SOME OBSERVATIONS ABOUT THE NETWORK CORE AND CONVEXITY

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I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Economics.

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ABSTRACT

SOME OBSERVATIONS ABOUT THE NETWORK CORE AND CONVEXITY

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In this study, we analyze the relationships between the value function - allocation rule setting and the TU game setting induced by value functions. As several different value functions may induce the same TU game, there is some information lost when passing to the TU game setting. We inquire in this study the impact of this lost information upon the preservation of the nonemptiness of the core when we pass from the network to the TU game setting. We pass from a value function to a TU game by associating with each coalition the maximal value of the graphs this coalition can form under the given value function. Conversely, we may associate with each TU game one of the value functions that induce the given TU game. Keeping this fixed, we define the network core as the collection of graphs where no coalition has an incentive to change the cooperation structure in itself, assuming that the rest of the society consists of isolated agents. Besides, we define convexity in the value function setting in an

analogous fashion to convexity for the TU games. As convexity implies the nonemptiness of the core in both settings (a well-known result in the TU setting, and a trivial one in the value function setting), we inquire if convexity is preserved in passing from one setting to the other. We find that convexity of the value function is equivalent to a stronger type of convexity of the induced game.

Keywords: value function, convexity, network core

ÖZET

AĞ ÇEKİRDEKLERİ VE DIŞBÜKEYLİK ÜZERİNE BAZI GÖZLEMLER

Yılmaz Koçer
Ekonomi, Yüksek Lisans
Tez Yöneticisi: Prof. Dr. Semih Koray

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Bu çalışmada, değer fonksiyonu - dağıtım kuralı kurulumuyla, değer fonksiyonunun doğurduğu aktarılabilir yarar oyunu kurulumu arasındakı ilişkiler incelenmiştir. Aynı aktarılabilir yarar oyununu doğuran çok sayıda değer fonksiyonu bulunabildiğinden, aktarılabilir yarar oyunu kurulumuna geçerken belli bir bilgi kaybı söz konusudur. Bu çalışmada araştırılan şey bu bilgi kaybının, çekirdeğin boş olmaması gibi temel unsurların bu aktarım sırasında kaybolup kaybolmadığıdır. Verilen bir değer fonksiyonundan aktarılabilir yarar oyununa, her koalisyona o koalisyonun, verilen değer fonksiyonu altında oluşturabileceği en yüksek değerli çizgenin değerini eşleştirerek geçeriz. Aynı şekilde verilen bir aktarılabilir yarar oyununu da, onu doğuran bir değer fonksiyonlarından biri ile eşleştirebiliriz. Verilen bir değer fonksiyonuna göre *ağ çekirdeği* ise, herhangi bir koalisyonun, geri kalanlar yalıtılmış halde iken, kendi içlerinde daha yüksek değer yaratacak yeni bir çizgeyi oluşturamayacakları çizgeler topluluğu olarak tanımlanır. Değer fonksiyonu kurulumunda dışbükeylik de, aktarılabilir yarar oyunlarındakine benzer bir biçimde tanımlanır.

Dışbükeylik her iki kurulumda da çekirdeğin boş olmamasını gerektirdiğinden, (aktarılabilir yarar oyunlarında bilinen bir teoremden, değer fonksiyonları kurulumunda ise lemma 3 den dolayı) bu iki kurulumdaki dışbükeylikler arasındaki gerekirlik ilişkisi incelenir. Değer fonksiyonunun dışbükeyliği, doğurduğu aktarılabilir yarar oyununda daha güçlü bir dışbükeyliğe denktir.

Anahtar Kelimeler : değer fonksiyonu, dışbükeylik, ağ çekirdeği

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CHAPTER I

INTRODUCTION

The use of network structures in economic research yielded some important results in many diverse fields. Essentially, in several economic settings agents' interaction between each other, unilaterally or bilaterally, can be represented by a suitable network structure. For example, buyer and seller networks, the symmetric connections model provide a framework, where the underlying network structure can be analyzed through some efficiency and stability concepts.

In Myerson's seminal paper (1977) he made extensive use of the notion of a cooperation structure underlying the communication possibilities of a set of agents. Given a fixed network g and a TU game to represent the value creating ability of a coalition, a graph induced game is introduced. The value of a coalition in the induced game is defined as the sum of the values of its communicating subcoalitions of under the initial value function and the given cooperation structure. Jackson (1996) extended this framework to a more general setting where the value function itself is originally defined on the set of all possible graphs, whereas in Myerson's setting, only the communicating coalitions would matter, not the particular way they are linked with each other. This generalization in the setting has been useful in revealing new facts regarding efficiency and stability.

In this study, we analyze the relationships between the value function setting and the cooperative game setting where we induce a TU game from the value function by associating with each coalition the maximal value of the graphs it can form under the given value function. Similarly we can associate a value function with a given TU game by choosing a value function that induces that TU game. We define the *network core* with respect to a value function as the collection of all graphs where no coalition has any incentives to alter the current structure in itself, assuming the rest of the society to consist of isolated agents. The network core is a key concept in this study, which we try to relate to the core of the associated TU game. When we pass from a value function to the induced TU game, obviously some information gets lost, as we consider only the maximal values that coalitions are able to create. In this study we inquire whether the information lost by passing to a TU game is important or not. We are mainly interested in the preservation of some basic attributes about the original situation, such as the nonemptiness of the core. In general, the nonemptiness of the core on either side, may it be network core or the core of the TU game, does not necessarily imply the nonemptiness of the same on the other side. But when we restrict ourselves to "convex structures" some positive results are obtained. Defining convexity for a value function analogously to the convexity of a TU game, convexity of the value function turns out to be equivalent to the "per capita" convexity of the associated TU game, which is stronger than convexity.

A second result we establish here is about the relationship between the network core and the allocation rule. We find that the network core with respect to a value function is the intersection of the network cores with respect to a value function and allocation rule pair for a quite large class of allocation rules, namely the *value monotonic allocation rules*.

CHAPTER II

LITERATURE SURVEY

There are some contributions to economic theory using network structures for many diverse fields including employment search, internal organization of firms, systems compatibility, information transmission and the structure of airline routes. There is also a game theoretic literature on network structures including the marriage problem setting, games of flow, and games with communication structures. In his seminal paper, Myerson (1977) studied cooperative games with communication structures. Given a set of agents N, a graph g on N is a set of unordered doubletons from the set of agents N, where each such doubleton is interpreted as a link between its members. Thus g can be regarded as representing a communication structure that determines who is communication-wise connected to whom in the set of agents, where a link between two agents means that these can communicate with each other. Considering indirect communications also as feasible, if there is a sequence of links between a pair of agents, we infer that they can communicate under g. Thus g partitions the set of agents into communicating coalitions; the vertices of the maximal connected subgraphs form the communicating coalitions induced by g. Similarly any given coalition is partitioned into sub-coalitions whose agents can communicate with each other under g. Given a cooperative game (N,v), and a graph g representing the underlying communication structure, Myerson derives another TU game, namely the *graph-restricted game* v^g in which the value of each coalition S is simply defined as the sum of the values of the communicating subcoalitions of S induced by g according to the original cooperative game v. The idea was that, if the new cooperative game is to measure the potential value of a coalition S under the communication structure, as the units that generate value are the connected subcoalitions, then these values can be summed up to give the value for the coalition S. Myerson defined for networks an analogue of the value operator concept in the context of cooperative games, namely the "*allocation rule*" to distribute the total value generated in the graph restricted game among the agents. He characterized his allocation rule by two fairness conditions: *equal bargaining power* and *component balancedness.* The first implies that, if a new link is added to the existing graph, the agents adding the link should benefit equally from the addition of the new link, that is the marginal contribution or loss of adding the new link should be the same for the two agents. The second is that the value generated by a component should be distributed to the members of that component; value transfers between components are ruled out..

The essential contribution of Myerson's setting is the dependence of the value distributed to a coalition on the underlying communication structure. A coalition can generate different values under different communication structures. But this graph-dependence of the value that is distributed to a coalition is limited in the sense that it does not matter how the agents in a coalition are connected; only the set of connected players matters. In order to obtain a more general setting, Jackson & Wolinsky (1996) start with an explicit value function defined on the set of networks that can be formed.

Thus, in this setting, the particular connections between the agents are recognized; the value to different networks may differ even if they connect the same sets of players directly or indirectly. An "allocation rule" is a function defined on pairs of possible graphs and value functions associating a unique payoff vector representing the value distributed to each agent for each such pair. Put in this way, the value function and the allocation rule concepts are natural extensions of the notions of a characteristic function and a value operator from cooperative game theory. But in the network setting, they depend on the full network structure rather than just the partition of players into connected subsets via the network structure.

Jackson & Wolinsky (1996), characterize an allocation rule, which is again based on the Shapley value, with the similar motivation to Myerson's allocation rule. They define efficiency and stability notions in their generalized setting. Efficiency is obtained when a graph maximizes the total value; stability is attained when no pair of agents wants to add a new link to the graph, and no agent wants to sever an existing link incident with himself from the current graph. In both the symmetric connections model and the co-authorship model, they focus on the tension between efficiency and stability. They find that the compatibility of overall societal welfare (efficiency) with individual incentives to form and sever links (stability) depends on the cost parameter in the symmetric connections model and is absent in the co-authorship model, as in this model stable networks will always tend to be over-connected from an efficiency perspective. For the generalized model they show that, no component-balanced and anonymous allocation rule can guarantee this compatibility, that is the existence of an efficient graph that is pairwise stable for each given value function. They show this by constructing a particular “problematic” value function. The approach here is to take the value function as exogenously given, whereas the allocation rule is the designer's object. In a later paper by Jackson (2003a), he finds that the incompatibility prevails even if we relax anonymity to "equal treatment of equals property" and efficiency to "constrained efficiency". He also finds that when the value function is anonymous and there exists an efficient graph where each agent is linked to at least two agents, the compatibility is reconciled. On the other hand, when graphs with directed links between the agents are considered, the incompatibility continues to prevail. In this setting, pairwise stability is replaced by individual stability; that is, agents can only control the links emanating from themselves, but cannot control any other links, including the ones directed to them (Jackson & Dutta, 2000). As a last note for this discussion, Dutta & Mutuswami (1997) show that restricting the problem to a particular class of value functions solves the compatibility problem again.

When we consider the “allocation rule” as an object to be designed, another important problem arises. Many common allocation rules including the Myerson allocation rule implicitly or explicitly take the network structure as fixed when distributing value to the agents. Jackson (2003b) criticizes this approach by saying that, as in many situations the network structure can be altered by the agents, the allocation rule should be sensitive to the alternative network structures when deciding to distribute value at a given network. For example, Myerson allocation rule considers only the values of subgraphs but not the alternative networks. Jackson presents a new family of allocation rules that takes alternative networks into account when distributing value on a given network, and calls this class “flexible network rules”. Flexible network rules take into account the contribution of a given link (or player) to various networks is deciding on what the contribution of a link (or player) to the given graph is.

Many authors have used this framework to model different economic situations. Currarini (2002) studies organizations and externalities under the network setting, where value of a graph depends only on the partition of players induced by the components of the graph. Though it seems that the other aspects of graph structure are not important in this approach, it should also be noted that, if some group of players changes their links, the resulting network and thus the partition of agents and the value associated with the new graph depends on how the players were connected at the initial graph, and not just on what the communicating coalitions were. In this setting he defines positive and negative externalities based on whether the value increases or decreases as the partition of players becomes finer. He finds that, positive externalities favor the adoption of dispersed organizations, while negative externalities favor the formation of more cohesive structures. Kranton & Minehart (2001) introduce a model for buyer-seller networks. They try to answer the question when buyers and sellers can form efficient network structures while acting non-cooperatively. They draw attention to efficient allocation of goods and claim that the buyer earns the marginal surplus from exchange, aligning buyers' incentives with social welfare. They show evidence for this fact in studies of industrial-supply networks.

Up to now, we have surveyed some of the literature that analyzed the efficiency and stability matters statically on a given graph, without any reference to the formation of a network starting from the empty graph. Thus a very important direction of research is to consider social and economic networks dynamically; that is, to investigate what happens when networks form in time according to a particular network formation process, modeled as a network formation game. When networks are viewed as ultimate structures that are formed through such a process, they can be analyzed utilizing the equilibrium concepts associated with the game through which the formation takes place.

Aumann & Myerson (1988) were the first to model network formation explicitly as a game. In their extensive form game, given an exogenous ranking of pairs of players, the pairs in turn decide whether or not to form a link knowing the decisions of all the pairs before them and forecasting the decisions that will come after them. A decision to form a link is binding and cannot be undone. In equilibrium, the payoffs to the players are determined by the Shapley value of the graph restricted game v^g, where g is the final network and v is the initially given cooperative game. They find that the subgame perfect Nash equilibria (SPNE) of this game may lead to inefficient graphs, which is again an example of the incompatibility problem we mentioned before.

Following Myerson (1991), Slikker & Nouweland (2000) modeled network formation as a normal form game where the players simultaneously announce the set of agents they want to form links with. The mutual consent of the two agents is necessary to form a link. The payoffs to players are determined as in the Aumann & Myerson (1988) setting. In this game, as the strategy profile consisting of the empty set for all agents will always be a Nash equilibrium (NE) of this game regardless of the underlying cooperative game, and as the formation of a link requires the consent of both players, they use refinements of NE such as strong Nash equilibrium (SNE) and the coalition proof Nash equilibrium (CPNE) in addition to undominated Nash equilibrium (UNE) and trembling hand perfection. In their model, there are fixed costs to form links and thus the model yields a cost extended version of Myerson's communication structure setting. The very surprising result of their study is that if the underlying cooperative game is convex but not superadditive, increasing costs to links may yield more links to be formed in many NE refinements. They conclude that "Authorities wishing to promote more cooperation cannot always rely on subsidies to accomplish this goal. In fact, such subsidies may have an adverse effect" (Slikker & Nouweland, 2000: 357).

Dutta, Nouweland & Tijs (1998) study the same normal form game setting, without a cost assigned to links and this time payoffs being determined by an allocation rule satisfying some appealing fairness axioms. They show that, if the underlying cooperative game is superadditive, the UNE, CPNE, SNE of the network formation game all lead to the complete network or a network structure that is payoff equivalent to it for the whole class of possible allocation rules satisfying the predetermined fairness axioms.

Currarini & Morelli (2000) try to endogenize the allocation rule, using the sequential game version of Aumann & Myerson (1988) network formation game. Here, players in turn announce the set of agents they want to form links with and a demand payoff on the ultimate graph that will form. Links form if there is mutual consent to form them and if the sum of demands of a component's agents does not exceed the value generated by that component. If it exceeds, then the whole component fails, no links form and all agents in the component get zero payoff. This extensive form game with complete information has a subgame perfect equilibrium. They show that the SPNE of this game leads to efficient networks for the class of size monotonic value functions.

Dutta & Mutuswami (1997) study Myerson's normal form game (Myerson, 1991). They adopt a mechanism design approach in the sense that they consider the allocation rule as an object to be designed. Thus they construct an allocation rule, which will ensure that, under mild restrictions imposed on the value functions, efficient graphs will form when self-interested agents decide to form and break links. The allocation rule is anonymous on the stable graphs, thus it causes no ethical problems if "ethical properties" are considered only at equilibria, which correspond to the SNE of the network formation game.

As a last note on network formation, the studies mentioned here all assume that the agents are myopic in the sense that they do not consider future reactions of the other agents to their decisions at a given stage of the network formation. Some attempts to model farsighted agents in network formation include Watts (2002). In her extensive form game where the value function and the allocation rule are imported from the symmetric connections model by Jackson & Wolinsky (1996), agents take possible future networks that may form in the following stages into account and decide whether to form a link or not in accordance with the maximization problem in which future benefits are discounted to the present by a constant. She finds that forward-looking agents may form a network shaped like a circle, when one starts from the empty network, although the cost of forming the first link exceeds its benefit.

CHAPTER 3

THE MODEL

Let $N = \{1,......,n\}$ be the set of agents. A graph g on N is a set of unordered doubletons from the set N, which we call links. The vertices are the agents. For example, let $N = \{1,2,3\}$ and $g = \{\{1,2\},\{2,3\}\}$. Here agent 1 is linked with agent 2 and agent 2 is linked with agent 3, however agents 1 and 3 are not linked under g. Thus we represent bilateral relations between two agents by a link between them. Here the links are not directed and they do not carry any intensity information about the communication possibilities of the agents. In this setting, a graph is just a set of links.

Let $g^N = \{\{i,j\} \mid i \in N , j \in N, \ i \neq j \}$. Here g^N is the set of all subsets of N of cardinality 2. Note that a graph can also be defined as any subset g of g^N. We refer g^N as the complete graph on N. In a similar fashion, we will denote the complete graph on S as g^S where $S \subset N$.

Hereafter, for ease of notation we will use ij to denote $\{i,j\}$, the link between the agents i and j. Let $G = \{g \mid g \subset g^N \}$ be the set of all possible graphs on N.

Given a graph $g \in G$, let $N(g) = \{i \,/\, \exists\, j \in N \ \ s.t.\ ij \in g\}$, that is the set of individuals who are involved in at least one link in g. The agents not in $N(g)$ are called *isolated players* under g.

Given a graph $g \in G$, a sequence of individuals $i_1, \ldots, i_K \in N$ such that $i_K i_{K+1} \in g$ for each $k \in \{1, \ldots, K\text{-}1\}$, with $i_1 = i$ and $i_K = j$, is called a *path* in g between the individuals i and j.

Given a graph $g \in G$ and $S \subset N$, $g|_S = g \cap g^S = \{ ij \mid ij \in g \text{ and } i,j \in S \}$ and we call $g|_S$ *the restriction of g to S.*

Given a graph $g \in G$, any nonempty subgraph $g' \subset g$ such that

1) there exists a path in g' between i and j for any $i, j \in N(g')$ with $i \neq j$, and

2) there does not exist a path in g between i and j for any $i \in N(g')$ and $j \in N \setminus N(g')$,

is called a *component* of g.

The components of a graph are its maximal connected subgraphs. We will denote the set of components of a graph g by $C(g)$.

A function $v : G \rightarrow \Re$ is called a *value function*.

The value function represents the total value created through the formation of a graph. It assigns to each possible network structure a real number. Note that the value function is a more general framework to evaluate the total value generated by a graph than the graph restricted game framework of Myerson (1977). In the latter what only matters is what the connected components are; in the former, however, it also matters how the agents within a component are connected. We assume that $v(\emptyset) = 0$, that is no value can be created in the absence of any connections between the agents.

We will denote the set of all value functions, that is all functions of the form $v : G \rightarrow \Re$, by V. But in our study we will mostly be interested in settings where the value is strictly positive except at the empty graph, as in Dutta & Mutuswami (1997). Let V^{+} stands for this class of value functions, i.e., $V^{+} = \{ v \in V \mid v(\emptyset) = 0 \text{ and } v(g) > 0 \text{ for all } g \in G \setminus \{\{\emptyset\}\}$.

Given a value function $v \in V$, we say that a network g^{*} is *efficient* with respect to v iff $v(g^{*}) \geq v(g)$ for all $g \in G$. As graphs on N are finite, there exists at least one efficient network with respect to v.

Given a permutation of agents π (i.e., a bijection on N) and any $g \in G$, we let $\pi(g) = \{ \pi(i)\, \pi(j) \mid ij \in g \}$. Thus $\pi(g)$ has the same architecture as g, with the vertices permuted under π. We say that v is *anonymous* iff for any permutation π and any $g \in G$, one has $v(\pi(g)) = v(g)$.

We say that v is *monotonic* iff for all $g, g' \in G$ with $g \subset g'$ we have $v(g') \geq v(g)$.

A natural interpretation for monotonicity is that more cooperation leads to more value. In this case, we assume that link formation is not costly, or at least the link costs do not have a substantial effect on the value created.

v is *convex* iff for all g, g'$\in$G, $v(g) + v(g') \leq v(g \cup g') + v(g \cap g')$.

Convexity of a value function implies that each link creates a higher marginal increase in value when added to a greater network structure.

v is *component additive* iff for all $g \in G$, $v(g) = \sum_{h \in C(g)} v(h)$. Note that component additivity requires that value generated by a component should not depend on the structure of the rest of the network.

Now we define the *network core* $NC(v)$ *with respect to a value function* $v \in V$ by $NC(v) = \{g \in G \mid v(g|_S) \geq v(g') \text{ for all } S \text{ and for all } g' \subset g^S\}$.

A natural interpretation for the network core is that it chooses the graphs with optimally connected agents in the value maximizing sense. If we assume the existence of a possible way to distribute the generated value among the agents while aligning individual incentives with efficiency, then we should expect that the graphs formed will be such that no coalition can do better on its own. That is, from a stability perspective, at a given state if in a particular graph there is a coalition S with a better communication opportunity than the present one in itself, then we can naturally expect that coalition to deviate from the existing structure to a better one for itself. Of course, here we assume that the coalition members are able to find a way to distribute the marginal increase in value among themselves, in a satisfactory manner.

A function $Y: G \times V \rightarrow \Re^N$ such that $\sum_i Y_i(g, v) = v(g)$ for all $v \in V$ and $g \in G$, is called an *allocation rule*. Let Y be the set of all such allocation rules.

An allocation rule distributes the total value to the agents for each graph - value function pair. The balancedness condition is built into the definition, that is the allocation rule distributes exactly the value of the given graph under the particular value function. In a sense, an allocation rule is analogus to a value operator in cooperative game theory. In many contexts allocation rules are considered as objects to be designed rather than naturally arising rules. Generally allocation rules are characterized through some "fairness" axioms, as is exemplified by Jackson (1996) who uses equal bargaining power and component balancedness for this purpose. In some models, the structure of the value function is such that it is naturally associated with an allocation rule as in the symmetric connections model in Jackson (1996).

EXAMPLE 1

Let $N = \{1, 2, 3, 4, 5\}$. Let v be anonymous and $v(12) = 1$, $v(12, 13) = 3$, $v(12, 13, 23) = 5$, $v(g^N) = 10$ and $v(g) = 0$ for all other graphs $g \in G$ not covered above.

Now consider $\overline{g} = \{12, 13, 23\}$, where $N(\overline{g}) = \{1, 2, 3\}$ and agents *4, 5* are isolated players. There is one component of this graph, $C(\overline{g}) = \{\overline{g}\}$. Now consider the egalitarian allocation rule $Y_i(g,v) = \dfrac{v(g)}{|N|}$ for all $i \in N$ and $g \in G$, $v \in V$. Under this allocation rule every agent gets a payoff of 1 at $\overline{g}$.

An allocation rule Y is said to be *component balanced* if for any component additive v, any $g \in G$, and any $h \in C(g)$

$$\sum_{i \in N(h)} Y_i(g,v) = v(h) \qquad \text{holds.}$$

An allocation rule is *proportional* iff for each $i \in N$ and $v \in V$ either $Y_i(g,v) = 0$ for all g, or for any g and g' such that $v(g') \neq 0$, $\dfrac{Yi(g,v)}{Yi(g',v)} = \dfrac{v(g)}{v(g')}$.

If the allocation rule is proportional, then for any given v, the allocation rule distributes the generated value to the agents with respect to a particular proportion; it is only the total value that changes with the graph. Trivially a proportional rule aligns individual incentives with overall efficiency. Actually the egalitarian allocation rule is a very special example of proportional rules.

An allocation rule satisfies the *dummy axiom* iff for any g, v pair, we have $Y_i(g,v) = 0$ whenever $i \notin N(g)$. The dummy axiom only requires that isolated players will always be given zero payoff, that is to be given a positive payoff, an agent should be involved in some kind of cooperation.

An allocation rule Y is said to be *value-monotonic* iff for any v, and any g such that $v(g) \geq v(g')$ for all $g' \subset g^S$ for some $S \supset N(g)$, there does not exist any $g'' \subset g^S$ with $Y_i(g'',v)) \geq Y_i(g,v)$ for all $i \in S$ holding strictly for at least one $j \in S$.

An immediate explanation is that an allocation rule is value – monotonic iff the following holds: When a coalition S is optimally connected in the value maximizing sense, then altering the graph by a rearrangement of links among the coalition members should not make the coalition better off, with at least one agent being strictly better off, under the given allocation rule. This property of the allocation rule reflects a minimal incentive for the agents to align their interests with overall efficiency. Let Ψ stand for the set of all value-monotonic allocation rules.

This class of allocation rules actually covers all of the most prominent allocation rules. For example the egalitarian allocation rule, the component-wise egalitarian allocation rule and Myerson Shapley allocation rule belong to this class (Jackson 2003b). Any allocation rule that satisfies the dummy axiom or is proportional is also in this class as shown in lemma 1 below. First we will define them.

The *egalitarian allocation rule* is defined by $Y_i^E(g,v) = \dfrac{v(g)}{|N|}$ for all g, v pairs, and for all $i \in N$.

The *component-wise egalitarian allocation rule* is given through $Y_i^{CE}(g,v) = \frac{v(h)}{|N(h)|}$ when v is component additive, $i \in N(h)$ and $h \in C(g)$. If there does not exist $h \in C(g)$ such that $i \in N(h)$, that is i is an isolated player under g, then i gets 0 payoff. When v is not component additive, then $Y_i^{CE}(g,v) = Y_i^{E}(g,v)$ for all $i \in N$.

The *Myerson-Shapley allocation rule* is defined as

$$Y_i^{MV}(g,v) = \sum_{S \subset N \setminus \{i\}} (v(g \mid S \cup \{i\}) - v(g|S)) \left(\frac{\#S!(n-\#S-1)!}{n!} \right)$$

for all $i \in N$ and for all g, v pairs.

As final note concerning the value-monotonic allocation rules is that that Ψ contains some allocation rules that do not exhibit even a minimal amount of anonymity. For example, consider the allocation rule which assigns the value of the graph to agent 1 and assigns zero payoff to all the other agents at every graph for any value function. This dictatorial allocation rule which is utterly non-anonymous is clearly value-monotonic.

EXAMPLE 2: *An allocation rule that is not value monotonic.*

To see that there exist non-value-monotonic allocation rules, consider the allocation rule which distributes all the value evenly among the isolated players at each graph for every value function. Let it be equal to the egalitarian allocation rule on graphs with no isolated players; that is, when there are no isolated players, the allocation rule distributes the value evenly among the set N of all players. Let $N = \{1,2,3,4\}$ and $v(12) = v(13) = v(23) = 1$ and let the "value of all other links" equal 0. Also let the value function v be additive, that is, for all $g, g' \in G$ with $g \cap g' = \emptyset$, $v(g \cup g') = v(g) + v(g')$. Note that

$g^* = \{12, 13, 23\} \in NC(v)$ as clearly $v(g^*|_S) \geq v(g')$ for all S and for all $g' \subset g^S$. Now we claim that Y is not value monotonic. Let $S = \{1,2,3\}$ and $g'=\{12\}$. We have $Y_i(g',v)= Y_i(g^*,v) = 0$ for $i = 1$ and 2, and $Y_3(g',v) = 0.5 > 0 = Y_3(g^*,v)$ .Thus Y is not value monotonic.

Now we will define the network core with respect to a value function and an allocation rule.

$NC(v,Y) = \{g \in G \mid \neg \exists (g', S)$: $S \subset N$ and $g' \subset g^S$ such that $Y_i(g',v)) \geq Y_i(g|_S,v)$ for all $i \in S$ and for at least one $j \in S$ the inequality holds strictly$\}$

A little explanation may be appropriate. The *network core with respect to the value function v and the allocation rule Y* is the set of graphs upon which no coalition can improve in the Pareto sense (all get better off and at least one agent of the coalition gets strictly better off) by deviating from the initial graph through adding or severing links with vertices in the coalition. As seen, here we take the approach that any coalition has the right to arrange the links with both vertices in the coalition, assuming that the allocation rule is applied to the graph obtained from the original one by deleting all the links with at least one vertex outside the coalition. Then the coalition will get payoffs with respect to the remaining graph under the original value function and allocation rule. As the deviation considerations of a coalition involve allocations at a graph where the rest of the society is considered as isolated agents, this approach proves to be useful to the extent that the value function is additive or at least component additive. Such restrictions on the value function make our approach according to which we consider the values generated by the restrictions of the graph to various coalitions meaningful. What we find is that, for any value function, the set of the graphs that are supported as network cores with respect to any allocation rule with a minimal tendency to align individuals with overall efficiency we use, coincides with the set of network cores with respect to the value function *v* (lemma 2).

A function $w : 2^N \setminus \{\emptyset\} \rightarrow \Re$ is called a *transferable utility (TU) game*.

Here, 2^N denotes the set of all possible coalitions in the society. In Myerson's communication structures setting, he begins with a TU game and a graph representing the communication possibilities among the agents. He assumes that if the agents of a coalition can communicate somehow, then the value for that coalition in the initial TU game can be generated regardless of the way the coalition is connected. Given a graph, he evaluates the value of a coalition by summing the values of the connected subcoalitions of the original coalition under the given TU game. Jackson (1996) proposes another framework where the value generated directly depends on the whole communication structure, not only the set of coalitions that can communicate within themselves. In Myerson's setting, we directly start with a TU game which determines the potential value of a coalition when all its agents can communicate with each other. Now in Jackson's framework, we should derive a function that measures the value generating potential of each coalition under a given value function that assigns to each graph the value it generates. We do this by taking the maximal value of the graphs that a coalition can form by links with both vertices in the coalition. Here we assume that any two agents can form a link between themselves by mutual consent, regardless of the remaining agents in the society. Thus a coalition can arrange in such a way that the graph that forms is one that maximizes the value among all possible graphs this coalition can form.

Given any value function v, we derive a TU game w_v associated with v in the following way. For any $S \subset N$, we set $w_v(S) = \max_{g \subset g^S} v(g)$

Thus from the way w_v is generated, we infer that it is monotonic, that is, if $S, T \subset N$ with $S \supset T$ then $w_v(S) \geq w_v(T)$.

We say that a TU game w is *convex* iff, $\forall\ S, T \subset N$, we have

$w(S \cup T) + w(S \cap T) \geq w(S) + w(T)$.

From cooperative game theory we know that this definition is equivalent to the following:

w is convex iff $\forall\ S, T \subset N$ with $T \subset S$ and $\forall i \notin S$, we have $w(S \cup \{i\}) - w(S) \geq w(T \cup \{i\}) - w(T)$.

CHAPTER 4

RESULTS

Lemma 1: Y^E , Y^{CE} , Y^{MV} $\in$ Ψ. *Also, if Y satisfies the dummy axiom or is proportional, then* $Y \in \Psi$.

<u>Proof:</u> Take any v, and any g with $v(g) \geq v(g')$ for all $g' \subset g^S$ for some $S \supset N(g)$ Assume to the contrary to the proposition that $\exists g'' \subset g^S$ with $Y_i(g'',v)) \geq Y_i(g,v)$ for all $i \in S$,holding strictly for at least one $j \in S$. As $Y_i^E(g,v) = \dfrac{v(g)}{|N|}$ for all $i \in N$, we have $Y_j^E(g'',v) = \dfrac{v(g")}{|N|} > \dfrac{v(g)}{|N|} = Y_j^E(g,v)$, which implies $v(g'') > v(g)$, a contradiction. Thus $Y_i^E \in \Psi$.

For Y_i^{CE} , if v is not component additive, the proof is the same. If v is component additive, $Y_i^{CE}(g,v) = \dfrac{v(h)}{|N(h)|}$ when $i \in N(h)$ where $h \in C(g)$. If there does not exist $h \in C(g)$ such that $i \in N(h)$, that is, i is an isolated player under g, then i gets 0 payoff. Now assume that $\exists g$ such that $v(g) \geq v(g')$ for all $g' \subset g^S$ for some $S \supset N(g)$, and also $\exists g'' \subset g^S$ with $Y_i(g'',v)) \geq Y_i(g,v)$ for all $i \in S$ and the inequality holds strictly for at least one

$j \in S$. Note that from component additivity of v, $v(g) = \sum_{h \in C(g)} v(h)$. Also note that $\sum_{i \in N(h)} Y_i^{CE}(h,v) = \frac{v(h)}{|N(h)|}$. $|N(h)| = v(h)$, showing the component balancedness of Y_i^{CE}. Now we have that, as $S \supset N(g)$, $\sum_{h \in C(g)} \sum_{i \in N(h)} Y_i^{CE}(h,v) = \sum_{i \in S} Y_i^{CE}(g,v) = v(g) < v(g'') = \sum_{i \in S} Y_i^{CE}(g'',v)$. Notice that in the double summation, isolated players are not taken into account as they will get zero payoff. A contradiction, implying that $Y^{CE} \in \Psi$.

Clearly Y^{MV} satisfies the dummy axiom, as all graph restrictions with or without the isolated agents will yield the same values in the Shapley calculations. Take any allocation rule satisfying the dummy axiom. Assume it is not value-monotonic. Then $\exists g \in$ G such that $v(g) \geq v(g')$ for all $g' \subset g^S$ for some $S \supset N(g)$, and also $\exists g'' \subset g^S$ with $Y_i(g'',v)) \geq Y_i(g,v)$ for all $i \in S$ where the inequality is strict for at least one $j \in S$. As we know that $N \setminus S$ consists of isolated agents, from the balancedness condition built in the definition of the allocation rule, summing both sides over i, we get that $\sum_{i \in N} Y_i(g,v) = \sum_{i \in S} Y_i(g,v) + \sum_{i \in N/S} Y_i(g,v) = v(g) = \sum_{i \in S} Y_i(g,v) < \sum_{i \in S} Y_i(g'',v) = v(g'')$, a contradiction again. Thus, if Y satisfies the dummy axiom, then $Y \in \Psi$.

Now assume Y is proportional. If $Y_i(g,v) = 0$ for all i and g, then clearly Y is value-monotonic. For the other case, assume on the contrary that $\exists g$ such that

$v(g) \geq v(g')$ for all $g' \subset g^S$ for some $S \supset N(g)$, and also $\exists g'' \subset g^S$ with $Y_i(g'',v) \geq Y_i(g,v)$ for all $i \in S$ with the inequality being strict for at least one $j \in S$. Then as $\frac{Yj(g'',v)}{Yj(g,v)} = \frac{v(g'')}{v(g)}$ from proportionality, we infer $v(g'') > v(g)$, which is a contradiction. Note that in case both $v(g)$ and $v(g'')$ are zero, we are in the first case where $Y_i(g,v) = 0$ for all g, as otherwise the proportionality condition would be violated. If one of them is zero, we consider it in the upper side of the fraction. Thus, if Y is proportional, then we have $Y \in \Psi$.

Lemma 2 : For any $v \in V$, $NC(v) = \bigcap_{Y \in \Psi} NC(v, Y) = NC(v, Y^E)$.

Proof : We will show that $NC(v) \subset \bigcap_{Y \in \Psi} NC(v, Y) \subset NC(v, Y^E) \subset NC(v)$.

First inclusion: Take any $g \in NC(v)$. Assume $\exists\, Y \in \Psi$ such that $g \notin NC(v, Y)$. Then $\exists\ S \subset N$, $g' \subset g^S$ with $\forall i \in S \quad Y_i(g',v)) \geq Y_i(g|_S, v)$, where the inequality holds stricly for at least one $j \in S$. But we know that $v(g|_S) \geq v(g')$ as $g \in NC(v)$ and $g' \subset g^S$. Thus Y is not value-monotonic, a contradiction.

Second inclusion: We already know that $Y^E \in \Psi$.

Third inclusion: Take any $g \in NC(v, Y^E)$. Assume on the contrary that $\exists\ S \subset N$, $g' \subset g^S$ with $v(g') > v(g|_S)$, that is $g \notin NC(v)$. Now $Y_i^E(g',v) = \frac{v(g')}{|N|} > \frac{v(g|_S)}{|N|} = Y_i^E(g|_S, v)$

for all $i \in N$, thus for all $i \in S$. Now, if we take S and g', the condition that $g \in NC(v, Y^E)$ is violated, a contradiction. thus $NC(v, Y^E) \subset NC(v)$.

Lemma 3: For any $v \in V^+$, $NC(v) \neq \emptyset \Leftrightarrow w_V{}^*(S) = v(g^S)$.

<u>Proof :</u> ($\Rightarrow$) Take any $g \in NC(v)$. Asume $ij \notin g$. Now setting S = $\{i,j\}$, $v(\{ij\}) > 0 = v(\emptyset) = v(g|_S)$, a contradiction. Thus $g = g^n$. Now $\forall\ S, \forall g' \in g^S : v(g^S) = v(g|_S) \geq v(g')$; thus $w_V(S) = v(g^S)$ for all $S \subset N$. For the other direction, it suffices to note that the complete graph is in the network core.

Lemma 4 : If $v \in V$ v is convex, then w_V is convex

<u>Proof :</u> Take any $S, T \subset N$. Assume on the contrary that $w_V(S \cup T) + w_V(S \cap T) < w_V(S) + w_V(T)$. Now $\exists\ g', g''$ with $g' \subset g^S$ and $g'' \subset g^T$ such that $w_V(S) = v(g')$, $w_V(T) = v(g'')$. Note that we have $w_V(S \cup T) \geq v(g' \cup g'')$ and $w_V(S \cap T) \geq v(g' \cap g'')$ as $g' \cup g'' \subset g^{S \cup T}$, $g' \cap g'' \subset g^{S \cap T}$, thus implying that

$v(g') + v(g'') = w_V(S) + w_V(T) > w_V(S \cup T) + w_V(S \cap T) \geq$

$v(g' \cup g'') + v(g' \cap g'')$, a contradiction as v is convex.

Note that the converse need not be true. Consider $N = \{1, 2, 3\}$ and $w_V(S) = |S| - 1$ for all $\emptyset \neq S \subset N$, which is clearly convex. Consider any v that induces this TU game. Let $g_0 = \emptyset$, $g_1 = \{12\}$, $g_2 = \{12, 13\}$. Now $w_V(12) = 1 = v(g_1)$. Let $v(g_2) = a$. Now $w_V(1, 2, 3) = 2 = v(g^N)$ because if some network with two links would have the maximal value, this would violate the convexity of v. To see this note that adding the last link to a two-link graph would not result in a strictly positive marginal increase in value while it would lead to a strictly positive marginal increase in value when it is added to the empty graph. Now from convexity of v, $2 - a \geq a - 1 \geq 1 - 0$ should be satisfied. It is a contradiction, thus there does not exist a convex value function inducing v.

Now we will look more closely to the implications of convexity of a value function on the TU game it induces.

Lemma 5 : Let $v \in V^{+}$. If v is anonymous and convex, then w_V is symmetric and

$\dfrac{\mathrm{w}\,(\#S) - w(\#S-1)}{\#S}$ is increasing in $\#S$ for $\#S \geq 1$.

<u>**Proof :**</u> First of all, note that the convexity of $v \in V^{+}$ implies the monotonicity of v, because we have $v(ij) > 0$ as $v \in V^{+}$ and the convexity of v implies that

$v(g \cup ij)) - v(g) \geq v(ij) - v(\emptyset) > 0$ for all $g \in G$ and all $i, j \in N$ with $ij \notin g$. Thus we will use $w_V(S) = v(g^S)$ hereafter in the proof.

<u>*Claim1:*</u> $\forall\, S, T \subset N$ with $T \subset S$ and $\forall\, i,j,k,l \in N$ with $i \in S, j \notin S, k \in T, l \notin S,$

we have $v(g^S \cup ij) - v(g^S) \geq v(g^T \cup kl) - v(g^T)$.

<u>*Proof of claim 1:*</u> Firstly note that for any permutation π on N, and for any $g \in G$

$\pi(g \cup g') = \{\pi(i)\pi(j) \mid ij \in g \cup g'\} = \{\pi(i)\pi(j) \mid ij \in g \text{ or } ij \in g'\} =$

$\{\pi(i)\pi(j) \mid ij \in g\} \cup \{\pi(i)\pi(j) \mid ij \in g'\} = \pi(g) \cup \pi(g')$.

Now consider the permutation π such that $\pi(i) = k$, $\pi(k) = i$ and $\pi(m) = m$ for all $m \in N \setminus \{ i,k \}$. Hereafter let us denote this type of permutations, i.e., transpositions, by $\pi : i \leftrightarrow k$. Now

$v(g^S \cup ij) - v(g^S) = v(\pi(g^S \cup ij)) - v(\pi(g^S)) = v(\pi(g^S) \cup \pi(ij)) - v(\pi(g^S))$

$= v(g^S \cup kj) - v(g^S)$ as $k, i \in S$.

Now by considering the transposition $\mu : j \leftrightarrow l$,

$v(g^S \cup kj) - v(g^S) = v(\mu(g^S \cup kj)) - v(\mu(g^S)) = v(\mu(g^S) \cup \mu(kj)) - v(\mu(g^S))$

$= v(g^S \cup kl) - v(g^S) \geq v(g^T \cup kl) - v(g^T)$ as $l, j \notin$ S and v is convex.

In what we did above, we used the fact that, a complete graph on S remains the same under a transposition, if the transposed vertices both belong or both do not belong to the coalition S.

<u>*Claim 2:*</u> For all $S \subset$ N, for any $\{i_1, i_2, i_3, ..., i_m\} \subset$ S, $\{j_1, j_2, j_3, ..., j_m\} \subset$ S and any $k \neq l$ with $k, l \notin$ S, we have

$$v(g^S \cup \{i_1k, i_2k, i_3k ..., i_mk\}) = v(g^S \cup \{j_1l, j_2l, j_3l, ..., j_ml\}).$$

<u>*Proof of claim 2:*</u> Consider the permutation π with $\pi(i_p) = j_p$ and $\pi(j_p) = i_p$ for all $p \in \{1,2, ...m\}$ and $\pi(k) = l$, $\pi(l) = k$, where π acts as the identity permutation on the rest of the society. Notice that this is possible, as trivially this permutation is the composition of m+1 transpositions.

Now under this permutation,

$v(g^S \cup \{i_1k, i_2k, i_3k \ldots, i_mk\}) = v(\pi\, (g^S \cup \{i_1k, i_2k, i_3k \ldots, i_mk\})) =$

$v(\pi\, (g^S) \cup \pi\{i_1k, i_2k, i_3k \ldots, i_mk\})) = v(g^S \cup \{ j_1l,\ j_2l,\ j_3l,\ \ldots, j_ml \})$.

The last equality is justified by the fact that the permutation only switches pairs which either belong to S or both not belong to S. It can also be justified as this particular permutation is a composition of transpositions; each satisfying the same equality, the composite function satisfies the equality.

Claim 3: For all $S \subset N$, for any $\{i_1, i_2, i_3, \ldots, i_m, \ldots\} \subset S$, $\{j_1, j_2, j_3, \ldots, j_m, \ldots\} \subset S$ and any $k, l \notin S$, we have, for any $m > 0$

$v(g^S \cup \{i_1k, i_2k, i_3k \ldots, i_{m+2}k\}) - v(g^S \cup \{i_1k, i_2k, i_3k \ldots, i_{m+1}k\}) \geq$

$v(g^S \cup \{ j_1l,\ j_2l,\ j_3l,\ \ldots, j_{m+1}l \}) - v(g^S \cup \{ j_1l,\ j_2l,\ j_3l,\ \ldots, j_ml \})$

and in particular in the boundary case,

$v(g^S \cup \{i_1k, i_2k\}) - v(g^S \cup \{i_1k\}) \ \geq\ v(g^S \cup \{ j_1l \}) - v(g^S)$.

Proof of claim3: For the m=0 boundary case (with an abuse of notation); we have $v(g^S \cup \{i_1k, i_2k\}) - v(g^S \cup \{i_1k\}) = v(g^S \cup \{j_1l, j_2l\}) - v(g^S \cup \{i_2l\}) \geq v(g^S \cup \{ j_1l \}) - v(g^S)$.

Note that the first equality is a direct corollary of claim 2. Similarly for $m \geq 1$;

$v(g^S \cup \{i_1k, i_2k, i_3k \ldots, i_{K+2}k\}) - v(g^S \cup \{i_1k, i_2k, i_3k \ldots, i_{K+1}k\})$

$= v(g^S \cup \{ j_1l,\ j_2l,\ j_3l,\ \ldots, j_{K+2}l \}) - v(g^S \cup \{ j_1l,\ j_2l,\ j_3l,\ \ldots, j_Kl, j_{K+2}l \})$

$\geq\ v(g^S \cup \{ j_1l,\ j_2l,\ j_3l,\ \ldots, j_{K+1}l \}) - v(g^S \cup \{ j_1l,\ j_2l,\ j_3l,\ \ldots, j_Kl \})$, where the first equality follows from claim 2 and the inequality follows from the convexity of *v*.

Now turning to our lemma, obviously w_V is symmetric as *v* is anonymous. Also as v is convex, the following inequalities yield the desired result.

Let $S = \{ i_1, i_2, i_3, ..., i_K \}$, and i_{K+1} , $i_{K+2} \in N \setminus S$. Note that

$$\frac{w(\#S+2) - w(\#S+1)}{\#S+1} = \frac{v(g^{\#S+2}) - v(g^{\#S+1})}{\binom{\#S+2}{2} - \binom{\#S+1}{2}}$$

$$= \frac{\sum_{t=0}^{k}\left[v(g^{\#S+1} \cup \{i_1 i_{k+2}, i_2 i_{k+2}, ..., i_{t+1} i_{k+2}\}) - v(g^{\#S+1} \cup \{i_1 i_{k+2}, i_2 i_{k+2}, ..., i_t i_{k+2}\}) \right]}{\binom{\#S+2}{2} - \binom{\#S+1}{2}}$$

$$\geq \frac{\sum_{t=0}^{k}\left[v(g^{\#S} \cup \{i_1 i_{k+1}, i_2 i_{k+1}, ..., i_{t+1} i_{k+1}\}) - v(g^{\#S} \cup \{i_1 i_{k+1}, i_2 i_{k+1}, ..., i_t i_{k+1}\}) \right]}{\binom{\#S+1}{2} - \binom{\#S}{2}}$$

$$= \frac{v(g^{\#S+1}) - v(g^{\#S})}{\binom{\#S+1}{2} - \binom{\#S}{2}} = \frac{w(\#S+1) - w(\#S)}{\#S}$$

Now, the first and the last equalities follow from monotonicity of v. The middle equalities are written with an abuse of notation, for t = 0 would imply the nonexistence of the $i_t\, i_{k+1}$ or the $i_t\, i_{k+1}$ link in the summation. If the summation is carried out, “the last minus the first terms” will remain, while the middle terms cancel out.

The inequality in the middle follows from claims, 1,2 and 3. Thus we are done.

In this lemma, we have shown that, convexity of the value function implies a property of the induced TU game which is stronger than mere convexity. The converse is also true.

Lemma 6 : Let w be a monotonic and symmetric TU game, where $\forall S \subset N \setminus \{\emptyset\}$, $w(S) \geq 0$ and $w(S) = 0$ if $\#S = 1$ (the associated normalization for TU games induced from value functions).

If $\dfrac{w(\#S) - w(\#S-1)}{\#S}$ is increasing in $\#S$ for $\#S \geq 1$ then there exists a value function $v \in V^+$ inducing w, where v is anonymous and convex.

__**Proof :**__ We will construct a suitable v :

For g=Ø assign v (g)=0. For any $g \in G \setminus \{\emptyset\}$, there exists a unique $k \in Z^+$ with $n \geq k \geq 2$ such that $\binom{k}{2} > \#g \geq \binom{k-1}{2}$ where #g stands for the number of links in g.

Now set

$$v(g) = w(k-1) + [w(k)-w(k-1)] \frac{[\#g - \binom{k-1}{2}]}{\binom{k}{2} - \binom{k-1}{2}}$$

Now our value function is well defined. Clearly $v \in V^+$ as v (Ø)=0 and for any g∈G with #g >0 we have v(g) ≥ w(k) ≥ w(2) > 0. As v is defined through the number of links g contains, v is surely anonymous. We will show that it is convex.

Now let f: $\{0,1,2,\ldots,\binom{n}{2}\} \to \Re$ be defined by f (#g) = v (g) Note that , by construction of v, f is a convex function in #g, which follows from the increasing coefficients of #g, which, in turn, is implied by the condition on w that $\dfrac{[\mathrm{w(k) - w(k-1)}]}{\binom{k}{2} - \binom{k-1}{2}}$ is increasing in k. Thus convexity of v follows from the convexity of f.

Note that in both lemma 5 and 6 anonymity (respectively symmetricity) is crucial in the sense that neither may hold if we delete anonymity.

CHAPTER V

CONCLUSION

Following the setting in Jackson (1996), we defined the value function on the set of all possible graphs. What we tried to do is to inquire which properties of the value function are inherited by the TU game it induces. This is important as we know much more about TU games and their solutions, whereas we have less knowledge networks. What we find is that, under anonymity, convexity of the value function is equivalent to a stronger form of convexity of the induced TU game. Our last two lemmata establish a bridge to associate the well known "nonemptiness of the core" results in both the network and the TU game settings, when we restrict ourselves to appropriately convex structures in both contexts. So this is a towards associating our knowledge in cooperative game theory with the network setting of Jackson (1996). Moreover, we defined the *network core* with respect to a value function (and then also with respect to a value function and an allocation rule pair) in an attempt introduce the counter part of the core for TU games in the network framework. What we found is that the network core with respect to a value function consists of graphs that are in the network core with respect to the same value function and any allocation rule in a quite wide class (namely the class of value monotonic allocation rules), which, in turn is the same as the core corresponding to the egalitarian allocation rule only.

From this point on, further study can be focused especially on relaxing the structure that we imposed on the evaluation of the benefits of being in a coalition. In this study, we assumed the rest of the society to be isolated when we evaluated the possible deviations from a particular cooperation structure on the part of a coalition. This is a very restrictive assumption, given the generality we permit concerning the class of value functions and allocation rules. The next step could be redefining the network core concept in α- or β-core sense, where the rest of the society also comes into play, actively. Similarly, in this study we defined the potential of a coalition merely as the maximal value of a network that coalition can form by itself. This also is vulnerable to the same logic we discussed above; the rest of the society is considered as isolated. Thus, the way we induce the TU game can be similarly redefined in various ways.

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