FIXED ORDER CONTROLLER DESIGN VIA PARAMETRIC METHODS

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DOCTOR OF PHILOSOPHY

By

Karim Saadaoui

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Prof. Dr. A. Bülent Özgüler (Supervisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Prof. Dr. M. Erol Sezer

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Prof. Dr. Hitay Özbay

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Prof. Dr. Mefharet Kocatepe

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Prof. Dr. Kemal Leblebicioğlu

Approved for the Institute of Engineering and Sciences:

Prof. Dr. Mehmet Baray
Director of the Institute of Engineering and Sciences

ABSTRACT

FIXED ORDER CONTROLLER DESIGN VIA PARAMETRIC METHODS

Karim Saadaoui

Ph.D. in Electrical and Electronics Engineering

Supervisor: Prof. Dr. A. Bülent Özgüler

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In this thesis, the problem of parameterizing stabilizing fixed-order controllers for linear time-invariant single-input single-output systems is studied. Using a generalization of the Hermite-Biehler theorem, a new algorithm is given for the determination of stabilizing gains for linear time-invariant systems. This algorithm requires a test of the sign pattern of a rational function at the real roots of a polynomial. By applying this constant gain stabilization algorithm to three subsidiary plants, the set of all stabilizing first-order controllers can be determined. The method given is applicable to both continuous and discrete time systems. It is also applicable to plants with interval type uncertainty. Generalization of this method to high-order controller is outlined. The problem of determining all stabilizing first-order controllers that places the poles of the closed-loop system in a desired stability region is then solved. The algorithm given relies on a generalization of the Hermite-Biehler theorem to polynomials with complex coefficients. Finally, the concept of local convex directions is studied. A necessary and sufficient condition for a polynomial to be a local convex direction of another Hurwitz stable polynomial is derived. The condition given constitutes a generalization of Rantzer's phase growth condition for global convex directions. It is used to determine convex directions for certain subsets of Hurwitz stable polynomials.

Keywords: Hermite-Biehler theorem, First-order controllers, Stability, Stabilization, Regional pole placement, Local convex directions.

ÖZET

PARAMETRİK YÖNTEMLE SABİT MERTEBEDEN DENETLEYİCİ TASARIMI

Karim Saadaoui
Elektrik ve Electronik Mühendisliği Doktora
Tez Yöneticisi: Prof. Dr. A. Bülent Özgüler
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Bu tezde, doğrusal, zamanla-değişmeyen, tek-giriş ve tek-çıkışlı sistemleri kararlı hale getiren sabit mertebeden denetleyicilerin parametrizasyonu problemi incelenmektedir. Hermite-Biehler teoreminin bir genellemesi kullanılarak, doğrusal, zamanla-değişmeyen sistemleri kararlılaştıran sabit kazançların belirlenmesi için yeni bir algoritma geliştirilmiştir. Bu algoritma rasyonel bir fonksiyonun gerçek bir polinomun köklerindeki değerlerinin işaret dizgesinin testine dayanmaktadır. Bu sabit kazanç algoritmasını üç yardımcı sisteme uygulayarak, verilen bir sistemi kararlı hale getiren birinci mertebeden denetleyiciler kümesi hesaplanabilir. Önerilen yöntem sürekli-zaman ve kesikli-zaman sistemlerine olduğu gibi parametreleri bir aralıkta değer alabilen belirsiz sistemler kümesine de uygulanabilir. Önerilen yöntemin herhangi bir mertebeden denetleyicilerin hesaplanmasına genellemesi de verilmiştir. Daha sonra, bir kapalı-çevrim sisteminde istenilen kutup atamayı elde edebilen tüm birinci mertebeden denetleyicilerin hesaplanması problemi çözülmüştür. Bu amaçla verilen algoritma Hermite-Biehler teoreminin kompleks katsayılı polinomlara bir genellemesine dayanmaktadır. Son olarak, yerel konveks yönler kavramı incelenmektedir. Verilen bir polinomun başka bir Hurwitz-kararlı polinomun konveks yönü olması için bir gerek ve yeter koşul verilmiştir. Bu koşul, Rantzer'in global konveks yön için verdiği koşulun bir genellemesi olarak düşünülebilir. Verilen koşul, çeşitli Hurwitz-kararlı polinom kümeleri için konveks yönler bulmakta kullanılabilir.

Anahtar kelimeler: Hermite-Biehler teoremi, Birinci-mertebeden denetleyiciler, Kararlılık, Kararlı hale getirme, Bölgesel kutup atama, Yerel konveks yönler.

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To my family.

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Chapter 1

Introduction

Controllers are designed to make certain physical variables of a system behave in a desired way by manipulating some input variables. In any controller design, a first and essential step in the design process is to guarantee stability of the resulting closed-loop system. Therefore, one natural approach to the synthesis problem is to find the set of all stabilizing controllers for a given system and then determine within this set controllers that satisfy extra design requirements. In fact, parameterization of all stabilizing controllers for linear, time-invariant plants was given in [1, 2] and it is known as the YJBK parameterization [3, 4]. Many synthesis techniques such as H_∞, H_2, and l^1 optimal control [5, 6] are based on YJBK parameterization. However, an important disadvantage of YJBK parameterization is that the order or the structure of the controller can not be fixed a priori. As a result, H_∞ and H_2 design techniques usually yield controllers of high-order in comparison to the order of the plant to be controlled [7, 8, 9, 10].

Simple low-order controllers are usually preferred to complex high-order controllers. It is known that more than 90% of the controllers used in industry are of low-order being proportional-integral-derivative (PID) or first-order lead/lag controllers [11]. The widespread use of these low-order controllers is due to their simplicity and practicality since in many cases a satisfactory behavior of the closed-loop system is achieved by adjusting only three parameters. Many of the elegant results of optimal control are rarely used in industry and this is an important gap between the well established theory of optimal control and applications. For these reasons, there is a need to design low-order controllers for high-order plants. There are mainly three different approaches to do this: (i) Design a high-order controller then approximate it with a low-order one (see [7] for different techniques of controller reduction). (ii) Reduce the order of the plant model so that a controller of low-order is obtained (see [12, 13, 14] and the references therein). (iii) Fix the order of the controller and search parameters that minimize a performance index. The main subject of this thesis falls into this third category.

In addition to fixing the order of the controller, fixing the structure of the controller may be desired in some applications. In [15], an H_2 optimal synthesis method of controllers with relative degree 2 is suggested. The advantage of stabilizing with a controller of relative degree 2 as advocated in [15] is the need for the frequency response to roll-off as quickly as possible after the gain cross-over frequency so that unmodeled high-frequency plant dynamics are not excited by the controller dynamics. A linear programming approach that attempts to meet the desired closed-loop specifications with fixed-order controllers was given in [16]. In [17], sufficient conditions for the synthesis of H_∞ fixed-order controllers are derived. These conditions convert the controller design problem into a linear matrix inequality feasibility problem. Synthesis of fixed-order controllers that minimize an upper bound on the peak magnitude of the tracking error was given in [18]. In [19], sufficient conditions for characterizing robust full and reduced order controllers with worst case H_2 performance bound were derived. We refer the interested reader to [20]-[24] for more state-space design methods with fixed-order controllers.

An alternative design strategy would be to (a) parameterize all fixed-order, fixed-structure stabilizing controllers and (b) among those that are obtained search the ones which satisfy a specified performance. The solution to problem (a) is an essential and a challenging first step. Designing an optimal low-order controller, PID or first-order, can not be achieved without solving problem (a). It also gives an answer to the best performance that can be achieved by these controllers for a given plant. A step in this direction was taken in [25] parameterizing the set of all stabilizing PID controllers. In fact, a lot of research has been done for finding parameters of PID controllers that lead to a satisfactory performance, see [26]-[33] and the references therein, but only a limited number of results have been reported to find the set of all stabilizing PID controllers and, hence, to find a compromising approach between the well established H_∞, H_2, and l^1 optimal techniques and the more practical low-order compensation methods.

In [25], a computational characterization of all stabilizing proportional-integral (PI) and PID controllers was derived. This method is based on an extension of the Hermite-Biehler theorem reported in [34], see [35]. The computational method of [25] has been extended to compute all stabilizing PID gains for discrete time systems in [36]. In [37], using the Nyquist plot an alternative method for determining the set of all stabilizing PID controllers is developed. The problem of determining all stabilizing PID controllers was also studied in [38, 39] using graphical methods. In [40], it was shown that for a fixed value of the proportional term the Hurwitz stability boundaries in the parameter space of the integral and derivative terms are hyperplanes and the stability regions are convex polyhedra. In [41], the problem of synthesizing PID controllers for which the closed-loop system is internally stable and the H_∞ norm of a related transfer function is less than a prescribed level was addressed. Recently, a computational characterization of all admissible PID controllers for robust performance was provided in [42]. None of the studies above give a clue to extend the results to first-order controllers which are structurally different and hence need to be considered separately.

The quest for an analytic design method for first-order controllers (e.g. phase-lead, phase-lag) controllers has been around for decades. Many classical control textbooks such as [43], [44] contain attempts to deductively obtain a first-order stabilizing controller. In [43], for example, an analytic method for designing a first-order controller is suggested although the authors emphasize that the design is not guaranteed to succeed and it may lead to an unstable system.

In this thesis, we first study the problem of parameterizing the set of all stabilizing first-order controllers. Although the number of parameters involved in both PID and first-order controllers is the same, structures of these controllers are different and the results found for PID controllers can not be directly applied to first-order controllers. We also establish that our method, unlike other methods, can be extended to higher order controllers. An alternative approach to the problem of determining all stabilizing first-order controllers for discrete time systems was also taken in [45]. The solution given in [45] is based on a Chebyshev representation of the characteristic equation on the unit circle. The method relies on arbitrarily fixing one of the controller parameters and generating the root distribution invariant regions in the space of the remaining two parameters. Once these regions are determined, a stability test has to be performed to determine the stabilizing region. Unlike our method, no hint is given on how to fix the first parameter. Hence, in order to determine the set of all stabilizing first-order controllers by the approach of [45], one has to carry out the method for an infinite range of the first parameter. The boundaries of the root distribution invariant regions are found by sweeping over all the frequencies ($w \geq 0$ for continuous time systems), hence another sweep over an infinite range has to be carried out for the method to be applicable to continuous time systems. Note also that this method can not be extended to higher-order controllers without arbitrarily fixing all but two of the controller parameters. This is due to the fact that the stability boundaries are obtained by setting to zero the imaginary and the real parts of the characteristic equation evaluated at a fixed frequency. The computational method proposed in this thesis is free of these drawbacks.

The second problem studied in this thesis is the determination of local convex directions for Hurwitz stable polynomials. The main motivation for studying convex directions for Hurwitz stable polynomials comes from the edge theorem [46] which states that, under mild conditions, it is enough to establish the stability of the edges of a polytope of polynomials in order to conclude the stability of the entire polytope. Each edge is a convex combination $\lambda r(s)+(1-\lambda)q(s),\ \ \lambda \in [0,1]$ of two vertex polynomials $r(s),\ q(s)$. If the difference polynomial $p(s) = r(s) - q(s)$ is a convex direction for $q(s)$, then the stability of the entire edge can be inferred from the stability of the vertex polynomials. In [47], Rantzer gave a condition which is necessary and sufficient for a given polynomial to be a convex direction for the set of *all* Hurwitz stable polynomials. However, this global requirement is unnecessarily restrictive when examining the stability of a particular segment of polynomials. It is of more interest to determine conditions for a polynomial to be a convex direction for a given polynomial, or still better, for specified subsets of Hurwitz stable polynomials.

Various solutions to the edge stability problem are already well-known [48]-[52]. Bialas [53] gave a solution in terms of the Hurwitz matrices associated with $r(s)$ and $q(s)$. The segment lemma of [54] gives another condition which requires checking the signs of two functions at some fixed points. In [55], [34] and [56], various definitions of local convex directions have been used. Among these, the following geometric characterization of [55] is the most relevant one to edge stability we have described above: A polynomial $p(s)$ is called a (local) convex direction for $q(s)$ if the set of $\alpha > 0$ for which $q(s) + \alpha p(s)$ is Hurwitz stable is a single interval on the real line. Note that, if $p(s)$ is a convex direction in this sense, the stability of $q(s)$ and $p(s)+q(s)$ implies the stability of $q(s)+\alpha p(s)$ for all $\alpha \in [0,1]$ but not vice versa, i.e., the main definition used in [55] and [34] is more stringent than the one concerning the edge stability. In this thesis we will use the definition given in [56]; namely, a local convex direction with respect to $q(s)$ is a polynomial $p(s)$ such that all polynomials which belong to the convex combination of $q(s)$ and $q(s)+p(s)$ are Hurwitz stable.

One motivation for deriving an alternative condition to those of [53] and [54] is to make contact with Ranzter's condition starting with the less stringent definition of local convexity. A second motivation is that none of the above local results seem to be suitable in determining convex directions for *subsets* of Hurwitz stable polynomials. Our main result is shown to be suitable for obtaining convex directions for certain subsets of Hurwitz stable polynomials. The condition provided also gives Rantzer's condition in a rather straightforward manner when it is satisfied by every Hurwitz stable polynomial. It is thus one natural local version of the global condition of Rantzer.

Although our two main problems (1) parameterizing stabilizing controllers with fixed-order and fixed-structure and (2) determining local convex directions for Hurwitz stable polynomials are two different problems, one contribution of this thesis is to show that they can be treated in the unifying framework of the Hermite-Biehler theorem and its extensions.

Contents of the thesis can be summarized as follows: In Chapter 2, we review the Hermite-Biehler theorem and its generalizations. In [34] a generalization of the Hermite-Biehler theorem, applicable to not necessarily Hurwitz stable polynomials, was given. The generalized theorem gives the root distribution of a real polynomial with respect to the imaginary axis. Based on this generalization, we show how to determine the number of distinct real negative roots of a real polynomial without explicitly calculating them. This will prove fundamental in parameterizing different types of controllers that stabilizes a given linear, time-invariant plant. In [41], a generalization of the Hermite-Biehler theorem to polynomials with complex coefficients was given. We also use this result to compute the number of real roots of a real polynomial, which is in turn used to solve the problem of stabilization with guaranteed damping.

In Chapter 3, we give the non-graphical method of [34] for the determination of stabilizing gains for linear, time-invariant, single input, single output systems. This method requires a test of the sign pattern of a rational function at the real roots of a polynomial. Thereafter, we simplify this method and give an algorithm which avoids the need for a search in an exponentially increasing set to determine the solution. From a computational complexity point of view, our method requires $\mathbf{O}(n^2)$ arithmetic operations, whereas using Neimark D-decomposition [57] the same problem can be solved with $\mathbf{O}(n^3)$. We compare this method with the recent Nyquist based method of [37]. We show how the algorithm developed in this chapter can be applied to determine local convex directions.

In Chapter 4, a new method is given for the computation of the set of all stabilizing proper first-order controllers for linear, time-invariant, scalar plants. For clarity, we first solve the problem for plants having either all zeros or all poles in the closed right-half plane. This restrictive assumption is then removed and a solution is given for general plants with no restrictions on pole or zero locations. The method requires the application of a modified constant gain stabilizing algorithm to three subsidiary plants. It is applicable to both continuous and discrete time systems. Using this characterization of all stabilizing first-order controller, we give a design example where several time domain performance indices of the closed-loop system are evaluated. We then solve the problem of determining all stabilizing first-order controllers that achieve a desired damping ratio for the closed-loop system. The algorithms given in this chapter can be applied to plants with interval type uncertainty. Finally in this chapter, we give an algorithm that computes all stabilizing second-order controllers.

In Chapter 5, we use one version of the Hermite-Biehler theorem to study of local convex directions [58]. A new condition for a polynomial $p(s)$ to be a local convex direction for a Hurwitz stable polynomial $q(s)$ is derived. The condition is in terms of polynomials associated with the even and odd parts of $p(s)$ and $q(s)$ and constitutes a generalization of Rantzer's phase growth condition for global convex directions. It is then used to determine convex directions for certain subsets of Hurwitz stable polynomials.

Finally, Chapter 6 contains some concluding remarks and directions for further research.

Chapter 2

The Hermite-Biehler Theorem

In this chapter, we review the Hermite-Biehler theorem and its generalizations. It is well known that studying stability of a dynamical system is one of the most fundamental problems in control theory. For linear time-invariant systems this is equivalent to finding conditions under which all the roots of a polynomial are in the open left-half complex plane. Routh-Hurwitz criterion is one of the first and most known tests for checking Hurwitz stability of a polynomial. See [59, 60, 61, 62, 63] for a detailed description of Routh-Hurwitz test and various other methods for checking stability of continuous as well as discrete time systems. Among these methods the Hermite-Biehler theorem seems to have several advantages. In addition to its use as a test for stability of polynomials, the Hermite-Biehler theorem played a central role in the first proof of the Kharitonov theorem pertaining to interval polynomials [64]. In [34] a generalization of the Hermite-Biehler theorem, applicable to not necessarily Hurwitz stable polynomials, was given. The generalized theorem gives the root distribution of a given real polynomial with respect to the imaginary axis. This will prove fundamental in parameterizing different types of controllers that stabilizes a given linear, time-invariant plant.

2.1 The Hermite-Biehler Theorem

In this section, we state the Hermite-Biehler theorem which gives a necessary and sufficient condition for Hurwitz stability of a given polynomial of real coefficients. We first review some elementary facts on polynomials and Hurwitz stable polynomials.

Let us denote the set of real numbers by $\mathbf{R}$, the set of complex numbers by $\mathbf{C}$, and let $\mathbf{C}_-$, $\mathbf{C}_0$, $\mathbf{C}_+$ denote the points in the open left-half, $j\omega$-axis, and the open right-half of the complex plane, respectively. Also let $\mathbf{C}_{0+}$ denote the points in the closed right-half complex plane. Let $\mathbf{R}[s]$ denote the set of real polynomials in s and $deg\,\psi$ the degree in s of a non-zero polynomial ψ. Given a set of polynomials $\psi_1, ..., \psi_k \in \mathbf{R}[s]$ not all zero and $k > 1$, their *greatest common divisor* (with highest coefficient 1) is unique and it is denoted by $gcd\,\{\psi_1, ..., \psi_k\}$. If $gcd\,\{\psi_1, ..., \psi_k\} = 1$, then we say $(\psi_1, ..., \psi_k)$ is *coprime*. The derivative of ψ is denoted by ψ'. The set $\mathcal{H}$ of Hurwitz stable polynomials are

$$\mathcal{H} = \{\psi \in \mathbf{R}[s] : \psi(s) = 0 \Rightarrow s \in \mathbf{C}_-\}.$$

The constant non-zero polynomials, i.e., the non-zero elements of $\mathbf{R}$, are thus considered Hurwitz stable. The *signature* $\sigma(\psi)$ *of a polynomial* $\psi \in \mathbf{R}[s]$ is the difference between the number of its $\mathbf{C}_-$ roots and $\mathbf{C}_+$ roots. The signature thus disregards the $j\omega$-axis zeros of the polynomial. Nevertheless, $\psi \in \mathcal{H} \Leftrightarrow \sigma(\psi) = deg\,\psi$ holds. If $\{r_1, ..., r_t\}$ are a finite number of real numbers and $\mathcal{I}$ is a subset of $\{1, ..., t\}$, then

$$\max_{i \in \mathcal{I}} r_i, \;\; \min_{i \in \mathcal{I}} r_i$$

denote the maximum and the minimum of the numbers r_i as i runs in $\mathcal{I}$. If $\mathcal{I}$ is the empty set, then the maximum is taken as $-\infty$ and the minimum is taken as $+\infty$, for convenience. We will also use the notation $r(\pm\infty)$ for the limit as $s \rightarrow \pm\infty$ of a real rational function $r(s)$.

Given $\psi \in \mathbf{R}[s]$, *the even-odd components* (a, b) *of* $\psi(s)$ are the unique polynomials $a, b \in \mathbf{R}[u]$ such that $\psi(s) = a(s^2) + sb(s^2)$. The even-odd components of a polynomial and the real and imaginary parts of $\psi(j\omega)$, $\tilde{a}(\omega) := Re\{\psi(j\omega)\}$ and $\tilde{b}(\omega) := Im\{\psi(j\omega)\}$, are related by

$$\tilde{a}(\omega) = a(-\omega^2), \;\; \tilde{b}(\omega) = \omega b(-\omega^2).$$

Also note that

$$\begin{array}{lcl} deg\,\psi \text{ is even} & \Rightarrow & \left\{ \begin{array}{l} deg\,a = \frac{deg\,\psi}{2} \\ deg\,b < \frac{deg\,\psi}{2} \end{array} \right\}, \\ deg\,\psi \text{ is odd} & \Rightarrow & \left\{ \begin{array}{l} deg\,a \leq \frac{deg\,\psi - 1}{2} \\ deg\,b = \frac{deg\,\psi - 1}{2} \end{array} \right\}. \end{array} \tag{2.1}$$

If $\psi \neq 0$, then $d := gcd\{a, b\}$ is well-defined. Since $d(u_0) = 0$ for $u_0 \in \mathbf{C}$ if and only if $s_1 = \sqrt{u_0}$ and $s_2 = -\sqrt{u_0}$ are both roots of $\psi(s)$, the roots of $d(s^2)$ correspond to roots of $\psi(s)$ which are symmetrically located with respect to the origin in the complex plane. As a consequence, if $d(u) \neq 0 \;\; \forall u \leq 0$, then $\psi(s)$ has no roots on $\mathbf{C}_0$ *except* possibly a simple zero (i.e., a zero of multiplicity 1) at the origin. Also note that if $\psi(s) \in \mathcal{H}$, then $d = 1$ since otherwise there would be at least one root of $\psi(s)$ in $\mathbf{C}_{0+}$. It is actually possible to state a necessary and sufficient condition for the Hurwitz stability of $\psi(s)$ in terms of its even-odd components (a, b). This result is the Hermite-Biehler theorem for real polynomials. We state it in a suitable form for our purpose. Let us define the *signum function* $\mathcal{S} : \mathbf{R} \rightarrow \{-1, 0, 1\}$ by

$$\mathcal{S}r = \begin{cases} -1 & \text{if } r < 0 \\ 0 & \text{if } r = 0 \\ 1 & \text{if } r > 0. \end{cases}$$

The proof of the following result can be found in [49, 59, 65]. See also [66] for several results related to the Hermite-Biehler theorem.

Proposition 2.1 *[59] A non-zero polynomial $\psi \in \mathbf{R}[s]$ is Hurwitz stable if and only if its even-odd components (a, b) are such that $b \not\equiv 0$ and at the distinct real negative roots $v_1 > v_2 > ... > v_k$ of b the following holds:*

$$deg\,\psi = \begin{cases} \mathcal{S}b(0)[\mathcal{S}a(0) - 2\mathcal{S}a(v_1) + 2\mathcal{S}a(v_2) - \dots \\ +(-1)^{k-1}2\mathcal{S}a(v_{k-1}) + (-1)^k 2\mathcal{S}a(v_k)] & \text{for } deg\,\psi \text{ odd} \\ \mathcal{S}b(0)[\mathcal{S}a(0) - 2\mathcal{S}a(v_1) + 2\mathcal{S}a(v_2) - \dots \\ +(-1)^k 2\mathcal{S}a(v_k) + (-1)^{k+1}\mathcal{S}a(-\infty)] & \text{for } deg\,\psi \text{ even.} \end{cases} \tag{2.2}$$

A pair of polynomials (a, b) is said to be a *positive pair* ([59], §XV, 14) if $\mathcal{S}a(0) = \mathcal{S}b(0)$, and the roots $\{u_i\}$ of $a(u)$ and $\{v_j\}$ of $b(u)$ are all real, negative, simple, and satisfy

$0 > u_1 > v_1 > u_2 > v_2 > ... > u_k > v_k$ when $k := deg\,b = deg\,a$,
$0 > u_1 > v_1 > u_2 > v_2 > ... > u_k > v_k > u_{k+1}$ when $k = deg\,b = deg\,a - 1$.

Theorem 2.1 *[59] A non-zero polynomial $\psi \in \mathbf{R}[\mathrm{s}]$ is Hurwitz stable if and only if its even-odd components (a, b) form a positive pair.*

Consider Proposition 2.1. By (2.1), if $deg\,\psi$ is odd, then $deg\,b = (deg\,\psi - 1)/2$ so that $deg\,\psi \geq 2k + 1$. However, the maximum value the right hand side of (2.2) can attain is also $2k + 1$. Similarly, if $deg\,\psi$ is even, then it is easy to see by (2.1) that $deg\,\psi \geq 2k + 2$ which is the maximum value the right hand side of (2.2) can attain. It follows that (2.2) is satisfied if only if $k = deg\,b$, $\mathcal{S}a(0) = \mathcal{S}b(0)$, and in each interval $(v_1, 0)$, $(v_2, v_1), ..., (v_k, v_{k-1})$ (or $(v_1, 0), (v_2, v_1), ..., (-\infty, v_k)$), the polynomial $a(u)$ has exactly one root. Proposition 2.1 then reads: $\psi \in \mathcal{H}$ *if and only if* (a, b) *is a positive pair.*

We now give an example to show the application of Proposition 2.1 to a Hurwitz stable polynomial.

Example 2.1 *Consider the real polynomial*

$$\psi(s) = s^7 + 2s^6 + 4s^5 + 5.4s^4 + 4.69s^3 + 3.58s^2 + 1.47s + 0.306.$$

The even-odd components (a, b) *of* $\psi(s)$ *are given by*

$$\begin{aligned} a(u) &= 2u^3 + 5.4u^2 + 3.58u + 0.306, \\ b(u) &= u^3 + 4u^2 + 4.69u + 1.47. \end{aligned}$$

Plots of $a(u)$ *and* $b(u)$ *are shown in the figure below. We can easily see that* (a, b) *form a positive pair. In fact,* $a(u)$ *and* $b(u)$ *have the following roots:*

$$\begin{aligned} &u_1 = -0.1, \quad u_2 = -0.9, \quad u_3 = -1.7, \\ &v_1 = -0.5, \quad v_2 = -1.4, \quad v_3 = -2.1. \end{aligned}$$

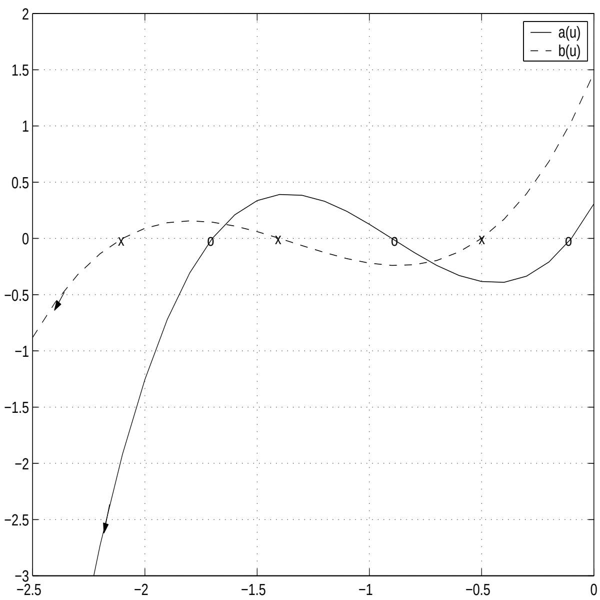

Figure 2.1: Plots of even-odd parts (a, b) *of* $\psi(s)$.

As deg ψ *is odd, we use first equation in (2.2),* $\mathcal{S}b(0) = 1, \quad \mathcal{S}a(0) = 1, \quad \mathcal{S}a(v_1) = -1, \quad \mathcal{S}a(v_2) = 1, \quad \mathcal{S}a(v_3) = -1$. *Hence*

$$\mathcal{S}b(0)[\mathcal{S}a(0) - 2\mathcal{S}a(v_1) + 2\mathcal{S}a(v_2) - 2\mathcal{S}a(v_3)] = 7.$$

To verify that $\psi(s)$ is indeed a Hurwitz stable polynomial, we give the roots of $\psi(s)$:

$$-0.0295 \pm j1.3041, \quad -0.1101 \pm j0.9508, \quad -0.3334 \pm j0.2740, \quad -1.0541.$$

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The "root interlacing condition" can be replaced by positivity of certain polynomials of u. Consider the polynomials

$$\begin{aligned} V_{\psi}(u) &:= a'(u)b(u) - a(u)b'(u), \\ V_{s\psi}(u) &:= a(u)b(u) - u[a'(u)b(u) - a(u)b'(u)]. \end{aligned} \tag{2.3}$$

Lemma 2.1 *[67] Let $a, b \in \mathbf{R}[u]$ be coprime with deg $a =$ deg $b \geq 1$ or with deg $a =$ deg $b + 1 \geq 1$. Then, (a, b) is a positive pair if and only if*

$$\begin{aligned} &(i) && \textit{all roots of } a \textit{ and } b \textit{ are real and negative,} \\ &(ii) && V_{\psi}(u) > 0 \ \forall u < 0, && (2.4) \\ &(iii) && V_{s\psi}(u) > 0 \ \forall u < 0. && (2.5) \end{aligned}$$

Proof. Let $k = deg\ a$ and $l = deg\ b$. Let $u_1 > u_2 > \ldots > u_k$ and $v_1 > v_2 > \ldots > v_l$ be the roots of a and b, respectively. By hypothesis, $u_i,\ v_i$ are real and either $k = l \geq 1$ or $k = l + 1 \geq 1$.

[Only if] By definition, if (a, b) is a positive pair, then $a(0)b(0) > 0$ and

$$(i) \quad k = l \text{ and } \quad 0 > u_1 > v_1 > u_2 > v_2 > \ldots > u_k > v_l, \tag{2.6}$$

$$(ii) \quad k = l + 1 \text{ and } \quad 0 > u_1 > v_1 > u_2 > v_2 > \ldots > v_l > u_k. \tag{2.7}$$

By partial fraction expansion

$$\frac{b(u)}{a(u)} = \alpha_0 + \sum_{i=1}^{k} \frac{\alpha_i}{u - u_i}, \tag{2.8}$$

$$\frac{a(u)}{ub(u)} = \beta_0 + \frac{\beta_1}{u} + \sum_{j=1}^{l} \frac{\beta_{j+1}}{u - v_j}, \tag{2.9}$$

where $\alpha_0 = 0$ if $k = l + 1$ and $\beta_0 = 0$ if $k = l$ and where

$$\alpha_i = \frac{b(u_i)}{a'(u_i)},\ i = 1, \ldots, k, \tag{2.10}$$

$$\beta_1 = \frac{a(0)}{b(0)},\ \beta_{j+1} = \frac{a(v_j)}{v_j b'(v_j)},\ j = 1, \ldots, l. \tag{2.11}$$

As all $u_i,\ v_j$ are real and negative, we have $\mathcal{S}a'(u_i) = (-1)^{i-1}\mathcal{S}a(0)$ and $\mathcal{S}b'(v_j) = (-1)^{j-1}\mathcal{S}b(0)$ for all $i = 1, \ldots, k;\ j = 1, \ldots, l$. By (2.6) and (2.7), we also have $\mathcal{S}a(v_j) = (-1)^{j-1}\mathcal{S}a(0)$ and $\mathcal{S}b(u_i) = (-1)^{i-1}\mathcal{S}b(0)$ for all $i = 1, \ldots, k;\ j = 1, \ldots, l$. It follows that

$$\alpha_i = |\alpha_i|\mathcal{S}\frac{b(0)}{a(0)},\ i = 1, \ldots, k,\ \beta_{j+1} = -|\beta_{j+1}|\mathcal{S}\frac{a(0)}{b(0)},\ j = 1, \ldots, l.$$

By differentiating (2.8) and (2.9) and multiplying by $a(u)^2$ and $u^2 b(u)^2$, respectively, we obtain

$$V_\psi(u) = a(u^2)\sum_{i=1}^{k}\frac{\alpha_i}{(u-u_i)^2} = a(u)^2\sum_{i=1}^{k}\frac{|\alpha_i|}{(u-u_i)^2}\mathcal{S}\frac{b(0)}{a(0)}, \tag{2.12}$$

$$\begin{aligned} V_{s\psi}(u) &= b(u)^2\beta_1 + u^2 b(u)^2\sum_{j=1}^{l}\frac{\beta_{j+1}}{(u-v_j)^2} \\ &= b(u)^2\frac{a(0)}{b(0)} + u^2 b(u)^2\sum_{j=1}^{l}\frac{|\beta_{j+1}|}{(u-v_j)^2}\mathcal{S}\frac{a(0)}{b(0)}. \end{aligned} \tag{2.13}$$

The conditions (2.4) and (2.5) follow.

[If] If (2.5) (resp., (2.4)) holds, then the roots of $a(u)$ are distinct; since if say $a(u) = (u-u_0)^2 a_1(u)$ for some $u_0 < 0$ and $a_1 \in \mathbf{R}[u]$, then $a(u_0) = a'(u_0) = 0$, which contradicts (2.5) (resp., (2.4)). Similarly, if $b(u)$ has a negative root of multiplicity greater than one, then (2.5) (resp., (2.4)) is contradicted. Since all roots of $a(u)$ and $b(u)$ are real, negative, and distinct, it follows that the equalities (2.9), (2.11) and (2.13) hold. By (2.5) and (2.13), we have

$$\beta_1 b(u)^2 + \sum_{i=1}^{l}\beta_{j+1}\frac{u^2 b(u)^2}{(u-v_j)^2} > 0\ \forall\ u < 0. \tag{2.14}$$

Evaluating the left hand side at $v_1, \ldots, v_l$, respectively, we obtain $\beta_j > 0,\ j = 2 \ldots .l+1$. This yields $\mathcal{S}b'(v_j) = -\mathcal{S}a(v_j)$ for $j = 2, \ldots, l+1$ by (2.11). On the other hand, as $u \to 0$, the left hand side of (2.14) approaches $\beta_1 b(0)^2 = a(0)b(0)$ by (2.11), so that $b(0)a(0) > 0$. Since all roots of $b(u)$ are real and negative, we have $\mathcal{S}b'(v_j) = (-1)^{j-1}\mathcal{S}b(0),\ j = 1, \ldots, l$ so that $\mathcal{S}a(v_j) = (-1)^j \mathcal{S}b(0)$ for $j = 1, \ldots, l$. This means that there are an odd number of roots of $a(u)$ between each pair of roots of $ub(u)$. Since the degrees k and l can differ by at most 1 however, the interval (v_j, v_{j+1}) must contain exactly one root of $a(u)$ for $j = 0, 1, \ldots, l$ where $v_0 := 0,\ v_{l+1} := -\infty$. The interlacing property (2.6) or (2.7) follows. ∎

Lemma 2.1 is an alternative statement of the Hermite-Biehler theorem, which is suitable for studying convex directions. It was used in [67] to construct new convex directions for Hurwitz stable polynomials. We will use this form of the Hermite-Biehler theorem in Chapter 6 to study local convex directions. Finally, **root sensitivities** of some polynomials can be computed in terms of V_ψ and $V_{s\psi}$. Consider

$$\begin{aligned} \phi_1(\alpha, u) &:= a(u) + \alpha b(u), \\ \phi_2(\alpha, u) &:= ub(u) + \alpha a(u), \end{aligned}$$

for $\alpha \in \mathbf{R}$. The equation $\phi_1(\alpha, u) = 0$ implicitly defines a function $u(\alpha)$. The root sensitivity of $\phi_1(\alpha, u)$ is defined by $\alpha \frac{du}{d\alpha}$, and gives a measure of the variation in the root location of $\phi_1(\alpha, u)$ with respect to percentage variations in α. The root sensitivities of $\phi_1(\alpha, u)$ and $\phi_2(\alpha, u)$, respectively, are easily computed to be

$$\begin{aligned} S_\psi(u) &:= \tfrac{a(u)b(u)}{V_\psi(u)}, \\ S_{s\psi}(u) &:= \tfrac{ua(u)b(u)}{V_{s\psi}(u)}. \end{aligned}$$

2.2 Generalized Hermite-Biehler Theorem

In the previous section, Hermite-Biehler theorem was used to check Hurwitz stability of real polynomials. This theorem can be generalized to give more information about the root distribution of a polynomial with respect to the imaginary axis. This result will be used to determine the set of all stabilizing constant gains for a given continuous time plant. The generalized Hermite-Biehler theorem was first derived in [34]. The same result was then reproduced, see [35], in [68], see also [69, 70]. The generalization of the Hermite-Biehler theorem to polynomials with complex coefficients was given in [71].

We first state the following lemma needed in the proof of Theorem 2.2 below. Let $\psi(j\omega) = \tilde{a}(\omega) + j\tilde{b}(\omega)$, and $\theta(\omega) = \arctan[\frac{\tilde{b}(\omega)}{\tilde{a}(\omega)}]$. Also, let $\triangle_0^\infty \theta$ denote the net change in the argument of $\psi(j\omega)$ as ω varies from 0 to ∞. Then we can state the following lemma of [59]:

Lemma 2.2 *Let $\psi(s)$ be a real polynomial with no roots on the imaginary axis. Then*

$$\triangle_0^\infty \theta = \frac{\pi}{2}\sigma(\psi).$$

We now state and prove the generalized Hermite-Biehler theorem.

Theorem 2.2 *[34] Let a non-zero polynomial $\psi \in \mathbf{R}[s]$ have the even-odd components (a, b). Suppose $b \not\equiv 0$ and (a, b) is coprime. Then, $\sigma(\psi) = r$ if and only if at the real negative roots of odd multiplicities $v_1 > v_2 > ... > v_k$ of b the following holds:*

$$r = \begin{cases} \mathcal{S}b(0_-) & [\mathcal{S}a(0) - 2\mathcal{S}a(v_1) + 2\mathcal{S}a(v_2) + \dots \\ & +(-1)^{k-1}2\mathcal{S}a(v_{k-1}) + (-1)^k 2\mathcal{S}a(v_k)] \text{ for deg}\,\psi \text{ odd} \\ \mathcal{S}b(0_-) & [\mathcal{S}a(0) - 2\mathcal{S}a(v_1) + 2\mathcal{S}a(v_2) + \dots \\ & +(-1)^k 2\mathcal{S}a(v_k) + (-1)^{k+1}\mathcal{S}a(-\infty)] \text{ for deg}\,\psi \text{ even,} \end{cases} \tag{2.15}$$

where $b(0_-) := (-1)^{m_0} b^{(m_0)}(0)$, m_0 *is the multiplicity of* $u = 0$ *as a root of* $b(u)$, *and* $b^{(m_0)}(0)$ *denotes the value at* $u = 0$ *of the* m_0*-th derivative of* $b(u)$.

Proof. [34] We first consider the case $\psi(0) \neq 0$. Since (a, b) is coprime, $\psi(s)$ has no zeros on $\mathbf{C}_0$ and $a(0) \neq 0$. Let the real negative roots (if any) with odd multiplicities of $a(u)$ be

$$u_1 > u_2 > \cdots > u_l$$

and define

$$U := \begin{cases} \{u_j\}_{j=1}^{l} & \text{if } m \text{ is even} \\ \{u_j\}_{j=1}^{l} \bigcup \{u_{l+1} = -\infty\} & \text{if } m \text{ is odd,} \end{cases} \tag{2.16}$$

$$V := \begin{cases} \{v_i\}_{i=1}^{k} \bigcup \{v_0 = 0,\, v_{k+1} = -\infty\} & \text{if } m \text{ is even} \\ \{v_i\}_{i=1}^{k} \bigcup \{v_0 = 0\} & \text{if } m \text{ is odd,} \end{cases} \tag{2.17}$$

where $m := deg\, \psi$. We now order the elements of $U \cup V$ as

$$0 = t_1 > t_2 > \cdots > t_{k+l+2} = -\infty$$

and define the index sets I and J which distinguishes certain elements in $\{t_j\}$:

$$\begin{aligned} i \in I \Leftrightarrow \quad & t_i \in V \text{ and } t_{i+1} \in U \quad \text{ for } \quad i = 1, 2, \ldots, k+l+1, \\ j \in J \Leftrightarrow\ & t_j \in U \text{ and } t_{j+1} \in V \quad \text{ for } \quad j = 1, 2, \ldots, k+l+1. \end{aligned}$$

By either tracing the Leonhard locus of $\psi(j\omega)$ ([72], §V.1) or by Cauchy index ([59], XV.3) considerations, it is now easy to compute the net change in $\theta(\omega) = arg\, \psi(j\omega)$ as ω increases from 0 to ∞ as

$$\Delta_0^\infty\ \theta(\omega) = \frac{\pi}{2}\Big(\sum_{i \in I} \mathcal{S}a(t_i)\mathcal{S}b(t_{i+1}) - \sum_{j \in J} \mathcal{S}b(t_j)\mathcal{S}a(t_{j+1})\Big).$$

By Lemma 2.2, $\sigma(\psi) = \frac{2}{\pi}\Delta_0^\infty\ \theta(\omega)$ and we obtain

$$\sigma(\psi) = \sum_{i \in I} \mathcal{S}a(t_i)\mathcal{S}b(t_{i+1}) - \sum_{j \in J} \mathcal{S}b(t_j)\mathcal{S}a(t_{j+1}). \tag{2.18}$$

We now show that the right hand sides of (2.15) and (2.18) are the same. Suppose first that $deg\ \psi$ is even. The right hand side of (2.15) can be written as

$$\mathcal{S}b(0_-)\sum_{i=0}^{k}(-1)^i(\mathcal{S}a(v_i)-\mathcal{S}a(v_{i+1})). \tag{2.19}$$

Let μ_i denote the number of $\{u_j\}$ between v_i and v_{i+1} for $i = 0, 1, \ldots, k+1$. Hence, we can rewrite (2.19) as

$$\mathcal{S}b(0_-)\sum_{i=0}^{k}2(\mu_i\ mod\ 2)(-1)^i\mathcal{S}a(v_i). \tag{2.20}$$

On the other hand, the right hand side of (2.18) can be written as

$$\sum_{i:u_i\neq 0}(\mathcal{S}a(v_i)\mathcal{S}b(v_{i-})-\mathcal{S}b(v_{i-})\mathcal{S}a(v_{i+1})\ ). \tag{2.21}$$

By noting that $\mathcal{S}a(v_i) = \mathcal{S}a(v_{i+1})$ when μ_i is even for $i = 0, 1, \ldots, k$, we obtain that

$$\sigma(\psi) = \sum_{i\ :\ u_i\ odd} 2\mathcal{S}a(v_i)\mathcal{S}b(v_{i-}). \tag{2.22}$$

We also have $\mathcal{S}b(v_{i-}) = (-1)^i\mathcal{S}b(0_-)$, since $b(u)$ have i zeros between v_{i-} and 0_- for $i = 0, 1, \ldots, k$. Hence, the right hand sides of (2.20) and (2.22) are equal. For the case $deg\ \psi$ is odd, the equality of the right hand sides of (2.15) and (2.18) can be shown similarly.

We now consider the case $\psi(0) = 0$. In this case by coprimeness of (a, b), $\psi(s)$ has a simple zero at the origin. Using

$$\sigma(\psi) = \frac{2}{\pi}\Delta_{0+}^{\infty}\,\theta(\omega)$$

and repeating all the above arguments by appropriate modifications it is possible to show that r given by (2.15) is again equal to $\sigma(\psi)$. Here we only give a heuristic argument. Let $a_1(u)$ be a polynomial obtained by a slight perturbation of the coefficients of $a(u)$ and let $\psi_1(s) := a_1(s^2) + sb(s^2)$. If the perturbations are sufficiently small, then $\psi_1(s)$ is such that $\mathcal{S}a(v_i) = \mathcal{S}a_1(v_i)$ for $i = 1, ..., k+1$

and the root at $s = 0$ of $\psi(s)$ moves either to $\mathbf{C}_-$ or to $\mathbf{C}_+$. In either case, $r_1 := \sigma(\psi_1) = r \pm 1$. By what has been proved, (2.15) holds with r, a replaced by r_1, a_1. Using the fact that $\mathcal{S}a(v_i) = \mathcal{S}a_1(v_i)$ for $i = 1, ..., k+1$, we obtain that (2.15) holds with $\mathcal{S}a(0) = 0$. ■

Another way of reaching the result in Theorem 2.2 is by using phase arguments and making the following observations [68].

- For two consecutive roots v_i and v_{i+1} of $b(u)$ we have

$$\triangle_{v_i}^{v_{i+1}}\theta = \frac{\pi}{2}[\mathcal{S}a(v_i) - \mathcal{S}a(v_{i+1})]\mathcal{S}b(v_i^-)$$

  where $v_i^- = v_i - \epsilon$, $\epsilon > 0$.

- If $deg(\psi)$ is odd then

$$\triangle_{v_k}^{\infty}\theta = \frac{\pi}{2}\mathcal{S}a(v_k)\mathcal{S}b(v_k^-)$$

-

$$\mathcal{S}b(v_{i+1}^-) = -\mathcal{S}b(v_i^-),\ i = 1, \ldots, k-1,$$

  and

$$\mathcal{S}b(0^-) = \mathcal{S}b(0_-)$$

  where $b(0_-) := (-1)^{m_0} b^{(m_0)}(0)$, m_0 is the multiplicity of $u = 0$ as a root of $b(u)$, and $b^{(m_0)}(0)$ denotes the value at $u = 0$ of the m_0-th derivative of $b(u)$.

Using these observations, we can show that (2.15) holds. We show it for *deg* ψ odd, the case *deg* ψ is even follows similar arguments and is omitted. We have

$$\begin{aligned}
\triangle_0^{v_1} &= \frac{\pi}{2}\mathcal{S}b(0_-)[\mathcal{S}a(0) - \mathcal{S}a(v_1)], \\
\triangle_{v_1}^{v_2} &= -\frac{\pi}{2}\mathcal{S}b(0_-)[\mathcal{S}a(v_1) - \mathcal{S}a(v_2)],
\end{aligned}$$

$$\begin{aligned}
&\vdots \\
\triangle_{v_i}^{v_{i+1}} &= (-1)^i \frac{\pi}{2} \mathcal{S}b(0_-)[\mathcal{S}a(v_i) - \mathcal{S}a(v_{i+1})], \\
&\vdots \\
\triangle_{v_k}^{\infty} &= (-1)^k \frac{\pi}{2} \mathcal{S}b(0_-)\mathcal{S}a(v_k).
\end{aligned}$$

Since

$$\triangle_0^{\infty} = \triangle_0^{v_1} + \triangle_{v_1}^{v_2} + \ldots + \triangle_{v_i}^{v_{i+1}} + \ldots + \triangle_{v_k}^{\infty},$$

we have

$$\triangle_0^{\infty} = \frac{\pi}{2} \mathcal{S}b(0_-)[\mathcal{S}a(0) - 2\mathcal{S}a(v_1) + 2\mathcal{S}a(v_2) + \ldots + (-1)^k \mathcal{S}a(v_k)] \text{ for } deg\ \psi \text{ odd},$$

and (2.15) follows.

Example 2.2 *Consider the real polynomial*

$$\psi(s) = s^7 + 2s^6 + 4s^5 - 5.4s^4 - 4.69s^3 + 3.58s^2 + 1.47s + 0.306.$$

The even-odd components (a, b) *of* $\psi(s)$ *are given by*

$$\begin{aligned}
a(u) &= 2u^3 - 5.4u^2 + 3.58u + 0.306, \\
b(u) &= u^3 + 4u^2 - 4.69u + 1.47.
\end{aligned}$$

The polynomial $b(u)$ *has only one real negative root with odd multiplicity at* $v_1 = -4.9974$. *In addition, we have* $\mathcal{S}b(0_-) = 1$, $\mathcal{S}a(0) = 1$, *and* $\mathcal{S}a(v_1) = -1$. *As degree of* $\psi(s)$ *is odd, we use first equation in (2.15),*

$$\mathcal{S}b(0)[\mathcal{S}a(0) - 2\mathcal{S}a(v_1)] = 3.$$

To verify that $\psi(s)$ *has signature equal to* 3, *we give the roots of* $\psi(s)$*:*

$$-1.2703 \pm j2.1732, \ -0.1674 \pm j0.1858, \ -0.8980, \ 0.8867 \pm j0.2714.$$

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2.3 Using the Generalized Hermite-Biehler Theorem to Find the Number of Real Negative Roots of a Real Polynomial

Based on the generalized Hermite-Biehler Theorem, we state and prove the following result which enables us to compute the number of real negative roots of a real polynomial. This problem is transformed to a signature computation of a new constructed polynomial. Using the generalized Hermite-Biehler theorem the transformed problem can be easily solved.

Lemma 2.3 *A non-zero polynomial $\psi \in \mathbf{R}[u]$, such that $\psi(0) \neq 0$, has r real negative roots without counting the multiplicities if and only if the signature of the polynomial $\psi(s^2)+s\psi'(s^2)$ is $2r$. All roots of ψ are real, negative, and distinct if and only if $\psi(s^2)+s\psi'(s^2) \in \mathcal{H}$.*

Proof. We first assume that (ψ, ψ') is coprime. Suppose that $\psi(u)$ has r real negative distinct roots $u_1 > u_2 > \ldots > u_r$. Since $\psi'(u)$ is the derivative of $\psi(u)$, it follows that between any two consecutive real negative roots u_i and u_{i+1} of $\psi(u)$ there is an odd number of real negative roots of $\psi'(u)$: $v_{i1} > v_{i2} > \ldots > v_{ij}$, where j is an odd integer. Since

$$\mathcal{S}\psi(v_{i1}) = \mathcal{S}\psi(v_{i2}) = \ldots = \mathcal{S}\psi(v_{ij}),$$

it follows that

$$2\mathcal{S}\psi(v_{i1}) - 2\mathcal{S}\psi(v_{i2}) + \ldots + (-1)^j 2\mathcal{S}\psi(v_{ij}) = 2\mathcal{S}\psi(v_{i1}).$$

In the interval $(-\infty, u_r)$, $\psi'(u)$ must have an even number or real roots otherwise $\psi(u)$ have a real root in this interval contradicting the fact that $\psi(u)$ has r real negative roots. Assume that $\psi(0) > 0$. If $\psi'(u)$ has an even number, k, of real roots $v_{01}, v_{02}, \ldots, v_{0k}$, between 0 and u_1, then $\psi'(0_-) > 0$ and

$$2\mathcal{S}\psi(v_{01}) - 2\mathcal{S}\psi(v_{02}) + \ldots + (-1)^k 2\mathcal{S}\psi(v_{0k}) = 0.$$

Finally, $\mathcal{S}\psi(0) = 1$, $\mathcal{S}\psi(v_{11}) = -1$, $\mathcal{S}\psi(v_{21}) = 1$, ..., $\mathcal{S}\psi(-\infty) = (-1)^r$. Using these facts in (2.15) of Theorem 2.2, we get

$$\begin{aligned}
&\mathcal{S}\psi'(0_-)[\mathcal{S}\psi(0) - 2\mathcal{S}\psi(v_{01}) + \ldots - 2\mathcal{S}\psi(v_{11}) + \ldots + (-1)^r\mathcal{S}\psi(-\infty)] \\
&= \mathcal{S}\psi(0) - 2\mathcal{S}\psi(v_{11}) + 2\mathcal{S}\psi(v_{21}) - 2\mathcal{S}\psi(v_{31}) + \ldots + (-1)^r\mathcal{S}\psi(-\infty) \\
&= 2r
\end{aligned}$$

If $\psi'(u)$ has an odd number of roots between 0 and u_1, then $\psi'(0_-) < 0$. In this case, we obtain again the same result

$$\begin{aligned}
&\mathcal{S}\psi'(0_-)[\mathcal{S}\psi(0) - 2\mathcal{S}\psi(v_{01}) + \ldots + 2\mathcal{S}\psi(v_{11}) - \ldots + (-1)^{r+1}\mathcal{S}\psi(-\infty)] \\
&= -[\mathcal{S}\psi(0) - 2\mathcal{S}\psi(v_{01}) + 2\mathcal{S}\psi(v_{11}) - 2\mathcal{S}\psi(v_{21}) + \ldots + (-1)^{r+1}\mathcal{S}\psi(-\infty)] \\
&= 2r
\end{aligned}$$

Similar arguments apply in the case $\psi(0) < 0$ to give the same result; namely,

$$\mathcal{S}\psi'(0_-)[\mathcal{S}\psi(0) - 2\mathcal{S}\psi(v_{01}) + \ldots + 2\mathcal{S}\psi(v_{11}) - \ldots + (-1)^{r+1}\mathcal{S}\psi(-\infty)] = 2r.$$

Therefore, by Theorem 2.2, signature of $\psi(s^2)+s\psi'(s^2)$ is $2r$. Conversely, suppose that the signature of $\psi(s^2)+s\psi'(s^2)$ is $2r$. Using the second equation of (2.15) in Theorem 2.2, it follows that $\psi(u)$ changes sign exactly r times for $u < 0$. Hence, $\psi(u)$ has r real negative roots.

Now, let us examine the case of non-coprime pair (ψ, ψ'). Since complex roots of $\psi(u)$ and $\psi'(u)$ do not affect the signature of $\psi(s^2) + s\psi'(s^2)$, we consider only the case of common real negative roots. Assume that $\psi(u)$ and $\psi'(u)$ have a common real negative root u_1, then $\psi(u) = (u - u_1)\psi_1(u)$ and $\psi'(u) = \psi_1(u) + (u-u_1)\psi_1'(u_1)$. Since u_1 is also a root of $\psi'(u_1)$, it follows that u_1 is a root of $\psi_1(u)$. This shows that whenever (ψ, ψ') are not coprime, $\psi(u)$ has a root of multiplicity greater than 1. Let $\psi(u)$ have a real negative root u_1 with multiplicity greater than 1. Repeating the same analysis as above, using the fact that u_1 is also a root of $\psi'(u_1)$, and that $\mathcal{S}\psi(u_1) = 0$, it follows that $\psi(u)$ has r real negative roots without counting the multiplicities if and only if the signature of $\psi(s^2) + s\psi'(s^2)$ is $2r$.

If $\psi(u)$ has all its roots real, negative, and distinct, then $r = deg\ \psi$. By the part we have just proved, signature of $\psi(s^2) + s\psi'(s^2)$ is $2r$ which is the degree of $\psi(s^2) + s\psi'(s^2)$. Hence, $\psi(s^2) + s\psi'(s^2) \in \mathcal{H}$. The converse follows by Hermite-Biehler theorem. ■

2.4 Generalized Hermite-Biehler Theorem: Complex Case

In this section, a generalization of the Hermite-Biehler theorem to polynomials with complex coefficients [41] is presented. This result will be used to solve the problem of stabilization with guaranteed damping. We also use this result to compute the number of real roots of a real polynomial.

Given $\psi \in \mathbf{C}[s]$, the real and imaginary parts $(\tilde{a}, \tilde{b})$ of $\psi(s)$ are the unique polynomials $\tilde{a},\ \tilde{b} \in \mathbf{R}[\omega]$ such that

$$\psi(j\omega) = \tilde{a}(\omega) + j\tilde{b}(\omega).$$

Theorem 2.3 *[25] Let a non-zero polynomial* $\psi \in \mathbf{C}[s]$ *of degree* n *have the real-imaginary components* $(\tilde{a}, \tilde{b})$*. Suppose* $\tilde{b} \not\equiv 0$ *and* $(\tilde{a}, \tilde{b})$ *is coprime. Let* $\omega_1 < \omega_2 < ... < \omega_k$ *be the real, distinct finite roots of* $\tilde{b}$ *with odd multiplicities. Also let*

$\omega_0 = -\infty$, $\omega_{k+1} = \infty$, and ξ_n be the leading coefficient of $\psi(s)$. Then

$$\sigma(\psi) = \begin{cases} \frac{1}{2}\{\mathcal{S}\tilde{a}(\omega_0)(-1)^k + 2\sum_{i=1}^{k}\mathcal{S}\tilde{a}(\omega_i)(-1)^{k-i} - \mathcal{S}\tilde{a}(\omega_{k+1})\}\mathcal{S}\tilde{b}(\infty) \\ \text{if } n \text{ is even and } \xi_n \text{ is purely real,} \\ \text{or } n \text{ is odd and } \xi_n \text{ is purely imaginary.} \\ \frac{1}{2}\{2\sum_{i=1}^{k}\mathcal{S}\tilde{a}(\omega_i)(-1)^{k-i}\}\mathcal{S}\tilde{b}(\infty) \\ \text{if } n \text{ is even and } \xi_n \text{ is not purely real,} \\ \text{or } n \text{ is odd and } \xi_n \text{ is not purely imaginary.} \end{cases} \tag{2.23}$$

Proof. See [25, 41]. ■

The following result transforms the problem of determining the number of real roots of a real polynomial to an equivalent problem of finding the signature of a complex polynomial.

Lemma 2.4 *A non-zero polynomial $\psi \in \mathbf{R}[u]$, has r real roots without counting the multiplicities if and only if the signature of the complex polynomial $\bar{\psi}(s)$ is $-r$, where $\bar{\psi}(j\omega) = \psi(w) + j\psi'(w)$.*

Proof. We first assume that (ψ, ψ') is coprime. If $deg\ \psi = n$, then $deg\ \psi' = n-1$, $deg\ \bar{\psi} = n$, and the highest coefficient $\bar{\xi}_n$ of $\bar{\psi}(s)$ depends only on the highest coefficient ξ_n of $\psi(\omega)$. If n is even, then $(j\omega)^n$ is real. As $\xi_n = (j\omega)^n\bar{\xi}_n$ is real, it follows that $\bar{\xi}_n$ is real. If n is odd, then $(j\omega)^n$ is imaginary and using similar arguments it follows that $\bar{\xi}_n$ is imaginary. In both cases, n even or odd, we use the first equation of (2.23) in Theorem 2.3 to calculate the signature of $\bar{\psi}(s)$. Let $\psi(\omega)$ have r real distinct roots $\omega_1 < \omega_2 < \ldots < \omega_r$. Since $\psi'(w)$ is the derivative of $\psi(w)$, it follows that between any two consecutive real roots ω_i and ω_{i+1} of $\psi(\omega)$ there is an odd number of real roots of $\psi'(\omega)$: $v_{i1} < v_{i2} < \ldots < v_{ij}$, where j is an odd integer. Since

$$\mathcal{S}\psi(v_{i1}) = \mathcal{S}\psi(v_{i2}) = \ldots = \mathcal{S}\psi(v_{ij}),$$

it follows that

$$2\mathcal{S}\psi(v_{i1}) - 2\mathcal{S}\psi(v_{i2}) + \ldots + (-1)^j 2\mathcal{S}\psi(v_{ij}) = 2\mathcal{S}\psi(v_{i1}).$$

In the interval $(-\infty, \omega_1)$ or (ω_r, ∞), $\psi'(\omega)$ has an even number of real roots which do not affect the signature as the sign of ψ is the constant throughout the interval. Finally note that $\mathcal{S}\psi(\infty)\mathcal{S}\psi'(\infty) = 1$, $\ldots$, $\mathcal{S}\psi(v_{01})\mathcal{S}\psi'(\infty) = (-1)^{r-1}$, $\mathcal{S}\psi(-\infty)\mathcal{S}\psi'(\infty) = (-1)^r$. Using these facts in (2.23) of Theorem 2.3, we get

$$\begin{aligned} \sigma(\bar{\psi}) &= \frac{1}{2}\{\mathcal{S}\psi(-\infty)(-1)^{r-1} + 2\mathcal{S}\psi(v_{01})(-1)^{r-2} + \ldots - \mathcal{S}\psi(\infty)\}\mathcal{S}\psi'(\infty) \\ &= -r \end{aligned}$$

Therefore, by Theorem 2.3, signature of $\bar{\psi}(s)$ is $-r$. Conversely, let the signature of $\bar{\psi}(s)$ be $-r$. Using the first equation of (2.23) in Theorem 2.3, it follows that $\psi(\omega)$ changes sign exactly r times . Hence, $\psi(\omega)$ has r real roots. for non-coprime pair (ψ, ψ'), repeating similar arguments it is easy to prove that $\psi(\omega)$ has r real roots without counting the multiplicities if and only if the signature of $\bar{\psi}(s)$ is $-r$. ■

Chapter 3

Stabilizing Feedback Gains

In this chapter, we present a non-graphical method of [34] for the determination of stabilizing gains for linear, time-invariant, single input, single output systems. This method requires a test of the sign pattern of a rational function at the real roots of a polynomial. Thereafter, we simplify this method and give an algorithm which avoids the need for a search in an exponentially increasing set to determine the solution. It has been shown based on the method of [34], that the set of all stabilizing PID controllers can be calculated [25]. Finally in this chapter, we compare these methods with the recent Nyquist based method of [37].

3.1 Introduction

In [34] the following old problem of control was considered:

Given coprime polynomials $p(s)$, $q(s)$ *with real coefficients, determine conditions under which a real number* α *exists such that* $\phi(s,\alpha) = q(s) + \alpha p(s)$ *has degree in* s *equal to the degree of* q *and is Hurwitz stable, i.e., has all its roots in* *the open left-half complex plane. Determine the set of all such α if one exists.*

If we define

$$A(p,q) := \{\alpha \in \mathbf{R} : \phi(s,\alpha) = q(s) + \alpha p(s) \in \mathcal{H} \ , \ deg\,\phi = deg\,q\},$$

then the problem is to determine under what conditions $A(p,q) \neq \emptyset$ and to give a description of $A(p,q)$ if it is not empty.

There are several classical solutions to this problem. *Evans root-locus method* and *Nyquist stability criterion* are among the most widely used graphical solutions. The method of *Hurwitz determinants* as refined in [72] and *Neimark D-decomposition*, [57], can be considered as non-graphical solutions. The last three methods are based on the following. Let $q(j\omega) = \tilde{h}(\omega) + j\tilde{g}(\omega)$ and $p(j\omega) = \tilde{f}(\omega) + j\tilde{e}(\omega)$. Consider the roots $\omega_i, \ i = 1, ..., \tilde{k}$ in $[0, \infty)$ of

$$\tilde{g}(\omega)\tilde{f}(\omega) - \tilde{h}(\omega)\tilde{e}(\omega) = 0 \tag{3.1}$$

and define

$$\alpha_i = \begin{cases} -\frac{\tilde{h}(\omega_i)}{\tilde{f}(\omega_i)} & \text{if } \tilde{f}(\omega_i) \neq 0 \\ \\ -\frac{\tilde{g}(\omega_i)}{\tilde{e}(\omega_i)} & \text{if } \tilde{e}(\omega_i) \neq 0. \end{cases}$$

If $\tilde{f}(\omega_i) = 0$ and $\tilde{e}(\omega_i) = 0$, then let $\alpha_i := \infty$. The values α_i so defined partition the real axis into a finite number of intervals. Each (open) interval belongs to $A(p,q)$ if and only if at one point α of this interval $\phi(s,\alpha)$ is Hurwitz stable. The method thus requires finding the roots of (3.1) and applying stability tests such as Nyquist or Routh-Hurwitz at one point in each obtained interval.

3.2 A Simple Case

In order to display the main ideas and techniques used in [34], it is appropriate to consider the relatively simple case when $p(s)$ is either a non-zero constant or has all its roots in the open right-half complex plane, i.e.,

$$p(s) = 0 \;\Rightarrow\; s \in \mathbf{C}_+. \tag{3.2}$$

In this case the set $A(p, q)$ can be obtained using Proposition 2.1 in a straightforward manner.

Let (h, g) and (f, e) be the even-odd components of q and p, respectively, so that

$$\begin{aligned} q(s) &= h(s^2) + sg(s^2), \\ p(s) &= f(s^2) + se(s^2). \end{aligned}$$

Then,

$$\psi(s, \alpha) \;:=\; \phi(s, \alpha)p(-s) = q(s)p(-s) + \alpha p(s)p(-s)$$

has even and odd components $a(u) := H(u) + \alpha F(u)$ and $b(u) := G(u)$, where

$$\begin{aligned} H(u) &= h(u)f(u) - ug(u)e(u), \\ F(u) &= f(u)^2 - ue(u)^2, \\ G(u) &= g(u)f(u) - h(u)e(u). \end{aligned}$$

Let $v_0 := 0$, $v_{k+1} := -\infty$, and let $v_1 > v_2 > ... > v_k$ be the real negative roots with odd multiplicities of $G(u)$. Since $p(-s)$ is Hurwitz stable, $\phi(s, \alpha) \in \mathcal{H}$ if and only if $\psi(s, \alpha) \in \mathcal{H}$.

We now apply Proposition 2.1 of Chapter 2 to $\psi(s, \alpha)$. Suppose for some $\alpha \in \mathbf{R}$, $\psi(s, \alpha) \in \mathcal{H}$. Then, $a = H + \alpha F$ and $b = G$ satisfies the conditions of Proposition 2.1. Here, $deg\,\psi = n + m$ is odd if and only if the relative degree $n - m$ of p/q is odd. Let us first suppose that $n - m$ is odd. By Proposition 2.1, $G(u) \not\equiv 0$, $k = deg\,G = (n + m - 1)/2$, i.e., $G(u)$ has $(n + m - 1)/2$ roots all of which are real, negative, simple, and

$$\mathcal{S}[H(v_i) + \alpha F(v_i)] = (-1)^i \mathcal{S}G(0),\; i = 0, 1, ..., k. \tag{3.3}$$

Using the fact that $F(v_i) > 0$ for all $i = 0, 1, ..., k$, it is easy to see that (3.3) implies

$$\underline{\alpha} := \max_{\{i\ even\}}\{-\frac{H}{F}(v_i)\} < \alpha < \bar{\alpha} := \min_{\{i\ odd\}}\{-\frac{H}{F}(v_i)\} \text{ for } G(0) > 0, \tag{3.4}$$

$$\underline{\alpha} := \max_{\{i\ odd\}}\{-\frac{H}{F}(v_i)\} < \alpha < \bar{\alpha} := \min_{\{i\ even\}}\{-\frac{H}{F}(v_i)\} \text{ for } G(0) < 0, \tag{3.5}$$

where $i = 0, 1, ..., k$ and $\underline{\alpha}$, $\bar{\alpha}$ are $-\infty$, $+\infty$, respectively, whenever the associated set of indices is empty. It follows that if $\alpha \in A(p, q)$, then α is in the interval $(\underline{\alpha}, \bar{\alpha})$. Conversely, suppose $G(u)$ has $k = (n+m-1)/2$ real, negative, and simple roots $v_1 > v_2 > ... > v_k$ and α satisfies (3.4) or (3.5). Then, α is easily seen to satisfy (3.3) so that, by Proposition 2.1, $\psi(s, \alpha) \in \mathcal{H}$.

Let us now suppose that $n-m$ is even. Suppose for some $\alpha \in \mathbf{R}$, $\psi(s, \alpha) \in \mathcal{H}$. Then, by Proposition 2.1, $G(0) \not\equiv 0$, $k = deg\, G = (n + m)/2 - 1$, i.e., $G(u)$ has $(n + m)/2 - 1$ roots all of which are real, negative, simple, (3.3) holds, and $\mathcal{S}(H + \alpha F)(-\infty) = (-1)^{k+1}\mathcal{S}G(0)$. By (2.1), we have $deg\, H = (m + n)/2$, $deg\, F = m$ which yields

$$\begin{aligned}
&m = n \;\&\; (-1)^m \mathcal{S}G(0) > 0 \;\Rightarrow \alpha > -\tfrac{H}{F}(-\infty),\\
&m = n \;\&\; (-1)^m \mathcal{S}G(0) < 0 \;\Rightarrow \alpha < -\tfrac{H}{F}(-\infty),\\
&m < n \;\Rightarrow\; \mathcal{S}H(-\infty) = (-1)^{k+1}\mathcal{S}G(0).
\end{aligned}$$

With the convention, $v_{k+1} = -\infty$, the first two conditions imply that α satisfies (3.4) or (3.5) for $i = 1, ..., k + 1 = n$ whenever $m = n$. The third condition fixes the sign of $H(-\infty)$. Conversely, suppose $G(u)$ has $k = (n + m)/2$ real, negative, and simple roots $v_1 > v_2 > ... > v_k$ and α satisfies (3.4) or (3.5) for $i = 1, ..., k+1$ when $n = m$ and satisfies (3.4) or (3.5) for $i = 1, ..., k$ when $n > m$ together with the condition $\mathcal{S}H(-\infty) = (-1)^{k+1}\mathcal{S}G(0)$. Then, α is easily seen to satisfy (3.3) so that, by Proposition 2.1, $\psi(s, \alpha) \in \mathcal{H}$.

We can summarize the results above as follows.

Proposition 3.1 *Let $p(s)$ satisfy (3.2). If $n-m$ is odd, then $A(p,q)$ is non-empty if and only if $k = deg\, G = (n+m-1)/2$,*

$$\underline{\alpha} = \max_{\{i\ even\}} \{-\frac{H}{F}(v_i)\} < \bar{\alpha} = \min_{\{i\ odd\}} \{-\frac{H}{F}(v_i)\} \ \textit{for}\ G(0) > 0, \tag{3.6}$$

$$\underline{\alpha} = \max_{\{i\ odd\}} \{-\frac{H}{F}(v_i)\} < \bar{\alpha} = \min_{\{i\ even\}} \{-\frac{H}{F}(v_i)\} \ \textit{for}\ G(0) < 0, \tag{3.7}$$

where $i \in \{0, 1, ..., (n+m-1)/2\}$. If $n = m$, then $A(p,q)$ is non-empty if and only if $k = deg\, G = n-1$ and (3.6) or (3.7) holds for $i \in \{0, 1, ..., n\}$. If $n-m$ is even and $n > m$, then $A(p,q)$ is non-empty if and only if $k = deg\, G = (n+m)/2 - 1$, $\mathcal{S}H(-\infty) = (-1)^{k+1}\mathcal{S}G(0)$, and (3.6) or (3.7) holds for $i \in \{0, 1, ..., (n+m)/2 - 1\}$. In case $A(p,q)$ is non-empty, $A(p,q) = (\underline{\alpha}, \bar{\alpha})$.

The main idea is thus to apply Proposition 2.1 to $\psi(s,\alpha)$ rather than to $\phi(s,\alpha)$ since the odd component of the former is independent of α. The simplicity of the case considered in this section is due to the fact that $\alpha \in A(p,q)$ if and only if $\psi(s,\alpha)$ is Hurwitz stable. In general $\psi(s,\alpha)$ will have roots in $\mathbf{C}_{0+}$ even though $\phi(s,\alpha)$ is Hurwitz stable. This necessitates the use of Theorem 2.2 and the analysis is considerably more involved.

3.3 The General Case

Let $p, q \in \mathbf{R}[s]$ be non-zero, with $m = deg\, p$ and $n = deg\, q$ and satisfy

(A1) $n \geq m,\ \ n \geq 1.$

(A2) (p,q) is coprime.

In this section a description of $A(p,q)$ is given in Theorem 3.1 [34], under assumptions (A1) and (A2). Note that if (A1) fails, then either $n < m$ in which case $A(p,q)=\emptyset$ or $n=m=0$ in which case $A(p,q)=\mathbf{R}-\{-\frac{p}{q}\}$. On the other hand, if (A2) fails, then with $t := gcd\{p,q\}$, we have $q=t\bar{q}$ and $p=t\bar{p}$ for coprime polynomials $(\bar{q},\bar{p})$. Then, $A(p,q)\neq\emptyset$ if and only if $t\in\mathcal{H}$ and $A(\bar{p},\bar{q})\neq\emptyset$, in which case $A(p,q)=A(\bar{p},\bar{q})$. Consequently, we can assume (A1) and (A2) without loss of generality.

Let (h,g) and (f,e) be the even-odd components of $q(s)$ and $p(s)$, respectively. By (A1), $f(u)$ and $e(u)$ are not both zero and $d := gcd\,\{f,e\}$ is well-defined. Let

$$f=d\bar{f},\ \ e=d\bar{e}$$

for coprime polynomials $\bar{f}$, $\bar{e}\in\mathbf{R}[u]$. Then, the polynomial

$$\bar{p}(s) := \bar{f}(s^2)+s\bar{e}(s^2)=p(s)/d(s^2) \tag{3.8}$$

is free of $\mathbf{C}_0$ roots except possibly a simple root at $s=0$. Let (H,G) be the even-odd components of $q(s)\bar{p}(-s)$. Also let $F(s^2) := p(s)\bar{p}(-s)$. By a simple computation, it follows that

$$\begin{aligned} H(u) &= h(u)\bar{f}(u)-ug(u)\bar{e}(u),\\ G(u) &= g(u)\bar{f}(u)-h(u)\bar{e}(u),\\ F(u) &= f(u)\bar{f}(u)-ue(u)\bar{e}(u). \end{aligned} \tag{3.9}$$

These polynomials are related to $q(j\omega)/p(j\omega)$ by

$$\frac{H}{F}(-\omega^2)=Re\{\frac{q(j\omega)}{p(j\omega)}\},\ \ -\omega\frac{G}{F}(-\omega^2)=Im\{\frac{q(j\omega)}{p(j\omega)}\}$$

whenever defined. If $G\not\equiv 0$ and if they exist, let the *real negative zeros with odd multiplicities of* $G(u)$ be $\{v_1,...,v_k\}$ with the ordering

$$0>v_1>v_2>\cdots>v_k, \tag{3.10}$$

with $v_0 := 0$ and $v_{k+1} := -\infty$ for notational convenience, and let the *real negative zeros with even multiplicities of* $G(u)$ be $\{u_1,...,u_l\}$.

Theorem 3.1 *[34] Let $p, q \in \mathbf{R}[s]$ satisfy the assumptions (A1), (A2) and let F, G, H, $\{v_i\}$ be defined by (3.9), (3.10).*

[**Existence**] *The set $A(p,q)$ is non-empty if and only if*

(i) $G \not\equiv 0$,

(ii) (F, G, H) *is coprime,*

(iii) There exists a sequence of signums

$$\mathcal{I} = \begin{cases} \{i_0, i_1, \dots, i_k\} & \text{for odd } n-m \\ \{i_0, i_1, \dots, i_{k+1}\} & \text{for even } n-m, \end{cases}$$

where $i_0 \in \{-1, 0, 1\}$ and $i_j \in \{-1, 1\}$ for $j = 1, \dots, k+1$ satisfying (1)-(3):

(1)

$$\begin{aligned} F(v_j) = 0 \qquad & \Rightarrow \; i_j = \mathcal{S}H(v_j)\mathcal{S}G(0_-), \;\; j = 0, 1, ..., k \;, \\ n-m \text{ even} \,\&\, n > m \; & \Rightarrow \; i_{k+1} = \mathcal{S}H(v_{k+1})\mathcal{S}G(0_-), \end{aligned}$$

(2)

$$n - \sigma(p) = \begin{cases} i_0 - 2i_1 + 2i_2 + \cdots + 2(-1)^k i_k & \text{for odd } n-m \\ i_0 - 2i_1 + 2i_2 + \cdots + 2(-1)^k i_k + (-1)^{k+1} i_{k+1} & \text{for even } n-m. \end{cases}$$

(3)

$$\max_{j \in \mathcal{J}^-} \frac{H}{F}(v_j) \;\; < \;\; \min_{j \in \mathcal{J}^+} \frac{H}{F}(v_j),$$

where

$$\begin{aligned} \mathcal{J}^+ &:= \{j : i_j \in \mathcal{I}_{free}, \, i_j \mathcal{S}F(v_j)\mathcal{S}G(0_-) = 1\}, \\ \mathcal{J}^- &:= \{j : i_j \in \mathcal{I}_{free}, \, i_j \mathcal{S}F(v_j)\mathcal{S}G(0_-) = -1\}, \end{aligned}$$

$\mathcal{I}_{free}$ denotes the set of signums not fixed by (1), and where $G(0_-) := (-1)^{m_0} G^{(m_0)}(0)$ with m_0 being the multiplicity of $u = 0$ as a root of $G(u)$.

[**Determination**] *Let (i)-(iii) hold. Let* $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_\mu$ *be the set of all signum sequences that satisfy (iii) and let* $\mathcal{J}_t^{\pm} := \{j : i_j \in \mathcal{I}_{t,free},\ i_j \mathcal{S}F(v_j)\mathcal{S}G(0_-) = \pm 1\}$ *for* $t = 1, ..., \mu$. *Consider the* μ *open intervals defined by*

$$A_t := (-\min_{j\in\mathcal{J}_t^+} \frac{H}{F}(v_j), \quad -\max_{j\in\mathcal{J}_t^-} \frac{H}{F}(v_j)) \tag{3.11}$$

for $t = 1, 2, \cdots, \mu$ *and the set of points*

$$\hat{A} := \{-\frac{H}{F}(u_j) : F(u_j) \neq 0\}$$

Then,

$$A(p, q) = \bigcup_{t=1}^{\mu} A_t \setminus (\hat{A} \cap A_t). \tag{3.12}$$

Proof. For completeness of presentation we present the proof given in [34].
[Only if] Suppose $A(p, q) \neq \emptyset$ and let $\alpha \in A(p, q)$. Let $\psi(s, \alpha) := \phi(s, \alpha)\bar{p}(-s)$ which has even-odd components $(H + \alpha F, G)$. Thus, $\sigma(\phi) = n$, $\sigma(\psi) = n - \sigma(\bar{p})$, and $deg\,\psi$ is odd if and only if $n - m$ is odd. Suppose $u_0 \in \mathbf{C}$ is a root of $gcd\{H+\alpha F, G\}$. Since $(H+\alpha F, G)$ are the even-odd components of $\phi(s, \alpha)\bar{p}(-s)$, it follows that $s_0 = \mp\sqrt{u_0}$ (or 0 with multiplicity 2) are both roots of $\psi(s, \alpha)$. If $Re\,\{s_0\} = 0$, then as $\phi(s, \alpha)$ is Hurwitz stable $\bar{p}(-s)$ must have two roots on $\mathbf{C}_0$. This is not possible since $\bar{p}(s)$ has no zeros in $\mathbf{C}_0$ except possibly a simple zero at $s = 0$. Hence $Re\,\{s_0\} \neq 0$ and one of the roots, say $s_0 = -\sqrt{u_0}$, is in $\mathbf{C}_+$. Since ϕ is Hurwitz stable, s_0 is a root of $\bar{p}(-s)$. Since $gcd\,(\bar{f}, \bar{e}) = 1$, $-s_0$ can not also be a root of $\bar{p}(-s)$ so that it is a root of $\phi(s, \alpha)$. But $\phi(-s_0, \alpha) = q(-s_0) + \alpha p(-s_0) = 0$ implies by $\bar{p}(-s_0) = 0$ that $q(-s_0) = 0$. This contradicts the assumption (A2). Therefore, $(H + \alpha F, G)$ and hence (F, G, H) is coprime. Now if $G \equiv 0$, then by coprimeness of $(H + \alpha F, G)$, $\psi(s, \alpha)$ is a constant. This implies that $n = 0$ which contradicts the assumption (A1). Hence, (i) and (ii) hold and $\sigma(\psi) = n - \sigma(\bar{p})$, where $\psi(s, \alpha) = \phi(s, a)\bar{p}(-s)$. By Theorem 2.2, at the roots v_j of $G(u)$, (2.15) holds with $r = n - \sigma(\bar{p})$, $a(u) := H(u) + \alpha F(u)$, and $b(u) := G(u)$. Therefore,

the sequence of signums $\mathcal{I} = \{i_j\}$ defined by

$$i_j := \mathcal{S}(H + \alpha\, F)(v_j)\mathcal{S}G(0_-) \tag{3.13}$$

for $j = 0, 1, \ldots, k(, k+1)$ satisfies (2) of condition (iii). Note that, by coprimeness of $(H + \alpha F, G)$, $i_j \neq 0$ for $j = 1, ..., k, k+1$. Moreover, $i_0 = 0$ if and only if $(H + \alpha F)(0) = \phi(0, \alpha)\bar{p}(0) = 0$. This can happen if and only if $\bar{p}(0) = 0$ so that $i_j \in \{-1, 1\}$ for $j = 1, ..., k+1$ and $i_0 \in \{-1, 0, 1\}$, where $i_0 = 0$ if and only if $\bar{p}(0) = 0$. To prove that (1) and (3) of condition (iii) are satisfied, let us first suppose $n - m$ is even. By $n \geq m$ and by (2.1), it follows that $deg\, H \geq deg\, F$, where equality holds if and only if $n = m$. Thus for $j = k+1$, (3.13) gives $i_{k+1} = \mathcal{S}H(-\infty)$ when $n > m$, $\alpha > -\frac{H}{F}(-\infty)$ when $i_{k+1}\mathcal{S}F(-\infty)\mathcal{S}G(0_-) = 1$, and $\alpha < -\frac{H}{F}(-\infty)$ when $i_{k+1}\mathcal{S}F(-\infty)\mathcal{S}G(0_-) = -1$. For $j = 0, 1, ..., k$, (3.13) gives $i_j = \mathcal{S}H(v_j)\mathcal{S}G(0_-)$ when $F(v_j) = 0$ and

$$\alpha > -\frac{H}{F}(v_j) \quad \text{for all } v_j \text{ for which } i_j\mathcal{S}F(v_j)\mathcal{S}G(0_-) = 1,$$

$$\alpha < -\frac{H}{F}(v_j) \quad \text{for all } v_j \text{ for which } i_j\mathcal{S}F(v_j)\mathcal{S}G(0_-) = -1.$$

It follows that

$$\max_{\{j\,:\,i_j\mathcal{S}F(v_j)\mathcal{S}G(0_-))=1\}} -\frac{H}{F}(v_j) \;<\; \alpha \;<\; \min_{\{j\,:\,i_j\mathcal{S}F(v_j)\mathcal{S}G(0_-)=-1\}} -\frac{H}{F}(v_j),$$

or equivalently,

$$-\min_{\{j\,:\,i_j\mathcal{S}F(v_j)\mathcal{S}G(0_-)=1\}} \frac{H}{F}(v_j) \;<\; \alpha \;<\; -\max_{\{j\,:\,i_j\mathcal{S}F(v_j)\mathcal{S}G(0_-)=-1\}} \frac{H}{F}(v_j).$$

This yields the inequality in (3). When $n - m$ is odd, similar arguments applied to $j = 0, 1, ..., k$ give (iii). This proves the "only if" part of the "existence" statement. By coprimeness of $(H + \alpha F, G)$, $(H + \alpha F)(u_j) \neq 0$ so that $\alpha \notin \hat{A}$. Therefore, $A(p, q) \subset A$, where A denotes the right hand side of (3.12).

[If] Suppose (i)-(iii) are satisfied. We prove that $A \subset A(p, q)$ establishing the "if" part of the "existence" statement as well as the description for $A(p, q)$. Let us first consider

$$A_c := A \cap \{\alpha \in \mathbf{R} : (H + \alpha F, G) \text{ is coprime}\}.$$

By the definition of the set A_c, $(H + \alpha F, G)$ is coprime for all $\alpha \in A_c$ and, by (i), $G \not\equiv 0$. Let $\alpha \in A_c$ belong to the interval A_ν obtained by a signum set $\mathcal{I}_\nu$ for some $\nu \in \{1, ..., \mu\}$. Thus, using (2) and noting that (3) holds for $\mathcal{J}_\nu^-$ and $\mathcal{J}_\nu^+$, it is easy to show that $\mathcal{S}(H + \alpha F)(v_j) = i_j \mathcal{S}G(0_-)$ for all $i_j \in \mathcal{I}_\nu$. By (2) of (iii), it follows that $a := H + \alpha F,\ \ b := G$ satisfy (2.15) of Theorem 2.2 so that $\sigma(\phi(s, \alpha)\bar{p}(-s)) = n - \sigma(\bar{p}(s))$. It follows that $\sigma(\phi(s, \alpha) = n$ and hence $A_c \subset A(p, q)$. We now show that the set $A \setminus A_c$ of *finite number of points* is empty. Suppose $\alpha_0 \in A \setminus A_c$ so that there exists $u_0 \in \mathbf{C}$ satisfying $H(u_0) + \alpha_0 F(u_0) = 0,\ G(u_0) = 0$. If $F(u_0) = 0$, then *gcd* $\{F, G, H\} \neq 0$ which contradicts (ii). Thus, $F(u_0) \neq 0$. We consider two cases. First, suppose u_0 is real and non-positive. Then, $u_0 \in \{v_0, ..., v_k, u_1, ..., u_l\}$ and $\alpha_0 = -H(u_0)/F(u_0)$. This contradicts the fact that $\alpha_0 \in A$. Second, suppose that u_0 is either a real positive number or a non-real complex number. It follows that $\phi(\pm\sqrt{u_0}, \alpha_0)\bar{p}(\mp\sqrt{u_0}) = 0$ since u_0 is a common zero of the even-odd components of $\phi(s, \alpha_0)\bar{p}(-s)$. Note that both $\pm\sqrt{u_0}$ can not be roots of $\bar{p}(s)$ since the latter has coprime even-odd components. On the other hand, if $\bar{p}(\pm\sqrt{u_0}) = 0$ and $\phi(\mp\sqrt{u_0}) = 0$, then (p, q) is not coprime and (A2) is contradicted. Hence, both of $\pm\sqrt{u_0}$ are the roots of $\phi(s, \alpha_0)$. Note that $Re\{\sqrt{u_0}\} \neq 0$ as u_0 is either real positive or non-real complex. Consequently, $\phi(s, \alpha_0)$ has a root in $\mathbf{C}_+$. But, since A_c is dense in A, any neighborhood in A of α_0 contains $\alpha_1 \in A_c$ for which $\phi(s, \alpha_1)$ is Hurwitz stable. By the continuity of the roots of ϕ with respect to α and by the fact that $\mathbf{C}_- \cap \mathbf{C}_+ = \emptyset$, such an α_0 can not exist. We have thus shown that $A \setminus A_c$ is empty and hence $A \subset A(p, q)$.

$\blacksquare$

Remark 3.1 *The condition (2) of Theorem 3.1 together with the degree restriction on $G(u)$ limits k. By (2.1) and by condition (2) of the theorem, respectively,*

$$k \leq deg\, G \leq \begin{cases} \frac{n+deg\,\bar{p}-1}{2}, & n-m \;\; odd \\ \frac{n+deg\,\bar{p}}{2}-1, & n-m \;\; even \end{cases}, \quad n-\sigma(p) \leq \begin{cases} 2k+1, & n-m \;\; odd \\ 2k+2, & n-m \;\; even. \end{cases}$$

Hence, in order for $A(p,q)$ to be non-empty, it is necessary that

$$\begin{array}{ll} \frac{n-\sigma(p)-1}{2} \leq k \leq \frac{n+deg\,\bar{p}-1}{2}, & n-m \;\; odd \\ \frac{n-\sigma(p)}{2}-1 \leq k \leq \frac{n+deg\,\bar{p}}{2}-1, & n-m \;\; even. \end{array} \tag{3.14}$$

$\triangle$

Remark 3.2 *Let us determine the possible cases where the stabilizing values of α can belong to infinite intervals, i.e., $A(p,q) = (-\infty, a_1)$ and/or $A(p,q) = (a_2, \infty)$ where a_1, a_2 are real numbers. Recall that $n = deg\ q$, $m = deg\ p$, and let $r = n - m$. We assume in what follows that $r \geq 1$. From root-locus arguments, whenever $r \geq 3$, stabilizing values of α can not include an infinite interval. This can be easily seen from the asymptotes of the root-locus. Moreover, as the roots of $q(s) + \alpha p(s)$ tends to the roots of $p(s)$ as $\alpha \to \pm\infty$, whenever $p(s)$ has a root in $\mathbf{C}_+$ stabilizing values of α can not include an infinite interval. Hence, the only possible case of obtaining an infinite stabilizing interval is when*

$$\begin{cases} r \leq 2 \\ \& \\ p(s) \text{ has no roots on } \mathbf{C}_+. \end{cases}$$

Now, using Theorem 3.1 we give a rigorous proof to the fact that whenever $r \geq 3$ or $p(s)$ has a root in $\mathbf{C}_+$, $A(p,q)$ can not include an infinite interval. We first assume that $F(u) \neq 0\ \forall u \leq 0$ (this means $p(s)$ has no roots on the $j\omega$-axis). Let us also assume that $G(0_-) > 0$, the case of $G(0_-) < 0$ follows similar arguments.
Case 1*: we consider the case $n - m = 3$. Suppose that $\sigma(p) = m$ (in this case all roots of $p(s)$ are in the open left-half plane). Then, $n - \sigma(p) = 3$. Let $v_1, \ldots, v_k$ be*

the real negative roots of $G(u)$*, with odd multiplicities. Since all* $v_i,\ i = 0, \dots, k$ *are finite, with* $v_0 = 0$*, values of* $\frac{H(v_i)}{F(v_i)}$ $i = 0, \dots, k$ *are also finite. Hence, an infinite stabilizing range can occur if and only if* $\mathcal{J}^+$ *or* $\mathcal{J}^-$ *is an empty set which means that the signums must have the same sign. By a simple calculation, the right-hand side of the first equation in (2) of Theorem 3.1 can either be* 1 *or* -1 *depending on whether* k *is even or odd and the signums being* 1*'s or* -1*'s. Hence, the signature* $n - \sigma(p) = 3$ *can not be achieved with a such a sequence of signums.*

Case 2*: we consider the case* $n-m = 4$*. Since* $n-m$ *is even, we have* $v_{k+1} = -\infty$ *and* $\frac{H(v_{k+1})}{F(v_{k+1})} = \pm\infty$*. Suppose that* $\sigma(p) = m$*, then* $n - \sigma(p) = 4$*. If all the signums are alike (*1 *or* -1*), then* $n - \sigma(p) = 0$ *and a signature of* 4 *can not be achieved. We consider four different cases where the signums are not of the same sign:*

Case 2.1 $\frac{H(v_{k+1})}{F(v_{k+1})} = \infty$ *and* $i_{k+1} = 1$*. Since* $i_{k+1} \in \mathcal{J}^+$*, the only possibility of an infinite interval is when* $\min_{j\in\mathcal{J}^+} \frac{H(v_j)}{F(v_j)} = \infty$*. This fixes all* $i_j,\ j = 0, \dots, k$ *to* -1 *otherwise* $\min_{j\in\mathcal{J}^+} \frac{H(v_j)}{F(v_j)} \neq \infty$*. In such a case* $n - \sigma(p) = -2$ *when* k *is even and* $n - \sigma(p) = 2$ *when* k *is odd. Hence a signature of* 4 *can not be achieved.*

Case 2.2 $\frac{H(v_{k+1})}{F(v_{k+1})} = \infty$ *and* $i_{k+1} = -1$*. Since* $i_{k+1} \in \mathcal{J}^-$*, the only possibility of an infinite interval is when* $\max_{j\in\mathcal{J}^-} \frac{H(v_j)}{F(v_j)} = \infty$*. However, condition 3*

$$\max_{j\in\mathcal{J}^-} \frac{H(v_j)}{F(v_j)} < \min_{j\in\mathcal{J}^+} \frac{H(v_j)}{F(v_j)}$$

in Theorem 3.1 can not be satisfied as $\frac{H(v_j)}{F(v_j)}, j = 0, \dots, k$ *are finite. Hence, an infinite stabilizing interval can not exist in this case.*

Case 2.3 $\frac{H(v_{k+1})}{F(v_{k+1})} = -\infty$ *and* $i_{k+1} = -1$*. Since* $i_{k+1} \in \mathcal{J}^-$*, the only possibility of an infinite interval is when* $\max_{j\in\mathcal{J}^-} \frac{H(v_j)}{F(v_j)} = -\infty$*. This fixes all* $i_j,\ j = 0, \dots, k$ *to* 1 *otherwise* $\max_{j\in\mathcal{J}^-} \frac{H(v_j)}{F(v_j)} \neq -\infty$*. In such a case* $n - \sigma(p) = 2$ *when* k *is even and* $n - \sigma(p) = 0$ *when* k *is odd. Hence a signature of* 4 *can not be achieved.*

Case 2.4 $\frac{H(v_{k+1})}{F(v_{k+1})} = -\infty$ *and* $i_{k+1} = 1$*. Since* $i_{k+1} \in \mathcal{J}^+$*, the only possibility of* *an infinite interval is when* $\min_{j\in\mathcal{J}^+} \frac{H(v_j)}{F(v_j)} = -\infty$*. However, condition 3*

$$\max_{j\in\mathcal{J}^-} \frac{H(v_j)}{F(v_j)} < \min_{j\in\mathcal{J}^+} \frac{H(v_j)}{F(v_j)}$$

in Theorem 3.1 can not be satisfied as $\frac{H(v_j)}{F(v_j)}, j = 0, \ldots, k$ *are finite. Hence, an infinite stabilizing interval can not exist in this case.*

Case 3: *We now consider the case of* $p(s)$ *having at least one root in the open right-half plane,* $\sigma(p) = m - 2$*. If* $n - m = 1$*, then* $n - \sigma(p) = 3$ *and by case 1 an infinite stabilizing interval can not exist. If* $n - m = 2$*, then* $n - \sigma(p) = 4$ *and by case 2 an infinite stabilizing interval can not exist. Note that whenever* $n - m \geq 4$ *or* $p(s)$ *has more than one root in the open right-half plane, similar arguments hold and an infinite stabilizing interval can not exist. Now, we show that when* $n = m$ *and* $p(s)$ *has a root in* $\mathbf{C}_+$ *a similar conclusion holds. In this case,* $n - \sigma(p) = 2$*,* H *and* F *have the same degree, and* $\frac{H(-\infty)}{F(-\infty)}$ *is finite. Hence, an infinite stabilizing interval can occur if and only if* $\mathcal{J}^+$ *or* $\mathcal{J}^-$ *is an empty set which means that the signums must have the same sign. However, for these sequences of signums* $n - \sigma(p) = 0$ *and a signature of* 2 *can not be achieved.* $\triangle$

Example 3.1 *Consider*

$$q(s) = s^6 + 2s^5 + 5s^4 + 5s^3 + s^2 + 0.5s - 0.05,$$
$$p(s) = s^6 + 4s^5 + 30s^4 + 60s^3 + 150s^2 + 100s + 100.$$

To determine $A(p,q)$*, we employ Theorem 3.1. By the method of Hurwitz determinants, it is easy to see that* p *is Hurwitz stable, i.e.,* $\sigma(p) = 6$*. Using (3.9), we have*

$$F(u) = u^6 + 44u^5 + 720u^4 + 4800u^3 + 16500u^2 + 20000u + 10000,$$
$$G(u) = -2u^5 - 15u^4 + 46.5u^3 + 405.2u^2 + 478u + 55,$$
$$H(u) = u^6 + 27u^5 + 161u^4 + 377.95u^3 + 118.5u^2 + 42.5u - 5.$$

The polynomial $G(u)$ *has one positive and four negative real zeros which are*

$$v_1 = -0.1289,\ v_2 = -1.3783,\ v_3 = -3.7921,\ v_4 = -7.5823.$$

Now, $G(0_-) = G(0) = 55 > 0$, $F(v_i) > 0$ *for* $i = 0, ..., 5$, *and*

$$\tfrac{H}{F}(v_0) = -0.0005,\ \tfrac{H}{F}(v_1) = -0.0012,\ \tfrac{H}{F}(v_2) = -0.1041,$$

$$\tfrac{H}{F}(v_3) = -0.1471,\ \tfrac{H}{F}(v_4) = -0.6207,\ \tfrac{H}{F}(v_5) = 1.$$

The signum sequences

$$\mathcal{I}_1 = \{1,1,1,1,1,1\},\ \mathcal{I}_2 = \{1,1,1,-1,-1,1\},$$
$$\mathcal{I}_3 = \{1,-1,-1,-1,-1,1\},\ \mathcal{I}_4 = \{-1,-1,-1,-1,-1,-1\}$$

satisfy (3) in Theorem 3.1.iii. We obtain the four intervals

$$A_1 = (0.6207, +\infty),\ A_2 = (0.1041, 0.1471),\ A_3 = (0.0005, 0.0012),\ A_4 = (-\infty, -1)$$

and $\hat{A} = \emptyset$ *so that* $A(p,q) = A_1 \cup A_2 \cup A_3 \cup A_4$.

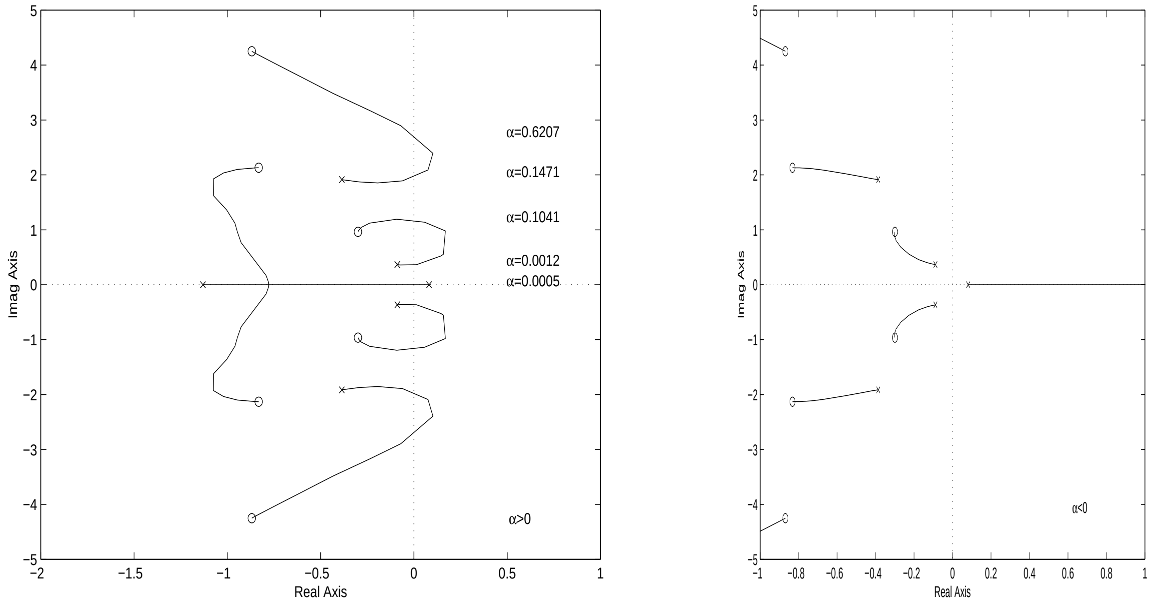

Figure 3.1: Root-loci of $\phi(s, \alpha)$.

•

Example 3.2 *In this example, we illustrate how fixed signums can arise in the candidate signum sequences. Consider*

$$q(s) = s^6 + s^5 + 11s^4 + 2s^3 + 19s^2 + 12,$$
$$p(s) = s^5 + 3s^4 + 4s^3 + 6s^2 + 4s.$$

We have $\bar{p} = s^3+3s^2+2s$, $\sigma(\bar{p}) = 2$, $G(u) = -(u+1)(u+2)(u+3)(u+4)$, $F(u) = -u(u-1)(u-4)(u+2)$, $H(u) = u(2u^3 + 29u^2 + 53u + 36)$. *The zeros of* $G(u)$ *are* $v_1 = -1, v_2 = -2, v_3 = -3, v_4 = -4$. *Since* $F(v_0) = 0, F(v_2) = 0$, *by (1) of Theorem 3.1.iii,* $i_0 = 0$ *and* $i_2 = 1$ *are fixed. We also have* $n - \sigma(p) = 4$ *and the signum sequences* $\mathcal{I}_1 = \{0, -1, 1, 1, 1\}$, $\mathcal{I}_2 = \{0, -1, 1, -1, -1\}$, $\mathcal{I}_3 = \{0, 1, 1, -1, 1\}$ *are the only ones that satisfy (2) of Theorem 3.1.iii. Moreover,* $\mathcal{S}G(0_-) = -1$, $\mathcal{S}F(v_1) = 1$, $\mathcal{S}F(v_3) = \mathcal{S}F(v_4) = -1$ *and we have* $\mathcal{J}_1^- = \emptyset, \mathcal{J}_1^+ = \{1, 3, 4\}, \mathcal{J}_2^- = \{3, 4\}, \mathcal{J}_2^+ = \{1\}, \mathcal{J}_3^- = \{1, 3\}, \mathcal{J}_3^+ = \{4\}$. *Finally,* $\frac{H}{F}(v_1) = -1$, $\frac{H}{F}(v_3) = 3$, $\frac{H}{F}(v_4) = 2$ *and the only signum sequence satisfying the third item of Theorem 3.1 turns out to be* $\mathcal{I}_1$ *which yields* $A(p, q) = (1, +\infty)$. •

3.4 The Dual Case

Let us now consider the set

$$B(p, q) := \{\beta \in \mathbf{R} : \phi(s, \beta) = \beta q(s) + p(s) \in \mathcal{H} \ , \ deg\,\theta = deg\,q\}.$$

If (A1) and (A2) hold, then the following relation between $A(p, q)$ and $B(p, q)$ is immediate. If $\alpha \in A(p, q)$ and $\alpha \neq 0$, then $\beta := \alpha^{-1}$ is in $B(p, q)$. If $0 \in A(p, q)$, then $q \in \mathcal{H}$ and the intervals (β_1, ∞), $(-\infty, -\beta_2)$ are contained in $B(p, q)$ for some β_1, $\beta_2 > 0$. If $\beta \in B(p, q)$ and $\beta \neq 0$, then $\alpha := \beta^{-1}$ is in $A(p, q)$. If $0 \in B(p, q)$, then $n = m$, $p \in \mathcal{H}$, and the intervals (α_1, ∞), $(-\infty, -\alpha_2)$ are contained in $A(p, q)$ for some α_1, $\alpha_2 > 0$.

We now state a counterpart to Theorem 3.1 which states conditions for $B(p,q)$ to be non-empty and gives a description of $B(p,q)$.

By (A1), h and g are not both zero and $b := gcd\,\{h,g\}$ is well-defined. Let

$$h = b\bar{h},\ \ g = b\bar{g}$$

for coprime polynomials $\bar{h},\ \bar{g} \in \mathbf{R}[u]$. Then, the polynomial

$$\bar{q}(s) := \bar{h}(s^2) + s\bar{e}(s^2) = q(s)/b(s^2) \tag{3.15}$$

is free of $\mathbf{C}_0$ roots except possibly a simple root at $s = 0$. Let (E, D) be the even-odd components of $p(s)\bar{q}(-s)$ and let $C(s^2) := \bar{q}(s)\bar{q}(-s)$. Similar to (3.9), we have

$$\begin{aligned} E(u) &= \bar{h}(u)f(u) - u\bar{g}(u)e(u), \\ D(u) &= \bar{h}(u)e(u) - \bar{g}(u)f(u), \\ C(u) &= \bar{h}(u)h(u) - u\bar{g}(u)g(u). \end{aligned} \tag{3.16}$$

If $D \not\equiv 0$ and if they exist, let the *real negative zeros with odd multiplicities of* $D(u)$ be $\{x_1, ..., x_k\}$ with the ordering

$$x_1 > x_2 > \cdots > x_k, \tag{3.17}$$

with $x_0 := 0$ and $x_{k+1} := -\infty$ for notational convenience, and let the *real negative zeros with even multiplicities of* $D(u)$ be $\{y_1, ..., y_l\}$.

Theorem 3.2 *[34] Let $p, q \in \mathbf{R}[s]$ satisfy the assumptions (A1), (A2) and let C, D, E, $\{x_j\}$ be defined by (3.16), (3.17).*

[Existence] *The set $B(p,q)$ is non-empty if and only if*

(i) $D \not\equiv 0$,

(ii) (C, D, E) is coprime,

(iii) There exists a sequence of signums

$$\mathcal{I} = \{i_0, i_1, \ldots, i_{k+1}\}$$

where $i_0 \in \{-1, 0, 1\}$ *and* $i_j \in \{-1, 1\}$ *for* $j = 1, \ldots, k+1$ *satisfying (1)-(3):*

(1)

$$C(x_j) = 0 \quad \Rightarrow \quad i_j = \mathcal{S}E(x_j)\mathcal{S}D(0_-), \;\; j = 0, 1, ..., k.$$

(2)

$$n - \sigma(q) = i_0 - 2i_1 + 2i_2 + \cdots + 2(-1)^k i_k + (-1)^{k+1} i_{k+1}.$$

(3)

$$\max_{j \in \mathcal{J}^-} \frac{E}{C}(x_j) \quad < \quad \min_{j \in \mathcal{J}^+} \frac{E}{C}(x_j) \quad \text{if} \quad D(0_-) > 0,$$

$$\max_{j \in \mathcal{J}^+} \frac{E}{C}(x_j) \quad < \quad \min_{j \in \mathcal{J}^-} \frac{E}{C}(x_j) \quad \text{if} \quad D(0_-) < 0,$$

where $\mathcal{J}^+ := \{j : i_j \in \mathcal{I},\; i_j\mathcal{S}C(x_j) = 1\}$ *and* $\mathcal{J}^- := \{j : i_j \in \mathcal{I},\; i_j\mathcal{S}C(x_j) = -1\}$ *and where* $D(0_-) := (-1)^{n_0} D^{(n_0)}(0)$ *with* n_0 *being the multiplicity of* $u = 0$ *as a root of* $D(u)$.

[Determination] *Let (i)-(iii) hold. Let* $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_\mu$ *be the set of all signum sequences that satisfy (iii) and let* $\mathcal{J}_t^{\pm} := \{j : i_j \in \mathcal{I}_t,\; i_j\mathcal{S}C(v_j) = \pm 1\}$ *for* $t = 1, ..., \mu$. *Consider* μ *open intervals defined by*

$$B_t := \begin{cases} (-\min\limits_{j \in \mathcal{J}_t^+} \dfrac{E}{C}(x_j), \;\; -\max\limits_{j \in \mathcal{J}_t^-} \dfrac{E}{C}(x_j)) & \text{if} \quad D(0_-) > 0 \\ \\ (-\min\limits_{j \in \mathcal{J}_t^-} \dfrac{E}{C}(x_j), \; -\max\limits_{j \in \mathcal{J}_t^+} \dfrac{E}{C}(x_j)) & \text{if} \quad D(0_-) < 0 \end{cases}$$

for $t = 1, 2, \cdots, \mu$ *and the set of points*

$$\hat{B} := \begin{cases} \{-\frac{E}{C}(y_j) : C(y_j) \neq 0\} \cup \{0\} & \textit{if} \quad n > m \\ \\ \{-\frac{E}{C}(y_j) : C(y_j) \neq 0\} \cup \{-\frac{q}{p}(\infty)\} & \textit{if} \quad n = m \, . \end{cases}$$

Then,

$$B(p,q) = \bigcup_{t=1}^{\mu} B_t \setminus (\hat{B} \cap B_t). \tag{3.18}$$

3.5 An Improved Algorithm

The following algorithm, which is based on Theorem 3.1, determines whether $A_r(p,q)$ is empty or not and outputs its elements when it is not empty, where $A_r(p,q) := \{\alpha \in \mathbf{R} : \ \sigma[\phi(s,\alpha)] = \sigma[q(s) + \alpha p(s)] = r\}$ is the set of all real α such that $\phi(s,\alpha)$ has signature equal to r. Recall that the real negative zeros with odd multiplicities of $G(u)$ are denoted by $\{v_1, ..., v_k\}$ with the ordering $0 > v_1 > v_2 > \cdots > v_k$, with $v_0 := 0$ and $v_{k+1} := -\infty$. For simplicity let us assume that $G(0_-) > 0$.

Algorithm 3.1 *1. Consider all the sequences of signums*

$$\mathcal{I} = \begin{cases} \{i_0, i_1, \ldots, i_k\} & \textit{for odd } r - m \\ \{i_0, i_1, \ldots, i_{k+1}\} & \textit{for even } r - m, \end{cases}$$

where $i_0 \in \{-1, 0, 1\}$ *and* $i_j \in \{-1, 1\}$ *for* $j = 1, \ldots, k+1$. *If* $F(v_j) = 0$, *then* $i_j = \mathcal{S}H(v_j)$.

2. Choose all sequences that satisfy

$$r - \sigma(p) = \begin{cases} i_0 - 2i_1 + \cdots + 2(-1)^k i_k & \textit{for odd } r - m \\ i_0 - 2i_1 + \cdots + 2(-1)^k i_k + (-1)^{k+1} i_{k+1} & \textit{for even } r - m. \end{cases}$$

3. *For each sequence of signums $\mathcal{I} = \{i_j\}$ that satisfy step 2, let*

$$\alpha_{max} = \max\{-\frac{H}{F}(v_j)\} \text{ for all } v_j \text{ for which } i_j \mathcal{S}F(v_j) = 1,$$

and

$$\alpha_{min} = \min\{-\frac{H}{F}(v_j)\} \text{ for all } v_j \text{ for which } i_j \mathcal{S}F(v_j) = -1.$$

The set $A_r(p,q)$ is non-empty if and only if for at least one signum sequence $\mathcal{I}$ satisfying step 2, $\alpha_{max} < \alpha_{min}$ holds.

4. *$A_r(p,q)$ is equal to the union of intervals $(\alpha_{max}, \alpha_{min})$ for each sequence of signums $\mathcal{I}$ that satisfy step 3. The set of points $\hat{A} := \{-\frac{H}{F}(u_j),\ j = 1, \ldots, l : F(u_j) \neq 0\}$ must be excluded from $A_r(p,q)$ as they correspond to values of α for which $q(s) + \alpha p(s)$ has zeros on the $jw-$axis.*

From a computational point of view, application of algorithm 3.1 is expensive. The main disadvantage comes from checking condition 2. In order to find the suitable signum sequences, we have to check condition 2 for 2^{k+2} different candidate signum sequences in case $p(s)$ has no roots in $\mathbf{C_0}$ and $n-m$ is even. In case $p(s)$ has no roots in $\mathbf{C_0}$ and $n-m$ is odd, the number of sequences is 2^{k+1}. Therefore, the number of sequences explodes exponentially as k increases. Since some sequences that satisfy condition 2 fail to satisfy condition 3, it is possible to improve Algorithm 3.1. In order to reduce the number of arithmetic operations needed in algorithm 3.1, we have to first identify the signum sequences for which condition 3 holds then proceed to check condition 2. We can show that two different signum sequences $\mathcal{I}_1$, $\mathcal{I}_2$ can not correspond to the same interval. Let us define the following sets:

$$\begin{aligned}
\mathcal{J}_1^+ &:= \{j :\ i_j \in \mathcal{I}_1,\ i_j \mathcal{S}F(v_j) = 1\}, \\
\mathcal{J}_1^- &:= \{j :\ i_j \in \mathcal{I}_1,\ i_j \mathcal{S}F(v_j) = -1\}, \\
\mathcal{J}_2^+ &:= \{j :\ i_j \in \mathcal{I}_2,\ i_j \mathcal{S}F(v_j) = 1\}, \\
\mathcal{J}_2^- &:= \{j :\ i_j \in \mathcal{I}_2,\ i_j \mathcal{S}F(v_j) = -1\}.
\end{aligned}$$

Since $\mathcal{I}_1 \neq \mathcal{I}_2$, it follows that $\mathcal{J}_1^+ \neq \mathcal{J}_2^+$ and $\mathcal{J}_1^- \neq \mathcal{J}_2^-$. Using condition 3 in Algorithm 3.1

$$\max_{j\in\mathcal{J}_1^-} \frac{H}{F}(v_j) \neq \max_{j\in\mathcal{J}_2^-} \frac{H}{F}(v_j),$$

and/or

$$\min_{j\in\mathcal{J}_1^+} \frac{H}{F}(v_j) \neq \min_{j\in\mathcal{J}_2^+} \frac{H}{F}(v_j).$$

In both cases $\mathcal{I}_1$ and $\mathcal{I}_2$ correspond to two different intervals as the endpoints of the intervals are different.

Algorithm 3.2 *1. If $F(v_j) \neq 0$, then calculate*

$$\alpha_i = \begin{cases} -\frac{H}{F}(v_i), i = 0, \ldots, k & \text{for odd } r - m \\ -\frac{H}{F}(v_i), i = 0, \ldots, k+1 & \text{for even } r - m, \end{cases}$$

and sort them in ascending order

$$\bar{\alpha}_0 < \bar{\alpha}_1 < \ldots < \bar{\alpha}_{k+2} < \bar{\alpha}_{k+3}$$

where $\bar{\alpha}_0 = -\infty$ and $\bar{\alpha}_{k+3} = \infty$.

2. Identify all the sequences of signums

$$\mathcal{I} = \begin{cases} \{i_0, i_1, \ldots, i_k\} & \text{for odd } r - m \\ \{i_0, i_1, \ldots, i_{k+1}\} & \text{for even } r - m, \end{cases}$$

where $i_0 \in \{-1, 0, 1\}$ and $i_j \in \{-1, 1\}$ for $j = 1, \ldots, k+1$, that correspond to the intervals $(\bar{\alpha}_i, \bar{\alpha}_{i+1})$ for $i = 0, \ldots, k+2$. If $F(v_j) = 0$, then $i_j = \mathcal{S}H(v_j)$.

3. For each signum sequence $\mathcal{I}_i$ from step 2, if

$$r - \sigma(p) = \begin{cases} i_0 - 2i_1 + 2i_2 - 2i_3 + \cdots + 2(-1)^k i_k & \text{for odd } r - m \\ i_0 - 2i_1 + 2i_2 - 2i_3 + \cdots + (-1)^{k+1} i_{k+1} & \text{for even } r - m. \end{cases}$$

holds, then $(\bar{\alpha}_i, \bar{\alpha}_{i+1}) \in A_r(p, q)$. The set of points $\hat{A} := \{-\frac{H}{F}(u_j),\ j = 1, \ldots, l : F(u_j) \neq 0\}$ must be excluded from $A_r(p, q)$ as they correspond to values of α for which $q(s) + \alpha p(s)$ has zeros on the $jw-$axis.

In step 2 above it is easy to identify the signum sequences that lead to the different intervals. Since α_i's are ordered in ascending order and $\mathcal{S}F(v_j)$, $j = 1, \ldots, k+1$ are known, we can determine $\mathcal{J}^-$ and $\mathcal{J}^+$ for a particular interval $(\bar{\alpha}_i, \bar{\alpha}_{i+1})$. This is equivalent to determining whether $i_j = 1$ or $i_j = -1$ for $j = 0, 1, \ldots, k+1$ and therefore identifying $\mathcal{I}$ for that particular interval. Algorithm 3.2 is similar to Neimark D-decomposition described in the introduction with the advantage that the application of some stability criterion at one interior point of each interval is replaced by step 3. Using Neimark D-decomposition the problem can be solved with $\mathbf{O}(n^3)$ arithmetic operations whereas Algorithm 3.2 requires only $\mathbf{O}(n^2)$ arithmetic operations.

The algorithm above is easily specialized to determine all stabilizing proportional controllers $c(s) = \alpha$ for the plant $g(s) = \frac{p(s)}{q(s)}$. This is achieved by replacing r in step 3 of the algorithm by n, the degree of $\phi(s, \alpha)$.

Remark 3.3 *By Step 3 of Algorithm 3.2, a necessary condition for the existence of an* $\alpha \in A_r(p, q)$ *is that the odd part of* $[q(s) + \alpha p(s)]\bar{p}(-s)$ *has at least* $\bar{r} = \max\{0, \lfloor \frac{|r-\sigma(p)|-1}{2} \rfloor\}$ *real negative roots with odd multiplicities. When solving a constant stabilization problem, this lower bound is* $\bar{r} = \max\{0, \lfloor \frac{n-\sigma(p)-1}{2} \rfloor\}$. $\triangle$

Example 3.3 *In order to see the differences between Algorithm 3.1 and Algorithm 3.2, let us consider the same plant in example 3.1 given by*

$$q(s) = s^6 + 2s^5 + 5s^4 + 5s^3 + s^2 + 0.5s - 0.05,$$
$$p(s) = s^6 + 4s^5 + 30s^4 + 60s^3 + 150s^2 + 100s + 100.$$

Table 1 summarizes the different steps needed in Algorithm 3.1. From the results below, we need to check 64 different signum sequences for condition 2 of Algorithm 3.1. Among these sequences 12 satisfy this condition. We have also to check the 12 sequences for condition 3. All this redundancy can be avoided *by applying Algorithm 3.2. Table 2 summarizes the steps of Algorithm 3.2.*

	i_0	i_1	i_2	i_3	i_4	i_∞	$i_0 - 2i_1 + 2i_2 - 2i_3 + 2i_4 - i_\infty$	*Interval*
1	*-1*	*-1*	*-1*	*-1*	*-1*	*-1*	**0**	$(-\infty, -1)$
2	*1*	*-1*	*-1*	*-1*	*-1*	*-1*	*2*	*No*
3	*-1*	*1*	*-1*	*-1*	*-1*	*-1*	*-4*	*No*
4	*1*	*1*	*-1*	*-1*	*-1*	*-1*	*-2*	*No*
5	*-1*	*-1*	*1*	*-1*	*-1*	*-1*	*4*	*No*
6	*1*	*-1*	*1*	*-1*	*-1*	*-1*	*6*	*No*
7	*-1*	*1*	*1*	*-1*	*-1*	*-1*	**0**	*No*
8	*1*	*1*	*1*	*-1*	*-1*	*-1*	*2*	*No*
9	*-1*	*-1*	*-1*	*1*	*-1*	*-1*	*-4*	*No*
10	*1*	*-1*	*-1*	*1*	*-1*	*-1*	*-2*	*No*
11	*-1*	*1*	*-1*	*1*	*-1*	*-1*	*-8*	*No*
12	*1*	*1*	*-1*	*1*	*-1*	*-1*	*-6*	*No*
13	*-1*	*-1*	*1*	*1*	*-1*	*-1*	**0**	*No*
14	*1*	*-1*	*1*	*1*	*-1*	*-1*	*2*	*No*
15	*-1*	*1*	*1*	*1*	*-1*	*-1*	*-4*	*No*
16	*1*	*1*	*1*	*1*	*-1*	*-1*	*-2*	*No*
17	*-1*	*-1*	*-1*	*-1*	*1*	*-1*	*4*	*No*
18	*1*	*-1*	*-1*	*-1*	*1*	*-1*	*6*	*No*
19	*-1*	*1*	*-1*	*-1*	*1*	*-1*	**0**	*No*
20	*1*	*1*	*-1*	*-1*	*1*	*-1*	*2*	*No*
21	*-1*	*-1*	*1*	*-1*	*1*	*-1*	*8*	*No*
22	*1*	*-1*	*1*	*-1*	*1*	*-1*	*10*	*No*
23	*-1*	*1*	*1*	*-1*	*1*	*-1*	*4*	*No*
24	*1*	*1*	*1*	*-1*	*1*	*-1*	*6*	*No*
25	*-1*	*-1*	*-1*	*1*	*1*	*-1*	**0**	*No*
26	*1*	*-1*	*-1*	*1*	*1*	*-1*	*2*	*No*

	i_0	i_1	i_2	i_3	i_4	i_∞	$i_0 - 2i_1 + 2i_2 - 2i_3 + 2i_4 - i_\infty$	*Interval*
27	*-1*	*1*	*-1*	*1*	*1*	*-1*	*-4*	*No*
28	*1*	*1*	*-1*	*1*	*1*	*-1*	*-2*	*No*
29	*-1*	*-1*	*1*	*1*	*1*	*-1*	*4*	*No*
30	*1*	*-1*	*1*	*1*	*1*	*-1*	*6*	*No*
31	*-1*	*1*	*1*	*1*	*1*	*-1*	**0**	*No*
32	*1*	*1*	*1*	*1*	*1*	*-1*	*2*	*No*
33	*-1*	*-1*	*-1*	*-1*	*-1*	*1*	*-2*	*No*
34	*1*	*-1*	*-1*	*-1*	*-1*	*1*	**0**	*(0.0005,0.0012)*
35	*-1*	*1*	*-1*	*-1*	*-1*	*1*	*-6*	*No*
36	*1*	*1*	*-1*	*-1*	*-1*	*1*	*-4*	*No*
37	*-1*	*-1*	*1*	*-1*	*-1*	*1*	*2*	*No*
38	*1*	*-1*	*1*	*-1*	*-1*	*1*	*4*	*No*
39	*-1*	*1*	*1*	*-1*	*-1*	*1*	*- 2*	*No*
40	*1*	*1*	*1*	*-1*	*-1*	*1*	**0**	*(0.1041,01.471)*
41	*-1*	*-1*	*-1*	*1*	*-1*	*1*	*-6*	*No*
42	*1*	*-1*	*-1*	*1*	*-1*	*1*	*-4*	*No*
43	*-1*	*1*	*-1*	*1*	*-1*	*1*	*-10*	*No*
44	*1*	*1*	*-1*	*1*	*-1*	*1*	*-8*	*No*
45	*-1*	*-1*	*1*	*1*	*-1*	*1*	*-2*	*No*
46	*1*	*-1*	*1*	*1*	*-1*	*1*	**0**	*No*
47	*-1*	*1*	*1*	*1*	*-1*	*1*	*-6*	*No*
48	*1*	*1*	*1*	*1*	*-1*	*1*	*-4*	*No*
49	*-1*	*-1*	*-1*	*-1*	*1*	*1*	*2*	*No*
50	*1*	*-1*	*-1*	*-1*	*1*	*1*	*4*	*No*
51	*-1*	*1*	*-1*	*-1*	*1*	*1*	*-2*	*No*
52	*1*	*1*	*-1*	*-1*	*1*	*1*	**0**	*No*
53	*-1*	*-1*	*1*	*-1*	*1*	*1*	*6*	*No*
54	*1*	*-1*	*1*	*-1*	*1*	*1*	*8*	*No*
55	*-1*	*1*	*1*	*-1*	*1*	*1*	*2*	*No*
56	*1*	*1*	*1*	*-1*	*1*	*1*	*4*	*No*
57	*-1*	*-1*	*-1*	*1*	*1*	*1*	*-2*	*No*
58	*1*	*-1*	*-1*	*1*	*1*	*1*	**0**	*No*
59	*-1*	*1*	*-1*	*1*	*1*	*1*	*-6*	*No*
60	*1*	*1*	*-1*	*1*	*1*	*1*	*-4*	*No*
61	*-1*	*-1*	*1*	*1*	*1*	*1*	*2*	*No*
62	*1*	*-1*	*1*	*1*	*1*	*1*	*4*	*No*
63	*-1*	*1*	*1*	*1*	*1*	*1*	*-2*	*No*
64	*1*	*1*	*1*	*1*	*1*	*1*	**0**	$(0.6207, \infty)$

Table 3.1: Summary of the results of Algorithm 3.1.

Interval	*Sequence*	$i_0 - 2i_1 + 2i_2 - 2i_3 + 2i_4 - i_\infty$
$(-\infty, -1)$	{*-1,-1,-1,-1,-1,-1*}	**0**
$(-1, 0.0005)$	{*-1,-1,-1,-1,-1,1*}	*-2*
$(0.0005, 0.0012)$	{*1,-1,-1,-1,-1,1*}	**0**
$(0.0012, 0.1041)$	{*1,1,-1,-1,-1,1*}	*-4*
$(0.1041, 0.1471)$	{*1,1,1,-1,-1,1*}	**0**
$(0.1471, 0.6207)$	{*1,1,1,1,-1,1*}	*4*
$(0.6207, \infty)$	{*1,1,1,1,1,1*}	**0**

Table 3.2: Results of Algorithm 3.2.

3.6 Nyquist Plot Based Method

In [37, 75], using the Nyquist plot an alternative method for determining the set of all stabilizing gains is developed. The method is based on calculating the location and direction of crossings of the Nyquist plot with the real axis. The method is extended to calculate the set of all stabilizing PID controllers. In what follows we summarize the method and compare it with the previously studied methods that are based on an extension of the Hermite-Biehler theorem.

Consider a linear time-invariant system given by a proper rational transfer function $g(s) = \frac{p(s)}{q(s)}$, where $p(s)$ and $q(s)$ are real polynomials and $q(s)$ has no roots on the imaginary axis. Let

$$g(j\omega) = \frac{p(j\omega)}{q(j\omega)} = \frac{\tilde{f}(\omega) + j\tilde{e}(\omega)}{\tilde{h}(\omega) + j\tilde{g}(\omega)}$$

so that $\tilde{f}(\omega) := Re\{p(j\omega)\}$, $\tilde{e}(\omega) := Im\{p(j\omega)\}$, $\tilde{h}(\omega) := Re\{q(j\omega)\}$ and $\tilde{g}(\omega) := Im\{q(j\omega)\}$. Note that

$$\begin{aligned}
\tilde{f}(\omega) &= f(-\omega^2),\\
\tilde{e}(\omega) &= \omega e(-\omega^2),\\
\tilde{h}(\omega) &= h(-\omega^2),\\
\tilde{g}(\omega) &= \omega g(-\omega^2),
\end{aligned}$$

where (h, g) are the even-odd components of $q(s)$ and (f, e) are the even-odd components of $p(s)$. By a simple computation, it follows that

$$\begin{aligned} g(j\omega) &= \frac{f(\omega) + j\omega e(\omega)}{h(\omega) + j\omega g(\omega)} \\ &= \frac{X(\omega^2)}{Z(\omega^2)} + j\omega \frac{Y(\omega^2)}{Z(\omega^2)} \end{aligned}$$

where

$$\begin{aligned} X(\omega^2) &:= h(-\omega^2)f(-\omega^2) + \omega^2 g(-\omega^2)e(-\omega^2) \\ Y(\omega^2) &:= h(-\omega^2)e(-\omega^2) - g(-\omega^2)f(-\omega^2) \\ Z(\omega^2) &:= h(-\omega^2)^2 + \omega^2 g(-\omega^2)^2 \end{aligned}$$

Let $v := \omega^2$. By noting that the imaginary part of $g(j\omega)$ is given by

$$Im[g(j\omega)] = \omega \frac{Y(\omega^2)}{Z(\omega^2)},$$

we can find the real axis crossings of the Nyquist plot of $g(j\omega)$. Let v_i for $i = 1, \ldots, k$ denote the real positive roots of $Y(v)$, also let $v_0 = 0$ and $V_{k+1} = \infty$. Then, the real axis crossing points are $\alpha_i = \frac{X(v_i)}{Z(v_i)}$ for $i = 0, \ldots, k+1$. Since, the closed-loop system characteristic equation is given by

$$1 + \alpha g(s) = 0,$$

the closed-loop system has a pole on the border of the stability region if and only if

$$1 + \alpha^* g(j\omega^*) = 0.$$

Since α^* is real and

$$\alpha^* = -\frac{1}{g(j\omega^*)},$$

the imaginary part of $g(j\omega^*)$ must be zero. Now, arranging the α_i's in ascending order it is easy to see that for $\alpha \in (-\frac{1}{\alpha_i}, -\frac{1}{\alpha_{i+1}})$ the number of unstable poles of the closed-loop system remains constant. By calculating the number of unstable poles of the open-loop system and the direction of crossing at the critical frequencies. we can find the number of unstable poles of the closed-loop system for each interval $(-\frac{1}{\alpha_i}, -\frac{1}{\alpha_{i+1}})$. The following algorithm was given in [75].

Algorithm 3.3 *1. Find the frequencies v_i's, $i = 1, \ldots, k$.*

2. Calculate the points $\alpha_i = \frac{X(v_i)}{Z(v_i)}$, $i = 1, \ldots, k$.

3. Relabel α_i such that $\alpha_i > \alpha_{i+1}$.

4. Find the direction of crossing using either numeric or algebraic methods.

5. Calculate the number of unstable poles of the closed-loop system.

6. Form the intervals I_i and for each interval determine the number of unstable poles of the closed-loop system from the previous step.

7. Return the intervals (if any) for which there is no unstable pole.

In step 4 above, the direction of crossing d_i is calculated as follows [75]:

$$d_i = \begin{cases} (1 - (-1)^l)\ \mathcal{S}Y^l(v_i) & \text{if } 0 < v_i < \infty, \\ \mathcal{S}y_o & \text{if } v_i = 0, \\ \mathcal{S}y_1 & \text{if } v_i = \infty, \end{cases}$$

where $Y^l(v)$ is the first non-zero derivative of $Y(v)$ at the point v_i, y_0 is is the last non-zero coefficient of $Y(v)$, and y_1 is the first coefficient of $Y(v)$.

This method was later extended to compute all stabilizing PID controllers $c(s) = \frac{k_d s^2 + k_p s + k_i}{s}$ in [37, 76]. By fixing K_p, values of k_i and k_d are found. It is shown that the resulting stabilizing PID compensators form a finite number of disjoint polyhedral sets in the parameter space.

We can see that Algorithm 3.2 and Algorithm 3.3 are similar. Algorithm 3.2 is based on an extension of the Hermite-Biehler theorem whereas Algorithm 3.3 is based on the Nyquist plot. Similarity of the algorithms can be seen from the equivalences of H and X, G and Y, and F and Z. Also, from a computational complexity point of view both algorithms require the same computational effort.

In Algorithm 3.2 the number of unstable poles is calculated by a simple addition of the signum of sequence that lead to that particular interval. In Algorithm 3.3 we can keep track of the number of unstable poles of the closed-loop system by calculating the direction of crossing at the critical frequencies.

3.7 PI and PID Controllers

The method described for finding stabilizing gains can be extended to a "sweeping algorithm" for determining PI controllers [25, 77]. A PI controller

$$c(s) = \alpha_1 + \frac{\alpha_2}{s} = \frac{\alpha_1 s + \alpha_2}{s},$$

applied to a plant $g(s) = \frac{p(s)}{q(s)}$, gives the closed-loop characteristic polynomial

$$\phi(s, \alpha_1, \alpha_2) = sq(s) + (\alpha_1 s + \alpha_2)p(s).$$

Multiplying $\phi(s, \alpha_1, \alpha_2)$ by $\bar{p}(-s)$, we obtain

$$\begin{aligned} \psi(s, \alpha_1, \alpha_2) &= \phi_0(s, \alpha_1, \alpha_2)\bar{p}(-s) \\ &= s^2 G(s^2) + \alpha_2 F(s^2) + s[H(s^2) + \alpha_1 F(s^2)]. \end{aligned}$$

Note that α_1 appears only in the odd part and α_2 appears only in the even part. For every fixed value of α_1, an application of the proportional controller algorithm above yields the set of all α_2 for which $\phi(s, \alpha_1, \alpha_2)$ is Hurwitz stable. This PI controller algorithm of [25] thus relies on finding a suitable range for α_1 over which the "sweeping" should be done. Such a range can be determined by Remark 3.3.

The method described for PI controllers can be applied to PID controllers with some modifications, [25]. A PID controller

$$c(s) = \alpha_1 + \frac{\alpha_2}{s} + \alpha_3 s = \frac{\alpha_3 s^2 + \alpha_1 s + \alpha_2}{s}$$

applied to $g(s)$ gives

$$\begin{aligned} \psi(s, \alpha_1, \alpha_2, \alpha_3) &= \phi(s, \alpha_1, \alpha_2, \alpha_3)\bar{p}(-s) \\ &= s^2 G(s^2) + \alpha_3 s^2 F(s^2) + \alpha_2 F(s^2) + s[H(s^2) + \alpha_1 F(s^2)]. \end{aligned} \tag{3.19}$$

Note that α_1 appears only in the odd part. Therefore, a range of suitable α_1 can be found as described above. Since now two parameters α_2, α_3 appear linearly in the even part, a modification of the algorithm in previous section is necessary for obtaining the proper values of α_2 and α_3. For each admissible value of α_1, a linear programming problem has to be solved. In order to highlight the modification in the algorithm, we consider a simple example. For a fixed value of α_1, suppose that the odd part of $\psi(s)$ has three real negative roots with odd multiplicities v_1, v_2, v_3. Also, suppose that the sequence of signums $\{1, -1, -1, 1\}$ gives the correct signature and recall that $v_0 = 0$. Then, values of (α_2, α_3) are obtained by solving the following set of linear inequalities:

$$\begin{cases} v_0 G(v_0) + \alpha_3 v_0 F(v_0) + \alpha_2 F(v_0) > 0, \\ v_1 G(v_1) + \alpha_3 v_1 F(v_1) + \alpha_2 F(v_1) < 0, \\ v_2 G(v_2) + \alpha_3 v_2 F(v_2) + \alpha_2 F(v_2) < 0, \\ v_3 G(v_3) + \alpha_3 v_3 F(v_3) + \alpha_2 F(v_3) > 0. \end{cases}$$

3.8 Application to Stability Robustness

In this section, we study the pairs of polynomials (p, q) for which $A(p, q)$ is either empty or a single interval, i.e., those pairs having the property:

(CC) $\alpha_1, \alpha_2 \in A(p, q)$ for some $\alpha_1 < \alpha_2$ in $\mathbf{R} \Rightarrow \alpha \in A(p, q) \ \forall \ \alpha \in [\alpha_1, \alpha_2]$.

The condition (CC) is a degree invariance and *convexity condition* on the family of polynomials $(q + \mathbf{R}p) \cap \mathcal{H}$, where $(q + \mathbf{R}p) := \{q(s) + \alpha p(s) : \alpha \in \mathbf{R}\}$. We refer the reader to [47], [74], [48] for motivations of studying (CC) when $q(s)$ is a stable polynomial.

By Theorem 3.1, we have the following characterization of (CC). *Let* $p, q \in \mathbf{R}[s]$ *satisfy the assumptions (A1), (A2). The pair* (p, q) *satisfies (CC) if and only if (i), (ii) of Theorem 3.1 hold,* $\mu \leq 1$*, and whenever* $\mu = 1$ *it holds that* $A_\mu \cap \hat{A} = \emptyset$*.* Here, we identify an interesting class of pairs (p, q) satisfying (CC) by a direct application of Theorem 3.1.

Corollary 3.1 *Let* $p, q \in \mathbf{R}[s]$ *satisfy the assumptions (A1), (A2),* $n > m$*, and*

$$n - \sigma(p) \geq \begin{cases} 2k - 1, & \textit{if } n - m \textit{ is even} \\ 2k, & \textit{if } n - m \textit{ is odd.} \end{cases} \tag{3.20}$$

Then, there is at most one signum sequence satisfying (1) and (2) of Theorem 3.1.

Proof. By (3.14) and (3.20), $n - \sigma(p)$ can have the values $\{2k + 1, 2k, 2k - 1\}$ when $n - m$ is odd and the values $\{2k + 2, 2k + 1, 2k\}$ when $n - m$ is even. The first values are the maximum values the right hand side of (2) can attain and the alternating sequence $i_j = (-1)^j \mathcal{S}G(0_-)$, $j = 0, 1, ...$ yields these values. Considering the second values, we see that $n - \sigma(p)$ is required to be even (resp., odd) when $n - m$ is odd (resp., even). This is possible only if $i_0 = 0$. In this case the sequence $i_0 = 0, i_j = (-1)^j \mathcal{S}G(0_-)$, $j = 1, 2, ...$ is the only sequence that achieves these values. If $n - \sigma(p) = 2k - 1$ when $n - m$ is odd, then the unique sequence satisfying (2) is easily seen to be $i_0 = -\mathcal{S}G(0_-), i_j = (-1)^j \mathcal{S}G(0_-)$, $j = 1, ..., k$. If $n - \sigma(p) = 2k$ when $n - m$ is even, then the two sequences $i_j = (-1)^j \mathcal{S}G(0_-)$, $j = 1, ..., k$ and $i_0 = \pm \mathcal{S}G(0_-), i_\infty = \mp \mathcal{S}G(0_-)$ both satisfy (2) of Theorem 3.1. By our assumption $n > m$, the signum i_∞ is fixed by (1) of Theorem 3.1 so that also in this case there is only one signum sequence satisfying (1) and (2). ∎

By (3.14), the condition (3.20) is easily seen to hold just in case

$$\text{number of } \mathbf{C}_- \text{ roots of } p(s) \;\leq\; deg\, G - k + 1, \tag{3.21}$$

whether $n-m$ is even or odd. If $p(s)$ is either constant or has all its roots in $\mathbf{C}_+$, then (3.21) holds. Moreover, by Proposition 2.1 applied to $(q+\alpha p)(s)p(-s)$, we have $deg\, G = k$ so that $\hat{A} = \emptyset$ and Corollary 3.1 yields the result of Proposition 3.1 in case $n > m$. To see other concrete examples of "one-interval" cases, suppose $\bar{p}(s)$ satisfies

(A3) $\sigma(\bar{p}) \leq -deg\, \bar{p} + 2$.

By (3.8), the polynomial $\bar{p}(s)$ is free of $\mathbf{C}_0$ roots except possibly a simple root at the origin. Thus, (A3) holds if and only if either of the following three holds:
(A3.i) $\bar{p}(s) = 0 \;\Rightarrow\; s \in \mathbf{C}_+$,
(A3.ii) $\bar{p}(s)$ has one root at 0 and the rest in $\mathbf{C}_+$,
(A3.iii) $\bar{p}(s)$ has one root in $\mathbf{C}_-$ and the rest in $\mathbf{C}_+$.
Note that if (A3) holds, then by (3.14) the inequality (3.20) also holds. Also by (3.14), $k \geq deg\, G - 1$ so that $\hat{A} = \emptyset$. We thus have the following re-discovery of the best known "Rantzer polynomials", see [78]. These classes are of course also easily obtained from Theorem 2 in [47].

Corollary 3.2 *If* $p \in \mathbf{R}[s]$ *satisfies (A3), then A(p,q) is an interval for all* $q \in \mathbf{R}[s]$ *satisfying (A1), (A2), and* $n > m$. ■

The following example shows that Corollary 3.1 covers many other non-trivial pairs (p, q) satisfying (CC) with $p(s)$ not a Rantzer polynomial.

Example 3.4 *Consider* $p(s) = s^2 + 2s + 1$. *Since* $p(s)$ *is a second degree Hurwitz stable polynomial, by [47], there are Hurwitz stable* $q(s)$ *for which (CC) does not* *hold. However, the polynomial* $q(s) = s^5+s^4+4s^3-s-1$ *is such that the condition of Corollary 1 holds with* $G(u) = (u+1)^3$, $n - \sigma(p) = 2k+1 = 3$. *Consequently, the pair* (p, q) *satisfies (CC). In fact,* $A(p, q)$ *is the interval* $(1, 2)$. •

Now, let us restrict our attention to $q \in \mathcal{H}$. In [55], the following definitions are given for local convex directions:

Definition 3.1 (Analytic) *Given a real Hurwitz stable polynomial* $q(s)$ *of degree* n*, a real polynomial* $p(s)$ *with deg* $p < n$ *is said to be a convex direction for* $q(s)$ *if all the roots* $S_j(\alpha)$, $j = 1, \dots, n$ *of* $q_\alpha(s) = q(s) + \alpha p(s)$, $\alpha \geq 0$ *on the punctured real imaginary axis* $j\mathbf{R} \setminus \{0\}$ *are simple and satisfy* $Re\{S_j^{'}(\alpha)\} > 0$.

Definition 3.2 (Geometric) *Given a real Hurwitz stable polynomial* $q(s)$ *of degree* n*, a real polynomial* $p(s)$ *with deg* $p < n$ *is said to be a convex direction for* $q(s)$ *if the intersection of the ray* $q + \mathbf{R}_+ p$ *with the set* $\mathcal{H}_n$ *of real Hurwitz polynomials of degree* n *is convex.*

We note that (CC) is a slight generalization (to unstable $q(s)$) of the *geometric local concept of convex directions* introduced in [55]. In particular, when $q(s)$ is Hurwitz stable, [55] gives conditions on the root-locus and the Nyquist plot of $\frac{p(s)}{q(s)}$ for (CC) to hold on the positive (or negative) real-axis.

Fact 3.1 *[55] Suppose that* $q(s)$ *is a Hurwitz stable polynomial. A real polynomial* $p(s)$ *with deg*$(p) <$ *deg*(q) *is a convex direction (analytic sense) if and only if the Nyquist plot* $r(s) = \frac{p(s)}{q(s)}$ *on* $j\mathbf{R}_+$ *crosses the negative real axis* $\mathbb{R}_-$ *only in the clockwise direction, i.e., for every* $w \geq 0$

$$r(jw) \in (-\infty, 0) \Longrightarrow \frac{\partial arg(r(jw))}{\partial w} < 0.$$

The global version of (CC) was introduced in [47] and can be shown to be equivalent to characterizing the set of $p(s)$ for which (p, q) satisfies (CC) *for any* Hurwitz stable $q(s)$. In Theorem 2 of [47], such $p(s)$ are characterized by a phase growth condition. In [48] and the references therein, one can find applications of the concept of convex directions to stability robustness of various families of polynomials.

Note that Fact 3.1 is equivalent to Corollary 3.1 applied to $q \in \mathcal{H}$. Recall that

$$q(s)p(-s) = H(s^2) + sG(s^2), \tag{3.22}$$

$$\frac{p(s)}{q(s)} = \frac{H(s^2) - sG(s^2)}{C(s^2)} \tag{3.23}$$

where $H(s^2)$ and $G(s^2)$ are given in (3.9) and $C(s^2)$ is given in (3.16). Now, let r_1 be the minimum number of real negative roots of $G(u)$ required for the existence of a solution to $A(p, q)$. If k the number of real negative roots of $G(u)$ is equal to r_1 or $r_1 + 1$, then only one alternating sequence of signums leads to the signature $n - \sigma(p)$. As $q \in \mathcal{H}$, the signature of the polynomial $q(s)p(-s)$ is given by $n - \sigma(p)$. Since $\phi(s, \alpha)$ and $q(s)p(-s)$ have the same odd part $G(u)$, the same signature, and only one alternating sequence of signums that leads to this signature, it is possible to give a solution to the analytic version of local convex directions problem in terms of the Nyquist plot of $\frac{p(s)}{q(s)}$ using (3.22) and (3.23). Hence the equivalence between Corollary 3.1 and Fact 3.1 follow. Characterizing $p(s)$ for which the geometric definition holds is more involved. We have to include the case where $k \geq r_1 + 2$. The following examples, show two cases for which $k \geq r_1 + 2$ and local convexity condition holds in one case and fails in the other.

Example 3.5

$$q(s) = s^6 + 2s^5 + 5s^4 + 5s^3 + s^2 + 0.5s + 0.005,$$

$$p(s) = s^5 + 4s^4 + 30s^3 + 60s^2 + 150s + 100.$$

we have $n - \sigma(p) = 1$*,* $k = 2$*, and* $r_1 = 0$*. The solution is* $A(p, q) = (-0.001, 0.005) \cup (12.2489, +\infty)$ *and the corresponding sequence of signums are* $\{1, -1, -1\}, \{1, 1, 1\}$*, hence* $p(s)$ *is not a local convex direction for* $q(s)$*.* $\bullet$

Example 3.6 *[74]*

$$\begin{aligned} q(s) &= s^5 + 3.2s^4 + 250.3s^3 + 75001.6s^2 + 7500.2s + 2500, \\ p(s) &= s^4 - 10s^3 + 2525s^2 + 23500s + 325000. \end{aligned}$$

we have $n - \sigma(p) = 5$*,* $k = 4$*, and* $r_1 = 2$*. The solution is* $A(p, q) = (-0.0077, 0.0815)$ *and the corresponding sequence of signums are* $\{1, -1, -1, -1, 1\}$*. Although the condition of Fact 3.1 does not hold,* $p(s)$ *is a local convex direction for* $q(s)$*.* $\bullet$

Chapter 4

Computation of First and Second Order Controllers

In this chapter, a new method is given for determining the set of all stabilizing proper first-order controllers for linear, time-invariant, scalar plants. We first solve the problem for plants with either all its zeros or all its poles in the closed right-half plane. This restrictive assumption is then removed and a solution is given for plants with no restrictions on the location of its poles or zeros. The method is based on a generalized Hermite-Biehler theorem and the application of a modified constant gain stabilizing algorithm to three subsidiary plants. It is applicable to both continuous and discrete time systems. Using this characterization of all stabilizing first-order controller, we give a design example where several time domain performance indices of the closed-loop system are evaluated. We also show that the algorithm given in this chapter can be applied to plants with interval type uncertainty. Finally, we extend the algorithm given for computing all stabilizing first-order controllers to high-order controllers. This method is also based on a generalized Hermite-Biehler theorem and the successive application of a modified constant stabilization algorithm to a number of auxiliary plants.

4.1 Introduction

In this chapter, we consider the problem of determining stabilizing proper first-order controllers. The plants are constrained to those having either all zeros or all poles in the closed right-half plane excluding the origin. The algorithm that will be given consists of a repeated application of the constant gain algorithm of Chapter 3 to appropriate subsidiary plants. It is, hence, similar to the computational algorithms of [25]. For constant gain, PI, or PID stabilization it is possible to modify the characteristic polynomial in such a way that only one of the controller parameters enter into the odd part (or the even part). This is crucial for an algorithmic application of the constant gain result of [34]. In case of proper first-order controllers or any controller of higher order, a reduction in the number of parameters appearing in the even or the odd part of a modified characteristic polynomial has not been obvious, as pointed out in [36]. In section 4.2, this difficulty is resolved for the particular class of plants described above yielding a method of determination for general first-order proper controllers. This special class of plants are considered first because the method is easy to follow. In later sections, this restrictive assumption is removed and the general problem is solved. We then show how to apply our method to plants with interval type uncertainty. Finally, we solve the problem of determining the set of all stabilizing controllers of a given degree for an arbitrary plant. We will solve the problem for a second-order controller and show how to extend the algorithm to high-order controllers. The method developed is again based on the application of a modified proportional controller algorithm to a number of auxiliary plants.

We have seen in Chapter 3 that there are several classical solutions to the problem of finding the set of all stabilizing proportional controllers. However, extensions of these methods to high-order controllers is not obvious. (i) Root-locus method: this is the most widely used graphical solution to the problem of finding the set of all stabilizing proportional controllers. However, as the order of the controller increases the number of parameters increases accordingly. Hence, it is difficult to use this method to solve the problem at hand. (ii) Routh-Hurwitz criterion: with a first-order controller, an example can show that solving the problem with this method is very difficult because we have to solve a highly non-linear set of inequalities. (iii) Neimark D-decomposition: this method was briefly described in Chapter 3. Since the number of parameters increases for a high-order controller, a direct application of this method to determine high-order controllers is not obvious.

In order to show the difficulties one might face when trying to solve this problem with classical methods, let us consider the following example.

Example 4.1 *Consider the plant* $g(s) = \frac{p(s)}{q(s)}$ *where*

$$\begin{aligned} q(s) &= s^5 + 8s^4 + 32s^3 + 46s^2 - 46s + 17, \\ p(s) &= s^3 - 4s^2 + s + 2. \end{aligned}$$

This plant is to be stabilized by a first-order controller $c(s) = \frac{\alpha_2 s + \alpha_3}{s + \alpha_1}$ *and all stabilizing* $(\alpha_1, \alpha_2, \alpha_3)$ *values are to be found. The closed loop characteristic polynomial is*

$$\begin{aligned} \phi(s) &= (s + \alpha_1)q(s) + (\alpha_2 s + \alpha_3)p(s) \\ &= s^6 + (\alpha_1 + 8)s^5 + (8\alpha_1 + \alpha_2 + 32)s^4 + (32\alpha_1 - 4\alpha_2 + \alpha_3 + 46)s^3 \\ &\quad +(46\alpha_1 + \alpha_2 - 4\alpha_3 + 46)s^2 + (46\alpha_1 + 2\alpha_2 + \alpha_3 + 17)s + 17 + 2\alpha_3. \end{aligned}$$

If we use Routh-Hurwitz criteria to solve this problem, then the following set of *inequalities must hold:*

(i) $8 + \alpha_1 > 0$

(ii) $46\alpha_1 + 8\alpha_1^2 + \alpha_1\alpha_2 - \alpha_3 + 12\alpha_2 + 210 > 0$

(iii) $-336\alpha_2 + 160\alpha_1\alpha_2 - 48\alpha_2^2 + 16\alpha_2\alpha_3 + 16\alpha_2\alpha_3 + 6852 + 6369\alpha_1$
$+428\alpha_3 + 1680\alpha_1^2 + 97\alpha_1\alpha_3 + 210\alpha_1^3 + 12\alpha_1^2\alpha_3 - 4\alpha_1\alpha_2^2 + \alpha_1\alpha_2\alpha_3$
$-\alpha_3^2 - \alpha_1^2\alpha_2 > 0$

(iv) $270346\alpha_1 - 29706\alpha_2 + 142\alpha_1\alpha_2\alpha_3 - 10882\alpha_3 + 205596\alpha_1^2 - 38402\alpha_1\alpha_2$
$3237\alpha_1\alpha_3 + 52776\alpha_1^3 - 3924\alpha_2^2 + 2127\alpha_2\alpha_3 - 1491\alpha_1^2\alpha_2 - 1988\alpha_1^2\alpha_3$
$-3183\alpha_1\alpha_2^2 - 1775\alpha_3^2 + 48\alpha_1^2\alpha_2\alpha_3 + 16\alpha_1\alpha_2^2\alpha_3 - 4\alpha_1\alpha_2\alpha_3^2 - 700\alpha_1^3\alpha_2$
$-336\alpha_1^3\alpha_3 - 263\alpha_1^2\alpha_2^2 - 48\alpha_1^2\alpha_3^2 + 198\alpha_2^2\alpha_3 - 6\alpha_2^3\alpha_3 - 64\alpha_2\alpha_3^2 + 6716\alpha_1^4$
$-71\alpha_2^3 + 4\alpha_3^3 + 235479 > 0$

(v) $81860800\alpha_1 + 779508\alpha_2 + 120212\alpha_1\alpha_2\alpha_3 - 982537\alpha_3 + 11747212\alpha_1^2$
$-882462\alpha_1\alpha_2 - 1554909\alpha_1\alpha_3 + 9378587\alpha_1^3 - 108168\alpha_2^2 + 104065\alpha_2\alpha_3$
$-1332384\alpha_1^2\alpha_2 - 559452\alpha_1^2\alpha_3 - 270619\alpha_1\alpha_2^2 - 99727\alpha_3^2 + 7685\alpha_1^2\alpha_2\alpha_3$
$+11417\alpha_1\alpha_2^2\alpha_3 - 4375\alpha_1\alpha_2\alpha_3^2 + 2704\alpha_1^3\alpha_2\alpha_3 + 821\alpha_1^2\alpha_2^2\alpha_3 - 196\alpha_1^2\alpha_2\alpha_3^2$
$-142501\alpha_1\alpha_3^2 + 40944\alpha_1^3\alpha_2 - 169206\alpha_1^3\alpha_3 - 151729\alpha_1^2\alpha_2^2 - 33484\alpha_1^2\alpha_3^2$
$-18768\alpha_1^4\alpha_2 - 21444\alpha_1^4\alpha_3 - 7712\alpha_2^2\alpha_3 - 10052\alpha_1\alpha_2^3 - 67\alpha_2\alpha_3^2 - 13498\alpha_1^3\alpha_2^2$
$-3720\alpha_1^3\alpha_3^2 - 802\alpha_1^2\alpha_2^3 - 380\alpha_1\alpha_3^3 + 2433884\alpha_1^4 + 308936\alpha_1^5 - 12336\alpha_2^3$
$-2470\alpha_3^3 - 6\alpha_1\alpha_2^3\alpha_3 + 24\alpha_1\alpha_2^2\alpha_3^2 - 6\alpha_1\alpha_2\alpha_3^3 - 60\alpha_2^3\alpha_3 - 12\alpha_1\alpha_2^4$
$+294\alpha_2^2\alpha_3^2 - 96\alpha_2\alpha_3^3 - 72\alpha_1^2\alpha_3^3 - 144\alpha_2^4 + 6\alpha_3^4 - 1017569 > 0$

(vi) $17 + 2\alpha_3 > 0$

Clearly the above inequalities are highly non-linear and there is no easy method for obtaining a solution. Other classical methods such as the root-locus is graphical in nature and therefore can not be used to solve the problem at hand. •

4.2 All stabilizing First-Order Controllers for a Special Class of Plants

Before giving the details of the algorithm that determines the set of all stabilizing first-order controllers, recall the following results proved in Chapter 2.

Lemma 4.1 *A non-zero polynomial $\psi \in \mathbf{R}[u]$, such that $\psi(0) \neq 0$, has r real negative roots without counting the multiplicities if and only if the signature of the polynomial $\psi(s^2)+s\psi'(s^2)$ is $2r$. All roots of ψ are real, negative, and distinct if and only if $\psi(s^2)+s\psi'(s^2) \in \mathcal{H}$.*

We now give the details of an algorithm that computes all stabilizing first-order controllers for a special class of plants. A first-order controller

$$c(s) = \frac{\alpha_2 s + \alpha_3}{s + \alpha_1},$$

applied to $g(s) = \frac{p(s)}{q(s)}$ gives the closed loop characteristic polynomial

$$\begin{aligned}
\phi_0(s, \alpha_1, \alpha_2, \alpha_3) &= (s+\alpha_1)q(s) + (\alpha_2 s + \alpha_3)p(s), \\
&= q_0(s) + \alpha_3 p_0(s),
\end{aligned}$$

where

$$\begin{aligned}
q_0(s, \alpha_1, \alpha_2) &= (s+\alpha_1)q(s) + \alpha_2 s p(s), \\
p_0(s) &= p(s).
\end{aligned}$$

Multiplying $\phi_0(s, \alpha_1, \alpha_2, \alpha_3)$ by $\bar{p}_0(-s)$ we obtain

$$\begin{aligned}
\psi_1(s, \alpha_1, \alpha_2, \alpha_3) &= \phi_0(s, \alpha_1, \alpha_2, \alpha_3)\bar{p}_0(-s) \\
&= s^2 G(s^2) + \alpha_1 H(s^2) + \alpha_3 F(s^2) \\
&\quad + s[H(s^2) + \alpha_1 G(s^2) + \alpha_2 F(s^2)].
\end{aligned} \tag{4.1}$$

Note that α_1, α_2 appear in the odd part and α_1, α_3 appear in the even part. As pointed out in [36], it is no longer possible to exploit the results given in the previous chapter and proceed. A major modification in the PID algorithm of [25] is hence needed.

Let us restrict the attention to plants $g(s) = \frac{p(s)}{q(s)}$ such that

$$\bar{p}(-s) = 0 \Rightarrow s \in \mathbf{C}_-.$$

We consider such plants because the algorithm is simple and easy to follow. The general case will be given in the next section. In this case $p(s)$ has all its roots in the closed right-half plane (with no zeros of odd multiplicity at the origin). We need to find values of $(\alpha_1, \alpha_2, \alpha_3)$ such that $\psi_1(s, \alpha_1, \alpha_2, \alpha_3)$ is a Hurwitz stable polynomial. By Hermite-Biehler theorem, $H(u) + \alpha_1 G(u) + \alpha_2 F(u)$ must have all its roots real, negative, and distinct. By Lemma 4.1, it follows that

$$\phi_1(s, \alpha_1, \alpha_2) = H(s^2) + \alpha_1 G(s^2) + \alpha_2 F(s^2) + s[H'(s^2) + \alpha_1 G'(s^2) + \alpha_2 F'(s^2)] \quad (4.2)$$

is Hurwitz stable. The algorithm given below exploits this necessary condition.

Let $B := gcd\{F, F'\}$ so that $F = B\bar{F}$, $F' = B\bar{F}'$ for coprime polynomials $\bar{F}, \bar{F}' \in \mathbf{R}[u]$. Let $\bar{p}_1(s) := \bar{F}(s^2) + s\bar{F}'(s^2)$. Then, by a straightforward computation,

$$\begin{aligned} \psi_2(s, \alpha_1, \alpha_2) &= \phi_1(s, \alpha_1, \alpha_2)\bar{p}_1(-s) \\ &= H_{2e}(s^2) + \alpha_1 G_{2e}(s^2) + \alpha_2 F_{2e}(s^2) + s[H_{2o}(s^2) + \alpha_1 G_{2o}(s^2)], \end{aligned}$$

where

$$\begin{aligned} H_{2e}(u) &= H(u)\bar{F}(u) - uH'(u)\bar{F}'(u), \\ G_{2e}(u) &= G(u)\bar{F}(u) - uG'(u)\bar{F}'(u), \\ F_{2e}(u) &= \bar{F}(u)\bar{F}(u) - u\bar{F}'(u)\bar{F}'(u), \\ H_{2o}(u) &= H'(u)\bar{F}(u) - H(u)\bar{F}'(u), \\ G_{2o}(u) &= G'(u)\bar{F}(u) - G(u)\bar{F}'(u). \end{aligned} \quad (4.3)$$

By Remark 3.3, it follows that the odd part of $\psi_2(s,\alpha_1,\alpha_2)$ should have at least r real negative roots with odd multiplicities. Now the set of $\alpha_1 \in \mathbf{R}$ which achieves r real negative roots with odd multiplicities in $H_{2o}(u)+\alpha_1 G_{2o}(u)$ can be determined by applying Algorithm 3.2 to

$$\begin{aligned} q_2(s) &= H_2(s) = H_{2o}(s^2) + sH'_{2o}(s^2), \\ p_2(s) &= G_2(s) = G_{2o}(s^2) + sG'_{2o}(s^2). \end{aligned}$$

The following algorithm determines all gains $\alpha_1, \alpha_2, \alpha_3$ such that $\psi_1(s,\alpha_1,\alpha_2,\alpha_3) \in \mathcal{H}$:

Algorithm 4.1 *1. Using Remark 3.3 and Algorithm 3.2, calculate the admissible ranges for α_1.*

(a) Fix an α_1 in the admissible range.

(b) Apply the proportional controller algorithm (Algorithm 3.2) to $q_1(s) = H(s^2)+sH'(s^2)+\alpha_1[G(s^2)+sG'(s^2)]$ replacing $q(s)$ and $p_1(s) = F(s^2)+sF'(s^2)$ replacing $p(s)$. (This calculates admissible values of α_2 such that $\phi_1(s)$ is in $\mathcal{H}$.)

i. Fix an α_2 from the range determined in 1.b.

ii. Apply the proportional controller algorithm (Algorithm 3.2) to $q_0(s) = (s+\alpha_1)q(s) + \alpha_2 sp(s)$ and $p_0(s) = p(s)$. (This calculates all admissible values of α_3 such that $\phi_0(s)$ is in $\mathcal{H}$.)

iii. Increment α_2 and go to step 1.b.i.

(c) Increment α_1 and go to step 1.a.

The Algorithm 3.2 is repeatedly used on three auxiliary plants:

$$\begin{aligned} g_2(s) = \tfrac{p_2(s)}{q_2(s)} &= \tfrac{G_2(s)}{H_2(s)}, \\ g_1(s) = \tfrac{p_1(s)}{q_1(s)} &= \tfrac{F(s^2)+sF'(s^2)}{H(s^2)+sH'(s^2)+\alpha_1[G(s^2)+sG'(s^2)]}, \\ g_0(s) = \tfrac{p_0(s)}{q_0(s)} &= \tfrac{p(s)}{(s+\alpha_1)q(s)+\alpha_2 sp(s)}. \end{aligned} \tag{4.4}$$

Noting that the odd part $H(u)+\alpha_1 G(u)+\alpha_2 F(u)$ of $[q_0(s)+\alpha_3 p_0(s)]\bar{p}(-s)$ must have all its roots real, negative, and distinct, there is only one sign pattern that satisfies step 2 of Algorithm 3.2. Therefore, *a very simple version of the constant gain stabilization problem is solved in step 1.b.ii for the third auxiliary plant* for each fixed (α_1, α_2).

Remark 4.1 *The above first-order controller algorithm can be applied to plants with poles in* $\mathbf{C}_{0+}$ *(except a pole of odd multiplicity at the origin), i.e.,*

$$\bar{q}(-s) = 0 \Rightarrow s \in \mathbf{C}_-,$$

where $\bar{q}(s) := \bar{h}(s^2) + s\bar{g}(s^2) = q(s)/l(s^2)$*, and* $l := gcd\{h, g\}$*. Consider a controller of the form* $c(s) = \frac{s+\alpha_1}{\alpha_2 s+\alpha_3}$*. Multiplying* $\phi_0(s, \alpha_1, \alpha_2, \alpha_3)$ *by* $\bar{q}_0(-s)$*, we obtain*

$$\begin{aligned}
\psi_1(s, \alpha_1, \alpha_2, \alpha_3) &= \phi_0(s, \alpha_1, \alpha_2, \alpha_3)\bar{q}_0(-s) \\
&= s^2 D(s^2) + \alpha_1 E(s^2) + \alpha_3 C(s^2) \\
&\quad + s[E(s^2) + \alpha_1 D(s^2) + \alpha_2 C(s^2)],
\end{aligned}$$

where

$$\begin{aligned}
E(u) &= f(u)\bar{h}(u) - ue(u)\bar{g}(u), \\
D(u) &= e(u)\bar{h}(u) - f(u)\bar{g}(u), \\
C(u) &= h(u)\bar{h}(u) - ug(u)\bar{g}(u).
\end{aligned}$$

As α_1, α_2 *appear in the odd part and* α_1, α_3 *appear in the even part, the method described above can be directly used with* C, D, E *replacing* F, G, H *to calculate the parameters of all stabilizing controllers of the form* $c(s) = \frac{s+\alpha_1}{\alpha_2 s+\alpha_3}$. △

Example 4.2 *We illustrate the details of the method on a fifth order plant. Consider a proper first-order controller to stabilize the plant* $g(s) = \frac{p(s)}{q(s)}$ *where*

$$\begin{aligned}
q(s) &= s^5 + 3s^4 + 29s^3 + 15s^2 - 3s + 60, \\
p(s) &= s^3 - 6s^2 + 2s - 1.
\end{aligned}$$

The roots of $q(s)$ *are* $\{-1.2576 \pm j5.1476,\ -1.5574,\ 0.5363 \pm j1.0414\}$ *and those of* $p(s)$ *are* $\{0.1606 \pm j0.3877, 5.6788\}$. *Using (3.9), we have*

$$\begin{aligned}
H(u) &= -u^4 - 49u^3 - 148u^2 - 369u - 60, \\
G(u) &= -9u^3 - 196u^2 - 101u - 117, \\
F(u) &= -u^3 + 32u^2 + 8u + 1.
\end{aligned}$$

The first step in the algorithm is to find values of α_1 *for which* $H_{2o}(u) + \alpha_1 G_{2o}(u)$ *has the necessary number of real negative roots. To this end we consider*

$$\phi_1(s, \alpha_1, \alpha_2) = H(s^2) + sH'(s^2) + \alpha_1[G(s^2) + sG'(s^2)] + \alpha_2[F(s^2) + sF'(s^2)].$$

As $gcd(F, F') = 1$*, we multiply* $\phi_1(s)$ *by* $p_1(-s) = F(s^2) - sF'(s^2)$. *Since* $deg\ \phi_1 - deg\ p_1 = 2$ *is even and* $deg\ \phi_1 - \sigma(p_1) = 8$*, the odd part of* $\psi_2(s)$ *must have at least 3 real negative roots. This lower bound is met only by values of* α_1 *in* $(-1.9251, 1.8190)$. *Now, we can fix* α_1 *and solve a constant gain stabilization problem by considering* $q_1(s)$ *and* $p_1(s)$ *of step 1.b in the algorithm to find admissible values of* α_2. *For these values of* α_2*, use step 1.b.ii to calculate admissible values of* α_3 *such that* $\phi_0(s) \in \mathcal{H}$. *With* $\alpha_1 = 1$ *and an increment of* 0.01 *of* α_2 *in step 1.b.iii, we obtain the stabilizing values of* (α_2, α_3) *shown in Figure 4.1. Figure 4.2 shows values of* (α_1, α_2) *for which* $H(u) + \alpha_1 G(u) + \alpha_2 F(u)$ *has all its roots real, negative, and distinct and Figure 4.3 shows the stabilizing set of* $(\alpha_1, \alpha_2, \alpha_3)$ *values.* •

4.3 The General Case

We now remove the restrictive assumption of the previous section and solve the problem for an arbitrary plant of a given degree [82]. Recall that

$$\begin{aligned}
\phi_0(s, \alpha_1, \alpha_2, \alpha_3) &= (s + \alpha_1)q(s) + (\alpha_2 s + \alpha_3)p(s) \\
&= q_0(s) + \alpha_3 p_0(s)
\end{aligned}$$

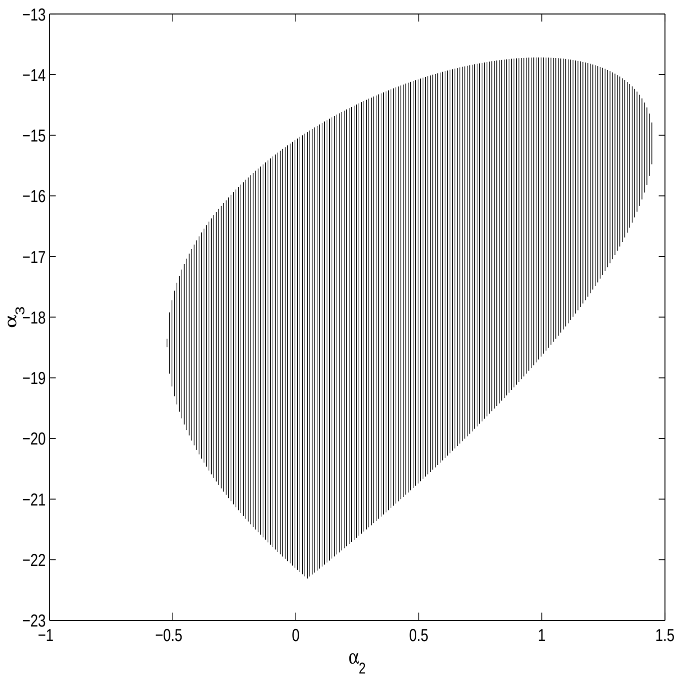

Figure 4.1: Stabilizing set of (α_2, α_3) values for $\alpha_1 = 1$ for Example 4.2.

where

$$\begin{aligned} q_0(s,\alpha_1,\alpha_2) &= (s+\alpha_1)q(s)+\alpha_2 sp(s), \\ p_0(s) &= p(s). \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} \psi_1(s,\alpha_1,\alpha_2,\alpha_3) &= \phi_0(s,\alpha_1,\alpha_2,\alpha_3)\bar{p}_0(-s) \\ &= s^2G(s^2)+\alpha_1 H(s^2)+\alpha_3 F(s^2) \\ &\quad +s[H(s^2)+\alpha_1 G(s^2)+\alpha_2 F(s^2)]. \end{aligned} \tag{4.6}$$

The reasoning behind the algorithm which determines the set of parameters $\alpha_1,\ \alpha_2,\ \alpha_3$ of a stabilizing first-order controller can be explained as follows. Suppose $\phi_0(s)$ is Hurwitz stable for some $\alpha_1,\ \alpha_2,\ \alpha_3 \in \mathbf{R}$. By Remark 3.3, it follows that the odd part $H(u)+\alpha_1 G(u)+\alpha_2 F(u)$ of $\psi_1(s)$ has at least $r_1 = \lfloor \frac{n-\sigma(p_0)}{2} \rfloor$ real negative roots with odd multiplicities. Suppose $H(u)+\alpha_1 G(u)+\alpha_2 F(u)$ has r_1 real negative roots with odd multiplicities. By Lemma 4.1, $\sigma[\phi_1(s)] = 2r_1$, where

$$\begin{aligned} \phi_1(s,\alpha_1,\alpha_2) &= H_1(s)+\alpha_1 G_1(s)+\alpha_2 F_1(s) \\ &= q_1(s)+\alpha_2 p_1(s) \end{aligned} \tag{4.7}$$

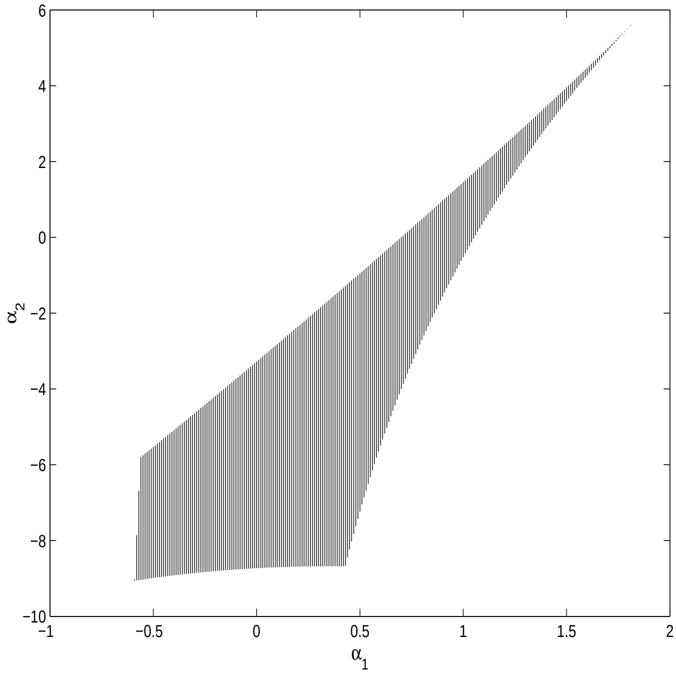

Figure 4.2: Values of (α_1, α_2) for which the odd part has all its roots real, negative, and distinct for Example 4.2.

and

$$\begin{aligned} H_1(s) &= H(s^2) + sH'(s^2), \\ G_1(s) &= G(s^2) + sG'(s^2), \\ F_1(s) &= F(s^2) + sF'(s^2), \\ q_1(s, \alpha_1) &= H_1(s) + \alpha_1 G_1(s), \\ p_1(s) &= F_1(s). \end{aligned}$$

In order to find the suitable ranges of α_1 and α_2, we modify $\phi_1(s, \alpha_1, \alpha_2)$ as follows. Let $B := gcd\{F, F'\}$ so that $F = B\bar{F}$, $F' = B\tilde{F}'$ [1] for coprime polynomials $\bar{F}, \tilde{F}' \in \mathbf{R}[u]$. Also let $\bar{p}_1(s) := \bar{F}(s^2) + s\tilde{F}'(s^2)$. By a simple computation, it follows that

$$\begin{aligned} \psi_2(s, \alpha_1, \alpha_2) = \phi_1(s, \alpha_1, \alpha_2)\bar{p}_1(-s) &= H_{2e}(s^2) + \alpha_1 G_{2e}(s^2) + \alpha_2 F_{2e}(s^2) \\ &\quad + s[H_{2o}(s^2) + \alpha_1 G_{2o}(s^2)], \end{aligned}$$

[1] The prime notation is still kept in $\tilde{F}'$ although strictly speaking, $\tilde{F}'$ is not the derivative of any of the polynomials above.

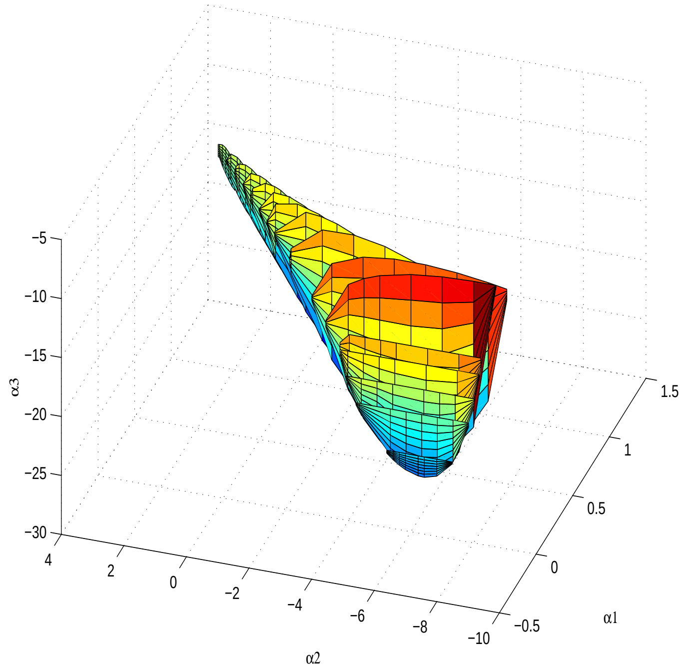

Figure 4.3: Stabilizing set of $(\alpha_1, \alpha_2, \alpha_3)$ values for example 4.2.

where

$$
\begin{aligned}
H_{2e}(u) &= H(u)\bar{F}(u) - uH'(u)\tilde{F}'(u),\\
G_{2e}(u) &= G(u)\bar{F}(u) - uG'(u)\tilde{F}'(u),\\
F_{2e}(u) &= F(u)\bar{F}(u) - uF'(u)\tilde{F}'(u),\\
H_{2o}(u) &= H'(u)\bar{F}(u) - H(u)\tilde{F}'(u),\\
G_{2o}(u) &= G'(u)\bar{F}(u) - G(u)\tilde{F}'(u).
\end{aligned}
\tag{4.8}
$$

Once more by Remark 3.3, since $\sigma[\phi_1(s)p_1(-s)] = 2r_1 - \sigma[p_1(s)]$ the odd part of $\phi_1(s)\bar{p}_1(-s)$ should have at least $r_2 = \lfloor \frac{|2r_1 - \sigma(p_1)| - 1}{2} \rfloor$ real negative roots with odd multiplicities . Now the set of $\alpha_1 \in \mathbf{R}$ which achieves r_2 real negative roots with odd multiplicities in $H_{2o}(u) + \alpha_1 G_{2o}(u)$ can be determined by applying Algorithm 3.2 to

$$
\begin{aligned}
q_2(s) &= H_2(s) = H_{2o}(s^2) + sH'_{2o}(s^2),\\
p_2(s) &= G_2(s) = G_{2o}(s^2) + sG'_{2o}(s^2).
\end{aligned}
$$

The algorithm below traces the above steps backwards by repetition of the steps (i)-(iii) below:

(i) Pick a value of α_1 such that the number of real negative roots with odd multiplicities of $H_{2o}(u) + \alpha_1 G_{2o}u)$ is r_2 or greater.
(ii) Determine using Algorithm 3.2 all $\alpha_2 \in \mathbf{R}$ such that $\sigma[\phi_1(s)] = 2r_1$. By Lemma 4.1 and Remark 4.2, this is equivalent to determining values of α_2 such that $H(u) + \alpha_1 G(u) + \alpha_2 F(u)$ has r_1 real negative roots with odd multiplicities.
(iii) For every α_2 determined, find using Algorithm 3.2 again, all α_3 such that $\phi_1(s)$ is Hurwitz stable.

Algorithm 4.2 1. *Partition the real axis into intervals (or union of intervals) such that the number of real negative roots with odd multiplicities of $H_{2o}(u) + \alpha_1 G_{2o}(u)$ is constant in each interval.*

2. *Fix $r_1 = \lfloor \frac{n-\sigma(p_0)}{2} \rfloor$.*
   (a) *Find admissible range of α_1 from the intervals found in the first step.*
      i. *Fix an α_1 in the admissible range.*
      ii. *Apply Algorithm 3.2 to $q_1(s)$ and $p_1(s)$. (This calculates admissible values of α_2 such that $H(u)+\alpha_1 G(u)+\alpha_2 F(u)$ has r_1 real negative roots with odd multiplicities.)*
         A. *Fix an α_2 from the range determined in 2.a.ii.*
         B. *Apply Algorithm 3.2 to $q_0(s)$ and $p_0(s)$. (This calculates all admissible values of α_3 such that $\phi_0(s)$ is in $\mathcal{H}$.)*
         C. *Increment α_2 and go to step 2.a.ii.B.*
      iii. *Increment α_1 and go to step 2.a.ii.*
   (b) *If $r_1 <$ deg(H), then increment r_1 by one and go to step 2.a.*

Once again, Algorithm 3.2 is used on three auxiliary plants given by (4.4) to obtain the admissible values of $(\alpha_1,\ \alpha_2,\ \alpha_3)$.

Remark 4.2 *Lemma 4.1 gives a signature condition to count the number of distinct real negative roots, whereas in step 2.a.ii of the above algorithm we employ Theorem 3.2 to ensure a certain signature for $\phi_2(s)$. This way, the Algorithm 3.2 does not distinguish those parameters that ensures real negative roots of odd multiplicities. However, Algorithm 3.2 misses only a finite number of parameter values for the following reason: If $H(u) + \alpha_1 G(u) + \alpha_2 F(u)$ has a real negative root u_0 of even multiplicity, then u_0 is also a root of $H'(u) + \alpha_1 G'(u) + \alpha_2 F'(u)$ with odd multiplicity. This corresponds to a conjugate pair of roots (with odd multiplicity) of $\phi_2(s)$ on the jw-axis. Values of α_2 leading to this situation are excluded from the solution set by Algorithm 3.2. If $H(u) + \alpha_1 G(u) + \alpha_2 F(u)$ has a real negative root u_1 with odd multiplicity (not a simple root), then $\phi_2(s)$ has a conjugate pair of roots (with even multiplicity) on the jw-axis. We can easily modify step 3 in Algorithm 3.2 such that values of α_2 leading to the latter situation are included in the solution set. The modification consists of including (instead of excluding) the finite set of points $\hat{A}$ in step 3 of Algorithm 3.2.* $\triangle$

Example 4.3 *Consider determining proper first-order controllers to stabilize the plant $g(s) = \frac{p(s)}{q(s)}$, where*

$$q(s) = s^5 + 3s^4 + 29s^3 + 15s^2 - 3s + 60,$$
$$p(s) = s^3 - 6s^2 + 2s + 1.$$

The roots of $q_0(s)$ are $\{-1.2576 \pm j5.1476, \ -1.5574, \ 0.5363 \pm j1.0414\}$ and those of $p_0(s)$ are $\{-0.2705, \ 0.6587, \ 5.6119\}$ so that this is an unstable and non-minimum phase plant. Using (3.9), we have

$$H(u) = -u^4 - 49u^3 - 142u^2 - 339u + 60,$$
$$G(u) = -9u^3 - 194u^2 - 43u - 123,$$
$$F(u) = -u^3 + 32u^2 - 16u + 1.$$

A necessary condition for the existence of a stabilizing first-order controller is that $H(u) + \alpha_1 G(u) + \alpha_2 F(u)$ has at least $r_1 = \lfloor \frac{n-\sigma(p_0)}{2} \rfloor = 3$ real negative roots *with odd multiplicities. As* $gcd(F, F') = 1$*, we multiply* $\phi_1(s)$ *by* $p_1(-s)$*. For* $r_1 = 3$*,* $\sigma(\phi_1) - \sigma(p_1) = 6$ *and the odd part of* $\phi_1(s)p_1(-s)$ *must have at least* $r_2 = \lfloor \frac{|2r_1 - \sigma(p_2)| - 1}{2} \rfloor = 2$ *real negative roots with odd multiplicities. Using Algorithm 3.2,* $\alpha_1 \in (-2.2917, 0.3088)$*. Similarly, for* $r_1 = 4$*, we find* $r_2 = 3$ *and* $\alpha_1 \in (0.3088, 3.6000)$*. Now let us follow the steps of Algorithm 4.2 for a fixed value of* α_1 *from the above intervals. For* $\alpha_1 = 1$*, we have*

$$\begin{aligned} q_1(s) &= -s^8 - 4s^7 - 58s^6 - 174s^5 - 336s^4 - 672s^3 - 382s^2 - 382s - 63, \\ p_1(s) &= -s^6 - 3s^5 + 32s^4 + 64s^3 - 16s^2 - 16s + 1. \end{aligned}$$

Using step 2.a.ii in Algorithm 4.2, the range of admissible values of α_2 *for which* $H(u) + \alpha_1 G(u) + \alpha_2 F(u)$ *has 4 negative roots is* $\alpha_2 \in (-3.1602, 1.3297)$*. With* $\alpha_2 = 1$*, we obtain*

$$\begin{aligned} q_0(s) &= s^6 + 4s^5 + 33s^4 + 38s^3 + 14s^2 + 58s + 60, \\ p_0(s) &= s^4 - 6s^3 + 2s + 1. \end{aligned}$$

Step 2.a.ii.B in Algorithm 4.2 gives the following solution $\alpha_3 \in (-17.0988, -11.5621)$ *for* $\alpha_1 = \alpha_2 = 1$ *. Application of Algorithm 4.2, with a* 0.05 *increment of* α_2 *in step 2.a.ii.C and a* 0.1 *increment of* α_1 *in step 2.a.iii, results in the set of stabilizing* $(\alpha_1, \alpha_2, \alpha_3)$ *values shown in figure 4.4.* •

Remark 4.3 *The method can also be applied to discrete time plants using a bilinear transformation of the complex plane. Let the controller transfer function be*

$$c(z) = \frac{\alpha_2 z + \alpha_3}{\alpha_1 z + 1}.$$

By the bilinear transformation $z = \frac{w+1}{w-1}$*, we get*

$$c(w) = \frac{(\alpha_2 + \alpha_3)w + (\alpha_2 - \alpha_3)}{(\alpha_1 + 1)w + (\alpha_1 - 1)}.$$

For a $c(w)$ *in this form,* α_1*,* α_2*, and* α_3 *appear both in the even and odd parts of* $\psi(w, \alpha_1, \alpha_2, \alpha_3) = \phi(w, \alpha_1, \alpha_2, \alpha_3)\bar{p}(-w)$*. Let* $\bar{\alpha}_2 = \alpha_2 + \alpha_3$ *and* $\bar{\alpha}_3 = \alpha_2 - \alpha_3$*. By*

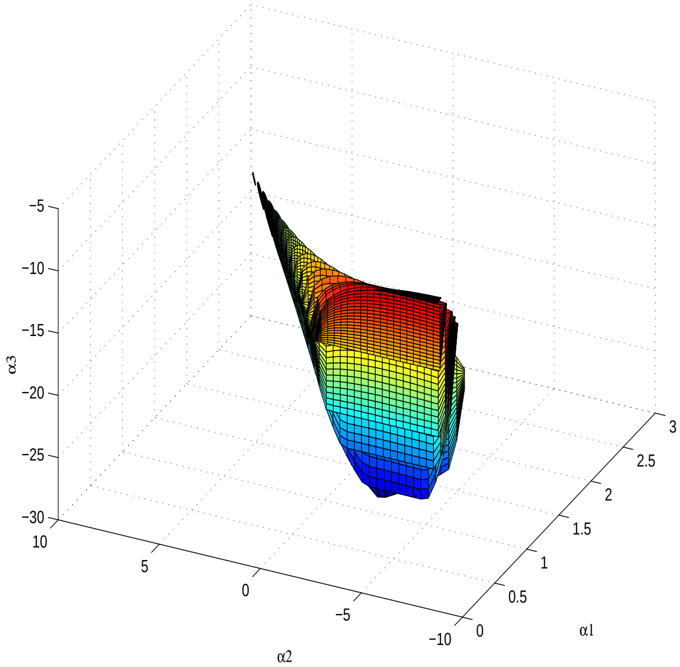

Figure 4.4: Stabilizing set of $(\alpha_1, \alpha_2, \alpha_3)$ values for Example 1.

a simple computation, it follows that

$$\begin{aligned}\psi(w, \alpha_1, \bar{\alpha}_2, \bar{\alpha}_3) &= w^2G(w^2) - H(w^2) + \alpha_1[w^2G(w^2) + H(w^2)] + \bar{\alpha}_3F(w^2) \\ &\quad + w[H(w^2) - G(w^2) + \alpha_1(H(w^2) + G(w^2)) + \bar{\alpha}_2F(w^2)].\end{aligned}$$

Stabilizing controller parameters $\alpha_1, \bar{\alpha}_2, \bar{\alpha}_3$ *and* $\alpha_2 = \frac{\bar{\alpha}_2+\bar{\alpha}_3}{2}$, $\alpha_3 = \frac{\bar{\alpha}_2-\bar{\alpha}_3}{2}$ *are thus obtained. The method hence applies to discrete time plants of arbitrary order.* $\triangle$

Remark 4.4 *If linear programming is used, then it is possible to extend the algorithm to cover PID controllers. Let*

$$c(s) = \frac{\alpha_1 s^2 + \alpha_2 s + \alpha_3}{s + \alpha_4}$$

so that

$$\begin{aligned}\psi_1(s, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &= s^2G(s^2) + \alpha_1 s^2F(s^2) + \alpha_3F(s^2) + \alpha_4H(s^2) \\ &\quad + s[H(s^2) + \alpha_2F(s^2) + \alpha_4G(s^2)].\end{aligned}$$

Applying the steps 1 and 2.a.ii of the first-order controller algorithm to appropriate polynomials, one first finds all admissible values of α_2, α_4. *Then, step 2.a.ii.B should be modified to determine values of* α_1, α_3 *using linear programming. Note that this controller specializes to a proportional controller for* $\alpha_1 = \alpha_3 = \alpha_4 = 0$, *PI controller for* $\alpha_1 = \alpha_4 = 0$, *PD controller for* $\alpha_3 = \alpha_4 = 0$, *PID controller for* $\alpha_4 = 0$, *and to a first-order controller for* $\alpha_1 = 0$.

*By the same amount of effort, second order, type-*1 *controllers of the form*

$$c(s) = \frac{\alpha_1 s^2 + \alpha_2 s + \alpha_3}{s(s+\alpha_4)}. \tag{4.9}$$

can also be determined. Such a controller applied to $g_(s)$ *gives*

$$\begin{aligned}\psi_1(s,\alpha_1,\alpha_2,\alpha_3,\alpha_4) &= s^2H(s^2) + \alpha_4 s^2 G(s^2) + \alpha_1 s^2 F(s^2) + \alpha_3 F(s^2) \\ &\quad + s[s^2G(s^2) + \alpha_4 H(s^2) + \alpha_2 F(s^2)],\end{aligned}$$

to which the algorithm is applicable. Note that (4.9) is a realizable (proper) PID controller for large positive values of α_4. △

Remark 4.5 *Let us assume that* $n = deg\ q > m = deg\ p$ *and identify the possibilities of obtaining infinite ranges for the stabilizing values of* $(\alpha_1, \alpha_2, \alpha_3)$.
Case 1: *Infinite range for* α_1. *The characteristic polynomial of the closed-loop system can be written as*

$$\begin{aligned}\psi(s) &= sq(s) + (\alpha_2 s + \alpha_3)p(s) + \alpha_1 q(s), \\ &= \tilde{q}(s) + \alpha_1 \tilde{p}(s),\end{aligned}$$

where

$$\begin{aligned}\tilde{q}(s) &= sq(s) + (\alpha_2 s + \alpha_3)p(s), \\ \tilde{p}(s) &= q(s).\end{aligned}$$

Using the fact that deg $\tilde{q} -$ *deg* $\tilde{p} = 1$ *and Remark 3.2,* α_1 *can have an infinite stabilizing range only if* $q(s)$ *has no roots on* $\mathbf{C}_+$.

Case 2: *Infinite range for α_2. The characteristic polynomial of the closed-loop system can be written as*

$$\begin{aligned}\psi(s) &= (s+\alpha_1)q(s)+\alpha_3 p(s)+\alpha_2 sp(s),\\ &= \tilde{q}(s)+\alpha_1\tilde{p}(s),\end{aligned}$$

where

$$\begin{aligned}\tilde{q}(s) &= (s+\alpha_1)q(s)+\alpha_3 p(s),\\ \tilde{p}(s) &= sp(s).\end{aligned}$$

Using the fact that deg $\tilde{q}$ − deg $\tilde{p} \geq 1$ and Remark 3.2, α_2 can have an infinite stabilizing range only if

$$\begin{cases} deg\ q - deg\ p = 1, 2 \\ \& \\ p(s)\ has\ no\ roots\ on\ \mathbf{C}_+. \end{cases}$$

Case 3: *Infinite range for α_3. The characteristic polynomial of the closed-loop system can be written as*

$$\begin{aligned}\psi(s) &= (s+\alpha_1)q(s)+\alpha_2 sp(s)+\alpha_3 p(s),\\ &= \tilde{q}(s)+\alpha_1\tilde{p}(s),\end{aligned}$$

where

$$\begin{aligned}\tilde{q}(s) &= (s+\alpha_1)q(s)+\alpha_2 sp(s),\\ \tilde{p}(s) &= p(s).\end{aligned}$$

Using the fact that deg $\tilde{q}$ − deg $\tilde{p} \geq 2$ and Remark 3.2, α_1 can have an infinite stabilizing range only if

$$\begin{cases} deg\ q - deg\ p = 1 \\ \& \\ p(s)\ has\ no\ roots\ on\ \mathbf{C}_+. \end{cases}$$

Infinite stabilizing ranges of $(\alpha_1, \alpha_2, \alpha_3)$ *causes problems in applying Algorithm 4.2, as we have to sweep over infinite ranges. However, by the above observations, this happens only in case deg* $q -$ *deg* $p = 1, 2$ *and* $p(s)$ *and* $q(s)$ *have all roots in* $\mathbf{C}_-$. *Note that in such a situation an infinite set of stabilizing first-order controllers exist. This can be seen from the fact that placing the zero and the pole of the controller anywhere in the left-half plane, there always exists a value of* α_2 *such that the closed-loop system is stable. In this case, we can solve the alternative problem of placing the roots of the closed-loop system in a new restricted stability region. This problem is solved in Section 4.5. In this way, in addition to avoiding the infinite ranges of the controller parameters, we solve the more realistic problem of stabilizing and achieving a desired performance for the step response of the closed-loop system.* $\triangle$

Remark 4.6 *Remark 3.3 gives only a necessary condition for the existence of a solution. Inherently this leads to some disadvantages. Not all values of of* $\alpha_1 \in I_1$ *found in step 1 of Algorithm 4.2 are stabilizing values. In order to reduce the effect of this disadvantage to a minimum, we can apply similar arguments to the even part* $s^2G(s^2) + \alpha_1 H(s^2) + \alpha_3 F(s^2)$ *of* $\psi_1(s)$. *This will give another interval* $\alpha_1 \in I_2$. *In addition, with*

$$\phi_0(s) = s^2g(s^2)+\alpha_1h(s^2)+\alpha_2s^2e(s^2)+\alpha_3f(s^2)+s[h(s^2)+\alpha_1g(s^2)+\alpha_2f(s^2)+\alpha_3e(s^2)]$$

all the roots of the even and odd parts must be real, negative, and distinct. Using similar arguments, we can compute two new intervals I_3 *and* I_4. *Hence* $\alpha_1 \in I_1 \cap I_2 \cap I_3 \cap I_4$. *Finally, in Algorithm 4.2 we first compute* α_1, *then* α_2 *and at last* α_3. *The order in which the computation of* α_i*'s is done can be changed and this can be seen from (4.6).* $\triangle$

4.4 Design Example

In this section, we give a design example. Using the characterization of all stabilizing first-order controllers, we can evaluate the performance of the closed-loop system with respect to controller parameters. Several time domain performance specifications such as overshoot, rise time, settling time, and steady-state error can be evaluated. In addition, H_∞ and H_2 norms of some closed-loop transfer function can be minimized over the set of all stabilizing parameters of the first-order controller. Before proceeding any further, we first present some standard H_∞ and H_2 designs.

For comparison reasons, we consider the following example given in [70]. Let

$$G(s) = \frac{s-1}{s^2 + 0.8s - 0.2}$$

be the transfer function of the plant to be stabilized. Note that this plant has a pole and a zero in the right-half of the complex plane. In [70], an optimal H_∞ robust controller was designed to minimize $||WT||_\infty$, where $W(s)$ is a high-pass filter given by

$$W(s) = \frac{s+0.1}{s+1},$$

and $T(s)$ is the complementary sensitivity function. The authors also designed a controller that minimizes $||WGS||_2$ where $S(s)$ is the sensitivity function. The aim of the latter design is to minimize the H_2 norm of a weighted transfer function from a disturbance input to the output. Both of these designs were then compared to the performance of PI controller.

Using YJBK parameterization, all proper controllers which stabilize the plant were found [70]. Then, the parameter $Q(s)$ was selected to minimize $||WT||_\infty$. The optimal value is

$$\begin{aligned} v_{opt} &= \inf_{Q(s)\ \text{stable}} ||WT||_\infty \\ &= 0.375 \end{aligned}$$

where

$$Q(s) = \frac{-5(s+1)(0.075s - 0.195)}{s+0.1}.$$

As $Q(s)$ is not proper, it was divided by $\tau s + 1$ where $\tau = 0.01$ to give the sub-optimal controller

$$c(s) = \frac{-39.3s^3 - 114.48s^2 - 112.68s - 37.5}{s^3 + 141.6s^3 + 275s + 137.5}.$$

With this controller the minimum is

$$||WT||_\infty = 0.391.$$

For the H_2 minimization problem, the same $Q(s)$ was obtained, namely

$$Q(s) = \frac{-5(s+1)(0.075s - 0.195)}{s+0.1}$$

and the minimum value is

$$\begin{aligned} v_{opt} &= \inf_{Q(s) \text{ stable}} ||WGS||_2 \\ &= 0.972. \end{aligned}$$

Repeating the same procedure to make $Q(s)$ proper, the following controller was obtained

$$c(s) = \frac{-39.3s^3 - 114.48s^2 - 112.68s - 37.5}{s^3 + 141.6s^3 + 275s + 137.5}$$

and the minimum value is

$$||WGS||_2 = 0.973.$$

Using a first-order controller of the form

$$c(s) = \frac{\alpha_2 s + \alpha_3}{s + \alpha_1}$$

we can study the transient response of the closed-loop system. In order to minimize the steady state error to ramp inputs, we chose $\alpha_1 = 0.005$ so that the

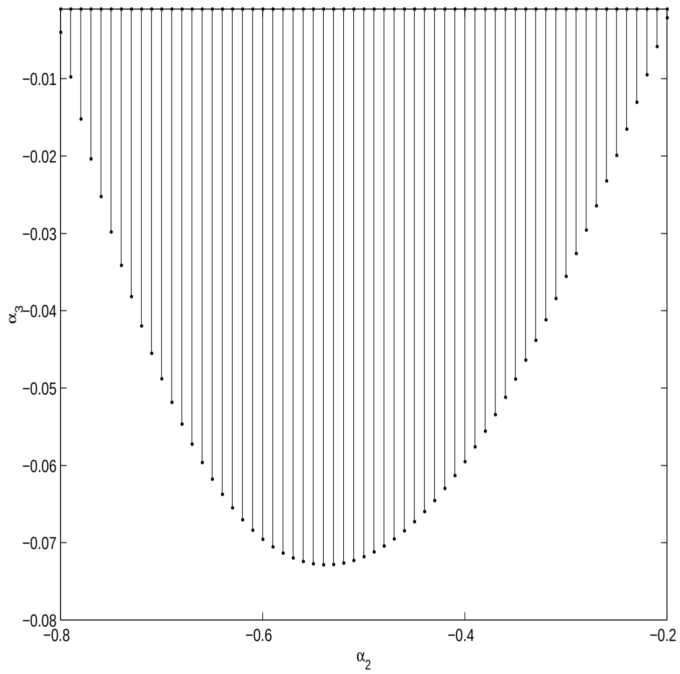

Figure 4.5: Stabilizing set of (α_2, α_3) values for $\alpha_1 = 0.005$.

controller has a pole close to the origin and this controller behaves like a PI controller. Figure 4.5 shows the stabilizing values of (α_2, α_3) for $\alpha_1 = 0.005$.

In Figure 4.6, the plot of $||WT||_\infty$ versus stabilizing values of (α_2, α_3) is given. The minimum value of $||WT||_\infty$ is 0.578 obtained at $\alpha_2 = -0.25$ and $\alpha_3 = -0.002$. Figure 4.7 shows the plot of $||WGS||_2$ for which the minimum is 1.054 obtained at $\alpha_2 = -0.3$ and $\alpha_3 = -0.002$. Hence, we can evaluate the performance achievable by this fixed-order and fixed-structure controller.

Fixing $\alpha_1 = 0.005$ and using the stabilizing values of (α_2, α_3), we can obtain the plots of several time domain performance specifications versus the stabilizing parameters of the controller.

- Overshoot: Figure 4.8 shows the plot of the percent maximum overshoot over stabilizing values of (α_2, α_3). The minimum percent maximum overshoot is 20.8% obtained at $\alpha_2 = -0.45$ and $\alpha_3 = -0.002$.

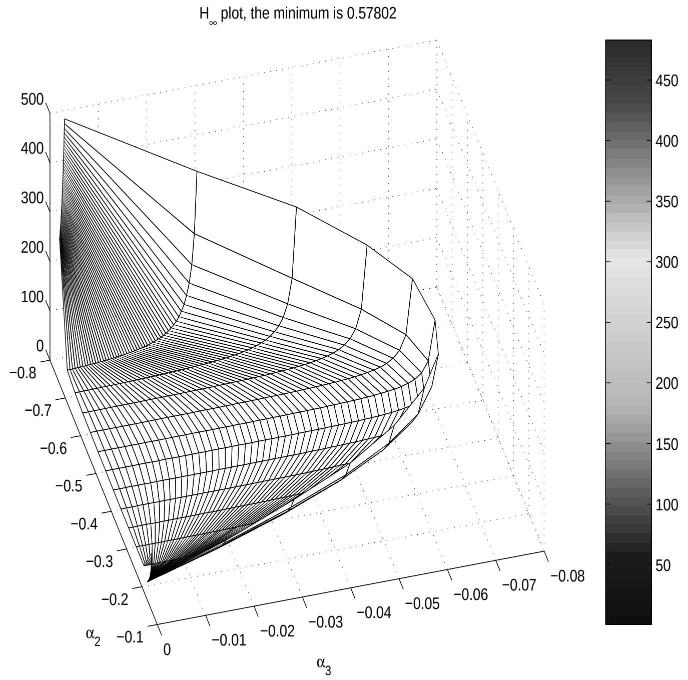

Figure 4.6: H_∞ norm of $W(s)T(s)$, minimum occurs at $\alpha_2 = -0.25$ and $\alpha_3 = -0.002$.

- Settling time: Figure 4.10 shows the plot of the settling time over stabilizing values of (α_2, α_3). The minimum settling time is $19.6s$ obtained at $\alpha_2 = -0.4$ and $\alpha_3 = -0.002$.

- Rise time: Figure 4.12 shows the plot of the rise time over stabilizing values of (α_2, α_3). The minimum rise time is $2.5s$ obtained at $\alpha_2 = -0.75$ and $\alpha_3 = -0.0272$.

- Steady state error: Figure 4.14 shows the plot of the percent steady state error over stabilizing values of (α_2, α_3). The minimum percent steady state error is 0.85% obtained at $\alpha_2 = -0.4$ and $\alpha_3 = -0.0562$.

We can alternatively generate the level curves for the different time domain performance indices, see Figures 4.9, 4.11, 4.13, and 4.15. Suppose that we are given the following performance specifications:

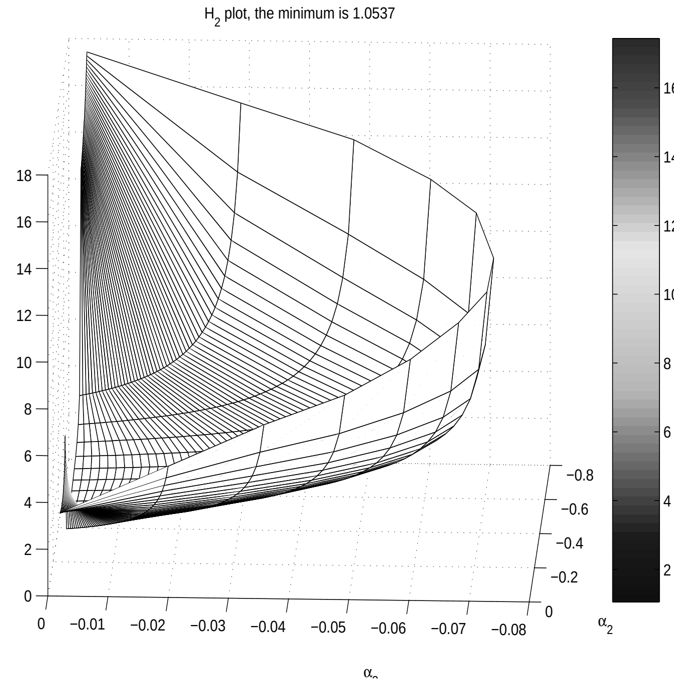

Figure 4.7: H_2 norm of $W(s)G(s)S(s)$, minimum occurs at $\alpha_2 = -0.3$ and $\alpha_3 = -0.002$.

- Percent overshoot is less than 25%.
- Settling time is less than or equal to $25s$.

By superimposing the level curves of the settling time and percent overshoot, we can determine whether a stabilizing controller satisfying these requirement exists or not.

Figures 4.16 through 4.18 shows the step responses for several values of α_2 and α_3. In Figure 4.16, the values of the stabilizing controller parameters are chosen randomly to be $\alpha_2 = -0.2$ and $\alpha_3 = -0.002$. Figure 4.17 shows the step response with the controller that leads to the minimum settling time and Figure 4.18 shows the step response of the controller that leads to minimum percent steady state error.

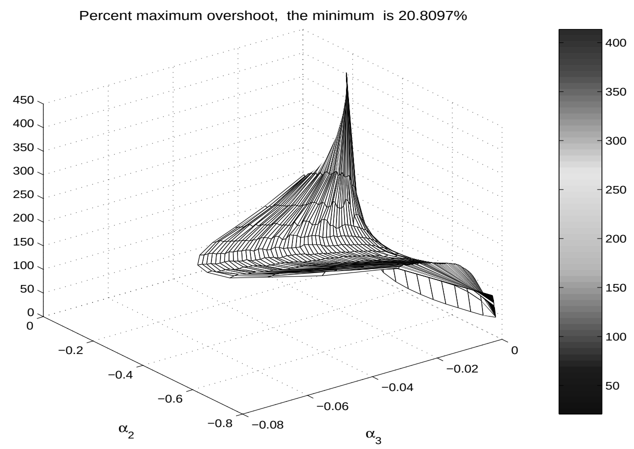

Figure 4.8: Overshoot, the minimum occurs at $\alpha_2 = -0.45$ and $\alpha_3 = -0.002$.

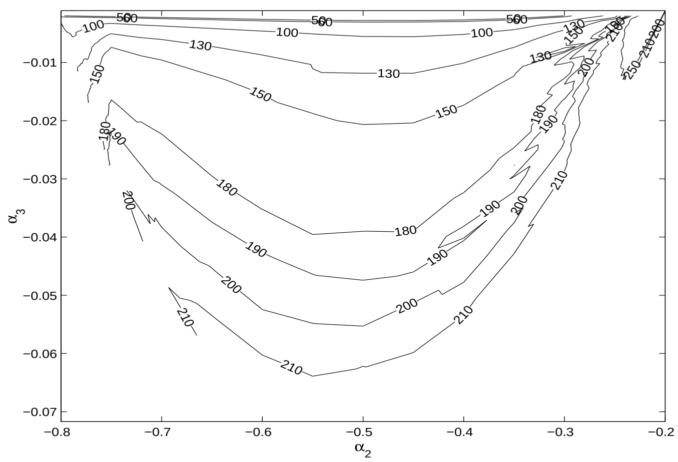

Figure 4.9: Overshoot level curves.

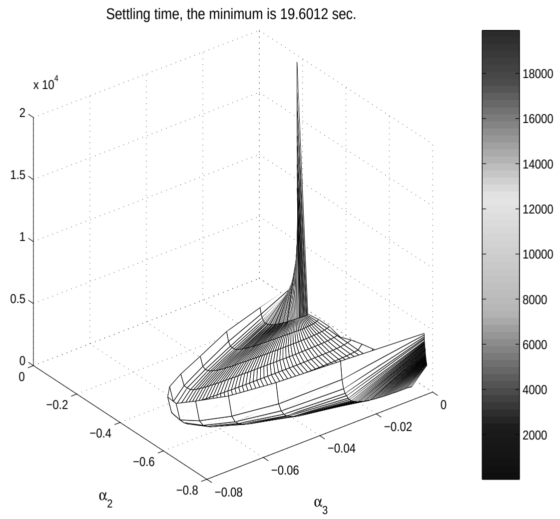

Figure 4.10: Settling time, the minimum occurs at $\alpha_2 = -0.4$ and $\alpha_3 = -0.002$.

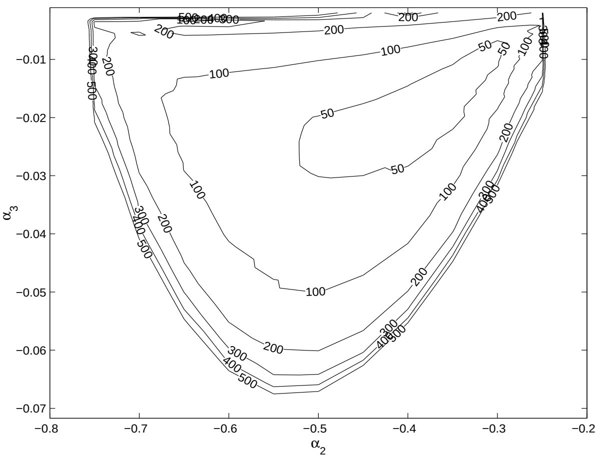

Figure 4.11: Settling time level curves.

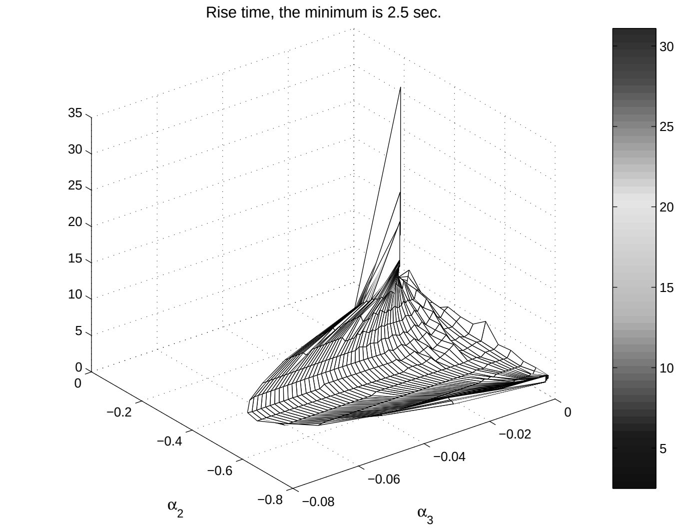

Figure 4.12: Rise time, the minimum occurs at $\alpha_2 = -0.75$ and $\alpha_3 = -0.0272$.

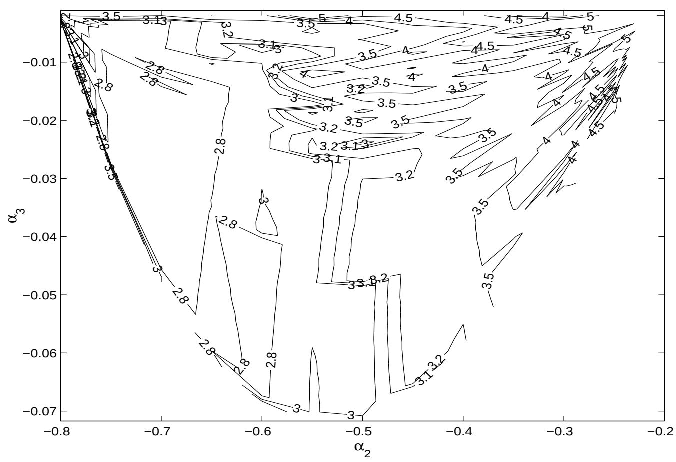

Figure 4.13: Rise time level curves.

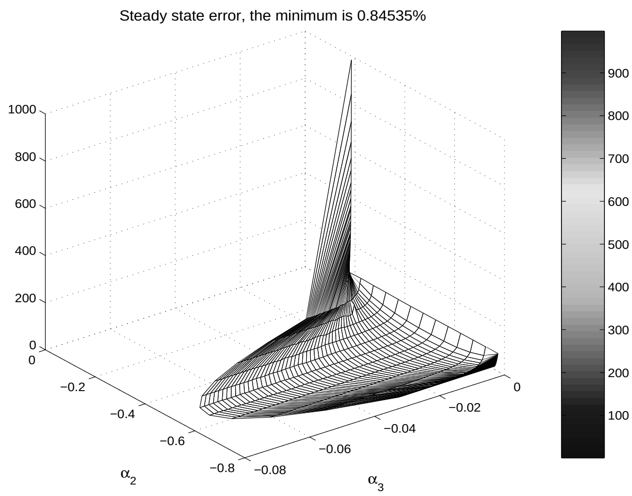

Figure 4.14: Steady state error, the minimum occurs at $\alpha_2 = -0.4$ and $\alpha_3 = -0.0562$.

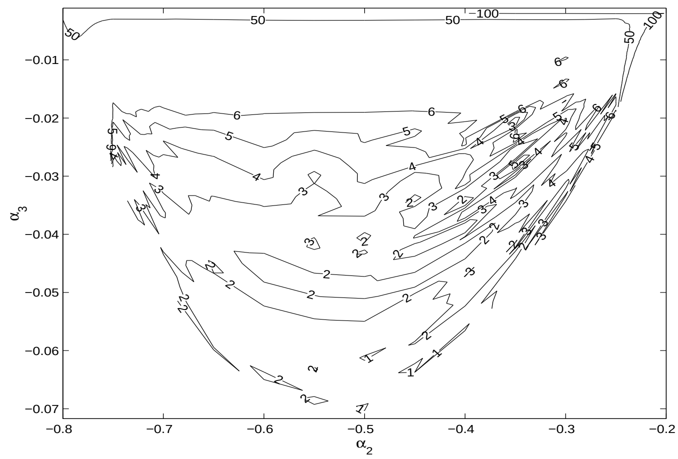

Figure 4.15: Steady state error level curves.

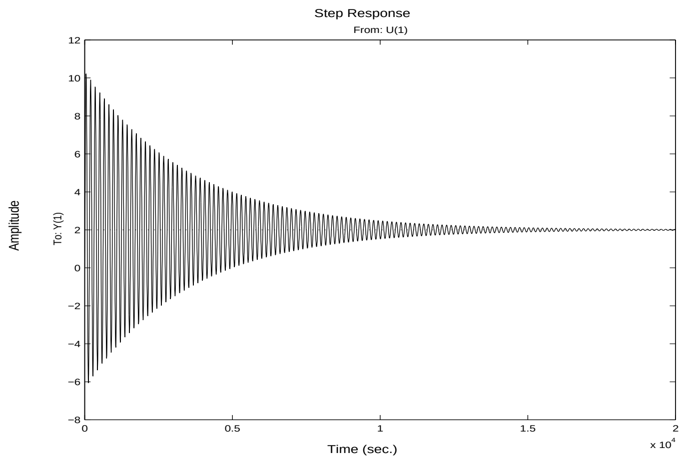

Figure 4.16: Step response using $\alpha_2 = -0.2$ and $\alpha_3 = -0.002$.

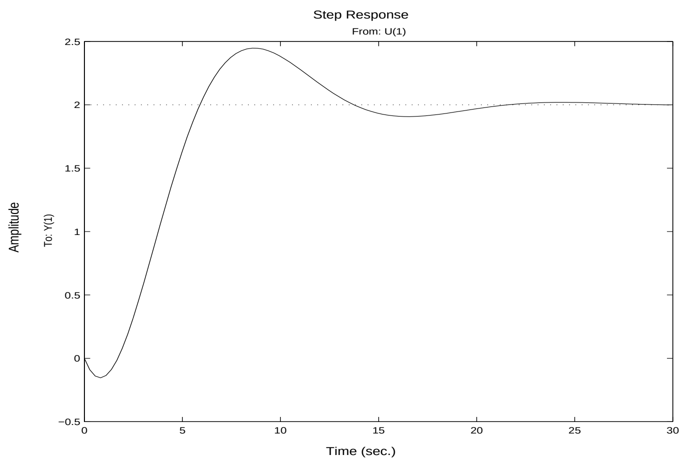

Figure 4.17: Step response using $\alpha_2 = -0.4$ and $\alpha_3 = -0.002$.

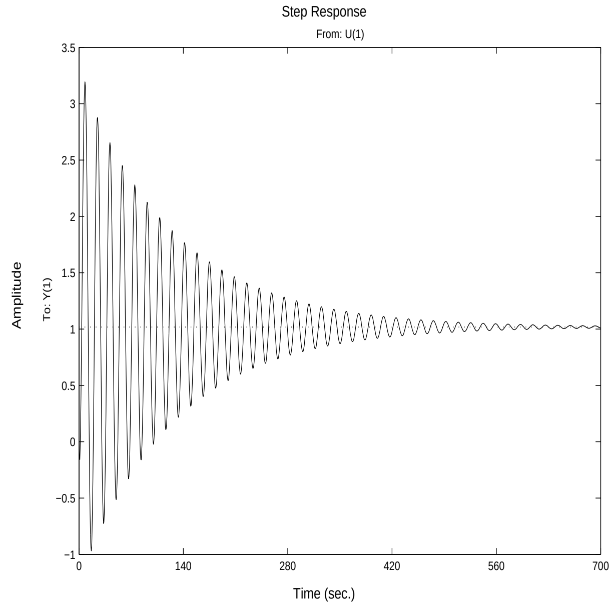

Figure 4.18: Step response using $\alpha_2 = -0.4$ and $\alpha_3 = -0.0562$.

4.5 Stabilizing First-order Controllers with Desired Stability Region

In many applications, stability of the closed-loop system is not enough, and usually it is required that the poles of the closed-loop system lie in a more restrictive stability regions. In this section, we use the generalized Hermite-Biehler theorem applicable to polynomials with complex coefficients and Lemma 2.4 to solve the problem of determining stabilizing first-order controllers that place the poles of the closed-loop system in a desired stability region. It is known that time domain specifications for a closed-loop system can be translated into desired closed-loop pole locations in the frequency domain. These are specified in terms of the damping ratio and damped natural frequency of the closed-loop poles. A desired stability region S in the complex plane is shown in Figure 4.19. The region S is the intersection of three regions $S_{-\gamma}$, S_{θ}, and $S_{-\theta}$ where

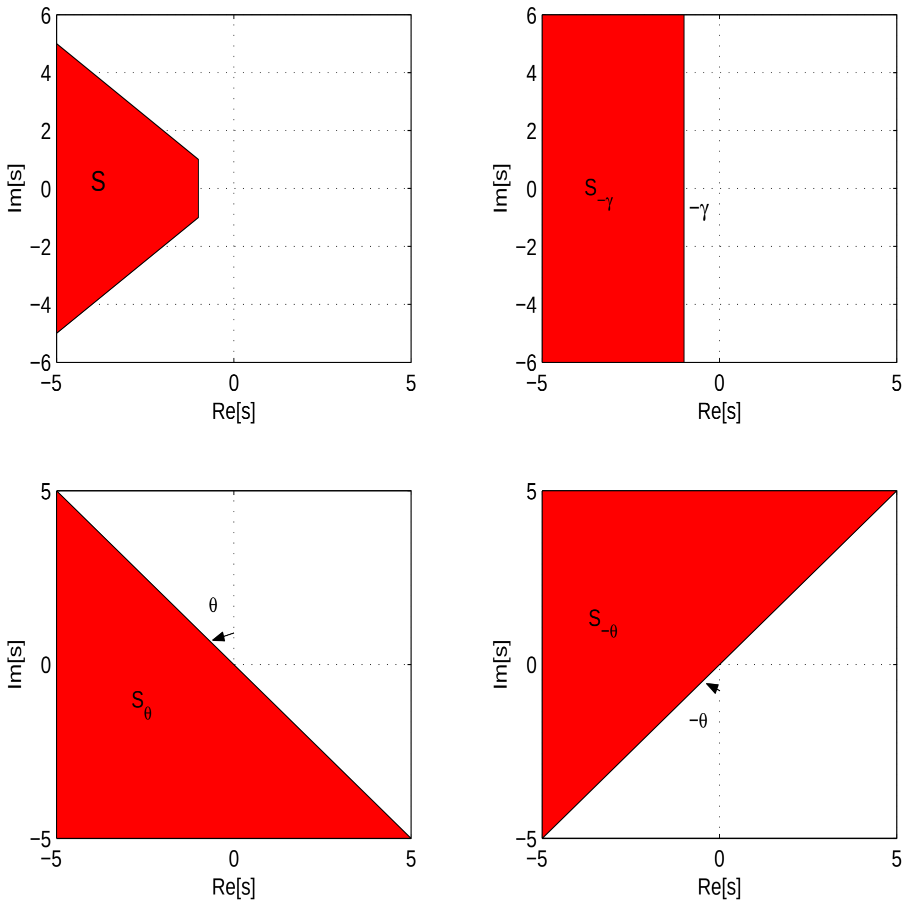

Figure 4.19: Stability region S.

- $S_{-\gamma} := \{s : s \in \mathbf{C},\ Re[s] < -\gamma\}$.
- $S_{\theta} := \{s : s \in \mathbf{C},\ Re[se^{-j\theta}] < 0\}$.
- $S_{-\theta} := \{s : s \in \mathbf{C},\ Re[se^{j\theta}] < 0\}$.

$S_{-\gamma}$ is a shifted Hurwitz stability region, S_θ and $S_{-\theta}$ are rotated Hurwitz stability regions. In [83], it is stated that if if all the poles of the closed-loop system lie in the region S, then the step response of the compensated system exhibits a settling time of no more than $4/\gamma$ and a maximum overshoot corresponding to the angle θ. In [84], the region S is approximated by a circular region and a design procedure that combines linear-quadratic optimal control with regional pole placement is given. See also [85]-[94] for different methods of regional pole placement with static full-state feedback controllers. Recently, a method for determining the set of all proportional controllers that places the closed-loop poles in the region S was given in [25]. Note, however, that using root-locus techniques the same problem can be solved as shown in [95]. In what follows we give a method to determine the set of all low-order dynamic controllers that places the poles of the closed-loop system in the region S.

Given a plant $g(s) = \frac{p(s)}{q(s)}$ and a first-order controller $c(s) = \frac{\alpha_2 s+\alpha_3}{s+\alpha_1}$, our objective is to find all values of $(\alpha_1, \alpha_2, \alpha_3)$ such that the closed-loop characteristic polynomial

$$\phi(s, \alpha_1, \alpha_2, \alpha_3) = (s + \alpha_1)q(s) + (\alpha_2 s + \alpha_3)p(s)$$

has all its roots in the region S. This is equivalent to solving three subproblems using the stability regions $S_{-\gamma}$, S_θ, and $S_{-\theta}$ and finding the intersection of the solution sets.

Let us first solve the problem for the shifted Hurwitz stability region $S_{-\gamma}$. Let $s = s_1 - \gamma$ then

$$\begin{aligned}\phi(s, \alpha_1, \alpha_2, \alpha_3) &= \phi(s_1 - \gamma, \alpha_1, \alpha_2, \alpha_3), \\ &= \phi_\gamma(s_1, \alpha_1, \alpha_2, \alpha_3).\end{aligned}$$

By this change of variable, we solve the usual stabilization problem for the new characteristic polynomial $\phi_\gamma(s_1, \alpha_1, \alpha_2, \alpha_3)$ with $\tilde{q}(s_1) = q(s_1 - \gamma)$ and $\tilde{p}(s_1) = p(s_1 - \gamma)$. Since we are using a dynamic controller, the new characteristic polynomial is given by

$$\phi_\gamma(s_1, \alpha_1, \alpha_2, \alpha_3) = (s_1 + \alpha_1 - \gamma)\tilde{q}(s_1) + (\alpha_2 s_1 + \alpha_3 - \alpha_2\gamma)\tilde{p}(s_1).$$

Multiplying $\phi_\gamma(s_1,\alpha_1,\alpha_2,\alpha_3)$ by $\bar{p}(-s_1)$ we obtain

$$\begin{aligned}\psi_\gamma(s_1,\alpha_1,\alpha_2,\alpha_3) &= \phi_\gamma(s_1,\alpha_1,\alpha_2,\alpha_3)\bar{p}(-s_1)\\ &= s_1^2G(s_1^2)-\gamma H(s_1^2)+\alpha_1 H(s_1^2)-\alpha_2\gamma F(s_1^2)+\alpha_3F(s_1^2)\\ &\quad +s_1[H(s_1^2)-\gamma G(s_1^2)+\alpha_1G(s_1^2)+\alpha_2F(s_1^2)].\end{aligned}$$

We can use the method discussed in Section 4.3 to find stabilizing values of $(\alpha_1,\alpha_2,\alpha_3)$.

Now let us consider the problem of determining the stabilizing values of $(\alpha_1,\alpha_2,\alpha_3)$ for the stability region S_θ. Let $s=s_1e^{j\theta}$, then

$$\begin{aligned}\phi(s,\alpha_1,\alpha_2,\alpha_3) &= (s+\alpha_1)q(s)+(\alpha_2 s+\alpha_3)p(s),\\ &= (s_1e^{j\theta}+\alpha_1)q(s_1e^{j\theta})+(\alpha_2s_1e^{j\theta}+\alpha_3)p(s_1e^{j\theta}).\end{aligned}$$

Since θ is constant, we have $e^{j\theta}=a+jb$, $q(s_1e^{j\theta})=\tilde{q}(s_1)$, and $p(s_1e^{j\theta})=\tilde{p}(s_1)$ where $\tilde{q}(s_1)$ and $\tilde{p}(s_1)$ are polynomials with complex coefficients. The new characteristic polynomial is given by

$$\phi_\theta(s_1,\alpha_1,\alpha_2,\alpha_3)=(s_1(a+jb)+\alpha_1)\tilde{q}(s_1)+(\alpha_2s_1(a+jb)+\alpha_3)\tilde{p}(s_1).$$

Roots of $\phi(s,\alpha_1,\alpha_2,\alpha_3)$ in stability region S_θ is equivalent to roots of $\phi_\theta(s_1,\alpha_1,\alpha_2,\alpha_3)$ in the open left-half complex plane. Using the generalized Hermite-Biehler theorem applicable to complex polynomials and Lemma 2.4, we outline in what follows a method to compute all values of $(\alpha_1,\alpha_2,\alpha_3)$ such that $\phi_\theta(s_1,\alpha_1,\alpha_2,\alpha_3)$ is Hurwitz stable. Let

$$\begin{aligned}\tilde{q}(j\omega) &= \tilde{h}(\omega)+j\tilde{g}(\omega),\\ \tilde{p}(j\omega) &= \tilde{f}(\omega)+j\tilde{e}(\omega),\\ \tilde{p}^*(j\omega) &= \tilde{f}(\omega)-j\tilde{e}(\omega),\end{aligned}$$

then

$$\begin{aligned}\tilde{q}(j\omega)\tilde{p}^*(j\omega) &= \tilde{H}(\omega)+j\tilde{G}(\omega),\\ \tilde{p}(j\omega)\tilde{p}^*(j\omega) &= \tilde{F}(\omega),\end{aligned}$$

where

$$\begin{aligned}\tilde{H}(\omega) &= \tilde{h}(\omega)\tilde{f}(\omega)+\tilde{g}(\omega)\tilde{e}(\omega),\\ \tilde{G}(\omega) &= \tilde{f}(\omega)\tilde{g}(\omega)-\tilde{h}(\omega)\tilde{e}(\omega),\\ \tilde{F}(\omega) &= \tilde{f}^2(\omega)+\tilde{e}^2(\omega).\end{aligned}$$

Multiplying $\phi_\theta(j\omega,\alpha_1,\alpha_2,\alpha_3)$ by $\tilde{p}^*(j\omega)$ we obtain

$$\begin{aligned}\psi_\theta(j\omega,\alpha_1,\alpha_2,\alpha_3) &= \phi_\theta(j\omega,\alpha_1,\alpha_2,\alpha_3)\tilde{p}^*(j\omega)\\ &= [-\omega(b\tilde{H}(\omega)+a\tilde{G}(\omega))+\alpha_1\tilde{H}(\omega)-\alpha_2\omega b\tilde{F}(\omega)+\alpha_3\tilde{F}(\omega)]\\ &\quad +j[\omega(a\tilde{H}(\omega)-b\tilde{G}(\omega))+\alpha_1\tilde{G}(\omega)+\alpha_2\omega a\tilde{F}(\omega)].\end{aligned}$$

Since only α_1 and α_2 appear in the imaginary part of $\psi_\theta(j\omega,\alpha_1,\alpha_2,\alpha_3)$, we can use the arguments developed in Section 4.3 to find stabilizing values of $(\alpha_1,\alpha_2,\alpha_3)$. As we are dealing with complex polynomials, we have to use Theorem 2.3 and Lemma 2.4 instead of Theorem 2.2 and Lemma 2.3.

The last stability region is $S_{-\theta}$. It was shown in [25], for the case of proportional controllers, that $S_{-\theta}$ and S_θ have exactly the same set of stabilizing controllers. This conclusion holds for first-order controllers. To see this, suppose that for a given triplet $(\bar{\alpha}_1,\bar{\alpha}_2,\bar{\alpha}_3)$, s_0 is a root of $\phi(s,\alpha_1,\alpha_2,\alpha_3)$, then

$$(s_0e^{j\theta}+\bar{\alpha}_1)q(s_0e^{j\theta})+(\bar{\alpha}_2 s_0e^{j\theta}+\bar{\alpha}_3)p(s_0e^{j\theta})=0.$$

As $q(s)$ and $p(s)$ are real polynomials, it follows that

$$(s_0^*e^{-j\theta}+\bar{\alpha}_1)q(s_0^*e^{-j\theta})+(\bar{\alpha}_2 s_0^*e^{-j\theta}+\bar{\alpha}_3)p(s_0^*e^{-j\theta})=0$$

where s_0^* is the complex conjugate of s_0. Since s_0^* and s_0 have the same real part, it follows that $(\bar{\alpha}_1,\bar{\alpha}_2,\bar{\alpha}_3)$ is stabilizing triplet for the stability region $S_{-\theta}$ if and only if it is stabilizing triplet for the stability region S_θ.

Example 4.4 *Consider a first-order controller to stabilize the plant* $g(s)=\frac{p(s)}{q(s)}$ *where*

$$q(s) = s^5+3s^4+29s^3+15s^2-3s+60,$$

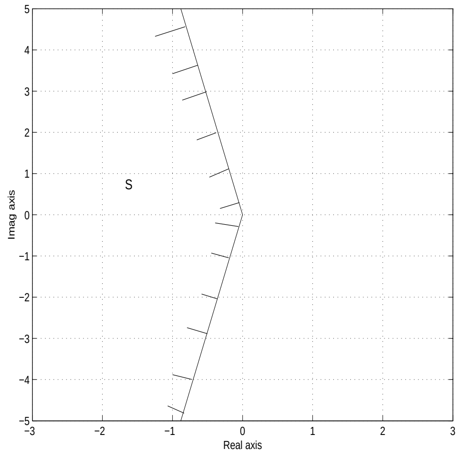

Figure 4.20: Stability region S.

$$p(s) = s^3 - 6s^2 + 2s - 1,$$

and the stability region S is given in Figure 4.20. This region is the intersection of $S_{\frac{\pi}{18}}$ and $S_{-\frac{\pi}{18}}$ as described at the beginning of the section. Let $s = s_1 e^{j\frac{\pi}{18}}$,then

$$\begin{aligned}\tilde{q}(s_1) &= (0.6428 + 0.7660j)s_1^5 + (2.2981 + 1.9284j)s_1^4 + (25.1147 + 14.5000j)s_1^3 \\ &\quad +(14.0954 + 5.1303j)s_1^2 - (2.9544 + 0.5209j)s_1 + 60, \\ \tilde{p}(s_1) &= (0.8660 + 0.5000j)s_1^3 - (5.6382 + 2.0521j)s_1^2 + (1.9696 + 0.3473j)s_1 - 1.\end{aligned}$$

Using the method developed in Section 4.3 together with Theorem 2.3 and Lemma 2.4, the stabilizing values of $(\alpha_1, \alpha_2, \alpha_3)$ are obtained as shown in Figure 4.21. From the results obtained, for $\alpha_1 = 0.2$ and $\alpha_2 = -4.1982$, roots of $\phi(s, \alpha_1, \alpha_2, \alpha_3)$ are in the stability region S for values of $\alpha_3 \in (-15.9491, -11.7427)$. The root-locus for the values of α_3 in this interval is shown in Figure 4.22. For Hurwitz stability, with $\alpha_1 = 0.2$ and $\alpha_2 = -4.1982$, we find $\alpha_3 \in (-22.5956, -9.548)$. The root-locus for the values of α_3 in this interval is shown in Figure 4.23. •

Remark 4.7 *The method of this section can be applied to PI and PID controllers. Let*

$$c(s) = \frac{\alpha_1 s^2 + \alpha_2 s + \alpha_3}{s},$$

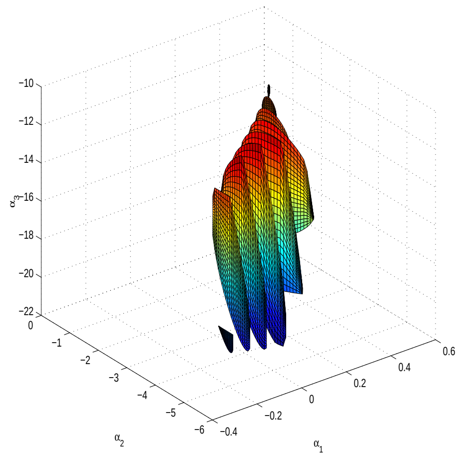

Figure 4.21: Stabilizing values $(\alpha_1, \alpha_2, \alpha_3)$.

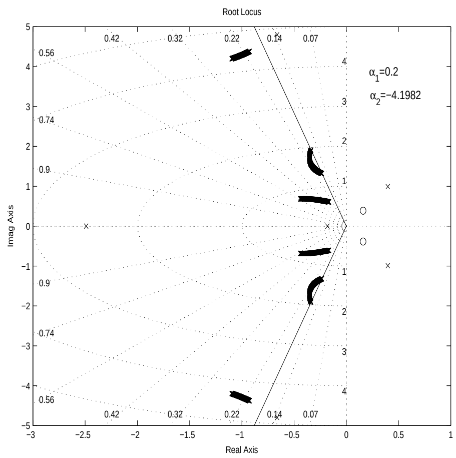

Figure 4.22: Attainable roots with respect to region S.

then we obtain

$$\begin{aligned}\psi_\gamma(s_1,\alpha_1,\alpha_2,\alpha_3) &= \phi_\gamma(s_1,\alpha_1,\alpha_2,\alpha_3)\bar{p}(-s_1)\\ &= s_1^2G(s_1^2)-\gamma H(s_1^2)+\alpha_1 s^2F(s_1^2)+\alpha_1\gamma^2F(s_1^2)-\alpha_2\gamma F(s_1^2)\\ &\quad +\alpha_3F(s_1^2)+s_1[H(s_1^2)-\gamma G(s_1^2)-\alpha_1 2\gamma F(s_1^2)+\alpha_2F(s_1^2)].\end{aligned}$$

and

$$\begin{aligned}\psi_\theta(j\omega,\alpha_1,\alpha_2,\alpha_3) &= \phi_\theta(j\omega,\alpha_1,\alpha_2,\alpha_3)\tilde{p}^*(j\omega)\\ &= [-\omega(b\tilde{H}(\omega)+a\tilde{G}(\omega))-\alpha_1\omega^2(a^2-b^2)\tilde{F}(\omega)-\alpha_2\omega b\tilde{F}(\omega)\\ &\quad +\alpha_3\tilde{F}(\omega)]+j[\omega(a\tilde{H}(\omega)-b\tilde{G}(\omega))-\alpha_1\omega^2 2ab\tilde{F}(\omega)\end{aligned}$$

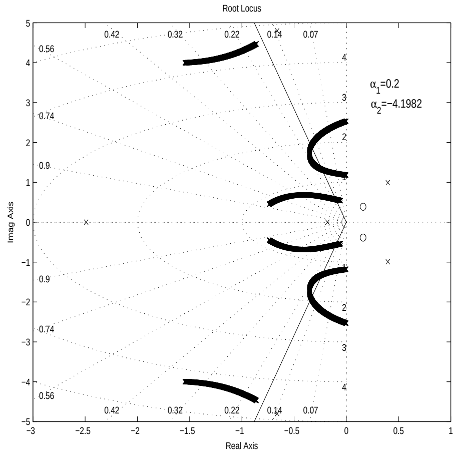

Figure 4.23: Attainable roots with respect to $\mathbf{C}_-$.

$$+\alpha_2 \omega a \tilde{F}(\omega)].$$

Since two parameters (α_1, α_2) *appear in the odd part of* $\psi_\gamma(s_1, \alpha_1, \alpha_2, \alpha_3)$, *imaginary part of* $\psi_\theta(s_1, \alpha_1, \alpha_2, \alpha_3)$, *we can directly apply the method developed for first-order controllers.* $\triangle$

Example 4.5 *Consider a PI controller* $c(s) = \frac{\alpha_1 s + \alpha_2}{s}$ *to stabilize the plant* $g(s) = \frac{p(s)}{q(s)}$ *where*

$$\begin{aligned} q(s) &= s^3 + 3s^2 + 4s, \\ p(s) &= s^2 + 2s - 2. \end{aligned}$$

The stability region S *is given in Figure 4.19 and specified by the parameters* $\gamma = 0.5$ *and* $\theta = \frac{\pi}{6}$. *For the rotated Hurwitz stability regions* S_θ *and* $S_{-\theta}$, *let* $s = s_1 e^{j\frac{\pi}{6}}$, *then*

$$\begin{aligned} \tilde{q}_1(s_1) &= js_1^3 + (1.5 + 2.5981j)s_1^2 + (3.4641 + 2j)s_1, \\ \tilde{p}_1(s_1) &= (0.5 + 0.866j)s_1^2 + (1.7321 + j)s_1 - 2. \end{aligned}$$

For the shifted Hurwitz stability regions $S_{-\gamma}$, *let* $s = s_1 - 0.5$, *then*

$$\begin{aligned} \tilde{q}_2(s_1) &= s_1^3 + 1.5s_1^2 + 1.75s_1 - 1.375, \\ \tilde{p}_2(s_1) &= s_1^2 + s_1 - 2.75. \end{aligned}$$

Using the new polynomials $\tilde{q}_1(s_1)$, $\tilde{p}_1(s_1)$, $\tilde{q}_2(s_1)$, and $\tilde{p}_2(s_1)$ and the method described in this section, we obtain the stabilizing values of (α_1, α_2) as shown in Figure 4.24. For $\alpha_1 = -0.7599$, if we consider the rotated Hurwitz stability regions

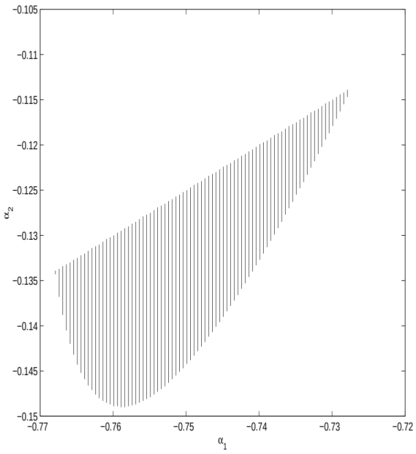

Figure 4.24: Stabilizing values (α_1, α_2).

S_θ and $S_{-\theta}$ only, then we obtain $(-0.1738, -0.0598)$ as the stabilizing interval for α_2. The root-locus for the values of α_2 in this interval is shown in Figure 4.25. With $\alpha_1 = -0.7599$ and stability region S, we obtain $(-0.1489, -0.13)$ as

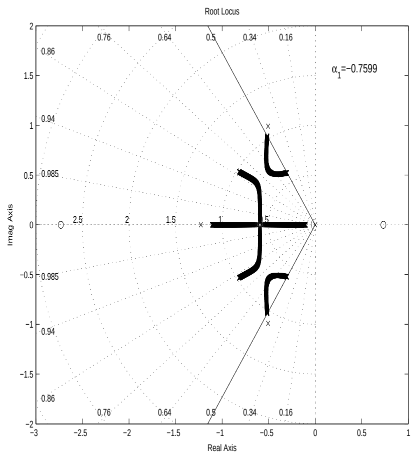

Figure 4.25: Attainable roots with respect to regions S_θ and $S_{-\theta}$.

the stabilizing interval for α_2. The root-locus for the values of α_2 in this interval is shown in Figure 4.26. •

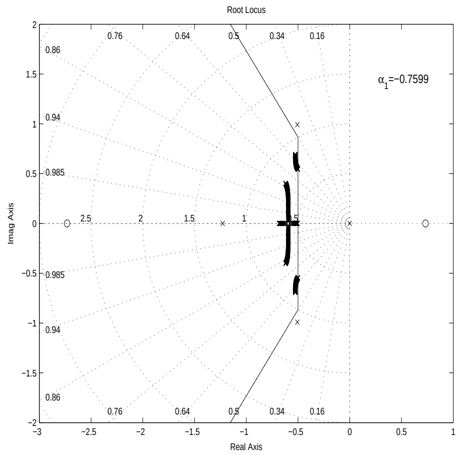

Figure 4.26: Attainable roots with respect to region S.

4.6 Uncertain Systems

The method described in the previous sections can be applied to plants with interval type uncertainty [96]. Let $g(s)$ be the transfer function of an uncertain system

$$g(s) = \frac{p(s)}{q(s)} = \frac{\sum_{i=0}^{m} x_i s^i}{\sum_{j=0}^{n} y_j s^j} \tag{4.10}$$

where $n > m$, $x_m \neq 0$, $y_n \neq 0$, and $x_i \in [x_{i-}, x_{i+}]$ $i = 1, \ldots, m$ and $y_i \in [y_{i-}, y_{i+}]$ $j = 1, \ldots, n$. Let $p_k(s)$ and $q_l(s)$, $k, l = 1, 2, 3, 4$ be the four Kharitonov polynomials corresponding to $p(s)$ and $q(s)$, respectively. Let $p_k^\lambda(s)$, $k = 1, 2, 3, 4$ be the four Kharitonov segments of $p(s)$, i.e.,

$$\begin{aligned}
p_1^\lambda(s) &= (1-\lambda)p_1(s) + \lambda p_2(s) \\
p_2^\lambda(s) &= (1-\lambda)p_1(s) + \lambda p_3(s) \\
p_3^\lambda(s) &= (1-\lambda)p_2(s) + \lambda p_4(s) \\
p_4^\lambda(s) &= (1-\lambda)p_3(s) + \lambda p_4(s)
\end{aligned}$$

where $\lambda \in [0, 1]$. The four Kharitonov segments $q_l^\lambda(s)$, $l = 1, 2, 3, 4$ of $q(s)$ can be defined similarly. Let $g_{seg}(s)$ denote the family of 32 segment plants

$$g_{seg}(s) = \{g_{kl}(s, \lambda) \mid g_{kl}(s, \lambda) = \frac{p_k^\lambda(s)}{q_l(s)}$$

$$\text{or} \quad g_{kl}(s,\lambda) = \frac{p_k(s)}{q_l^\lambda(s)}, k,l = 1,2,3,4, \text{ and } \lambda \in [0,1]\}.$$

It is well known [48] that the family $g(s)$ is stabilized by a particular controller, if and only if the 32 segment plants g_{seg} are stabilized by the same controller. Let $\tilde{g}_{seg}(s)$ denote the family of 16 segment plants

$$\tilde{g}_{seg}(s) = \{g_{kl}(s,\lambda) \mid g_{kl}(s,\lambda) = \frac{p_k^\lambda(s)}{q^l(s)}, k,l = 1,2,3,4, \text{ and } \lambda \in [0,1]\}.$$

It is shown in [97] ([98]) that "the entire family $g(s)$ is stabilized by a particular PID controller, if and only if each segment plant $g_{kl}(s) \in \tilde{g}_{seg}(s)$ is stabilized by that same PID controller". In reaching this result the structure of the PID controller was used to reduce the 32 segment plants to only 16. Since we are considering first-order controllers, the numerator and denominator of the controller are convex directions [48]. It is shown in [48] that stabilizing an interval plant $g(s)$ by a first-order controller is equivalent to stabilizing 16 vertex plants; namely,

$$g_v(s) = \{g_{kl}(s) \mid g_{kl}(s) = \frac{p_k(s)}{q_l(s)}, \ k,l = 1,2,3,4\}.$$

The stabilizing controller, if any, can be determined by first calculating α_1 which is the intersection of α_1's found for the 16 plants mentioned above. We can then apply the algorithm of the previous section for the 16 vertex plants to find α_2 and α_3.

Example 4.6 *Consider a proper first-order controller to stabilize the interval plant* $g(s) = \frac{p(s)}{q(s)}$ *where*

$$\begin{aligned} q(s) &= s^5 + y_4 s^4 + y_3 s^3 + y_2 s^2 + y_1 s + y_0, \\ p(s) &= s^3 + x_2 s^2 + x_1 s + x_0, \end{aligned}$$

and

$$\begin{aligned} &x_0 \in [-1,-2] \ x_1 \in [2,2], \ x_2 \in [-6,-5], \\ &y_0 \in [60,65], \ y_1 \in [-5,-3], \ y_2 \in [14,15], \\ &y_3 \in [29,29], \ y_4 \in [3,4]. \end{aligned}$$

We get the following Kharitonov polynomials

$$
\begin{aligned}
q_1(s) &= s^5 + 3s^4 + 29s^3 + 15s - 5s + 60, \\
q_2(s) &= s^5 + 3s^4 + 29s^3 + 15s - 3s + 60, \\
q_3(s) &= s^5 + 4s^4 + 29s^3 + 14s - 3s + 65, \\
q_4(s) &= s^5 + 4s^4 + 29s^3 + 14s - 5s + 65, \\
p_1(s) &= p_3(s) = s^3 - 6s^2 + 2s - 1, \\
p_2(s) &= p_4(s) = s^3 - 5s^2 + 2s - 2,
\end{aligned}
$$

a suitable range of α_1 *was determined to be* $\alpha_1 \in (-1.54, 0.97)$. *This is the intersection of suitable ranges of* α_1 *for the 16 vertex plants. Using Algorithm 4.2 for the 16 vertex plants, the set of stabilizing* $(\alpha_1, \alpha_2, \alpha_3)$ *values are shown in Figure 4.27.* •

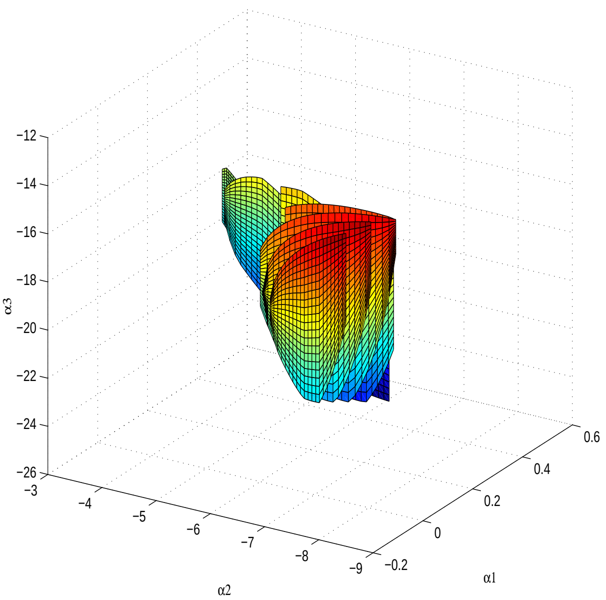

Figure 4.27: Stabilizing set of $(\alpha_1, \alpha_2, \alpha_3)$ *values.*

4.7 Second-Order Controllers

In this section, we show that Algorithm 4.2 can be extended to compute all stabilizing parameters of a high-order controller. We give a detailed derivation of the second-order controller case and show how to find the j-th parameter in a l-th-order controller. Now, we describe an algorithm that determines the set of all stabilizing second-order controllers for a given plant. A second-order controller

$$c(s) = \frac{\alpha_3 s^2 + \alpha_4 s + \alpha_5}{s^2 + \alpha_1 s + \alpha_2},$$

applied to $g(s) = \frac{p(s)}{q(s)}$ gives the closed loop characteristic polynomial

$$\begin{aligned} \phi_0(s, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) &= (s^2 + \alpha_1 s + \alpha_2)q(s) + (\alpha_3 s^2 + \alpha_4 s + \alpha_5)p(s) \\ &= q_0(s) + (\alpha_3 s^2 + \alpha_5)p_0(s), \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} q_0(s) &= (s^2 + \alpha_1 s + \alpha_2)q(s) + \alpha_4 s p(s), \\ p_0(s) &= p(s). \end{aligned} \tag{4.12}$$

Multiplying $\phi_0(s, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ by $\bar{p}_0(-s)$ we obtain

$$\begin{aligned} \psi_1(s, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) &= \phi_0(s, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)\bar{p}_0(-s) \\ &= s^2 H(s^2) + \alpha_1 s^2 G(s^2) + \alpha_2 H(s^2) + \alpha_3 s^2 F(s^2) \\ &\quad + \alpha_5 F(s^2) + s[s^2 G(s^2) + \alpha_1 H(s^2) + \alpha_2 G(s^2) \\ &\quad + \alpha_4 F(s^2)]. \end{aligned} \tag{4.13}$$

The reasoning behind the algorithm which determines the set of parameters $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4$, and α_5 of a stabilizing second-order controller can be explained as follows. Suppose $\phi_0(s)$ is Hurwitz stable for some $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5 \in \mathbf{R}$. By Remark 3.3, it follows that the odd part $uG(u) + \alpha_1 H(u) + \alpha_2 G(u) + \alpha_4 F(u)$ of $\psi_1(s)$ has at least $r_1 = \lfloor \frac{n+1-\sigma(p_0)}{2} \rfloor$ real negative roots with odd multiplicities.

Suppose $uG(u)+\alpha_1 H(u)+\alpha_2 G(u)+\alpha_3 F(u)$ has r_1 real negative roots with odd multiplicities. By Lemma 4.1, $\sigma[\phi_1(s)] = 2r_1$, where

$$\begin{aligned}\phi_1(s,\alpha_1,\alpha_2,\alpha_4) &= G_1^u(s)+\alpha_1 H_1(s)+\alpha_2 G_1(s)+\alpha_4 F_1(s)\\ &= q_1(s)+\alpha_4 p_1(s)\end{aligned}$$

and

$$\begin{aligned}H_1(s) &= H(s^2)+sH'(s^2),\\ G_1(s) &= G(s^2)+sG'(s^2),\\ F_1(s) &= F(s^2)+sF'(s^2),\\ G_1^u(s) &= s^2G(s^2)+s[G'(s^2)+s^2G(s^2)],\\ q_1(s) &= G_1^u(s)+\alpha_1 H_1(s)+\alpha_2 G_1(s),\\ p_1(s) &= F_1(s).\end{aligned} \tag{4.14}$$

In order to find the suitable ranges of α_1, α_2 and α_4, we modify $\phi_1(s)$ as follows. Let $B := gcd\{F, F'\}$ so that $F = B\bar{F}$, $F' = B\tilde{F}'$ for coprime polynomials $\bar{F}, \tilde{F}' \in \mathbf{R}[u]$. Let $\bar{p}_1(s) := \bar{F}(s^2)+s\tilde{F}'(s^2)$. By a simple computation, it follows that

$$\begin{aligned}\psi_2(s,\alpha_1,\alpha_2,\alpha_4) &= \phi_1(s,\alpha_1,\alpha_2,\alpha_4)\bar{p}_1(-s)\\ &= G_{2e}^u(s^2)+\alpha_1 H_{2e}(s^2)+\alpha_2 G_{2e}(s^2)+\alpha_4 F_{2e}(s^2)\\ &\quad +s[G_{2o}^u(s^2)+\alpha_1 H_{2o}(s^2)+\alpha_2 G_{2o}(s^2)],\end{aligned}$$

where

$$\begin{aligned}G_{2e}^u(u) &= uG(u)\bar{F}(u)-u[G(u)+uG'(u)]\tilde{F}'(u),\\ G_{2o}^u(u) &= [G(u)+uG'(u)]\bar{F}(u)-uG(u)\tilde{F}'(u),\\ H_{2e}(u) &= H(u)\bar{F}(u)-uH'(u)\tilde{F}'(u),\\ H_{2o}(u) &= H'(u)\bar{F}(u)-H(u)\tilde{F}'(u),\\ G_{2e}(u) &= G(u)\bar{F}(u)-uG'(u)\tilde{F}'(u),\\ G_{2o}(u) &= G'(u)\bar{F}(u)-G(u)\tilde{F}'(u),\\ F_{2e}(u) &= F(u)\bar{F}(u)-uF'(u)\tilde{F}'(u).\end{aligned} \tag{4.15}$$

Again by Remark 3.3, it follows that the odd part $G_{2o}^u(s^2)+\alpha_1 H_{2o}(s^2)+\alpha_2 G_{2o}(s^2)$ has at least $r_2 = \lfloor \frac{|2r_1-\sigma(p_1)|-1}{2} \rfloor$ real negative roots with odd multiplicities. Repeating the same procedure once more, suppose that $G_{2o}^u(s^2)+\alpha_1 H_{2o}(s^2)+\alpha_2 G_{2o}(s^2)$

has r_2 real negative roots with odd multiplicities. By Lemma 4.1, $\sigma[\phi_2(s)] = 2r_2$, where

$$\begin{aligned} \phi_2(s,\alpha_1,\alpha_2) &= G_2^u(s) + \alpha_1 H_2(s) + \alpha_2 G_2(s) \\ &= q_2(s) + \alpha_2 p_2(s) \end{aligned}$$

and

$$\begin{aligned} G_2^u(s) &= G_{2o}^u(s^2) + sG_{2o}^{u'}(s^2) \\ H_2(s) &= H_{2o}(s^2) + sH_{2o}'(s^2) \\ G_2(s) &= G_{2o}(s^2) + sG_{2o}'(s^2) \\ q_2(s) &= G_2^u(s) + \alpha_1 H_2(s), \\ p_2(s) &= G_2(s). \end{aligned} \tag{4.16}$$

The same steps above are repeated for $\phi_2(s)$. Let $C := gcd\{G_{2o}, G_{2o}'\}$ so that $G_{2o} = C\bar{G}_{2o},\ G_{2o}' = C\tilde{G}_{2o}'$ for coprime polynomials $\bar{G}_{2o}, \tilde{G}_{2o}' \in \mathbf{R}[u]$. Let $\bar{p}_2(s) := \bar{G}_{2o}(s^2) + s\tilde{G}_{2o}'(s^2)$. Multiplying $\phi_2(s)$ by $p_2(-s)$, we get

$$\begin{aligned} \psi_3(s,\alpha_1,\alpha_2) &= \phi_2(s,\alpha_1,\alpha_2)\bar{p}_2(-s) \\ &= G_{3e}^u(s^2) + \alpha_1 H_{3e}(s^2) + \alpha_2 G_{3e}(s^2) \\ &\quad + s[G_{3o}^u(s^2) + \alpha_1 H_{3o}(s^2)], \end{aligned}$$

where

$$\begin{aligned} G_{3e}^u(u) &= G_{2o}^u(u)\bar{G}_{2o}(u) - uG_{2o}^{u'}(u)\tilde{G}_{2o}'(u), \\ G_{3o}^u(u) &= G_{2o}^{u'}(u)\bar{G}_{2o}(u) - G_{2o}^u(u)\tilde{G}_{2o}'(u), \\ H_{3e}(u) &= H_{2o}(u)\bar{G}_{2o}(u) - uH_{2o}'(u)\tilde{G}_{2o}'(u), \\ H_{3o}(u) &= H_{2o}'(u)\bar{G}_{2o}(u) - H_{2o}(u)\tilde{G}_{2o}'(u), \\ G_{3e}(u) &= G_{2o}(u)\bar{G}_{2o}(u) - uG_{2o}'(u)\tilde{G}_{2o}'(u). \end{aligned} \tag{4.17}$$

Once more by Remark 3.3, the odd part of $\psi_3(s)$ has at least $r_3 = \lfloor \frac{|2r_2 - \sigma(p_2)| - 1}{2} \rfloor$ real negative roots with odd multiplicities . Now the set of $\alpha_1 \in \mathbf{R}$ which achieves r_3 real negative roots with odd multiplicities in $G_{3o}^u(u) + \alpha_1 H_{3o}(u)$ can be determined by applying Algorithm 3.2 to

$$\begin{aligned} q_3(s) &= G_3^u(s) = G_{3o}^u(s^2) + sG_{3o}^{u'}(s^2), \\ p_3(s) &= H_3(s) = H_{3o}(s^2) + sH_{3o}'(s^2). \end{aligned}$$

The algorithm below traces the above steps backwards by repetition of the steps (i)-(iv) below:

(i) Pick a value of α_1 such that the number of real negative roots with odd multiplicities of $G^u_{3o}(u) + \alpha_1 H_{3o}(u)$ is r_3 or greater.

(ii) Determine using Algorithm 3.2 all $\alpha_2 \in \mathbf{R}$ such that $\sigma[\phi_2(s)] = 2r_2$. By Lemma 4.1 and Remark 3.3, this is equivalent to determining values of α_2 such that $G^u_{2o}(u) + \alpha_1 H_{2o}(u) + \alpha_2 G_{2o}(u)$ has r_2 real negative roots with odd multiplicities.

(iii) For every α_2 found, determine using Algorithm 3.2 all $\alpha_4 \in \mathbf{R}$ such that $\sigma[\phi_1(s)] = 2r_1$. By Lemma 4.1 and Remark 3.3, this is equivalent to determining values of α_4 such that $uG(u) + \alpha_1 H(u) + \alpha_2 G(u) + \alpha_4 F(u)$ has r_1 real negative roots with odd multiplicities.

(iv) For every α_4 determined, find using extension of Algorithm 3.2, all α_3, α_5 such that $\phi_0(s)$ is Hurwitz stable.

The following algorithm determines all α_1, α_2, α_3, α_4, α_5 such that $\phi(s, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \in \mathcal{H}$:

Algorithm 4.3

- *Partition the real axis into intervals (or union of intervals) such that the number of real negative roots with odd multiplicities of $G^u_{3o}(u) + \alpha_1 H_{3o}(u)$ is constant in each interval.*
- *Fix $r_1 = \lfloor \frac{n+1-\sigma(p_0)}{2} \rfloor$.*
  1. *Fix $r_2 = \lfloor \frac{2r_1-\sigma(p_1)}{2} \rfloor$.*
  2. *Find admissible range of α_1 from the intervals found in the first step.*
     (a) *Fix an α_1 in the admissible range.*
     (b) *Apply Algorithm 3.2 to $q_2(s)$ and $p_2(s)$ given by (4.16). (This calculates admissible values of α_2 such that $G^u_{2o}(u) + \alpha_1 H_{2o}(u) + \alpha_2 G_{2o}(u)$ has r_2 real negative roots with odd multiplicities.)*

i. *Fix an* α_2 *from the range determined in 2.b.*

ii. *Apply Algorithm 3.2 to* $q_1(s)$ *and* $p_1(s)$ *given by (4.14). (This calculates all admissible values of* α_4 *such that* $uG(u) + \alpha_1 H(u) + \alpha_2 G(u) + \alpha_4 F(u)$ *has* r_1 *real negative roots with odd multiplicities.)*

  A. *Fix an* α_4 *from the range determined in 2.b.ii.*

  B. *Apply modified Algorithm 3.2 to* $q_0(s)$ *and* $p_0(s)$ *given by (4.12). (This calculates all admissible values of* α_3 *and* α_5 *such that* ϕ_0 *of (4.11) is in* $\mathcal{H}$*.)*

  C. *Increment* α_4 *and go to step 2.b.ii.A.*

iii. *Increment* α_2 *and go to step 2.b.i.*

(c) *Increment* α_1 *and go to step 2.a.*

3. *If* $r_2 < deg(G^u_{2o})$*, then increment* r_2 *by one and go to step 2.*

- *If* $r_1 < deg(uG)$ *then increment* r_1 *by one and go to step 1.*

Algorithm 3.2 is repeatedly used on four auxiliary plants:

$$\begin{aligned}
g_0(s) = \frac{p_0(s)}{q_0(s)} &= \frac{p(s)}{(s^2+\alpha_1 s+\alpha_2)q(s)+\alpha_4 sp(s)}, \\
g_1(s) = \frac{p_1(s)}{q_1(s)} &= \frac{F_1(s)}{G^u_1(s)+\alpha_1 H_1(s)+\alpha_2 G_1(s)}, \\
g_2(s) = \frac{p_2(s)}{q_2(s)} &= \frac{G_2(s)}{G^u_2(s)+\alpha_1 H_2(s)}, \\
g_3(s) = \frac{p_3(s)}{q_3(s)} &= \frac{H_3(s)}{G^u_3(s)},
\end{aligned}$$

to give the admissible values of $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$.

Remark 4.8 *The method can also be applied to discrete time plants using a bilinear transformation of the complex plane. Let the controller transfer function be*

$$c(z) = \frac{\alpha_3 z^2 + \alpha_4 z + \alpha_5}{\alpha_1 z^2 + \alpha_2 z + 1}.$$

By the bilinear transformation $z = \frac{w+1}{w-1}$, *we get*

$$c(w) = \frac{(\alpha_3+\alpha_4+\alpha_5)w^2 + (2\alpha_3 - 2\alpha_5)w + \alpha_3 - \alpha_4 + \alpha_5}{(\alpha_1+\alpha_2+1)w^2 + (2\alpha_1 - 2)w + \alpha_1 - \alpha_2 + 1}.$$

For a $c(w)$ *in this form,* α_1, α_2, α_3, *and* α_5 *appear both in the even and odd parts of* $\psi_1(w,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) = \phi_0(w,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)\bar{p}_0(-w)$. *Let* $\bar{\alpha}_3 = \alpha_3 + \alpha_4 + \alpha_5$, $\bar{\alpha}_4 = \alpha_3 - \alpha_5$ *and* $\bar{\alpha}_5 = \alpha_3 - \alpha_4 + \alpha_5$. *Then, by a simple computation it follows that*

$$\begin{aligned}\psi_1(w) \quad &= w^2H(w^2) + H(w^2) - 2w^2G(w^2) + \alpha_1[w^2H(w^2) + H(w^2) + 2w^2G(w^2)] \\ &+\alpha_2[w^2H(w^2) - H(w^2)] + \bar{\alpha}_3 w^2F(w^2) + \bar{\alpha}_5F(w^2) + w[w^2G(w^2) - 2H(w^2) \\ &+G(w^2) + \alpha_1(w^2G(w^2) + 2H(w^2) + G(w^2)) + \alpha_2(w^2G(w^2) - G(w^2) + \bar{\alpha}_4F(w^2)].\end{aligned}$$

Stabilizing controller parameters α_1, α_2, $\bar{\alpha}_3$, $\bar{\alpha}_4$, *and* $\bar{\alpha}_5$ *can be calculated using Algorithm 4.3. Since*

$$\begin{bmatrix}\bar{\alpha}_3\\ \bar{\alpha}_4\\ \bar{\alpha}_5\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\ 1 & 0 & -1\\ 1 & -1 & 1\end{bmatrix} \cdot \begin{bmatrix}\alpha_3\\ \alpha_4\\ \alpha_5\end{bmatrix}$$

and the linear transformation is invertible, we can calculate the values of α_3, α_4 *and* α_5 *as follows:*

$$\begin{bmatrix}\alpha_3\\ \alpha_4\\ \alpha_5\end{bmatrix} = \begin{bmatrix}\frac{1}{4} & \frac{1}{2} & \frac{1}{4}\\ \frac{1}{2} & 0 & -\frac{1}{2}\\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4}\end{bmatrix} \cdot \begin{bmatrix}\bar{\alpha}_3\\ \bar{\alpha}_4\\ \bar{\alpha}_5\end{bmatrix}.$$

The method hence applies to discrete time plants of arbitrary order. $\triangle$

Example 4.7 *Consider determining a strictly proper second-order controllers*

$$c(s) = \frac{\alpha_3 s + \alpha_4}{s^2 + \alpha_1 s + \alpha_2}$$

to stabilize the plant $g(s) = \frac{p(s)}{q(s)}$, *where*

$$\begin{aligned}q(s) &= s^5 + 4s^4 + 29s^3 + 15s^2 - 3s + 60,\\ p(s) &= s^3 - 6s^2 + 2s + 1.\end{aligned}$$

The roots of $q_0(s)$ *are* $\{-1.2576 \pm j5.1476,\ -1.5574,\ 0.5363 \pm j1.0414\}$ *and those of* $p_0(s)$ *are* $\{-0.2705,\ 0.6587,\ 5.6119\}$ *so that this is an unstable and non-minimum phase plant. Using (3.9), we have*

$$\begin{aligned} H(u) &= -u^4 - 49u^3 - 142u^2 - 339u + 60, \\ G(u) &= -9u^3 - 194u^2 - 43u - 123, \\ F(u) &= -u^3 + 32u^2 - 16u + 1. \end{aligned}$$

A necessary condition for the existence of a stabilizing second-order controller is that $uG(u) + \alpha_1 H(u) + \alpha_2 G(u) + \alpha_3 F(u)$ *has at least* $r_1 = \lfloor \frac{n+1-\sigma(p_o)}{2} \rfloor = 3$ *real negative roots with odd multiplicities. As* $gcd(F, F') = 1$*, we multiply* $\phi_1(s)$ *by* $p_1(-s) = F(s^2) - sF'(s^2)$*. For* $r_1 = 3$, $\sigma(\phi_1) - \sigma(p_1) = 6$ *and the odd part of* $\phi_1(s)p_1(-s)$ *must have at least* $r_2 = \lfloor \frac{|2r_1 - \sigma(p_1)| - 1}{2} \rfloor = 2$ *real negative roots with odd multiplicities. In a similar way we can determine* $r_3 = \lfloor \frac{|2r_2 - \sigma(p_2)| - 1}{2} \rfloor = 1$*. For* $r_1 = 4$ *we obtain* $r_2 = 3$ *and* $r_3 = 2$*. Now let us follow the steps of Algorithm 4.3 for a fixed value of* α_1*. For* $\alpha_1 = 1$*, using step 2.b in Algorithm 4.3, the range of admissible values of* α_2 *for which* $G^u_{2o}(u) + \alpha_1 H_{2o}(u) + \alpha_2 G_{2o}(u)$ *has at least 2 negative real roots is* $(-14.3402, 1.5032)$*. With* $\alpha_2 = 0.5$*, we obtain*

$$\begin{aligned} q_1(s) &= -10s^8 - 40s^7 - 247.5s^6 - 742.5s^5 - 282s^4 - 564s^3 - 483.5s^2 \\ &\quad -483.58s - 1.5, \\ p_1(s) &= -s^6 - 3s^5 + 32s^4 + 64s^3 - 16s^2 - 16s + 1. \end{aligned}$$

Step 2.b.ii in Algorithm 4.3 gives the following solution $\alpha_3 \in (-15.8926, -8.5154)$ *for* $\alpha_1 = 1$ *and* $\alpha_2 = 0.5$ *. With* $\alpha_3 = -10$*, we obtain*

$$\begin{aligned} q_0(s) &= s^7 + 4s^6 + 32.5s^5 + 35.5s^4 + 86.5s^3 + 44.5s^2 + 48.5s + 30, \\ p_0(s) &= s^3 - 6s^2 + 2s + 1. \end{aligned}$$

Step 2.b.ii.A in Algorithm 4.3 gives the following solution $\alpha_4 \in (-4.0566, -2.8786)$ *for* $\alpha_1 = 1$, $\alpha_2 = 0.5$ *and* $\alpha_3 = -10$ *. The solution set for* $\alpha_1 = 1$ *is shown in Figure 4.28. Figure 4.29 and Figure 4.30 shows the results for* $\alpha_1 = 5$ *and* $\alpha_1 = 15$*, respectively.* •

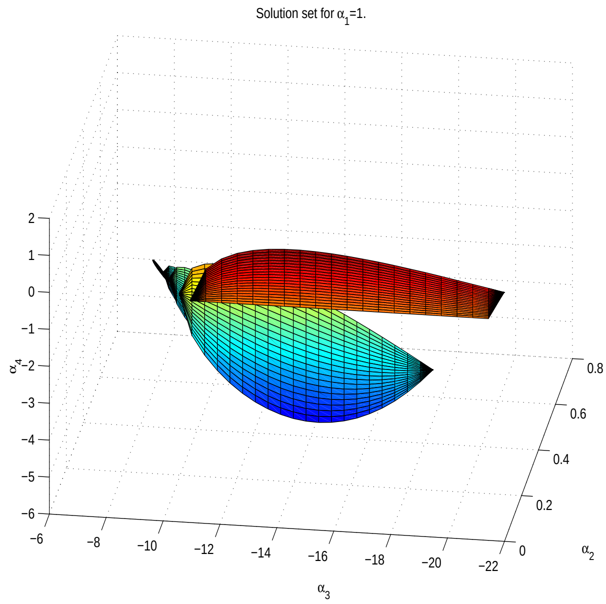

Figure 4.28: Stabilizing set of $(\alpha_2, \alpha_3, \alpha_4)$ values for $\alpha_1 = 1$.

Remark 4.9 *In this section, we gave a complete derivation of an algorithm that determines all stabilizing second-order controllers for a given plant. Algorithm 3.2 is repeatedly applied to a number of auxiliary plants ($g_0(s) = \frac{p_0(s)}{q_0(s)}$, $g_1(s) = \frac{p_1(s)}{q_1(s)}$, $g_2(s) = \frac{p_2(s)}{q_2(s)}$, and $g_3(s) = \frac{p_3(s)}{q_3(s)}$). The above algorithm can be extended to high-order controllers. As the number of parameters of the controller increases, the number of auxiliary plants increases accordingly. For an l-th order controller (we assume here that l is even and let $k = \frac{3l}{2}$)*

$$c(s) = \frac{s[\alpha_{l+1}s^{l-2} + \alpha_{l+2}s^{l-4} + \ldots + \alpha_k] + \alpha_{k+1}s^l + \alpha_{k+2}s^{l-2} + \ldots + \alpha_{2l+1}}{s^l + \alpha_1 s^{l-1} + \alpha_2 s^{l-2} + \ldots + \alpha_l},$$

we can determine recursively ϕ_i's and ψ_i's as follows:

$$\begin{aligned}
\phi_0(s) &= (s^l + \alpha_1 s^{l-1} + \alpha_2 s^{l-2} + \ldots + \alpha_l)q(s) + s[\alpha_{l+1}s^{l-2} + \alpha_{l+2}s^{l-4} + \ldots + \alpha_k]p(s) \\
&\quad + [\alpha_{k+1}s^l + \alpha_{k+2}s^{l-2} + \ldots + \alpha_{2l+1}]p(s) \\
&= q_0(s) + [\alpha_{k+1}s^l + \alpha_{k+2}s^{l-2} + \ldots + \alpha_{2l+1}]p_0(s) \\
\psi_1(s) &= \phi_0(s)\bar{p}_0(s) \\
&= \psi_{1e}(s^2) + s\psi_{1o}(s^2)
\end{aligned}$$

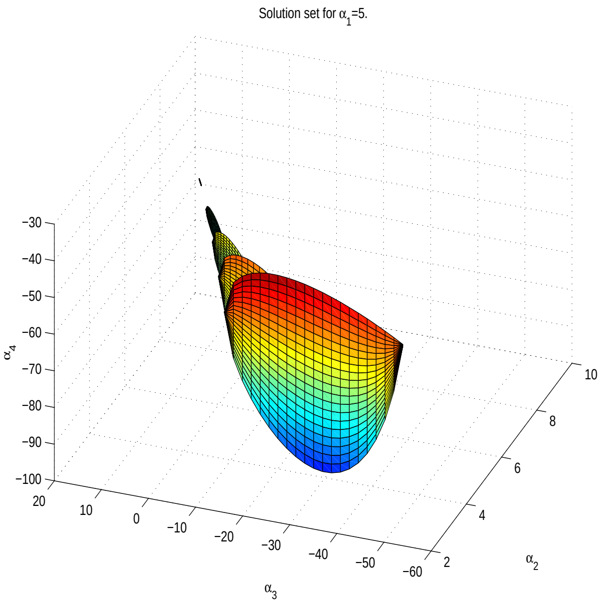

Figure 4.29: Stabilizing set of $(\alpha_2, \alpha_3, \alpha_4)$ values for $\alpha_1 = 5$.

$$\begin{aligned}
\phi_1(s) &= \psi_{1o}(s^2) + s\psi'_{1o}(s^2) \\
&= q_1(s) + \alpha_1 p_1(s) \\
&\vdots \\
\psi_i(s) &= \phi_{i-1}(s)\bar{p}_{i-1}(s) \\
&= \psi_{ie}(s^2) + s\psi_{io}(s^2) \\
\phi_i(s) &= \psi_{io}(s^2) + s\psi'_{io}(s^2) \\
&= q_i(s) + \alpha_i p_i(s) \\
&\vdots \\
\phi_k(s) &= q_k(s) + \alpha_k p_k(s)
\end{aligned}$$

Hence, at each step we can determine $p_i(s)$ and $q_i(s)$ for $i = 0, 1, \ldots, k$. It is also possible to determine r_i's recursively, i.e., $r_0 = \lfloor \frac{n+l-\sigma(p_0)}{2} \rfloor$ and $r_i = \lfloor \frac{|2r_{i-1}-\sigma(p_{i-1})|-1}{2} \rfloor$ for $i = 1, 2, 3, \ldots, k$. At the j-th step of the algorithm as $q_j(s)$, $p_j(s)$, and r_j are all known, we can determine α_j using Algorithm 3.2. $\triangle$

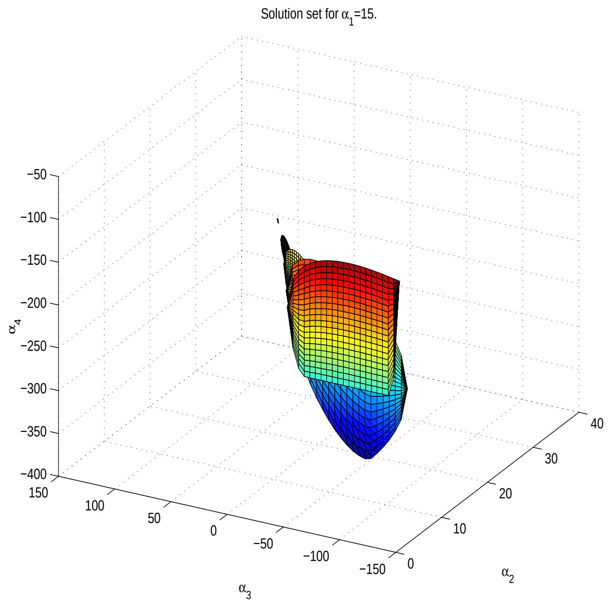

Figure 4.30: Stabilizing set of $(\alpha_2, \alpha_3, \alpha_4)$ values for $\alpha_1 = 15$.

Chapter 5

Local Convex Directions

In Chapters 3 and 4, we saw that the concept of convex directions plays an important role in studying stability of uncertain parameter systems. For plants with interval type parameter uncertainty, extreme point results can be obtained whenever Ranzter's growth condition is satisfied. In [67], a new version of the Hermite-Biehler theorem was derived and used to construct new convex directions. In this chapter, we use this new version to study local convex directions [58]. A new condition for a polynomial $p(s)$ to be a local convex direction for a Hurwitz stable polynomial $q(s)$ is derived. The condition is in terms of polynomials associated with the even and odd parts of $p(s)$ and $q(s)$ and constitutes a generalization of Rantzer's phase growth condition for global convex directions. It is used to determine convex directions for certain subsets of Hurwitz stable polynomials.

5.1 Local Convex Directions

A polynomial $p(s)$ is called a **global convex direction** (for all Hurwitz stable polynomials of degree n) if for any Hurwitz stable polynomial q(s) the implication

$$q(s) + p(s) \text{ is Hurwitz stable and } deg\ q + \lambda p = n\ \forall \lambda \in [0,1]$$
$$\Rightarrow q(s) + \lambda p(s) \text{ is Hurwitz stable } \forall \lambda \in (0,1)$$

holds. Rantzer in [47] has shown that a polynomial $p(s)$ is a convex direction if and only if it satisfies the **phase growth condition** ([47], [48])

$$\theta_p^{'}(w) \leq |\frac{\sin(2\theta_p(w))}{2w}| \quad \forall w > 0, \tag{5.1}$$

whenever $\theta_p(w) := arg\ p(jw) \neq 0$. The condition (5.1) is in a sense a complement of the phase increasing property of Hurwitz stable polynomials. For a Hurwitz stable polynomial $q(s)$ the rate of change of the argument satisfies

$$\theta_q^{'}(w) \geq |\frac{\sin(2\theta_q(w))}{2w}| \quad \forall w > 0,$$

where the inequality is strict if $deg\ q \geq 2$. This property also given in [47] seems to be known in network theory as pointed out by [60] (see also [99] for a proof based on Hermite-Biehler Theorem and [66] for related conditions).

Our main result in this section yields a characterization of polynomials $p(s),\ q(s)$ which satisfy the **local convexity condition**

$$(LCC)\ q,\ q+p \in \mathcal{H}\ and\ deg\ q+\lambda p = deg\ q\ \forall \lambda \in [0,1] \Rightarrow q+\lambda p \in \mathcal{H}\ \forall \lambda \in (0,1).$$

Let (h, g) and (f, e) be the even-odd parts of $q(s)$ and $p(s)$, respectively. Recall that

$$\begin{aligned} V_p(u) &:= f'(u)e(u) - f(u)e'(u), \\ V_{sp}(u) &:= f(u)e(u) - u[f'(u)e(u) - f(u)e'(u)], \end{aligned}$$

and

$$\begin{aligned} S_p(u) &:= \tfrac{f(u)e(u)}{V_p(u)}, \\ S_{sp}(u) &:= \tfrac{uf(u)e(u)}{V_{sp}(u)}. \end{aligned}$$

The following theorem gives a test for LCC in terms of polynomials associated with the even-odd parts of $p(s)$ and the vertex polynomials $q(s)$, $q(s)+p(s)$. This result is suitable for obtaining convex directions for certain subsets of Hurwitz stable polynomials. It also gives Rantzer's condition in a rather straightforward manner when it is satisfied by every Hurwitz stable polynomial. It is thus one natural local version of the global condition of Rantzer.

Theorem 5.1 *Let $p(s)$, $q(s)$ be polynomials with $n := deg\ q > 1$. Then, LCC holds if and only if*

$$\begin{aligned} &V_p(u) < (\sqrt{V_{p+q}(u)} + \sqrt{V_q(u)})^2 \quad \forall \ \ u \in \{u < 0 : \ f(u)e(u) \geq 0\}, \\ &V_{sp}(u) < (\sqrt{V_{s(p+q)}(u)} + \sqrt{V_{sq}(u)})^2 \quad \forall \ \ u \in \{u < 0 : \ f(u)e(u) < 0\}. \end{aligned} \tag{5.2}$$

Proof. [Only if] If $q + \lambda p \in \mathcal{H}$ for all $\lambda \in [0,1]$, then $(h + \lambda f, g + \lambda e)$ is a positive pair for all $\lambda \in [0,1]$. By lemma 2.1, $V_{q+\lambda p}(u) > 0$ and $V_{s(q+\lambda p)}(u) > 0$ $\forall u < 0$ and $\forall \lambda \in [0,1]$. The following identities are obtained by an easy computation.

$$V_{q+\lambda p}(u) = (1-\lambda)V_q(u) + \lambda(\lambda - 1)V_p(u) + \lambda V_{q+p}(u), \tag{5.3}$$

$$V_{s(q+\lambda p)}(u) = (1-\lambda)V_{sq}(u) + \lambda(\lambda - 1)V_{sp}(u) + \lambda V_{s(q+p)}(u). \tag{5.4}$$

Suppose for some $u < 0$, the first condition in (5.2) fails. For this value of u,

$$\lambda := \frac{\sqrt{V_q(u)}}{\sqrt{V_{q+p}(u)} + \sqrt{V_q(u)}} \in (0,1),$$

achieves the value

$$V_{q+\lambda p}(u) = \frac{[(\sqrt{V_{p+q}(u)} + \sqrt{V_q(u)})^2 - V_p(u)]\sqrt{V_{p+q}(u)V_q(u)}}{[\sqrt{V_{p+q}(u)} + \sqrt{V_q(u)}]^2}.$$

By our hypothesis, the right hand side is nonpositive which contradicts the fact that $V_{q+\lambda p}(u) > 0$. Thus the first condition in (5.2) must hold. Similarly, using (5.4), the second condition in (5.2) is obtained.

[If] Consider the identities

$$V_{q+\lambda p}(u) = (1-\lambda)^2 V_q(u) + \lambda(1-\lambda)A(u) + \lambda^2 V_{q+p}(u), \tag{5.5}$$

$$V_{s(q+\lambda p)}(u) = (1-\lambda)^2 V_{sq}(u) + \lambda(1-\lambda)B(u) + \lambda^2 V_{s(q+p)}(u), \tag{5.6}$$

where

$$\begin{aligned} A(u) &:= V_{q+p}(u) + V_q(u) - V_p(u), \\ B(u) &:= V_{s(q+p)}(u) + V_{sq}(u) - V_{sp}(u). \end{aligned}$$

If $u < 0$ is such that $A(u) \geq 0$, then as $V_q(u) > 0$, $V_{q+p}(u) > 0$, the right hand side of (5.5) is positive for all $\lambda \in [0,1]$. If $u < 0$ satisfies $A(u) < 0$, then

$$A(u) - 2\sqrt{V_{p+q}(u)V_q(u)} = (\sqrt{V_{q+p}(u)} - \sqrt{V_q(u)})^2 - V_p(u) < 0$$

and by (5.2)

$$\begin{aligned} &[(\sqrt{V_{q+p}(u)} + \sqrt{V_q(u)})^2 - V_p(u)][(\sqrt{V_{q+p}(u)} - \sqrt{V_q(u)})^2 - V_p(u)] \\ &= [A(u)]^2 - 4V_{p+q}(u)V_q(u) < 0 \end{aligned}$$

for all $u \in \{u < 0 : f(u)e(u) \geq 0\}$ for which $A(u) < 0$. But then for such u, the right hand side of (5.5) is nonzero for all $\lambda \in (0,1)$ so that it is positive for all $\lambda \in [0,1]$. This implies that

$$V_{q+\lambda p}(u) > 0 \; \forall u \in \{u < 0 : f(u)e(u) \geq 0\}, \; \forall \lambda \in [0,1]. \tag{5.7}$$

By similar arguments, the identity (5.6) and the condition (5.2) imply that

$$V_{s(q+\lambda p)}(u) > 0 \; \forall u \in \{u < 0 : f(u)e(u) < 0\}, \; \forall \lambda \in [0,1]. \tag{5.8}$$

We now show that (5.7) and (5.8) imply $q + \lambda p \in \mathcal{H}$ for all $\lambda \in (0,1)$. Suppose for some $\lambda_0 \in (0,1)$, $q + \lambda_0 p$ is not in $\mathcal{H}$. Then, as $q,\ q+p \in \mathcal{H}$ and $deg\ q + \lambda p$ is constant for $\lambda \in [0,1]$, by the continuity of the roots of $q + \lambda p$ with respect to λ, there exists $0 < \lambda_1 \leq \lambda_2 < 1$ such that $q + \lambda p \in \mathcal{H}, \;\; \forall \lambda \in [0, \lambda_1) \cup (\lambda_2, 1]$ and one of the following two cases happen

(*i*) $q_0 + \lambda_1 p_0 = 0$ and $q_0 + \lambda_2 p_0 = 0$

(*ii*) $(q + \lambda_1 p)(jw_0) = 0$, or $(q + \lambda_2 p)(jw_1) = 0$ where $w_0 \neq 0$ or $w_1 \neq 0$, with $q_0 := q(0),\ p_0 := p(0)$.

(i) Note that if $\lambda_1 \neq \lambda_2$, then $q_0 = 0$ contradicting the fact that $q \in \mathcal{H}$. Hence with $\lambda_0 := \lambda_1 = \lambda_2$, we have $\lambda_0(q_0 + p_0) + (1 - \lambda_0)q_0 = 0$ implying that q_0 and $q_0 + p_0$ have different signs. say $q_0 > 0$ and $q_0 + p_0 < 0$. Since $q + \lambda p \in \mathcal{H}\ \forall\lambda \in [0, \lambda_0) \cup (\lambda_0, 1]$, it follows that $q_0 + \lambda p_0 > 0,\ \forall\lambda \in [0, \lambda_0)$ and $q_0 + \lambda p_0 < 0,\ \forall\lambda \in (\lambda_0, 1]$. Since all coefficients of a Hurwitz stable polynomial are of the like sign, it follows that all coefficients of $q + \lambda p$ for $\lambda \in [0, \lambda_0)$ are positive and that all coefficients of $q + \lambda p$ for $\lambda \in (\lambda_0, 1]$ are negative. This implies that $q + \lambda_0 p \equiv 0$ contradicting the hypothesis that *deg* $q + \lambda p = n$.

(ii) Suppose without loss of generality that $u_0 := -w_0^2 < 0$. Then, we have $h(u_0) + \lambda_1 f(u_0) = 0$ and $g(u_0) + \lambda_1 e(u_0) = 0$ which contradicts either (5.7) or (5.8) depending on whether $f(u_0)e(u_0) \geq 0$ or $f(u_0)e(u_0) < 0$. ■

Remark 5.1 *The following alternative statement eliminates the square roots in (5.2): Under the assumptions of Theorem 5.1, $q + \lambda p \in \mathcal{H}$ for all $\lambda \in (0, 1)$ if and only if*

$$u < 0 : f(u)e(u) \geq 0, A(u) < 0 \ \Rightarrow\ A(u)^2 < 4V_{p+q}(u)V_q(u), \tag{5.9}$$

$$u < 0 : f(u)e(u) < 0, B(u) < 0 \ \Rightarrow\ B(u)^2 < 4V_{s(p+q)}(u)V_{sq}(u). \tag{5.10}$$

△

It is easy to see that given a polynomial $p(s) = f(s^2) + se(s^2)$, it is a local convex direction for any Hurwitz stable polynomial $q(s) = h(s^2) + sg(s^2)$ whenever (h, e) and (f, g) form positive pairs. This follows by $A(u) \geq 0,\ B(u) \geq 0\ \ \forall u < 0$ and by Remark 5.1. In what follows, we identify other sets of Hurwitz stable polynomials for which $p(s)$ is a local convex direction. Consider the control system in Figure 5.1. Given a family of plants

$$\mathcal{G} = \{g(s, \lambda) = \frac{g(s^2) + \lambda}{h(s^2) + \lambda}\ :\ \lambda \in [0, 1]\},$$

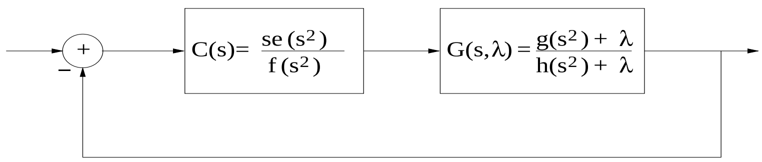

Figure 5.1: A robust stabilization problem for plants of even transfer functions.

it is easy to see that if a controller $c(s) = \frac{se(s^2)}{f(s^2)}$ stabilizes $g(s,0)$ then it stabilizes the whole family if and only if $p(s) = f(s^2) + se(s^2)$ is a local convex direction for $q(s) = h(s^2)f(s^2) + se(s^2)g(s^2)$. In order to get more concrete conditions using Theorem 5.1, we restrict $h(s^2)$ and $g(s^2)$ to be of first order. We thus consider certain subsets of polynomials obtained by adding zeros to even and/or odd part of a candidate convex direction $p(s) = f(s^2) + se(s^2)$. Consider

$$Q_p = \{q(s) = (ks^2+1)f(s^2) + s(ls^2+1)e(s^2) : \; k > l \geq 0\}, \tag{5.11}$$

we assume here that $p(s) \in \mathcal{H}$ so that $Q_p \subset \mathcal{H}$ for a majority of values of k and l. The case of $l > k \geq 0$ follows similar arguments and therefore it is omitted. In what follows, we use Theorem 5.1 to find conditions in terms of sensitivity functions $S_p(u)$ and $S_{sp}(u)$ such that $p(s)$ is a local convex direction for Q_p.

Corollary 5.1 *Let $p(s)$ be a Hurwitz stable polynomial and Q_p as defined in (5.11). The polynomial $p(s)$ is a local convex direction for Q_p if and only if k and l satisfy the following conditions:*

$$\begin{aligned} u < 0 : f(u)e(u) \geq 0, \quad & S_p(u) < \tfrac{2klu^2+3(k+l)u+4}{3(l-k)} \\ & \Rightarrow \tfrac{(l-k)u}{2\sqrt{kl}+k+l} < S_p(u) < \tfrac{(k-l)u}{2\sqrt{kl}-(k+l)} \end{aligned} \tag{5.12}$$

$$u < 0 : f(u)e(u) < 0 \;\Rightarrow\; S_{sp}(u) \leq \tfrac{2klu^2+3(k+l)u+4}{3(k-l)} \tag{5.13}$$

Proof. For

$$q(s) = (ks^2+1)f(s^2) + s(ls^2+1)e(s^2),$$

we have

$$\begin{aligned} A(u) &= (2klu + 3(k+l)u + 4)V_p(u) + 3(k-l)f(u)e(u), \\ B(u) &= (2klu + 3(k+l)u + 4)V_{sp}(u) - 3(k-l)uf(u)e(u). \end{aligned}$$

It is lengthy but straightforward to verify that (5.12) is equivalent to (5.9). If

$$\forall u < 0 : f(u)e(u) < 0, B(u) < 0$$

then

$$\frac{(k-l)u}{2\sqrt{kl}+k+l} < S_{sp}(u) < \frac{(k-l)u}{(k+l)-2\sqrt{kl}} \leq 0$$

must be satisfied for LCC to hold. This is impossible as

$$S_{sp}(u) > 0 \ \forall u < 0, f(u)e(u) < 0.$$

Condition (5.13) is hence equivalent to the following condition

$$\forall u < 0 : f(u)e(u) < 0 \ \Rightarrow B(u) \geq 0.$$

The result follows by Remark 5.1. ■

Remark 5.2 *Setting $l = 0$ in Corollary 5.1, we get*

$$Q_p = \{q(s) = (ks^2+1)f(s^2) + se(s^2)\}.$$

$A(u) \geq 0$ and $B(u) \geq 0$ reduce to

$$\begin{aligned} S_p(u) &\geq \tfrac{-3ku-4}{3k}, \\ S_{sp} &\leq \tfrac{2ku+4}{3k}, \end{aligned}$$

which can be shown to hold for every $q(s) \in \mathcal{H}$. Hence $p(s)$ is a local convex direction for all $q(s) = (ks^2+1)f(s^2) + se(s^2)$ such that $q(s) \in \mathcal{H}$. This simple result is equivalent to the following robust stabilization result. Consider the family of Hurwitz stable plants

$$\mathcal{P} = \{g(s,\gamma) = \frac{f(s^2)s^2}{\gamma(f(s^2) + se(s^2))} : \gamma \in [1,2]\}.$$

Any constant feedback gain which stabilizes the vertex plant $g(s,1) = \frac{f(s^2)s^2}{f(s^2)+se(s^2)}$ also stabilizes the whole family. △

5.2 Convex Directions for all Hurwitz Stable Polynomials

In this section, we investigate the relation between the local condition of Theorem 5.1 and the phase growth condition of Rantzer [47] which characterizes those polynomials $p(s)$ which satisfy LCC for all $q(s) \in \mathcal{H}$. In Theorem 5.2 below, we give an alternative proof of Rantzer's result. One part of this proof (the "if" part) is particularly straightforward and makes the connection between the local condition and the phase growth condition very clear.

The other direction of the proof requires a construction and hence it is not straightforward. We first prove a lemma used in this part of the proof of Theorem 5.2. The claim is that given any point $j\omega_0$ on the imaginary axis and any numerator polynomial $p(s)$ such that $p(j\omega_0) \neq 0$, one can design a stable denominator polynomial $\bar{r}(s)$ such that the root-locus (or the complementary root-locus) of $\frac{p(s)}{\bar{r}(s)}$ passes through $j\omega_0$.

Lemma 5.1 *Given a polynomial $p(s)$ with deg $p > 1$ and a real positive number ω_0 such that $p(j\omega_0) \neq 0$, there exists a Hurwitz stable polynomial $\bar{r}(s)$ with deg $\bar{r} \geq$ deg p and a real number α for which $(\bar{r} + \alpha p)(j\omega_0) = 0$.*

Proof. Let $u_0 := -\omega_0^2$. Since $p(j\omega_0) \neq 0$, the polynomials $p(s),\ s - j\omega_0$ are coprime so that given any $r_0 \in \mathbf{C}[s]$, there exists $c \in \mathbf{C}$ and $n \in \mathbf{C}[s]$ such that

$$(s - j\omega_0)n(s) + p(s)c = r_0(s) \tag{5.14}$$

by Euclidean algorithm in $\mathbf{C}[s]$. We can in particular choose a Hurwitz stable polynomial $r_0(s)$ with real coefficients such that *deg* $r_0 \geq$ *deg* p and such that the even-odd components (h_0, g_0) of $r_0(s)$ satisfy

$$\begin{array}{ll} \frac{g_0(u_0)}{h_0(u_0)} > \frac{f(u_0)}{u_0 e(u_0)} > \frac{e(u_0)}{f(u_0)} & \text{if} \quad f(u_0)e(u_0) < 0 \text{ or } f(u_0) = 0, \\ \frac{h_0(u_0)}{g_0(u_0)} < \frac{u_0 e(u_0)}{f(u_0)} < \frac{f(u_0)}{e(u_0)} & \text{if} \quad f(u_0)e(u_0) > 0 \text{ or } e(u_0) = 0. \end{array} \tag{5.15}$$

Let $c = c_r + jc_i$ for $c_r,\ c_i \in \mathbf{R}$ and let

$$n(s) = n_r(s) + jn_i(s)$$

for $n_r,\ n_i \in \mathbf{R}[s]$. Note that $c \neq 0$ in (5.14), since otherwise $r_0(s)$ would not be Hurwitz stable. If $c_i = 0$, then $r_0 - cp \in \mathbf{R}[s]$ and $\bar{r}(s) := r_0(s)$ is the desired polynomial. If $c_i \neq 0$, we proceed as follows. Multiplying both sides of (5.14) by $(s + j\omega_0)(c_r - jc_i)$ and equating the real and imaginary parts, we have

$$(s^2 - u_0)m(s) - \alpha p(s) = (c_i s - c_r \omega_0) r_0(s) =: \bar{r}(s)$$

where

$$\begin{aligned} m(s) &:= c_i n_r(s) - c_r n_i(s), \\ \alpha &:= \omega_0 (c_r^2 + c_i^2), \end{aligned}$$

and where we used the fact that $p,\ r_0 \in \mathbf{R}[s]$. To complete the proof, we show that $\bar{r}(s)$ is Hurwitz stable. This requires showing that $\mathcal{S}(c_r c_i) = -1$. Evaluating (5.14) at $s = j\omega_0$, we have

$$c_r + jc_i = \frac{r_0(j\omega_0)}{p(j\omega_0)} = \frac{H(u_0)}{F(u_0)} + j\omega_0 \frac{G(u_0)}{F(u_0)},$$

where

$$\begin{aligned} H(u) &:= h_0(u)f(u) - u g_0(u) e(u), \\ G(u) &:= g_0(u) f(u) - h_0(u) e(u), \\ F(u) &:= f(u)^2 - u e(u)^2. \end{aligned}$$

Since $p(j\omega_0) \neq 0$ by assumption, $f(u)$ and $e(u)$ can not be simultaneously zero at u_0. In all four possible cases

1. $f(u_0) = 0$, $e(u_0) \neq 0$,

2. $f(u_0) \neq 0$, $e(u_0) = 0$,

3. $\mathcal{S}[f(u_0)e(u_0)] = +1$,

4. $\mathcal{S}[f(u_0)e(u_0)] = -1$,

it is straightforward to show using (5.15) that $\mathcal{S}[H(u_0)G(u_0)] = -1$. Since $F(u_0) > 0$, this yields that $\mathcal{S}(c_r c_i) = -1$ and the proof is complete. ∎

In [100], Rantzer's phase growth condition is translated into conditions on $V_p(u)$ and $V_{sp}(u)$. This new form of Rantzer's condition was then used to construct new convex directions for Hurwitz stable polynomials. In what follows we state the phase growth condition in this form: $p(s)$ is a global convex direction if and only if

$$\begin{aligned} V_p(u) \leq 0 \ \forall u < 0 \quad &\text{such that} \quad f(u)e(u) \geq 0, \\ V_{sp}(u) \leq 0 \ \forall u < 0 \quad &\text{such that} \quad f(u)e(u) < 0. \end{aligned}$$

If a given $p(s)$ need not be a convex direction for the set of all Hurwitz stable polynomials, then it is natural that the upper bounds on $V_p(u)$ and $V_{sp}(u)$ are relaxed. In the extreme case of a single polynomial $q(s)$, these bounds turn out to be the ones given by (5.2).

Theorem 5.2 *Given a polynomial $p(s)$, the local convexity condition (LCC) holds for all $q(s) \in \mathcal{H}$ if and only if*

$$\begin{aligned} V_p(u) \leq 0 \quad \forall \quad u \in \{u < 0 : \ f(u)e(u) \geq 0\}, \\ V_{sp}(u) \leq 0 \quad \forall \quad u \in \{u < 0 : \ f(u)e(u) < 0\}. \end{aligned} \tag{5.16}$$

Proof. [If] If $deg\ p \leq 1$, then for $q(s)$ such that $deg\ q \leq 1$, LCC is easily seen to hold. For $q(s)$ such that $deg\ q > 1$, if (5.16) holds then the conditions in (5.2) hold for all $q \in \mathcal{H}$ such that $q + p \in \mathcal{H}$. By Theorem 5.1, LCC holds for all $q \in \mathcal{H}$.

If *deg* $p > 1$, then *deg* $q > 1$ in order for *deg* $q + \lambda p =$ *deg* q for all $\lambda \in [0,1]$. For such $q(s)$, if (5.16) holds, then again by Theorem 5.1 LCC is satisfied.

[Only if] If *deg* $p \leq 1$, then by direct computation it easy to see that (5.16) holds. We can therefore assume *deg* $p > 1$. Suppose for some $u_0 < 0$, one of the conditions in (5.16) fails. We construct $q \in \mathcal{H}$ for which LCC fails. Suppose that $V_p(u_0) > 0$ and $f(u_0)e(u_0) \geq 0$. Note that $f(u_0)$ and $e(u_0)$ can not simultaneously be zero since otherwise $V_p(u_0) = 0$. Hence, with $\omega_0 = \sqrt{-u_0}$, we have

$$p(j\omega_0) = f(u_0) + j\omega_0 e(u_0) \neq 0.$$

By Lemma 5.1, there exists $\bar{r} \in \mathcal{H}$, *deg* $\bar{r} \geq$ *deg* p such that $(\bar{r} + \alpha p)(j\omega_0) = 0$ for some $\alpha \in \mathbf{R}$. Since $\bar{r}(s)$ is Hurwitz stable , $\alpha \neq 0$. If we let $(\bar{k}(u),\ \bar{l}(u))$ be the even-odd components of $\bar{r}(s)$, then by $(\bar{r} + \alpha p)(j\omega_0) = 0$ and $\alpha \neq 0$, we have

$$(\bar{k}e - \bar{l}f)(u_0) = 0. \tag{5.17}$$

Let

$$r(s) := -\lambda_0 p(s) + (s^2 + \omega_0^2)\bar{r}(s) \tag{5.18}$$

for some arbitrary but fixed $\lambda_0 \in (0,1)$. If we let $(k(u),\ l(u))$ be the even-odd components of $r(s)$, we have $(k + \lambda_0 f)(u_0) = 0$ and $(l + \lambda_0 e)(u_0) = 0$ so that

$$V_{r+\lambda_0 p}(u_0) = 0,\ V_{s(r+\lambda_0 p)}(u_0) = 0. \tag{5.19}$$

We now show that, there exists $\epsilon > 0$ such that

$$V_{r+\lambda p}(u) > 0,\ V_{s(r+\lambda p)}(u) > 0\ \forall \lambda \in [\lambda_0 - \epsilon, \lambda_0) \cup (\lambda_0, \lambda_0 + \epsilon],\ \forall u < 0.$$

Note that

$$\begin{aligned} V_{r+\lambda p}(u) &= (k + \lambda f)'(u)(l + \lambda e)(u) - (k + \lambda f)(u)(l + \lambda e)'(u), \\ V_{s(r+\lambda p)}(u) &= (k + \lambda f)(u)(l + \lambda e)(u) - uV_{r+\lambda p}(u). \end{aligned}$$

By (5.18),

$$\begin{aligned} (k + \lambda f)(u) &= (u - u_0)\bar{k}(u), \\ (l + \lambda e)(u) &= (u - u_0)\bar{l}(u), \end{aligned}$$

so that

$$\begin{aligned} V_{r+\lambda p}(u) &= V_{r+\lambda_0 p}(u) + (\lambda-\lambda_0)(\bar{k}e - \bar{l}f)(u) + (\lambda-\lambda_0)(\bar{k}'e \\ &\quad -\bar{k}e' - \bar{l}'f + \bar{l}f')(u-u_0) + (\lambda-\lambda_0)^2 V_p(u). \end{aligned}$$

Hence using (5.17) and (5.19), we have

$$\begin{aligned} V_{r+\lambda p}(u_0) &= (\lambda-\lambda_0)^2 V_p(u_0), \\ V_{r+\lambda_0 p}(u) &= (u-u_0)^2 V_{\bar{r}}(u). \end{aligned}$$

Similarly,

$$\begin{aligned} V_{s(r+\lambda p)}(u_0) &= (\lambda-\lambda_0)^2 V_{sp}(u_0), \\ V_{s(r+\lambda_0 p)}(u) &= (u-u_0)^2 V_{s\bar{r}}(u). \end{aligned}$$

Since $\bar{r} \in \mathcal{H}$ and $deg(\bar{r}) \geq 2$, we can apply Lemma 2.1 to obtain

$$\begin{aligned} &V_{\bar{r}}(u) > 0, \\ &V_{s\bar{r}}(u) > 0, \end{aligned}$$

for all $u < 0$. Hence,

$$\begin{aligned} &V_{r+\lambda_0 p}(u) > 0, \\ &V_{s(r+\lambda_0 p)}(u) > 0, \end{aligned}$$

for all u such that $0 > u \neq u_0$. By our assumption,

$$V_p(u_0) > 0, \text{ and } f(u_0)e(u_0) \geq 0.$$

Hence,

$$V_{sp}(u_0) = f(u_0)e(u_0) - u_0 V_p(u_0) > 0.$$

Consequently,

$$\begin{aligned} &V_{r+\lambda p}(u_0) > 0, \\ &V_{s(r+\lambda p)}(u_0) > 0, \end{aligned}$$

for all $\lambda \in [0,1]$ such that $\lambda \neq \lambda_0$. It follows that, for some sufficiently small $\epsilon_1 > 0$, we have

$$\begin{aligned} &V_{r+\lambda p}(u) > 0 \ \forall u < 0 \ \forall \lambda \in [\lambda_0 - \epsilon_1, \lambda_0) \cup (\lambda_0, \lambda_0 + \epsilon_1], \\ &V_{s(r+\lambda p)}(u) > 0 \ \forall u < 0 \ \forall \lambda \in [\lambda_0 - \epsilon_1, \lambda_0) \cup (\lambda_0, \lambda_0 + \epsilon_1]. \end{aligned}$$

We now note, by

$$\begin{aligned}(k+\lambda_0 f)(u) &= (u-u_0)\bar{k}(u),\\ (l+\lambda_0 e)(u) &= (u-u_0)\bar{l}(u),\end{aligned}$$

and the fact that $(\bar{k},\bar{l})$ is a positive pair, that all the roots of $k+\lambda f$ and $l+\lambda e$ are real and negative for all $\lambda \in [\lambda_0-\epsilon_2, \lambda_0+\epsilon_2]$ for some sufficiently small ϵ_2. Therefore, for all $\lambda \in [\lambda_0-\epsilon, \lambda_0) \cup (\lambda_0, \lambda_0+\epsilon]$ with $\epsilon := \min\{\epsilon_1, \epsilon_2\}$, we have that $(k+\lambda f, l+\lambda e)$ is a positive pair by Lemma 2.1 so that $r+(\lambda_0+\epsilon)p,\ r+(\lambda_0-\epsilon)p \in \mathcal{H}$. If we now define

$$q(s) := \frac{1}{2\epsilon}[r(s)+(\lambda_0-\epsilon)p(s)],$$

then $q,\ q+p \in \mathcal{H}$, $deg(q+\lambda p) = deg(q)\ \forall\lambda \in [0,1]$, but

$$(q+0.5p)(j\omega_0) = (\frac{1}{2\epsilon})(r+\lambda_0 p)(j\omega_0) = 0$$

and LCC fails for this $q(s)$. If $u_0 < 0$ is such that $V_{sp}(u_0) > 0$ and $f(u_0)e(u_0) < 0$, then

$$u_0 V_p(u_0) = f(u_0)e(u_0) - V_{sp}(u_0) < 0$$

so that $V_p(u_0) > 0$. The construction of $q(s)$ for which LCC fails is exactly the same as above. ■

Example 5.1 *Consider*

$$\begin{aligned}p(s) &= 2s^5+9s^3+4s^2+6s+3,\\ q(s) &= 0.4s^5+2.1s^4+1.9s^3+4.2s^2+1.6s+1.6.\end{aligned}$$

We can easily check that $q(s)$ and $p(s)+q(s)$ are Hurwitz stable. For $u < -2$, $V_p(u) < 0$ and $V_{sp}(u) < 0$. From Figure 5. 2 we can see that the first and second condition of Theorem 5.2 fail in the intervals $[-1.183, -0.8139] \cup [-0.75, -0.317]$ and $[-0.8139, -0.75]$, respectively. Hence $p(s)$ is not a global convex direction. On the other hand, from Figure 5.3, we can see that the conditions of Theorem 5.1 are satisfied in the whole interval $[-2, 0]$. Hence LCC holds for the pair (p,q).

•

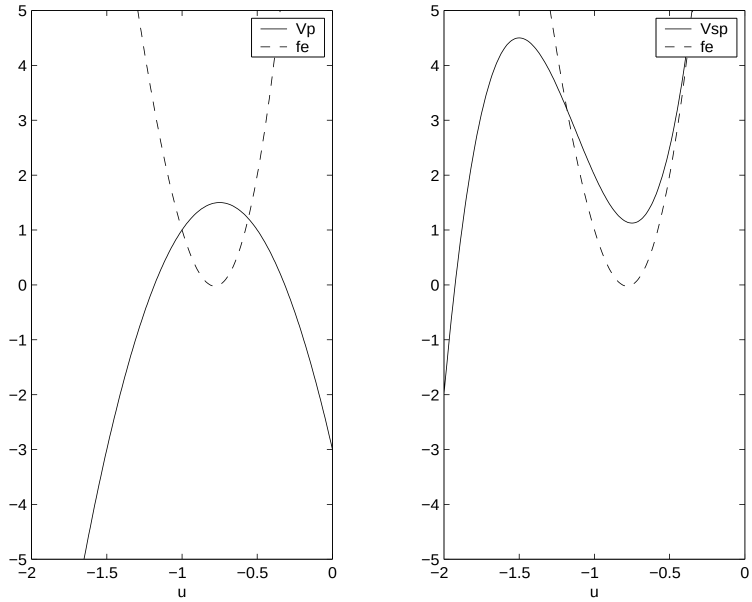

Figure 5.2: Checking conditions of Theorem 5.2.

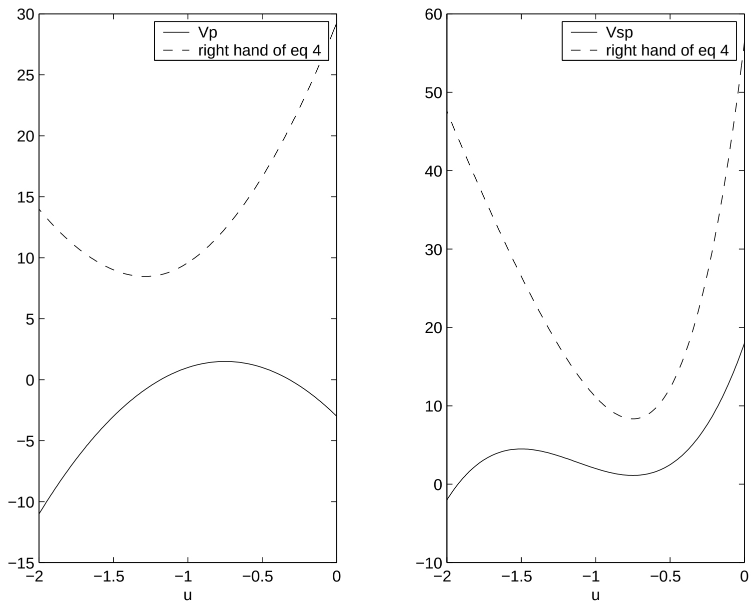

Figure 5.3: Checking conditions of Theorem 5.1.

Chapter 6

Conclusions

In this work, we studied the problem of determining all stabilizing controllers with fixed-order and fixed-structure, for a given single-input single-output, linear, time-invariant plant. Most synthesis problem can be posed as follows: given a plant, design a controller such that the feedback system is stable and an additional desired property hold, for example, the output tracks a step input. This problem can be solved using YJBK parameterization method. The importance of YJBK parameterization comes from the fact that this problem is decoupled and a two steps solution is given. First the set of all stabilizing controllers are computed, then a controller in this set is sought such that the desired second property holds. Although the problem of determining the set of all rational, proper, stabilizing controllers can be solved using YJBK parameterization method, it is important to note that this method can not accommodate fixing the order or the structure of the controller. This disadvantage leads to the synthesis of high-order controllers generally comparable to the order of the plant. Hence, there is a need to develop alternative methods which incorporate fixing the order and the structure of the controller.

The results obtained in this thesis are based on an extension of the well known Hermite-Biehler theorem. A generalization of this theorem enables us to compute the signature of a given polynomial. It was used in [34] to obtain a new method for the determination of stabilizing feedback gains for a given plant. We modified this result to determine the gains for which the closed loop system has a fixed signature. We also simplified the algorithm and the need for a search in an exponentially growing set is avoided. As an application of this algorithm, we studied the problem of characterizing local convex directions which arises in robust control. In Chapter 6, this problem is considered in more depth. Using a modified version of the Hermite-Biehler theorem, a necessary and sufficient condition is given for a polynomial to be a local convex direction of another polynomial. The relation between this result and the global convex direction concept of Rantzer is given. The new condition is also useful in determining subsets of polynomials for which local convexity condition holds.

In Chapter 4, a solution is given to the problem of determining all stabilizing first-order controllers for a given linear, time-invariant, scalar plant. The algorithm given consists of applying the stabilizing proportional controller result to a number of auxiliary plants. Once all stabilizing "gains" of the first-order controller are determined, several performance criteria such us maximum overshoot, settling time, and rise time can be evaluated . Although this method is computationally demanding as we have to calculate the performance indices for all the stabilizing controllers, in view of the recent results given in [41], first-order controllers for which the closed loop system is stable and the H_∞-norm of a related transfer function is less than a prescribed level, can be determine efficiently. We believe that further research is needed to develop similar results for other performance indices. The algorithm is then used to determine stabilizing first-order controllers for interval plants. It is also applicable to discrete-time systems by using a bilinear transformation of the complex plane. Using an extension of the

Hermite-Biehler theorem applicable to complex polynomials, the problem of stabilization with first-order controllers while achieving a desired degree of damping was solved.

Extension of these results to high-order controllers is outlined and the case of second-order controllers is studied in detail. A line for future research are systems with time delay. Since there is generalization of the Hermite-Biehler theorem applicable to time delay systems, we anticipate that similar results can be developed.

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