THEOREMS ON DOUBLE LARGE ECONOMIES AND ON THE INTEGRAL OF BANACH SPACE VALUED CORRESPONDENCES

A Master's Thesis

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by

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Economics.

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ABSTRACT

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September 2004

In this study we analyze Pareto optimal and core allocations of an exchange economy containing a Banach space of commodities and a measure space of traders. We show that in such an economy *E*, if a coalition *C* blocks an allocation, then a sufficiently small perturbation of *C* will also block the allocation. It is also shown that the Pareto set and the core of *E* are closed subsets of the Banach space of all integrable mappings of the consumer space into the commodity space. Provided that the commodity space of *E* is separable, we give a strengthening of this result by considering a particular form of convergence of a sequence of economies. To obtain these theorems on double large economies we establish several results related to the integral of B-space valued correspondences.

Keywords: Infinite Dimensional Economy, Large Economy, Core, Pareto Set, Bochner Integral, Correspondence

ÖZET

İKİL BÜYÜK EKONOMİLER VE BANACH UZAYINDA ÇOKDEĞERLİ FONKSİYONLARIN İNTEGRALİ HAKKINDA SAVLAR

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İktisat Bölümü

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Bu çalışmada bir Banach emtia uzayı ve bir ölçüm tüketici uzayını içeren değişim ekonomilerinin Pareto optimal ve çekirdek dağılımlarını inceliyoruz. Böyle bir *E* ekonomisinde, bir *C* koalisyonunun bir dağılımı bloke etmesi durumunda, ölçümü *C* koalisyonununkine yeterince yakın herhangi bir koalisyonun da bu dağılımı bloke ettiğini gösteriyoruz. Ayrıca, *E*'nin Pareto kümesinin ve çekirdeğinin tüketici uzayından emtia uzayına tanımlı bütün integrallenebilir fonksiyonlardan oluşan Banach uzayının kapalı altkümeleri olduğunu gösteriyoruz. Emtia uzayı ayrılabilir ise, bu sonucu bir ekonomiler dizisinin belirli bir yakınsaklık biçimini düşünmek suretiyle güçlendiriyoruz. İkil büyük ekonomiler hakkındaki bu savları elde etmek için Banach uzayında çokdeğerli fonksiyonların integraliyle ilgili çeşitli sonuçlar ispatlıyoruz.

Anahtar Sözcükler: Sonsuz Boyutlu Ekonomi, Büyük Ekonomi, Çekirdek, Pareto Kümesi, Bochner İntegrali, Çokdeğerli Fonksiyon

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To my Parents

Contents

Chapter 1

Introduction

Aumann (1964) and (1966) was the first author to construct a general equilibrium model that contains a measure space of traders instead of a finite number of traders. In these works, he suggested that a mathematical model appropriate to the notion of perfect competition should contain a continuum of traders and an atomless measure on the set of traders, so that no single trader can influence prices as the price taking assumption postulates; and he succeeded to show that in such economies the set of core allocations and the set of competitive equilibria are nonempty sets which coincide with each other. Following Aumann, a huge literature devoted to the study of large economies, i.e. economies with a measure space of traders, emerged (see, e.g. Hildenbrand, 1968; Kannai, 1970; and Schmeidler, 1969). Obviously, a most general way of modelling an economy involves the assumption of an arbitrary measure space of traders which is not necessarily atomless, so that individual influences may or may not be negligible. That is the approach we will use in our thesis.

Another major break-through in the theory of general equilibrium was the works of Peleg and Yaari (1970) and Bewley (1972) where the authors proved the existence of competitive equilibria for economies with specific infinite dimensional commodity spaces. The main qualification of infinite dimensional models is their ability to capture the effects of variations in time, location, and state of the world which point to an infinite variability in a market. In a typical infinite dimensional model, a commodity bundle x is a mapping of a set A, which is interpreted as the time or the state set, into an Euclidean space, and $x(a)$ represents consumption at a. Another model that operates in an infinite dimensional setting was developed within the theory of competitive product differentiation advanced by Mas-Colell (1975). In this model, a commodity bundle x is a mapping of a collection of subsets of a set of characteristics A into the interval $[0, 1]$, and for a set B contained in A, $x(B)$ represents the share of commodities in the bundle x whose characteristics lie in B. A thorough discussion of general equilibrium theory in infinite dimensional spaces can be found in Aliprantis et. al. (1990), or Mas-Colell and Zame (1991).

The main aim of this study is to show that some pleasant properties possessed by the core and Pareto optimal allocations of a finite dimensional exchange economy continue to hold for those of an infinite dimensional economy as well. Our first result is on the stability of blocking coalitions. We will show that in a large economy with a Banach space of commodities if a coalition blocks an allocation, then any coalition that slightly differs from the original one will still block the allocation. A finite dimensional version of this theorem is due to Hüsseinov (2003). As he observes, since in a large economy the precise formation of a particular coalition can hardly be expected, and since it eliminates the possibility of turning a blocking coalition into a nonblocking one by tempting a tiny group within the coalition with a small amount of resource transfer, this result contributes to the practical relevance of the concept of the core.

Another result which will be generalized here is related to the continuity of the core and Pareto optimal allocations. It will be shown that for an exchange economy with a Banach space S of commodities and a measure space (T, Σ, μ) of traders, the core and the Pareto set are closed subsets of the Banach space $L_1(T, \Sigma, \mu; S)$ of all μ-integrable mappings of T into S. A finite dimensional version of this theorem first appeared in Grodal (1971), and was generalized by Hüsseinov (2003) to include economies with nonconvex preferences.

Finally, we shall show that for a sequence $\{\xi_n\}$ of economies and an economy ξ all containing the same measure space (T, Σ, μ) of traders and the same separable Banach space S of commodities, if the components of the economies ξ_n converge to the corresponding components of ξ in a particular sense, and if there exists a convergent sequence $\{x_n\}$ of allocations in $L_1(T, \Sigma, \mu; S)$ such that x_n is in the core of ξ_n for all n, then $\lim_n x_n$ is in the core of ξ. This theorem generalizes Kannai's (1970) continuity result to an infinite dimensional environment which was previously generalized by Grodal (1971) to the case of economies including atoms, and by Hüsseinov (2003) to the case of economies with nonconvex preferences.

Proofs of the finite dimensional versions of the above results were based on Grodal's (1971) theorems on the integral of correspondences with values in an Euclidean space. Following the same approach, to obtain our results on infinite dimensional economies we will generalize these theorems to include correspondences with values in an arbitrary Banach space.

The thesis is organized as follows. In Chapter 2, we list the preliminary mathematical facts and definitions that will be used in the rest of this study.

In Chapter 3, our results on the integral of correspondences are presented. In Chapter 4, we shall introduce double large economies and prove our results on such economies. Finally, in Chapter 5, we present the conclusion.

Chapter 2

Mathematical Preliminaries

In this chapter, we will introduce some basic concepts and facts related to the theory of integration of B-space (Banach space) valued mappings and correspondences. Throughout the thesis we confine ourselves to real B-spaces. Mostly we will follow Dunford and Schwartz (1988), and Aliprantis and Border (1994). For the sake of brevity, we shall assume that the reader is familiar with basic set theory, topology, and elementary facts about vector spaces. It should also be noted that we will just cover what will be used in the rest of this study; a detailed discussion of the material can be found in Dunford and Schwartz (1988), and Aliprantis and Border (1994).

2.1 Measure Theory

In this section, we give a brief introduction to the basic notions of measure theory. As usual, for a pair of sets E and F, $E \backslash F$ denotes the set theoretic difference of E from F.

Definition 2.1 Let T be an arbitrary set. *A σ-field* Σ of subsets of T is a collection of subsets of T with the following properties

1. $\emptyset \in \Sigma$;

2. for all $E \in \Sigma$, we have $T \backslash E \in \Sigma$;

3. Σ is closed under countable unions, i.e. for any countable collection $\{E_i : i \in I\} \subset \Sigma$, we have $\bigcup_{i \in I} E_i \in \Sigma$.

A *measurable space* (T, Σ) consists of a set T, and *a* σ-field Σ of subsets of T.

Note that De Morgan's laws imply that a σ-field is closed under countable intersections. Therefore, for any two sets E and F in a σ-field Σ of subsets of a set T, $E \backslash F = (T \backslash F) \cap E$ belongs to Σ as well.

Clearly, the intersection of an arbitrary collection of σ-field's is again a σ-field. Moreover, the collection of all subsets of a set T is a σ-field. Thus, any collection $\mathfrak{G}$ of subsets of a set T is included in a smallest σ-field which is defined by

$$\sigma(\mathfrak{G}) = \bigcap \{\Sigma : \Sigma \text{ is a } \sigma\text{-field of subsets of } T,\, \mathfrak{G} \subset \Sigma\},$$

and which is called the *σ-field generated by* $\mathfrak{G}$.

Definition 2.2 The *Borel σ-field* of a topological space S is the σ-field generated by the collection of all open subsets of S. This σ-field will be denoted by $\mathfrak{B}(S)$.

Notice that the σ-field $\mathfrak{B}(S)$ is also generated by the collection of all closed subsets of S.

Let $\{(T_i, \Sigma_i) : i = 1, ..., n\}$ be a finite set of measurable spaces. $\Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_n$ will denote the collection $\{E_1 \times E_2 \times \cdots \times E_n : E_i \in \Sigma_i, i = 1, ..., n\}$, where $E_1 \times E_2 \times \cdots \times E_n$ is the Cartesian product of E_i for $i = 1, ..., n$. The σ-field generated by the collection $\Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_n$ will be denoted with

$\Sigma_1 \otimes \Sigma_2 \otimes \cdots \otimes \Sigma_n$. It is easily seen that $\Sigma_i \otimes \Sigma_j \otimes \Sigma_k = \Sigma_i \otimes (\Sigma_j \otimes \Sigma_k)$ for any $i, j, k = 1, ..., n$.

For any pair of topological spaces S and R, we endow $S \times R$ with the product topology. The *projection* of $S \times R$ into S is the mapping $\mathrm{proj}_S : S \times R \to S$ defined by $\mathrm{proj}_S(x, y) = x$ for all $(x, y) \in S \times R$. If f is a mapping of T into S, and if $\mathfrak{G}$ is a collection of subsets of S, we denote the collection $\{f^{-1}(E) : E \in \mathfrak{G}\}$ with $f^{-1}(\mathfrak{G})$.

Proposition 2.3 *Let (S, τ_S) and (R, τ_R) be topological spaces. Then $\mathfrak{B}(S) \otimes \mathfrak{B}(R) \subset \mathfrak{B}(S \times R)$. If S and Y are both second countable, the converse is also true.*

Proof. For the first part observe that $\mathfrak{B}(S) \times \mathfrak{B}(R) = \mathrm{proj}_S^{-1}(\mathfrak{B}(S)) \cap \mathrm{proj}_R^{-1}(\mathfrak{B}(R))$, and $\mathrm{proj}_S^{-1}(\mathfrak{B}(S)) = \mathrm{proj}_S^{-1}(\sigma(\tau_S)) = \sigma\left(\mathrm{proj}_S^{-1}(\tau_S)\right)$, and similarly for the collection $\mathrm{proj}_R^{-1}(\mathfrak{B}(R))$. Now, by definition of the product topology, proj_S and proj_T are continuous functions, and hence, the collections $\mathrm{proj}_S^{-1}(\tau_S)$ and $\mathrm{proj}_R^{-1}(\tau_R)$ are contained in $\mathfrak{B}(S \times R)$. Clearly, this ends the proof of the first part. For the second part, simply notice that the collection of all sets of the form $G_S \times G_R$, where G_S (resp. G_R) is a member of a fixed countable base for τ_S (resp. τ_R), is a countable base for the product topology $\tau_{S \times R}$, and so, $\tau_{S \times R}$ is contained in the σ-field generated by this collection. ■

The elements of a collection of sets $\{E_i : i \in I\}$ are said to be *mutually disjoint*, if $E_i \cap E_j = \emptyset$ for all $i, j \in I$ with $i \neq j$. We are now ready to introduce the concept of a measure which grew out of the efforts to generalize the notions of length, area, and volume.

Definition 2.4 A *measure space* is a triplet (T, Σ, μ) where T is a set, Σ is a σ-field of subsets of T, and μ is a *measure* on Σ, i.e. a nonnegative, extended real valued mapping of Σ with the following properties

1. $\mu(\emptyset) = 0$;

2. μ is countably additive, that is $\mu\left(\bigcup_{i\in I} E_i\right) = \sum_{i\in I} \mu(E_i)$ for any countable collection of mutually disjoint sets $\{E_i : i \in I\} \subset \Sigma$.

Note that for a measure space (T, Σ, μ), we have $\mu(E) = \mu(F) + \mu(E \backslash F) \geq \mu(F)$ whenever $E, F \in \Sigma$ and $F \subset E$. This property is known as *monotonicity.*

Let (T, Σ, μ) be a measure space. Elements of the collection Σ are called *μ-measurable sets.* Clearly, if E is a μ-measurable set then the collection $\Sigma_E = \{F \in \Sigma : F \subset E\}$ is a σ-field, and the restriction of μ to Σ_E is a measure. We denote this new measure space by (E, Σ_E, μ_E). A set $E \in \Sigma$ is said to be an *atom* of the measure space (T, Σ, μ) if for each μ-measurable subset F of E, either $\mu(F) = 0$ or $\mu(E \backslash F) = 0$ is true. (T, Σ, μ) is said to be an *atomless* measure space if it does not contain any atoms.

A measure space (T, Σ, μ) with $\mu(T) = 1$ is known as a *probability space.* (T, Σ, μ) is said to be *finite* if $\mu(T) < \infty$, and is said to be *σ-finite* if there exists a sequence $\{E_n\}$ of mutually disjoint, μ-measurable sets such that $\bigcup_{n\in\mathbb{N}} E_n = T$ and $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$. For any sequence of sets $\{E_n\}$ and for a set E, $E_n \downarrow E$ means $E_{n+1} \subset E_n$ for all $n \in \mathbb{N}$, and $\bigcap_{n\in\mathbb{N}} E_n = E$. Similarly, for any sequence of real numbers $\{\gamma_n\}$ and for a number γ, $\gamma_n \downarrow \gamma$ means $\gamma_{n+1} \leq \gamma_n$ for all $n \in \mathbb{N}$, and $\gamma_n \to \gamma$. The following theorem is known as *continuity of measure.*

Theorem 2.5 *Let (T, Σ, μ) be a finite measure space. Then $\mu(E_n) \downarrow 0$ for any sequence of measurable sets $\{E_n\}$ with $E_n \downarrow \emptyset$.*

Proof. See Dudley (1989: 64, Theorem 3.1.1). ■

Let us denote the set of extended real numbers with $\overline{\mathbb{R}}$. We shall now give an extension theorem.

Theorem 2.6 *Let (T, Σ, μ) be a measure space, and let Σ^* consist of all sets of the form $E \cup N$ where E is in Σ and N is a subset of a μ-measurable set M with $\mu(M) = 0$. Define $\mu^* : \Sigma^* \to \overline{\mathbb{R}}$ by $\mu^*(E \cup N) = \mu(E)$. Then μ^* is well defined, and (T, Σ^*, μ^*) is a measure space.*

Proof. See Dunford and Schwartz (1988: 142, Theorem III.5.17). ■

Note that in the above theorem the collection Σ is contained in the collection Σ^*, and $\mu^*(E) = \mu(E)$ for all $E \in \Sigma$. Thus, μ^* is an extension of μ from Σ to Σ^*. Moreover, the collection $\{E \in \Sigma^* : \mu^*(E) = 0\}$ equals to the collection $\{E \in \Sigma^* : E \subset M \text{ for some } M \in \Sigma \text{ with } \mu(M) = 0\}$. So if $\mu^*(E) = 0$ for some $E \in \Sigma^*$, then any subset of E also belongs to Σ^*. Such measure spaces are known as *complete* measure spaces, and (T, Σ^*, μ^*) (resp. μ^*) is called the *completion* of (T, Σ, μ) (resp. μ).

We can now give a concrete example of a measure on the real line. For any interval $I \subset \mathbb{R}$ let $m(I)$ be the length of I. If an interval is not bounded, its length is defined to be ∞. For each $B \in \mathfrak{B}(\mathbb{R})$, set $\mu(B) = \inf \sum_{n=1}^{\infty} m(I_n)$ where the infimum is taken over all sequences $\{I_n\}$ of intervals whose union contains B. Then it can be shown that μ is countably additive on $\mathfrak{B}(\mathbb{R})$. The completion of the measure μ is known as the *Lebesgue measure.*

Before introducing the concept of measurability of functions we must give a further extension result and a few definitions about different convergence notions.

Proposition 2.7 *Let (T, Σ, μ) be a measure space. For each $F \subset T$ let*

$$\widehat{\mu}(F) = \inf \sum_{n=1}^{\infty} \mu(E_n)$$

where the infimum is taken over all sequences $\{E_n\}$ *of sets in* Σ *whose union contains* F*. Then* $\widehat{\mu}(E) = \mu^*(E)$ *for all* $E \in \Sigma^*$.

Proof. See Dunford and Schwartz (1988: 134, Lemma III.5.5). ■

Notice that in the above proposition the mapping $\widehat{\mu}$ is monotone.

For a measure space (T, Σ, μ) and for a set $E \subset T$, a property P is said to hold μ*-almost everywhere* on E, or equivalently for μ*-almost every* $t \in E$, if the set $E_P^c = \{t \in E : P \text{ is not true for } t\}$ belongs to Σ^*, and $\mu^*(E_P^c) = 0$. Both the phrases "almost everywhere" and "almost every" will be abbreviated as "a.e.".

Definition 2.8 Let (T, Σ, μ) be a measure space, and let $(S, \|\cdot\|)$ be a B-space. Let furthermore $\{f_n\}$ be a sequence of mappings of T into S. Then the sequence $\{f_n\}$ is said to *converge in* μ*-measure to a mapping* $f : T \to S$ provided that

$$\widehat{\mu}(\{t \in T : \|f_n(t) - f(t)\| \geq \varepsilon\}) \longrightarrow 0 \qquad \text{for all } \varepsilon > 0,$$

where $\widehat{\mu}$ is defined as in Proposition 2.7.
For $E \in \Sigma$, the sequence $\{f_n\}$ is said to *converge* μ*-a.e. on* E to f, if $\|f_n(t) - f(t)\| \longrightarrow 0$ for μ-a.e. $t \in E$.

Let E be a subset of a set T. The *characteristic function of* E is the mapping $\chi_E : T \to \{0, 1\}$ defined by $\chi_E(t) = 1$ for $t \in E$ and $\chi_E(t) = 0$ for $t \in T \backslash E$. Now we are ready to proceed with the concept of measurability of functions.

Definition 2.9 Let (T, Σ, μ) be a measure space, and let S be a B-space. A mapping f of T into S is said to be μ*-simple* if it takes finitely many distinct values $y_1, ..., y_n \in S$ and $f^{-1}(y_i) \in \Sigma$ for all $i \in \{1, ..., n\}$. A mapping g of T

into S is said to be *totally μ-measurable* if there exists a sequence $\{f_n\}$ of μ-simple mappings which converge in μ-measure to g. The mapping g is said to be *measurable* if $\chi_E g$ is totally measurable for each $E \in \Sigma$ with $\mu(E) < \infty$.

The following useful characterization of measurable functions is known as the *Pettis measurability criterion.*

Theorem 2.10 *Let (T, Σ, μ) be a measure space, and let S be a B-space. Then a mapping f of T into S is μ-measurable if and only if for each $E \in \Sigma$ with $\mu(E) < \infty$ both of the following conditions hold*

1. *$f(E \backslash N)$ is separable for some $N \in \Sigma^*$ with $\mu^*(N) = 0$;*
2. *$f^{-1}(B) \cap E$ belongs to the collection Σ^* for each $B \in \mathfrak{B}(S)$.*

Proof. See Dunford and Schwartz (1988: 148, Theorem III.6.10). ■

We can use the above characterization to define measurability of extended real valued mappings of a measure space in the following way.

Definition 2.11 An extended real valued mapping f of a measure space (T, Σ, μ) is said to be μ-measurable if the set $f^{-1}(A) \cap E$ belongs to the collection Σ^* for each measurable set E with $\mu(E) < \infty$, and for each set A of the form $A = B \cup C$ where $B \in \mathfrak{B}(\mathbb{R})$, and C is a subset of the set $\{+\infty, -\infty\}$.

Since the collection of all intervals of the form (α, ∞) $(\alpha \in \mathbb{R})$ generates the Borel σ-field of the real line, in the above definition, the borel set B can be replaced by a set of the form $\{t \in T : f(t) > \alpha\}$, or $\{t \in T : f(t) < \alpha\}$. Now let f and g be extended real valued mappings of a set T. Then, for any $t \in T$ and for any $\alpha \in \mathbb{R}$, $\min\{f(t), g(t)\} > \alpha$ if and only if $f(t) > \alpha$ and $g(t) > \alpha$. So, we have the following proposition.

Proposition 2.12 *If f and g are extended real valued, μ-measurable mappings of a measure space (T, Σ, μ), then $\min\{f, g\}$ and $\max\{f, g\}$ are μ-measurable mappings.*

The next result is a direct consequence of Theorem 2.10 and Definition 2.11.

Corollary 2.13 *Let (T, Σ, μ) be a σ-finite measure space.*

1. *A mapping f of T into a separable B-space S is μ-measurable if and only if $f^{-1}(B) \in \Sigma^*$ for each $B \in \mathfrak{B}(S)$.*

2. *An extended real valued mapping f of T is μ-measurable if and only if $f^{-1}(A) \in \Sigma^*$ for each set A of the form $A = B \cup C$ where $B \in \mathfrak{B}(\mathbb{R})$, and C is a subset of the set $\{+\infty, -\infty\}$.*

The following two propositions list a few properties of measurable mappings.

Proposition 2.14 *Let (T, Σ, μ) be a measure space, and let $(S, \|\cdot\|)$ be a B-space. Let furthermore the mappings f and g of T into S and the mapping $\gamma : T \to \mathbb{R}$ be μ-measurable (totally μ-measurable). Then the mappings γf, $\|f(\cdot)\|$, and $f + g$ are μ-measurable (totally μ-measurable). Moreover, if g is a real valued continuous mapping of $\gamma(T)$, then $g \circ \gamma$ is μ-measurable (totally μ-measurable).*

Proof. See Dunford and Schwartz (1988: 106, Lemma III.2.11 and Lemma III.2.12). ■

Proposition 2.15 *Let (T, Σ, μ) be a measure space, and let S be a B-space. If a sequence $\{f_n\}$ of μ-measurable (totally μ-measurable) mappings of T into S converges in μ-measure to a mapping $f : T \to S$, then f is also μ-measurable (totally μ-measurable).*

Proof. See Dunford and Schwartz (1988: 106, Lemma III.2.11). ■

Now we shall give two well-known theorems about convergence of measurable mappings.

Theorem 2.16 (Egoroff) *Let (T, Σ, μ) be a finite measure space, and let $(S, \|\cdot\|)$ be a B-space. A sequence $\{f_n\}$ of measurable mappings of T into S converges μ-a.e. on T to a mapping $f : T \to S$ if and only if for each $\varepsilon > 0$ there is a μ-measurable set E on which $\|f(t) - f_n(t)\|$ converges to 0 uniformly in t, and such that $\mu(T \backslash E) < \varepsilon$.*

Proof. See Dunford and Schwartz (1988: 149, Theorem III.6.12). ■

Theorem 2.17 *Let (T, Σ, μ) be a measure space, and let S be a B-space. Let furthermore $\{f_n\}$ be a sequence of measurable mappings of T into S.*

1. *If the sequence $\{f_n\}$ converges to a mapping $f : T \to S$ in μ-measure, then a subsequence of $\{f_n\}$ converges to f μ-a.e. on T.*

2. *If (T, Σ, μ) is finite and the sequence $\{f_n\}$ converges to a mapping $f : T \to S$ μ-a.e. on T, then $\{f_n\}$ converges to f in μ-measure.*

Proof. See Dunford and Schwartz (1988: 150, Corollary III.6.13). ■

The next result follows from the second part of Theorem 2.17 and Proposition 2.15.

Corollary 2.18 *Let (T, Σ, μ) be a measure space, and let S be a B-space. If a sequence $\{f_n\}$ of μ-measurable mappings of T into S converges to a mapping $f : T \to S$ μ-a.e. on T, then f is also μ-measurable.*

We conclude this section with the following corollary which is an immediate consequence of the first part of Theorem 2.17 and the definition of totally measurable mappings.

Corollary 2.19 *Let (T, Σ, μ) be a measure space, and let S be a B-space. If a mapping $f : T \to S$ is totally μ-measurable, then there exists a sequence $\{f_n\}$ of μ-simple mappings of T into S which converge to f μ-a.e. on T.*

2.2 Integration

We are now ready to present some relevant aspects of the theory of integration of vector valued functions. The integral to be used here is known as the Bochner integral. For the readers who are familiar with the integration of real valued mappings, we should emphasize that Bochner integral is a simple abstraction of the Lebesgue integral.

The following lemma is needed for the definition of the integral of μ-simple mappings. Throughout the thesis, we define $0 * \infty$ to be 0.

Lemma 2.20 *Let (T, Σ, μ) be a measure space, and let S be a B-space. Assume that for a pair of collections $\{F_i : 1 \le i \le n\}$, $\{E_j : 1 \le j \le m\} \subset \Sigma$, and for a pair of sets $\{x_i : 1 \le i \le n\}$, $\{y_j : 1 \le j \le m\} \subset S$ the following conditions hold*

1. $\sum_{i=1}^{n} \chi_{F_i} x_i = \sum_{j=1}^{m} \chi_{E_j} y_j$;
2. *if $i \ne i'$ then $F_i \cap F_{i'} = \emptyset$, and if $j \ne j'$ then $E_j \cap E_{j'} = \emptyset$;*
3. *$x_i \ne 0$ implies $\mu(F_i) < \infty$, and $y_j \ne 0$ implies $\mu(E_j) < \infty$ for $i = 1, ..., n$ and $j = 1, ..., m$.*

Then $\sum_{i=1}^{n} \mu(F_i \cap E) x_i = \sum_{j=1}^{m} \mu(E_j \cap E) y_j$ for any $E \in \Sigma$.

Proof. Condition (3) ensures that these sums are well defined for any $E \in \Sigma$. Let $E \in \Sigma$. Set $I_+ = \{i : x_i \ne 0\}$ and $J_+ = \{j : y_j \ne 0\}$. By conditions (1) and (2), it is easily seen that $\bigcup_{i \in I_+} F_i = \bigcup_{j \in J_+} E_j$. Moreover, again by conditions (1) and (2), for any $i \in \{1, ..., n\}$ and any $j \in \{1, ..., m\}$, if $F_i \cap E_j$ is nonempty, then $x_i = y_j$. Thus,

$$\begin{aligned}
\sum_{i \in I_+} \mu (F_i \cap E) x_i &= \sum_{i \in I_+} \sum_{j \in J_+} \mu (E_j \cap F_i \cap E) x_i \\
&= \sum_{j \in J_+} \sum_{i \in I_+} \mu (E_j \cap F_i \cap E) y_j = \sum_{j \in J_+} \mu (E_j \cap E) y_j.
\end{aligned}$$

■

Definition 2.21 Let (T, Σ, μ) be a measure space, and let $(S, \|\cdot\|)$ be a B-space. A simple mapping f of T into S is said to be *μ-integrable simple* if f is of the form $f = \sum_{i=1}^{n} \chi_{E_i} x_i$ for a collection of mutually disjoint sets $\{E_i : 1 \leq i \leq n\} \subset \Sigma$, and for a set $\{x_i : 1 \leq i \leq n\} \subset S$ such that $\mu (E_i) < \infty$ whenever $x_i \neq 0$. In this case, the *μ-integral of f over* $E \in \Sigma$ is defined as

$$\int_E f(t)\, d\mu(t) = \sum_{i=1}^{n} \mu (E_i \cap E) x_i.$$

In view of Lemma 2.20, we see that the integral of f is independent of the particular representation of f. The reader should notice that if f and g are μ-integrable simple mappings of T into S, then so is $\|f(\cdot)\| : T \to \mathbb{R}$, and $\alpha f + \beta g$ for any pair of real numbers α and β. The following lemma will enable us to define the integral of a more general class of mappings.

Lemma 2.22 *Let (T, Σ, μ) be a measure space, and let $(S, \|\cdot\|)$ be a B-space. If $\{f_n^1\}$ and $\{f_n^2\}$ are sequences of μ-integrable simple mappings of T into S both converging in μ-measure to the same limit and if*

$$\lim_{n,m} \int_T \left\| f_n^i (t) - f_m^i (t) \right\| d\mu(t) = 0 \qquad \text{for } i = 1, 2,$$

then for $E \in \Sigma$ the limits $\lim_n \int_E f_n^i (t)\, d\mu(t)$ exist for $i = 1, 2$, and are equal.

Proof. See Dunford and Schwartz (1988: 111, Lemma III.2.16). ■

Definition 2.23 Let (T, Σ, μ) be a measure space, and let $(S, \|\cdot\|)$ be a B-space. A mapping f of T into S is said to be *μ-integrable* if there exists a sequence $\{f_n\}$ of μ-integrable simple mappings of T into S converging in μ-measure to f and satisfying in addition the condition

$$\lim_{n,m} \int_T \|f_n(t) - f_m(t)\| \, d\mu(t) = 0.$$

In this case we say that the sequence $\{f_n\}$ *determines* f, and define the *$\mu-$integral of f over* $E \in \Sigma$ as

$$\int_E f(t) \, d\mu(t) = \lim_n \int_E f_n(t) \, d\mu(t).$$

Lemma 2.22 shows that this limit exists and is independent of the particular sequence $\{f_n\}$ of μ-integrable simple mappings. Equivalence of this definition with Definition 2.21 for μ-integrable simple mappings follows again from Lemma 2.22.

We proceed with some simple properties of the integral.

Theorem 2.24 *Let f and g be μ-integrable mappings of a measure space (T, Σ, μ) into a B-space $(S, \|\cdot\|)$.*

1. $\|f(\cdot)\|$ *is μ-integrable and* $\left\|\int_E f(t) \, d\mu(t)\right\| \leq \int_E \|f(t)\| \, d\mu(t)$ *for* $E \in \Sigma$.

2. *For any pair of real numbers α, β, the mapping $\alpha f + \beta g$ is μ-integrable and* $\int_E \alpha f + \beta g(t) \, d\mu(t) = \alpha \int_E f(t) \, d\mu(t) + \beta \int_E g(t) \, d\mu(t)$ *for* $E \in \Sigma$.

3. *For $E \in \Sigma$ the mapping $\chi_E f$ is μ-integrable and* $\int_T \chi_E f(t) \, d\mu(t) = \int_E f(t) \, d\mu(t)$.

4. *For each $\varepsilon > 0$ there is a measurable set E with $\mu(E) < \infty$ such that* $\int_{T \setminus E} \|f(t)\| \, d\mu(t) < \varepsilon$.

5. *(Absolute continuity) For each* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that* $\int_F \|f(t)\| \, d\mu(t) < \varepsilon$ *whenever* $F \in \Sigma$ *and* $\mu(F) < \delta$.

6. $\int_T \|f(t)\| \, d\mu(t) = 0$ *if and only if* $f(t) = 0$ *a.e. on* T*. In particular, if* f *is real valued,* $f(t) \geq 0$ μ*-a.e. on* T*, and* $f(t) > 0$ μ*-a.e. on a* μ*-measurable set* E *with* $\mu(E) > 0$*, then* $\int_T f(t) \, d\mu(t) > 0$.

Proof. For the proof of (1) see Dunford and Schwartz (1988: 113, Lemma III.2.18); for the proof of (2) and (3) see Dunford and Schwartz (1988: 113, Theorem III.2.19); for the proof of (4), (5), and (6) see Dunford and Schwartz (1988: 114, Theorem III.2.20). ■

In line with Theorem 2.24(3), it should be clear that if $f : (T, \Sigma, \mu) \to S$ is μ-integrable, then the restriction of f to $E \in \Sigma$ is μ_E-integrable over the measure space (E, Σ_E, μ_E), and $\int_E f \mid_E (t) \, d\mu_E(t) = \int_E f(t) \, d\mu(t)$, where $f \mid_E$ stands for the restriction of f to E.

Before moving to next section we shall give a useful criterion for integrability of measurable functions.

Theorem 2.25 *Let* (T, Σ, μ) *be a measure space, and let* $(S_1, \|\cdot\|_1)$, $(S_2, \|\cdot\|_2)$ *be a pair of B-spaces. If* f_2 *is a* μ*-integrable mapping of* T *into* S_2 *and if* f_1 *is a* μ*-measurable mapping of* T *into* S_1 *such that* $\|f_1(t)\|_1 \leq \|f_2(t)\|_2$ *for a.e.* $t \in T$*, then* f_1 *is* μ*-integrable. In particular, the* μ*-measurable mapping* f_1 *is* μ*-integrable if and only if* $\|f_1(\cdot)\|_1$ *is* μ*-integrable.*

Proof. See Dunford and Schwartz (1988: 117, Theorem III.2.22). ■

2.3 $\mathcal{L}_p$ Spaces

We shall define spaces of integrable mappings of a measure space and discuss a few properties of these spaces.

Definition 2.26 Let p be a natural number, and let (T, Σ, μ) be a measure space. Then for a B-space $(S, \|\cdot\|)$, $\mathcal{L}_p^0 (T, \Sigma, \mu; S)$ denotes the vector space of all μ-measurable mappings f of T into S such that $\|f(\cdot)\|^p$ is μ-integrable.

For each $f \in \mathcal{L}_p^0 (T, \Sigma, \mu; S)$ set $\|f\|_p = \left(\int_T \|f(t)\|^p \, d\mu(t) \right)^{1/p}$. Using Hölder's Inequality (see Dunford and Schwartz, 1988: 119, Lemma III.3.2) it can be shown that $\|\cdot\|_p$ is a seminorm on $\mathcal{L}_p^0 (T, \Sigma, \mu; S)$, i.e. for any f and g in $\mathcal{L}_p^0 (T, \Sigma, \mu; S)$ and for any real number α, we have $\|\alpha f\|_p = |\alpha| \, \|f\|_p$ and $\|f + g\|_p \leq \|f\|_p + \|g\|_p$. However, as a consequence of Theorem 2.24(6), $\|\cdot\|_p$ is not a norm on $\mathcal{L}_p^0 (T, \Sigma, \mu; S)$. Thus, to turn $\mathcal{L}_p^0 (T, \Sigma, \mu; S)$ into a normed space we shall consider mappings which equal to each other μ-a.e. on T as identical. To this end, for any f and g in $\mathcal{L}_p^0 (T, \Sigma, \mu; S)$, let us write $f \sim' g$ if and only if $f(t) = g(t)$ μ-a.e. on T. It is easily seen that $\sim'$ is an equivalence relation. For all $f \in \mathcal{L}_p^0 (T, \Sigma, \mu; S)$, put $[f] = \left\{ g \in \mathcal{L}_p^0 (T, \Sigma, \mu; S) : g \sim' f \right\}$. The set of all such equivalence classes is denoted by $\mathcal{L}_p (T, \Sigma, \mu; S)$. Now we can introduce a vector structure to $\mathcal{L}_p (T, \Sigma, \mu; S)$ by declaring $[f] + [g]$ to be the equivalence class of $f + g$, and $\alpha [f]$ to be the equivalence class of αf, for any $[f]$ and $[g]$ in $\mathcal{L}_p (T, \Sigma, \mu; S)$ and for any real number α. If we set $\|[f]\|_p = \|g\|_p$ where g is an arbitrary member of $[f]$, in view of Theorem 2.24, we see that $\|\cdot\|_p$ is a well-defined norm on $\mathcal{L}_p (T, \Sigma, \mu; S)$. It is customary to speak of the elements of $\mathcal{L}_p (T, \Sigma, \mu; S)$ as if they are functions rather than equivalence classes. Thus, we shall write f instead of $[f]$.

Theorem 2.27 *Let (T, Σ, μ) be a measure space, and let S be a B-spaces. Then for any natural number p, $\mathcal{L}_p (T, \Sigma, \mu; S)$ is norm complete and thus a B-space.*

Proof. See Dunford and Schwartz (1988: 146, Theorem III.6.6). ■

If the range space is the real line, we can define another B-space of measurable mappings on a measure space.

Definition 2.28 Let (T, Σ, μ) be a measure space. A real valued mapping f of T is said to be *essentially bounded* if the set $\{t \in T \backslash N : f(t)\} \subset \mathbb{R}$ is bounded for a set $N \in \Sigma^*$ with $\mu^*(N) = 0$. $\mathcal{L}_\infty(T, \Sigma, \mu)$ denotes the vector space of equivalence classes $[f]$ of all essentially bounded, μ-measurable mappings f of (T, Σ, μ) into $\mathbb{R}$.

Now for all $[f] \in \mathcal{L}_\infty(T, \Sigma, \mu)$ set

$$\|[f]\|_\infty = \inf\{M > 0 : |f(t)| \leq M \ \mu\text{-a.e. on } T\}.$$

It is easily seen that $\|\cdot\|_\infty$ defines a norm on $\mathcal{L}_\infty(T, \Sigma, \mu)$. Also notice that by Corollary 2.18 $\mathcal{L}_\infty(T, \Sigma, \mu)$ is norm complete. As in the previous definition we will write f instead of $[f]$.

We conclude this section with the next theorem.

Theorem 2.29 *Let (T, Σ, μ) be a measure space, and let S be a B-space. For a natural number p, let $\{f_n\}$ be a sequence in $\mathcal{L}_p(T, \Sigma, \mu; S)$. Then a mapping $f : T \to S$ is in $\mathcal{L}_p(T, \Sigma, \mu; S)$ and $\|f - f_n\|_p$ converges to 0 if and only if the following three conditions hold*

1. *$\{f_n\}$ converges to f in μ-measure;*

2. *For each $\varepsilon > 0$ there exists a $\delta > 0$ such that, for each natural number n, $\int_F \|f_n(t)\|^p d\mu(t) < \varepsilon$ whenever $F \in \Sigma$ and $\mu(F) < \delta$.*

3. *For each $\varepsilon > 0$ there is a μ-measurable set E_ε with $\mu(E_\varepsilon) < \infty$ such that $\int_{T \backslash E_\varepsilon} \|f_n(t)\|^p d\mu(t) < \varepsilon$ for all $n \in \mathbb{N}$.*

Proof. See Dunford and Schwartz (1988: 122, Theorem III.3.6). ■

2.4 Correspondences

In this section, we present a brief introduction to integration and measurability of correspondences.

Let T and S be a pair of sets. A *correspondence*, or a *multifunction*, φ from T into S is a function such that $\varphi(t)$ is a subset of S for all $t \in T$. The notation $\varphi : T \Rightarrow S$ means "φ is a correspondence from T into S". The graph of a correspondence $\varphi : T \Rightarrow S$ is defined as $Gr_\varphi = \{(t, y) \in T \times \mathbb{S} : y \in \varphi(t)\}$. If the domain of a correspondence φ is a measure space one can talk about measurability of φ. In the literature there are several definitions for measurable correspondences. Here, we will use the most common one.

Definition 2.30 Let (T, Σ, μ) be a measure space and let S be a topological space. A correspondence $\varphi : T \Rightarrow S$ is said to be *μ-measurable* if the set $\{t \in T : G \cap \varphi(t) \neq \emptyset\}$ belongs to the collection Σ^* for each open set $G \subset S$.

For a subset A of a topological space, $\overline{A}$ will denote the closure of A. Let φ be a correspondence from T into S, where T and S are as in the above definition. We define the *complement* of φ as $\varphi^c(t) = S \backslash \varphi(t)$ for $t \in T$, and the *closure* of φ as $\overline{\varphi}(t) = \overline{\varphi(t)}$ for $t \in T$. *Interior* of φ is denoted by $\operatorname{int} \varphi$, and it associates interior of the set $\varphi(t)$ to each $t \in T$. Clearly, the complement of φ is μ-measurable if and only if the set $\{t \in T : G \subset \varphi(t)\}$ belongs to the collection Σ^* for each open set $G \subset S$. We say that the graph of a correspondence φ from T into S is μ-measurable if Gr_φ belongs to the collection $\Sigma^* \otimes \mathfrak{B}(S)$. It is clear the graph of a correspondence φ is μ-measurable if and only if so is the graph of φ^c. A standard practice in economics is to work with correspondences which have measurable graphs. As we shall see in the following results, if the range space is a separable metric space, this property is a powerful tool, though not in general.

Theorem 2.31 (Projection Theorem) *Let (T, Σ, μ) be a σ-finite measure space and let S be a separable metric space. Then for any set $A \in \Sigma^* \otimes \mathfrak{B}(S)$, the set $\operatorname{proj}_T(A)$ belongs to the collection Σ^*.*

Proof. See Klein and Thompson (1984: 147, Theorem 12.3.4). ■

For a metric space (S, d), *distance* of a point $x \in S$ from a subset A of S is defined as $\operatorname{dist}(x, A) = \inf_{y \in A} d(x, y)$. Note that $\operatorname{dist}(x, \emptyset) = \infty$ for all $x \in S$.

Proposition 2.32 *Let (T, Σ, μ) be a σ-finite measure space, and let S be a separable metric space. If $\varphi : T \Rightarrow S$ is a correspondence such that $Gr_\varphi \in \Sigma^* \otimes \mathfrak{B}(S)$, then*

1. *φ is μ-measurable;*

2. *the graph of the closure correspondence $\overline{\varphi}$ belongs to $\Sigma^* \otimes \mathfrak{B}(S)$;*

3. *the graph of the interior correspondence $\operatorname{int}\varphi$ belongs to $\Sigma^* \otimes \mathfrak{B}(S)$.*

Proof. (1) Notice that for any subset B of S

$$\{t \in T : B \cap \varphi(t) \neq \emptyset\} = \operatorname*{proj}_T([T \times B] \cap Gr_\varphi).$$

Thus, proof of this part follows from Theorem 2.31 immediately.

(2) Let Z be a countable dense subset of S, and observe that

$$Gr_{\overline{\varphi}} = \bigcap_{m=1}^{\infty} \bigcup_{z \in Z} \left\{t \in T : \operatorname{dist}(z, \varphi(t)) < \frac{1}{m}\right\} \times \left\{x \in S : d(x, z) < \frac{1}{m}\right\}.$$

Now as the ball $B_{\frac{1}{m}}(z) = \left\{x \in S : d(x, z) < \frac{1}{m}\right\}$ is open, it suffices to show that $\left\{t \in T : \operatorname{dist}(z, \varphi(t)) < \frac{1}{m}\right\} = \left\{t \in T : \varphi(t) \cap B_{\frac{1}{m}}(z) \neq \emptyset\right\} \in \Sigma^*$ for $m \in \mathbb{N}$, and for $z \in Z$. But this is a direct consequence of the first part.

(3) As the correspondence $\operatorname{int}\varphi$ equals to the complement of the correspondence $\overline{\varphi^c}$, (3) follows from (2). ■

Definition 2.33 Let (T, Σ, μ) be a measure space, and let S and R be topological spaces. A mapping $f : T \times S \to R$ is said to be a *Carathéodory mapping* if both of the following holds

1. for each $t \in T$ the mapping $f(t, \cdot) : S \to R$ is continuous;

2. $\{t \in T : f(t, x) \in B\} \in \Sigma^*$ for all $x \in S$, and for all $B \in \mathfrak{B}(R)$.

Lemma 2.34 *Let (T, Σ, μ) be a measure space, S a separable metric space, and R a metric space. Let furthermore, $f : T \times S \to R$ be a Carathéodory mapping. Then $f^{-1}(B) \in \Sigma^* \otimes \mathfrak{B}(S)$ for all $B \in \mathfrak{B}(R)$.*

Proof. See Aliprantis and Border (1994: 499, Lemma 14.75). ■

Theorem 2.35 *Let (T, Σ, μ) be a measure space, and let (S, d) be a separable metric space. Let furthermore, $\varphi : T \Rightarrow S$ be a nonempty valued correspondence. Define $\delta : T \times S \to \mathbb{R}$ by $\delta(t, x) = \operatorname{dist}(x, \varphi(t))$ for $t \in T$, and for $x \in S$.*

1. *The correspondence φ is μ-measurable if and only if δ is a Carathéodory mapping. In particular, if $Gr_\varphi \in \Sigma^* \otimes \mathfrak{B}(S)$, then δ is a Carathéodory mapping.*

2. *If the correspondence φ is μ-measurable, then the graph of the closure correspondence $\overline{\varphi}$ belongs to $\Sigma^* \otimes \mathfrak{B}(S)$.*

Proof. See Aliprantis and Border (1994: 501, Theorem 14.78). ■

For a subset A of a metric space (S, d), and for $\varepsilon > 0$, put $B_\varepsilon(A) = \{x \in S : \operatorname{dist}(x, A) < \varepsilon\}$. Note that $B_\varepsilon(\{x\})$ is nothing but the ball with the center $x \in S$, and of radius ε. Instead of $B_\varepsilon(\{x\})$ we will write $B_\varepsilon(x)$. For a pair of subsets $A, E \subset S$ the *Hausdorff distance* between A and E is defined by $\sigma(A, E) = \inf\{\varepsilon > 0 : A \subset B_\varepsilon(E), E \subset B_\varepsilon(A)\}$.

Proposition 2.36 *Let (T, Σ, μ) be a σ-finite measure space, and let (S, d) be a separable metric space. If the correspondences $\varphi : T \to S$ and $\psi : T \to S$ are nonempty valued and μ-measurable, then the function $\gamma : T \to \overline{\mathbb{R}}$ defined by $\gamma(t) = \sigma(\varphi(t), \psi(t))$ is μ-measurable.*

Proof. First notice that for any $t \in T$, and for any $\varepsilon > 0$, we have that $\{t \in T : \gamma(t) < \varepsilon\} = \bigcup_q \{t \in T : \varphi(t) \subset B_q(\psi(t)), \psi(t) \subset B_q(\varphi(t))\}$, where the union is taken over all rationals q with $0 < q < \varepsilon$. Hence, it suffices to show that $E_q = \{t \in T : \varphi(t) \subset B_q(\psi(t))\} \in \Sigma^*$ for each such q. Define the correspondence $B_q : T \Rightarrow S$ as $B_q(t) = B_q(\psi(t))$. Notice that $T \backslash E_q = \left\{t \in T : \varphi(t) \cap B_q^c(t) \neq \emptyset\right\} = \mathrm{proj}_T\left(Gr_\varphi \cap Gr_{B_q^c}\right)$. Hence, by Theorem 2.31, it suffices to show that $Gr_{B_q^c}$, or equivalently Gr_{B_q}, belongs to $\Sigma^* \otimes \mathfrak{B}(S)$. Now observe that $Gr_{B_q} = \{(t, x) : \mathrm{dist}(x, \psi(t)) < q\} = \delta^{-1}[(-\infty, q)]$, where δ is the mapping defined as in Theorem 2.35. So, δ is a Carathéodory mapping by Theorem 2.35, and the proof follows from Lemma 2.34. ■

For a correspondence φ of a measure space (T, Σ, μ) into a B-space S, a mapping $f : T \to S$ is said to be a *selector* of φ provided that $f(t) \in \varphi(t)$ μ-a.e. on T. The set of all μ-measurable selectors of φ will be denoted by $\mathfrak{M}_\varphi$. We proceed with a theorem on measurable selectors.

Theorem 2.37 (Aumann) *Let (T, Σ, μ) be a σ-finite measure space, and let S be a separable B-space. Let furthermore $\varphi : T \Rightarrow S$ be a nonempty valued correspondence such that $Gr_\varphi \in \Sigma^* \otimes \mathfrak{B}(S)$. Then $\mathfrak{M}_\varphi$ is nonempty.*

Proof. See Aumann (1969). ■

The following definition of the integral of a correspondence is due to Aumann (1965).

Definition 2.38 Let φ be a correspondence from a measure space (T, Σ, μ)

into a B-space S. For $E \in \Sigma$ the *integral of* φ *over* E is defined by

$$\int_E \varphi\,(t)\,d\mu_E\,(t) = \left\{ \int_E f\,(t)\,d\mu_E(t) : \; f \in \mathcal{L}_{\varphi|_E} \right\},$$

where $\mathcal{L}_{\varphi|_E}$ stands for the set of all μ_E-integrable mappings f of the measure space (E, Σ_E, μ_E) into S such that $f(t) \in \varphi(t)$ μ_E-a.e. on E. We will write $\mathcal{L}_\varphi$ instead of $\mathcal{L}_{\varphi|_T}$.

It is important to observe that the integral of a correspondence is always well defined. It should also be clear that even if the integral of a correspondence over a set $E \subset T$ is nonempty, the integral over T might be empty.

We close this chapter with the next theorem which is known as the Theorem on Convexity.

Theorem 2.39 *Let* (T, Σ, μ) *be a probability space, and let* S *be a B-space. Let furthermore,* X *be a closed and convex subset of* S*. Define* $X' : T \Rightarrow S$ *as* $X'(t) = X$ *for all* $t \in T$*. Then* $\int_T X'(t)\,d\mu\,(t) = X$*.*

Proof. See Bourbaki (1965: 203, Chapter IV, § 6, Theorem 1). ■

Chapter 3

Theorems on Correspondences

In this chapter, we will present our results on the integral of B-space valued correspondences.

Let (T, Σ, μ) be a measure space, and let S be a B-space. To simplify the notation, when the measure μ is understood, instead of "μ-simple" we will write "simple", and similarly for the terms "μ-measurable", "μ-integrable" and "μ-a.e.". Remember that for an integrable mapping $f : T \to S$ and for $E \in \Sigma$, the vectors $\int_E f(t)d\mu(t)$, $\int_T \chi_E f(t)d\mu(t)$, and $\int_E f \mid_E (t)\, d\mu_E(t)$ coincide with one another. We denote all these vectors as $\int_E f(t)d\mu(t)$, and when μ is understood, simply as $\int_E f$. Again if a confusion is unlikely, for a correspondence $X : T \rightrightarrows S$ and a set $E \in \Sigma$, instead of $\int_E X(t)\, d\mu_E(t)$ we will write $\int_E X$. $\int X$ will stand for $\int_T X$. For a set $A \subset S$, $\operatorname{int} A$ and $\operatorname{co} A$ will denote the interior of A and the convex hull of A, respectively. As usual, a *convex body* is a convex subset of S with nonempty interior. For any pair of sets $A, B \in \Sigma^*$ with $\mu^* [(A \backslash B) \cup (B \backslash A)] = 0$, we say that A and B are *equivalent*, and write $A \sim B$. Rest of the notations to be used in this chapter is the same with notations of Chapter 3.

Our main aim here is to generalize the following two theorems of Grodal (1971) to B-space valued correspondences (for generalized versions see Theorem 3.8 and Theorem 3.14 below).

Theorem 3.1 *Let (T, Σ, μ) be a σ-finite measure space, and let $X : T \Rightarrow R^l$ be a correspondence such that $Gr_X \in \Sigma^* \otimes \mathfrak{B}\left(\mathbb{R}^l\right)$. If $f \in \mathcal{L}_X$ and there exists a set $E \in \Sigma$ with $\mu(E) > 0$ such that $f(t) \in \operatorname{int} X(t)$ a.e. on E, then $\int f \in \operatorname{int}\left(\int X\right)$.*

Theorem 3.2 *Let (T, Σ, μ) be a σ-finite measure space, and let $X : T \Rightarrow R^l$ be a convex valued correspondence such that $Gr_X \in \Sigma^* \otimes \mathfrak{B}\left(\mathbb{R}^l\right)$. If $\varphi : T \Rightarrow R^l$ is a correspondence such that $\varphi(t)$ is a relative open subset of $X(t)$ a.e. on T and $Gr_\varphi \in \Sigma^* \otimes \mathfrak{B}\left(\mathbb{R}^l\right)$, then*

$$\operatorname{int}\left(\int X\right) \cap \left(\int \varphi\right) = \operatorname{int}\left(\int \varphi\right).$$

This version of Theorem 3.2 which drops the additional assumption of convexity of the set $\int \varphi$ is due to Hüsseinov (2003).

A natural question related to Theorem 3.2 is whether, under the same assumptions, the theorem can be strengthened to read as $\int \varphi$ being relative open in $\int X$. In Hüsseinov (2003), it is shown that in case $X(t) = \mathbb{R}^l_+$ a.e. on T, the answer of the above question is affirmative, though not in general. We provide an example to show that such a strengthening is not possible for infinite dimensional Banach lattices (see Example 3.18 below).

We start with the following simple results which will be used repeatedly.

Claim 3.3 *Let (T, Σ, μ) be a σ-finite measure space, and let $\varepsilon : T \to \overline{\mathbb{R}}$ be a measurable mapping. If there exists a set $E \in \Sigma$ with $\mu(E) > 0$ such that $\varepsilon_t > 0$ a.e. on E, then there is a number $\overline{\varepsilon} > 0$, and a measurable subset E_1 of E with $0 < \mu(E_1) < \infty$ such that $\varepsilon_t > \overline{\varepsilon}$ for all $t \in E_1$.*

Proof. For $m \in \mathbb{N}$, set $F_m = \left\{t \in E : \varepsilon_t > \frac{1}{m}\right\}$. Then $E \sim \bigcup_{m\in\mathbb{N}} F_m$. Hence, there exists a natural number $\overline{m}$ such that $\mu^*(F_{\overline{m}}) > 0$. Let $\{T_n\}$ be a sequence in Σ such that $\bigcup_{n\in\mathbb{N}} T_n = T$, and $\mu(T_n) < \infty$ for all $n \in \mathbb{N}$. Then $F_{\overline{m}} = \bigcup_{n\in\mathbb{N}} F_{\overline{m}} \cap T_n$. Thus, there exists a natural number $\overline{n}$ such that $\mu^*(F_{\overline{m}} \cap T_{\overline{n}}) > 0$. To complete the proof let $\overline{\varepsilon} = \frac{1}{\overline{m}}$ and pick a μ-measurable subset E_1 of $F_{\overline{m}} \cap T_{\overline{n}}$ with $\mu(E_1) = \mu^*(F_{\overline{m}} \cap T_{\overline{n}})$. ■

Claim 3.4 *Let (T, Σ, μ) be a σ-finite measure space, and let $\varepsilon : T \to \overline{\mathbb{R}}$ be a measurable mapping such that $\varepsilon_t > 0$ a.e. on T. Then there exists an integrable mapping $\gamma : T \to \mathbb{R}$ such that $0 < \gamma(t) < \varepsilon_t$ a.e. on T.*

Proof. First we shall show that there exists an integrable mapping $\gamma' : T \to \mathbb{R}$ such that $\gamma'(t) > 0$ for all $t \in T$. If $\mu(T) < \infty$, we can let γ' be any strictly positive, constant mapping. Now assume $\mu(T) = \infty$, and let $\{T_n\}$ be a sequence of mutually disjoint, measurable sets such that $\bigcup_{n\in\mathbb{N}} T_n = T$, and $\mu(T_n) < \infty$ for all $n \in \mathbb{N}$. Then, as $\sum_{n\in\mathbb{N}} \mu(T_n) = \mu(T) = \infty$, by passing to a subsequence if necessary, we can assume that $\mu(T_n) > 0$ for all $n \in \mathbb{N}$. Put $\gamma'_m = \sum_{n\leq m} \chi_{T_n} \left(2^n \mu(T_n)\right)^{-1}$ for $m \in \mathbb{N}$, and $\gamma' = \sum_{n\in\mathbb{N}} \chi_{T_n} \left(2^n \mu(T_n)\right)^{-1}$. It can easily be checked that the sequence of simple functions $\{\gamma'_m\}$ determines γ'.

Now let us define $\gamma : T \to \mathbb{R}$ as $\gamma(t) = \min\left\{\frac{1}{2}\varepsilon_t, \gamma'(t)\right\}$ for all $t \in T$. Then by Proposition 2.12 and Theorem 2.25 γ is integrable, and $0 < \gamma(t) < \varepsilon_t$ a.e. on T. ■

Claim 3.5 *Let (T, Σ, μ) be a measure space, and let $\varepsilon : T \to \overline{\mathbb{R}}$ be a measurable mapping such that $\varepsilon_t > 0$ a.e. on T. Then there exist a measurable mapping $\gamma : T \to \mathbb{R}$ which takes countably many values, and which satisfies $0 < \gamma(t) < \varepsilon_t$ a.e. on T.*

Proof. For each $m \in \mathbb{N}$ set $F_m = \left\{t \in T : \varepsilon_t > \frac{1}{m}\right\}$. Set furthermore

$E_1 = F_1$, and $E_m = F_m \backslash \bigcup_{n<m} F_n$ for each natural number $m \geq 2$. Define the mapping $\gamma : T \to \mathbb{R}$ as $\gamma(t) = \frac{1}{m}$ if $t \in E_m$ for some $m \in \mathbb{N}$ and $\gamma(t) = 0$ otherwise. Clearly, γ is well defined and satisfies properties listed in the claim. ■

Lemma 3.6 *Let each element x of a B-space $(S, \|\cdot\|)$ be a real valued mapping of a set P and satisfy $|x(p)| \leq \|x\|$ for all $p \in P$. Assume moreover that the vector structure of S is defined pointwisely. Let furthermore, $f : t \to f_t$ be a μ-integrable mapping of a measure space (T, Σ, μ) into S. Then for each $p \in P$ the real valued mapping $f(p) : t \to f_t(p)$ is μ-integrable, and the number $\int_T f_t(p)\, d\mu(t)$ equals to $F(p)$, where $F = \int_T f_t d\mu(t) \in S$.*

Proof. First assume that f is a μ-integrable simple mapping. Let $\{E_i : 1 \leq i \leq n\}$ be a collection of mutually disjoint, μ-measurable sets, and let $\{x_i : 1 \leq i \leq n\}$ be a subset of S such that $\mu(E_i) < \infty$ whenever $x_i \neq 0$, and $f = \sum_{i=1}^{n} \chi_{E_i} x_i$. Now note that for any $p \in P$, and for any $x \in S$, $x(p) \neq 0$ implies $x \neq 0$. Thus, the mapping $f(p) = \sum_{i=1}^{n} \chi_{E_i} x_i(p)$ is μ-integrable simple, and by definition of the integral, $\int_T f_t(p)\, d\mu(t) = \sum_{i=1}^{n} \mu(E_i)\, x_i(p) = F(p)$.

For the general case, let $\{f^n\}$ be a sequence of μ-integrable simple mappings that determines f, and let $p \in P$. For each $n \in \mathbb{N}$ put $F^n = \int_T f_t^n d\mu(t) \in S$, and note that by the first part of the proof

$$F^n(p) = \int_T f_t^n(p)\, d\mu(t) \quad \text{for all } n \in \mathbb{N}. \tag{3.1}$$

Moreover, as $|F(p) - F^n(p)| \leq \|F - F^n\|$ for all $n \in \mathbb{N}$, and as $\|F - F^n\| \longrightarrow 0$ by definition of the integral,

$$F(p) = \lim_n F^n(p). \tag{3.2}$$

Now since $|f_t(p) - f_t^n(p)| \leq \|f_t - f_t^n\|$ for all $n \in \mathbb{N}$, and for all $t \in T$, and since $\{f^n\}$ converges in μ-measure to f, $\{f^n(p)\}$ converges in μ-measure to

$f(p)$. Note that by the first part of the proof $|f^n(p) - f^m(p)|$ is μ-integrable simple, and clearly,

$$\int_T |f_t^n(p) - f_t^m(p)|\, d\mu(t) \leq \int_T \|f_t^n - f_t^m\|\, d\mu(t) \quad \text{for all } n, m \in \mathbb{N}.$$

So, the sequence $\{f^n(p)\}$ determines $f(p)$, and by definition of the integral,

$$\int_T f_t(p)\, d\mu(t) = \lim_n \int_T f_t^n(p)\, d\mu(t).$$

Hence, by (3.1) and (3.2) the proof is complete. ∎

We proceed with a generalization of Theorem 3.1.

Theorem 3.7 *Let (T, Σ, μ) be a σ-finite measure space, and let $(S, \|\cdot\|)$ be a Banach space. If the complement of a correspondence $X : T \Rightarrow S$ is measurable and there exists a set $E \in \Sigma$ with $\mu(E) > 0$ such that X is open valued a.e. on E, then $\int X$ is open.*

Proof. Let $\overline{x} \in \int X$. Then $\overline{x} = \int f$ for some selector $f \in \mathcal{L}_X$. For $t \in E$, put $\varepsilon_t = \sup\{\varepsilon > 0 : B_\varepsilon(f(t)) \subset X(t)\}$. As $X(t)$ is open valued a.e. on E, $\varepsilon_t > 0$ a.e. on E. Clearly, for $\alpha > 0$,

$$\{t \in E : \varepsilon_t > \alpha\} = \bigcup_{\substack{q>\alpha \\ q\in\mathbb{Q}}} \{t \in E : B_q(f(t)) \subset X(t)\}.$$

By Corollary 2.19, there exists a sequence $\{f_n\}$ of simple mappings of T into S such that $f_n(t) \longrightarrow f(t)$ a.e. on T. Then it is easily seen that, for $q > 0$, $\{t \in E : B_q(f(t)) \subset X(t)\}$ is equivalent to

$$\bigcap_{\substack{0<r<q \\ r\in\mathbb{Q}}} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \{t \in E : B_r(f_n(t)) \subset X(t)\}.$$

Thus, as f_n is simple for $n \in \mathbb{N}$, measurability of the mapping $t \to \varepsilon_t$ follows from measurability of the complement of X. So, by Claim 3.3, there is a number $\overline{\varepsilon} > 0$, and a measurable subset E_1 of E with $0 < \mu(E_1) < \infty$ such that $\varepsilon_t > \overline{\varepsilon}$ for $t \in E_1$. Now we shall show that

$$B_{\overline{\varepsilon}\mu(E_1)}\left(\int_{E_1} f\right) \subset \left\{\int_{E_1} h : h \in \mathcal{L}_X\right\}. \tag{3.3}$$

Let $z \in B_{\overline{\varepsilon}\mu(E_1)}\left(\int_{E_1} f\right)$. Then as $\frac{\left\|z-\int_{E_1} f\right\|}{\mu(E_1)} < \overline{\varepsilon}$, $f(t)+\frac{z-\int_{E_1} f}{\mu(E_1)}$ belongs to $X(t)$ for $t \in E_1$. Define $h : T \to S$ by

$$h(t) = \begin{cases} f(t) + \frac{z-\int_{E_1} f}{\mu(E_1)} & \text{for } t \in E_1, \\ f(t) & \text{for } t \in T \backslash E_1. \end{cases}$$

Clearly, $h \in \mathcal{L}_X$ and $\int_{E_1} h = z$. This establishes (3.3).

Since $\int X = \left\{\int_{E_1} h : h \in \mathcal{L}_X\right\} + \left\{\int_{T\backslash E_1} h : h \in \mathcal{L}_X\right\}$, from (3.3), we see that the ball $B_{\overline{\varepsilon}\mu(E_1)}(\overline{x}) = B_{\overline{\varepsilon}\mu(E_1)}\left(\int_{E_1} f\right) + \int_{T\backslash E_1} f$ is contained in $\int X$. ■

Theorem 3.8 *Let (T, Σ, μ) be a σ-finite measure space, and let $(S, \|\cdot\|)$ be a Banach space. Let furthermore, the complement of a correspondence $X : T \Rightarrow S$ be measurable. If $f \in \mathcal{L}_X$ and there exists a set $E \in \Sigma$ with $\mu(E) > 0$ such that $f(t) \in \operatorname{int} X(t)$ a.e. on E, then $\int f \in \operatorname{int}\left(\int X\right)$.*

Proof. Define $X' : T \Rightarrow S$ as $X'(t) = \operatorname{int} X(t)$ for $t \in E$, and $X'(t) = X(t)$ for $t \in T\backslash E$. Then $\int f \in \int X' \subset \int X$, and $\int X'$ is open by Theorem 3.7. ■

Remark 3.9 **(1)** The reader should remember that for a correspondence X from (T, Σ, μ) into a separable metric space S, if graph of X belongs to the collection $\Sigma^* \otimes \mathfrak{B}(S)$, then so does the graph of the complement of X, and by Proposition 2.32(1), the complement of X is measurable. In the rest of the study we will make use of this fact without further mention.

(2) If X is a correspondence satisfying conditions of Theorem 3.7, Theorem 3.8 implies that every element of $\int X$ is an interior point. Thus, in fact, these two theorems are equivalent.

The following result generalizes Proposition 1 of Hüsseinov (2003), and its proof is almost the same with that of the mentioned proposition. For the sake completeness we repeat this proof.

Proposition 3.10 *Let (T, Σ, μ) be a σ-finite measure space, and let $(S, \|\cdot\|)$ be a Banach space. Let furthermore, the complement of a correspondence $X : T \Rightarrow S$ be measurable, $\mathfrak{M}_{\operatorname{int} X} \neq \emptyset$, and $X(t)$ be convex a.e. on T. Then*

$$\operatorname{int}\left(\int X\right) = \int \operatorname{int} X.$$

Proof. Since the other inclusion is immediate from Theorem 3.7, it suffices to show that $\operatorname{int}\left(\int X\right) \subset \int \operatorname{int} X$. Note that $\int \operatorname{int} X$ is an open convex set, and hence, $\int \operatorname{int} X = \operatorname{int}\left(\overline{\int \operatorname{int} X}\right)$. So, if we can show that $\int X$ is contained in $\overline{\int \operatorname{int} X}$, the proof will be complete. To this end, let $x = \int f \in \int X$ where $f \in \mathcal{L}_X$. We shall first show that $\mathcal{L}_{\operatorname{int} X}$ is nonempty. Pick $g \in \mathfrak{M}_{\operatorname{int} X}$. Set $E = \{t \in T : \|g(t) - f(t)\| > 0\}$. By Claim 3.4, there exists a μ_E^*-integrable mapping $\gamma : E \to \mathbb{R}$ such that $0 < \gamma_t < \|g(t) - f(t)\|$ a.e. on E. Define, $h : T \to S$ as

$$h(t) = \begin{cases} f(t) + \frac{\gamma_t (g(t) - f(t))}{\|g(t) - f(t)\|} & \text{for } t \in E, \\ g(t) & \text{for } t \in T \backslash E. \end{cases}$$

Note that $\|h(t) - f(t)\| \leq \varepsilon_t$ a.e. on T, where $\varepsilon : t \to \varepsilon_t$ is the integrable mapping defined by $\varepsilon_t = \gamma_t$ for $t \in E$ and $\varepsilon_t = 0$ for $t \in T \backslash E$. Thus, h belongs to $\mathcal{L}_{\operatorname{int} X}$ by Theorem 2.25.

Now, for all $m \in \mathbb{N}$ set $f_m = f + \frac{1}{m}(h - f)$. Then $\left\{\int f_m\right\}$ is a convergent sequence in $\int \operatorname{int} X$, and $\lim_m \int f_m = \int f = x$. Thus, x belongs to the set $\overline{\int \operatorname{int} X}$. ■

Remark 3.11 In Proposition 3.10, provided that S is separable, $Gr_X \in \Sigma^* \otimes \mathfrak{B}(S)$ and $\operatorname{int} X(t) \neq \emptyset$ a.e. on T, existence of a measurable selector of $\operatorname{int} X$ is guaranteed by Proposition 2.32(3) and Theorem 2.37.

The reader should note that measurability of the complement of a correspondence $X : T \Rightarrow S$ bears no information about whether $\mathfrak{M}_{\operatorname{int} X}$ is empty or not. The following example underlines this point.

Example 3.12 Let $I = [0,1]$, and let L_∞ be the set of all essentially bounded real functions on I endowed with the usual norm $\|\cdot\|_\infty$. For $t \in I$, define $f_t \in L_\infty$ as $f_t(x) = 1$ if $x \le t$ and $f_t(x) = -1$ if $x > t$. Put $F(t) = \operatorname{co}\left[0 \cup \overline{B_1(f_t)}\right]$ for $t \in I$. Clearly, for any $t, t' \in I$ with $t \neq t'$ $\|f_t - f_{t'}\|_\infty = 2$, and therefore, $\operatorname{int} F(t) \cap \operatorname{int} F(t') = \emptyset$. So, for any nonempty and open set $G \subset L_\infty$, the set $\{t \in I : G \subset F(t)\}$ is either empty or a singleton. Thus, the complement of F is Lebesgue measurable. Now let $g : I \to L_\infty$ be an arbitrary mapping such that $g(t) \in \operatorname{int} F(t)$ for $t \in I$, and let $P \subset I$ be a nonmeasurable set. Put $V = \bigcup_{t\in P} \operatorname{int} F(t)$. Then $g^{-1}(V) = P$, and therefore, $\mathfrak{M}_{\operatorname{int} F} = \emptyset$. Also note that $\mathfrak{L}_F \neq \emptyset$. In particular, $0 \in F(t)$ for $t \in I$.

The next claim will be useful in generalization of Theorem 3.2.

Claim 3.13 *Let $(S, \|\cdot\|)$ be a normed space, and let ε be positive number. Let furthermore, $\{x_n\}$ and $\{s_n\}$ be convergent sequences in S. Set $x = \lim_n x_n$, $s = \lim_n s_n$, and let φ be a subset of S which contains x.*

1. *For each pair of numbers q, r with $0 < q < r < \varepsilon$ there exists a natural number $\overline{m}$ such that $\left(B_r(x_n) \backslash \overline{B_q(x_n)}\right) \cap \operatorname{co}[x_n \cup B_r(s_n)] \subset B_\varepsilon(x) \cap \operatorname{co}[x \cup B_\varepsilon(s)]$ for all $n \ge \overline{m}$.*

2. *Conversely, if for each pair of rational numbers q, r with $0 < q < r < \varepsilon$ there exists a natural number $\overline{m}$ such that for all $n \ge \overline{m}$ the set $\left(B_r(x_n) \backslash \overline{B_q(x_n)}\right) \cap \operatorname{co}[x_n \cup B_r(s_n)]$ is contained in φ, then $B_\varepsilon(x) \cap \operatorname{co}[x \cup B_\varepsilon(s)] \subset \varphi$.*

Proof. **(1)** Fix a pair of numbers q, r with $0 < q < r < \varepsilon$. Pick a natural number $\overline{m}$ such that for all $n \geq \overline{m}$

$$\begin{aligned} \|x_n - x\| &< \min\left\{\varepsilon - r, r, \frac{q\left(\frac{\varepsilon - r}{2}\right)}{\|x - s\| + 2r + \left(\frac{\varepsilon - r}{2}\right)}\right\}, \qquad (3.4) \\ \|s_n - s\| &< \frac{\varepsilon - r}{2}. \end{aligned}$$

Let z be a point in the set $\left(B_r(x_n) \backslash \overline{B_q(x_n)}\right) \cap \operatorname{co}[x_n \cup B_r(s_n)]$ for some $n \geq \overline{m}$. First, note that $\|z - x\| \leq \|z - x_n\| + \|x_n - x\| < r + \varepsilon - r = \varepsilon$. So, it suffices to show that $z \in \operatorname{co}[x \cup B_\varepsilon(s)]$. Since $z \in \operatorname{co}[x_n \cup B_r(s_n)] \backslash \overline{B_q(x_n)}$, there exist a point $\overline{y} \in B_r(s_n)$ and a number $\gamma \in [0, 1)$ such that $z = \gamma x_n + (1 - \gamma)\overline{y}$. Moreover,

$$(1 - \gamma)\|x_n - \overline{y}\| = \|x_n - z\| > q. \qquad (3.5)$$

Set $y = \frac{z - \gamma x}{1 - \gamma}$. We shall complete the proof by showing that y belongs to $B_\varepsilon(s)$. Now, as $\|y - s\| \leq \|y - \overline{y}\| + \|\overline{y} - s\|$, and

$$\|\overline{y} - s\| \leq \|\overline{y} - s_n\| + \|s_n - s\| < r + \frac{\varepsilon - r}{2}, \qquad (3.6)$$

it suffices to show that $\|y - \overline{y}\| \leq \frac{\varepsilon - r}{2}$. Now, by (3.4), (3.5), and (3.6),

$$\begin{aligned} \|y - \overline{y}\| &= \frac{\gamma}{(1 - \gamma)}\|x_n - x\| \\ &< \frac{\gamma}{q}\|x_n - \overline{y}\|\,\|x_n - x\| \\ &\leq \frac{\gamma}{q}\left(\|x_n - x\| + \|x - s\| + \|s - \overline{y}\|\right)\|x_n - x\| \\ &\leq \frac{\gamma}{q}\left(r + \|x - s\| + r + \frac{\varepsilon - r}{2}\right)\|x_n - x\| \\ &< \frac{\gamma}{q}q\left(\frac{\varepsilon - r}{2}\right) = \gamma\left(\frac{\varepsilon - r}{2}\right) < \left(\frac{\varepsilon - r}{2}\right). \end{aligned}$$

This completes the proof of (1).

(2) Let $w \in B_\varepsilon(x) \cap \operatorname{co}[x \cup B_\varepsilon(s)]$. We need to show that $w \in \varphi$. If $w = x$, this is true by hypotheses, so assume that is not the case. Set

$w = \delta x + (1-\delta)\, y$ where $y \in B_\varepsilon(s)$, and $\delta \in [0,1)$. Pick rational numbers r and q such that $\max\{\|w-x\|, \|y-s\|\} < r < \varepsilon$, and $0 < q < \|w-x\|$. By hypotheses, there exists a number $\overline{m} \in \mathbb{N}$ such that $\left(B_r(x_n) \setminus \overline{B_q(x_n)}\right) \cap \operatorname{co}[x_n \cup B_r(s_n)] \subset \varphi$ for all $n \geq \overline{m}$. Pick a natural number $n_0 \geq \overline{m}$ such that

$$\begin{aligned} \|x_{n_0} - x\| &< \min\left\{r - \|w-x\|, \|w-x\| - q, (1-\delta)\frac{r - \|y-s\|}{2}\right\}, \\ \|s_{n_0} - s\| &< \frac{r - \|y-s\|}{2}. \end{aligned}$$

Then, $\|w - x_{n_0}\| \leq \|w-x\| + \|x - x_{n_0}\| < r$. Moreover, $\|w - x_{n_0}\| \geq \|w-x\| - \|x_{n_0} - x\| > q$. Hence, w belongs to the set $\left(B_r(x_{n_0}) \setminus \overline{B_q(x_{n_0})}\right)$. Thus, what remains to show is that $w \in \operatorname{co}[x_{n_0} \cup B_r(s_{n_0})]$. Set $\overline{y} = \frac{w - \delta x_{n_0}}{1-\delta}$. Then, $\|y - \overline{y}\| = \frac{\delta}{1-\delta}\|x_{n_0} - x\| \leq \frac{\|x_{n_0} - x\|}{1-\delta} < \frac{r - \|y-s\|}{2}$. Hence, $\|\overline{y} - s\| \leq \|\overline{y} - y\| + \|y - s\| < \frac{r + \|y-s\|}{2}$. Thus, $\|\overline{y} - s_{n_0}\| \leq \|\overline{y} - s\| + \|s - s_{n_0}\| < \frac{r+\|y-s\|}{2} + \frac{r - \|y-s\|}{2} = r$. So, $\overline{y} \in B_r(s_{n_0})$, and this completes the proof. ■

We now show that Theorem 3.2 is valid for B-space valued correspondences.

Theorem 3.14 *Let (T, Σ, μ) be a σ-finite measure space, and let $(S, \|\cdot\|)$ be a Banach space. Let furthermore, the complement of a correspondence $X : T \rightrightarrows S$ be measurable, $\mathfrak{M}_{\operatorname{int} X} \neq \emptyset$, and $X(t)$ be convex a.e. on T. If the complement of a correspondence $\varphi : T \rightrightarrows S$ is measurable and $\varphi(t)$ is a relative open subset of $X(t)$ a.e. on T, then*

$$\operatorname{int}\left(\int X\right) \cap \left(\int \varphi\right) = \operatorname{int}\left(\int \varphi\right). \tag{3.7}$$

Proof. As the other inclusion is trivial, it suffices to show that $\operatorname{int}\left(\int X\right) \cap \left(\int \varphi\right) \subset \operatorname{int}\left(\int \varphi\right)$. Let $\overline{x} \in \operatorname{int}\left(\int X\right) \cap \left(\int \varphi\right)$. Then there exists a selector $f \in \mathcal{L}_\varphi$ such that $\overline{x} = \int f$. Since X satisfies conditions of Proposition 3.10, there exists a further selector $g \in \mathcal{L}_{\operatorname{int} X}$ such that $\overline{x} = \int g$.

For $t \in T$ put

$$\varepsilon_t = \sup \{\varepsilon > 0 : B_\varepsilon (f(t)) \cap \operatorname{co} [f(t) \cup B_\varepsilon(g(t))] \subset \varphi (t)\} .$$

Since $g \in \mathcal{L}_{\operatorname{int} X}$, X is convex valued, and $\varphi (t)$ is relative open in $X(t)$, $\varepsilon_t > 0$ a.e. on T. To show that the mapping $t \to \varepsilon_t$ is measurable, as in the proof of Theorem 3.7, it suffices to show that $T^\varepsilon \in \Sigma^*$ for each $\varepsilon > 0$, where $T^\varepsilon = \{t \in T : B_\varepsilon (f(t)) \cap \operatorname{co} [f(t) \cup B_\varepsilon(g(t))] \subset \varphi (t)\}$. By Corollary 2.19, there exist a pair of sequences $\{f_n\}$, $\{g_n\}$ of simple mappings of T into S such that $f_n (t) \longrightarrow f (t)$ and $g_n (t) \longrightarrow g (t)$ a.e. on T. From Claim 3.13 it follows that, for $\varepsilon > 0$, T^ε is equivalent to

$$\bigcap_{\substack{0<q<r<\varepsilon \\ r,q\in\mathbb{Q}}} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \left\{t \in T : \left(B_r (f_n(t)) \setminus \overline{B_q (f_n(t))}\right) \cap C_{n,r} (t) \subset \varphi (t)\right\},$$

where $C_{n,r} (t) = \operatorname{co} [f_n(t) \cup B_r(g_n(t))]$ for $t \in T$, $n \in \mathbb{N}$, and $r \in \mathbb{Q}$. Thus, measurability of the mapping $t \to \varepsilon_t$ follows from measurability of the complement of φ. So, by Claim 3.3, there is a number $\overline{\varepsilon} > 0$, and a set $T_0 \in \Sigma$ with $0 < \mu(T_0) < \infty$ such that

$$\varepsilon_t > \overline{\varepsilon} \quad \text{for } t \in T_0. \tag{3.8}$$

Put $h^m(t) = f(t) + \frac{1}{m}(g(t) - f(t))$ for $t \in T$, and $m \in \mathbb{N}$. Then, by continuity of measure (Theorem 2.5), $\mu^* \left(\left\{t \in T_0 : \|h^m(t) - f(t)\| \geq \frac{\overline{\varepsilon}}{2}\right\}\right) \downarrow 0$. So, there exists a number $m_0 \in \mathbb{N}$ and a measurable subset T_1 of T_0 with $\mu(T_1) > 0$ such that

$$\|h^m(t) - f(t)\| < \frac{\overline{\varepsilon}}{2} \quad \text{for } t \in T_1, \text{ and for } m \geq m_0. \tag{3.9}$$

Since $g - f$ is integrable over T, by Theorem 2.24(4), there exists a set $\widetilde{T} \in \Sigma$ with $\mu(\widetilde{T}) < \infty$ such that

$$\int_{T\setminus\widetilde{T}} \|g - f\| < \frac{\overline{\varepsilon}\mu(T_1)}{4}. \tag{3.10}$$

Put $A_m = \{t \in T : \|h^m(t) - f(t)\| \geq \varepsilon_t\}$ and $\delta_m = \mu^*\left(A_m \cap \widetilde{T}\right)$ for $m \in \mathbb{N}$. Clearly, for each $t \in T$, there exists a number $m(t) \in \mathbb{N}$ such that $\left\|h^{m(t)}(t) - f(t)\right\| < \varepsilon_t$, and so $\bigcap_{m \in \mathbb{N}} A_m = \emptyset$. Hence, as $\mu(\widetilde{T}) < \infty$, by continuity of measure, $\delta_m \longrightarrow 0$.

By absolute continuity of integral (Theorem 2.24(5)), there exists a number $\delta > 0$ such that

$$F \in \Sigma \text{ and } \mu(F) < \delta \Longrightarrow \int_F \|g - f\| < \frac{\overline{\varepsilon}\mu(T_1)}{4}. \tag{3.11}$$

Pick a number $m_1 \in \mathbb{N}$ such that $m_1 > \max\{2, m_0\}$ and $\delta_{m_1} < \delta$. Now we will show that

$$B_{\frac{\overline{\varepsilon}}{m_1}}\left(h^{m_1}(t)\right) \subset \varphi(t) \quad \text{for } t \in T_1. \tag{3.12}$$

Let $\overline{t} \in T_1$, and let $y \in B_{\frac{\overline{\varepsilon}}{m_1}}\left(h^{m_1}(\overline{t})\right)$. Then since $\left\|y - h^{m_1}(\overline{t})\right\| < \frac{\overline{\varepsilon}}{m_1}$, $\left\|f(\overline{t}) + m_1(y - f(\overline{t})) - g(\overline{t})\right\| < \overline{\varepsilon}$, that is, $f(\overline{t}) + m_1(y - f(\overline{t})) \in B_{\overline{\varepsilon}}\left(g(\overline{t})\right)$. Thus, y belongs to $\operatorname{co}\left[f(\overline{t}) \cup B_{\overline{\varepsilon}}\left(g(\overline{t})\right)\right]$. Moreover, by (3.9), $\left\|y - f(\overline{t})\right\| \leq \left\|y - h^{m_1}(\overline{t})\right\| + \left\|h^{m_1}(\overline{t}) - f(\overline{t})\right\| < \overline{\varepsilon}$. So, by (3.8), y belongs to $\varphi(\overline{t})$. This proves (3.12).

Define $z : T \to S$ as

$$z(t) = \begin{cases} f(t) & \text{for } t \in A_{m_1}, \\ h^{m_1}(t) + \frac{u}{\mu(T_1)} & \text{for } t \in T_1, \\ h^{m_1}(t) & \text{for } t \in T \backslash \left(A_{m_1} \cup T_1\right), \end{cases}$$

where $u = \int_{A_{m_1}} (h^{m_1} - f)$. Note that by (3.8) and (3.9), $A_{m_1} \cap T_1 = \emptyset$, and hence, z is correctly defined. Since

$$\begin{aligned} \|u\| &\leq \textstyle\int_{A_{m_1} \cap \widetilde{T}} \|h^{m_1} - f\| + \int_{A_{m_1} \backslash \widetilde{T}} \|h^{m_1} - f\| \\ &= \tfrac{1}{m_1}\left(\textstyle\int_{A_{m_1} \cap \widetilde{T}} \|g - f\| + \int_{A_{m_1} \backslash \widetilde{T}} \|g - f\|\right), \end{aligned}$$

by (3.10) and (3.11), $\|u\| < \frac{\overline{\varepsilon}\mu(T_1)}{2m_1}$. Thus, by (3.12), $z(t) \in \operatorname{int} \varphi(t)$ for $t \in T_1$. Clearly, $z(t) \in \varphi(t)$ a.e. on T, and so, we have $z \in \mathcal{L}_\varphi$. Thus, from Theorem 3.8, $\int z = \int h^{m_1} = \overline{x} \in \operatorname{int}\left(\int \varphi\right)$. ■

As we will see in the following examples, in Theorem 3.14, convexity of the values of the correspondence $X : T \Rightarrow S$ and the condition "$\mathfrak{M}_{\mathrm{int}\, X} \neq \emptyset$", which reduces to "$\mathrm{int}\, X(t) \neq \emptyset$ a.e. on T" in separable case, are indispensable conditions.

Example 3.15 Let l be the vector space of all real sequences $x = \{x_n\}$ such that $\sum_{n=1}^{\infty} n |x_n| < \infty$. Clearly, l is complete with respect to norm $\|\cdot\|_l$ which is defined as $\|x\|_l = \sum_{n=1}^{\infty} n |x_n|$ for $x \in l$. μ will denote the Lebesgue measure on $I = (0, 1]$. Define the correspondences $X : I \Rightarrow l$ and $\varphi : I \Rightarrow l$ as

$$\begin{aligned} X(t) &= \{x \in l : x_n \geq 0, \ \forall n > m\} \text{ for } t \in \left(\frac{1}{m+1}, \frac{1}{m}\right] \ (m \in \mathbb{N}), \\ \varphi(t) &= X(t) \cap \{x \in l : x_n > 0, \ \forall n \leq m\} \text{ for } t \in \left(\frac{1}{m+1}, \frac{1}{m}\right] \ (m \in \mathbb{N}). \end{aligned}$$

Obviously, the values of X are closed and convex sets with empty interior. Moreover, $\varphi(t)$ is open relative to $X(t)$ for $t \in I$. Since both X and φ are constant on $\left(\frac{1}{m+1}, \frac{1}{m}\right]$ for $m \in \mathbb{N}$, Gr_X and Gr_φ belong to $\Sigma \otimes \mathcal{B}(l)$, where Σ is the collection of μ-measurable subsets of I. We shall now show that (3.7) does not hold due to violation of the condition "$\mathfrak{M}_{\mathrm{int}\, X} \neq \emptyset$".

Let $x \in l$. Define $f : I \to l$ as $f(t) = \sum_{n=1}^{m} n x_n e^n$ for $t \in \left(\frac{1}{m+1}, \frac{1}{m}\right]$ ($m \in \mathbb{N}$), where $e^n \in l$ is the n^{th} unit coordinate vector. Then $f(t) \in X(t)$ for all $t \in I$. Moreover, if $x_n > 0$ for all $n \in \mathbb{N}$, then $f(t) \in \varphi(t)$ for all $t \in I$. So, if can show that f is μ-integrable and $\int_T f(t)\, d\mu(t) = x$, we can conclude that $\int_T X(t)\, d\mu(t) = l$, and in view of Lemma 3.6, $\int_T \varphi(t)\, d\mu(t) = \{x \in l : x_n > 0, \ \forall n \in \mathbb{N}\}$. Then as $\mathrm{int}\left(\int_T \varphi(t)\, d\mu(t)\right) = \emptyset$, the example will be complete.

To this end, set $f_1 = \chi_I x_1 e^1$, and for each natural number $m \geq 2$ inductively define the simple mapping $f_m : I \to l$ as

$$f_m(t) = \begin{cases} f_{m-1}(t) & \text{for } t \in \left(\frac{1}{m}, 1\right], \\ f_{m-1}(t) + m x_m e^m & \text{for } t \in \left(0, \frac{1}{m}\right]. \end{cases}$$

Now as $\int_T f_1(t) d\mu(t) = x_1 e^1$, for each natural number $m \geq 2$

$$\int_T f_m(t) d\mu(t) = \int_T f_{m-1}(t) d\mu(t) + x_m e^m = \sum_{n \leq m} x_n e^n.$$

Thus, $\lim_m \int_T f_m(t) d\mu(t) = x$. Hence, what remains to show is that the sequence $\{f_m\}$ determines f. Now note that for a fixed $t \in \left(\frac{1}{\overline{m}+1}, \frac{1}{\overline{m}}\right]$

$$f_m(t) = f_{\overline{m}}(t) = \sum_{n=1}^{\overline{m}} n x_n e^n = f(t) \quad \text{for all } m > \overline{m}.$$

Thus, $f_m(t) \longrightarrow f(t)$ for all $t \in T$. So, by Theorem 2.17(2), $\{f_m\}$ converges to f in μ-measure. Moreover, for all $t \in I$, and for all $k, h \in \mathbb{N}$ with $h > k$ $\|f_h(t) - f_k(t)\|_l \leq \sum_{m=k+1}^{h} \|f_m(t) - f_{m-1}(t)\|_l$. Hence,

$$\begin{aligned} \int_T \|f_h(t) - f_k(t)\|_l \, d\mu(t) &\leq \sum_{m=k+1}^{h} \int_T \|f_m(t) - f_{m-1}(t)\|_l \, d\mu(t) \\ &= \sum_{m=k+1}^{h} \frac{1}{m} \|m x_m e^m\|_l = \sum_{m=k+1}^{h} m |x_m|. \end{aligned}$$

So, as $\sum_{n=1}^{\infty} n |x_n| < \infty$, we conclude that $\{f_m\}$ determines f.

Example 3.16 Here we show that convexity of values of the correspondence X is an indispensable condition in Theorem 3.14. Let $a = (2,0)$, $b = (1,2)$, and $c = (1,1)$ be points in $\mathbb{R}^2$, and let $X' = \text{co}\{0, a, b\} \setminus [\text{int}(\text{co}\{0, a, c\}) \cup (0, a)]$. μ will denote the Lebesgue measure on $[0,1]$. Put $X(t) = X'$ for $t \in [0,1]$, and

$$\varphi(t) = \begin{cases} X' \cap H_1 & \text{for } t \in \left[0, \frac{1}{2}\right], \\ X' \cap H_2 & \text{for } t \in \left(\frac{1}{2}, 1\right], \end{cases}$$

where $H_1 = \left\{x \in \mathbb{R}^2 : x_1 < 1, x_2 < \frac{1}{2}\right\}$, and $H_2 = \left\{x \in \mathbb{R}^2 : x_1 > 1, x_2 < \frac{1}{2}\right\}$. Then, it can easily be shown that the set $\frac{1}{2}\left[(0,c) \cap H_1\right] + \frac{1}{2}\left[(a,b) \cap H_2\right]$ is contained in $\left[\operatorname{int} \int X(t)\, d\mu(t) \cap \int \varphi(t)\, d\mu(t)\right] \setminus \operatorname{int} \int \varphi(t)\, d\mu(t)$.

Corollary 3.17 *Let (T, Σ, μ) be a probability space, and let $(S, \|\cdot\|)$ be a Banach space. If the complement of a correspondence $\varphi : T \Rightarrow S$ is measurable and $\varphi(t)$ is a relative open subset of a closed convex set $X \subset S$ a.e. on T, then* $\operatorname{int} X \cap \int \varphi = \operatorname{int} \int \varphi$.

Proof. Define $X' : T \Rightarrow S$ by $X'(t) = X$ for $t \in T$. Then, by Theorem 2.39, $\int X' = X$. If $\operatorname{int} X = \emptyset$, the equality holds trivially. If $\operatorname{int} X \neq \emptyset$, we can apply Theorem 3.14. ■

We now show that, in contrast to the finite dimensional case, for the nonnegative cone X of an infinite dimensional Banach lattice, Corollary 3.17 cannot be strengthened to read as $\int \varphi$ being relative open in X.

Example 3.18 Let C be the set of all continuous real functions on $[0,1]$ endowed with the usual norm $\|\cdot\|_\infty$, and let C_+ be the nonnegative cone of C with respect to usual order on C. For $B \subset C_+$, $\operatorname{ri} B$ will denote the interior of B relative to C_+, and μ will denote the Lebesgue measure. Let $x(s) \equiv s$ on $[0,1]$, and $T = (0,1]$. Define $F : T \Rightarrow C$ as $F(t) = B_t(x) \cap C_+$ for $t \in (0,1]$. Observe that

$$Gr_F = \{(t,y) : \|x - y\|_\infty < t\} \cap (T \times C_+)$$

and

$$\begin{aligned}
\{(t,y) : \|x - y\|_\infty < t\} &= \bigcup_{\substack{q>0 \\ q \in \mathbb{Q}}} \{(t,y) : \|x - y\|_\infty < q\} \cap \{(t,y) : q < t\} \\
&= \bigcup_{\substack{q>0 \\ q \in \mathbb{Q}}} \left[T \times B_q(x)\right] \cap \left[(q,1] \times C\right].
\end{aligned}$$

Hence, Gr_F belongs to $\Sigma \otimes \mathfrak{B}(C)$, where Σ is the Lebesgue measurable subsets of $(0,1]$. Clearly, F satisfies also other conditions of Theorem 3.14. However, $B = \int_{(0,1]} F(t)\, d\mu(t) \subset C_+$ is not open in C_+. In particular, from Lemma 3.6 it follows that $g(x) > 0$ for each $g \in B$, and for each $x \in (0,1]$, and therefore, f belongs to $B \backslash \operatorname{ri} B$.

We will conclude this chapter with a result that strengthens Proposition 3.10. But first we have to prove the following claim.

Claim 3.19 *Let $(S, \|\cdot\|)$ be a normed space, and let ε a be positive number. Let furthermore, $\{x_n\}$ and $\{s_n\}$ be convergent sequences in S. Set $x = \lim_n x_n$, $s = \lim_n s_n$, and let G be an open subset of S.*

1. *For each pair of numbers q, r with $q, r - \varepsilon > 0$ there exists a natural number $\overline{m}$ such that $\operatorname{co}[x \cup B_\varepsilon(s)] \subset \bigcap_{n \geq \overline{m}} \operatorname{co}[B_q(x_n) \cup B_r(s_n)]$.*

2. *Conversely, if for each pair of rational numbers q, r with $q, r - \varepsilon > 0$ there exists a natural number $\overline{m}$ such that G is contained in the set $\bigcap_{n \geq \overline{m}} \operatorname{co}[B_q(x_n) \cup B_r(s_n)]$, then G is also contained in $\operatorname{co}[x \cup B_\varepsilon(s)]$.*

Proof. **(1)** Fix a pair of numbers q, r with $q, r - \varepsilon > 0$. Pick a number $\overline{m} \in \mathbb{N}$ such that $x \in B_q(x_n)$ and $\|s - s_n\| < r - \varepsilon$ for all $n \geq \overline{m}$. Then, clearly, $B_\varepsilon(s) \subset B_r(s_n)$ for all $n \geq \overline{m}$, and so, $\operatorname{co}[x \cup B_\varepsilon(s)] \subset \bigcap_{n \geq \overline{m}} \operatorname{co}[B_q(x_n) \cup B_r(s_n)]$. This proves (1).

(2) Notice that since $\operatorname{co}[x \cup B_\varepsilon(s)]$ is a convex body, $\operatorname{int}(\operatorname{co}[x \cup B_\varepsilon(s)]) = \operatorname{int}\left(\overline{\operatorname{co}[x \cup B_\varepsilon(s)]}\right)$. So, as G is open, $G \subset \operatorname{co}[x \cup B_\varepsilon(s)]$ if and only if $G \subset \overline{\operatorname{co}[x \cup B_\varepsilon(s)]}$. Moreover, it is easily seen that $\operatorname{co}\left[x \cup \overline{B_\varepsilon(s)}\right]$ is closed, and hence, $\operatorname{co}\left[x \cup \overline{B_\varepsilon(s)}\right] = \overline{\operatorname{co}[x \cup B_\varepsilon(s)]}$. Thus, it suffices to show that $G \subset \operatorname{co}\left[x \cup \overline{B_\varepsilon(s)}\right]$.

To this end, let z be a point of G. Pick a pair of sequences of rational numbers $\{q_k\}, \{r_k\}$ such that $q_k \downarrow 0$, $r_k \downarrow \varepsilon$, and $q_k, r_k - \varepsilon > 0$ for all

$k \in \mathbb{N}$. By hypotheses, for each $k \in \mathbb{N}$ there exists a number $n_k \in \mathbb{N}$ such that $z \in \operatorname{co}\left[B_{q_k}(x_{n_k}) \cup B_{r_k}(s_{n_k})\right]$. Moreover, clearly, we can choose these numbers such that $n_{k+1} > n_k$ for all $k \in \mathbb{N}$. Set $z = \gamma_k x'_k + (1-\gamma_k)\, s'_k$ where $\gamma_k \in [0,1]$, $x'_k \in B_{q_k}(x_{n_k})$, and $s'_k \in B_{r_k}(s_{n_k})$ for all $k \in \mathbb{N}$. Note that $x'_k \to x$. Furthermore, by passing to a subsequence if necessary, we can assume that $\{\gamma_k\}$ converges to some $\gamma \in [0,1]$. Now, since $\|s'_k\| \le \|s_{n_k}\| + r_k \le \|s_{n_k}\| + r_1$ for all $k \in \mathbb{N}$, and since $\{s_{n_k}\}$ is norm bounded as a convergent sequence, we can conclude that $\{s'_k\}$ is a norm bounded sequence. Hence, $\gamma = 1$ would imply that $z = x$. So, without loss of generality we can assume $\gamma < 1$ and $1 - \gamma_k > 0$ for all $k \in \mathbb{N}$. Then the sequence$\{s'_k\} = \left\{\frac{z-\gamma_k x'_k}{1-\gamma_k}\right\}$ is convergent and $\lim_k s'_k = \frac{z-\gamma x}{1-\gamma}$. Set $s' = \frac{z-\gamma x}{1-\gamma}$, and observe that

$$\|s' - s\| \le \limsup_k \left(\|s' - s'_k\| + \|s'_k - s_{n_k}\| + \|s_{n_k} - s\|\right) \le \varepsilon.$$

Thus, $\frac{z-\gamma x}{1-\gamma}$ belongs to $\overline{B_\varepsilon(s)}$, and this completes the proof. ■

Proposition 3.20 *Let (T, Σ, μ) be a σ-finite measure space, and let $(S, \|\cdot\|)$ be a Banach space. Let furthermore, the complement of a correspondence $X : T \rightrightarrows S$ be measurable, $\mathfrak{M}_{\operatorname{int} X} \neq \emptyset$, and $X(t)$ be convex a.e. on T. If the complement of a correspondence $\varphi : T \rightrightarrows S$ is measurable and $\varphi(t)$ is a relative open subset of $X(t)$ a.e. on T, then*

$$\operatorname{int}\left(\int \varphi\right) = \int \operatorname{int} \varphi.$$

Proof. By Theorem 3.7, $\int \operatorname{int} \varphi \subset \operatorname{int}\left(\int \varphi\right)$, so, it is sufficient to show the inverse inclusion. Let $\overline{x} \in \operatorname{int}\left(\int \varphi\right)$. Then there exists a selector $f \in \mathcal{L}\varphi$ such that $\overline{x} = \int f$. Let $g \in \mathfrak{M}_{\operatorname{int} X}$. For $t \in T$ and $\varepsilon > 0$, put $C_\varepsilon(t) = \operatorname{co}\left[f(t) \cup B_\varepsilon(g(t))\right]$. Define

$$\varepsilon_t = \{\sup \varepsilon > 0 : B_\varepsilon(g(t)) \subset X(t),\, B_\varepsilon(f(t)) \cap C_\varepsilon(t) \subset \varphi(t)\}$$

for $t \in T$. Clearly, $\varepsilon_t > 0$ a.e. on T. As in proofs of Theorems 3.7 and 3.14, for $\varepsilon > 0$, the set $\{t \in T : B_\varepsilon(g(t)) \subset X(t), B_\varepsilon(f(t)) \cap C_\varepsilon(t) \subset \varphi(t)\}$ belongs to the collection Σ^*. Thus, the mapping $t \to \varepsilon_t$ is measurable. By Claim 3.5, there exists a measurable mapping $\gamma : T \to \mathbb{R}$ which takes countably many values y^k $(k \in \mathbb{N})$ and which satisfies $0 < \gamma_t < \varepsilon_t$ a.e. on T. Let $T_k = \gamma^{-1}\left(y^k\right)$ for $k \in \mathbb{N}$. Define $Y : T \rightrightarrows S$ as $Y(t) = \mathrm{co}\left[f(t) \cup B_{\gamma_t}(g(t))\right]$ for $t \in T$. By Corollary 2.19, there exist a pair of sequences $\{f_n\}$, $\{g_n\}$ of simple mappings of T into S such that $f_n(t) \longrightarrow f(t)$ and $g_n(t) \longrightarrow g(t)$ a.e. on T. From Claim 3.19 it follows that, for each open set $G \subset S$,

$$\begin{aligned}\{t \in T : G \subset Y(t)\} &= \bigcup_{k=1}^{\infty} \left\{t \in T_k : G \subset \mathrm{co}\left[f(t) \cup B_{y^k}(g(t))\right]\right\} \\ &\sim \bigcup_{k=1}^{\infty} \bigcap_{\substack{q>0 \\ r>y^k \\ q,r\in\mathbb{Q}}} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \{t \in T_k : G \subset C_{n,q,r}(t)\},\end{aligned}$$

where $C_{n,q,r}(t) = \mathrm{co}\left[B_q(f_n(t)) \cup B_r(g_n(t))\right]$ for $t \in T$, $n \in \mathbb{N}$, and $q, r \in \mathbb{Q}$. Thus, the complement of Y is measurable. As $Y(t)$ is a relative open subset of $X(t)$ a.e. on T, by Theorem 3.14,

$$\mathrm{int}\left(\int X\right) \cap \left(\int Y\right) = \mathrm{int}\left(\int Y\right). \tag{3.13}$$

For $t \in T$, put $\psi(t) = B_{\gamma_t}(f(t)) \cap Y(t)$. Clearly, the complement of ψ is measurable and $\psi(t)$ is relative open in $Y(t)$ a.e. on T. Define, $h : T \to S$ as

$$h(t) = \begin{cases} f(t) + \frac{\gamma_t(g(t)-f(t))}{2\|g(t)-f(t)\|} & \text{if } g(t) \neq f(t), \\ g(t) & \text{if } g(t) = f(t). \end{cases}$$

Then, $h \in \mathfrak{M}_{\mathrm{int}\,\psi} \neq \emptyset$. Thus, by Theorem 3.14 and Proposition 3.10,

$$\mathrm{int}\left(\int Y\right) \cap \left(\int \psi\right) = \mathrm{int}\left(\int \psi\right) = \int \mathrm{int}\,\psi. \tag{3.14}$$

Since $\overline{x} \in \mathrm{int}\left(\int \varphi\right) \subset \mathrm{int}\left(\int X\right)$, from (3.13) and (3.14), $\overline{x} \in \int \mathrm{int}\,\psi$. As $\int \mathrm{int}\,\psi \subset \int \mathrm{int}\,\varphi$, the proof is complete. ■

Chapter 4

Double Large Economies

We are now ready to introduce a double large exchange economy and state our results on such economies. We shall keep using the notation and terminology introduced in previous chapters.

An exchange economy $\xi = \{(T, \Sigma, \mu), S, X, e, \succ\}$ consists of the following components:

(T, Σ, μ) is a measure space of consumers and S is a Banach space of commodities which is endowed with the norm $\|\cdot\|$. We shall endow $S \times S$ with the norm $\|(x, y)\|_{\times} = \|x\| + \|y\|$.

$X : T \Rightarrow S$ is a nonempty valued correspondence, where $X(t)$ denotes the set of a priori possible consumption bundles of consumer $t \in T$, and is called consumption set of consumer $t \in T$.

$e : T \rightarrow S$ is a an integrable mapping, where $e(t)$ represents the initial endowment of commodities of consumer $t \in T$.

$\succ : T \Rightarrow S \times S$ is a correspondence, where for $t \in T$, $\succ_t \subset X(t) \times X(t)$ is a binary relation on $X(t)$ which represents preferences of consumer t over consumption set $X(t)$. Instead of $(x, y) \in \succ_t$ and $(x, y) \in$

$(X(t) \times X(t)) \setminus \succ_t$ we will write $x \succ_t y$ and $x \not\succ_t y$, respectively. For $t \in T$,

$\succ_t$ is said to be *complete*, if for all $x, y \in X(t)$ we have either $x \succ_t y$ or $y \succ_t x$;

$\succ_t$ is said to be *reflexive*, if $x \succ_t x$ for all $x \in X(t)$;

$\succ_t$ is said to be *continuous* if $\{x \in X(t) : x \succ_t y\}$, $\{x \in X(t) : y \succ_t x\}$ are relative open subsets of $X(t)$ for each $y \in X(t)$;

$\succ_t$ is said to be *asymmetric*, if $x \succ_t y$ implies $y \not\succ_t x$;

$\succ_t$ is said to be *transitive* (resp. *negatively transitive*) if $y \succ_t x$ (resp. $y \not\succ_t x$) and $x \succ_t z$ (resp. $x \not\succ_t z$) imply $y \succ_t z$ (resp. $y \not\succ_t z$);

$\succ_t$ is said to be *locally nonsatiated* if for all $x \in X(t)$ and for all $\varepsilon > 0$, there exists a $y \in B_\varepsilon(x) \cap X(t)$ such that $y \succ_t x$.

Remember that a subset C of S is said to be *star-shaped around* 0 if $x \in C$ and $\gamma \in [0,1]$ imply $\gamma x \in C$. For any star-shaped around 0 set $C \subset S$ with $\operatorname{int} C \neq \emptyset$, we shall say that $\succ_t$ is C*-monotone* if $x \in X(t)$ and $y - x \in \operatorname{int} C$ imply $y \succ_t x$. Notice that if $\succ_t$ is C-monotone, then $X(t) + \operatorname{int} \widehat{C} \subset X(t)$, where $\widehat{C}$ is the cone generated by C. What is more, if $\succ_t$ satisfies transitivity in addition to C-monotonicity, then $x \in X(t)$ and $y - x \in \operatorname{int} \widehat{C}$ imply $y \succ_t x$. If S is an ordered vector space the nonnegative cone S_+ of which has nonempty interior, then S_+-monotonicity reduces to the usual definition of monotonicity.

Whenever we say that $\succ$ is continuous we will mean that $\succ_t$ is continuous a.e. on T, and similarly for the other definitions above. Provided that $\succ_t$ is known for some $t \in T$, the relation $\{(x,y) \in X(t) \times X(t) : y \not\succ_t x\}$ will be denoted by $\succsim_t$. Notice that if $\succ_t$ is asymmetric and negatively transitive, then $\succsim_t$ is a complete preorder, i.e. reflexive, transitive, and complete.

Let $\xi = \{(T, \Sigma, \mu), S, X, e, \succ\}$ be an exchange economy. A *coalition* E is an element of Σ with $\mu(E) > 0$. An *allocation* f is an element of $\mathcal{L}_X$. An *attainable allocation* f is an allocation such that $\int f = \int e$. A coalition E is said to *block an allocation* f *via* g if there exists an integrable mapping g of E into S such that $g(t) \succ_t f(t)$ a.e. on E and $\int_E g = \int_E e$. The *Pareto set,* $P(\xi)$, is the set of attainable allocations which are not blocked by the *grand coalition* T. The *core, Core* (ξ), is the set of attainable allocations which are not blocked by any coalition.

Due to obvious reasons, any model with a measure space of consumers has to employ some measurability assumptions. These are the ones to be used here.

AMC The complement of X is measurable, that is, $\{t \in T : G \subset X(t)\} \in \Sigma^*$ for each open set $G \subset S$.

AMP1 For each allocation g, the correspondences $\widetilde{L}_g : t \Rightarrow \{x \in X(t) : x \not\succ_t g(t)\}$, and $\widetilde{U}_g : t \Rightarrow \{x \in X(t) : g(t) \not\succ_t x\}$ are measurable, in other words, the sets $\left\{t \in T : G \cap \widetilde{L}_g(t) \neq \emptyset\right\}$ and $\left\{t \in T : G \cap \widetilde{U}_g(t) \neq \emptyset\right\}$ belong to Σ^* for each open set $G \subset S$.

AMP2 For each allocation g, the complements of the correspondences $L_g : t \Rightarrow \{x \in X(t) : g(t) \succ_t x\}$ and $U_g : t \Rightarrow \{x \in X(t) : x \succ_t g(t)\}$ are measurable, that is, the sets $\{t \in T : G \subset L_g(t)\}$, $\{t \in T : G \subset U_g(t)\}$ belong to Σ^* for each open set $G \subset S$.

AMP3 For any pair of allocations g and f, the set $\{t \in T : f(t) \succ_t g(t)\}$ belongs to Σ^*.

We would like to stress that all of the above axioms are technical and they do not bear any economic meaning. What is more, they do not cause a significant loss of generality, since in economic models the set of consumers is commonly endowed with the Lebesgue measure, for which nonmeasurable sets are very pathological, or the counting measure, for which measurability is not a problem at all. When the commodity space is separable, the usual practice in the theory of large economies is to assume that $Gr_{\succ} \in \Sigma^* \otimes \mathfrak{B}(S) \otimes \mathfrak{B}(S)$ and $Gr_X \in \Sigma^* \otimes \mathfrak{B}(S)$, so that thanks to the Projection Theorem all of the above are satisfied. Unfortunately, in the nonseparable case there is no such canonical assumption. As we shall see in the next lemma, under some mild additional conditions, AMP1 is stronger than both AMP2 and AMP3.

Lemma 4.1 *Let $\xi = \{(T, \Sigma, \mu), S, X, e, \succ\}$ be an exchange economy.*

1. *If ξ satisfies AMC and AMP1, then it also satisfies AMP2.*
2. *If ξ satisfies AMP1, $\succ$ is continuous, and $X(t)$ is closed a.e. on T, then AMP3 is also satisfied.*

Proof. **(1)** Let g be an allocation, and G be an open subset of S. Now notice that the set $\{t \in T : G \cap L_g^c(t) \neq \emptyset\}$ equals to $\left\{t \in T : G \cap \widetilde{U}_g(t) \neq \emptyset\right\} \cup \{t \in T : G \cap X^c(t) \neq \emptyset\}$, and hence, the complement of L_g is measurable, and similarly for the complement of U_g.

(2) Let g and f be a pair of allocations, and observe that we have $T \backslash \{t \in T : f(t) \succ_t g(t)\} \sim \left\{t \in T : f(t) \in \widetilde{L}_g(t)\right\}$. Now note that since $X(t)$ is closed and $U_g(t)$ is relative open in $X(t)$, and as $\widetilde{L}_g(t) = X(t) \backslash U_g(t)$, $\widetilde{L}_g(t)$ is closed a.e. on T. Let $\{f_n\}$ be a sequence of simple functions such that $\lim_n f_n(t) = f(t)$ a.e. on T. If we can show that

$$\left\{t \in T : f(t) \in \widetilde{L}_g(t)\right\} \sim \bigcap_{\substack{r>0 \\ r \in \mathbb{Q}}} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \left\{t \in T : B_r(f_n(t)) \cap \widetilde{L}_g(t) \neq \emptyset\right\},$$

the proof will be complete by measurability of $\widetilde{L}_g$. Let $t \in T$ be a point such that $\lim_n f_n(t) = f(t)$. First assume $f(t) \in \widetilde{L}_g(t)$, and note that for each $r > 0$ we will have that $f(t) \in B_r(f_n(t))$, and hence, $B_r(f_n(t)) \cap \widetilde{L}_g(t) \neq \emptyset$ for all sufficiently large $n \in \mathbb{N}$. Conversely, assume that for each rational number $r > 0$ there exists a natural number m_r such that $B_r(f_n(t)) \cap \widetilde{L}_g(t) \neq \emptyset$ for all $n \geq m_r$. Pick a sequence of rational numbers $\{r_k\}$ such that $r_k \downarrow 0$, and $r_k > 0$ for all $k \in \mathbb{N}$. By hypothesis, for each $k \in \mathbb{N}$ there exists a number $n_k \in \mathbb{N}$ such that $B_{r_k}(f_{n_k}(t)) \cap \widetilde{L}_g(t) \neq \emptyset$. Moreover, clearly, we can choose these numbers such that $n_{k+1} > n_k$ for all $k \in \mathbb{N}$. Now for each $k \in \mathbb{N}$ pick a $z_k \in B_{r_k}(f_{n_k}(t)) \cap \widetilde{L}_g(t)$. Then, obviously, $\lim_k z_k = \lim_k f_{n_k}(t) = f(t)$. Hence, as $\widetilde{L}_g(t)$ is closed, we conclude that $f(t) \in \widetilde{L}_g(t)$, and this completes the proof. $\blacksquare$

Now, we state and prove our result on closedness of the Pareto set and the core for the particular case of finite measure space of consumers. The proof we provide is a modification of the proof of Grodal's (1971) Theorem 3.

Theorem 4.2 *Let an exchange economy $\xi = \{(T, \Sigma, \mu), S, X, e, \succ\}$ satisfy AMC and AMP1 in addition to the following conditions:*

1. *(T, Σ, μ) is a finite measure space, and S is a Banach space;*
2. *$X(t)$ is closed and convex a.e. on T;*
3. *$e(t) \in \operatorname{int} X(t)$ a.e. on T;*
4. *$\succ$ is continuous, transitive, and C-monotone for a star-shaped around 0 set $C \subset S$ with $\operatorname{int} C \neq \emptyset$.*

Then the Pareto set and the core are closed subsets of $L_1(T, \Sigma, \mu; S)$.

Proof. We shall prove that $Core\,(\xi)$ is closed; closedness of $P\,(\xi)$ can be shown analogously.

Let $\{f_n\}$ be a convergent sequence of core allocations, and let $f = \lim_n f_n$ in L_1. First, we shall show that f is an attainable allocation. To this end, notice that by Theorem 2.29(1) and Theorem 2.17(1), there exists a subsequence $\{f_{n_k}\}$ such that $\lim_k f_{n_k}(t) = f(t)$ a.e. on T. Now as $f_{n_k}(t) \in X(t)$ for all $k \in \mathbb{N}$, and for a.e. $t \in T$, and as $X(t)$ is closed a.e. on T, we see that $f(t) \in X(t)$ a.e. on T. Moreover, as $\lim_n \left\| \int f_n - \int f \right\| \le \lim_n \|f_n - f\|_1 = 0$, and as $\int f_n = \int e$ for all $n \in \mathbb{N}$, we conclude that $\int f = \int e$. So, f is an attainable allocation as we claimed.

Now suppose $f \notin Core\,(\xi)$. Then there exists a coalition E, and a mapping $h \in \mathcal{L}_{U_f|_E}$ with $\int_E h = \int_E e$. Hence, $\int_E e$ belongs to $\int_E U_f$. Note that assumption (3) together with Theorem 3.8 implies that $\int_E e \in \operatorname{int}\left(\int_E X\right)$. From Lemma 4.1(1) it follows that the complement of the correspondence U_f is measurable. So, we can apply Theorem 3.14, and conclude that $\int_E e \in \operatorname{int}\left(\int_E U_f\right)$. Pick a point $z \in \operatorname{int} C$ with $\int_E e - z \in \int_E U_f$, and a number $\delta > 0$ such that $B_{2\delta}(z) \subset \operatorname{int} C$. Now observe that

$$x \in B_\delta\left(\int_E e\right) \text{ and } y \in B_\delta\left(\int_E e - z\right) \Longrightarrow x - y \in \operatorname{int} C. \tag{4.1}$$

Pick a selector $g \in \mathcal{L}_{U_f|_E}$ such that $\int_E g = \int_E e - z$. By Theorem 2.29(2) and by absolute continuity of integral (Theorem 2.24(5)), there exists a $\delta' > 0$ such that

$$F \in \Sigma \text{ and } \mu(F) < \delta' \Longrightarrow \left\| \int_F g \right\|, \left\| \int_F f_n \right\| < \delta \quad \text{for all } n \in \mathbb{N}. \tag{4.2}$$

For each $k \in \mathbb{N}$ put $E_k = \cup_{l \ge k} \{t \in E : g(t) \not\succ_t f_{n_l}(t)\}$. Note that as $g(t) \succ_t f(t)$ a.e. on E, by continuity of $\succ$, for a.e. $t \in E$ there exists a number $k_t \in \mathbb{N}$ such that $g(t) \succ_t f_{n_l}(t)$ for all $l \ge k_t$. Hence $E_k \downarrow \emptyset$. Moreover, from Lemma 4.1(2) it follows that $E_k \in \Sigma^*$ for all $k \in \mathbb{N}$. So, $\mu^*(E_k) \downarrow 0$. Pick a

k_0 with $\mu^*(E_{k_0}) < \delta'$, and a μ-measurable subset E_0 of E such that $E_0 \supset E_{k_0}$ and $\mu(E_0) = \mu^*(E_{k_0})$. We will finish the proof by showing that E blocks $f_{n_{k_0}}$, which is in contradiction with $f_{n_{k_0}} \in Core(\xi)$.

First assume $\mu(E_0) > 0$. Define $g_0 : E \to S$ as

$$g_0(t) = \begin{cases} g(t) & \text{for } t \in E \backslash E_0, \\ f_{n_{k_0}}(t) + \frac{u}{\mu(E_0)} & \text{for } t \in E_0, \end{cases}$$

where $u = \int_E e - \int_{E_0} f_{n_{k_0}} - \int_{E \backslash E_0} g$. Obviously, $\int_E g_0 = \int_E e$. Moreover, by (4.2),

$$\begin{aligned} \left\| \left(\int_E e - \int_{E_0} f_{n_{k_0}} \right) - \int_E e \right\| &= \left\| \int_{E_0} f_{n_{k_0}} \right\| < \delta, \\ \left\| \int_{E \backslash E_0} g - \left(\int_E e - z \right) \right\| &= \left\| \int_{E \backslash E_0} g - \int_E g \right\| = \left\| \int_{E_0} g \right\| < \delta. \end{aligned}$$

So, from (4.1) it follows that $u \in \operatorname{int} C$, and hence by C-monotonicity and transitivity, $g_0(t) \succ_t f_{n_{k_0}}(t)$ a.e. on E.

Now assume $\mu(E_0) = 0$, and define $g_0 : E \to S$ as $g_0(t) = g(t) + \frac{z}{\mu(E)}$ for $t \in E$. Then, clearly we have $\int_E g_0 = \int_E e$, and by C-monotonicity and transitivity, $g_0(t) \succ_t f_{n_{k_0}}(t)$ a.e. on E. ■

Our next theorem on the continuity of core correspondences is at the same time a strengthening of Theorem 4.2 for the case of separable commodity spaces. Apart from some details, the proof we provide is the same with the proof of Grodal's (1971) Theorem 4.

Theorem 4.3 *Let exchange economies* $\xi = \{(T, \Sigma, \mu), S, X, e, \succ\}$ *and* $\xi_n = \{(T, \Sigma, \mu), S, X_n, e_n, \succ_n\}$ $(n \in \mathbb{N})$ *satisfy the following conditions:*

1. (T, Σ, μ) *is a finite measure space, and* S *is a separable Banach space;*

2. $X(t)$ *is closed and convex a.e. on* T*;*

3. $e(t) \in \operatorname{int} X(t)$ *a.e. on* T*;*

4. $\succ$ *is asymmetric, negatively transitive and continuous, moreover,* $\succ_n$ *are asymmetric, negatively transitive and* C*-monotone for a star-shaped around* 0 *subset* C *of* S *with* $\operatorname{int} C \neq \emptyset$ $(n \in N)$*;*

5. $Gr_{\succ}, Gr_{\succ_n} \in \Sigma^* \otimes \mathfrak{B}(S) \otimes \mathfrak{B}(S)$ *and* $Gr_X, Gr_{X_n} \in \Sigma^* \otimes \mathfrak{B}(S)$ $(n \in \mathbb{N})$.

If $\sigma(\succsim_{t,n}, \succsim_t) \longrightarrow 0$ *a.e. on* T *and* $e_n \longrightarrow e$ *in* $L_1(T, \Sigma, \mu; S)$*, where* σ *denotes the Hausdorff distance, then* $f_n \in Core(\xi_n)$ $(n \in \mathbb{N})$ *and* $f_n \longrightarrow f$ *in* $L_1(T, \Sigma, \mu; S)$ *imply* $f \in Core(\xi)$.

Proof. We shall first show that f is an attainable allocation in ξ. To this end, notice that by Theorem 2.29(1) and Theorem 2.17(1), there exists a subsequence $\{f_{n_k}\}$ such that $\lim_k f_{n_k}(t) = f(t)$ a.e. on T. Now fix a $t \in T$ such that $f_{n_k}(t) \in X_{n_k}(t)$ for all $k \in \mathbb{N}$, and $\lim_n \sigma(\succsim_{t,n}, \succsim_t) = \lim_k \|f_{n_k}(t) - f(t)\| = 0$. Assume furthermore, $\succ_{t,n_k}$ is asymmetric for all $k \in \mathbb{N}$. Let $\varepsilon > 0$, and pick a $k \in \mathbb{N}$ such that $\succsim_{t,n_k} \subset B_{\frac{\varepsilon}{2}}(\succsim_t)$ and $\|f_{n_k}(t) - f(t)\| < \frac{\varepsilon}{2}$. Observe that $\succsim_{t,n_k}$ is reflexive, so, $(f_{n_k}(t), f_{n_k}(t)) \in \succsim_{t,n_k}$. Thus, there exists a point $(x, y) \in \succsim_t$ such that $\|f_{n_k}(t) - x\| + \|f_{n_k}(t) - y\| < \frac{\varepsilon}{2}$. Then, clearly $\|f(t) - x\| < \varepsilon$, and hence, $f(t)$ is in the closure of $X(t)$. So, by assumption (2), we see that $f(t) \in X(t)$ a.e. on T. Moreover, as $\int f = \lim_n \int f_n = \lim_n \int e_n = \int e$, we conclude that f is an attainable allocation as we claimed.

Now suppose $f \notin Core(\xi)$. Then there exists a coalition E, and a mapping $h \in \mathcal{L}_{U_f|E}$ with $\int_E h = \int_E e$. Hence, $\int_E e$ belongs to $\int_E U_f$. Note that assumption (3) together with Theorem 3.8 implies that $\int_E e \in \operatorname{int}\left(\int_E X\right)$. Observe that $Gr_{U_f} = \{(t, x) : x \succ_t f(t)\} = \omega^{-1}(Gr_{\succ})$, where the mapping $\omega : T \times S \to T \times S \times S$ is defined by $\omega(t, x) = (t, x, f(t))$. Using measurability of f it can easily be shown that $\omega^{-1}(B) \in \Sigma^* \otimes \mathfrak{B}(S)$ for each $B \in \Sigma^* \otimes \mathfrak{B}(S) \otimes \mathfrak{B}(S)$. Thus, we see that $Gr_{U_f} \in \Sigma^* \otimes \mathfrak{B}(S)$. So, we can apply Theorem 3.14, and conclude that $\int_E e \in \operatorname{int}\left(\int_E U_f\right)$. Pick a point $z \in \operatorname{int} C$ with $\int_E e - z \in \int_E U_f$, and a number $\delta > 0$ such that $B_{2\delta}(z) \subset \operatorname{int} C$. Observe that

$$x \in B_\delta\left(\int_E e\right) \text{ and } y \in B_\delta\left(\int_E e - z\right) \Longrightarrow x - y \in \operatorname{int} C. \tag{4.3}$$

Pick a selector $g \in \mathcal{L}_{U_f|_E}$ such that $\int_E g = \int_E e - z$. By absolute continuity of integral (Theorem 2.24(5)), there exists a $\delta' > 0$ such that

$$F \in \Sigma \text{ and } \mu(F) < \delta' \Longrightarrow \left\|\int_F g\right\|, \left\|\int_F e\right\| < \frac{\delta}{4}. \tag{4.4}$$

Now we shall show that $Gr_{\succsim} \in \Sigma^* \otimes \mathfrak{B}(S) \otimes \mathfrak{B}(S)$. Define the correspondence $\Psi : T \Rightarrow S \times S$ as $\Psi(t) = X(t) \times X(t)$, and note that $Gr_\Psi = (Gr_X \times S) \cap \varkappa^{-1}(Gr_X)$, where the mapping $\varkappa : T \times S \times S \to T \times S$ is defined by $\varkappa(t, x, y) = (t, y)$. So, as $Gr_X \in \Sigma^* \otimes \mathfrak{B}(S)$, clearly we have that $Gr_\Psi \in \Sigma^* \otimes \mathfrak{B}(S) \otimes \mathfrak{B}(S)$. Notice that $Gr_{\succsim} = Gr_\Psi \cap \rho^{-1}(Gr_{\succ}^c)$, where the mapping $\rho : T \times S \times S \to T \times S \times S$ is defined by $\rho(t, x, y) = (t, y, x)$. Hence, as $Gr_{\succ} \in \Sigma^* \otimes \mathfrak{B}(S) \otimes \mathfrak{B}(S)$, clearly we can conclude that $Gr_{\succsim} \in \Sigma^* \otimes \mathfrak{B}(S) \otimes \mathfrak{B}(S)$, and similarly for the sets $Gr_{\succsim_n}$ $(n \in \mathbb{N})$. So, from Proposition 2.36 it follows that the mapping $t \to \sigma(\succsim_{t,n}, \succsim_t)$ is measurable for each $n \in \mathbb{N}$.

Set $\delta_0 = \frac{1}{2} \min\{\delta', \mu(E)\}$. By Egoroff Theorem (Theorem 2.16) there exists a set $E_1 \in \Sigma$ such that $\mu(E_1) < \delta_0$ and $\sigma(\succsim_{t,n}, \succsim_t) \longrightarrow 0$ and $\|f_{n_k}(t) - f(t)\| \longrightarrow 0$ uniformly on $T \backslash E_1$. Now, as $X(t)$ is closed, it can easily be seen that the assumption (4) ensures that $\succsim_t$ is closed in $S \times S$ a.e. on T. So, for a.e. $t \in E$ there exists an $\varepsilon_t > 0$ such that $[B_{\varepsilon_t}(f(t)) \times B_{\varepsilon_t}(g(t))] \cap \succsim_t = \emptyset$. For $\varepsilon > 0$ put $E_\varepsilon = \{t \in E : [B_\varepsilon(f(t)) \times B_\varepsilon(g(t))] \cap \succsim_t = \emptyset\}$. It can easily be shown that

$$E \backslash E_\varepsilon = \bigcup_{\substack{0<q<\varepsilon \\ q \in \mathbb{Q}}} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \{t \in E : [B_q(s_n(t)) \times B_q(g_n(t))] \cap \succsim_t \neq \emptyset\},$$

where $\{s_n\}, \{g_n\}$ are sequences of simple mappings with $\lim_n s_n(t) = f(t)$ and $\lim_n g_n(t) = g(t)$ a.e. on T. Now as $Gr_{\succsim} \in \Sigma^* \otimes \mathfrak{B}(S) \otimes \mathfrak{B}(S)$, we can conclude that $E_\varepsilon \in \Sigma^*$ for $\varepsilon > 0$. Thus, by continuity of measure, there exists a number ε_0 such that $\mu^*(E \backslash E_{\varepsilon_0}) < \delta_0$ and $0 < \varepsilon_0 < \frac{\delta}{\mu(E)}$. Note that $\mu^*(E_{\varepsilon_0}) = \mu^*(E) - \mu^*(E \backslash E_{\varepsilon_0}) > \delta_0$.

Pick a μ-measurable set $\overline{E}$ such that $\overline{E} \subset E_{\varepsilon_0} \backslash E_1$ and $\mu(\overline{E}) = \mu^*(E_{\varepsilon_0} \backslash E_1)$. Then $\mu(\overline{E}) = \mu^*(E_{\varepsilon_0} \backslash E_1) \geq \mu^*(E_{\varepsilon_0}) - \mu^*(E_1) > 0$, and

$$\mu\left(E \backslash \overline{E}\right) = \mu^*\left[E \backslash (E_{\varepsilon_0} \backslash E_1)\right] \leq \mu^*\left[E \backslash E_{\varepsilon_0}\right] + \mu^*(E_1) < 2\delta_0 \leq \delta'. \tag{4.5}$$

Determine $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} \sigma\left(\succsim_{t,n_0}, \succsim_t\right) &< \frac{\varepsilon_0}{2} \quad \text{for all } t \in \overline{E}, \\ \|f_{n_0}(t) - f(t)\| &< \frac{\varepsilon_0}{2} \quad \text{for all } t \in \overline{E}, \\ \left\| \int_{\overline{E}} e_{n_0} - \int_{\overline{E}} e \right\| &< \frac{\delta}{4}. \end{aligned} \tag{4.6}$$

We will finish the proof by showing that $\overline{E}$ blocks f_{n_0} in the economy ξ_{n_0}, which is in contradiction with $f_{n_0} \in \xi_{n_0}$.

Now note that by assumption (4), $\succsim_t$ is reflexive a.e. on T. So, for a.e. $t \in \overline{E}$, we have that $(g(t), g(t)) \in \succsim_t \subset B_{\frac{\varepsilon_0}{2}}(\succsim_{t,n_0})$, and hence $B_{\frac{\varepsilon_0}{2}}(g(t)) \cap X_{n_0}(t) \neq \emptyset$. Observe that graph of the correspondence $\Phi : t \Rightarrow B_{\frac{\varepsilon_0}{2}}(g(t)) \cap X_{n_0}(t)$ is the set $\left\{(t, x) \in T \times S : \text{dist}(x, g(t)) < \frac{\varepsilon_0}{2}\right\} \cap Gr_{X_{n_0}}$. Therefore, by Theorem 2.35(1), Lemma 2.34 and assumption (5), $Gr_\Phi \in \Sigma^* \otimes \mathfrak{B}(S)$. So, by Theorem 2.37 there exists an integrable mapping $g_0 : \overline{E} \to S$ such that $g_0(t) \in \Phi(t)$ a.e. on $\overline{E}$. Note that if for some $t \in \overline{E}$ $(f_{n_0}(t), g_0(t)) \in \succsim_{t,n_0}$, then as $\sigma(\succsim_{t,n_0}, \succsim_t) < \frac{\varepsilon_0}{2}$, there is a point $(x, y) \in \succsim_t$ such that $\|f_{n_0}(t) - x\|$, $\|g_0(t) - y\| < \frac{\varepsilon_0}{2}$. This together with $\|f_{n_0}(t) - f(t)\|$, $\|g_0(t) - g(t)\| < \frac{\varepsilon_0}{2}$ a.e. on $\overline{E}$ imply that $(f_{n_0}(t), g_0(t)) \notin \succsim_{t,n_0}$ for a.e. $t \in \overline{E}$; otherwise we would obtain a contradiction to definition of E_{ε_0}. Hence, $g_0(t) \succ_{t,n_0} f_{n_0}(t)$ a.e. on

$\overline{E}$. Moreover, by (4.4), (4.5) and (4.6),

$$\begin{aligned}
\left\| \int_{\overline{E}} e_{n_0} - \int_E e \right\| &\leq \left\| \int_{\overline{E}} e_{n_0} - \int_{\overline{E}} e \right\| + \left\| \int_{E \setminus \overline{E}} e \right\| < \frac{\delta}{2}, \\
\left\| \int_{\overline{E}} g_0 - \left(\int_E e - z \right) \right\| &= \left\| \int_{\overline{E}} g_0 - \int_E g \right\| \\
&\leq \left\| \int_{\overline{E}} g_0 - \int_{\overline{E}} g \right\| + \left\| \int_{E \setminus \overline{E}} g \right\| \\
&< \frac{\varepsilon_0}{2} \mu\left(\overline{E}\right) + \frac{\delta}{4} < \delta.
\end{aligned}$$

So, from (4.3) it follows that $\int_{\overline{E}} e_{n_0} - \int_{\overline{E}} g_0 \in \operatorname{int} C$. Since assumption (4) implies that $\succ_{n_0}$ is transitive, and since $\succ_{n_0}$ is C-monotone, we can conclude that $\overline{E}$ blocks f_{n_0} in the economy ξ_{n_0} via the mapping $\widetilde{g} : \overline{E} \to S$ defined as $\widetilde{g}(t) = g_0(t) + \frac{\int_{\overline{E}} e_{n_0} - \int_{\overline{E}} g_0}{\mu(\overline{E})}$ for $t \in \overline{E}$. ■

The following theorem is the result on the stability of blocking coalitions.

Theorem 4.4 *Let an exchange economy $\xi = \{(T, \Sigma, \mu), S, X, e, \succ\}$ satisfy AMC and AMP2 in addition to the following conditions:*

1. *(T, Σ, μ) is σ-finite, and S is a Banach space;*

2. *$X(t)$ is convex a.e. on T;*

3. *$\succ$ is continuous and C-monotone for a star-shaped around 0 subset C of S with $\operatorname{int} C \neq \emptyset$.*

If a coalition E with $\int_E e \in \operatorname{int}\left(\int_E X\right)$ blocks an allocation f, then there exists a number $\delta > 0$ such that every coalition E' with $\mu(E \triangle E') < \delta$ also blocks f.

Proof. By AMP2 the complement of U_f is measurable, moreover, by continuity of $\succ$, $U_f(t)$ is relative open in $X(t)$ a.e. on T. Let $\gamma : T \to (0, 1]$ be an integrable mapping, and let $B_\varepsilon(z) \subset \operatorname{int} C$ for some $z \in \operatorname{int} C$ and

$\varepsilon > 0$. Define $g : T \to S$ as $g(t) = \gamma(t) z + f(t)$ for $t \in T$. Then, by C-monotonicity of $\succ$, $\gamma(t) B_\varepsilon(z) + f(t)$ is a neighborhood of $g(t)$ which is contained in $U_f(t)$ a.e. on T. So, $g \in \mathcal{L}_{\operatorname{int} U_f} \subset \mathfrak{M}_{\operatorname{int} X}$. Moreover, since E blocks f, we have $\int_E e \in \int_E U_f$. So, as $\int_E e \in \operatorname{int}\left(\int_E X\right)$ by hypothesis, from Theorem 3.14 and Proposition 3.20, $\int_E e \in \operatorname{int}\left(\int_E U_f\right) = \int_E \operatorname{int} U_f$. Then there exists a selector $h \in \mathcal{L}_{\operatorname{int} U_f|E}$ such that $\int_E h = \int_E e$. Define $s \in \mathcal{L}_{\operatorname{int} U_f}$ by $s(t) = h(t)$ for $t \in E$, and $s(t) = g(t)$ for $t \in T \backslash E$. Then $\int_E s = \int_E e$. Moreover, since $s \in \mathcal{L}_{\operatorname{int} U_f}$, as in the proof of Theorem 3.7, there exists a number $r > 0$, and a measurable subset E_1 of E with $0 < \mu(E_1) < \infty$ such that

$$B_r(s(t)) \subset U_f(t) \quad \text{for } t \in E_1. \tag{4.7}$$

By absolute continuity of integral (Theorem 2.24(5)), there exists a positive number $\delta < \frac{\mu(E_1)}{2}$ such that

$$E' \in \Sigma \text{ and } \mu(E \triangle E') < \delta \Longrightarrow \left\| \int_{E'} e - \int_{E'} s \right\| < \frac{r\mu(E_1)}{2}. \tag{4.8}$$

Let $E' \in \Sigma$ with $\mu(E \triangle E') < \delta$. We will complete the proof by showing that E' blocks f. Define $z : E' \to S$ as

$$z(t) = \begin{cases} s(t) + a & \text{for } t \in E_1 \cap E', \\ s(t) & \text{for } t \in E' \backslash E_1, \end{cases}$$

where $a = \frac{\int_{E'} e - \int_{E'} s}{\mu(E_1 \cap E')}$. Note that, as $\mu(E \triangle E') < \delta$, $\mu(E_1 \cap E') > \frac{\mu(E_1)}{2}$, and therefore, from (4.8), $\|a\| < r$. Hence, from (4.7), $z(t) \in U_f(t)$ for $t \in E'$. Thus, as $\int_{E'} z = \int_{E'} e$, coalition E' blocks f via z. ■

Remark 4.5 In Theorem 4.4, if the commodity space S is separable, monotonicity of preferences can be replaced by the weaker assumption of local nonsatiation. An analysis of the theorem's proof shows that monotonicity is used just because it implies that, for any allocation f the correspondence $\operatorname{int} U_f : t \Rightarrow \operatorname{int}\{x \in X(t) : x \succ_t f(t)\}$ has an integrable selector. It can easily be shown that if f is blocked by a coalition satisfying the hypothesis of the theorem, then $\mathfrak{L}_{\operatorname{int} U_f} \neq \emptyset$ provided that:

1. (T, Σ, μ) is σ-finite, and S is a separable Banach space;

2. $Gr_X \in \Sigma^* \otimes \mathfrak{B}(S)$, $X(t)$ is convex and $\operatorname{int} X(t) \neq \emptyset$ a.e. on T;

3. $Gr_\succ \in \Sigma^* \otimes \mathfrak{B}(S) \otimes \mathfrak{B}(S)$, $\succ$ is continuous and locally nonsatiated.

We conclude with an example which shows that, in contrast to its finite dimensional version (see Hüsseinov, 2003: Theorem 4-b), Theorem 4.4 cannot be extended to include blocking coalitions with a total endowment vector which belongs to the boundary of the total consumption set.

Example 4.6 Let C be the set of all continuous real functions on $[0, 1]$ endowed with the usual norm $\|\cdot\|_\infty$, and let C_+ be the nonnegative cone of C with respect to usual order on C. Put $a_0 = 0$, and $a_n = \sum_{i=1}^{n} 2^{-i}$ $(n \in \mathbb{N})$. Define $\{x_n : n \in \mathbb{Z}_+\} \subset C_+$ as

$$x_n(s) = \begin{cases} \frac{2}{a_{n+2}-a_n}(s - a_n) & \text{for } s \in [a_n, \frac{a_n+a_{n+2}}{2}], \\ \frac{2}{a_{n+2}-a_n}(-s + a_{n+2}) & \text{for } s \in (\frac{a_n+a_{n+2}}{2}, a_{n+2}), \\ 0 & \text{for } s \in [a_{n+2}, 1], \end{cases}$$

for $n \in \mathbb{N}$, and $x_0(s) = \max\left\{\frac{-s}{a_2} + 1, 0\right\}$ for $s \in [0, 1]$.

Let furthermore $I = (0, 1)$, $I_0 = (a_0, a_1)$, and $I_n = [a_n, a_{n+1})$ $(n \in \mathbb{N})$. Consumption set of each consumer $t \in I$ is assumed to be C_+. Define endowment function $e : I \to C_+$ by $e_t = x_n$ for $t \in I_n$ $(n \in \mathbb{Z}_+)$. It is easily seen that e is integrable over I, and $e_E = \sum_{n=0}^{\infty} \mu(I_n \cap E)\, x_n$ for $E \in \Sigma$, where μ stands for the Lebesgue measure, Σ for Lebesgue measurable subsets of I, and $e_E \in C_+$ for the integral of e over E. Also note that $e_I = \sum_{n=0}^{\infty} 2^{-n-1} x_n$. For $t \in I$, define binary relation $\succ_t$ on C_+ as

$$x \succ_t y \Longleftrightarrow x(s) > y(s) \quad \forall s \in [0, 1-t].$$

Obviously, for $t \in I$, preference relation $\succ_t$ is continuous and C_+-monotone. Furthermore, it can easily be shown that

$$Gr_{\succ} = \bigcup_{\substack{0<q<r<1 \\ q,r \in \mathbb{Q}}} \left[(q,r) \times \{(x,y) \in C_+ \times C_+ : x(s) > y(s), \forall s \in [0, 1-q]\} \right],$$

where $Gr_{\succ} = \{(t,x,y) \in T \times C \times C : x \succ_t y\}$. Clearly, for any $q \in \mathbb{Q}$, the set $\{(x,y) \in C_+ \times C_+ : x(s) > y(s), \forall s \in [0, 1-q]\}$ is relative open in $C_+ \times C_+$. This shows that $Gr_{\succ} \in \Sigma^* \otimes \mathfrak{B}(S) \otimes \mathfrak{B}(S)$.

For any $x, y \in C$ define $x \vee y \in C$ as $x \vee y(s) = \max\{x(s), y(s)\}$ for $s \in [0,1]$. Set $\mathbf{1} \in C$ to be the mapping which is identically 1 on $[0,1]$, and similarly for $\mathbf{0}$. Finally, define the mapping $f : I \to C_+$ as $f_t = \mathbf{0} \vee (e_I - t\mathbf{1})$ for $t \in I$. Note that, the mapping $g : t \to e_I - t\mathbf{1}$ is integrable. Let $\{g^n\}$ be a sequence of simple-integrable mappings which determines g. Then for any $t \in I$, $\|\mathbf{0} \vee g_t^n - \mathbf{0} \vee g_t\|_\infty \leq \|g_t^n - g_t\|_\infty$, and $\|\mathbf{0} \vee g_t^n - \mathbf{0} \vee g_t^m\|_\infty \leq \|g_t^n - g_t^m\|_\infty$ for any $n, m \in \mathbb{N}$. Hence, the sequence of simple-integrable mappings $\{f^n\}$ determines f, where $f^n : t \to \mathbf{0} \vee g_t^n$. This shows that f is an allocation. Moreover, as $e_I \succ_t f_t$ for $t \in I$, the grand coalition I blocks f via mapping $t \to e_I$. But, for $m \in \mathbb{Z}_+$, coalition $J_m = (0, a_{m+1})$ does not block f. In fact, if h is an integrable mapping of J_m to C_+ with $h_t \succ_t f_t$ $(t \in J_m)$, then from Lemma 3.6 it follows that $h_{J_m}(s) > 0$ for $s \in [0,1)$. However, $e_{J_m}(s) = \sum_{n=0}^{m} 2^{-n-1} x_n(s) = 0$ for $s \in [a_{m+2}, 1]$.

Chapter 5

Conclusion

The most important conclusion of our work on double large economies is related to robustness of blocking behavior of coalitions in such economies. Specifically, Theorem 4.4 shows that if a coalition C blocks an allocation f, then any coalition which can be formed by removal of a small portion of C from C, or addition of a small set to C will still block the allocation f. The importance of this result is twofold. The solution concept core relies on the premise that allocations which are blocked by a coalition are not likely to occur. However, in large economies it is hard to believe that all interested consumers can come together and form a coalition in a precise way. Theorem 4.4 suggests that such a precision is unnecessary; once it is known that there exists a coalition which blocks an allocation, then formation of a coalition that is sufficiently close to the original one will be sufficient for the allocation to be blocked. Secondly, in case consumers who are happy with a given blocked allocation have the intention of turning a blocking coalition into a nonblocking one, in view of this stability result, we see that their task is relatively costly since it requires a considerable amount of resource transfer for temptation of a relatively large portion of the coalition.

The basic use of our continuity results is related to the problem of identification of core and Pareto optimal allocations. In light of Theorem 4.2, to conclude that an allocation f is a core (resp. Pareto optimal) allocation it is sufficient to find a sequence of core (resp. Pareto optimal) allocations that converges to f in a particular sense. Moreover, Theorem 4.3 suggests that in order to identify the core allocations of an economy ξ, one can use a sequence of economies which can probably be more easily analyzed, and which converges to ξ in a certain sense. Though it is hard to argue that these continuity results can directly be applied to practical problems, we believe that future theoretical works about double large economies might contain areas of application of these theorems.

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