_2 - m_b$ is the probability of no purchase. The arrival rates ℓ_1 , ℓ_2 for the two products and the bundle ℓ_b are then given as $\ell_i = \lambda * m_i$ $i = 1, 2, b$. The difference between the reservation price and the actual price is known as consumer surplus in the economics literature. To maximize the retailer's profit, consumer surplus should be zero. However, pricing each product differently considering each customer's reservation price is not an easy task. Therefore bundling can give an alternative way to handle this problem to maximize the profit.

Customers' reservation prices are random variables. Reservation price of Product 1 is R_1 and reservation price of Product 2 is R_2 . Mean and variance parameters of the reservation price distribution are (μ_1, σ_1) and (μ_2, σ_2) , for Product 1 and Product 2 respectively. Reservation price of Bundle, R_b , is stated as $R_b = R_1 + R_2$.

In the next section, expressions used to calculate expected revenue for a given price set and a fixed number of bundles are presented. These expressions are used to find the optimal number of bundles to be formed and their price to maximize

the expected profit.

2.2 Problem Formulation

2.2.1 Purchasing Probabilities

Let $f_{R_1, R_2}(r_1, r_2)$ denotes the joint reservation price density for the two products. When all products are available, purchasing probabilities are calculated by comparing the customer reservation price with the product price. Customer buys either a single product or a bundle or leaves the store without buying any product. Probability expressions of these events are stated below:

Probability of No Purchase: A customer will leave the store without buying any product when his reservation prices for each product and the bundle are lower than their corresponding sales prices. Therefore, the probability of no purchase is stated as,

$$\begin{aligned}
 m_0 &= P(R_1 < p_1, R_2 < p_2, R_b < p_b) \\
 &= P(R_1 < p_1, R_2 < p_2, R_1 + R_2 < p_b) \\
 &= P(R_1 < p_1, R_2 < \min\{p_2, p_b - R_1\}) \\
 &= \int_{-\infty}^{p_1} \int_{-\infty}^{a_1} f_{R_1, R_2}(r_1, r_2) dr_1 dr_2
 \end{aligned}$$

where $a_1 = \min\{p_2, p_b - r_1\}$.

Probability of Purchasing Product 1: Purchase probabilities are calculated by comparing the consumer surplus, which is the difference between the reservation price and the product price. A customer will purchase Product 1 if his surplus from Product 1 is positive and greater than his surplus values from Product 2 and the bundle separately. Thus the probability of purchasing Product 1 is stated as,

$$m_1 = P(R_1 > p_1, R_1 - p_1 > R_2 - p_2, R_1 - p_1 > R_b - p_b)$$

$$\begin{aligned}
&= P(R_1 > p_1, R_1 - p_1 > R_2 - p_2, R_1 - p_1 > R_1 + R_2 - p_b) \\
&= P(R_1 > p_1, R_2 < \min \{R_1 - p_1 + p_2, p_b - p_1\}) \\
&= \int_{p_1}^{\infty} \int_{-\infty}^{a_2} f_{R_1, R_2}(r_1, r_2) dr_1 dr_2
\end{aligned}$$

where $a_2 = \min \{r_1 - p_1 + p_2, p_b - p_1\}$.

Probability of Purchasing Product 2: The probability of purchasing Product 2 is similarly stated as,

$$\begin{aligned}
m_2 &= P(R_2 > p_2, R_2 - p_2 > R_1 - p_1, R_2 - p_2 > R_b - p_b) \\
&= P(R_2 > p_2, R_2 - p_2 > R_1 - p_1, R_2 - p_2 > R_1 + R_2 - p_b) \\
&= P(R_2 > p_2, R_1 < \min \{R_2 - p_2 + p_1, p_b - p_2\}) \\
&= \int_{p_2}^{\infty} \int_{-\infty}^{a_3} f_{R_1, R_2}(r_1, r_2) dr_2 dr_1
\end{aligned}$$

where $a_3 = \min \{r_2 - p_2 + p_1, p_b - p_2\}$.

Probability of Purchasing a Bundle: Using the same reasoning we state the probability of purchasing a bundle as,

$$\begin{aligned}
m_b &= P(R_b > p_b, R_b - p_b > R_1 - p_1, R_b - p_b > R_2 - p_2) \\
&= P(R_1 + R_2 > p_b, R_1 + R_2 - p_2 > R_1 - p_1, R_1 + R_2 - p_b > R_2 - p_2) \\
&= P(R_1 > p_b - p_2, R_2 > \max \{p_b - R_1, p_b - p_1\}) \\
&= \int_{p_b - p_2}^{\infty} \int_{a_4}^{\infty} f_{R_1, R_2}(r_1, r_2) dr_1 dr_2
\end{aligned}$$

where $a_4 = \max \{p_b - r_1, p_b - p_1\}$

2.2.2 Switching Probabilities

We now consider the situation when one or two product types are not available.

One Type of Product Incurs Shortage

Probability of Switching from Product 1 to Bundle or to Product 2

Suppose Product 1 incurs shortage and the customer has the option to switch to the Bundle, Product 2 or to leave without purchase. Let θ_{1B} , θ_{12} and $1 - \theta_{1B} - \theta_{12}$ be the probability of these events respectively. Since the switching events follow after the first choices of the customers are made, we calculate these probabilities conditional on the event that the first preference of the customer was to buy Product 1. Then we have

$$\begin{aligned}
\theta_{1B} &= \Pr \left\{ R_b - p_b \geq R_2 - p_2; R_b \geq p_b \left| \begin{array}{l} R_1 - p_1 \geq R_b - p_b; \\ R_1 - p_1 \geq R_2 - p_2; R_1 \geq p_1 \end{array} \right. \right\} \\
&= \Pr \left\{ \begin{array}{l} R_1 + R_2 - p_b \geq R_2 - p_2; R_1 + R_2 \geq p_b; \\ R_1 - p_1 \geq R_1 + R_2 - p_b; \\ R_1 - p_1 \geq R_2 - p_2; R_1 \geq p_1 \end{array} \right\} / m_1 \\
&= \Pr \left\{ \begin{array}{l} R_1 \geq p_b - p_2; R_2 \geq p_b - R_1; \\ p_b - p_1 \geq R_2; R_1 - p_1 + p_2 \geq R_2; R_1 \geq p_1 \end{array} \right\} / m_1 \\
&= \Pr \left\{ \begin{array}{l} R_1 \geq \max(p_b - p_2, p_1); \\ \min(p_b - p_1, R_1 - p_1 + p_2) \geq R_2 \geq p_b - R_1 \end{array} \right\} / m_1 \\
&= \Pr \left\{ \begin{array}{l} R_1 \geq \max(p_b - p_2, p_1); \\ p_b - p_1 \geq R_2 \geq p_b - R_1 \end{array} \right\} / m_1 \\
&= \int_{\max(p_b - p_2, p_1)}^{\infty} \int_{p_b - r_1}^{p_b - p_1} f_{R_1, R_2}(r_1, r_2) dr_1 dr_2 / m_1
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\theta_{12} &= \Pr \left\{ R_2 - p_2 \geq R_b - p_b; R_2 \geq p_2 \left| \begin{array}{l} R_1 - p_1 \geq R_b - p_b; \\ R_1 - p_1 \geq R_2 - p_2; R_1 \geq p_1 \end{array} \right. \right\} \\
&= \Pr \left\{ \begin{array}{l} R_2 - p_2 \geq R_1 + R_2 - p_b; R_2 \geq p_2; \\ R_1 - p_1 \geq R_1 + R_2 - p_b; R_1 - p_1 \geq R_2 - p_2; R_1 \geq p_1 \end{array} \right\} / m_1
\end{aligned}$$

$$\begin{aligned}
&= \Pr \left\{ \begin{array}{l} p_b - p_2 \geq R_1; R_2 \geq p_2; \\ p_b - p_1 \geq R_2; p_2 + R_1 - p_1 \geq R_2; R_1 \geq p_1 \end{array} \right\} / m_1 \\
&= \Pr \left\{ \begin{array}{l} p_b - p_2 \geq R_1 \geq p_1; \\ \min(p_b - p_1, p_2 + R_1 - p_1) \geq R_2 \geq p_2 \end{array} \right\} / m_1 \\
&= \Pr \left\{ \begin{array}{l} p_b - p_2 \geq R_1 \geq p_1; \\ p_2 + R_1 - p_1 \geq R_2 \geq p_2 \end{array} \right\} / m_1 \\
&= \int_{p_1}^{p_b - p_2} \int_{p_2}^{p_2 + r_1 - p_1} f_{R_1, R_2}(r_1, r_2) dr_1 dr_2 / m_1
\end{aligned}$$

Probability of Switching from Product 2 to Bundle or to Product 1

Similar to the previous case let θ_{2B} , θ_{21} be the probability of switching to a Bundle or to Product 1 when Product 2 stocks out. We have

$$\begin{aligned}
\theta_{2B} &= \Pr \left\{ R_b - p_b \geq R_1 - p_1; R_b \geq p_b \left| \begin{array}{l} R_2 - p_2 \geq R_b - p_b; \\ R_2 - p_2 \geq R_1 - p_1; R_2 \geq p_2 \end{array} \right. \right\} \\
&= \Pr \left\{ \begin{array}{l} R_1 + R_2 - p_b \geq R_1 - p_1; R_1 + R_2 \geq p_b; \\ R_2 - p_2 \geq R_1 + R_2 - p_b; \\ R_2 - p_2 \geq R_1 - p_1; R_2 \geq p_2 \end{array} \right\} / m_2 \\
&= \Pr \left\{ \begin{array}{l} R_2 \geq p_b - p_1; R_1 \geq p_b - R_2; \\ p_b - p_2 \geq R_1; R_2 - p_2 + p_1 \geq R_1; R_2 \geq p_2 \end{array} \right\} / m_2 \\
&= \Pr \left\{ \begin{array}{l} R_2 \geq \max(p_b - p_1, p_2); \\ \min(p_b - p_2, R_2 - p_2 + p_1) \geq R_1 \geq p_b - R_2 \end{array} \right\} / m_2 \\
&= \Pr \left\{ \begin{array}{l} R_2 \geq \max(p_b - p_1, p_2); \\ p_b - p_2 \geq R_1 \geq p_b - R_2 \end{array} \right\} / m_2 \\
&= \int_{\max(p_b - p_1, p_2)}^{\infty} \int_{p_b - r_2}^{p_b - p_2} f_{R_1, R_2}(r_1, r_2) dr_2 dr_1 / m_2
\end{aligned}$$

and

$$\begin{aligned}
\theta_{21} &= \Pr \left\{ R_1 - p_1 \geq R_b - p_b; R_1 \geq p_1 \left| \begin{array}{l} R_2 - p_2 \geq R_b - p_b; \\ R_2 - p_2 \geq R_1 - p_1; R_2 \geq p_2 \end{array} \right. \right\} \\
&= \Pr \left\{ \begin{array}{l} R_1 - p_1 \geq R_1 + R_2 - p_b; R_1 \geq p_1; \\ R_2 - p_2 \geq R_1 + R_2 - p_b; \\ R_2 - p_2 \geq R_1 - p_1; R_2 \geq p_2 \end{array} \right\} / m_2
\end{aligned}$$

$$\begin{aligned}
&= \Pr \left\{ \begin{array}{l} p_b - p_1 \geq R_2; R_1 \geq p_1; \\ p_b - p_2 \geq R_1; R_2 - p_2 + p_1 \geq R_1; R_2 \geq p_2 \end{array} \right\} / m_2 \\
&= \Pr \left\{ \begin{array}{l} p_b - p_1 \geq R_2 \geq p_2; \\ R_2 - p_2 + p_1 \geq R_1 \geq p_1 \end{array} \right\} / m_2 \\
&= \int_{p_2}^{p_b - p_1} \int_{p_1}^{r_2 - p_2 + p_1} f_{R_1, R_2}(r_1, r_2) dr_2 dr_1 / m_2
\end{aligned}$$

Probability of Switching from Bundle to Product 1 or to Product 2

Now let θ_{B1} and θ_{B2} be the probability of switching to Product 1 or to Product 2 when Bundle stocks out. We have

$$\begin{aligned}
\theta_{B1} &= \Pr \left\{ R_1 - p_1 \geq R_2 - p_2; R_1 \geq p_1 \left| \begin{array}{l} R_b - p_b \geq R_1 - p_1; \\ R_b - p_b \geq R_2 - p_2; R_b \geq p_b \end{array} \right. \right\} \\
&= \Pr \left\{ \begin{array}{l} R_1 - p_1 + p_2 \geq R_2; R_1 \geq p_1; \\ R_1 + R_2 - p_b \geq R_1 - p_1; \\ R_1 + R_2 - p_b \geq R_2 - p_2; R_1 + R_2 \geq p_b \end{array} \right\} / m_b \\
&= \Pr \left\{ \begin{array}{l} R_1 - p_1 + p_2 \geq R_2; R_1 \geq p_1; \\ R_2 \geq p_b - p_1; R_1 \geq p_b - p_2; R_2 \geq p_b - R_1 \end{array} \right\} / m_b \\
&= \Pr \left\{ \begin{array}{l} R_1 \geq \max(p_1, p_b - p_2) \\ R_1 - p_1 + p_2 \geq R_2 \geq \max(p_b - p_1, p_b - R_1) \end{array} \right\} / m_b \\
&= \Pr \left\{ \begin{array}{l} R_1 \geq \max(p_1, p_b - p_2) \\ R_1 - p_1 + p_2 \geq R_2 \geq p_b - p_1 \end{array} \right\} / m_b \\
&= \int_{\max(p_1, p_b - p_2)}^{\infty} \int_{p_b - p_1}^{r_1 - p_1 + p_2} f_{R_1, R_2}(r_1, r_2) dr_1 dr_2 / m_b
\end{aligned}$$

and

$$\begin{aligned}
\theta_{B2} &= \Pr \left\{ R_2 - p_2 \geq R_1 - p_1; R_2 \geq p_2 \left| \begin{array}{l} R_b - p_b \geq R_1 - p_1; \\ R_b - p_b \geq R_2 - p_2; R_b \geq p_b \end{array} \right. \right\} \\
&= \Pr \left\{ \begin{array}{l} R_2 - p_2 + p_1 \geq R_1; R_2 \geq p_2; \\ R_1 + R_2 - p_b \geq R_1 - p_1; \\ R_1 + R_2 - p_b \geq R_2 - p_2; R_1 + R_2 \geq p_b \end{array} \right\} / m_b \\
&= \Pr \left\{ \begin{array}{l} R_2 - p_2 + p_1 \geq R_1; R_2 \geq p_2; \\ R_2 \geq p_b - p_1; R_1 \geq p_b - p_2; R_1 \geq p_b - R_2 \end{array} \right\} / m_b
\end{aligned}$$

$$\begin{aligned}
&= \Pr \left\{ \begin{array}{l} R_2 \geq \max(p_2, p_b - p_1) \\ R_2 - p_2 + p_1 \geq R_1 \geq \max(p_b - p_2, p_b - R_2) \end{array} \right\} / m_b \\
&= \Pr \left\{ \begin{array}{l} R_2 \geq \max(p_2, p_b - p_1) \\ R_2 - p_2 + p_1 \geq R_1 \geq p_b - p_2 \end{array} \right\} / m_b \\
&= \int_{\max(p_2, p_b - p_1)}^{\infty} \int_{p_b - p_2}^{r_2 - p_2 + p_1} f_{R_1, R_2}(r_1, r_2) dr_2 dr_1 / m_b
\end{aligned}$$

Two Types of Products Incur Shortage

Switching probabilities explained in this section are used when dedicated customer cannot find his preferred product and only one type of the remaining products is available. Therefore, customer can decide only to switch to that available product or to leave the store without buying any product.

Product 1 and Product 2 incur shortage:

Let θ_{1B}^- , θ_{2B}^- be the probability of switching from Product 1 or Product 2 to Bundle when only Bundle is available. Then we have,

$$\begin{aligned}
\theta_{1B}^- &= \Pr \left\{ R_b \geq p_b \left| \begin{array}{l} R_1 - p_1 \geq R_b - p_b; \\ R_1 - p_1 \geq R_2 - p_2; R_1 \geq p_1 \end{array} \right. \right\} \\
&= \Pr \left\{ \begin{array}{l} R_1 + R_2 \geq p_b; R_1 - p_1 \geq R_1 + R_2 - p_b; \\ R_1 - p_1 \geq R_2 - p_2; R_1 \geq p_1 \end{array} \right\} / m_1 \\
&= \Pr \left\{ \begin{array}{l} R_2 \geq p_b - R_1; p_b - p_1 \geq R_2; \\ R_1 - p_1 + p_2 \geq R_2; R_1 \geq p_1 \end{array} \right\} / m_1 \\
&= \Pr \left\{ \begin{array}{l} R_1 \geq p_1; \\ \min(p_b - p_1, R_1 - p_1 + p_2) \geq R_2 \geq p_b - R_1 \end{array} \right\} / m_1 \\
&= \int_{p_1}^{\infty} \int_{p_b - r_1}^{\min(p_b - p_1, r_1 - p_1 + p_2)} f_{R_1, R_2}(r_1, r_2) dr_1 dr_2 / m_1
\end{aligned}$$

and

$$\begin{aligned}
\theta_{2B}^- &= \Pr \left\{ R_b \geq p_b \left| \begin{array}{l} R_2 - p_2 \geq R_b - p_b; \\ R_2 - p_2 \geq R_1 - p_1; R_2 \geq p_2 \end{array} \right. \right\} \\
&= \Pr \left\{ \begin{array}{l} R_1 + R_2 \geq p_b; R_2 - p_2 \geq R_1 + R_2 - p_b; \\ R_2 - p_2 \geq R_1 - p_1; R_2 \geq p_2 \end{array} \right\} / m_2
\end{aligned}$$

$$\begin{aligned}
&= \Pr \left\{ \begin{array}{l} R_1 \geq p_b - R_2; p_b - p_2 \geq R_1; \\ R_2 - p_2 + p_1 \geq R_1; R_2 \geq p_2 \end{array} \right\} / m_2 \\
&= \Pr \left\{ \begin{array}{l} R_2 \geq p_2 \\ \min(R_2 - p_2 + p_1, p_b - p_2) \geq R_1 \geq p_b - R_2 \end{array} \right\} / m_2 \\
&= \int_{p_2}^{\infty} \int_{p_b - r_2}^{\min(r_2 - p_2 + p_1, p_b - p_2)} f_{R_1, R_2}(r_1, r_2) dr_2 dr_1 / m_2
\end{aligned}$$

Bundle and Product 2 incur shortage

Let θ_{B1}^- , θ_{21}^- be the probability of switching from Bundle or Product 2 to Product 1 when only Product 1 is available. Then

$$\begin{aligned}
\theta_{B1}^- &= \Pr \left\{ R_1 \geq p_1 \left| \begin{array}{l} R_b - p_b \geq R_1 - p_1; \\ R_b - p_b \geq R_2 - p_2; R_b \geq p_b \end{array} \right. \right\} \\
&= \Pr \left\{ \begin{array}{l} R_1 \geq p_1; R_1 + R_2 - p_b \geq R_1 - p_1; \\ R_1 + R_2 - p_b \geq R_2 - p_2; R_1 + R_2 \geq p_b \end{array} \right\} / m_b \\
&= \Pr \left\{ \begin{array}{l} R_1 \geq p_1; R_2 \geq p_b - p_1; \\ R_1 \geq p_b - p_2; R_2 \geq p_b - R_1 \end{array} \right\} / m_b \\
&= \Pr \left\{ \begin{array}{l} R_1 \geq \max(p_1, p_b - p_2) \\ R_2 \geq \max(p_b - p_1, p_b - R_1) \end{array} \right\} / m_b \\
&= \Pr \left\{ \begin{array}{l} R_1 \geq \max(p_1, p_b - p_2) \\ R_2 \geq p_b - p_1 \end{array} \right\} / m_b \\
&= \int_{\max(p_1, p_b - p_2)}^{\infty} \int_{p_b - p_1}^{\infty} f_{R_1, R_2}(r_1, r_2) dr_1 dr_2 / m_b
\end{aligned}$$

and we have

$$\begin{aligned}
\theta_{21}^- &= \Pr \left\{ R_1 \geq p_1 \left| \begin{array}{l} R_2 - p_2 \geq R_b - p_b; \\ R_2 - p_2 \geq R_1 - p_1; R_2 \geq p_2 \end{array} \right. \right\} \\
&= \Pr \left\{ \begin{array}{l} R_1 \geq p_1; R_2 - p_2 \geq R_1 + R_2 - p_b; \\ R_2 - p_2 \geq R_1 - p_1; R_2 \geq p_2 \end{array} \right\} / m_2 \\
&= \Pr \left\{ \begin{array}{l} R_1 \geq p_1; p_b - p_2 \geq R_1; \\ R_2 \geq R_1 - p_1 + p_2; R_2 \geq p_2 \end{array} \right\} / m_2 \\
&= \Pr \left\{ \begin{array}{l} p_b - p_2 \geq R_1 \geq p_1; \\ R_2 \geq \max(R_1 - p_1 + p_2, p_2) \end{array} \right\} / m_2
\end{aligned}$$

$$\begin{aligned}
&= \Pr \left\{ \begin{array}{l} p_b - p_2 \geq R_1 \geq p_1; \\ R_2 \geq R_1 - p_1 + p_2 \end{array} \right\} / m_2 \\
&= \int_{p_1}^{p_b - p_2} \int_{r_1 - p_1 + p_2}^{\infty} f_{R_1, R_2}(r_1, r_2) dr_1 dr_2 / m_2
\end{aligned}$$

Bundle and Product 1 incur shortage

Similar to the previous case let θ_{B2}^- and θ_{12}^- be the probability of switching when only Product 2 is available. Then,

$$\begin{aligned}
\theta_{B2}^- &= \Pr \left\{ R_2 \geq p_2 \left| \begin{array}{l} R_b - p_b \geq R_1 - p_1; \\ R_b - p_b \geq R_2 - p_2; R_b \geq p_b \end{array} \right. \right\} \\
&= \Pr \left\{ \begin{array}{l} R_2 \geq p_2; R_1 + R_2 - p_b \geq R_1 - p_1; \\ R_1 + R_2 - p_b \geq R_2 - p_2; R_1 + R_2 \geq p_b \end{array} \right\} / m_b \\
&= \Pr \left\{ \begin{array}{l} R_2 \geq p_2; R_2 \geq p_b - p_1; \\ R_1 \geq p_b - p_2; R_1 \geq p_b - R_2 \end{array} \right\} / m_b \\
&= \Pr \left\{ \begin{array}{l} R_2 \geq \max(p_2, p_b - p_1); \\ R_1 \geq \max(p_b - p_2, p_b - R_2) \end{array} \right\} / m_b \\
&= \Pr \left\{ \begin{array}{l} R_2 \geq \max(p_2, p_b - p_1); \\ R_1 \geq p_b - p_2 \end{array} \right\} / m_b \\
&= \int_{p_b - p_2}^{\infty} \int_{\max(p_2, p_b - p_1)}^{\infty} f_{R_1, R_2}(r_1, r_2) dr_1 dr_2 / m_b
\end{aligned}$$

and

$$\begin{aligned}
\theta_{12}^- &= \Pr \left\{ R_2 \geq p_2 \left| \begin{array}{l} R_1 - p_1 \geq R_b - p_b; \\ R_1 - p_1 \geq R_2 - p_2; R_1 \geq p_1 \end{array} \right. \right\} \\
&= \Pr \left\{ \begin{array}{l} R_2 \geq p_2; R_1 - p_1 \geq R_1 + R_2 - p_b; \\ R_1 - p_1 \geq R_2 - p_2; R_1 \geq p_1 \end{array} \right\} / m_1 \\
&= \Pr \left\{ \begin{array}{l} R_2 \geq p_2; p_b - p_1 \geq R_2; \\ R_1 \geq R_2 - p_2 + p_1; R_1 \geq p_1 \end{array} \right\} / m_1 \\
&= \Pr \left\{ \begin{array}{l} p_b - p_1 \geq R_2 \geq p_2; \\ R_1 \geq \max(R_2 - p_2 + p_1, p_1) \end{array} \right\} / m_1 \\
&= \Pr \left\{ \begin{array}{l} p_b - p_1 \geq R_2 \geq p_2; \\ R_1 \geq R_2 - p_2 + p_1 \end{array} \right\} / m_1
\end{aligned}$$

$$= \int_{p_2}^{p_b - p_1} \int_{r_2 - p_2 + p_1}^{\infty} f_{R_1, R_2}(r_1, r_2) dr_2 dr_1 / m_1$$

After explaining the purchase and switching probability calculations, we explain the probability calculations for different sale realizations in the next section.

2.3 Sales Probabilities and the Objective Function

Recall that Q_1 and Q_2 are the initial inventory levels of Product 1 and Product 2, respectively. Let n_b be the number of bundles formed and n_i $i = 1, 2$ be the remaining units of Product i , with $n_1 = Q_1 - n_b$ and $n_2 = Q_2 - n_b$.

Also let X_1 , X_2 and X_b be the number of dedicated customers for Product 1, Product 2 and the Bundle respectively. We also have the random variables corresponding to the number of dedicated customers that switch from one product to another. These variables are denoted as X_{ij} , where i is the dedicated customer's product type (first preference) and j is the type of the product that the customer switches to (substitutes for i).

The realized values of these random variables will be denoted by x_1 , x_2 , x_b , x_{1b} , x_{12} , x_{2b} , x_{21} , x_{b1} and x_{b2} . Let

$$P \left(\begin{array}{c} x_1, x_2, x_b, x_{1b}, x_{12}, \\ x_{2b}, x_{21}, x_{b1}, x_{b2} \end{array} \right) = P \left(\begin{array}{c} X_1 = x_1, X_2 = x_2, X_b = x_b, \\ X_{1b} = x_{1b}, X_{12} = x_{12}, X_{2b} = x_{2b}, \\ X_{21} = x_{21}, X_{b1} = x_{b1}, X_{b2} = x_{b2} \end{array} \right)$$

denote the joint probability mass function of those random variables. Note that for certain realizations we only need joint marginal probability mass function of only a subset of these variables.

When all dedicated demand can be satisfied, due to the independency property of the Poisson processes, we have

$$P(X_1 = x_1, X_2 = x_2, X_b = x_b) = P(X_1 = x_1)P(X_2 = x_2)P(X_b = x_b)$$

| | |
|---------|---|
| Case 1: | Nothing incurs shortage |
| Case 2: | Bundle incurs shortage |
| | a) All excess demand of the Bundle is satisfied |
| | b) Product 1 incurs shortage with the excess bundle demand |
| | c) Product 2 incurs shortage with the excess bundle demand |
| | d) Both products incur shortage with the excess bundle demand |
| Case 3: | Product 1 incurs shortage |
| | a) All excess demand of Product 1 is satisfied |
| | b) Product 2 incurs shortage with the excess demand of Product 1 |
| | c) Bundle incurs shortage with the excess demand of Product 1 |
| | d) Product 2 and Bundle incur shortage with the excess demand of Product 1 |
| Case 4: | Product 2 incurs shortage |
| | a) All excess demand of Product 2 is satisfied |
| | b) Product 1 incurs shortage with the excess demand of Product 2 |
| | c) Bundle incurs shortage with the excess demand of Product 2 |
| | d) Product 1 and Bundle incur shortage with the excess demand of Product 2 |
| Case 5: | Product 1 and the Bundle incur shortage |
| | a) All excess demand of Product 1 and the Bundle are satisfied |
| | b) Product 2 incurs shortage with the excess demand of Product 1 and the Bundle |
| Case 6: | Product 2 and the Bundle incur shortage |
| | a) All excess demand of Product 2 and the Bundle are satisfied |
| | b) Product 1 incurs shortage with the excess demand of Product 2 and the Bundle |
| Case 7: | Product 1 and Product 2 incur shortage |
| | a) All excess demand of the products are satisfied from the Bundle |
| | b) Bundle incurs shortage with the excess demand of two products |
| Case 8: | All products incur shortage |

Table 2.1: Cases of realizations to derive the expected profit

where X_i has a Poisson distribution with rate $\ell_i = \lambda * m_i$, $i = 1, 2, b$.

For the derivation of the expected profit, π , there are eight possible realization cases when the initial choices are only considered. After this first classification, sub-cases are defined to include the switching customer realizations. All realization cases are listed in Table 2.1.

Now we will provide the expressions of these realization cases.

Case 1: Nothing incurs shortage ($x_1 \leq n_1, x_2 \leq n_2, x_b \leq n_b$)

Expected profit in the region where all customers are satisfied by their first choice products is given by π_1 which is

$$\begin{aligned}\pi_1 &= \sum_{x_1=0}^{n_1} \sum_{x_2=0}^{n_2} \sum_{x_b=0}^{n_b} (x_1 p_1 + x_2 p_2 + x_b p_b - n_b c) P(x_1, x_2, x_b) \\ &= \sum_{x_1=0}^{n_1} \sum_{x_2=0}^{n_2} \sum_{x_b=0}^{n_b} (x_1 p_1 + x_2 p_2 + x_b p_b - n_b c) \frac{e^{-\ell_1} \ell_1^{x_1}}{x_1!} \frac{e^{-\ell_2} \ell_2^{x_2}}{x_2!} \frac{e^{-\ell_b} \ell_b^{x_b}}{x_b!}\end{aligned}$$

Case 2: Bundle incurs shortage ($x_1 \leq n_1, x_2 \leq n_2, x_b > n_b$)

Initial demand for the bundle is more than the available stock but initial demands for Product 1 and Product 2 are satisfied from the stocks. Excess demand of the Bundle customers can result in four sub-cases.

(a) All excess demand of the Bundle is satisfied

Let $x_b - n_b$ be the number of excess bundle customers. In this case, $x_b - n_b$ units are satisfied from the excess inventories of Product 1 and Product 2. That is, we have $x_1 + x_{b1} \leq n_1$, $x_2 + x_{b2} \leq n_2$, $x_1 \leq n_1$, $x_2 \leq n_2$, $x_b > n_b$ and the contribution of this case to the total expected profit is given by:

$$\begin{aligned}\pi_{2a} &= \sum_{x_1=0}^{n_1} \sum_{x_2=0}^{n_2} \sum_{x_b=n_b}^{\infty} \sum_{x_{b1}=0}^{x_b-n_b} \sum_{x_{b2}=0}^{x_b-n_b-x_{b1}} ((x_1 + x_{b1})p_1 + (x_2 + x_{b2})p_2 + n_b p_b - n_b c) \\ &\quad P(x_1, x_2, x_b, x_{b1}, x_{b2}) \\ &= \sum_{x_1=0}^{n_1} \sum_{x_2=0}^{n_2} \sum_{x_b=n_b}^{\infty} \sum_{x_{b1}=0}^{x_b-n_b} \sum_{x_{b2}=0}^{x_b-n_b-x_{b1}} ((x_1 + x_{b1})p_1 + (x_2 + x_{b2})p_2 + n_b p_b - n_b c) \\ &\quad \frac{e^{-\ell_1} \ell_1^{x_1}}{x_1!} \frac{e^{-\ell_2} \ell_2^{x_2}}{x_2!} \frac{e^{-\ell_b} \ell_b^{x_b}}{x_b!} \frac{(x_b - n_b)!}{x_{b1}! x_{b2}! x_{b0}!} (\theta_{B1}^-)^{x_{b1}} (\theta_{B2}^-)^{x_{b2}} (\theta_{B0}^-)^{x_{b0}}\end{aligned}$$

where $x_{b0} = x_b - n_b - x_{b1} - x_{b2}$ and $\theta_{B0}^- = 1 - \theta_{B1}^- - \theta_{B2}^-$

(b) *Product 1 incurs shortage with the excess bundle demand*

In this case, original demand for Product 1 is satisfied but the left over is not sufficient to satisfy the overflow from the Bundle customers. All demands for Product 2 are satisfied from the stock. For this case we have $x_1 + x_{b1} > n_1$, $x_2 + x_{b2} \leq n_2$, $x_1 \leq n_1$, $x_2 \leq n_2$, $x_b > n_b$ and the contribution is given by:

$$\begin{aligned} \pi_{2b} &= \sum_{x_1=0}^{n_1} \sum_{x_2=0}^{n_2} \sum_{x_b=n_b}^{\infty} \sum_{x_{b1}=n_1-x_1}^{x_b-n_b} \sum_{x_{b2}=0}^{x_b-n_b-x_{b1}} (n_1 p_1 + (x_2 + x_{b2}) p_2 + n_b p_b - n_b c) \\ &\quad P(x_1, x_2, x_b, x_{b1}, x_{b2}) \\ &= \sum_{x_1=0}^{n_1} \sum_{x_2=0}^{n_2} \sum_{x_b=n_b}^{\infty} \sum_{x_{b1}=n_1-x_1}^{x_b-n_b} \sum_{x_{b2}=0}^{x_b-n_b-x_{b1}} (n_1 p_1 + (x_2 + x_{b2}) p_2 + n_b p_b - n_b c) \\ &\quad \frac{e^{-\ell_1} \ell_1^{x_1}}{x_1!} \frac{e^{-\ell_2} \ell_2^{x_2}}{x_2!} \frac{e^{-\ell_b} \ell_b^{x_b}}{x_b!} \frac{(x_b - n_b)!}{x_{b1}! x_{b2}! x_{b0}!} (\theta_{B1}^-)^{x_{b1}} (\theta_{B2}^-)^{x_{b2}} (\theta_{B0}^-)^{x_{b0}} \end{aligned}$$

(c) *Product 2 incurs shortage with the excess bundle demand*

This case is similar to the above case, except that Product 2 incurs shortage. Hence we have $x_1 + x_{b1} \leq n_1$, $x_2 + x_{b2} > n_2$, $x_1 \leq n_1$, $x_2 \leq n_2$, $x_b > n_b$ and the contribution is given by:

$$\begin{aligned} \pi_{2c} &= \sum_{x_1=0}^{n_1} \sum_{x_2=0}^{n_2} \sum_{x_b=n_b}^{\infty} \sum_{x_{b1}=0}^{x_b-n_b} \sum_{x_{b2}=n_2-x_2}^{x_b-n_b-x_{b1}} ((x_1 + x_{b1}) p_1 + n_2 p_2 + n_b p_b - n_b c) \\ &\quad P(x_1, x_2, x_b, x_{b1}, x_{b2}) \\ &= \sum_{x_1=0}^{n_1} \sum_{x_2=0}^{n_2} \sum_{x_b=n_b}^{\infty} \sum_{x_{b1}=0}^{x_b-n_b} \sum_{x_{b2}=n_2-x_2}^{x_b-n_b-x_{b1}} ((x_1 + x_{b1}) p_1 + n_2 p_2 + n_b p_b - n_b c) \\ &\quad \frac{e^{-\ell_1} \ell_1^{x_1}}{x_1!} \frac{e^{-\ell_2} \ell_2^{x_2}}{x_2!} \frac{e^{-\ell_b} \ell_b^{x_b}}{x_b!} \frac{(x_b - n_b)!}{x_{b1}! x_{b2}! x_{b0}!} (\theta_{B1}^-)^{x_{b1}} (\theta_{B2}^-)^{x_{b2}} (\theta_{B0}^-)^{x_{b0}} \end{aligned}$$

(d) *Both products incur shortage with the excess bundle demand*

In this case the excess inventories of Product 1 and Product 2 are not sufficient to satisfy the overflow demand from the bundle customers. That is, we have $x_1 + x_{b1} > n_1$, $x_2 + x_{b2} > n_2$, $x_1 \leq n_1$, $x_2 \leq n_2$, $x_b > n_b$ and

$$\pi_{2d} = \sum_{x_1=0}^{n_1} \sum_{x_2=0}^{n_2} \sum_{x_b=n_b}^{\infty} \sum_{x_{b1}=n_1-x_1}^{x_b-n_b} \sum_{x_{b2}=n_2-x_2}^{x_b-n_b-x_{b1}} (n_1 p_1 + n_2 p_2 + n_b p_b - n_b c)$$

$$\begin{aligned}
& P(x_1, x_2, x_b, x_{b1}, x_{b2}) \\
= & \sum_{x_1=0}^{n_1} \sum_{x_2=0}^{n_2} \sum_{x_b=n_b}^{\infty} \sum_{x_{b1}=n_1-x_1}^{x_b-n_b} \sum_{x_{b2}=n_2-x_2}^{x_b-n_b-x_{b1}} (n_1 p_1 + n_2 p_2 + n_b p_b - n_b c) \\
& \frac{e^{-\ell_1} \ell_1^{x_1}}{x_1!} \frac{e^{-\ell_2} \ell_2^{x_2}}{x_2!} \frac{e^{-\ell_b} \ell_b^{x_b}}{x_b!} \frac{(x_b - n_b)!}{x_{b1}! x_{b2}! x_{b0}!} (\theta_{B1}^-)^{x_{b1}} (\theta_{B2}^-)^{x_{b2}} (\theta_{B0}^-)^{x_{b0}}
\end{aligned}$$

Expected profit for the Case 2 is calculated as $\pi_2 = \pi_{2a} + \pi_{2b} + \pi_{2c} + \pi_{2d}$

Case 3: Product 1 incurs shortage ($x_1 > n_1, x_2 \leq n_2, x_b \leq n_b$)

Initial demand for Product 1 is more than the available stock but initial demands for Product 2 and the Bundle are satisfied from the stocks. Excess demand of Product 1 customers can result in four sub-cases.

(a) All excess demand of Product 1 is satisfied

Demands for Product 2 and Bundle are satisfied including the switching customers from Product 1. We have $x_2 + x_{12} \leq n_2, x_b + x_{1b} \leq n_b, x_1 > n_1, x_2 \leq n_2, x_b \leq n_b$ and

$$\begin{aligned}
\pi_{3a} &= \sum_{x_1=n_1}^{\infty} \sum_{x_2=0}^{n_2} \sum_{x_b=0}^{n_b} \sum_{x_{12}=0}^{x_1-n_1} \sum_{x_{1b}=0}^{x_1-n_1-x_{12}} (n_1 p_1 + (x_2 + x_{12}) p_2 + (x_b + x_{1b}) p_b - n_b c) \\
& P(x_1, x_2, x_b, x_{12}, x_{1b}) \\
&= \sum_{x_1=n_1}^{\infty} \sum_{x_2=0}^{n_2} \sum_{x_b=0}^{n_b} \sum_{x_{12}=0}^{x_1-n_1} \sum_{x_{1b}=0}^{x_1-n_1-x_{12}} (n_1 p_1 + (x_2 + x_{12}) p_2 + (x_b + x_{1b}) p_b - n_b c) \\
& \frac{e^{-\ell_1} \ell_1^{x_1}}{x_1!} \frac{e^{-\ell_2} \ell_2^{x_2}}{x_2!} \frac{e^{-\ell_b} \ell_b^{x_b}}{x_b!} \frac{(x_1 - n_1)!}{x_{12}! x_{1b}! x_{10}!} (\theta_{12}^-)^{x_{12}} (\theta_{1B}^-)^{x_{1b}} (\theta_{10}^-)^{x_{10}}
\end{aligned}$$

where $x_{10} = x_1 - n_1 - x_{12} - x_{1b}$ and $\theta_{10}^- = 1 - \theta_{12}^- - \theta_{1B}^-$

(b) Product 2 incurs shortage with the excess demand of Product 1

In this case, we have $x_2 + x_{12} > n_2, x_b + x_{1b} \leq n_b, x_1 > n_1, x_2 \leq n_2, x_b \leq n_b$ and

$$\pi_{3b} = \sum_{x_1=n_1}^{\infty} \sum_{x_2=0}^{n_2} \sum_{x_b=0}^{n_b} \sum_{x_{12}=n_2-x_2}^{x_1-n_1} \sum_{x_{1b}=0}^{x_1-n_1-x_{12}} (n_1 p_1 + n_2 p_2 + (x_b + x_{1b}) p_b - n_b c)$$

$$\begin{aligned}
& P(x_1, x_2, x_b, x_{12}, x_{1b}) \\
= & \sum_{x_1=n_1}^{\infty} \sum_{x_2=0}^{n_2} \sum_{x_b=0}^{n_b} \sum_{x_{12}=n_2-x_2}^{x_1-n_1} \sum_{x_{1b}=0}^{x_1-n_1-x_{12}} (n_1p_1 + n_2p_2 + (x_b + x_{1b})p_b - n_b c) \\
& \frac{e^{-\ell_1} \ell_1^{x_1}}{x_1!} \frac{e^{-\ell_2} \ell_2^{x_2}}{x_2!} \frac{e^{-\ell_b} \ell_b^{x_b}}{x_b!} \frac{(x_1 - n_1)!}{x_{12}! x_{1b}! x_{10}!} (\theta_{12}^-)^{x_{12}} (\theta_{1B}^-)^{x_{1b}} (\theta_{10}^-)^{x_{10}}
\end{aligned}$$

(c) *Bundle incurs shortage with the excess demand of Product 1*

Hence the overflow Product 1 customers to Bundle are not satisfied with the excess stock of the bundle, but all demands for Product 2 are satisfied from the stock. Then, $x_2 + x_{12} \leq n_2$, $x_b + x_{1b} > n_b$, $x_1 > n_1$, $x_2 \leq n_2$, $x_b \leq n_b$ and

$$\begin{aligned}
\pi_{3c} &= \sum_{x_1=n_1}^{\infty} \sum_{x_2=0}^{n_2} \sum_{x_b=0}^{n_b} \sum_{x_{12}=0}^{x_1-n_1} \sum_{x_{1b}=n_b-x_b}^{x_1-n_1-x_{12}} (n_1p_1 + (x_2 + x_{12})p_2 + n_b p_b - n_b c) \\
& P(x_1, x_2, x_b, x_{12}, x_{1b}) \\
&= \sum_{x_1=n_1}^{\infty} \sum_{x_2=0}^{n_2} \sum_{x_b=0}^{n_b} \sum_{x_{12}=0}^{x_1-n_1} \sum_{x_{1b}=n_b-x_b}^{x_1-n_1-x_{12}} (n_1p_1 + (x_2 + x_{12})p_2 + n_b p_b - n_b c) \\
& \frac{e^{-\ell_1} \ell_1^{x_1}}{x_1!} \frac{e^{-\ell_2} \ell_2^{x_2}}{x_2!} \frac{e^{-\ell_b} \ell_b^{x_b}}{x_b!} \frac{(x_1 - n_1)!}{x_{12}! x_{1b}! x_{10}!} (\theta_{12}^-)^{x_{12}} (\theta_{1B}^-)^{x_{1b}} (\theta_{10}^-)^{x_{10}}
\end{aligned}$$

(d) *Product 2 and Bundle incur shortage with the excess demand of Product 1*

In this case the overflows to Product 2 and the Bundle from Product 1 are not satisfied by the excess inventories. That is, $x_2 + x_{12} > n_2$, $x_b + x_{1b} > n_b$, $x_1 > n_1$, $x_2 \leq n_2$, $x_b \leq n_b$ and

$$\begin{aligned}
\pi_{3d} &= \sum_{x_1=n_1}^{\infty} \sum_{x_2=0}^{n_2} \sum_{x_b=0}^{n_b} \sum_{x_{12}=n_2-x_2}^{x_1-n_1} \sum_{x_{1b}=n_b-x_b}^{x_1-n_1-x_{12}} (n_1p_1 + n_2p_2 + n_b p_b - n_b c) \\
& P(x_1, x_2, x_b, x_{12}, x_{1b}) \\
&= \sum_{x_1=n_1}^{\infty} \sum_{x_2=0}^{n_2} \sum_{x_b=0}^{n_b} \sum_{x_{12}=n_2-x_2}^{x_1-n_1} \sum_{x_{1b}=n_b-x_b}^{x_1-n_1-x_{12}} (n_1p_1 + n_2p_2 + n_b p_b - n_b c) \\
& \frac{e^{-\ell_1} \ell_1^{x_1}}{x_1!} \frac{e^{-\ell_2} \ell_2^{x_2}}{x_2!} \frac{e^{-\ell_b} \ell_b^{x_b}}{x_b!} \frac{(x_1 - n_1)!}{x_{12}! x_{1b}! x_{10}!} (\theta_{12}^-)^{x_{12}} (\theta_{1B}^-)^{x_{1b}} (\theta_{10}^-)^{x_{10}}
\end{aligned}$$

Expected profit for the Case 3 is calculated as $\pi_3 = \pi_{3a} + \pi_{3b} + \pi_{3c} + \pi_{3d}$

Case 4: Product 2 incurs shortage ($x_1 \leq n_1, x_2 > n_2, x_b \leq n_b$)

This case is similar to case 3, so we directly write the expressions of sub-cases.

(a) All excess demand of Product 2 is satisfied

We have $x_1 + x_{21} \leq n_1, x_b + x_{2b} \leq n_b, x_1 \leq n_1, x_2 > n_2, x_b \leq n_b$ and

$$\begin{aligned} \pi_{4a} &= \sum_{x_1=0}^{n_1} \sum_{x_2=n_2}^{\infty} \sum_{x_b=0}^{n_b} \sum_{x_{21}=0}^{x_2-n_2} \sum_{x_{2b}=0}^{x_2-n_2-x_{21}} ((x_1 + x_{21})p_1 + n_2p_2 + (x_b + x_{2b})p_b - n_b c) \\ &\quad P(x_1, x_2, x_b, x_{21}, x_{2b}) \\ &= \sum_{x_1=0}^{n_1} \sum_{x_2=n_2}^{\infty} \sum_{x_b=0}^{n_b} \sum_{x_{21}=0}^{x_2-n_2} \sum_{x_{2b}=0}^{x_2-n_2-x_{21}} ((x_1 + x_{21})p_1 + n_2p_2 + (x_b + x_{2b})p_b - n_b c) \\ &\quad \frac{e^{-\ell_1} \ell_1^{x_1}}{x_1!} \frac{e^{-\ell_2} \ell_2^{x_2}}{x_2!} \frac{e^{-\ell_b} \ell_b^{x_b}}{x_b!} \frac{(x_2 - n_2)!}{x_{21}! x_{2b}! x_{20}!} (\theta_{21}^-)^{x_{21}} (\theta_{2B}^-)^{x_{2b}} (\theta_{20}^-)^{x_{20}} \end{aligned}$$

where $x_{20} = x_2 - n_2 - x_{21} - x_{2b}$ and $\theta_{20}^- = 1 - \theta_{21}^- - \theta_{2B}^-$

(b) Product 1 incurs shortage with the excess demand of Product 2

$x_1 + x_{21} > n_1, x_b + x_{2b} \leq n_b, x_1 \leq n_1, x_2 > n_2, x_b \leq n_b$

$$\begin{aligned} \pi_{4b} &= \sum_{x_1=0}^{n_1} \sum_{x_2=n_2}^{\infty} \sum_{x_b=0}^{n_b} \sum_{x_{21}=n_1-x_1}^{x_2-n_2} \sum_{x_{2b}=0}^{x_2-n_2-x_{21}} (n_1p_1 + n_2p_2 + (x_b + x_{2b})p_b - n_b c) \\ &\quad P(x_1, x_2, x_b, x_{21}, x_{2b}) \\ &= \sum_{x_1=0}^{n_1} \sum_{x_2=n_2}^{\infty} \sum_{x_b=0}^{n_b} \sum_{x_{21}=n_1-x_1}^{x_2-n_2} \sum_{x_{2b}=0}^{x_2-n_2-x_{21}} (n_1p_1 + n_2p_2 + (x_b + x_{2b})p_b - n_b c) \\ &\quad \frac{e^{-\ell_1} \ell_1^{x_1}}{x_1!} \frac{e^{-\ell_2} \ell_2^{x_2}}{x_2!} \frac{e^{-\ell_b} \ell_b^{x_b}}{x_b!} \frac{(x_2 - n_2)!}{x_{21}! x_{2b}! x_{20}!} (\theta_{21}^-)^{x_{21}} (\theta_{2B}^-)^{x_{2b}} (\theta_{20}^-)^{x_{20}} \end{aligned}$$

(c) Bundle incurs shortage with the excess demand of Product 2

$x_1 + x_{21} \leq n_1, x_b + x_{2b} > n_b, x_1 \leq n_1, x_2 > n_2, x_b \leq n_b$

$$\begin{aligned} \pi_{4c} &= \sum_{x_1=0}^{n_1} \sum_{x_2=n_2}^{\infty} \sum_{x_b=0}^{n_b} \sum_{x_{21}=0}^{x_2-n_2} \sum_{x_{2b}=n_b-x_b}^{x_2-n_2-x_{21}} ((x_1 + x_{21})p_1 + n_2p_2 + n_b p_b - n_b c) \\ &\quad P(x_1, x_2, x_b, x_{21}, x_{2b}) \end{aligned}$$

$$\begin{aligned}
&= \sum_{x_1=0}^{n_1} \sum_{x_2=n_2}^{\infty} \sum_{x_b=0}^{n_b} \sum_{x_{21}=0}^{x_2-n_2} \sum_{x_{2b}=n_b-x_b}^{x_2-n_2-x_{21}} ((x_1+x_{21})p_1 + n_2p_2 + n_bp_b - n_b c) \\
&\quad \frac{e^{-\ell_1} \ell_1^{x_1}}{x_1!} \frac{e^{-\ell_2} \ell_2^{x_2}}{x_2!} \frac{e^{-\ell_b} \ell_b^{x_b}}{x_b!} \frac{(x_2-n_2)!}{x_{21}!x_{2b}!x_{20}!} (\theta_{21}^-)^{x_{21}} (\theta_{2B}^-)^{x_{2b}} (\theta_{20}^-)^{x_{20}}
\end{aligned}$$

(d) *Product 1 and Bundle incur shortage with the excess demand of Product 2*

$$x_1 + x_{21} > n_1, x_b + x_{2b} > n_b, x_1 \leq n_1, x_2 > n_2, x_b \leq n_b$$

$$\begin{aligned}
\pi_{4d} &= \sum_{x_1=0}^{n_1} \sum_{x_2=n_2}^{\infty} \sum_{x_b=0}^{n_b} \sum_{x_{21}=n_1-x_1}^{x_2-n_2} \sum_{x_{2b}=n_b-x_b}^{x_2-n_2-x_{21}} (n_1p_1 + n_2p_2 + n_bp_b - n_b c) \\
&\quad P(x_1, x_2, x_b, x_{21}, x_{2b}) \\
&= \sum_{x_1=0}^{n_1} \sum_{x_2=n_2}^{\infty} \sum_{x_b=0}^{n_b} \sum_{x_{21}=n_1-x_1}^{x_2-n_2} \sum_{x_{2b}=n_b-x_b}^{x_2-n_2-x_{21}} (n_1p_1 + n_2p_2 + n_bp_b - n_b c) \\
&\quad \frac{e^{-\ell_1} \ell_1^{x_1}}{x_1!} \frac{e^{-\ell_2} \ell_2^{x_2}}{x_2!} \frac{e^{-\ell_b} \ell_b^{x_b}}{x_b!} \frac{(x_2-n_2)!}{x_{21}!x_{2b}!x_{20}!} (\theta_{21}^-)^{x_{21}} (\theta_{2B}^-)^{x_{2b}} (\theta_{20}^-)^{x_{20}}
\end{aligned}$$

Expected profit for the Case 4 is calculated as $\pi_4 = \pi_{4a} + \pi_{4b} + \pi_{4c} + \pi_{4d}$

Case 5: Product 1 and the Bundle incur shortage ($x_1 > n_1, x_2 \leq n_2, x_b > n_b$)

Initial demands for Product 1 and the Bundle are greater than the respective stock amounts and only initial demand for Product 2 is satisfied from the stock. Excess demands of Product 1 and the Bundle result in two sub-cases.

(a) *All excess demand of Product 1 and the Bundle are satisfied*

Demand for Product 2 is satisfied including the switching customers from Product 1 and the Bundle. That is, we have $x_2 + x_{12} + x_{b2} \leq n_2, x_1 > n_1, x_2 \leq n_2, x_b > n_b$ and

$$\begin{aligned}
\pi_{5a} &= \sum_{x_1=n_1}^{\infty} \sum_{x_2=0}^{n_2} \sum_{x_b=n_b}^{\infty} \sum_{x_{12}=0}^{x_1-n_1} \sum_{x_{b2}=0}^{x_b-n_b} (n_1p_1 + (x_2 + x_{12} + x_{b2})p_2 + n_bp_b - n_b c) \\
&\quad P(x_1, x_2, x_b, x_{12}, x_{b2}) \\
&= \sum_{x_1=n_1}^{\infty} \sum_{x_2=0}^{n_2} \sum_{x_b=n_b}^{\infty} \sum_{x_{12}=0}^{x_1-n_1} \sum_{x_{b2}=0}^{x_b-n_b} (n_1p_1 + (x_2 + x_{12} + x_{b2})p_2 + n_bp_b - n_b c)
\end{aligned}$$

$$\frac{e^{-\ell_1} \ell_1^{x_1}}{x_1!} \frac{e^{-\ell_2} \ell_2^{x_2}}{x_2!} \frac{e^{-\ell_b} \ell_b^{x_b}}{x_b!} \binom{x_1 - n_1 + x_{12} - 1}{x_1 - n_1 - 1} (\theta_{12})^{x_1 - n_1} (1 - \theta_{12})^{x_{12}}$$

$$\binom{x_b - n_b + x_{b2} - 1}{x_b - n_b - 1} (\theta_{B2})^{x_b - n_b} (1 - \theta_{B2})^{x_{b2}}$$

(b) *Product 2 incurs shortage with the excess demand of Product 1 and the Bundle*

Initial Product 2 demand is satisfied but excess demand from Product 1 and Bundle cannot be satisfied with the Product 2 stock. We have $x_2 + x_{12} + x_{b2} > n_2$, $x_1 > n_1$, $x_2 \leq n_2$, $x_b > n_b$ and

$$\begin{aligned} \pi_{5b} &= \sum_{x_1=n_1}^{\infty} \sum_{x_2=0}^{n_2} \sum_{x_b=n_b}^{\infty} \sum_{x_{12}, x_{b2}: x_{12}+x_{b2} \geq n_2-x_2} (n_1 p_1 + n_2 p_2 + n_b p_b - n_b c) \\ &\quad P(x_1, x_2, x_b, x_{12}, x_{b2}) \\ &= \sum_{x_1=n_1}^{\infty} \sum_{x_2=0}^{n_2} \sum_{x_b=n_b}^{\infty} \sum_{x_{12}, x_{b2}: x_{12}+x_{b2} \geq n_2-x_2} (n_1 p_1 + n_2 p_2 + n_b p_b - n_b c) \\ &\quad \frac{e^{-\ell_1} \ell_1^{x_1}}{x_1!} \frac{e^{-\ell_2} \ell_2^{x_2}}{x_2!} \frac{e^{-\ell_b} \ell_b^{x_b}}{x_b!} \binom{x_1 - n_1 + x_{12} - 1}{x_1 - n_1 - 1} (\theta_{12})^{x_1 - n_1} (1 - \theta_{12})^{x_{12}} \\ &\quad \binom{x_b - n_b + x_{b2} - 1}{x_b - n_b - 1} (\theta_{B2})^{x_b - n_b} (1 - \theta_{B2})^{x_{b2}} \end{aligned}$$

Expected profit for the Case 5 is calculated as $\pi_5 = \pi_{5a} + \pi_{5b}$

Case 6: Product 2 and the Bundle incur shortage ($x_1 \leq n_1$, $x_2 > n_2$, $x_b > n_b$)

Initial demand for Product 2 and Bundle are greater than their respective stock amounts and only initial Product 1 demand is satisfied from the stock. We have again two sub-cases:

(a) All excess demand of Product 2 and the Bundle are satisfied

We have $x_1 + x_{21} + x_{b1} \leq n_1$, $x_1 \leq n_1$, $x_2 > n_2$, $x_b > n_b$ and

$$\begin{aligned}
\pi_{6a} &= \sum_{x_1=0}^{n_1} \sum_{x_2=n_2}^{\infty} \sum_{x_b=n_b}^{\infty} \sum_{x_{21}=0}^{x_2-n_2} \sum_{x_{b1}=0}^{x_b-n_b} ((x_1 + x_{21} + x_{b1})p_1 + n_2p_2 + n_b p_b - n_b c) \\
&\quad P(x_1, x_2, x_b, x_{21}x_{b1}) \\
&= \sum_{x_1=0}^{n_1} \sum_{x_2=n_2}^{\infty} \sum_{x_b=n_b}^{\infty} \sum_{x_{21}=0}^{x_2-n_2} \sum_{x_{b1}=0}^{x_b-n_b} ((x_1 + x_{21} + x_{b1})p_1 + n_2p_2 + n_b p_b - n_b c) \\
&\quad \frac{e^{-\ell_1} \ell_1^{x_1}}{x_1!} \frac{e^{-\ell_2} \ell_2^{x_2}}{x_2!} \frac{e^{-\ell_b} \ell_b^{x_b}}{x_b!} \binom{x_2 - n_2 + x_{21} - 1}{x_2 - n_2 - 1} (\theta_{21})^{x_2 - n_2} (1 - \theta_{21})^{x_{21}} \\
&\quad \binom{x_b - n_b + x_{b1} - 1}{x_b - n_b - 1} (\theta_{B1})^{x_b - n_b} (1 - \theta_{B1})^{x_{b1}}
\end{aligned}$$

(b) Product 1 incurs shortage with the excess demand of Product 2 and the Bundle

In this case we have $x_1 + x_{21} + x_{b1} > n_1$, $x_1 \leq n_1$, $x_2 > n_2$, $x_b > n_b$ and

$$\begin{aligned}
\pi_{6b} &= \sum_{x_1=0}^{n_1} \sum_{x_2=n_2}^{\infty} \sum_{x_b=n_b}^{\infty} \sum_{x_{21}, x_{b1}: x_{21}+x_{b1} \geq n_1-x_1} (n_1 p_1 + n_2 p_2 + n_b p_b - n_b c) \\
&\quad P(x_1, x_2, x_b, x_{21}x_{b1}) \\
&= \sum_{x_1=0}^{n_1} \sum_{x_2=n_2}^{\infty} \sum_{x_b=n_b}^{\infty} \sum_{x_{21}, x_{b1}: x_{21}+x_{b1} \geq n_1-x_1} (n_1 p_1 + n_2 p_2 + n_b p_b - n_b c) \\
&\quad \frac{e^{-\ell_1} \ell_1^{x_1}}{x_1!} \frac{e^{-\ell_2} \ell_2^{x_2}}{x_2!} \frac{e^{-\ell_b} \ell_b^{x_b}}{x_b!} \binom{x_2 - n_2 + x_{21} - 1}{x_2 - n_2 - 1} (\theta_{21})^{x_2 - n_2} (1 - \theta_{21})^{x_{21}} \\
&\quad \binom{x_b - n_b + x_{b1} - 1}{x_b - n_b - 1} (\theta_{B1})^{x_b - n_b} (1 - \theta_{B1})^{x_{b1}}
\end{aligned}$$

Expected profit for the Case 6 is calculated as $\pi_6 = \pi_{6a} + \pi_{6b}$

Case 7: Product 1 and Product 2 incur shortage ($x_1 > n_1$, $x_2 > n_2$, $x_b \leq n_b$)

Similar to previous cases we have two sub-cases.

(a) *All excess demand of the products are satisfied from the Bundle*

That is, $x_b + x_{1b} + x_{2b} \leq n_b$, $x_1 > n_1$, $x_2 > n_2$, $x_b \leq n_b$ and

$$\begin{aligned}
\pi_{7a} &= \sum_{x_1=n_1}^{\infty} \sum_{x_2=n_2}^{\infty} \sum_{x_b=0}^{n_b} \sum_{x_{1b}=0}^{x_1-n_1} \sum_{x_{2b}=0}^{x_2-n_2} (n_1 p_1 + n_2 p_2 + (x_b + x_{1b} + x_{2b}) p_b - n_b c) \\
&\quad P(x_1, x_2, x_b, x_{1b}, x_{2b}) \\
&= \sum_{x_1=n_1}^{\infty} \sum_{x_2=n_2}^{\infty} \sum_{x_b=0}^{n_b} \sum_{x_{1b}=0}^{x_1-n_1} \sum_{x_{2b}=0}^{x_2-n_2} (n_1 p_1 + n_2 p_2 + (x_b + x_{1b} + x_{2b}) p_b - n_b c) \\
&\quad \frac{e^{-\ell_1} \ell_1^{x_1}}{x_1!} \frac{e^{-\ell_2} \ell_2^{x_2}}{x_2!} \frac{e^{-\ell_b} \ell_b^{x_b}}{x_b!} \binom{x_1 - n_1 + x_{1b} - 1}{x_1 - n_1 - 1} (\theta_{1B})^{x_1 - n_1} (1 - \theta_{1B})^{x_{1b}} \\
&\quad \binom{x_2 - n_2 + x_{2b} - 1}{x_2 - n_2 - 1} (\theta_{2B})^{x_2 - n_2} (1 - \theta_{2B})^{x_{2b}}
\end{aligned}$$

(b) *Bundle incurs shortage with the excess demand of two products*

$x_b + x_{1b} + x_{2b} > n_b$, $x_1 > n_1$, $x_2 > n_2$, $x_b \leq n_b$

$$\begin{aligned}
\pi_{7b} &= \sum_{x_1=n_1}^{\infty} \sum_{x_2=n_2}^{\infty} \sum_{x_b=0}^{n_b} \sum_{x_{1b}, x_{2b}: x_{1b}+x_{2b} \geq n_b - x_b} (n_1 p_1 + n_2 p_2 + n_b p_b - n_b c) \\
&\quad P(x_1, x_2, x_b, x_{1b}, x_{2b}) \\
&= \sum_{x_1=n_1}^{\infty} \sum_{x_2=n_2}^{\infty} \sum_{x_b=0}^{n_b} \sum_{x_{1b}, x_{2b}: x_{1b}+x_{2b} \geq n_b - x_b} (n_1 p_1 + n_2 p_2 + n_b p_b - n_b c) \\
&\quad \frac{e^{-\ell_1} \ell_1^{x_1}}{x_1!} \frac{e^{-\ell_2} \ell_2^{x_2}}{x_2!} \frac{e^{-\ell_b} \ell_b^{x_b}}{x_b!} \binom{x_1 - n_1 + x_{1b} - 1}{x_1 - n_1 - 1} (\theta_{1B})^{x_1 - n_1} (1 - \theta_{1B})^{x_{1b}} \\
&\quad \binom{x_2 - n_2 + x_{2b} - 1}{x_2 - n_2 - 1} (\theta_{2B})^{x_2 - n_2} (1 - \theta_{2B})^{x_{2b}}
\end{aligned}$$

Expected profit for the Case 7 is calculated as $\pi_7 = \pi_{7a} + \pi_{7b}$

Case 8: All products incur shortage

As the last case, we consider the case where all products incur shortage with the initial dedicated customer demand. That is $x_1 > n_1$, $x_2 > n_2$, $x_b > n_b$ and

$$\pi_8 = \sum_{x_1=n_1}^{\infty} \sum_{x_2=n_2}^{\infty} \sum_{x_b=n_b}^{\infty} (n_1 p_1 + n_2 p_2 + n_b p_b) P(x_1, x_2, x_b)$$

$$= \sum_{x_1=n_1}^{\infty} \sum_{x_2=n_2}^{\infty} \sum_{x_b=n_b}^{\infty} (n_1 p_1 + n_2 p_2 + n_b p_b) \frac{e^{-\ell_1} \ell_1^{x_1}}{x_1!} \frac{e^{-\ell_2} \ell_2^{x_2}}{x_2!} \frac{e^{-\ell_b} \ell_b^{x_b}}{x_b!}$$

After calculating the expected profit for these cases we need to calculate the total expected profit. The total expected profit is calculated as adding all expected profits calculated for each cases. It can be formulated as:

$$\pi = \sum_{i=1}^8 \pi_i$$

where i is the index for the cases.

2.4 Superadditivity and Subadditivity of Reservation Prices

In the first part we considered strict additivity of customers' reservation prices where reservation price for the bundle is the sum of the reservation prices of individual products that form the bundle. However Venkatesh and Kamatura [28] consider the product substitutability and complementarity. If the products that form the bundle are substitute products then the reservation price for the bundle is less than the sum of the reservation prices of the individual products as the customers' willingness to buy both is less. However if the products are complements, customers' willingness to buy both products at the same time will be higher. In this section we investigate the effects of substitution and complementarity among products.

Venkatesh and Kamatura [28] refer to the degree of substitutability and complementarity as the degree of contingency, θ , and it is given as:

$$\theta = \frac{R_b - (R_1 + R_2)}{R_1 + R_2}$$

Degree of contingency changes the purchasing (m_i) and switching (θ_{ij}) probability formulations. However, sales probability and objective function formulations are still valid as defined in the previous section. We will directly give the

formulations of the purchasing and switching probabilities. Note that we replace R_b in our formulas with the following equation:

$$R_b = (1 + \theta)(R_1 + R_2)$$

2.4.1 Purchasing Probabilities

Probability of No Purchase:

$$\begin{aligned} m_0 &= P(R_1 < p_1, R_2 < p_2, R_b < p_b) \\ &= P(R_1 < p_1, R_2 < p_2, (1 + \theta)(R_1 + R_2) < p_b) \\ &= P(R_1 < p_1, R_2 < \min \{p_2, (p_b - (1 + \theta)R_1)/(1 + \theta)\}) \\ &= \int_{-\infty}^{p_1} \int_{-\infty}^{a_1} f_{R_1, R_2}(r_1, r_2) dr_1 dr_2 \end{aligned}$$

where $a_1 = \min \{p_2, (p_b - (1 + \theta)r_1)/(1 + \theta)\}$.

Probability of Purchasing Product 1:

$$\begin{aligned} m_1 &= P(R_1 > p_1, R_1 - p_1 > R_2 - p_2, R_1 - p_1 > R_b - p_b) \\ &= P(R_1 > p_1, R_1 - p_1 > R_2 - p_2, R_1 - p_1 > (1 + \theta)(R_1 + R_2) - p_b) \\ &= P(R_1 > p_1, R_2 < \min \{R_1 - p_1 + p_2, (p_b - p_1 - \theta R_1)/(1 + \theta)\}) \\ &= \int_{p_1}^{\infty} \int_{-\infty}^{a_2} f_{R_1, R_2}(r_1, r_2) dr_1 dr_2 \end{aligned}$$

where $a_2 = \min \{r_1 - p_1 + p_2, (p_b - p_1 - \theta r_1)/(1 + \theta)\}$.

Probability of Purchasing Product 2:

$$\begin{aligned} m_2 &= P(R_2 > p_2, R_2 - p_2 > R_1 - p_1, R_2 - p_2 > R_b - p_b) \\ &= P(R_2 > p_2, R_2 - p_2 > R_1 - p_1, R_2 - p_2 > (1 + \theta)(R_1 + R_2) - p_b) \\ &= P(R_2 > p_2, R_1 < \min \{R_2 - p_2 + p_1, (p_b - p_2 - \theta R_2)/(1 + \theta)\}) \\ &= \int_{p_2}^{\infty} \int_{-\infty}^{a_3} f_{R_1, R_2}(r_1, r_2) dr_2 dr_1 \end{aligned}$$

where $a_3 = \min \{r_2 - p_2 + p_1, (p_b - p_2 - \theta r_2)/(1 + \theta)\}$.

Probability of Purchasing a Bundle:

$$\begin{aligned}
m_b &= P(R_b > p_b, R_b - p_b > R_1 - p_1, R_b - p_b > R_2 - p_2) \\
&= P((1 + \theta)(R_1 + R_2) > p_b, (1 + \theta)(R_1 + R_2) - p_2 > R_1 - p_1, \\
&\quad (1 + \theta)(R_1 + R_2) - p_b > R_2 - p_2) \\
&= P(R_2 > \max((p_b - (1 + \theta)R_1)/(1 + \theta), \\
&\quad (p_b - p_1 - \theta R_1)/(1 + \theta), (p_b - p_2 - (1 + \theta)R_1)/\theta)) \\
&= \int_{p_b - p_2}^{\infty} \int_{a_4}^{\infty} f_{R_1, R_2}(r_1, r_2) dr_1 dr_2
\end{aligned}$$

where $a_4 = \max$

$\{(p_b - (1 + \theta)r_1)/(1 + \theta), (p_b - p_1 - \theta r_1)/(1 + \theta), (p_b - p_2 - (1 + \theta)r_1)/\theta\}$

2.4.2 Switching Probabilities

One Type of Product Incurs Shortage

Probability of Switching from Product 1 to Bundle or to Product 2

$$\begin{aligned}
\theta_{1B} &= \Pr \left\{ R_b - p_b \geq R_2 - p_2; R_b \geq p_b \left| \begin{array}{l} R_1 - p_1 \geq R_b - p_b; \\ R_1 - p_1 \geq R_2 - p_2; R_1 \geq p_1 \end{array} \right. \right\} \\
&= \Pr \left\{ \begin{array}{l} (1 + \theta)(R_1 + R_2) - p_b \geq R_2 - p_2; \\ (1 + \theta)(R_1 + R_2) \geq p_b; \\ R_1 - p_1 \geq (1 + \theta)(R_1 + R_2) - p_b; \\ R_1 - p_1 \geq R_2 - p_2; R_1 \geq p_1 \end{array} \right\} / m_1 \\
&= \Pr \left\{ \begin{array}{l} (1 + \theta)R_2 - R_2 \geq p_b - p_2 - (1 + \theta)R_1; \\ (1 + \theta)R_2 \geq p_b - (1 + \theta)R_1; \\ R_1 - (1 + \theta)R_1 - p_1 + p_b \geq (1 + \theta)R_2; \\ R_1 - p_1 + p_2 \geq R_2; R_1 \geq p_1 \end{array} \right\} / m_1
\end{aligned}$$

$$\begin{aligned}
&= \left\{ \begin{array}{l} \Pr \left\{ \begin{array}{l} R_2 \geq \frac{p_b - p_2 - (1+\theta)R_1}{\theta}; R_2 \geq \frac{p_b}{1+\theta} - R_1; \\ \frac{p_b - \theta R_1 - p_1}{1+\theta} \geq R_2; R_1 - p_1 + p_2 \geq R_2; R_1 \geq p_1 \end{array} \right\} / m_1 \quad \text{if } \theta > 0 \\ \Pr \left\{ \begin{array}{l} R_2 \leq \frac{p_b - p_2 - (1+\theta)R_1}{\theta}; R_2 \geq \frac{p_b}{1+\theta} - R_1; \\ \frac{p_b - \theta R_1 - p_1}{1+\theta} \geq R_2; R_1 - p_1 + p_2 \geq R_2; R_1 \geq p_1 \end{array} \right\} / m_1 \quad \text{if } \theta < 0 \end{array} \right\} \\
&= \left\{ \begin{array}{l} \Pr \left\{ \begin{array}{l} R_2 \geq \max \left(\frac{p_b - p_2 - (1+\theta)R_1}{\theta}, \frac{p_b}{1+\theta} - R_1 \right); \\ R_2 \leq \min \left(\frac{p_b - \theta R_1 - p_1}{1+\theta}, R_1 - p_1 + p_2 \right); R_1 \geq p_1 \end{array} \right\} / m_1 \quad \text{if } \theta > 0 \\ \Pr \left\{ \begin{array}{l} R_2 \geq \frac{p_b}{(1+\theta)} - R_1; R_1 \geq p_1 \\ R_2 \leq \min \left(\frac{p_b - p_2 - (1+\theta)R_1}{\theta}, \frac{p_b - \theta R_1 - p_1}{(1+\theta)}, R_1 - p_1 + p_2 \right) \end{array} \right\} / m_1 \quad \text{if } \theta < 0 \end{array} \right\} \\
&= \left\{ \begin{array}{l} \int_{p_1}^{\infty} \int_{\max \left(\frac{p_b - p_2 - (1+\theta)R_1}{\theta}, \frac{p_b}{1+\theta} - R_1 \right)}^{\min \left(\frac{p_b - \theta R_1 - p_1}{1+\theta}, R_1 - p_1 + p_2 \right)} f_{R_1, R_2}(r_1, r_2) dr_1 dr_2 / m_1 \quad \text{if } \theta > 0 \\ \int_{p_1}^{\infty} \int_{\frac{p_b}{(1+\theta)} - R_1}^{\min \left(\frac{p_b - p_2 - (1+\theta)R_1}{\theta}, \frac{p_b - \theta R_1 - p_1}{(1+\theta)}, R_1 - p_1 + p_2 \right)} f_{R_1, R_2}(r_1, r_2) dr_1 dr_2 / m_1 \quad \text{if } \theta < 0 \end{array} \right\}
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\theta_{12} &= \Pr \left\{ R_2 - p_2 \geq R_b - p_b; R_2 \geq p_2 \left| \begin{array}{l} R_1 - p_1 \geq R_b - p_b; \\ R_1 - p_1 \geq R_2 - p_2; R_1 \geq p_1 \end{array} \right. \right\} \\
&= \Pr \left\{ \begin{array}{l} R_2 - p_2 \geq (1+\theta)(R_1 + R_2) - p_b; R_2 \geq p_2; \\ R_1 \geq p_1; R_1 - p_1 \geq (1+\theta)(R_1 + R_2) - p_b; \\ R_1 - p_1 \geq R_2 - p_2; \end{array} \right\} / m_1 \\
&= \Pr \left\{ \begin{array}{l} p_b - (1+\theta)R_1 - p_2 \geq -R_2 + (1+\theta)R_2; R_2 \geq p_2; \\ R_1 \geq p_1; R_1 - (1+\theta)R_1 - p_1 + p_b \geq (1+\theta)R_2; \\ R_1 - p_1 + p_2 \geq R_2; \end{array} \right\} / m_1 \\
&= \left\{ \begin{array}{l} \Pr \left\{ \begin{array}{l} \frac{p_b - (1+\theta)R_1 - p_2}{\theta} \geq R_2; R_2 \geq p_2; R_1 \geq p_1 \\ \frac{p_b - \theta R_1 - p_1}{1+\theta} \geq R_2; R_1 - p_1 + p_2 \geq R_2 \end{array} \right\} / m_1 \quad \text{if } \theta > 0 \\ \Pr \left\{ \begin{array}{l} \frac{p_b - (1+\theta)R_1 - p_2}{\theta} \leq R_2; R_2 \geq p_2; R_1 \geq p_1 \\ \frac{p_b - \theta R_1 - p_1}{1+\theta} \geq R_2; R_1 - p_1 + p_2 \geq R_2 \end{array} \right\} / m_1 \quad \text{if } \theta < 0 \end{array} \right\} \\
&= \left\{ \begin{array}{l} \Pr \left\{ \begin{array}{l} R_2 \geq p_2; R_1 \geq p_1 \\ R_2 \leq \min \left(\frac{p_b - (1+\theta)R_1 - p_2}{\theta}, \frac{p_b - \theta R_1 - p_1}{1+\theta}, R_1 - p_1 + p_2 \right) \end{array} \right\} / m_1 \quad \text{if } \theta > 0 \\ \Pr \left\{ \begin{array}{l} R_2 \geq \max \left(\frac{p_b - (1+\theta)R_1 - p_2}{\theta}, p_2 \right); \\ R_2 \leq \min \left(\frac{p_b - \theta R_1 - p_1}{(1+\theta)}, R_1 - p_1 + p_2 \right); R_1 \geq p_1 \end{array} \right\} / m_1 \quad \text{if } \theta < 0 \end{array} \right\}
\end{aligned}$$

$$= \begin{cases} \int_{p_1}^{\infty} \int_{p_2}^{\min\left(\frac{p_b - (1+\theta)R_1 - p_2}{\theta}, \frac{p_b - \theta R_1 - p_1}{1+\theta}, R_1 - p_1 + p_2\right)} f_{R_1, R_2}(r_1, r_2) dr_1 dr_2 / m_1 & \text{if } \theta > 0 \\ \int_{p_1}^{\infty} \int_{\max\left(\frac{p_b - (1+\theta)R_1 - p_2}{\theta}, p_2\right)}^{\min\left(\frac{p_b - \theta R_1 - p_1}{1+\theta}, R_1 - p_1 + p_2\right)} f_{R_1, R_2}(r_1, r_2) dr_1 dr_2 / m_1 & \text{if } \theta < 0 \end{cases}$$

Probability of Switching from Product 2 to Bundle or to Product 1

$$\begin{aligned} \theta_{2B} &= \Pr \left\{ R_b - p_b \geq R_1 - p_1; R_b \geq p_b \left| \begin{array}{l} R_2 - p_2 \geq R_b - p_b; \\ R_2 - p_2 \geq R_1 - p_1; R_2 \geq p_2 \end{array} \right. \right\} \\ &= \begin{cases} \Pr \left\{ \begin{array}{l} R_1 \geq \max\left(\frac{p_b - p_1 - (1+\theta)R_2}{\theta}, \frac{p_b}{1+\theta} - R_2\right); \\ R_1 \leq \min\left(\frac{p_b - \theta R_2 - p_2}{1+\theta}, R_2 - p_2 + p_1\right); R_2 \geq p_2 \end{array} \right\} / m_2 & \text{if } \theta > 0 \\ \Pr \left\{ \begin{array}{l} R_1 \geq \frac{p_b}{1+\theta} - R_2; R_2 \geq p_2; \\ R_1 \leq \min\left(\frac{p_b - p_1 - (1+\theta)R_2}{\theta}, \frac{p_b - \theta R_2 - p_2}{1+\theta}, R_2 - p_2 + p_1\right) \end{array} \right\} / m_2 & \text{if } \theta < 0 \end{cases} \\ &= \begin{cases} \int_{p_2}^{\infty} \int_{\max\left(\frac{p_b - p_1 - (1+\theta)R_2}{\theta}, \frac{p_b}{1+\theta} - R_2\right)}^{\min\left(\frac{p_b - \theta R_2 - p_2}{1+\theta}, R_2 - p_2 + p_1\right)} f_{R_1, R_2}(r_1, r_2) dr_2 dr_1 / m_2 & \text{if } \theta > 0 \\ \int_{p_2}^{\infty} \int_{\frac{p_b}{1+\theta} - R_2}^{\min\left(\frac{p_b - p_1 - (1+\theta)R_2}{\theta}, \frac{p_b - \theta R_2 - p_2}{1+\theta}, R_2 - p_2 + p_1\right)} f_{R_1, R_2}(r_1, r_2) dr_2 dr_1 / m_2 & \text{if } \theta < 0 \end{cases} \end{aligned}$$

and

$$\begin{aligned} \theta_{21} &= \Pr \left\{ R_1 - p_1 \geq R_b - p_b; R_1 \geq p_1 \left| \begin{array}{l} R_2 - p_2 \geq R_b - p_b; \\ R_2 - p_2 \geq R_1 - p_1; R_2 \geq p_2 \end{array} \right. \right\} \\ &= \begin{cases} \Pr \left\{ \begin{array}{l} R_1 \geq p_1; R_2 \geq p_2; \\ R_1 \leq \min\left(\frac{p_b - p_1 - (1+\theta)R_2}{\theta}, \frac{p_b - \theta R_2 - p_2}{1+\theta}, R_2 - p_2 + p_1\right) \end{array} \right\} / m_2 & \text{if } \theta > 0 \\ \Pr \left\{ \begin{array}{l} R_1 \geq \max\left(\frac{p_b - p_1 - (1+\theta)R_2}{\theta}, p_1\right); R_2 \geq p_2; \\ R_1 \leq \min\left(\frac{p_b - \theta R_2 - p_2}{1+\theta}, R_2 - p_2 + p_1\right) \end{array} \right\} / m_2 & \text{if } \theta < 0 \end{cases} \\ &= \begin{cases} \int_{p_2}^{\infty} \int_{p_1}^{\min\left(\frac{p_b - p_1 - (1+\theta)R_2}{\theta}, \frac{p_b - \theta R_2 - p_2}{1+\theta}, R_2 - p_2 + p_1\right)} f_{R_1, R_2}(r_1, r_2) dr_2 dr_1 / m_2 & \text{if } \theta > 0 \\ \int_{p_2}^{\infty} \int_{\max\left(\frac{p_b - p_1 - (1+\theta)R_2}{\theta}, p_1\right)}^{\min\left(\frac{p_b - \theta R_2 - p_2}{1+\theta}, R_2 - p_2 + p_1\right)} f_{R_1, R_2}(r_1, r_2) dr_2 dr_1 / m_2 & \text{if } \theta < 0 \end{cases} \end{aligned}$$

Probability of Switching from Bundle to Product 1 or to Product 2

$$\begin{aligned} \theta_{B1} &= \Pr \left\{ R_1 - p_1 \geq R_2 - p_2; R_1 \geq p_1 \left| \begin{array}{l} R_b - p_b \geq R_1 - p_1; \\ R_b - p_b \geq R_2 - p_2; R_b \geq p_b \end{array} \right. \right\} \\ &= \Pr \left\{ \begin{array}{l} R_1 - p_1 + p_2 \geq R_2; R_1 \geq p_1; \\ (1 + \theta)(R_1 + R_2) - p_b \geq R_1 - p_1; \\ (1 + \theta)(R_1 + R_2) - p_b \geq R_2 - p_2; (1 + \theta)(R_1 + R_2) \geq p_b \end{array} \right\} / m_b \end{aligned}$$

$$\begin{aligned}
&= \Pr \left\{ \begin{array}{l} R_1 - p_1 + p_2 \geq R_2; R_1 \geq p_1; \\ (1 + \theta)R_2 \geq R_1 - (1 + \theta)R_1 - p_1 + p_b; \\ (1 + \theta)R_2 - R_2 \geq p_b - (1 + \theta)R_1 - p_2; \\ (1 + \theta)R_2 \geq p_b - (1 + \theta)R_1 \end{array} \right\} / m_b \\
&= \left\{ \begin{array}{l} \Pr \left\{ \begin{array}{l} R_1 - p_1 + p_2 \geq R_2; R_1 \geq p_1; R_2 \geq \frac{p_b - \theta R_1 - p_1}{1 + \theta}; \\ R_2 \geq \frac{p_b - (1 + \theta)R_1 - p_2}{\theta}; R_2 \geq \frac{p_b - (1 + \theta)R_1}{1 + \theta} \end{array} \right\} / m_b \quad \text{if } \theta > 0 \\ \Pr \left\{ \begin{array}{l} R_1 - p_1 + p_2 \geq R_2; R_1 \geq p_1; R_2 \geq \frac{p_b - \theta R_1 - p_1}{1 + \theta}; \\ R_2 \leq \frac{p_b - (1 + \theta)R_1 - p_2}{\theta}; R_2 \geq \frac{p_b - (1 + \theta)R_1}{1 + \theta} \end{array} \right\} / m_b \quad \text{if } \theta < 0 \end{array} \right\} \\
&= \left\{ \begin{array}{l} \Pr \left\{ \begin{array}{l} R_2 \geq \max \left(\frac{p_b - \theta R_1 - p_1}{1 + \theta}, \frac{p_b - (1 + \theta)R_1 - p_2}{\theta}, \frac{p_b - (1 + \theta)R_1}{1 + \theta} \right); \\ R_2 \leq R_1 - p_1 + p_2; R_1 \geq p_1 \end{array} \right\} / m_b \quad \text{if } \theta > 0 \\ \Pr \left\{ \begin{array}{l} R_2 \geq \max \left(\frac{p_b - \theta R_1 - p_1}{1 + \theta}, \frac{p_b - (1 + \theta)R_1}{1 + \theta} \right); \\ R_2 \leq \min \left(\frac{p_b - (1 + \theta)R_1 - p_2}{\theta}, R_1 - p_1 + p_2 \right); R_1 \geq p_1 \end{array} \right\} / m_b \quad \text{if } \theta < 0 \end{array} \right\} \\
&= \left\{ \begin{array}{l} \Pr \left\{ \begin{array}{l} R_2 \geq \max \left(\frac{p_b - \theta R_1 - p_1}{1 + \theta}, \frac{p_b - (1 + \theta)R_1 - p_2}{\theta} \right); \\ R_2 \leq R_1 - p_1 + p_2; R_1 \geq p_1 \end{array} \right\} / m_b \quad \text{if } \theta > 0 \\ \Pr \left\{ \begin{array}{l} R_2 \geq \frac{p_b - \theta R_1 - p_1}{1 + \theta}; \\ R_2 \leq \min \left(\frac{p_b - (1 + \theta)R_1 - p_2}{\theta}, R_1 - p_1 + p_2 \right), R_1 \geq p_1 \end{array} \right\} / m_b \quad \text{if } \theta < 0 \end{array} \right\} \\
&= \left\{ \begin{array}{l} \int_{p_1}^{\infty} \int_{\max \left(\frac{p_b - \theta R_1 - p_1}{1 + \theta}, \frac{p_b - (1 + \theta)R_1 - p_2}{\theta} \right)}^{R_1 - p_1 + p_2; R_1 \geq p_1} f_{R_1, R_2}(r_1, r_2) dr_1 dr_2 / m_b \quad \text{if } \theta > 0 \\ \int_{p_1}^{\infty} \int_{\frac{p_b - \theta R_1 - p_1}{1 + \theta}}^{\min \left(\frac{p_b - (1 + \theta)R_1 - p_2}{\theta}, R_1 - p_1 + p_2 \right)} f_{R_1, R_2}(r_1, r_2) dr_1 dr_2 / m_b \quad \text{if } \theta < 0 \end{array} \right\}
\end{aligned}$$

and

$$\begin{aligned}
\theta_{B2} &= \Pr \left\{ R_2 - p_2 \geq R_1 - p_1; R_2 \geq p_2 \left| \begin{array}{l} R_b - p_b \geq R_1 - p_1; \\ R_b - p_b \geq R_2 - p_2; R_b \geq p_b \end{array} \right. \right\} \\
&= \left\{ \begin{array}{l} \Pr \left\{ \begin{array}{l} R_1 \geq \max \left(\frac{-(1 + \theta)R_2 - p_1 + p_b}{\theta}, \frac{p_b - \theta R_2 - p_2}{1 + \theta} \right); \\ R_2 \leq R_2 - p_2 + p_1; R_2 \geq p_2 \end{array} \right\} / m_b \quad \text{if } \theta > 0 \\ \Pr \left\{ \begin{array}{l} R_1 \geq \frac{p_b - \theta R_2 - p_2}{1 + \theta}; R_2 \geq p_2; \\ R_1 \leq \min \left(\frac{-(1 + \theta)R_2 - p_1 + p_b}{\theta}, R_2 - p_2 + p_1 \right) \end{array} \right\} / m_b \quad \text{if } \theta < 0 \end{array} \right\} \\
&= \left\{ \begin{array}{l} \int_{p_2}^{\infty} \int_{\max \left(\frac{-(1 + \theta)R_2 - p_1 + p_b}{\theta}, \frac{p_b - \theta R_2 - p_2}{1 + \theta} \right)}^{R_2 - p_2 + p_1} f_{R_1, R_2}(r_1, r_2) dr_2 dr_1 / m_b \quad \text{if } \theta > 0 \\ \int_{p_2}^{\infty} \int_{\frac{p_b - \theta R_2 - p_2}{1 + \theta}}^{\min \left(\frac{-(1 + \theta)R_2 - p_1 + p_b}{\theta}, R_2 - p_2 + p_1 \right)} f_{R_1, R_2}(r_1, r_2) dr_2 dr_1 / m_b \quad \text{if } \theta < 0 \end{array} \right\}
\end{aligned}$$

Two Types of Products Incur Shortage

Product 1 and Product 2 incurs shortage:

$$\begin{aligned}
\theta_{1B}^- &= \Pr \left\{ R_b \geq p_b \left| \begin{array}{l} R_1 - p_1 \geq R_b - p_b; \\ R_1 - p_1 \geq R_2 - p_2; R_1 \geq p_1 \end{array} \right. \right\} \\
&= \Pr \left\{ \begin{array}{l} (1 + \theta)(R_1 + R_2) \geq p_b; R_1 - p_1 \geq (1 + \theta)(R_1 + R_2) - p_b; \\ R_1 - p_1 \geq R_2 - p_2; R_1 \geq p_1 \end{array} \right\} / m_1 \\
&= \Pr \left\{ \begin{array}{l} (1 + \theta)R_2 \geq p_b - (1 + \theta)R_1; \\ R_1 - (1 + \theta)R_1 - p_1 + p_b \geq (1 + \theta)R_2; \\ R_1 - p_1 + p_2 \geq R_2; R_1 \geq p_1 \end{array} \right\} / m_1 \\
&= \Pr \left\{ \begin{array}{l} R_2 \geq \frac{p_b}{1 + \theta} - R_1; \frac{p_b - \theta R_1 - p_1}{1 + \theta} \geq R_2; \\ R_1 - p_1 + p_2 \geq R_2; R_1 \geq p_1 \end{array} \right\} / m_1 \\
&= \Pr \left\{ \begin{array}{l} R_2 \geq \frac{p_b}{1 + \theta} - R_1; R_2 \leq \min \left(\frac{p_b - \theta R_1 - p_1}{1 + \theta}, R_1 - p_1 + p_2 \right); \\ R_1 \geq p_1 \end{array} \right\} / m_1 \\
&= \int_{p_1}^{\infty} \int_{\frac{p_b}{1 + \theta} - R_1}^{\min \left(\frac{p_b - \theta R_1 - p_1}{1 + \theta}, R_1 - p_1 + p_2 \right)} f_{R_1, R_2}(r_1, r_2) dr_1 dr_2 / m_1
\end{aligned}$$

and similarly

$$\begin{aligned}
\theta_{2B}^- &= \Pr \left\{ R_b \geq p_b \left| \begin{array}{l} R_2 - p_2 \geq R_b - p_b; \\ R_2 - p_2 \geq R_1 - p_1; R_2 \geq p_2 \end{array} \right. \right\} \\
&= \Pr \left\{ \begin{array}{l} (1 + \theta)(R_1 + R_2) \geq p_b; R_2 - p_2 \geq (1 + \theta)(R_1 + R_2) - p_b; \\ R_2 - p_2 \geq R_1 - p_1; R_2 \geq p_2 \end{array} \right\} / m_2 \\
&= \Pr \left\{ \begin{array}{l} (1 + \theta)R_1 \geq p_b - (1 + \theta)R_2; \\ R_2 - (1 + \theta)R_2 - p_2 + p_b \geq (1 + \theta)R_1; \\ R_2 - p_2 + p_1 \geq R_1; R_2 \geq p_2 \end{array} \right\} / m_2 \\
&= \Pr \left\{ \begin{array}{l} R_1 \geq \frac{p_b}{1 + \theta} - R_2; \frac{p_b - \theta R_2 - p_2}{1 + \theta} \geq R_1; \\ R_2 - p_2 + p_1 \geq R_1; R_2 \geq p_2 \end{array} \right\} / m_2 \\
&= \Pr \left\{ \begin{array}{l} R_1 \geq \frac{p_b}{1 + \theta} - R_2; R_1 \leq \min \left(\frac{p_b - \theta R_2 - p_2}{1 + \theta}, R_2 - p_2 + p_1 \right); \\ R_2 \geq p_2 \end{array} \right\} / m_2 \\
&= \int_{p_2}^{\infty} \int_{\frac{p_b}{1 + \theta} - R_2}^{\min \left(\frac{p_b - \theta R_2 - p_2}{1 + \theta}, R_2 - p_2 + p_1 \right)} f_{R_1, R_2}(r_1, r_2) dr_2 dr_1 / m_2
\end{aligned}$$

Bundle and Product 2 incur shortage

$$\begin{aligned}
\theta_{B1}^- &= \Pr \left\{ R_1 \geq p_1 \left| \begin{array}{l} R_b - p_b \geq R_1 - p_1; \\ R_b - p_b \geq R_2 - p_2; R_b \geq p_b \end{array} \right. \right\} \\
&= \Pr \left\{ \begin{array}{l} R_1 \geq p_1; (1 + \theta)(R_1 + R_2) - p_b \geq R_1 - p_1; \\ (1 + \theta)(R_1 + R_2) - p_b \geq R_2 - p_2; (1 + \theta)(R_1 + R_2) \geq p_b \end{array} \right\} / m_b \\
&= \Pr \left\{ \begin{array}{l} R_1 \geq p_1; (1 + \theta)(R_1 + R_2) - p_b \geq R_1 - p_1; \\ (1 + \theta)(R_1 + R_2) - p_b \geq R_2 - p_2; (1 + \theta)(R_1 + R_2) \geq p_b \end{array} \right\} / m_b \\
&= \Pr \left\{ \begin{array}{l} R_1 \geq p_1; (1 + \theta)R_2 \geq R_1 - (1 + \theta)R_1 - p_1 + p_b; \\ (1 + \theta)R_2 - R_2 \geq p_b - (1 + \theta)R_1 - p_2; \\ (1 + \theta)R_2 \geq p_b - (1 + \theta)R_1 \end{array} \right\} / m_b \\
&= \left\{ \begin{array}{l} \Pr \left\{ \begin{array}{l} R_1 \geq p_1; R_2 \geq \frac{p_b - \theta R_1 - p_1}{1 + \theta}; \\ R_2 \geq \frac{p_b - (1 + \theta)R_1 - p_2}{\theta}; R_2 \geq \frac{p_b - (1 + \theta)R_1}{1 + \theta} \end{array} \right\} / m_b \quad \text{if } \theta > 0 \\ \Pr \left\{ \begin{array}{l} R_1 \geq p_1; R_2 \geq \frac{p_b - \theta R_1 - p_1}{1 + \theta}; \\ R_2 \leq \frac{p_b - (1 + \theta)R_1 - p_2}{\theta}; R_2 \geq \frac{p_b - (1 + \theta)R_1}{1 + \theta} \end{array} \right\} / m_b \quad \text{if } \theta < 0 \end{array} \right\} \\
&= \left\{ \begin{array}{l} \Pr \left\{ \begin{array}{l} R_2 \geq \max \left(\frac{p_b - \theta R_1 - p_1}{1 + \theta}, \frac{p_b - (1 + \theta)R_1 - p_2}{\theta}, \frac{p_b - (1 + \theta)R_1}{1 + \theta} \right); \\ R_1 \geq p_1 \end{array} \right\} / m_b \quad \text{if } \theta > 0 \\ \Pr \left\{ \begin{array}{l} R_2 \geq \max \left(\frac{p_b - \theta R_1 - p_1}{1 + \theta}, \frac{p_b - (1 + \theta)R_1}{1 + \theta} \right); \\ R_2 \leq \frac{p_b - (1 + \theta)R_1 - p_2}{\theta}; R_1 \geq p_1 \end{array} \right\} / m_b \quad \text{if } \theta < 0 \end{array} \right\} \\
&= \left\{ \begin{array}{l} \Pr \left\{ \begin{array}{l} R_2 \geq \max \left(\frac{p_b - \theta R_1 - p_1}{1 + \theta}, \frac{p_b - (1 + \theta)R_1 - p_2}{\theta} \right); \\ R_1 \geq p_1 \end{array} \right\} / m_b \quad \text{if } \theta > 0 \\ \Pr \left\{ \begin{array}{l} R_2 \geq \frac{p_b - \theta R_1 - p_1}{1 + \theta}; \\ R_2 \leq \frac{p_b - (1 + \theta)R_1 - p_2}{\theta}; R_1 \geq p_1 \end{array} \right\} / m_b \quad \text{if } \theta < 0 \end{array} \right\} \\
&= \left\{ \begin{array}{l} \int_{p_1}^{\infty} \int_{\max \left(\frac{p_b - \theta R_1 - p_1}{1 + \theta}, \frac{p_b - (1 + \theta)R_1 - p_2}{\theta} \right)}^{\infty} f_{R_1, R_2}(r_1, r_2) dr_1 dr_2 / m_b \quad \text{if } \theta > 0 \\ \int_{p_1}^{\infty} \int_{\frac{p_b - (1 + \theta)R_1 - p_2}{\theta}}^{\frac{p_b - \theta R_1 - p_1}{1 + \theta}} f_{R_1, R_2}(r_1, r_2) dr_1 dr_2 / m_b \quad \text{if } \theta < 0 \end{array} \right\}
\end{aligned}$$

and we have

$$\begin{aligned}
\theta_{21}^- &= \Pr \left\{ R_1 \geq p_1 \left| \begin{array}{l} R_2 - p_2 \geq R_b - p_b; \\ R_2 - p_2 \geq R_1 - p_1; R_2 \geq p_2 \end{array} \right. \right\} \\
&= \Pr \left\{ \begin{array}{l} R_1 \geq p_1; R_2 - p_2 \geq (1 + \theta)(R_1 + R_2) - p_b; \\ R_2 - p_2 \geq R_1 - p_1; R_2 \geq p_2 \end{array} \right\} / m_2
\end{aligned}$$

$$\begin{aligned}
&= \Pr \left\{ \begin{array}{l} R_1 \geq p_1; R_2 - (1 + \theta)R_2 - p_2 + p_b \geq (1 + \theta)R_1; \\ R_2 - p_2 + p_1 \geq R_1; R_2 \geq p_2 \end{array} \right\} / m_2 \\
&= \Pr \left\{ \begin{array}{l} R_1 \geq p_1; \frac{p_b - \theta R_2 - p_2}{1 + \theta} \geq R_1; \\ R_2 - p_2 + p_1 \geq R_1; R_2 \geq p_2 \end{array} \right\} / m_2 \\
&= \Pr \left\{ \begin{array}{l} R_1 \leq \min \left(\frac{p_b - \theta R_2 - p_2}{1 + \theta}, R_2 - p_2 + p_1 \right); \\ R_1 \geq p_1; R_2 \geq p_2 \end{array} \right\} / m_2 \\
&= \int_{p_2}^{\infty} \int_{p_1}^{\min \left(\frac{p_b - \theta R_2 - p_2}{1 + \theta}, R_2 - p_2 + p_1 \right)} f_{R_1, R_2}(r_1, r_2) dr_2 dr_1 / m_2
\end{aligned}$$

Bundle and Product 1 incur shortage

Similar to previous cases we have

$$\begin{aligned}
\theta_{B2}^- &= \Pr \left\{ R_2 \geq p_2 \left| \begin{array}{l} R_b - p_b \geq R_1 - p_1; \\ R_b - p_b \geq R_2 - p_2; R_b \geq p_b \end{array} \right. \right\} \\
&= \left\{ \begin{array}{l} \Pr \left\{ \begin{array}{l} R_2 \geq p_2; \\ R_1 \geq \max \left(\frac{-(1 + \theta)R_2 - p_1 + p_b}{\theta}, \frac{p_b - \theta R_2 - p_2}{1 + \theta} \right) \end{array} \right\} / m_b \quad \text{if } \theta > 0 \\ \Pr \left\{ \begin{array}{l} R_2 \geq p_2; R_1 \geq \frac{p_b - \theta R_2 - p_2}{1 + \theta}; \\ R_1 \leq \frac{-(1 + \theta)R_2 - p_1 + p_b}{\theta} \end{array} \right\} / m_b \quad \text{if } \theta < 0 \end{array} \right. \\
&= \left\{ \begin{array}{l} \int_{p_2}^{\infty} \int_{\max \left(\frac{-(1 + \theta)R_2 - p_1 + p_b}{\theta}, \frac{p_b - \theta R_2 - p_2}{1 + \theta} \right)}^{\infty} f_{R_1, R_2}(r_1, r_2) dr_2 dr_1 / m_b \quad \text{if } \theta > 0 \\ \int_{p_2}^{\infty} \int_{\frac{p_b - \theta R_2 - p_2}{1 + \theta}}^{\frac{-(1 + \theta)R_2 - p_1 + p_b}{\theta}} f_{R_1, R_2}(r_1, r_2) dr_2 dr_1 / m_b \quad \text{if } \theta < 0 \end{array} \right.
\end{aligned}$$

and

$$\begin{aligned}
\theta_{12}^- &= \Pr \left\{ R_2 \geq p_2 \left| \begin{array}{l} R_1 - p_1 \geq R_b - p_b; \\ R_1 - p_1 \geq R_2 - p_2; R_1 \geq p_1 \end{array} \right. \right\} \\
&= \Pr \left\{ \begin{array}{l} R_2 \leq \min \left(\frac{p_b - \theta R_1 - p_1}{1 + \theta}, R_1 - p_1 + p_2 \right); \\ R_2 \geq p_2; R_1 \geq p_1 \end{array} \right\} / m_1 \\
&= \int_{p_1}^{\infty} \int_{p_2}^{\min \left(\frac{p_b - \theta R_1 - p_1}{1 + \theta}, R_1 - p_1 + p_2 \right)} f_{R_1, R_2}(r_1, r_2) dr_1 dr_2 / m_1
\end{aligned}$$

Chapter 3

NUMERICAL STUDY

In this chapter, we present the results of our numerical study to demonstrate the effects of various factors on the bundling and pricing problem. The factors considered are the correlation between the reservation price distributions, the variance of the reservation price distributions, initial inventory levels, the unit bundle formation cost, and the intensity of the customer arrivals. We also consider the effects of the degree of product complementarity and substitutability, also known as the degree of contingency, in the last part.

For the numerical study, we write a FORTRAN code to calculate the expected profit and to determine the optimum number of bundles to be formed and the optimum bundle price for a given parameter set.

Before providing the results, we will discuss the setup of the model used for the numerical study. The model defined in previous section contains a general bivariate continuous distribution for the customer reservation prices. In literature, personal choices and tastes are modeled using distributions of Gaussian family (Schmalensee [23]). Hence, for the numerical study, we choose bivariate normal distribution to model customer reservation prices of the two products. Another reason to use normal distribution in our model is that summation of two normally distributed random variables is also a random variable with normal distribution. It is also easy to capture correlation effects with bivariate normal distributions.

One disadvantage in using normal distribution is that it can take negative values. In our study we avoid this problem by using appropriate parameters that will give nonnegative valuations.

As stated before R_1 and R_2 denote the reservation prices of Product 1 and Product 2 and $f_{R_1, R_2}(r_1, r_2)$ is the joint probability density function of these two variables. The joint probability has a bivariate normal distribution and it is defined as,

$$f_{R_1, R_2}(r_1, r_2) = \frac{e^{-\Gamma(r_1, r_2)/2}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}},$$

where ρ is the correlation coefficient between the reservation prices and

$$\Gamma(r_1, r_2) = \frac{1}{1-\rho^2} \left[\left(\frac{r_1 - \mu_1}{\sigma_1} \right)^2 - 2\rho \left(\frac{r_1 - \mu_1}{\sigma_1} \right) \left(\frac{r_2 - \mu_2}{\sigma_2} \right) + \left(\frac{r_2 - \mu_2}{\sigma_2} \right)^2 \right].$$

In this setup, reservation price of Product 1 and Product 2 are normally distributed with distribution parameters of (μ_1, σ_1) and (μ_2, σ_2) , respectively. Then, for $i = 1, 2$ we have the following marginal distributions for the reservation prices R_i , with mean μ_i and standard deviation σ_i :

$$f_{R_i}(r) = \frac{e^{-(r-\mu_i)^2/2\sigma_i^2}}{\sigma_i\sqrt{2\pi}}$$

Since the problem contains bivariate normality, when $R_b = R_1 + R_2$ (bundle reservation price is strictly additive), R_b is normally distributed with mean $\mu_b = \mu_1 + \mu_2$ and standard deviation σ_b calculated as:

$$\sigma_b = (\sigma_1 + \sigma_2)\sqrt{1 - 2(1 - \rho)\omega(1 - \omega)}$$

where $\omega = \frac{\sigma_1}{\sigma_1 + \sigma_2}$.

In the second part of the study, we consider the substitutability and complementarity of the products. In this setup, reservation price of the bundle is not strictly additive, it is superadditive or subadditive.

$$R_b = (1 + \theta)(R_1 + R_2)$$

As seen from the above formula, reservation price of the bundle is superadditive for positive θ and subadditive for negative θ values. Customers value the bundle

with a higher reservation price than the summation of reservation prices of individual products when the individual products are complements of each other and lower reservation price when the products are substitutes of each other.

The degree of substitutability and complementarity is defined as the degree of contingency, θ , and it is given as:

$$\theta = \frac{R_b - (R_1 + R_2)}{R_1 + R_2}$$

To investigate the effects of factors defined at the beginning of the chapter, we fix the individual product prices p_1 and p_2 , and optimize the number of bundles formed, n_b and the bundle price, p_b . However the model defined in the previous chapter is a general model and can be used to jointly optimize all product prices.

Now we provide the findings of our study.

The Base Case

A base case is used to investigate the effects of the factors. The results of the various cases are compared with the base case results. Individual product prices, p_1 and p_2 are set to 10 for the base case. Reservation price distributions used in the base case are such that the marginals have mean value of $\mu_1 = \mu_2 = 10$ and standard deviation of $\sigma_1 = \sigma_2 = 2$. Degree of contingency, θ in this section is zero. Initial inventory level for individual products are $Q_1 = Q_2 = 5$. Arrival rate, λ is set to 10. We search optimal bundle price within the search interval with increments of 0.25. We take unit bundling cost c as 0, 1, 2 and 4. The results of the base case under the correlation coefficient, ρ , of -0.9, 0 and 0.9 are tabulated in Table 3.1. n_b^* is the optimal number of bundles and p_b^* is the optimal bundle price. The fifth column is the expected profit, and the next two columns represent the expected number of products sold. The last three columns are the arrival rate of customers dedicated to each product (ℓ_1, ℓ_2, ℓ_b) and the arrival rate of customers who would leave the store without buying any product (ℓ_0) at the beginning when all products are available.

We also demonstrate the results of the base case in Figure 3.1. It is observed that the correlation coefficient, ρ and the bundling cost, c have significant effects

| ρ | c | n_b^* | p_b^* | $E(Profit)$ | $E(x_1) = E(x_2)$ | $E(x_b)$ | $\ell_1 = \ell_2$ | ℓ_b | ℓ_0 |
|--------|-----|---------|---------|-------------|-------------------|----------|-------------------|----------|----------|
| -0.9 | 0 | 5 | 19.25 | 93.0645 | 0.00 | 4.52 | 3.34 | 2.82 | 0.50 |
| -0.9 | 1 | 4 | 19 | 88.3828 | 0.76 | 3.07 | 2.97 | 3.68 | 0.38 |
| -0.9 | 2 | 3 | 19 | 84.8695 | 1.50 | 2.37 | 2.97 | 3.68 | 0.38 |
| -0.9 | 4 | 1 | 19.25 | 80.6290 | 3.12 | 0.92 | 3.34 | 2.82 | 0.50 |
| 0 | 0 | 5 | 18.75 | 87.1901 | 0.00 | 3.55 | 1.33 | 5.13 | 2.21 |
| 0 | 1 | 4 | 18.75 | 83.0791 | 0.77 | 3.39 | 1.33 | 5.13 | 2.21 |
| 0 | 2 | 3 | 19 | 79.1586 | 1.53 | 2.73 | 1.54 | 4.61 | 2.31 |
| 0 | 4 | 2 | 19.25 | 73.3638 | 2.21 | 1.89 | 1.77 | 4.07 | 2.39 |
| 0.9 | 0 | 5 | 18.75 | 85.3171 | 0.00 | 4.54 | 0.06 | 6.19 | 3.68 |
| 0.9 | 1 | 5 | 18.75 | 80.3171 | 0.00 | 4.54 | 0.06 | 6.19 | 3.68 |
| 0.9 | 2 | 5 | 18.75 | 75.3171 | 0.00 | 4.54 | 0.06 | 6.19 | 3.68 |
| 0.9 | 4 | 4 | 18.75 | 67.0220 | 0.59 | 3.79 | 0.06 | 6.19 | 3.68 |

Table 3.1: Base case

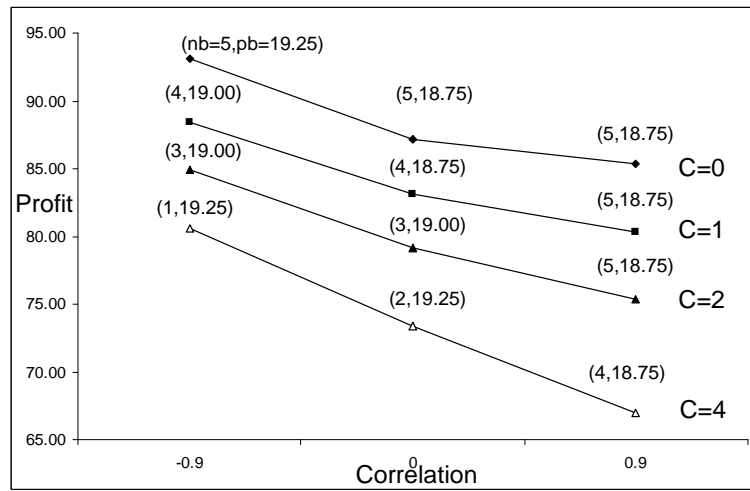


Figure 3.1: Base case - Profit vs. correlation

on the optimal number of bundles formed, the bundle price and the expected profit. The expected profit decreases as the correlation coefficient increases, since the standard deviation of the bundle, σ_b gets larger as the correlation increases. Obviously, increase in bundling cost results in decrease in profit. With negative correlation, bundling cost has a significant impact on the number of bundles formed. However with positive correlation, the solution calls for forming all inventory into the bundle, and only when $c = 4$, one unit of each product is spared for individual purchase. Another finding is that the optimal bundle price is a decreasing function of correlation coefficient.

After explaining the base case and some findings on it we now continue to study the effects of the factors defined previously.

The impact of individual product prices

To investigate the effects of individual product prices, we set product prices equal to each other ($p_1 = p_2$) and have values 8, 10 and 12. The values of 8 and 12 are chosen to represent one standard deviation above and below the mean. The other parameters are same as the base case ($\mu_1 = \mu_2 = 10$, $\sigma_1 = \sigma_2 = 2$, $Q_1 = Q_2 = 5$, $\lambda = 10$, $\rho = -0.9, 0, 0.9$, $c = 0, 1, 2, 4$). The results are tabulated in Table A.1. We provide three figures extracted from Table A.1. Figure 3.2 is the profit versus price graph under the correlation coefficient of -0.9 and Figure 3.3 and Figure 3.4 are the same graphs under the correlation coefficient of 0 and 0.9 respectively.

When the product prices are below the mean value of the reservation price distribution, the optimal bundle price is the maximum of the possible bundle prices which is the sum of individual product prices ($8+8=16$). In this case, the retailer sells individual products as well as the bundle (mixed bundling). When the individual products are high, the retailer charges a bundle price such that only bundles are sold (pure bundling). Maximum expected profit for high product prices is very close to the expected profits for medium prices. However expected profit decrease severely, as the bundling cost increases. Similar to our previous results, increase in correlation coefficient results in decrease in the expected profit value for all price levels and the optimal bundle price is a decreasing function of the correlation coefficient.

The impact of initial inventory level

We now study the effect of initial inventory levels on the expected profit. We set up the model with equal individual product inventory levels $Q_1 = Q_2$ with values 5 and 10. The other parameters are same as the base case ($\mu_1 = \mu_2 = 10$, $\sigma_1 = \sigma_2 = 2$, $p_1 = p_2 = 10$, $\lambda = 10$, $\rho = -0.9, 0, 0.9$, $c = 0, 1, 2, 4$). The results are tabulated in Table A.2.

From this analysis, we see that the expected profit increases with the increasing initial inventory level. This is obvious since we are not considering the cost of purchasing the items. With more inventories, the retailer reduces its bundle price,

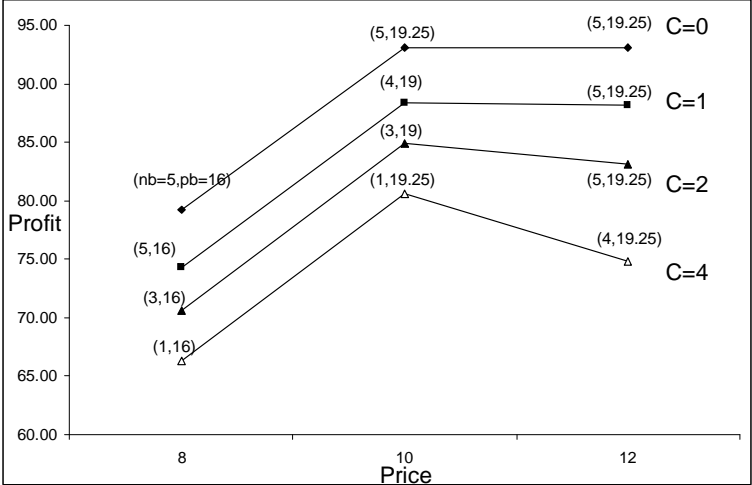


Figure 3.2: Profit vs. product price for $\rho = -0.9$

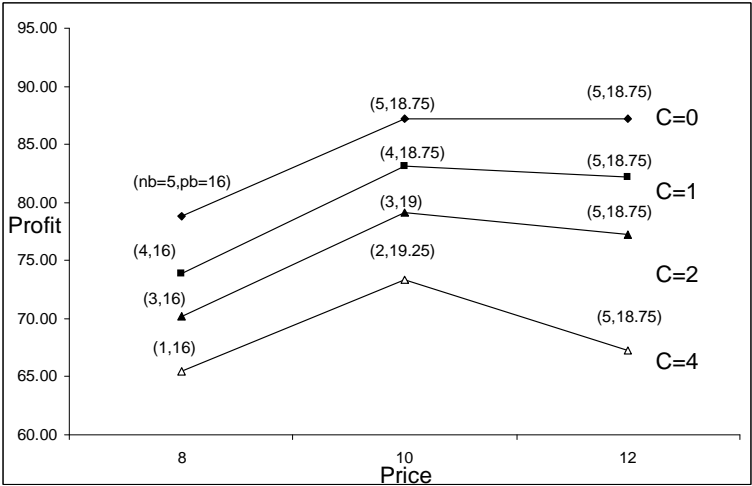
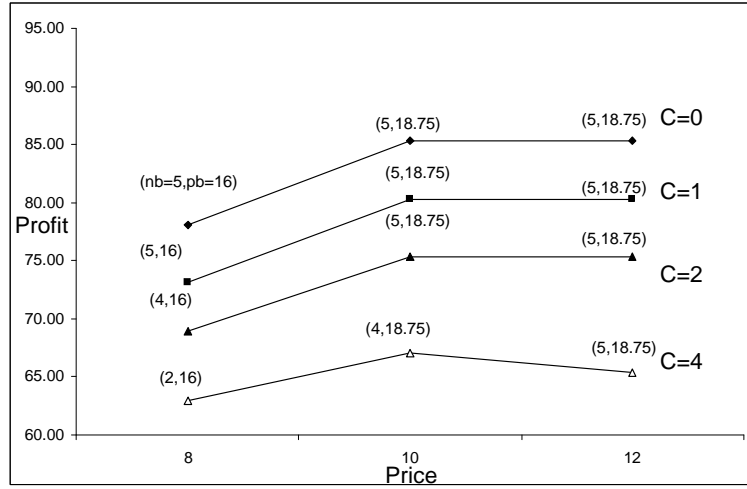


Figure 3.3: Profit vs. product price for $\rho = 0$

Figure 3.4: Profit vs. product price for $\rho = 0.9$

and also offers more bundles for sale as the customer arrival rate is constant. The results are particularly interesting for $\rho = -0.9$. In this case the retailer converts all inventory to the bundles regardless of the bundle formation cost, as bundling is very effective with negative correlation. For high initial inventory levels, the optimal bundle price decreases more rapidly than the case of low initial inventory levels as the correlation coefficient increases.

The impact of variance of reservation price distribution

Another factor we investigate in our study is the variance of the reservation price distribution. We run our model for the equal standard deviation ($\sigma_1 = \sigma_2$) values of 1, 2 and 3 to investigate its effect. The other parameters are same as the base case ($\mu_1 = \mu_2 = 10$, $p_1 = p_2 = 10$, $Q_1 = Q_2 = 5$, $\lambda = 10$, $\rho = -0.9, 0, 0.9$, $c = 0, 1, 2, 4$). The results are tabulated in Table A.3.

The obvious finding of this analysis is that the expected profit decreases as the variance of the reservation price distribution increases. For the same bundling cost value, the difference between the expected profit at $\sigma = 1$ and the expected profit at $\sigma = 3$ increases as the correlation coefficient increases. In other words, the effect of the variance of the reservation price distribution on the expected profit is much more pronounced when there is a positive correlation between the reservation prices. Figures 3.5, 3.6 and 3.7 demonstrate the effect of the standard

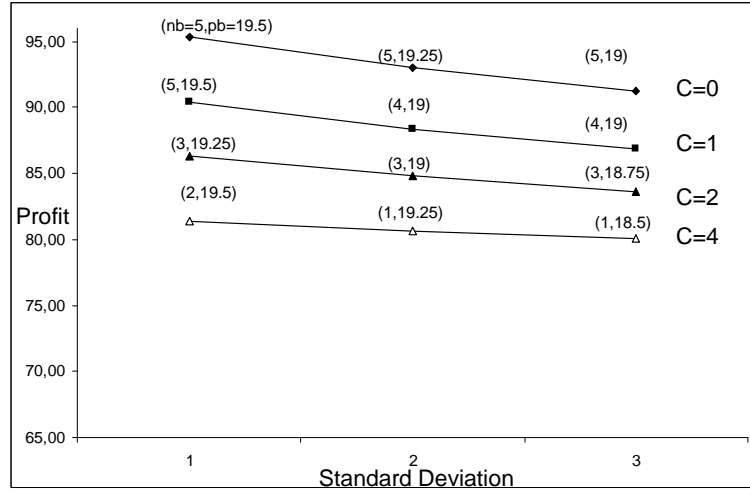


Figure 3.5: Profit vs. standard deviation for $\rho = -0.9$

deviation on the expected profit with correlation coefficient values of -0.9, 0 and 0.9, respectively. Again it confirms the previous findings that the expected profit decreases as the correlation coefficient increases. The optimal number of bundles formed decreases as the standard deviation increases. The optimal bundle price has different behavior with respect to the correlation coefficient. For negative correlation, optimal bundle price decreases as the standard deviation increases. However for positive correlation, the optimal bundle price is an increasing function of the standard deviation. For the zero correlation case, the optimal bundle price is a decreasing function of the standard deviation for small bundling cost values and it is an increasing function for large bundling cost values.

The impact of arrival rate

Finally, we investigate the effect of customers' arrival rate, λ . Arrival rate has three different values (5, 10 and 15) for this setup. All other parameters are set as the same as the base case ($\mu_1 = \mu_2 = 10$, $\sigma_1 = \sigma_2 = 2$, $p_1 = p_2 = 10$, $Q_1 = Q_2 = 5$, $\rho = -0.9, 0, 0.9$, $c = 0, 1, 2, 4$). The results are tabulated in Table A.4.

The clear finding of this case that can be seen from the Figures 3.8, 3.9 and 3.10 is that the expected profit is an increasing function of the arrival rate.

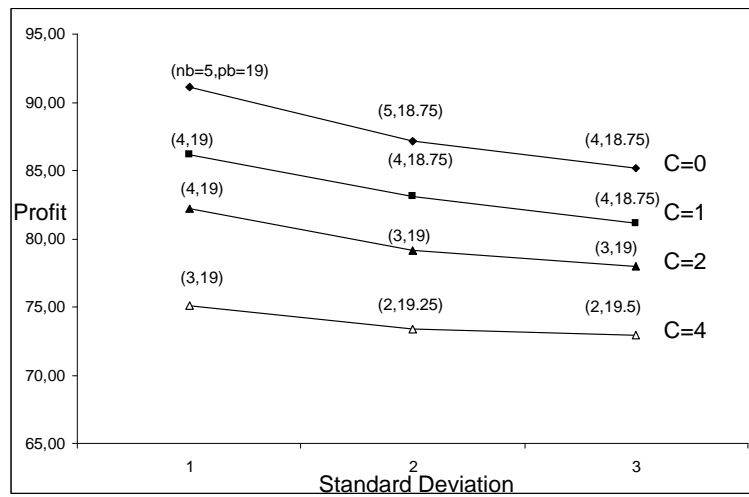


Figure 3.6: Profit vs. standard deviation for $\rho = 0$

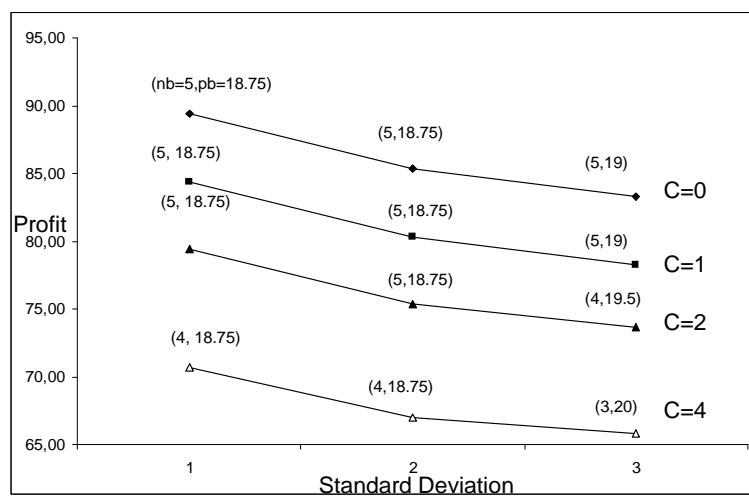
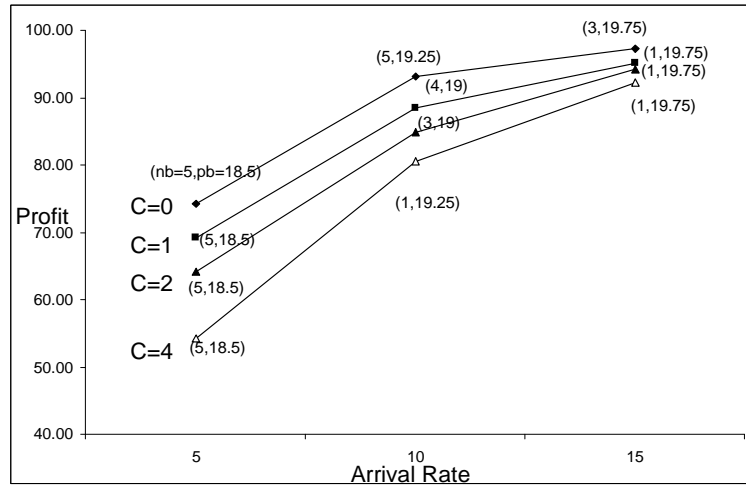


Figure 3.7: Profit vs. standard deviation for $\rho = 0.9$

Figure 3.8: Profit vs. arrival rate for $\rho = -0.9$

The number of bundle sales decreases and the number of individual product sales increases when the arrival rate increases since the need for bundling decreases; the retailer can easily sell its products individually. For the negative correlation case, decrease in the optimal number of bundles formed is more significant. The retailer offers less bundles and charges higher prices for them as the arrival rate increases.

All the analysis so far assumes that the bundle reservation price is strictly additive, i.e., the products are neither substitutable, nor complementary. In the next section we consider the effect of the product substitutability and complementarity.

Results with the degree of contingency

As stated before the products are substitutable when the degree of contingency, $\theta < 0$ (bundle reservation price is subadditive) and complementary when the degree of contingency, $\theta > 0$ (bundle reservation price is superadditive). We investigate the effects of product substitutability and complementarity in this section. Similar to the model setup of the previous sections, we fix the individual product prices p_1 and p_2 , and optimize the number of bundles formed, n_b and the bundle price, p_b . Individual product prices, p_1 and p_2 are set to 10. Reservation price distribution used in the base case has mean value of $\mu_1 = \mu_2 = 10$

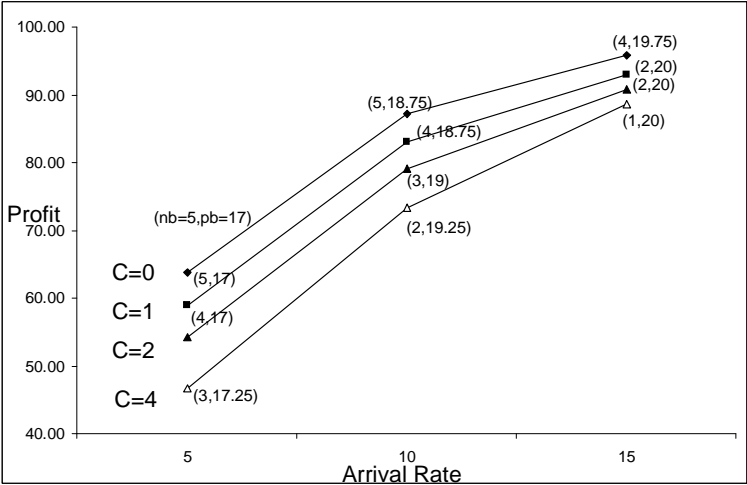


Figure 3.9: Profit vs. arrival rate for $\rho = 0$

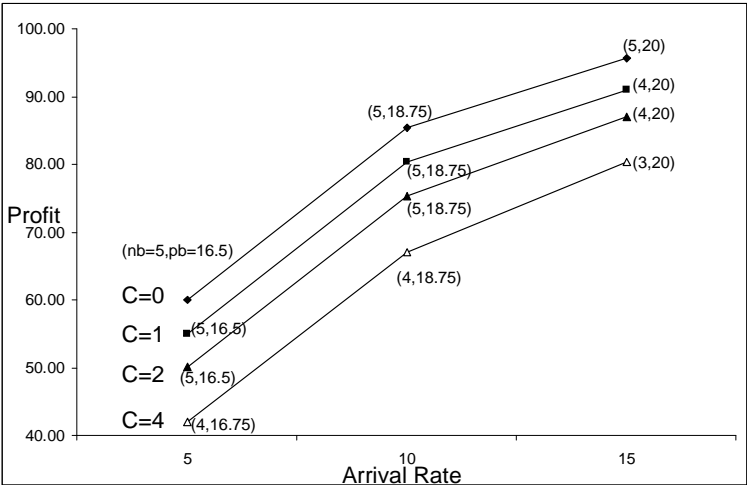


Figure 3.10: Profit vs. arrival rate for $\rho = 0.9$

and standard deviation of $\sigma_1 = \sigma_2 = 2$. Initial inventory levels for individual products are $Q_1 = Q_2 = 5$. Arrival process parameter, λ is set to 10. We search optimal bundle price within the search interval with increments of 0.25. We take unit bundling cost c as 0, 1, 2 and 4. The degree of contingency, θ considered in this section has five different values: -0.1, -0.05, 0, 0.05 and 0.1.

The results for the correlation coefficient, ρ , of -0.9, 0 and 0.9 are tabulated in Tables A.5, A.6 and A.7 respectively. n_b^* is the optimal number of bundles and p_b^* is the optimal bundle price. The fifth column is the expected profit, and the next two columns represent the expected number of products sold. The last three columns represent the arrival rates of the customers that intend to buy each product (ℓ_1, ℓ_2, ℓ_b) and arrival rate of customers that leaves the store without buying any product (ℓ_0) . Figures 3.11, 3.12 and 3.13 depict the same results graphically.

Expected profit is an increasing function of degree of contingency under all correlation coefficient values. For the positive degree of contingency case (super-additive reservation prices), customers' willingness to purchase bundle is higher than the negative degree of contingency case (subadditive reservation prices). Therefore for all correlation values and bundle formation costs, we see that the retailer is forming more bundles, or charging higher prices for the bundle or both as the degree of contingency increases. The highest jump in the profits occur when the degree of contingency increases to 0 from -0.05 which shows that the product substitutability has a significant impact on the efficiency of bundling and pricing. The impact of the degree of contingency is more pronounced when the product prices are uncorrelated and positively correlated. It seems that the retailer is already able to generate significant profits through bundling when the product reservation prices are negatively correlated and the impact of product complementarity is not significant.

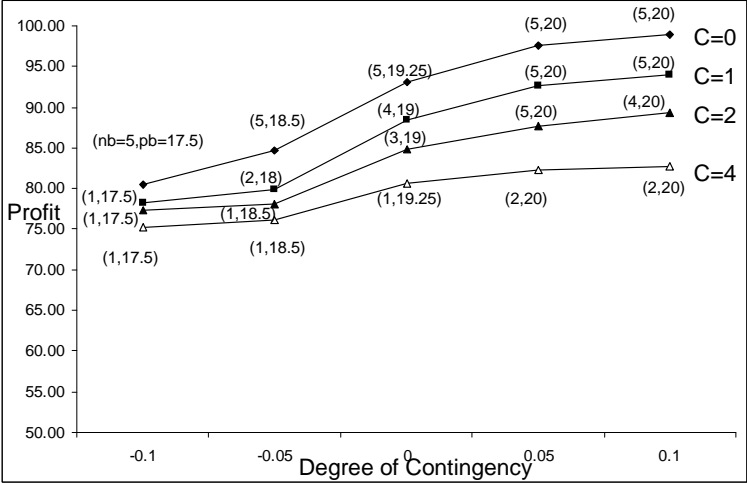


Figure 3.11: Profit vs. degree of contingency for $\rho = -0.9$

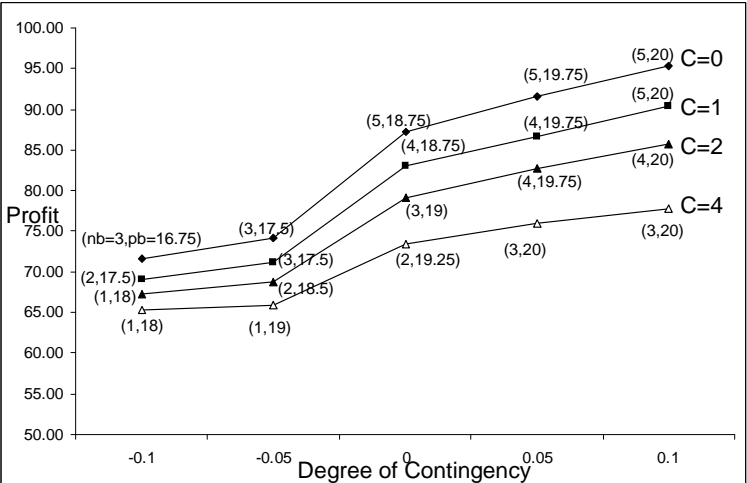


Figure 3.12: Profit vs. degree of contingency for $\rho = 0$

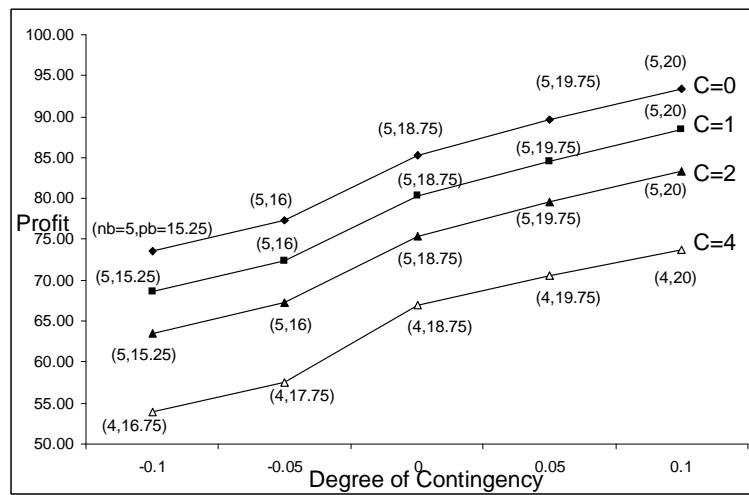


Figure 3.13: Profit vs. degree of contingency for $\rho = 0.9$

Chapter 4

CONCLUSION

In this study, we consider a retail firm that sells two types of perishable products in a single period not only as independent items but also as a bundle (mixed bundling). Our emphasis is on understanding the bundling practices on the inventory and pricing decisions of the firm. We study the bundle formation and pricing problem of two products facing random demand, under inventory constraints over a finite selling horizon. Product prices are set only at the beginning of the selling season and cannot be changed during the period. Although the prices are known by the customers, they do not know the stock amounts of that season before they come to the store. After the retailer decides the number of bundles to be formed at the beginning of the season, no new bundles are formed and none of the bundles are unbundled to offer individual products during the season. Unit bundle formation cost is also included into the model.

Customer arrival to the store follows a Poisson process with a fixed arrival rate. The purchasing behavior of the customers is governed by the reservation prices. Customers decide to buy products by comparing their reservation price with the product price. The reservation price of the bundle is the sum of the reservation prices of individual products that form the bundle. Considering the stock-out situations, the customer either switches to one of the other products or leaves the store without any purchase.

In our numerical study we model the effects of various factors on the bundling and pricing problem. The factors considered are the individual product prices, the correlation between the reservation price distributions, the variance of the reservation price distributions, the initial inventory levels, the unit bundle formation cost, and the intensity of the customer arrivals. We also consider the effects of the degree of product complementarity and substitutability, also known as the degree of contingency.

Through the results of this numerical study from the base case we observe that the expected profit decreases as the correlation coefficient increases and increase in bundling cost results in decrease in profit. With negative correlation, bundling cost has a significant impact on the number of bundles formed. However with positive correlation, this effect is negligible. Another finding is that optimal bundle price is a decreasing function of correlation coefficient.

After the base case analysis, we investigate the effects of other parameters comparing with the base case results. From the individual product prices analysis, we find that increase in correlation coefficient results in decrease in the expected profit value for all price levels and the optimal bundle price is a decreasing function of the correlation coefficient. When the product prices are below the mean value of the reservation price distribution, the optimal bundle price is the maximum of the possible bundle prices which is the sum of individual product prices and in this case, the retailer sells individual products as well as the bundle (mixed bundling). When the individual products are high, the retailer charges bundle price such that only bundles are sold (pure bundling).

The expected profit and the optimal number of bundles formed decreases as the variance of the reservation price distribution increases. The impact of the variance of the reservation price distribution on the expected profit is much more significant when there is a positive correlation between the reservation prices.

When we change the initial inventory levels, we see that when the initial inventory level is high, the retailer reduces bundle price and also offers more bundles for sale as the customer arrival rate is constant. For the high initial inventory level, the optimal bundle price decreases more rapidly than the low

initial inventory level as the correlation coefficient increases.

The expected profit is an increasing function of the arrival rate. The number of bundle sales decreases and the number of individual product sales increases when the arrival rate increases since the need for bundling decreases; the retailer can easily sell its products individually.

Finally we perform analysis to investigate the product substitutability and complementarity. For the positive degree of contingency case (superadditive reservation prices), customers' willingness to purchase bundle is higher than the negative degree of contingency case (subadditive reservation prices). Therefore for all correlation values and bundle formation costs, we see that the retailer is forming more bundles, or charging higher prices for the bundle or both as the degree of contingency increases. As a result, expected profit is an increasing function of degree of contingency.

Lastly, we will give some future research directions related to our study. In our study we include the bundling cost but it is not the only cost in a real life environment, therefore including other cost factors, such as retailer's product purchasing cost from the manufacturer, into the model can give interesting results. In our model our retailer is a monopoly and one interesting and complex extension of our model will be including the other retailers' decisions into the model with a game theoretical approach. Customer's preferences are modeled in our study with the reservation price and product price comparison. But customer demand is also affected by the remaining shelf life of products. Including product remaining shelf life will be an extension. As a future research direction, our single period model can be extended to a multi-period model and allowing product replenishments at the beginning of each period can be interesting for the inventory side of the problem.

Appendix A

TABLES

| $p_1 = p_2$ | ρ | c | n_b^* | p_b^* | $E(Profit)$ | $E(x_1) = E(x_2)$ | $E(x_b)$ | $\ell_1 = \ell_2$ | ℓ_b | ℓ_0 |
|-------------|--------|-----|---------|---------|-------------|-------------------|----------|-------------------|----------|----------|
| 8 | -0.9 | 0 | 5 | 16 | 79.2561 | 0.00 | 2.81 | 1.59 | 6.83 | 0.00 |
| 8 | -0.9 | 1 | 5 | 16 | 74.2561 | 0.00 | 2.81 | 1.59 | 6.83 | 0.00 |
| 8 | -0.9 | 2 | 3 | 16 | 70.5990 | 1.74 | 2.84 | 1.59 | 6.83 | 0.00 |
| 8 | -0.9 | 4 | 1 | 16 | 66.2855 | 3.39 | 1.00 | 1.59 | 6.83 | 0.00 |
| 8 | 0 | 0 | 5 | 16 | 78.7881 | 0.00 | 3.00 | 1.33 | 7.08 | 0.25 |
| 8 | 0 | 1 | 4 | 16 | 73.9089 | 0.81 | 3.44 | 1.33 | 7.08 | 0.25 |
| 8 | 0 | 2 | 3 | 16 | 70.1810 | 1.75 | 2.91 | 1.33 | 7.08 | 0.25 |
| 8 | 0 | 4 | 1 | 16 | 65.5052 | 3.35 | 1.00 | 1.33 | 7.08 | 0.25 |
| 8 | 0.9 | 0 | 5 | 16 | 78.0908 | 0.00 | 4.38 | 0.43 | 7.98 | 1.15 |
| 8 | 0.9 | 1 | 5 | 16 | 73.0908 | 0.00 | 4.38 | 0.43 | 7.98 | 1.15 |
| 8 | 0.9 | 2 | 4 | 16 | 68.9604 | 0.86 | 3.92 | 0.43 | 7.98 | 1.15 |
| 8 | 0.9 | 4 | 2 | 16 | 62.9023 | 2.44 | 2.00 | 0.43 | 7.98 | 1.15 |
| 10 | -0.9 | 0 | 5 | 19.25 | 93.0645 | 0.00 | 4.52 | 3.34 | 2.82 | 0.50 |
| 10 | -0.9 | 1 | 4 | 19 | 88.3828 | 0.76 | 3.07 | 2.97 | 3.68 | 0.38 |
| 10 | -0.9 | 2 | 3 | 19 | 84.8695 | 1.50 | 2.37 | 2.97 | 3.68 | 0.38 |
| 10 | -0.9 | 4 | 1 | 19.25 | 80.6290 | 3.12 | 0.92 | 3.34 | 2.82 | 0.50 |
| 10 | 0 | 0 | 5 | 18.75 | 87.1901 | 0.00 | 3.55 | 1.33 | 5.13 | 2.21 |
| 10 | 0 | 1 | 4 | 18.75 | 83.0791 | 0.77 | 3.39 | 1.33 | 5.13 | 2.21 |
| 10 | 0 | 2 | 3 | 19 | 79.1586 | 1.53 | 2.73 | 1.54 | 4.61 | 2.31 |
| 10 | 0 | 4 | 2 | 19.25 | 73.3638 | 2.21 | 1.89 | 1.77 | 4.07 | 2.39 |
| 10 | 0.9 | 0 | 5 | 18.75 | 85.3171 | 0.00 | 4.54 | 0.06 | 6.19 | 3.68 |
| 10 | 0.9 | 1 | 5 | 18.75 | 80.3171 | 0.00 | 4.54 | 0.06 | 6.19 | 3.68 |
| 10 | 0.9 | 2 | 5 | 18.75 | 75.3171 | 0.00 | 4.54 | 0.06 | 6.19 | 3.68 |
| 10 | 0.9 | 4 | 4 | 18.75 | 67.0220 | 0.59 | 3.79 | 0.06 | 6.19 | 3.68 |
| 12 | -0.9 | 0 | 5 | 19.25 | 93.1312 | 0.00 | 3.89 | 0.75 | 6.86 | 1.65 |
| 12 | -0.9 | 1 | 5 | 19.25 | 88.1312 | 0.00 | 3.89 | 0.75 | 6.86 | 1.65 |
| 12 | -0.9 | 2 | 5 | 19.25 | 83.1312 | 0.00 | 3.89 | 0.75 | 6.86 | 1.65 |
| 12 | -0.9 | 4 | 4 | 19.25 | 74.8394 | 0.64 | 3.79 | 0.75 | 6.86 | 1.65 |
| 12 | 0 | 0 | 5 | 18.75 | 87.2226 | 0.00 | 4.63 | 0.08 | 6.61 | 3.22 |
| 12 | 0 | 1 | 5 | 18.75 | 82.2226 | 0.00 | 4.63 | 0.08 | 6.61 | 3.22 |
| 12 | 0 | 2 | 5 | 18.75 | 77.2226 | 0.00 | 4.63 | 0.08 | 6.61 | 3.22 |
| 12 | 0 | 4 | 5 | 18.75 | 67.2226 | 0.00 | 4.63 | 0.08 | 6.61 | 3.22 |
| 12 | 0.9 | 0 | 5 | 18.75 | 85.3275 | 0.00 | 4.55 | 0.00 | 6.26 | 3.74 |
| 12 | 0.9 | 1 | 5 | 18.75 | 80.3275 | 0.00 | 4.55 | 0.00 | 6.26 | 3.74 |
| 12 | 0.9 | 2 | 5 | 18.75 | 75.3275 | 0.00 | 4.55 | 0.00 | 6.26 | 3.74 |
| 12 | 0.9 | 4 | 5 | 18.75 | 65.3275 | 0.00 | 4.55 | 0.00 | 6.26 | 3.74 |

$$\mu_1 = \mu_2 = 10, \sigma_1 = \sigma_2 = 2, \theta = 0, Q_1 = Q_2 = 5, \lambda = 10$$

Table A.1: The impact of product prices on the expected profit

| $Q_1 = Q_2$ | ρ | c | n_b^* | p_b^* | $E(Profit)$ | $E(x_1) = E(x_2)$ | $E(x_b)$ | $\ell_1 = \ell_2$ | ℓ_b | ℓ_0 |
|-------------|--------|-----|---------|---------|-------------|-------------------|----------|-------------------|----------|----------|
| 5 | -0,9 | 0 | 5 | 19,25 | 93,0645 | 0,00 | 4,52 | 3,34 | 2,82 | 0,50 |
| 5 | -0,9 | 1 | 4 | 19 | 88,3828 | 0,76 | 3,07 | 2,97 | 3,68 | 0,38 |
| 5 | -0,9 | 2 | 3 | 19 | 84,8695 | 1,50 | 2,37 | 2,97 | 3,68 | 0,38 |
| 5 | -0,9 | 4 | 1 | 19,25 | 80,6290 | 3,12 | 0,92 | 3,34 | 2,82 | 0,50 |
| 5 | 0 | 0 | 5 | 18,75 | 87,1901 | 0,00 | 3,55 | 1,33 | 5,13 | 2,21 |
| 5 | 0 | 1 | 4 | 18,75 | 83,0791 | 0,77 | 3,39 | 1,33 | 5,13 | 2,21 |
| 5 | 0 | 2 | 3 | 19 | 79,1586 | 1,53 | 2,73 | 1,54 | 4,61 | 2,31 |
| 5 | 0 | 4 | 2 | 19,25 | 73,3638 | 2,21 | 1,89 | 1,77 | 4,07 | 2,39 |
| 5 | 0,9 | 0 | 5 | 18,75 | 85,3171 | 0,00 | 4,54 | 0,06 | 6,19 | 3,68 |
| 5 | 0,9 | 1 | 5 | 18,75 | 80,3171 | 0,00 | 4,54 | 0,06 | 6,19 | 3,68 |
| 5 | 0,9 | 2 | 5 | 18,75 | 75,3171 | 0,00 | 4,54 | 0,06 | 6,19 | 3,68 |
| 5 | 0,9 | 4 | 4 | 18,75 | 67,0220 | 0,59 | 3,79 | 0,06 | 6,19 | 3,68 |
| 10 | -0,9 | 0 | 10 | 18,5 | 157,3871 | 0,00 | 8,34 | 2,23 | 5,36 | 0,17 |
| 10 | -0,9 | 1 | 10 | 18,5 | 147,3871 | 0,00 | 8,34 | 2,23 | 5,36 | 0,17 |
| 10 | -0,9 | 2 | 10 | 18,5 | 137,3871 | 0,00 | 8,34 | 2,23 | 5,36 | 0,17 |
| 10 | -0,9 | 4 | 10 | 18,5 | 117,3871 | 0,00 | 8,34 | 2,23 | 5,36 | 0,17 |
| 10 | 0 | 0 | 10 | 16,75 | 135,1856 | 0,00 | 7,95 | 0,26 | 8,38 | 1,10 |
| 10 | 0 | 1 | 10 | 16,75 | 125,1856 | 0,00 | 7,95 | 0,26 | 8,38 | 1,10 |
| 10 | 0 | 2 | 9 | 16,75 | 116,1115 | 0,40 | 7,52 | 0,26 | 8,38 | 1,10 |
| 10 | 0 | 4 | 8 | 17 | 99,7217 | 0,70 | 6,93 | 0,33 | 8,10 | 1,24 |
| 10 | 0,9 | 0 | 10 | 16,5 | 126,7549 | 0,00 | 7,68 | 0,00 | 8,15 | 1,85 |
| 10 | 0,9 | 1 | 10 | 16,5 | 116,7549 | 0,00 | 7,68 | 0,00 | 8,15 | 1,85 |
| 10 | 0,9 | 2 | 9 | 16,5 | 107,3775 | 0,18 | 7,38 | 0,00 | 8,15 | 1,85 |
| 10 | 0,9 | 4 | 8 | 16,5 | 90,1415 | 0,37 | 6,95 | 0,00 | 8,15 | 1,85 |

$$p_1 = p_2 = 10, \mu_1 = \mu_2 = 10, \sigma_1 = \sigma_2 = 2, \theta = 0, \lambda = 10$$

Table A.2: The impact of initial inventory levels on the expected profit

| $\sigma_1 = \sigma_2$ | ρ | c | n_b^* | p_b^* | $E(Profit)$ | $E(x_1) = E(x_2)$ | $E(x_b)$ | $\ell_1 = \ell_2$ | ℓ_b | ℓ_0 |
|-----------------------|--------|-----|---------|---------|-------------|-------------------|----------|-------------------|----------|----------|
| 1 | -0.9 | 0 | 5 | 19.5 | 95.3845 | 0.00 | 4.14 | 2.97 | 3.68 | 0.38 |
| 1 | -0.9 | 1 | 5 | 19.5 | 90.3845 | 0.00 | 4.14 | 2.97 | 3.68 | 0.38 |
| 1 | -0.9 | 2 | 3 | 19.25 | 86.3451 | 1.56 | 2.59 | 2.23 | 5.36 | 0.17 |
| 1 | -0.9 | 4 | 2 | 19.5 | 81.3399 | 2.34 | 1.75 | 2.97 | 3.68 | 0.38 |
| 1 | 0 | 0 | 5 | 19 | 91.0916 | 0.00 | 3.83 | 0.79 | 6.59 | 1.82 |
| 1 | 0 | 1 | 4 | 19 | 86.1757 | 0.81 | 3.76 | 0.79 | 6.59 | 1.82 |
| 1 | 0 | 2 | 4 | 19 | 82.1757 | 0.81 | 3.76 | 0.79 | 6.59 | 1.82 |
| 1 | 0 | 4 | 3 | 19 | 75.1321 | 1.55 | 2.94 | 0.79 | 6.59 | 1.82 |
| 1 | 0.9 | 0 | 5 | 18.75 | 89.4106 | 0.00 | 4.77 | 0.00 | 7.39 | 2.61 |
| 1 | 0.9 | 1 | 5 | 18.75 | 84.4106 | 0.00 | 4.77 | 0.00 | 7.39 | 2.61 |
| 1 | 0.9 | 2 | 5 | 18.75 | 79.4106 | 0.00 | 4.77 | 0.00 | 7.39 | 2.61 |
| 1 | 0.9 | 4 | 4 | 18.75 | 70.6432 | 0.67 | 3.91 | 0.00 | 7.39 | 2.61 |
| 2 | -0.9 | 0 | 5 | 19.25 | 93.0645 | 0.00 | 4.52 | 3.34 | 2.82 | 0.50 |
| 2 | -0.9 | 1 | 4 | 19 | 88.3828 | 0.76 | 3.07 | 2.97 | 3.68 | 0.38 |
| 2 | -0.9 | 2 | 3 | 19 | 84.8695 | 1.50 | 2.37 | 2.97 | 3.68 | 0.38 |
| 2 | -0.9 | 4 | 1 | 19.25 | 80.6290 | 3.12 | 0.92 | 3.34 | 2.82 | 0.50 |
| 2 | 0 | 0 | 5 | 18.75 | 87.1901 | 0.00 | 3.55 | 1.33 | 5.13 | 2.21 |
| 2 | 0 | 1 | 4 | 18.75 | 83.0791 | 0.77 | 3.39 | 1.33 | 5.13 | 2.21 |
| 2 | 0 | 2 | 3 | 19 | 79.1586 | 1.53 | 2.73 | 1.54 | 4.61 | 2.31 |
| 2 | 0 | 4 | 2 | 19.25 | 73.3638 | 2.21 | 1.89 | 1.77 | 4.07 | 2.39 |
| 2 | 0.9 | 0 | 5 | 18.75 | 85.3171 | 0.00 | 4.54 | 0.06 | 6.19 | 3.68 |
| 2 | 0.9 | 1 | 5 | 18.75 | 80.3171 | 0.00 | 4.54 | 0.06 | 6.19 | 3.68 |
| 2 | 0.9 | 2 | 5 | 18.75 | 75.3171 | 0.00 | 4.54 | 0.06 | 6.19 | 3.68 |
| 2 | 0.9 | 4 | 4 | 18.75 | 67.0220 | 0.59 | 3.79 | 0.06 | 6.19 | 3.68 |
| 3 | -0.9 | 0 | 5 | 19 | 91.2895 | 0.00 | 4.59 | 3.46 | 2.54 | 0.54 |
| 3 | -0.9 | 1 | 4 | 19 | 86.8313 | 0.89 | 3.42 | 3.46 | 2.54 | 0.54 |
| 3 | -0.9 | 2 | 3 | 18.75 | 83.6536 | 1.54 | 2.38 | 3.22 | 3.10 | 0.46 |
| 3 | -0.9 | 4 | 1 | 18.5 | 80.0610 | 3.18 | 0.95 | 2.97 | 3.68 | 0.38 |
| 3 | 0 | 0 | 4 | 18.75 | 85.1352 | 0.77 | 3.18 | 1.69 | 4.25 | 2.37 |
| 3 | 0 | 1 | 4 | 18.75 | 81.1352 | 0.77 | 3.18 | 1.69 | 4.25 | 2.37 |
| 3 | 0 | 2 | 3 | 19 | 77.9490 | 1.52 | 2.59 | 1.85 | 3.89 | 2.41 |
| 3 | 0 | 4 | 2 | 19.5 | 72.9087 | 2.21 | 1.79 | 2.17 | 3.18 | 2.48 |
| 3 | 0.9 | 0 | 5 | 19 | 83.2592 | 0.00 | 4.28 | 0.24 | 5.44 | 4.09 |
| 3 | 0.9 | 1 | 5 | 19 | 78.2592 | 0.00 | 4.28 | 0.24 | 5.44 | 4.09 |
| 3 | 0.9 | 2 | 4 | 19.5 | 73.6754 | 0.61 | 3.55 | 0.43 | 4.90 | 4.23 |
| 3 | 0.9 | 4 | 3 | 20 | 65.8112 | 1.17 | 2.72 | 0.72 | 4.28 | 4.28 |

$$p_1 = p_2 = 10, \mu_1 = \mu_2 = 10, \theta = 0, Q_1 = Q_2 = 5, \lambda = 10$$

Table A.3: The impact of variance of reservation price distributions on the expected profit

| λ | ρ | c | n_b^* | p_b^* | $E(Profit)$ | $E(x_1) = E(x_2)$ | $E(x_b)$ | $\ell_1 = \ell_2$ | ℓ_b | ℓ_0 |
|-----------|--------|-----|---------|---------|-------------|-------------------|----------|-------------------|----------|----------|
| 5 | -0.9 | 0 | 5 | 18.5 | 74.2297 | 0.00 | 3.89 | 1.12 | 2.68 | 0.09 |
| 5 | -0.9 | 1 | 5 | 18.5 | 69.2297 | 0.00 | 3.89 | 1.12 | 2.68 | 0.09 |
| 5 | -0.9 | 2 | 5 | 18.5 | 64.2297 | 0.00 | 3.89 | 1.12 | 2.68 | 0.09 |
| 5 | -0.9 | 4 | 5 | 18.5 | 54.2297 | 0.00 | 3.89 | 1.12 | 2.68 | 0.09 |
| 5 | 0 | 0 | 5 | 17 | 63.8536 | 0.00 | 3.73 | 0.17 | 4.05 | 0.62 |
| 5 | 0 | 1 | 5 | 17 | 58.8536 | 0.00 | 3.73 | 0.17 | 4.05 | 0.62 |
| 5 | 0 | 2 | 4 | 17 | 54.2706 | 0.35 | 3.25 | 0.17 | 4.05 | 0.62 |
| 5 | 0 | 4 | 3 | 17.25 | 46.5785 | 0.67 | 2.62 | 0.21 | 3.89 | 0.69 |
| 5 | 0.9 | 0 | 5 | 16.5 | 60.0115 | 0.00 | 3.64 | 0.00 | 4.08 | 0.92 |
| 5 | 0.9 | 1 | 5 | 16.5 | 55.0115 | 0.00 | 3.64 | 0.00 | 4.08 | 0.92 |
| 5 | 0.9 | 2 | 5 | 16.5 | 50.0115 | 0.00 | 3.64 | 0.00 | 4.08 | 0.92 |
| 5 | 0.9 | 4 | 4 | 16.75 | 41.9065 | 0.20 | 3.21 | 0.00 | 3.99 | 1.01 |
| 10 | -0.9 | 0 | 5 | 19.25 | 93.0645 | 0.00 | 4.52 | 3.34 | 2.82 | 0.50 |
| 10 | -0.9 | 1 | 4 | 19 | 88.3828 | 0.76 | 3.07 | 2.97 | 3.68 | 0.38 |
| 10 | -0.9 | 2 | 3 | 19 | 84.8695 | 1.50 | 2.37 | 2.97 | 3.68 | 0.38 |
| 10 | -0.9 | 4 | 1 | 19.25 | 80.6290 | 3.12 | 0.92 | 3.34 | 2.82 | 0.50 |
| 10 | 0 | 0 | 5 | 18.75 | 87.1901 | 0.00 | 3.55 | 1.33 | 5.13 | 2.21 |
| 10 | 0 | 1 | 4 | 18.75 | 83.0791 | 0.77 | 3.39 | 1.33 | 5.13 | 2.21 |
| 10 | 0 | 2 | 3 | 19 | 79.1586 | 1.53 | 2.73 | 1.54 | 4.61 | 2.31 |
| 10 | 0 | 4 | 2 | 19.25 | 73.3638 | 2.21 | 1.89 | 1.77 | 4.07 | 2.39 |
| 10 | 0.9 | 0 | 5 | 18.75 | 85.3171 | 0.00 | 4.54 | 0.06 | 6.19 | 3.68 |
| 10 | 0.9 | 1 | 5 | 18.75 | 80.3171 | 0.00 | 4.54 | 0.06 | 6.19 | 3.68 |
| 10 | 0.9 | 2 | 5 | 18.75 | 75.3171 | 0.00 | 4.54 | 0.06 | 6.19 | 3.68 |
| 10 | 0.9 | 4 | 4 | 18.75 | 67.0220 | 0.59 | 3.79 | 0.06 | 6.19 | 3.68 |
| 15 | -0.9 | 0 | 3 | 19.75 | 97.3646 | 1.75 | 2.58 | 6.01 | 1.94 | 1.03 |
| 15 | -0.9 | 1 | 1 | 19.75 | 95.2020 | 2.66 | 0.68 | 6.01 | 1.94 | 1.03 |
| 15 | -0.9 | 2 | 1 | 19.75 | 94.2020 | 2.66 | 0.68 | 6.01 | 1.94 | 1.03 |
| 15 | -0.9 | 4 | 1 | 19.75 | 92.2020 | 2.66 | 0.68 | 6.01 | 1.94 | 1.03 |
| 15 | 0 | 0 | 4 | 19.75 | 95.9375 | 0.64 | 2.50 | 3.38 | 4.51 | 3.73 |
| 15 | 0 | 1 | 2 | 20 | 92.8984 | 2.23 | 1.55 | 3.75 | 3.75 | 3.75 |
| 15 | 0 | 2 | 2 | 20 | 90.8984 | 2.23 | 1.55 | 3.75 | 3.75 | 3.75 |
| 15 | 0 | 4 | 1 | 20 | 88.6177 | 3.28 | 0.89 | 3.75 | 3.75 | 3.75 |
| 15 | 0.9 | 0 | 5 | 20 | 95.6171 | 0.00 | 3.43 | 1.08 | 6.42 | 6.42 |
| 15 | 0.9 | 1 | 4 | 20 | 90.9569 | 0.82 | 3.60 | 1.08 | 6.42 | 6.42 |
| 15 | 0.9 | 2 | 4 | 20 | 86.9569 | 0.82 | 3.60 | 1.08 | 6.42 | 6.42 |
| 15 | 0.9 | 4 | 3 | 20 | 80.2742 | 1.65 | 2.92 | 1.08 | 6.42 | 6.42 |

$$p_1 = p_2 = 10, \mu_1 = \mu_2 = 10, \sigma_1 = \sigma_2 = 2, \theta = 0, Q_1 = Q_2 = 5$$

Table A.4: The impact of arrival rate on the expected profit

| c | θ | n_b^* | p_b^* | $E(Profit)$ | $E(x_1) = E(x_2)$ | $E(x_b)$ | $\ell_1 = \ell_2$ | ℓ_b | ℓ_0 |
|-----|----------|---------|---------|-------------|-------------------|----------|-------------------|----------|----------|
| 0 | -0.1 | 5 | 17.5 | 80.4776 | 0.00 | 4.53 | 3.74 | 1.93 | 0.59 |
| 0 | -0.05 | 5 | 18.5 | 84.7307 | 0.00 | 4.51 | 3.72 | 1.97 | 0.60 |
| 0 | 0 | 5 | 19.25 | 93.0645 | 0.00 | 4.52 | 3.34 | 2.82 | 0.50 |
| 0 | 0.05 | 5 | 20 | 97.6682 | 0.00 | 4.13 | 2.96 | 3.68 | 0.40 |
| 0 | 0.1 | 5 | 20 | 98.9330 | 0.00 | 2.84 | 1.60 | 6.71 | 0.09 |
| 1 | -0.1 | 1 | 17.5 | 78.2782 | 3.09 | 0.75 | 3.74 | 1.93 | 0.59 |
| 1 | -0.05 | 2 | 18 | 79.9001 | 2.33 | 1.59 | 2.98 | 3.69 | 0.35 |
| 1 | 0 | 4 | 19 | 88.3828 | 0.76 | 3.07 | 2.97 | 3.68 | 0.38 |
| 1 | 0.05 | 5 | 20 | 92.6682 | 0.00 | 4.13 | 2.96 | 3.68 | 0.40 |
| 1 | 0.1 | 5 | 20 | 93.9330 | 0.00 | 2.84 | 1.60 | 6.71 | 0.09 |
| 2 | -0.1 | 1 | 17.5 | 77.2782 | 3.09 | 0.75 | 3.74 | 1.93 | 0.59 |
| 2 | -0.05 | 1 | 18.5 | 78.0645 | 3.09 | 0.76 | 3.72 | 1.97 | 0.60 |
| 2 | 0 | 3 | 19 | 84.8695 | 1.50 | 2.37 | 2.97 | 3.68 | 0.38 |
| 2 | 0.05 | 5 | 20 | 87.6682 | 0.00 | 4.13 | 2.96 | 3.68 | 0.40 |
| 2 | 0.1 | 4 | 20 | 89.3489 | 0.75 | 3.22 | 1.60 | 6.71 | 0.09 |
| 4 | -0.1 | 1 | 17.5 | 75.2782 | 3.09 | 0.75 | 3.74 | 1.93 | 0.59 |
| 4 | -0.05 | 1 | 18.5 | 76.0645 | 3.09 | 0.76 | 3.72 | 1.97 | 0.60 |
| 4 | 0 | 1 | 19.25 | 80.6290 | 3.12 | 0.92 | 3.34 | 2.82 | 0.50 |
| 4 | 0.05 | 2 | 20 | 82.2348 | 2.34 | 1.76 | 2.96 | 3.68 | 0.40 |
| 4 | 0.1 | 2 | 20 | 82.7798 | 2.53 | 1.98 | 1.60 | 6.71 | 0.09 |

$$p_1 = p_2 = 10, \mu_1 = \mu_2 = 10, \sigma_1 = \sigma_2 = 2, Q_1 = Q_2 = 5, \lambda = 10$$

Table A.5: The impact of degree of contingency on the expected profit for $\rho = -0.9$

| c | θ | n_b^* | p_b^* | $E(Profit)$ | $E(x_1) = E(x_2)$ | $E(x_b)$ | $\ell_1 = \ell_2$ | ℓ_b | ℓ_0 |
|-----|----------|---------|---------|-------------|-------------------|----------|-------------------|----------|----------|
| 0 | -0.1 | 3 | 16.75 | 71.6768 | 1.54 | 2.55 | 1.36 | 5.13 | 2.15 |
| 0 | -0.05 | 3 | 17.5 | 74.2086 | 1.55 | 2.71 | 1.14 | 5.67 | 2.05 |
| 0 | 0 | 5 | 18.75 | 87.1901 | 0.00 | 3.55 | 1.33 | 5.13 | 2.21 |
| 0 | 0.05 | 5 | 19.75 | 91.5326 | 0.00 | 3.54 | 1.32 | 5.13 | 2.23 |
| 0 | 0.1 | 5 | 20 | 95.3458 | 0.00 | 3.80 | 0.82 | 6.44 | 1.92 |
| 1 | -0.1 | 2 | 17.5 | 69.0528 | 2.21 | 1.61 | 2.13 | 3.31 | 2.44 |
| 1 | -0.05 | 3 | 17.5 | 71.2086 | 1.55 | 2.71 | 1.14 | 5.67 | 2.05 |
| 1 | 0 | 4 | 18.75 | 83.0791 | 0.77 | 3.39 | 1.33 | 5.13 | 2.21 |
| 1 | 0.05 | 4 | 19.75 | 86.6642 | 0.77 | 3.39 | 1.32 | 5.13 | 2.23 |
| 1 | 0.1 | 5 | 20 | 90.3458 | 0.00 | 3.80 | 0.82 | 6.44 | 1.92 |
| 2 | -0.1 | 1 | 18 | 67.2864 | 2.85 | 0.82 | 2.68 | 2.15 | 2.50 |
| 2 | -0.05 | 2 | 18.5 | 68.7396 | 2.21 | 1.65 | 2.06 | 3.43 | 2.45 |
| 2 | 0 | 3 | 19 | 79.1586 | 1.53 | 2.73 | 1.54 | 4.61 | 2.31 |
| 2 | 0.05 | 4 | 19.75 | 82.6642 | 0.77 | 3.39 | 1.32 | 5.13 | 2.23 |
| 2 | 0.1 | 4 | 20 | 85.7052 | 0.80 | 3.73 | 0.82 | 6.44 | 1.92 |
| 4 | -0.1 | 1 | 18 | 65.2864 | 2.85 | 0.82 | 2.68 | 2.15 | 2.50 |
| 4 | -0.05 | 1 | 19 | 65.8967 | 2.84 | 0.85 | 2.58 | 2.33 | 2.50 |
| 4 | 0 | 2 | 19.25 | 73.3638 | 2.21 | 1.89 | 1.77 | 4.07 | 2.39 |
| 4 | 0.05 | 3 | 20 | 75.9234 | 1.53 | 2.74 | 1.52 | 4.64 | 2.33 |
| 4 | 0.1 | 3 | 20 | 77.8449 | 1.55 | 2.94 | 0.82 | 6.44 | 1.92 |

$$p_1 = p_2 = 10, \mu_1 = \mu_2 = 10, \sigma_1 = \sigma_2 = 2, Q_1 = Q_2 = 5, \lambda = 10$$

Table A.6: The impact of degree of contingency on the expected profit for $\rho = 0$

| c | θ | n_b^* | p_b^* | $E(Profit)$ | $E(x_1) = E(x_2)$ | $E(x_b)$ | $\ell_1 = \ell_2$ | ℓ_b | ℓ_0 |
|-----|----------|---------|---------|-------------|-------------------|----------|-------------------|----------|----------|
| 0 | -0.1 | 5 | 15.25 | 73.5533 | 0.00 | 4.82 | 0.00 | 7.83 | 2.17 |
| 0 | -0.05 | 5 | 16 | 77.2984 | 0.00 | 4.83 | 0.00 | 7.91 | 2.09 |
| 0 | 0 | 5 | 18.75 | 85.3171 | 0.00 | 4.54 | 0.06 | 6.19 | 3.68 |
| 0 | 0.05 | 5 | 19.75 | 89.5730 | 0.00 | 4.52 | 0.07 | 6.13 | 3.73 |
| 0 | 0.1 | 5 | 20 | 93.3858 | 0.00 | 4.67 | 0.01 | 6.78 | 3.19 |
| 1 | -0.1 | 5 | 15.25 | 68.5533 | 0.00 | 4.82 | 0.00 | 7.83 | 2.17 |
| 1 | -0.05 | 5 | 16 | 72.2984 | 0.00 | 4.83 | 0.00 | 7.91 | 2.09 |
| 1 | 0 | 5 | 18.75 | 80.3171 | 0.00 | 4.54 | 0.06 | 6.19 | 3.68 |
| 1 | 0.05 | 5 | 19.75 | 84.5730 | 0.00 | 4.52 | 0.07 | 6.13 | 3.73 |
| 1 | 0.1 | 5 | 20 | 88.3858 | 0.00 | 4.67 | 0.01 | 6.78 | 3.19 |
| 2 | -0.1 | 5 | 15.25 | 63.5533 | 0.00 | 4.82 | 0.00 | 7.83 | 2.17 |
| 2 | -0.05 | 5 | 16 | 67.2984 | 0.00 | 4.83 | 0.00 | 7.91 | 2.09 |
| 2 | 0 | 5 | 18.75 | 75.3171 | 0.00 | 4.54 | 0.06 | 6.19 | 3.68 |
| 2 | 0.05 | 5 | 19.75 | 79.5730 | 0.00 | 4.52 | 0.07 | 6.13 | 3.73 |
| 2 | 0.1 | 5 | 20 | 83.3858 | 0.00 | 4.67 | 0.01 | 6.78 | 3.19 |
| 4 | -0.1 | 4 | 16.75 | 53.9068 | 0.60 | 3.78 | 0.05 | 6.33 | 3.57 |
| 4 | -0.05 | 4 | 17.75 | 57.4697 | 0.60 | 3.77 | 0.06 | 6.26 | 3.63 |
| 4 | 0 | 4 | 18.75 | 67.0220 | 0.59 | 3.79 | 0.06 | 6.19 | 3.68 |
| 4 | 0.05 | 4 | 19.75 | 70.5879 | 0.59 | 3.79 | 0.07 | 6.13 | 3.73 |
| 4 | 0.1 | 4 | 20 | 73.7592 | 0.63 | 3.86 | 0.01 | 6.78 | 3.19 |

$p_1 = p_2 = 10, \mu_1 = \mu_2 = 10, \sigma_1 = \sigma_2 = 2, Q_1 = Q_2 = 5, \lambda = 10$

Table A.7: The impact of degree of contingency on the expected profit for $\rho = 0.9$

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