THE GEOMETRY OF TANGENT BUNDLE AND ITS APPLICATIONS

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By
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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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In this thesis, we first give a brief summary of the Riemannian Geometry which is the extension of Euclidean Geometry. Later we introduce the Finsler Geometry and the geometry of tangent bundle. Finally we give the applications of the geometry of the tangent bundle to the physics. We find Schwarzschild-like spacetime solutions and modified red shift formula.

Keywords: Riemannian geometry, Finsler geometry, the geometry of tangent bundle, Schwarzschild-like spacetime.

ÖZET

TANJANT DEMETİ GEOMETRİSİ VE UYGULAMALARI

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Bu tezde, ilk önce Öklid Geometrisinin genelleştirilmişi olan Rieman Geometrisinin kısa bir özetini vereceğiz. Sonra Finsler Geometri ve tanjant demeti geometrisini tanımlıyacağız. Son olarak tanjant demeti geometrisinin fiziğe uygulaması olarak Schwarzschild-gibi çözümlerini ve değiştirilmiş kızıl kayma formülünü vereceğiz.

Anahtar sözcükler: Rieman geometrisi, Finsler geometri, tanjant demeti geometrisi, Schwarzchild-gibi uzay zaman geometrisi.

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Contents

Chapter 1

Introduction

Euclid constructed his geometry on flat spaces. Euclidean Geometry is based on the points, lines, planes, angles, etc. in $\mathbb{R}^n$ and on some fundamental relationships between them given by some propositions and theorems which themselves follow from some axioms. (Pythagoras' Theorem, formulas in trigonometry, etc.) To understand the nature we need to construct geometry on spaces which are not flat. Gauss introduced nonflat spaces by studying 2-surfaces in $\mathbb{R}^3$. He measured the inner angles of a triangle whose vertices the peaks of three high mountains far apart in Germany, and tried to guess which geometry reflected the nature or the real world. Later G. F. B. Riemann, in his Habilitationsschrift, "Über die Hypotheses, welche der Geometrie grund-liegen" in 1854, opened an era in Geometry and in other areas of Mathematics and also in other branches of science. He proposed the notion of "Mannigfaltigkeiten" (manifolds) which are locally homeomorphic to Euclidean spaces. Then he introduced the notion of Riemannian metric which is needed to measure the length of curves, distance between two points, angles between vectors, etc., on a manifold. Riemannian metric is defined by a positive definite inner product as

$$ds^2 = g_{ij}(u)du^i du^j,$$

where u^i are local coordinates of an open subset U of the manifold and $g_{ij} = g_{ji}$ are smooth functions on U. He introduced the notion of curvature which is a measure of how much a surface is curved. A zero curvature surface in Riemannian geometry can be thought as an Euclidean plane. Riemann's idea was developed later by T. Levi-Civita and Elwin Christoffel by introducing the notion of torsion-free and metric-compatible connection which is called Christoffel Levi-Civita connection. This connection is one of the main tools in the classical tensor calculus. Tensor calculus plays an important role in the general relativity theory which was developed by Einstein in 1916 [1]-[4].

Riemann constructed his metric as a quadratic differential form. He recognized that his metric is a special case and thought there must be a general case. Paul Finsler introduced a metric on an m dimensional manifold M in the general case. He introduced the Finsler metric in his thesis in 1918 as

$$ds = F(u; du),\ u = (u^1, \dots, u^m),\ du = (du^1, \dots, du^m),$$

where $F(u; X)$ is the Finsler function. Details about the Finsler function and Finsler Geometry are given in Chapter 3. There is a close relation between Finsler geometry and calculus of variations. As Riemannian geometry, Finsler geometry also plays an important role in other areas of mathematics and has applications to other branches of science like biology, control theory, engineering and physics [5]-[13].

Finsler geometry is based on the projectivised tangent bundle (PTM) which is obtained by using line bundles or sphere bundle (SM) of a Finsler manifold M instead of TM in Riemannian geometry. Berwald, Cartan and Chern defined connections on Finsler manifold in 1926, 1934 and 1948, respectively. Cartan connection is metric-compatible but it has torsion. Chern connection is torsion-free and almost metric-compatible which is the generalization of the Christoffel Levi-Civita connection. Chern connection differs from the Berwald connection by the term $\dot{A}$ which will be introduced in Chapter 3. In Finsler geometry we have horizontal-horizontal and horizontal-vertical curvatures. Vertical-vertical curvature vanishes identically. The Riemannian curvature is called as the horizontal-horizontal curvature [14], [15], [16].

K. Yano and E. T. Davies [18] constructed geometries on the tangent bundles of Finsler and Riemannian manifolds in 1963. By using the components of fundamental tensor g_{ij} they constructed a metric on the tangent bundle. For the ease of calculations they introduced the special frames and also the anholonomic basis. They defined the adapted Christoffel Levi-Civita connection and curvature on the tangent bundle. Components of that adapted Christoffel Levi-Civita connection and curvature tensor have extra terms different from the classical ones which come from the anholonomic basis.

Howard E. Brandt in 1991 [19] constructed a metric of the 8-dimensional spacetime tangent bundle by using the proper acceleration a, which is given in Einstein's general relativity theory. He adapted the results of K. Yano and E. T. Davies's work [18] and used them in the spacetime tangent bundle which is constructed from the spacetime and the four-velocity space. In the same work he also calculated the Riemannian scalar curvature of the spacetime tangent bundle.

Howard E. Brandt again in 1991 [20] defined an action on the bundle manifold and considered the maximal acceleration invariant fiber bundles which are defined on a special spacetime as Riemannian Schwarzschild-like spacetime. By using the results of [19] and the Euler-Lagrange equations of motion from this action, he found the Schwarzschild-like solutions. He obtained the modified red shift formula for a static emitter and observer in Schwarzschild-like spacetime.

In Chapter 2 we give a brief summary of the Riemannian geometry. We define a metric and a linear connection on a manifold for the calculation of curvature tensors and geodesics. Then, we introduce the curvature which is one of the invariants of a manifold.

In Chapter 3 we introduce the Finsler geometry and its properties. Firstly, we state the Euler's theorem on homogeneous functions. By using that we define the Finsler function F. Then we consider the projectivised tangent bundle and define the Finsler metric by using the Finsler function F. We give the definition of the Chern connection Γ, the Cartan tensor A and obtain the components of the connection matrix on the projectivised tangent bundle which satisfies the torsion-free and almost metric-compatible structure equations. By writing the Chern connection in natural coordinates we obtain formulas for Γ, first and second curvature tensors R and P in natural coordinates. We obtain symmetry relations of horizontal and vertical derivatives of R and P. We define the Geodesic spray, the Flag curvature and the Ricci curvature. We prove the Schur's lemma which is related to the scalar Flag curvature. Finally we give some special Finsler spaces and compare their structure equations.

In Chapter 4 we introduce the geometry of the tangent bundle. We first define a metric on that tangent bundle and almost complex structure, then we introduce a special frame which makes the metric and almost complex structure simpler. By using these special frames we define an anholonomic basis, the Levi-Civita connection and the curvature tensor on the tangent bundle. Finally we obtain some relations among the connection coefficients and components of the Riemannian curvature tensor.

In Chapter 5 we construct the spacetime tangent bundle by using the spacetime and the four-velocity space. We first define metric on the spacetime tangent bundle by using the proper acceleration a, and then we define connection and calculate the connection coefficients, the curvature tensor, the Ricci and the scalar curvatures. Then we define an action on the spacetime tangent bundle and considering the Riemannian Schwarzschild-like spacetime, we obtain the Euler-Lagrange equations of motion coming from the action and then find the Schwarzschild-like solutions. Finally we obtain the modified red shift formula on the Schwarzschild-like spacetime. We observe that the contribution of the tangent bundle metric to the red shift formula is at order of the square of the gravitational constant G.

Chapter 2

Riemannian Geometry

In this chapter we will give a brief summary of the Riemannian geometry. We will not give the proof of theorems. One can find the proof of theorems from any Differential Geometry book which includes Riemannian geometry [1], [2], [14]. Firstly, on manifold we define a metric. Secondly, we define a linear connection on this manifold for the calculation of curvature tensors and geodesics. Connection plays the role of differential calculus in Euclidean geometry. Then, we will define the curvature which is one of the invariants for the manifolds. In this thesis we use the Einstein's summation convention, i.e., if we have repeated indices in the same term, then they are summed up over the range of the indices.

2.1 Riemann Metric

Let M be an m-dimensional C^∞ manifold and u^i, $1 \leq i \leq m$ be a local coordinate system on an open subset $U \subset M$. $T_p(M)$ and $T_p^*(M)$ are respectively the tangent and cotangent spaces of M at the point $p \in M$.
$T(M)$ and $T^*(M)$ are tangent and cotangent bundles on M such that

$$T(M) = \bigcup_{p \in M} T_p(M), \qquad T^*(M) = \bigcup_{p \in M} T_p^*(M). \tag{2.1}$$

$T^r_s(p;M)$ is the (r,s)-type tensor space of M at point $p \in M$ such that

$$T^r_s(p;M) = \underbrace{T_p(M) \otimes \cdots \otimes T_p(M)}_{r} \otimes \underbrace{T^*_p(M) \otimes \cdots \otimes T^*_p(M)}_{s}, \tag{2.2}$$

where r and s are contravariant and covariant order respectively. $T^r_s(M)$ is an (r,s)-type tensor bundle on M such that

$$T^r_s(M) = \bigcup_{p \in M} T^r_s(p;M). \tag{2.3}$$

The natural basis $\{\frac{\partial}{\partial u^i}, 1 \leq i \leq m\}$ and $\{du^i, 1 \leq i \leq m\}$ are local frame field of $T(M)$ and local coframe field of $T^*(M)$ on U, respectively. Tangent and cotangent vector fields X and Z are written in local coordinates as $X = X^i \frac{\partial}{\partial u^i}$ and $Z = Z_i du^i$, respectively.

Suppose G is a symmetric, positive definite $(0,2)$-type tensor space. Here symmetric means,

$$G(X,Y) = G(Y,X), \tag{2.4}$$

and positive definite means,

$$G(X,X) > 0 \qquad \text{if} \qquad X \neq 0, \tag{2.5}$$

for all $X, Y \in T_p(M)$. G can be written in local coordinate system $(U; u^i)$ as

$$G = g_{ij}(u) du^i \otimes du^j, \tag{2.6}$$

where $g_{ij} = g_{ji}$ are smooth functions on U. G defines a smooth inner product on $T_p(M)$ at every point $p \in M$ as

$$\langle X, Y \rangle = G(X,Y), \tag{2.7}$$

for all $X, Y \in T_p(M)$. Then $X, Y \in T(M)$ can be written in coordinate base as

$$X = X^i \frac{\partial}{\partial u^i}, \qquad Y = Y^j \frac{\partial}{\partial u^j}, \tag{2.8}$$

and $G(X,Y)$ takes the form

$$G(X,Y) = g_{ij} du^i \otimes du^j (X^k \frac{\partial}{\partial u^k}, Y^l \frac{\partial}{\partial u^l}) = g_{ij} X^k Y^l \delta^i_k \delta^j_l = g_{ij} X^i Y^j, \tag{2.9}$$

since $du^i(\frac{\partial}{\partial u^j}) = \delta^i_j$. By taking $X^k = Y^l = 1$, we have

$$g_{ij} = \langle \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \rangle. \tag{2.10}$$

(2.6) can be written as

$$ds^2 = G = g_{ij} du^i du^j, \tag{2.11}$$

and is called **Riemannian metric**. Here g_{ij} are components of Riemannian metric in matrix form. g^{ij} is components of the inverse matrix $(g_{ij})^{-1}$. We will lower and raise the indices by g_{ij} and g^{ij}.
For example,

$$g_{ij}\Gamma^j{}_{kl} = \Gamma_{ikl}, \qquad g^{ik}\Gamma_{jkl} = \Gamma_j{}^i{}_l. \tag{2.12}$$

Definition 2.1.1 *Let M be an m-dimensional C^∞ manifold with metric G. Then M is called* **Riemannian manifold**.

2.2 Riemannian Connection

Definition 2.2.1 *Let M be an m-dimensional Riemannian manifold. An affine connection on M is a map*

$$D : \Gamma(T(M)) \longrightarrow \Gamma(T^*(M) \otimes T(M)), \tag{2.13}$$

which have the following properties

1) $D(\frac{\partial}{\partial u^i} + \frac{\partial}{\partial u^j}) = D\frac{\partial}{\partial u^i} + D\frac{\partial}{\partial u^j}$,

2) $D(f\frac{\partial}{\partial u^i}) = df \otimes \frac{\partial}{\partial u^i} + fD\frac{\partial}{\partial u^i}$,

for any $f \in C^\infty(M)$. And locally it is defined as

$$D\frac{\partial}{\partial u^i} = w_i{}^j \frac{\partial}{\partial u^j}, \qquad \textit{and} \qquad Ddu^i = -w_j{}^i du^j. \tag{2.14}$$

Here $\Gamma(T(M))$ denotes the infinitely differentiable vector fields. ${w_i}^j$ are the components of the connection matrix of D and has the form as,

$$ {w_i}^j = {\Gamma^j}_{ik} du^k, \tag{2.15} $$

where ${\Gamma^j}_{ik}$ are infinitely differentiable functions on U. For $X \in T(M)$, the absolute differential of X has the form

$$ \begin{aligned} DX &= (dX^i + X^j {w_j}^i) \otimes \frac{\partial}{\partial u^i} \\ &= \Big(\frac{\partial X^i}{\partial u^j} + X^k {\Gamma^i}_{kj}\Big) du^j \otimes \frac{\partial}{\partial u^i}. \end{aligned} \tag{2.16} $$

From now on D denotes the affine connection. $D_X \frac{\partial}{\partial u^i}$ and $D_X du^i$ denote the covariant derivative of $\frac{\partial}{\partial u^i}$ and du^i along the vector field $X \in T_p(M)$, respectively. $D_X \frac{\partial}{\partial u^i}$ satisfies the following properties

1) $D_{X+Y} \frac{\partial}{\partial u^i} = D_X \frac{\partial}{\partial u^i} + D_Y \frac{\partial}{\partial u^i}$,

2) $D_{fX} \frac{\partial}{\partial u^i} = f D_X \frac{\partial}{\partial u^i}$,

3) $D_X(\frac{\partial}{\partial u^i} + \frac{\partial}{\partial u^j}) = D_X \frac{\partial}{\partial u^i} + D_X \frac{\partial}{\partial u^j}$,

4) $D_X(f \frac{\partial}{\partial u^i}) = (Xf) \frac{\partial}{\partial u^i} + f D_X \frac{\partial}{\partial u^i}$,

for any $X, Y \in T(M)$ and $f \in C^\infty(M)$. Taking $X^i = 1$ and since $X = X^i \frac{\partial}{\partial u^i}$ we have

$$ D_{\frac{\partial}{\partial u^j}} \frac{\partial}{\partial u^i} = {w_i}^k(\frac{\partial}{\partial u^j}) \frac{\partial}{\partial u^k} = {\Gamma^k}_{il} du^l(\frac{\partial}{\partial u^j}) \frac{\partial}{\partial u^k} = {\Gamma^k}_{ij} \frac{\partial}{\partial u^k}, \tag{2.17} $$

and

$$ D_{\frac{\partial}{\partial u^j}} du^i = -{w_k}^i(\frac{\partial}{\partial u^j}) du^k = -{\Gamma^i}_{kl} du^l(\frac{\partial}{\partial u^j}) du^k = -{\Gamma^i}_{kj} du^k, \tag{2.18} $$

since $du^l(\frac{\partial}{\partial u^j}) = \delta^l_j$ and ${w_i}^k = {\Gamma^k}_{il} du^l$.

Definition 2.2.2 *Let T be a linear map as*

$$ T : \Gamma(T(M)) \times \Gamma(T(M)) \longrightarrow \Gamma(T(M)) \tag{2.19} $$

and can be written in tensor form locally as

$$T = T^k{}_{ij} \frac{\partial}{\partial u^k} \otimes du^i \otimes du^j, \tag{2.20}$$

where $T^k_{ij} = \Gamma^k{}_{ji} - \Gamma^k{}_{ij}$ *and* $\Gamma^k{}_{ji}$ *are components of the connection coefficients. Then* T *is called* **torsion tensor** *of the connection* D*. And* $T(X, Y)$ *is defined as*

$$T(X, Y) = D_X Y - D_Y X - [X, Y], \tag{2.21}$$

for any tangent vector fields X, Y*.*

Definition 2.2.3 *Let* M *be a Riemannian manifold with metric* G *and affine connection* D*. If (2.21) equal to zero such that*

$$D_X Y - D_Y X = [X, Y] \qquad (\textit{i.e., the torsion tensor vanishes}), \tag{2.22}$$

then D *is called* **torsion-free connection***.*

Definition 2.2.4 *Let* M *be a Riemannian manifold with metric* G *and affine connection* D*. If*

$$DG = D(g_{ij} du^i \otimes du^j) = (dg_{ij} - w_i{}^k g_{kj} - w_j{}^k g_{ik}) \otimes du^i \otimes du^j = 0, \tag{2.23}$$

then D *is called* **metric-compatible connection***.*

Theorem 2.2.1 (**Fundamental Theorem of Riemannian Geometry**)*. Let* M *be an* m*-dimensional Riemannian manifold with metric* G*. There exist a unique linear connection* D *on manifold* M *which is metric-compatible and torsion-free. This connection is called the* **Riemannian connection** *or* **Christoffel Levi-Civita connection***.*

From now on, the connection on the Riemannian manifold will be takes as the Christoffel Levi-Civita connection. Now, we will obtain the connection coefficient $\Gamma^i{}_{jk}$ interms of the components of the Riemannian metric g_{ij}. D is torsion-free i.e., $T^k_{ij} = \Gamma^k{}_{ji} - \Gamma^k{}_{ij} = 0$, so we have

$$\Gamma^k{}_{ji} = \Gamma^k{}_{ij}, \qquad \text{and also} \qquad \Gamma_{kji} = \Gamma_{kij}, \tag{2.24}$$

and D is metric-compatible, so we have

$$dg_{ij} = w_i{}^k g_{kj} + w_j{}^k g_{ik}, \tag{2.25}$$

or equivalently

$$\frac{\partial g_{ij}}{\partial u^l} du^l = \Gamma^k{}_{il} du^l g_{kj} + \Gamma^k{}_{jl} du^l g_{ik} = (\Gamma_{jil} + \Gamma_{ijl}) du^l. \tag{2.26}$$

From (2.26) and cycling the indices, we have

$$\frac{\partial g_{ij}}{\partial u^l} = \Gamma_{jil} + \Gamma_{ijl}, \tag{2.27}$$

$$\frac{\partial g_{il}}{\partial u^j} = \Gamma_{lij} + \Gamma_{ilj}, \tag{2.28}$$

$$\frac{\partial g_{jl}}{\partial u^i} = \Gamma_{lji} + \Gamma_{jli}, \tag{2.29}$$

By taking the sum of the last two equalities minus the first one and using $\Gamma_{jki} = \Gamma_{ikj}$ we get

$$\Gamma_{lij} = \frac{1}{2}\Big(\frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l}\Big), \tag{2.30}$$

and multiplying by g^{lk}, we obtain

$$\Gamma^k{}_{ij} = \frac{1}{2} g^{kl}\Big(\frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l}\Big). \tag{2.31}$$

Here (2.30) and (2.31) are called **Christoffel symbols of the first kind and second kind**, respectively.

We have made use of the natural frame field of M. We could use an arbitrary frame field. Suppose $\{e_i,\ 1 \le i \le m\}$ is a local frame field and $\{\theta^i,\ 1 \le i \le m\}$ is coframe field. The connection on frame field e_i can be defined as

$$De_i = \theta_i{}^j e_j \tag{2.32}$$

where θ_i^j is components of the connection matrix of D. Torsion-free condition of D is

$$d\theta^i = \theta^j \wedge \theta_j{}^i \tag{2.33}$$

and metric-compatible condition is

$$dg_{ij} = \theta_i{}^k g_{kj} + \theta_j{}^k g_{ik}, \tag{2.34}$$

which can be obtained from the fact that $G = g_{ij}\theta^i \otimes \theta^j$ and $DG = (dg_{ij} - \theta_i{}^k g_{kj} - \theta_j{}^k g_{ik}) \otimes \theta^i \otimes \theta^j = 0$.

2.3 Geodesic

Let C be a differentiable curve on M with local coordinate system u^i such that

$$C : t \longrightarrow u^i = u^i(t) = C(t) \in M, \qquad t \in [a, b] \subset \mathbb{R}. \tag{2.35}$$

The length of C is defined as

$$L(C) = \int_a^b \sqrt{g_{ij}\frac{du^i}{dt}\frac{du^j}{dt}}dt. \tag{2.36}$$

Definition 2.3.1 *Let C be a curve on M as (2.35) and $X(t)$ be a tangent vector field with $X(t) = X^i(t)\frac{\partial}{\partial u^i}$. $X(t)$ is called parallel along the curve C if its covariant derivative in the direction of the tangent vector to $C$is zero,*

$$\begin{aligned} 0 = D_{\frac{du}{dt}}X(t) &= D_{\frac{du^i}{dt}\frac{\partial}{\partial u^i}}X(t) = \frac{du^i}{dt}D_{\frac{\partial}{\partial u^i}}X^j(t)\frac{\partial}{\partial u^j} \\ &= \frac{du^i}{dt}dX^j\frac{\partial}{\partial u^i}\frac{\partial}{\partial u^j} + \frac{du^i}{dt}X^j\Gamma^k{}_{ij}\frac{\partial}{\partial u^k} \\ &= \Big(\frac{dX^i}{dt} + X^k\Gamma^j{}_{ik}\frac{du^i}{dt}\Big)\frac{\partial}{\partial u^j}, \end{aligned} \tag{2.37}$$

From there we can write the parallel condition along C for $X(t)$ as

$$\frac{dX^i}{dt} + X^k\Gamma^j{}_{ik}\frac{du^i}{dt} = 0. \tag{2.38}$$

Definition 2.3.2 *A differentiable parametrized curve $C(t)$ as in (2.35) is called* **geodesic** *if its tangent vectors are parallel along $C(t)$. Equivalently , C(t) is* **geodesic** *if and only if*

$$\frac{d^2u^i}{dt^2} + \Gamma^j{}_{ik}\frac{du^i}{dt}\frac{du^k}{dt} = 0. \tag{2.39}$$

And since $C(t)$ is on M, we call also **geodesic** *of the Riemannian manifold M.*

2.4 Curvature

Definition 2.4.1 $\Omega_i{}^j = dw_i{}^j - w_i{}^h \wedge w_h{}^j$ *are called the components of the* **curvature matrix** *of D.*

Definition 2.4.2 *Let R be a map as*

$$R : \Gamma(T(M)) \times \Gamma(T(M)) \times \Gamma(T(M)) \longrightarrow \Gamma(T(M)), \tag{2.40}$$

and can be written in tensor form locally as

$$R = R_i{}^j{}_{kl} \frac{\partial}{\partial u^j} \otimes du^i \otimes du^k \otimes du^l. \tag{2.41}$$

Then R is called **curvature tensor** *of the connection D. $R(X, Y)$ is defined as*

$$R(X, Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z, \tag{2.42}$$

for any tangent vector fields X, Y, Z.

We will obtain the coefficient of the curvature $R_i{}^j{}_{kl}$ in terms of the Christoffel symbols by using the curvature matrix. Consider the curvature matrix

$$\Omega_i{}^j = dw_i{}^j - w_i{}^h \wedge w_h{}^j. \tag{2.43}$$

By writing $w_i{}^j = \Gamma^j{}_{ik} du^k$, we get

$$\begin{aligned}
\Omega_i{}^j &= d(\Gamma^j{}_{ik} du^k) - (\Gamma^h{}_{il} du^l) \wedge (\Gamma^j{}_{hk} du^k) \\
&= \frac{\partial \Gamma^j{}_{ik}}{\partial u^l} du^l \wedge du^k - \Gamma^h{}_{il} \Gamma^j{}_{hk} du^l \wedge du^k \\
&= \frac{1}{2} \frac{\partial \Gamma^j{}_{ik}}{\partial u^l} du^l \wedge du^k - \frac{1}{2} \frac{\partial \Gamma^j{}_{ik}}{\partial u^l} du^k \wedge du^l + \\
&- \frac{1}{2} \Gamma^h{}_{il} \Gamma^j{}_{hk} du^l \wedge du^k + \frac{1}{2} \Gamma^h{}_{il} \Gamma^j{}_{hk} du^k \wedge du^l \\
&= \frac{1}{2} \left[\frac{\partial \Gamma^j{}_{ik}}{\partial u^l} - \frac{\partial \Gamma^j{}_{il}}{\partial u^k} + \Gamma^h{}_{ik} \Gamma^j{}_{hl} - \Gamma^h{}_{il} \Gamma^j{}_{hk} \right] du^l \wedge du^k \\
&\equiv \frac{1}{2} R_i{}^j{}_{lk} du^l \wedge du^k
\end{aligned} \tag{2.44}$$

From there

$$\Omega_i{}^j = \frac{1}{2} R_i{}^j{}_{lk} du^l \wedge du^k, \tag{2.45}$$

where

$$R_i{}^j{}_{lk} = \frac{\partial \Gamma^j{}_{ik}}{\partial u^l} - \frac{\partial \Gamma^j{}_{il}}{\partial u^k} + \Gamma^h{}_{ik} \Gamma^j{}_{hl} - \Gamma^h{}_{il} \Gamma^j{}_{hk}. \tag{2.46}$$

Multiplying both sides of (2.45) by g_{jh}, we get

$$\Omega_{ij} = \frac{1}{2} R_{ijlk} du^l \wedge du^k, \tag{2.47}$$

where

$$\Omega_{ij} = dw_{ij} + w_i{}^l \wedge w_{jl}. \tag{2.48}$$

In the above equation applying the procedure in (2.44), we get

$$R_{ijlk} = \frac{\partial \Gamma_{jik}}{\partial u^l} - \frac{\partial \Gamma_{jil}}{\partial u^k} + \Gamma^h{}_{il}\Gamma_{hjk} - \Gamma^h{}_{ik}\Gamma_{hjl}. \tag{2.49}$$

The curvature tensor can be written as

$$R = R_{ijkl} du^i \otimes du^i \otimes du^j \otimes du^l \otimes du^l. \tag{2.50}$$

Theorem 2.4.1 *The curvature tensor R_{ijkl} have the following properties:*

1) $R_{ijkl} = -R_{jikl} = -R_{ijlk}$,

2) $R_{ijkl} + R_{iklj} + R_{iljk} = 0$,

3) $R_{ijkl} = R_{klij}$.

By contracting (2.41) with $(Z, W)X$ and (2.50) with (X, Y, Z, W), respectively, we get

$$R(Z, W)X = R_i{}^j{}_{kl} X^i Z^k W^l \frac{\partial}{\partial u^j}, \tag{2.51}$$

and

$$R(X, Y, Z, W) = R_{ijkl} X^i Y^j Z^k W^l, \tag{2.52}$$

or equivalently

$$R(X, Y, Z, W) = (R(Z, W)X) \cdot Y, \tag{2.53}$$

where X, Y, Z, W are tangent vector fields and $du^i(\frac{\partial}{\partial u^j}) = \delta^i_j$.

By (Theorem 2.4.1) and (2.52) we have the followings

1) R(X,Y,Z,W)=-R(X,Y,W,Z)=-R(Y,X,Z,W),

2) R(X,Y,Z,W)+R(X,Z,W,Y)+R(X,W,Y,Z)=0,

3) R(X,Y,Z,W)=R(Z,W,X,Y).

For the Riemannian metric G we can define the following function

$$G(X,Y,Z,W) = G(X,Z)G(Y,W) - G(X,W)G(Y,Z). \tag{2.54}$$

Definition 2.4.3 *For each two dimensional subspace E of $T_p(M)$ we define the sectional curvature on E*

$$K(E) = -\frac{R(X,Y,X,Y)}{G(X,Y,X,Y)}, \tag{2.55}$$

where $X,Y \in T_p(M)$ are any vectors spanning E.

The term which is in the denominator of the sectional curvature, denotes the square of the area of parallelogram spanned by X,Y and so it nonzero, such that

$$G(X,Y,X,Y) = \|X\|^2\|Y\|^2 - \langle X,Y\rangle^2. \tag{2.56}$$

Definition 2.4.4 *An m-dimensional Riemannian manifold M is called* **wandering** *at point $p \in M$ if $K(E)$ is constant at p.*

Definition 2.4.5 *An m-dimensional Riemannian manifold M is called a* **constant curvature space** *if K(p) is constant and everywhere wandering.*

Theorem 2.4.2 (**F. Schur's Theorem**). *Suppose M is a connected m-dimensional Riemannian manifold that is everywhere wandering. If $m \geq 3$, then M is a constant curvature space.*

Definition 2.4.6 *The trace of the curvature tensor is called* **Ricci curvature***. The components of the Rici curvature are*

$$R_{ij} = R_{i\ \ kj}^{\ k}. \tag{2.57}$$

Definition 2.4.7 *The trace of the Ricci curvature is called* **scalar curvature***. The scalar curvature is*

$$R = R_{ij}g^{ij}. \tag{2.58}$$

Definition 2.4.8 *If Ricci curvature tensor is scalar multiple of the metric tensor, then Riemannian metric is called* **Einstein metric**. *The components of the Einstein metric are*

$$G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R. \tag{2.59}$$

2.5 Summary

Riemannian metric: $G = g_{ij}(u) du^i du^j, \ g_{ij} = \langle \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \rangle,$
Levi-Civita Connection: $D_{\frac{\partial}{\partial u^j}} \frac{\partial}{\partial u^i} = \Gamma^k{}_{ij} \frac{\partial}{\partial u^k}, \ D_{\frac{\partial}{\partial u^j}} du^i = -\Gamma^i{}_{kj} du^k,$
Connection coefficients:

$$\Gamma_{lij} = \frac{1}{2}\Big(\frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l}\Big),$$

$$\Gamma^k{}_{ij} = \frac{1}{2} g^{kl}\Big(\frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l}\Big),$$

Torsion tensor: $T(X, Y) = D_X Y - D_Y X - [X, Y],$
Torsion-free condition: $D_X Y - D_Y X - [X, Y] = 0,$
Metric compatibility condition: $dg_{ij} - w_i{}^k g_{kj} - w_j{}^k g_{ik} = 0,$ where $w_i{}^j = \Gamma^j{}_{ik} du^k,$
Curvature tensor: $R(X, Y) = D_X D_Y - D_Y D_X - D_{[X,Y]},$
Curvature matrix: $\Omega_i{}^j = dw_i{}^j - w_i{}^h \wedge w_h{}^j = \frac{1}{2} R_i{}^j{}_{lk} du^l \wedge du^k,$
Components of Curvature matrix:

$$R_i{}^j{}_{lk} = \frac{\partial \Gamma^j{}_{ik}}{\partial u^l} - \frac{\partial \Gamma^j{}_{il}}{\partial u^k} + \Gamma^h{}_{ik} \Gamma^j{}_{hl} - \Gamma^h{}_{il} \Gamma^j{}_{hk},$$

with properties

1) $R_{ijkl} = -R_{jikl} = -R_{ijlk},$

2) $R_{ijkl} + R_{iklj} + R_{iljk} = 0,$

3) $R_{ijkl} = R_{klij},$

Sectional curvature: $K(E) = -\frac{R(X,Y,X,Y)}{G(X,Y,X,Y)},$
$G(X, Y, Z, W) \equiv G(X, Z)G(Y, W) - G(X, W)G(Y, Z),$

Ricci curvature: $R_{ij} = R_i{}^k{}_{kj}$,
Scalar curvature: $R = R_{ij} g^{ij}$,
Einstein tensor: $G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R$,

Chapter 3

Finsler Geometry

In this chapter we will be interested in Finsler geometry and its properties [14], [15], [16]. Firstly, we state the Euler's theorem on homogeneous functions. By using that we define the Finsler function F. Then we consider projectivised tangent bundle and define Finsler metric by using Finsler function F. We define the Chern connection Γ, the Cartan tensor A and obtain the components of the connection matrix on projectivised tangent bundle which satisfy torsion-free and almost metric-compatible structure equations. By writing the Chern connection in natural coordinates we obtain formulas for Γ, first and second curvature tensors R and P in natural coordinates. We obtain some relations about symmetries, horizontal and vertical derivatives of R and P. We define Geodesic spray, Flag curvature and Ricci curvature in terms of the Flag curvature. We prove the Schur's lemma which is related to scalar Flag curvature. At the end we give some special Finsler spaces and compare their structure equations. Our convention is as follows: Latin indices denote the natural bases and run from 1 to m (except m). Greek indices denote the frame field and coframe field and run from 1 to m. Greek indices with bar run 1 to $m-1$. m is the dimension of the Finsler manifold.

3.1 Finsler Structure and Euler's Theorem

Let M be an m-dimensional C^∞ manifold and u^i , $1 \leq i \leq m$ be a local coordinates on open subset $U \subset M$. TM and T^*M are tangent and cotangent bundle with natural bases $\frac{\partial}{\partial u^i}$ and du^i, respectively on M. They have the natural projection as

$$\pi : TM \to M \qquad , \qquad \pi^* : T^*M \to M.$$

(u^i, X^i), $1 \leq i \leq m$ are local coordinates on an open subset $\pi^{-1}(U) \subset T(M)$. We can write any tangent and cotangent vector as

$$X = X^i \frac{\partial}{\partial u^i}, \qquad Z = Z_i du^i. \tag{3.1}$$

Definition 3.1.1 *Let $F : \mathbb{R}^m \longrightarrow \mathbb{R}$ be a real-valued function. $F(X^i)$ is homogeneous of degree k in X^i if*

$$F(\lambda X^i) = \lambda^k F(X^i), \qquad for\ \lambda \geq 0,\ i = 1, \ldots, m. \tag{3.2}$$

Theorem 3.1.1 (**Euler**). *Let $F : \mathbb{R}^m \longrightarrow \mathbb{R}$ be a real-vaued function. If F is homogeneous of degree one in X then*

$$X^i \frac{\partial F(X)}{\partial X^i} = F(X), \qquad X = (X^1, \ldots, X^m),\ i = 1, \ldots, m. \tag{3.3}$$

Proof: Taking $k = 1$ in (3.2), we get

$$F(\lambda X) = \lambda F(X). \tag{3.4}$$

Differentiating (3.4) with respect to X^i, we have

$$\frac{\partial F(\lambda X)}{\partial(\lambda X^i)} = \frac{\partial F(X)}{\partial X^i}. \tag{3.5}$$

Differentiating (3.4) with respect to λ, we have

$$\frac{\partial F(\lambda X)}{\partial(\lambda X^i)} X^i = F(X). \tag{3.6}$$

Inserting (3.5) in (3.6), we get

$$X^i \frac{\partial F(X)}{\partial X^i} = F(X), \qquad i = 1, \ldots, m. \tag{3.7}$$

$\boxdot$

From now on F_{X^i} and F_{u^i} denotes the partial dervative of F with respect to X^i and u^i, respectively. In that notation Euler's theorem take the form

$$X^i F_{X^i} = F \tag{3.8}$$

Differentiating (3.8) with respect to X^i and u^i, we obtain followings as corollary of Euler's theorem.

Corollary 3.1.1 *If F is homogeneous function of $X = (X^1, \ldots, X^m)$ with degree one such that $X^i F_{X^i} = F$ then we have*

1) $X^j F_{X^i X^j} = 0,$

2) $X^k F_{X^i X^j X^k} = -F_{X^i X^j},$

3) $X^l F_{X^i X^j X^k X^k} = -2F_{X^i X^j X^k},$

4) $X^i F_{X^i u^i} = F_{u^i}.$

Definition 3.1.2 *Suppose F is a function on tangent bundle $T(M)$ such that $F : TM \to [0, \infty)$ and has the following properties:*

1) F is C^∞ on $TM \setminus 0$,

2) Positive symmetrically homogeneous of degree one in the X's

$$F(u; \lambda X) = \lambda F(u; X),$$

where $\lambda \in R$, $u = (u^1, \ldots, u^m)$, $X = (X^1, \ldots, X^m)$,

3) F has the strong convexity property such that the $m \times m$ matrix

$$(g_{ij}) = \left(\frac{1}{2}F^2\right)_{X^i X^j}$$

is positive definite at every point of $TM \setminus 0$.

4) $F(u, X) > 0 \qquad for \qquad X \neq 0,$

5) $F(u, X) + F(u, Y) \geq F(u, X + Y)$.

Then F *is called* ***Finsler function or Finsler structure*** *of* M.

Riemannian geometry is constructed on the quadratic form as

$$ds^2 = F^2 = g_{ij}(u)du^i du^j. \tag{3.9}$$

We construct the Finsler geometry on

$$ds = F(u; du), \tag{3.10}$$

such that F does not satisfy the quadratic restriction (3.9). Here F is the Finsler function.

Definition 3.1.3 *Let* C *be a differentiable curve on* M *such that*

$$C : t \longrightarrow u^i = u^i(t) = C(t) \in M, \qquad t \in [a, b] \subset \mathbb{R}. \tag{3.11}$$

If the arclength of C *is defined as*

$$L(C) = \int ds = \int_a^b F(u, \frac{du}{dt})dt, \tag{3.12}$$

where $u = (u^1, \ldots, u^m)$, $\frac{du}{dt} = (\frac{du^1}{dt}, \ldots, \frac{du^m}{dt})$ *and* F *is Finsler function. Then* (M, F) *is called the* **Finsler manifold**.

3.2 Projective Tangent Bundle, Finsler Metric and Hilbert Form

Projectivised tangent bundle (PTM) of M is obtained from T(M), by identifying the non-zero vectors differening from each other by a real factor i.e. the bundles of line elements of M. (u^i, X^i), $1 \leq i \leq m$ are also local coordinates of PTM with

X^i's are homogeneous coordinates of degree one. On PTM, there are quantities which are homogeneous of degree zero in X. So F is not defined on PTM, since it is homogeneous of degree one in X^i.
Let p be a projection map such that $p : PTM \longrightarrow M$ and defined as

$$p(u^i, X^i) = (u^i). \tag{3.13}$$

p^*TM is the m-dimensional pulled-back tangent bundle with dual p^*T^*M. $T_u(M)$ and $T_u^*(M)$ are fibers of p^*TM and p^*T^*M in local coordinates u^i, respectively. PTM is $(2m-1)$-dimensional base manifold of p^*TM.

Definition 3.2.1 *The one-form on PTM*

$$w = F_{X^i} du^i \tag{3.14}$$

is called the **Hilbert form**.

F_{X^i} is homogeneous of degree zero in X^i. So w is homogeneous of degree zero in X^i and it is on PTM. By Euler's theorem, arclength of C can be written in terms of the Hilbert form as

$$L(C) = \int ds = \int_a^b F(u, \frac{du}{dt})dt = \int_a^b w. \tag{3.15}$$

Let $\frac{\partial}{\partial u^i}$ and du^i be bases of $T(M)$ and $T^*(M)$, respectively. By using these bases, we write the sections of p^*TM and p^*T^*M.

Let

$$e_\alpha = e_\alpha{}^i \frac{\partial}{\partial u^i}, \qquad \alpha = 1, \ldots, m. \tag{3.16}$$

be section of (or an orthonormal frame field on the bundle) p^*TM , and the differential one-form on PTM

$$w^\alpha = e^\alpha{}_i du^i, \qquad \alpha = 1, \ldots, m. \tag{3.17}$$

be section of p^*T^*M, which is coframe field of e_α.

These sections have the orthonormality and duality conditions as

$$(e_\alpha, e_\beta) \equiv e_\alpha{}^k e_\beta{}^i (\frac{\partial}{\partial u^k}, \frac{\partial}{\partial u^i}) = e_\alpha{}^k g_{ki} e_\beta{}^i = \delta_{\alpha\beta}. \tag{3.18}$$

where $(\frac{\partial}{\partial u^k}, \frac{\partial}{\partial u^i}) = g_{ki}$.
From (3.18), we can write

$$g^{lk} = e_\alpha{}^l \delta^{\alpha\beta} e_\beta{}^k, \qquad l, k = 1, \ldots, m. \tag{3.19}$$

and

$$\delta^{\alpha\beta} e_\alpha{}^i g_{ij} = e^\beta{}_j, \qquad \beta = 1, \ldots, m.\ j = 1, \ldots, m. \tag{3.20}$$

By using $e_\alpha{}^i e^\beta{}_i = \delta_\alpha^\beta$ we have the following

$$\begin{aligned} \langle e_\alpha, w^\beta \rangle &= \langle e_\alpha{}^k \frac{\partial}{\partial u^k}, e^\beta{}_i du^i \rangle = e_\alpha{}^k e^\beta{}_i \delta_k^i \\ &= e_\alpha{}^k e_k{}^\beta = \delta_\alpha^\beta, \end{aligned} \tag{3.21}$$

where $(e^\alpha{}_i)$ and $(e_\alpha{}^i)$ are inverse to each other and $\langle \frac{\partial}{\partial u^i}, du^j \rangle = \delta_i^j$. By using (3.16) and (3.17) we can write the inversion formula

$$\frac{\partial}{\partial u^i} = e_\alpha e^\alpha{}_i, \qquad du^i = w^\alpha e_\alpha{}^i, \qquad i = 1, \ldots, m. \tag{3.22}$$

Definition 3.2.2 *Suppose F is Finsler function and G is a symmetric, positive definite $(0,2)$ -type tensor such that*

$$G = g_{ij} du^i \otimes du^j \equiv \frac{\partial^2(\frac{1}{2}F^2)}{\partial X^i \partial X^j} du^i \otimes du^j. \tag{3.23}$$

Then G is called the ***Finsler metric (or fundamental tensor)****.*

Here g_{ij} has the form

$$g_{ij} = F F_{X^i X^j} + F_{X^i} F_{X^j}. \tag{3.24}$$

By contracting (3.24) with $X^i X^j$ and using Euler's theorem and its corollary, we obtain the following useful fact

$$g_{ij} X^i X^j = F F_{X^i X^j} X^i X^j + X^i F_{X^i} X^j F_{X^j} = F^2. \tag{3.25}$$

Here g_{ij} is the components of the metric tensor and homogeneous of degree zero in X^i's and it is defined on PTM. In Riemannian geometry, components of metric g_{ij} are only functions of local coordinates u^i of M. In Finsler geometry, g_{ij} are functions of local coordinates (u^i, X^i) of PTM. We now write the expressions for the global sections on p^*TM and p^*T^*M, respectively as

$$e_m = \frac{X^i}{F}\frac{\partial}{\partial u^i} = e_m^{\ i}\frac{\partial}{\partial u^i}, \tag{3.26}$$

from that

$$e_m^{\ i} = \frac{X^i}{F}, \tag{3.27}$$

and

$$w = w^m = F_{X^i} du^i = e^m_{\ i} du^i, \tag{3.28}$$

from that

$$e^m_{\ i} = F_{X^i}. \tag{3.29}$$

By using Euler's theorem, (3.27) and (3.29) we have

$$g_{ij} e_m^{\ i} = F F_{X^i X^j}\frac{X^i}{F} + \frac{X^i}{F} F_{X^i} F_{X^j} = F_{X^j} = e^m_{\ j}. \tag{3.30}$$

We can write the following useful relations

$$e^{\bar{\alpha}}_{\ i} X^i = 0, \qquad \bar{\alpha} = 1, \ldots, m-1 \tag{3.31}$$

since

$$e^{\bar{\alpha}}_{\ i}\frac{X^i}{F} = e^{\bar{\alpha}}_{\ i} e_m^{\ i} = \delta^{\bar{\alpha}}_m = 0,$$

and

$$F_{X^i} e_{\bar{\alpha}}^{\ i} = e_i^{\ m} e_{\bar{\alpha}}^{\ i} = \delta^m_{\bar{\alpha}} = 0, \qquad \bar{\alpha} = 1, \ldots, m-1. \tag{3.32}$$

To construct contact structure on PTM and torsion-free condition of Chern connection, we will obtain the exterior derivative of w^m on PTM. Taking the exterior derivative of w^m, we get

$$\begin{aligned} dw = dw^m &= F_{u^i X^k} du^i \wedge du^k + F_{X^i X^k} dX^j \wedge du^k \\ &= F_{u^i X^k} e_\alpha^{\ i} e_\beta^{\ k} w^\alpha \wedge w^\beta + F_{X^i X^k} e_\beta^{\ k} dX^j \wedge w^\beta. \end{aligned} \tag{3.33}$$

Expanding the summation indices as $\alpha = (\bar{\alpha}, m)$ and using the corallary of Euler's theorem we can write dw^m in a closed form as

$$dw^m = w^{\bar{\alpha}} \wedge w_{\bar{\alpha}}^{\ m}, \tag{3.34}$$

where one-forms ${w_{\bar{\alpha}}}^m$ has the following general form

$$\begin{aligned} {w_{\bar{\alpha}}}^m &= -{e_{\bar{\alpha}}}^i F_{X^i X^j} dX^j + \frac{{e_{\bar{\alpha}}}^i}{F}(F_{u^i} - X^j F_{X^i u^j}) w^m \\ &+ {e_{\bar{\alpha}}}^i {e_{\bar{\beta}}}^j F_{u^i X^j} w^{\bar{\beta}} + \lambda_{\bar{\alpha}\bar{\beta}} w^{\bar{\beta}}, \end{aligned} \tag{3.35}$$

where $\lambda_{\bar{\alpha}\bar{\beta}}$ are arbitrary but it must satisfy $\lambda_{\bar{\alpha}\bar{\beta}} = \lambda_{\bar{\beta}\bar{\alpha}}$. Later, we will obtain the $\lambda_{\bar{\alpha}\bar{\beta}}$ interms of F. We now prove the condition of having a contact structure by the following lemma.

Lemma 1 . *The Hilbert form $w = F_{X^i} du^i$ satisfies the following condition*

$$w \wedge (dw)^{m-1} \neq 0, \tag{3.36}$$

on PTM.

Proof: Denote $I = w \wedge (dw)^{m-1}$. Writing the expressions for dw and ${w_{\bar{\alpha}}}^m$ and using (3.24),(3.32) and the fact that wedge product of two one forms is zero, we obtain

$$\begin{aligned} I &= w \wedge (dw)^{m-1} = w \wedge dw \wedge \ldots dw \\ &= (-1)^{m-2} w \wedge w^{\bar{\alpha}_1} \wedge \ldots \wedge w^{\bar{\alpha}_{m-1}} \wedge ({w_{\bar{\alpha}_1}}^m \wedge \ldots {w_{\bar{\alpha}_{m-1}}}^m) \\ &= \mp \bigwedge_{\alpha} w^{\alpha} \bigwedge_{\bar{\alpha}} {w_{\bar{\alpha}}}^m \\ &= \mp \bigwedge_{\alpha} w^{\alpha} \bigwedge_{\bar{\alpha}} \left({e_{\bar{\alpha}}}^j F_{X^j X^k} dX^k \right) \\ &= \mp \bigwedge_{\alpha} w^{\alpha} \bigwedge_{\bar{\alpha}} \left({e_{\bar{\alpha}}}^j g_{jk} dX^k \right), \end{aligned} \tag{3.37}$$

since g_{ij} positive definite and ${e_{\bar{\alpha}}}^j$ are invertibles, so ${e_{\bar{\alpha}}}^j g_{jk} dX^k$ are linearly independent on PTM. The one-forms w^{α} are also linearly independent and does not have term dX^k. Thus $\bigwedge_{\alpha} w^{\alpha}$ and $\bigwedge_{\bar{\alpha}} \left({e_{\bar{\alpha}}}^j g_{jk} dX^k \right)$ are linearly independent. Wedge product of two linearly independent terms is non zero, so

$$I = \mp \bigwedge_{\alpha} w^{\alpha} \bigwedge_{\bar{\alpha}} \left({e_{\bar{\alpha}}}^j g_{jk} dX^k \right) \neq 0.$$

$\boxdot$

Definition 3.2.3 *If there is a one-form w which satisfies (3.36), then* $(2m-1)$-*dimensional manifold PTM have a contact structure and w is called a contact form.*

3.3 The Chern Connection

Let the connection one-from in the bundle PTM defined as

$$De_\alpha = w_\alpha{}^\beta e_\beta, \tag{3.38}$$

where e_α is the orthonormal frame field with dual coframe field w^α, and $w_\alpha{}^\beta$ are components of connection matrix (of one-form) on PTM. The connection is called **torsion-free** if

$$dw^\alpha = w^\beta \wedge w_\beta{}^\alpha, \tag{3.39}$$

since

$$D(e_\alpha \otimes w^\alpha) = w_\alpha{}^\beta e_\beta \wedge w^\alpha + e_\alpha dw^\alpha = e_\alpha(dw^\alpha - w^\beta \wedge w_\beta{}^\alpha) = 0. \tag{3.40}$$

3.3.1 Determination of the Torsion-Free Connection

We have the expression for dw^m in (3.34) and $w_{\bar\alpha}{}^m$ in (3.35). We choose $w_m{}^m = 0$. By differentiating $w^{\bar\alpha}$ in (3.17) and using (3.22), $e^{\bar\alpha}{}_k d(e_m{}^k) = e^{\bar\alpha}{}_k \frac{dX^k}{F}$, we obtain

$$\begin{aligned}
dw^{\bar\alpha} &= de^{\bar\alpha}{}_k \wedge du^k \\
&= e_\beta{}^k de^{\bar\alpha}{}_k \wedge w^\beta = d(e_\beta{}^k e^{\bar\alpha}{}_k) \wedge w^\beta - d(e_\beta{}^k) e^{\bar\alpha}{}_k \wedge w^\beta \\
&= -e^{\bar\alpha}{}_k de_\beta{}^k \wedge w^\beta = -e^{\bar\alpha}{}_k de_{\bar\beta}{}^k \wedge w^{\bar\beta} - e^{\bar\alpha}{}_k de_m{}^k \wedge w^m \\
&= w^{\bar\beta} \wedge (e^{\bar\alpha}{}_k de_{\bar\beta}{}^k) + w^m \wedge (\frac{1}{F} e^{\bar\alpha}{}_k dX^k).
\end{aligned} \tag{3.41}$$

$dw^{\bar\alpha}$ can be written in a closed form as

$$dw^{\bar\alpha} = w^{\bar\beta} \wedge w_{\bar\beta}{}^{\bar\alpha} + w^m \wedge w_m{}^{\bar\alpha}, \tag{3.42}$$

where $w_m{}^{\bar\alpha}$ and $w_{\bar\beta}{}^{\bar\alpha}$ have the general form

$$w_{\bar\beta}{}^{\bar\alpha} = e^{\bar\alpha}{}_k de_{\bar\beta}{}^k + \xi_{\bar\beta}{}^{\bar\alpha} w^m + \mu_{\bar\beta\bar\gamma}{}^{\bar\alpha} w^{\bar\gamma}, \tag{3.43}$$

$$w_m{}^{\bar\alpha} = \frac{1}{F} e^{\bar\alpha}{}_k dX^k + \xi_\nu{}^{\bar\alpha} w^\nu. \tag{3.44}$$

Here $\xi_{\nu}{}^{\bar{\alpha}}$ and $\mu_{\bar{\beta}\bar{\gamma}}{}^{\bar{\alpha}}$ are arbitrary but $\mu_{\bar{\beta}\bar{\gamma}}{}^{\bar{\alpha}}$ must be symmetric in the lower indices i.e. $\mu_{\bar{\beta}\bar{\gamma}}{}^{\bar{\alpha}} = \mu_{\bar{\gamma}\bar{\beta}}{}^{\bar{\alpha}}$. Again we will obtain the coefficients $\xi_{\nu}{}^{\bar{\alpha}}$ and $\mu_{\bar{\beta}\bar{\gamma}}{}^{\bar{\alpha}}$ interms of the Finsler function F.

Thus we found the connection forms $w_{\alpha}{}^{\beta}$, with components $w_{\bar{\alpha}}{}^{m}$ in (3.35) satisfying (3.34) and $w_{\bar{\beta}}{}^{\bar{\alpha}}$, $w_{m}{}^{\bar{\alpha}}$ in (3.43), (3.44) satisfying (3.42), satisfies the torsion-free condition. Using (3.35) and (3.44), we get

$$w_{\bar{\alpha}}{}^{m} + \delta_{\bar{\alpha}\bar{\sigma}} w_{m}{}^{\bar{\sigma}} = 0 \qquad (\text{mod } w^{\alpha}). \tag{3.45}$$

Substituting (3.35) and (3.44) in (3.45) and taking the terms parenthesis of dX^j, w^m and $w^{\bar{\beta}}$, we get

$$\begin{aligned} w_{\bar{\alpha}}{}^{m} + \delta_{\bar{\alpha}\bar{\sigma}} w_{m}{}^{\bar{\sigma}} &= \Big(\frac{\delta_{\bar{\alpha}\bar{\sigma}} e^{\bar{\sigma}}{}_{j}}{F} - e_{\bar{\alpha}}{}^{i} F_{X^i X^j}\Big) dX^j + \Big\{\delta_{\bar{\alpha}\bar{\sigma}} \xi_{m}{}^{\bar{\sigma}} + \frac{e_{\bar{\alpha}}{}^{i}}{F}\Big(F_{u^i} - X^j F_{X^i u^j}\Big)\Big\} w^m \\ &+ \Big(\delta_{\bar{\alpha}\bar{\sigma}} \xi_{\bar{\beta}}{}^{\bar{\sigma}} + e_{\bar{\alpha}}{}^{i} e_{\bar{\beta}}{}^{j} F_{u^i X^j} + \lambda_{\bar{\alpha}\bar{\beta}}\Big) w^{\bar{\beta}} = 0. \end{aligned} \tag{3.46}$$

By using (3.18), (3.24) and (3.32), the coefficient of dX^j vanishes. In (3.46), w^m and $w^{\bar{\beta}}$ are linearly independent one-forms, so their coefficients equal to zero. From (3.46) we obtain the expressions for $\xi_{m}{}^{\bar{\alpha}}$ and $\xi_{\bar{\beta}}{}^{\bar{\alpha}}$, as

$$\xi_{m}{}^{\bar{\alpha}} = -\frac{\delta^{\bar{\alpha}\bar{\sigma}} e_{\bar{\sigma}}{}^{i}}{F}\Big(F_{u^i} - X^j F_{X^i u^j}\Big), \tag{3.47}$$

$$\xi_{\bar{\beta}}{}^{\bar{\alpha}} = -\delta^{\bar{\alpha}\bar{\sigma}}\Big(e_{\bar{\sigma}}{}^{i} e_{\bar{\beta}}{}^{j} F_{X^j u^i} + \lambda_{\bar{\sigma}\bar{\beta}}\Big). \tag{3.48}$$

To complete the determination of connection forms w_{α}^{β}, we need to write $\lambda_{\bar{\alpha}\bar{\beta}}$ and $\mu_{\bar{\beta}\bar{\gamma}}{}^{\bar{\alpha}}$ in terms of the known terms. For this purpose we put the following condition

$$w_{\bar{\rho}}{}^{\bar{\alpha}} \delta_{\bar{\alpha}\bar{\sigma}} + w_{\bar{\sigma}}{}^{\bar{\alpha}} \delta_{\bar{\alpha}\bar{\rho}} = 0 \qquad \text{mod } w_{m}^{\bar{\beta}}. \tag{3.49}$$

This implies that, $w_{\bar{\alpha}}{}^{\bar{\beta}}$ can be chosen at most $(m-1)(m-2)/2$ of them to be linearly independent of each other. We will state a lemma without proof which gives the basis for $T^*(PTM)$.

Lemma 2 . *The $(2m-1) + (m-1)(m-2)/2$ Pfaffian forms w^{α} $(\alpha = 1, \dots, m)$, $w_{m}{}^{\bar{\alpha}}$ $(\bar{\alpha} = 1, \dots, m-1)$, and $w_{\bar{\alpha}}{}^{\bar{\beta}}$ $(\bar{\alpha}, \bar{\beta} = 1, \dots, m-1; \bar{\alpha} < \bar{\beta})$ are linearly independent and form basis for the space of coframes on PTM and given by (3.17), (3.18) and (3.28). The $(2m-1)$ Pfaffian forms w^{α} $(\alpha = 1, \dots, m)$, $w_{m}{}^{\bar{\alpha}}$ $(\bar{\alpha} = 1, \dots, m-1)$ at each point $p \in PTM$ form basis for $T_p^*(PTM)$.*

To write (3.49) in a compact form, we have to obtain the following expression. Consider

$$g_{ij} = e^{\alpha}_{\ i} e^{\beta}_{\ j}\delta_{\alpha\beta} = e^{\bar{\alpha}}_{\ i} e^{\bar{\beta}}_{\ j}\delta_{\bar{\alpha}\bar{\beta}} + e^{m}_{\ i} e^{m}_{\ j}\delta_{mm} = e^{\bar{\alpha}}_{\ i} e^{\bar{\beta}}_{\ j}\delta_{\bar{\alpha}\bar{\beta}} + F_{X^iX^j}. \tag{3.50}$$

By using the above equality and (3.24), we can write

$$e^{\bar{\alpha}}_{\ i} e^{\bar{\beta}}_{\ j}\delta_{\bar{\alpha}\bar{\beta}} = g_{ij} - F_{X^i}F_{X^j} = FF_{X^iX^j}. \tag{3.51}$$

By exterior differentiating, contracting with $e_{\bar{\sigma}}^{\ i}$ and $e_{\bar{\rho}}^{\ j}$ and using (3.31), (3.32), Euler's theorem and its corollary, we get the expression which will be used in (3.49) as

$$\delta_{\bar{\alpha}\bar{\sigma}} e^{\bar{\alpha}}_{\ i} de_{\bar{\rho}}^{\ i} + \delta_{\bar{\alpha}\bar{\rho}} e^{\bar{\alpha}}_{\ i} de_{\bar{\sigma}}^{\ i} = -e_{\bar{\sigma}}^{\ j} e_{\bar{\rho}}^{\ i} d\Big(FF_{X^iX^j}\Big). \tag{3.52}$$

By using (3.52), (3.48) and (3.43), the condition (3.49) takes the form

$$\begin{aligned} w_{\bar{\rho}}^{\ \bar{\alpha}}\delta_{\bar{\alpha}\bar{\sigma}} + w_{\bar{\sigma}}^{\ \bar{\alpha}}\delta_{\bar{\alpha}\bar{\rho}} &= \delta_{\bar{\alpha}\bar{\sigma}} e^{\bar{\alpha}}_{\ i} de_{\bar{\rho}}^{\ i} + \delta_{\bar{\alpha}\bar{\rho}} e^{\bar{\alpha}}_{\ i} de_{\bar{\sigma}}^{\ i} - e_{\bar{\sigma}}^{\ j} e_{\bar{\rho}}^{\ i}\Big\{F_{u^iX^j} + F_{u^jX^i}\Big\}w^m \\ &- 2\lambda_{\bar{\rho}\bar{\sigma}} w^m + (\delta_{\bar{\alpha}\bar{\sigma}}\mu_{\bar{\rho}\bar{\gamma}}^{\ \bar{\alpha}} + \delta_{\bar{\alpha}\bar{\sigma}}\mu_{\bar{\sigma}\bar{\gamma}}^{\ \bar{\alpha}})w^{\bar{\gamma}} \\ &= -e_{\bar{\sigma}}^{\ j} e_{\bar{\rho}}^{\ i}\Big\{d\Big(FF_{X^iX^j}\Big) + \Big(F_{u^iX^j} + F_{u^jX^i}\Big)w^m\Big\} \\ &- 2\lambda_{\bar{\rho}\bar{\sigma}} w^m + (\delta_{\bar{\alpha}\bar{\sigma}}\mu_{\bar{\rho}\bar{\gamma}}^{\ \bar{\alpha}} + \delta_{\bar{\alpha}\bar{\sigma}}\mu_{\bar{\sigma}\bar{\gamma}}^{\ \bar{\alpha}})w^{\bar{\gamma}} = 0 \qquad \mathrm{mod}\ w_m^{\bar{\beta}}. \end{aligned} \tag{3.53}$$

Since, $FF_{X^iX^j}$ is homogeneous of degree zero in X^i, so it is on PTM. And its differential forms $d(FF_{X^iX^j})$ is an element of $T^*(PTM)$, according to Lemma 2 we can write that as a linear combination of basis w^{α} and $w_m^{\bar{\beta}}$ as the following form

$$d(FF_{X^iX^j}) = S_{ij}^{\ \ \bar{\alpha}} w_{\bar{\alpha}}^{\ m} + G_{ij\beta}w^{\beta}. \tag{3.54}$$

We now determine $S_{ij}^{\ \ \bar{\alpha}}$, $G_{ij\beta}$, $\lambda_{\bar{\rho}\bar{\sigma}}$ and $\mu_{\bar{\rho}\bar{\sigma}}^{\ \bar{\alpha}}$. First, by contracting (3.54) with $Fe_{\bar{\beta}}^{\ k}\frac{\partial}{\partial X^k}$ and using (3.35) for $w_{\bar{\alpha}}^{\ m}$, $\langle du^{\beta}, \frac{\partial}{\partial X^k}\rangle = 0$, (3.24), (3.32) and (3.18), we determine $S_{ij}^{\ \ \bar{\alpha}}$ as

$$\langle d\Big(FF_{X^iX^j}\Big), Fe_{\bar{\beta}}^{\ k}\frac{\partial}{\partial X^k}\rangle = S_{ij}^{\ \ \bar{\alpha}}\langle w_{\bar{\alpha}}^{\ \ m}, Fe_{\bar{\beta}}^{\ k}\frac{\partial}{\partial X^k}\rangle + G_{ij\gamma}\langle w^{\bar{\gamma}}, Fe_{\bar{\beta}}^{\ k}\frac{\partial}{\partial X^k}\rangle$$

$$\begin{aligned} Fe_{\bar{\beta}}^{\ k}\frac{\partial}{\partial X^k}\Big(FF_{X^iX^j}\Big) &= -S_{ij}^{\ \ \bar{\alpha}} e_{\bar{\beta}}^{\ s} e_{\bar{\beta}}^{\ k} FF_{X^sX^k} \\ &= -S_{ij}^{\ \ \bar{\alpha}} e_{\bar{\beta}}^{\ s} e_{\bar{\beta}}^{\ k}(g_{ij} - F_{X^s}F_{X^r}) \\ &= -S_{ij}^{\ \ \bar{\alpha}}\delta_{\bar{\alpha}\bar{\beta}}. \end{aligned} \tag{3.55}$$

get ${S_{ij}}^{\bar{\alpha}}$ as

$$ {S_{ij}}^{\bar{\alpha}} = -F e^{\bar{\alpha}}_{\ l} g^{lk} \frac{\partial}{\partial X^k}\Big(F F_{X^i X^j}\Big). \tag{3.56} $$

We determine $G_{ij\beta}$ in two parts as G_{ijm} and $G_{ij\bar{\beta}}$. Contracting (3.54) with $e_m = \frac{X^s}{F}\frac{\partial}{\partial u^s}$ and using (3.35), (3.21), we obtain G_{ijm} as

$$ \langle d\Big(F F_{X^i X^j}\Big), \frac{X^s}{F}\frac{\partial}{\partial u^s}\rangle = {S_{ij}}^{\bar{\alpha}} \langle {w_{\bar{\alpha}}}^m, e_m\rangle + G_{ij\beta}\langle w^\beta, e_m\rangle, $$

$$ \begin{aligned} \frac{X^s}{F}\frac{\partial}{\partial u^s}\Big(F F_{X^i X^j}\Big) &= {S_{ij}}^{\bar{\alpha}} \langle \frac{{e_{\bar{\alpha}}}^s}{F}\Big(F_{u^s} - X^r F_{X^s u^r}\Big) w^m, e_m\rangle + G_{ijm} \\ &= {S_{ij}}^{\bar{\alpha}} \frac{{e_{\bar{\alpha}}}^s}{F}\Big(F_{u^s} - X^r F_{X^s u^r}\Big) + G_{ijm}. \end{aligned} \tag{3.57} $$

Thus from the above expression we get G_{ijm} as

$$ G_{ijm} = \frac{-{e_{\bar{\alpha}}}^s {S_{ij}}^{\bar{\alpha}}}{F}\Big(F_{u^s} - X^r F_{X^s u^r}\Big) + \frac{X^s}{F}\frac{\partial}{\partial u^s}\Big(F F_{X^i X^j}\Big). \tag{3.58} $$

To determine $\lambda_{\bar{\alpha}\bar{\beta}}$ and ${\mu_{\bar{\alpha}\bar{\beta}}}^{\bar{\gamma}}$, we need to write G_{ijm} in a simpler form. For that reason consider the following quantities which will help us.

$$ g^{kl} F_{X^l} = {e_\alpha}^k \delta^{\alpha\beta} {e_\beta}^l = {e_m}^k {e_m}^l F_{X^l} = \frac{X^k}{F}, \tag{3.59} $$

where we opened the summation indices and used Euler's theorem, (3.32) and (3.27).

$$ \frac{X^s}{F}(F_{u^s} - X^r F_{X^s u^r}) = \frac{X^s}{F} X^r F_{X^r u^s} - \frac{X^s X^r}{F} F_{X^s u^r} = 0, \tag{3.60} $$

where we used the corollary of Euler's theorem. Now consider

$$ \delta^j_i = e^\alpha_{\ i} {e_\alpha}^j = e^{\bar{\alpha}}_{\ i} {e_{\bar{\alpha}}}^j + e^m_{\ i} {e_m}^j, $$

which leads to

$$ e^{\bar{\alpha}}_{\ i} {e_{\bar{\alpha}}}^j = \delta^j_i - e^m_{\ i} {e_m}^j. \tag{3.61} $$

$$ \Big(\delta^s_l - \frac{X^s}{F} F_{X^l}\Big)\frac{X^l}{F} = \delta^s_l \frac{X^l}{F} - \frac{X^s}{F} F_{X^l}\frac{X^l}{F} = \frac{X^s}{F} - \frac{X^s}{F} = 0. \tag{3.62} $$

Here we used again Euler's theorem. By using the above quantities and (3.56) for ${S_{ij}}^{\bar\alpha}$, the first term of G_{ijm} can be expressed as

$$\begin{aligned}
& {e_{\bar\alpha}}^{s}{e^{\bar\alpha}}_{l}g^{lk}\frac{\partial}{\partial X^k}\Big(FF_{X^iX^j}\Big)\Big(F_{u^s}-X^rF_{X^su^r}\Big)\\
= & \Big(\delta^s_l-{e^m}_l{e_m}^s\Big)g^{kl}\frac{\partial}{\partial X^k}\Big(F_{X^iX^j}\Big)\Big(F_{u^s}-X^rF_{X^su^r}\Big)\\
= & \Big(\delta^s_l-F_{X^l}\frac{X^s}{F}\Big)g^{kl}\Big(F_{X^k}F_{X^iX^j}+FF_{X^iX^jX^k}\Big)\Big(F_{u^s}-X^rF_{X^su^r}\Big)\\
= & g^{ks}FF_{X^kX^iX^j}\Big(F_{u^s}-X^rF_{X^su^r}\Big). \qquad (3.63)
\end{aligned}$$

Thus by (3.58) and (3.63) we can write G_{ijm} in a simpler form as

$$\begin{aligned}
G_{ijm} &= g^{ks}FF_{X^kX^iX^j}\Big(F_{u^s}-X^rF_{X^su^r}\Big)\\
&+ X^sF_{u^sX^iX^j}+\frac{X^s}{F}F_{u^s}F_{X^iX^j}. \qquad (3.64)
\end{aligned}$$

Now, we obtain the expressions for the terms $\lambda_{\bar\alpha\bar\beta}$ and ${\mu_{\bar\beta\bar\gamma}}^{\bar\alpha}$. By using (3.54), (3.45), we can write (3.53) as

$$\begin{aligned}
{w_{\bar\rho}}^{\bar\alpha}\delta_{\bar\alpha\bar\sigma}+{w_{\bar\sigma}}^{\bar\alpha}\delta_{\bar\alpha\bar\rho} &= -{e_{\bar\sigma}}^{j}{e_{\bar\rho}}^{i}\Big\{{S_{ij}}^{\bar\alpha}{w_{\bar\alpha}}^{m}+G_{ij\bar\gamma}w^{\bar\gamma}+G_{ijm}w^m+\Big(F_{u^iX^j}+F_{u^jX^i}\Big)w^m\Big\}\\
&- 2\lambda_{\bar\rho\bar\sigma}w^m+(\delta_{\bar\alpha\bar\sigma}{\mu_{\bar\rho\bar\gamma}}^{\bar\alpha}+\delta_{\bar\alpha\bar\sigma}{\mu_{\bar\sigma\bar\gamma}}^{\bar\alpha})w^{\bar\gamma}=0 \qquad (\text{mod } {w_{\bar\beta}}^{m}). \qquad (3.65)
\end{aligned}$$

Equating the coefficient of w^m to zero in the above expression, we get

$$\lambda_{\bar\rho\bar\sigma}=-\frac{1}{2}{e_{\bar\rho}}^{i}{e_{\bar\sigma}}^{j}\Big(G_{ijm}+F_{X^ju^i}+F_{X^iu^j}\Big). \qquad (3.66)$$

We obtained expression for G_{ijm} and also we have to find $G_{ij\bar\beta}$. For this purpose, we contract (3.54) with $e_{\bar\beta}$ and obtain

$$G_{ij\bar\beta}=-{S_{ij}}^{\bar\alpha}\langle {w_{\bar\alpha}}^{m},e_{\bar\beta}\rangle+{e_{\bar\beta}}^{k}\frac{\partial}{\partial u^k}\Big(FF_{X^iX^j}\Big). \qquad (3.67)$$

By using (3.59), (3.61), (3.62), corollary of Euler's theorem, (3.56) for $S^{\bar\alpha}_{ij}$ and (3.66) for $\lambda_{\bar\alpha\bar\beta}$, we can write the first term on the right hand side of (3.67) as

$$\begin{aligned}
-{S_{ij}}^{\bar\alpha}\langle {w_{\bar\alpha}}^{m},e_{\bar\beta}\rangle &= F{e^{\bar\alpha}}_{l}g^{lk}\Big(F_{X^k}F_{X^iX^j}+FF_{X^iX^jX^k}\Big)\\
&\times \Big[{e_{\bar\alpha}}^{r}{e_{\bar\beta}}^{s}F_{u^rX^s}-\frac{1}{2}{e_{\bar\alpha}}^{r}{e_{\bar\beta}}^{s}\Big(G_{rsm}+F_{X^su^r}+F_{X^ru^s}\Big)\Big]\\
&= \frac{F}{2}{e^{\bar\alpha}}_{l}{e_{\bar\alpha}}^{r}{e_{\bar\beta}}^{s}g^{lk}\Big(F_{X^k}F_{X^iX^j}+FF_{X^iX^jX^k}\Big)
\end{aligned}$$

$$
\begin{aligned}
&\times \quad \Big[F_{u^r X^s} - G_{rsm} - F_{X^s u^r}\Big] \\
&= \quad \frac{F}{2} g^{lk} e_{\bar{\beta}}{}^{s} \Big(\delta_l^r - \frac{X^r}{F} F_{X^l}\Big) \\
&\times \quad \Big(F_{u^r X^s} - F_{X^r u^s} - G_{rsm}\Big)\Big(F_{X^k} F_{X^i X^j} + F F_{X^i X^j X^k}\Big) \\
&= \quad \frac{1}{2} e_{\bar{\beta}}{}^{s} \Big(F_{u^r X^s} - F_{X^r u^s} - G_{rsm}\Big) \\
&\times \quad \Big(F^2 g^{rk} F_{X^i X^j X^k} + X^r F_{X^i X^j}\Big).
\end{aligned} \tag{3.68}
$$

Thus, inserting that equation into (3.67) we obtain $G_{ij\bar{\beta}}$ as

$$
\begin{aligned}
G_{ij\bar{\beta}} \quad &= \quad e_{\bar{\beta}}{}^{s} \Big\{ \frac{1}{2} \Big(F_{u^r X^s} - F_{X^r u^s} - G_{rsm}\Big)\Big(F^2 g^{rk} F_{X^i X^j X^k} \\
&+ \quad X^r F_{X^i X^j}\Big) + F_{u^s} F_{X^i X^j} + F F_{u^s X^i X^j} \Big\}.
\end{aligned} \tag{3.69}
$$

And equating the coefficient of w^γ to zero in (3.65), we get

$$
\delta^{\bar{\alpha}\bar{\beta}} e_{\bar{\rho}}{}^{i} e_{\bar{\sigma}}{}^{j} G_{ij\bar{\beta}} = \delta^{\bar{\alpha}\bar{\beta}} \delta_{\bar{\nu}\bar{\sigma}} \mu^{\bar{\nu}}_{\bar{\rho}\bar{\beta}} + \delta^{\bar{\alpha}\bar{\beta}} \delta_{\bar{\nu}\bar{\sigma}} \mu^{\bar{\nu}}_{\bar{\sigma}\bar{\beta}}. \tag{3.70}
$$

Two similar equations can be obtained by commuting the index set (ρ, σ, β) in cyclic order. Adding these and subtracting (3.70) and using symmetry of $\mu_{\bar{\rho}\bar{\sigma}}{}^{\bar{\alpha}}$ i.e. $\mu_{\bar{\sigma}\bar{\rho}}{}^{\bar{\alpha}} = \mu_{\bar{\rho}\bar{\sigma}}{}^{\bar{\alpha}}$, we get

$$
\mu_{\bar{\rho}\bar{\sigma}}{}^{\bar{\alpha}} = \frac{1}{2} \delta^{\bar{\alpha}\bar{\beta}} \Big(e_{\bar{\beta}}{}^{i} e_{\bar{\rho}}{}^{j} G_{ij\bar{\sigma}} - e_{\bar{\rho}}{}^{i} e_{\bar{\sigma}}{}^{j} G_{ij\bar{\beta}} + e_{\bar{\sigma}}{}^{i} e_{\bar{\beta}}{}^{j} G_{ij\bar{\rho}} \Big). \tag{3.71}
$$

Thus, by $w_{\bar{\alpha}}{}^{m}$ in (3.35) with $\lambda_{\bar{\alpha}\bar{\beta}}$ in (3.66), $w_{\bar{\alpha}}{}^{\bar{\beta}}$ in (3.43) with $\xi_{\bar{\alpha}}{}^{\bar{\beta}}$ in (3.48) and $\mu_{\bar{\alpha}\bar{\beta}}{}^{\bar{\gamma}}$ in (3.71), $w_m{}^{\bar{\alpha}}$ in (3.44) with $\xi_m{}^{\bar{\alpha}}$ in (3.47) and $w_m{}^{m} = 0$, we determined the components of connection matrix $w_\alpha{}^{\beta}$ of Chern connection which satisfy the torsion-free condition.

3.3.2 The Cartan Tensor and Determination of the Almost Metric-Compatible Connection

In this section, we will investigate the metric-compatibility of the Chern connection. We first define the Cartan tensor.

Using (3.45), we can write (3.49) as

$$\begin{aligned} w_{\bar{\rho}}{}^{\bar{\alpha}}\delta_{\bar{\alpha}\bar{\sigma}} + w_{\bar{\sigma}}{}^{\bar{\alpha}}\delta_{\bar{\alpha}\bar{\rho}} &= -e_{\bar{\rho}}{}^{i}e_{\bar{\sigma}}{}^{j}S_{ij}{}^{\bar{\alpha}} \equiv 2A_{\bar{\rho}\bar{\sigma}}{}^{\bar{\alpha}}w_{\bar{\alpha}}{}^{m} \\ &= -2A_{\bar{\rho}\bar{\sigma}\bar{\alpha}}w_{m}{}^{\bar{\alpha}}, \end{aligned} \tag{3.72}$$

where

$$A_{\bar{\rho}\bar{\sigma}\bar{\alpha}} = A_{\bar{\rho}\bar{\sigma}}{}^{\bar{\beta}}\delta_{\bar{\beta}\bar{\alpha}}. \tag{3.73}$$

The indices $A_{\bar{\rho}\bar{\sigma}\bar{\alpha}}$ take the values 1 to $m-1$. We can write $A_{\rho\sigma\beta}$ which the indices take the values 1 to m with the condition

$$A_{\alpha\beta\gamma} = 0 \qquad \text{whenever any index has the value } m. \tag{3.74}$$

With the condition (3.74), we can write (3.72) as

$$w_{\alpha\beta} + w_{\beta\alpha} = -2A_{\alpha\beta\gamma}w_{m}{}^{\gamma}, \tag{3.75}$$

where

$$w_{\alpha\beta} = w_{\alpha}{}^{\gamma}\delta_{\gamma\beta}. \tag{3.76}$$

The (0,3) tensor with respect to w^{γ}

$$A = A_{\alpha\beta\gamma}w^{\alpha} \otimes w^{\beta} \otimes w^{\gamma}, \tag{3.77}$$

is called the **Cartan tensor**. We now obtain a formula for $A_{\alpha\beta\gamma}$ interms of F. By using (3.32) and (3.56) for $S^{\bar{\alpha}}_{ij}$ in (3.72), we can write $A_{\bar{\rho}\bar{\sigma}\bar{\alpha}}$ as

$$\begin{aligned} A_{\bar{\rho}\bar{\sigma}\bar{\alpha}} &= \frac{F}{2}e_{\bar{\sigma}}{}^{j}e_{\bar{\rho}}{}^{i}e_{\bar{\alpha}}{}^{k}\Big[F_{X^k}F_{X^iX^j} + FF_{X^iX^jX^k}\Big] \\ &= \frac{F}{2}e_{\bar{\sigma}}{}^{j}e_{\bar{\rho}}{}^{i}e_{\bar{\alpha}}{}^{k}\Big[F_{X^k}F_{X^iX^j} + F_{X^j}F_{X^iX^k} + F_{X^i}F_{X^jX^k} + FF_{X^iX^jX^k}\Big] \\ &= \frac{F}{2}\frac{\partial^3(\frac{F^2}{2})}{\partial X^i\partial X^j\partial X^k}e_{\bar{\sigma}}{}^{j}e_{\bar{\rho}}{}^{i}e_{\bar{\alpha}}{}^{k} = \frac{F}{2}\frac{\partial g_{ij}}{\partial X^k}e_{\bar{\sigma}}{}^{j}e_{\bar{\rho}}{}^{i}e_{\bar{\alpha}}{}^{k}. \end{aligned} \tag{3.78}$$

By using (3.24), Euler's theorem and its corollary we have

$$X^j\frac{\partial g_{lk}}{\partial X^j} = X^l\frac{\partial g_{lk}}{\partial X^j} = X^k\frac{\partial g_{lk}}{\partial X^j} = 0. \tag{3.79}$$

By using the above expressions, we can write (3.78) as an expression so that all indices take the values from 1 to m as

$$A_{\rho\sigma\alpha} = \frac{F}{2}\frac{\partial^3(\frac{F^2}{2})}{\partial X^i\partial X^j\partial X^k}e_{\sigma}{}^{j}e_{\rho}{}^{i}e_{\alpha}{}^{k} = \frac{F}{2}(g_{ij})_{,X^k}e_{\sigma}{}^{j}e_{\rho}{}^{i}e_{\alpha}{}^{k}. \tag{3.80}$$

Thus we get the Cartan tensor with components (3.80) with respect to w^α as

$$A = A_{\rho\sigma\gamma} w^\rho \otimes w^\sigma \otimes w^\gamma. \tag{3.81}$$

By using (3.17) we can write the Cartan tensor with respect to the natural basis du^i as

$$\begin{aligned} A &= A_{\rho\sigma\gamma} w^\rho \otimes w^\sigma \otimes w^\gamma \\ &= \frac{F}{2}(g_{rs})_{,X^t} e_\sigma^{\ s} e_\rho^{\ r} e_\gamma^{\ t} e^\rho_{\ i} e^\sigma_{\ j} e^\gamma_{\ k} du^i \otimes du^j \otimes du^k \\ &= \frac{F}{2}(g_{ij})_{,X^k} du^i \otimes du^j \otimes du^k \\ &\equiv A_{ijk} du^i \otimes du^j \otimes du^k. \end{aligned} \tag{3.82}$$

By using (3.79) and (3.82), we can write

$$X^i A_{ijk} = X^j A_{ijk} = X^k A_{ijk} = 0. \tag{3.83}$$

Above A_{ijk} and $A_{\rho\sigma\gamma}$ are both symmetric in all indices. By using (3.22) and (3.18), we can write G interms of w^α as

$$G = g_{ij} du^i \otimes du^j = g_{ij} e_\alpha^{\ i} e_\beta^{\ j} w^\alpha \otimes w^\beta = \delta_{\alpha\beta} w^\alpha \otimes w^\beta. \tag{3.84}$$

For the metric-compatibility, consider

$$DG = \delta_{\alpha\beta} dw^\alpha \otimes w^\beta + \delta_{\alpha\beta} w^\alpha \otimes dw^\beta. \tag{3.85}$$

Chern connection is torsion-free, so using $dw^\alpha = w^\gamma \wedge w_\gamma^{\ \alpha}$ in the above equation, we get

$$\begin{aligned} DG &= -\Big[\delta_{\alpha\beta} w_\gamma^{\ \alpha} \wedge w^\gamma \otimes w^\beta + \delta_{\alpha\beta} w_\gamma^{\ \beta} \wedge w^\alpha \otimes w^\gamma\Big] \\ &= -\Big[\delta_{\alpha\beta} w_\gamma^{\ \alpha} + \delta_{\alpha\beta} w_\beta^{\ \alpha}\Big] w^\gamma \otimes w^\beta \\ &= -\Big[w_{\gamma\beta} + w_{\beta\gamma}\Big] w^\gamma \otimes w^\beta = 2A_{\gamma\beta\alpha} w_m^{\ \alpha} w^\gamma \otimes w^\beta. \end{aligned} \tag{3.86}$$

Thus DG is not directly zero. It is zero if $A_{\gamma\beta\alpha}$ is zero. So the Chern connection is not metric-compatible. We can say, it is almost metric-compatible and **almost metric-compatibility condition** is

$$w_{\gamma\beta} + w_{\beta\gamma} = -2A_{\gamma\beta\alpha} w_m^{\ \alpha}. \tag{3.87}$$

We can write the following theorem for the Chern connection which summarize all calculations about the torsion-freeness and almost metric-compatibility.

Theorem 3.3.1 **(Chern)**. *Let M be an m-dimensional Finsler manifold. Then there exist unique torsion-free and almost metric-compatible connection D on p^*TM such that*

$$D : \Gamma(p^*TM) \longrightarrow \Gamma(p^*TM \otimes T^*(PTM)), \tag{3.88}$$

and defined as

$$De_\alpha = w_\alpha{}^\beta e_\beta, \tag{3.89}$$

where components of connection matrix $w_\alpha^{\ \beta}$ satisfy the torsion-free and almost metric-compatible structure equations, respectively as

$$dw^\alpha = w^\beta \wedge w_\beta{}^\alpha, \tag{3.90}$$

and

$$w_{\alpha\beta} + w_{\beta\alpha} = -2A_{\alpha\beta\gamma} w_m{}^\gamma. \tag{3.91}$$

Here e_α and w^α are sections of p^*TM and p^*T^*M, respectively. $A = A_{\alpha\beta\gamma} w^\alpha \otimes w^\beta \otimes w^\gamma$ is Cartan tensor with components $A_{\alpha\beta\gamma} = \frac{F}{2}(g_{ij})_{,X^k} e_\beta{}^j e_\alpha{}^i e_\gamma{}^k$ where F is Finsler function.

We know that Finsler metric is Riemannian if g_{ij} is independent of X^i. As a consequence of that fact we can state the following corollary.

Corollary 3.3.1 *The Finsler metric is Riemannian if and only if the Cartan tensor vanish i.e. $A_{\alpha\beta\gamma} = 0$.*

If the Finsler metric is Riemannian then almost metric-compatibility takes the form of metric-compatibility. In that case, Chern connection reduces to Christoffel-Levi-Civita cannection.

3.3.3 Chern Connection and Formulas for Connection Coefficients Γ in Natural Coordinates

In this section, we will consider the Chern connection in natural coordinates. By Lemma (2) w^α and $w_m{}^\alpha$ form bases for $T^*(PTM)$. w^α has the form in natural coordinates, $w^\alpha = e_i{}^\alpha du^i$. But we do not have yet a formula for $w_m{}^\alpha$ in natural coordinates. For that purpose, firstly we obtain $w_{\bar\alpha}{}^m$ in terms of the natural bases du^i and dX^i. Using (3.17), (3.28), (3.61) and (3.66)(for $\lambda_{\bar\rho\bar\sigma}$), we can write $w_{\bar\alpha}{}^m$ in (3.35) as

$$\begin{aligned} w_{\bar\alpha}{}^m &= -e_{\bar\alpha}{}^i F_{X^iX^j} dX^j \\ &+ e_{\bar\alpha}{}^i \Bigg[\frac{X^j}{2F} F_{X^k}\Big(G_{ijm} + F_{X^ju^i} - F_{X^iu^j}\Big) \\ &- \frac{1}{2}\Big(G_{ikm} + F_{X^iu^k} - F_{X^ku^i}\Big)\Bigg] du^k. \end{aligned} \tag{3.92}$$

To write the second term of the above expression in the right hand side, we define the following fact

$$\mathcal{G} \equiv \frac{1}{2}F^2 \tag{3.93}$$

$$\begin{aligned} \mathcal{G}_l &\equiv \frac{1}{2}\Big(X^s \frac{\partial^2 \mathcal{G}}{\partial u^s \partial X^l} - \frac{\partial \mathcal{G}}{\partial u^l}\Big) \\ &= \frac{1}{2}\Big(X^s F_{u^s} F_{X^l} + X^s F F_{X^l u^s} - F F_{u^l}\Big), \end{aligned} \tag{3.94}$$

$$\mathcal{G}^i \equiv g^{il}\mathcal{G}_l. \tag{3.95}$$

By using the above facts, (3.64) for G_{ijm}, Euler's theorem and corollary, (3.32) and after complicated calculations, we can write the second term of (3.92) as

$$\begin{aligned} & e_{\bar\alpha}{}^i \Bigg[\frac{X^j}{2F} F_{X^k}\Big(G_{ijm} + F_{X^ju^i} - F_{X^iu^j}\Big) \\ -& \frac{1}{2}\Big(G_{ikm} + F_{X^iu^k} - F_{X^ku^i}\Big)\Bigg] du^k \\ =& -e_{\bar\alpha}{}^i \frac{g_{ij}}{F}\frac{\partial \mathcal{G}^j}{\partial X^k} du^k. \end{aligned} \tag{3.96}$$

Thus we can write $w_{\bar\alpha}{}^m$ in natural bases as

$$w_{\bar\alpha}{}^m = -e_{\bar\alpha}{}^m \Big[\frac{g_{ij}}{F}\frac{\partial \mathcal{G}^j}{\partial X^k} du^k + F_{X^iX^k} dX^k\Big]. \tag{3.97}$$

We can write $G_{ij\beta}$ in terms of natural coordinates which gives the common formula for (3.64) and (3.69). By contracting (3.54) with $e_\beta = e_\beta{}^l \frac{\partial}{\partial u^l}$ and using (3.97) in (3.54), (3.22) and (3.21), we get

$$e_\beta{}^l \frac{\partial}{\partial u^l}\Big(F F_{X^iX^j}\Big) = S_{ij}{}^{\bar\alpha}\Big(- e_{\bar\alpha}{}^k g_{kl}\frac{\partial \mathcal{G}^l}{\partial X^t}\Big) + G_{ij\beta}. \tag{3.98}$$

By using (3.56) for $S_{ij}{}^{\bar{\alpha}}$, (3.61), (3.59), Euler's theorem and corollary in the above expression, we get

$$G_{ij\beta} = e_{\beta}{}^{l}\Big[\frac{\partial}{\partial u^l}\Big(FF_{X^iX^j}\Big) - \frac{\partial \mathcal{G}^r}{\partial X^l}\frac{\partial}{\partial X^r}\Big(FF_{X^iX^j}\Big)\Big]. \tag{3.99}$$

By using that formula we can write $\lambda_{\bar{\rho}\bar{\sigma}}$ and $\mu_{\bar{\rho}\bar{\sigma}}{}^{\bar{\alpha}}$ in natural coordinates as

$$\lambda_{\bar{\rho}\bar{\sigma}} = -\frac{1}{2}\Big(e_m{}^{l}\Big[\frac{\partial}{\partial u^l}(FF_{X^iX^j}) - \frac{\partial \mathcal{G}^r}{\partial X^l}\frac{\partial}{\partial X^r}(FF_{X^iX^j})\Big] + F_{X^ju^i} + F_{X^iu^j}\Big), \tag{3.100}$$

and

$$\mu_{\bar{\rho}\bar{\sigma}}{}^{\bar{\alpha}} = \frac{1}{2}\delta^{\bar{\alpha}\bar{\beta}}\Big(\frac{\partial}{\partial u^l}(FF_{X^iX^j}) - \frac{\partial \mathcal{G}^r}{\partial X^l}\frac{\partial}{\partial X^r}(FF_{X^iX^j})\Big)\Big[e_{\bar{\beta}}{}^{i}e_{\bar{\rho}}{}^{j}e_{\bar{\sigma}}{}^{l} - e_{\bar{\rho}}{}^{i}e_{\bar{\sigma}}{}^{j}e_{\bar{\beta}}{}^{l} + e_{\bar{\sigma}}{}^{i}e_{\bar{\beta}}{}^{j}e_{\bar{\rho}}{}^{l}\Big]. \tag{3.101}$$

We have w^α in natural coordinates as $w^\alpha = e^{\alpha}{}_{i}du^i$. By using (3.45), (3.24), (3.18), (3.32), (3.20) and (3.97), we can write $w_m{}^{\bar{\alpha}}$ in natural coordinate as

$$\begin{aligned} w_m{}^{\bar{\sigma}}\delta_{\bar{\alpha}\bar{\sigma}} &= e_{\bar{\alpha}}{}^{i}\frac{g_{ij}}{F}\frac{\partial \mathcal{G}^j}{\partial X^k}du^k + e_{\bar{\alpha}}{}^{i}(\frac{g_{ij}}{F} - \frac{F_{X^i}F_{X^j}}{F})dX^j \\ &= \frac{e^{\bar{\alpha}}{}_{j}}{F}\frac{\partial \mathcal{G}^j}{\partial X^k}du^k + \frac{e^{\bar{\alpha}}{}_{j}}{F}dX^j. \end{aligned} \tag{3.102}$$

From there we get

$$w_m{}^{\bar{\alpha}} = e^{\bar{\alpha}}{}_{j}\delta X^j, \tag{3.103}$$

where

$$N_j{}^{i} \equiv \frac{1}{F}\frac{\partial \mathcal{G}^i}{\partial X^j}, \tag{3.104}$$

and

$$\delta X^j \equiv \frac{dX^j}{F} + N_k{}^{j}du^k. \tag{3.105}$$

Thus we can write the bases w^α and $w_m{}^{\bar{\alpha}}$ of $T^*(PTM)$ in terms of the natural bases. The dual orthonormal vectors of w^α and $w_m{}^{\bar{\alpha}}$ have the forms respectively, as

$$\hat{e}_\alpha = e_\alpha{}^{i}\frac{\delta}{\delta u^i}, \qquad \alpha = 1, \ldots, m, \tag{3.106}$$

and

$$\hat{e}_{m+\bar{\alpha}} = e_{\bar{\alpha}}{}^{j}\frac{\delta}{\delta X^j}, \qquad \bar{\alpha} = 1, \ldots, m-1, \tag{3.107}$$

where

$$\frac{\delta}{\delta u^i} \equiv \frac{\partial}{\partial u^i} - FN_i{}^{j}\frac{\partial}{\partial X^j}, \tag{3.108}$$

and

$$\frac{\delta}{\delta X^i} \equiv F\frac{\partial}{\partial X^i}. \tag{3.109}$$

Here $\frac{\delta}{\delta u^i}$ and $\frac{\delta}{\delta X^i}$ are dual to du^i and δX^i form local bases for $T(TM\setminus 0)$ and $T^*(TM\setminus 0)$. $\{w^\alpha, w_m^{\bar\alpha}\}$ and $\{\hat e_{\ \bar\alpha}, \hat e_{m+\bar\alpha}\}$ form local bases for $T^*(PTM)$ and $T(PTM)$ and also dual to each other.

We now define the Chern connection in natural coordinates as follows

$$D:\Gamma(p^*TM)\to\Gamma(p^*TM\otimes T^*(TM\setminus 0)),$$

$$D\frac{\partial}{\partial u^i} = \theta_i^{\ j}\frac{\partial}{\partial u^j}, \qquad Ddu^i = -\theta_j^{\ i}du^j, \tag{3.110}$$

where $\theta_i^{\ j}$ are the components of the connection matrix in natural coordinates. To find the relation between $\theta_i^{\ j}$ and $w_\alpha^{\ \beta}$, considering

$$D\Big(e_\alpha^{\ i}\frac{\partial}{\partial u^i}\Big) = De_\alpha = w_\alpha^{\ \beta}e_\beta,$$

we get

$$de_\alpha^{\ l}\frac{\partial}{\partial u^l} + e_\alpha^{\ k}\theta_k^{\ l}\frac{\partial}{\partial u^l} = w_\alpha^{\ \beta}e_\beta^{\ l}\frac{\partial}{\partial u^l}. \tag{3.111}$$

Comparing the coefficients of $\frac{\partial}{\partial u^l}$ in (3.111) and then contracting the resulting equation with $e^\gamma_{\ l}$, we get the relation for $w_\alpha^{\ \beta}$ interms of $\theta_i^{\ j}$ as

$$w_\alpha^{\ \beta} = e^\beta_{\ i}(de_\alpha^{\ i} + e_\alpha^{\ j}\theta_j^{\ i}). \tag{3.112}$$

Comparing the coefficients of $\frac{\partial}{\partial u^l}$ in (3.111) and then contracting the result equation with $e^\alpha_{\ r}$, we get the relation for $\theta_i^{\ j}$ in terms of $w_\alpha^{\ \beta}$ as

$$\theta_i^{\ j} = e_\beta^{\ j}(de^\beta_{\ i} + e^\alpha_{\ i}w_\alpha^{\ \beta}). \tag{3.113}$$

By wedge product (3.112) with du^i and using (3.17), $e_\alpha^{\ i}e^\beta_{\ i} = \delta^\beta_\alpha$, the torsion-free condition (3.39) is equivalent to

$$\begin{aligned} du^i\wedge\theta_i^{\ j} &= du^i\wedge[e_\beta^{\ j}de^\beta_{\ i} + e_\beta^{\ j}e^\alpha_{\ i}w_\alpha^{\ \beta}] \\ &= e_\beta^{\ j}du^i de^\beta_{\ i} + w^\gamma e_\gamma^{\ i}e_\beta^{\ j}e^\alpha_{\ i}w_\alpha^{\ \beta} \\ &= e_\beta^{\ j}dw^\alpha + w^\gamma e_\beta^{\ j}w_\gamma^{\ \beta} \\ &= e_\beta^{\ j}(dw^\alpha + w^\gamma\wedge w_\gamma^{\ \beta}). \end{aligned} \tag{3.114}$$

Chern connection is torsion-free so $dw^\alpha + w^\gamma \wedge w_\gamma{}^\beta = 0$. From there

$$du^i \wedge \theta_i{}^j = 0. \tag{3.115}$$

We can say the above expression is equivalent to torsion-free condition of the Chern connection in natural coordinates. Wedge product of $\theta_i{}^j$ and du^i is zero in (3.115), so they are linearly dependent. We can write $\theta_i{}^j$ in terms of du^i as

$$\theta_i{}^j = \Gamma^j{}_{il} du^l. \tag{3.116}$$

By (3.115) we can write the following symmetry property

$$\Gamma^j{}_{il} = \Gamma^j{}_{li}. \tag{3.117}$$

By using (3.116), we can write (3.110) as

$$D\frac{\partial}{\partial u^i} = \Gamma^j{}_{il} du^l \otimes \frac{\partial}{\partial u^j}. \tag{3.118}$$

We obtain $\Gamma^j{}_{il}$ in terms of g_{ij} and $N_j{}^i$ by using the almost metric-compatibility of the Chern connection. Contracting (3.23) with $\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right)$, we get

$$g_{ij} = G\Big(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\Big). \tag{3.119}$$

Applying D to the above equation we have

$$dg_{ij} = (DG)\Big(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\Big) + G\Big(D\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\Big) + G\Big(\frac{\partial}{\partial u^i}, D\frac{\partial}{\partial u^j}\Big). \tag{3.120}$$

By (3.86), (3.17), (3.80) for $A_{\alpha\beta\gamma}$, (3.82) for A_{ijk} and (3.103),we get

$$\begin{aligned} DG &= -(w_{\alpha\beta} + w_{\beta\alpha}) w^\alpha \otimes w^\beta \\ &= 2A_{\alpha\beta\gamma} w_m{}^\gamma w^\alpha \otimes w^\beta \\ &= 2A_{ijk}\delta X^k du^i \otimes du^j. \end{aligned} \tag{3.121}$$

Thus, the first term of (3.120) is

$$DG\Big(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\Big) = 2A_{ijk}\delta X^k. \tag{3.122}$$

By using (3.23), (3.118), the second and third terms of (3.120), respectively are

$$\begin{aligned} G\Big(D\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\Big) &= g_{rs} du^r \otimes du^s \Gamma^k{}_{il} du^l \Big(\frac{\partial}{\partial u^k}, \frac{\partial}{\partial u^j}\Big) \\ &= \Gamma^k{}_{il} g_{kj} du^l = \Gamma_{jil} du^l, \end{aligned} \tag{3.123}$$

$$\begin{aligned} G\Big(\frac{\partial}{\partial u^i}, D\frac{\partial}{\partial u^j}\Big) &= g_{rs}du^r \otimes du^s \Gamma^k{}_{jl}du^l \Big(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^k}\Big) \\ &= \Gamma^k{}_{jl} g_{ik} du^l = \Gamma_{ijl} du^l. \end{aligned} \tag{3.124}$$

Inserting these terms and (3.122) into (3.120), we obtain the almost metric-compatibility condition in natural coordinates as

$$\begin{aligned} dg_{ij} &= g_{kj}\Gamma^k{}_{il}du^l + g_{ik}\Gamma^k{}_{jk}du^l + 2A_{ijk}\delta X^k \\ &= g_{kj}\theta_i{}^k + g_{ik}\theta_j{}^k + 2A_{ijk}\delta X^k. \end{aligned} \tag{3.125}$$

By using (3.105), (3.116) and $g_{ki}\Gamma^k{}_{jl} = \Gamma_{ijl}$, we get the equivalent equation of (3.125) as

$$\frac{\partial g_{ij}}{\partial u^l}du^l + \frac{\partial g_{ij}}{\partial X^l}dX^l = \Gamma_{ijl}du^l + \Gamma_{jil}du^l + 2A_{ijl}\frac{dX^l}{F} + 2A_{ijk}N_l{}^k du^l \tag{3.126}$$

Comparing the coefficients of du^l and dX^k in the above equation, we get respectively

$$\Gamma_{ijk} + \Gamma_{jik} = \frac{\partial g_{ij}}{\partial u^k} - 2A_{ijl}N_k{}^l, \tag{3.127}$$

$$A_{ijk} = \frac{F}{2}\frac{\partial g_{ij}}{\partial X^k}, \qquad \text{(Cartan tensor)}. \tag{3.128}$$

Consider the following expression

$$\Big(\Gamma_{ijk} + \Gamma_{jik}\Big) - \Big(\Gamma_{jki} + \Gamma_{kji}\Big) + \Big(\Gamma_{kij} + \Gamma_{ikj}\Big). \tag{3.129}$$

Using (3.117) and (3.127), we find

$$\begin{aligned} \Gamma_{ijk} &= \frac{1}{2}\Big(\frac{\partial g_{ij}}{\partial u^k} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{jk}}{\partial u^i}\Big) \\ &- \frac{F}{2}\Big(\frac{\partial g_{ij}}{\partial X^l}N_k{}^l + \frac{\partial g_{ki}}{\partial X^l}N_j{}^l - \frac{\partial g_{jk}}{\partial X^l}N_i{}^l\Big). \end{aligned} \tag{3.130}$$

Using the fact $\Gamma^l{}_{jk} = g^{li}\Gamma_{ijk}$, we can write the above equation as

$$\begin{aligned} \Gamma^r{}_{jk} &= \frac{g^{ri}}{2}\Big(\frac{\partial g_{ij}}{\partial u^k} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{jk}}{\partial u^i}\Big) \\ &- \frac{F}{2}g^{ri}\Big(\frac{\partial g_{ij}}{\partial X^l}N_k{}^l + \frac{\partial g_{ki}}{\partial X^l}N_j{}^l - \frac{\partial g_{jk}}{\partial X^l}N_i{}^l\Big). \end{aligned} \tag{3.131}$$

As in the Riemannian case, denoting

$$\gamma_{ijk} \equiv \frac{1}{2}\Big(\frac{\partial g_{ij}}{\partial u^k} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{jk}}{\partial u^i}\Big), \tag{3.132}$$

$$\gamma^{l}{}_{jk} \equiv \frac{g^{il}}{2}\Big(\frac{\partial g_{ij}}{\partial u^k} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{jk}}{\partial u^i}\Big). \tag{3.133}$$

which are called the **Cristoffel symbols of the first** and **second kinds**, respectively. We can write Γ_{jik} and Γ^{j}_{ik} respectively as

$$\Gamma_{ijk} = \gamma_{ijk} - \frac{F}{2}\Big(\frac{\partial g_{ij}}{\partial X^l}N_k{}^l + \frac{\partial g_{ki}}{\partial X^l}N_j{}^l - \frac{\partial g_{jk}}{\partial X^l}N_i{}^l\Big), \tag{3.134}$$

$$\Gamma^{r}{}_{jk} = \gamma^{r}{}_{jk} - \frac{F}{2}g^{ir}\Big(\frac{\partial g_{ij}}{\partial X^l}N_k{}^l + \frac{\partial g_{ki}}{\partial X^l}N_j{}^l - \frac{\partial g_{jk}}{\partial X^l}N_i{}^l\Big). \tag{3.135}$$

We can write $N_j{}^i$ in terms of Cartan tensor and Christoffel symbols of second kind as

$$N_j{}^i = \gamma^{i}{}_{jk}\frac{X^k}{F} - A^{i}{}_{jk}\gamma^{k}{}_{rs}\frac{X^r X^s}{F^2}. \tag{3.136}$$

Using (3.79)and (3.130) we can write

$$\Gamma^{i}{}_{jk}\frac{X^j}{F} = N_k{}^i, \tag{3.137}$$

or

$$N_k{}^i du^k = \theta_j{}^i \frac{X^j}{F}. \tag{3.138}$$

3.4 Curvature

In this section we investigate the properties of the curvature tensor Ω of the Chern connection. Ω splits into two parts R and P, where the R-part is the generalization of the Riemann curvature tensor.

3.4.1 Expressions for R and P in Natural Coordinates

Let θ^i and $\theta_i{}^j$, be basis one-form and connection one-form, then

$$\Omega_j{}^i = d\theta_j{}^i - \theta_j{}^k \wedge \theta_k{}^i, \tag{3.139}$$

are called the curvature two-form of the Chern connection in natural coordinates. Since it is a two-form on the manifold $TM \setminus 0$, we can express it as

$$\Omega_j{}^i = \frac{1}{2}R_j{}^i{}_{kl}du^k \wedge du^l + P_j{}^i{}_{kl}du^k \wedge \frac{\delta X^l}{F} + \frac{1}{2}Q_j{}^i{}_{kl}\frac{\delta X^k}{F} \wedge \frac{\delta X^l}{F} \tag{3.140}$$

Here the terms R, P and Q are respectively, horizontal-horizontal (h-h), horizontal-vertical (h-v) and vertical-vertical (v-v) curvature tensors of the Chern connection in natural coordinates. R and Q satisfy the following property

$$R_j{}^i{}_{kl} = -R_j{}^i{}_{lk}, \qquad Q_j{}^i{}_{kl} = -Q_j{}^i{}_{lk}. \tag{3.141}$$

By (3.115), we have

$$du^j \wedge \theta_j{}^i = 0.$$

Also from above, we can write

$$du^j \wedge \theta_j{}^k \wedge \theta_k{}^i = 0. \tag{3.142}$$

Exterior differentiation of (3.115), given

$$du^j \wedge d\theta_j{}^i = 0. \tag{3.143}$$

By subtracting (3.142) from (3.143), we get

$$du^j \left(d\theta_j{}^i - \theta_j{}^k \wedge \theta_k{}^i \right) = 0, \tag{3.144}$$

or equivalently

$$du^j \wedge \Omega_j{}^i = 0. \tag{3.145}$$

Substituting (3.140) into (3.145), we obtain

$$\begin{aligned} &\frac{1}{2} R_j{}^i{}_{kl} du^j \wedge du^k \wedge du^l + P_j{}^i{}_{kl} du^j \wedge du^k \wedge \frac{\delta X^l}{F} + \\ &\frac{1}{2} Q_j{}^i{}_{kl} du^j \wedge \frac{\delta X^k}{F} \wedge \frac{\delta^l X}{F} = 0 \end{aligned} \tag{3.146}$$

The three terms of (3.146) are different types, so they vanish. From the vanishing of the third term, Q must be symmetric. Together with (3.141) we obtain that

$$Q_j{}^i{}_{kl} = 0 \tag{3.147}$$

Thus curvature two-form take the form

$$\Omega_j{}^i = \frac{1}{2} R_j{}^i{}_{kl} du^i \wedge du^l + P_j{}^i{}_{kl} du^k \wedge \frac{\delta X^l}{F}. \tag{3.148}$$

From the vanishing of the second term, we have

$$P_j{}^i{}_{kl} = -P_k{}^i{}_{jl}, \tag{3.149}$$

and from the vanishing of the first term, we obtain the Bianchi identity

$$R_{j\ kl}^{\ i} du^j \wedge du^k \wedge du^l = 0. \tag{3.150}$$

By changing the indices and order of the wedge products, we have the following

$$R_{k\ lj}^{\ i} du^j \wedge du^k \wedge du^l = 0, \tag{3.151}$$

$$R_{l\ jk}^{\ i} du^j \wedge du^k \wedge du^l = 0. \tag{3.152}$$

Adding (3.150), (3.151) and (3.152), we get

$$\Big(R_{j\ kl}^{\ i} + R_{k\ lj}^{\ i} + R_{l\ jk}^{\ i}\Big) du^j \wedge du^k \wedge du^l = 0. \tag{3.153}$$

Thus we obtained the Bianchi identity as

$$R_{j\ kl}^{\ i} + R_{k\ lj}^{\ i} + R_{l\ jk}^{\ i} = 0. \tag{3.154}$$

We now obtain formulas for R and P in natural coordinates. Consider

$$d\theta_j^{\ i} - \theta_j^{\ h} \wedge \theta_h^{\ i} = \Omega_j^{\ i} = \frac{1}{2} R_{j\ kl}^{\ i} du^k \wedge du^l + P_{j\ kl}^{\ i} du^k \wedge \frac{\delta X^l}{F}. \tag{3.155}$$

By using (3.116), we can write $d\theta_j^{\ i}$ as

$$d\theta_j^{\ i} = d\Gamma^i_{\ jl} \wedge du^l. \tag{3.156}$$

Since $d\Gamma^i_{\ jl}$ is a one-form on $TM \setminus 0$, we can write it as

$$d\Gamma^i_{\ jl} = \frac{\delta \Gamma^i_{\ jl}}{\delta u^k} du^k + F \frac{\partial \Gamma^{\ i}_{\ jl}}{\partial X^k} \frac{\delta X^k}{F}. \tag{3.157}$$

Writing (3.157) into (3.156), we get

$$\begin{aligned} d\theta_j^{\ i} &= \frac{\delta \Gamma^i_{\ jl}}{\delta u^k} du^k \wedge du^l + F \frac{\partial \Gamma^i_{\ jl}}{\partial X^k} \frac{\delta X^k}{F} \wedge du^l \\ &= \frac{\delta \Gamma^i_{\ jl}}{\delta u^k} du^k \wedge du^l - F \frac{\partial \Gamma^i_{\ jk}}{\partial X^l} du^k \wedge \frac{\delta X^l}{F}. \end{aligned} \tag{3.158}$$

Consider the following term which is in (3.155)

$$-\theta_j^{\ h} \wedge \theta_h^{\ i} = \theta_h^{\ i} \wedge \theta_j^{\ h} = \Gamma^i_{\ hk} \Gamma^h_{\ jl} du^k \wedge du^l. \tag{3.159}$$

Substituting (3.158) and (3.159) into (3.155), we get

$$\Big(\frac{\delta\Gamma^i{}_{jl}}{\delta u^k}+\Gamma^i{}_{hk}\Gamma^h{}_{jl}\Big)du^k\wedge du^l-F\frac{\partial\Gamma^i{}_{jk}}{\partial X^l}du^k\wedge\frac{\delta X^l}{F}=$$
$$\frac{1}{2}R_j{}^i{}_{kl}du^k\wedge du^l+P_j{}^i{}_{kl}du^k\wedge\frac{\delta X^l}{F},\tag{3.160}$$

or equivalently

$$\frac{1}{2}\frac{\delta\Gamma^i{}_{jl}}{\delta u^k}du^k\wedge du^l-\frac{1}{2}\frac{\delta\Gamma^i{}_{jk}}{\delta u^l}du^k\wedge du^l+$$
$$\frac{1}{2}\Gamma^i{}_{hk}\Gamma^h{}_{jl}du^k\wedge du^l-\frac{1}{2}\Gamma^i{}_{hl}\Gamma^h{}_{jk}du^k\wedge du^l-F\frac{\partial\Gamma^i{}_{jk}}{\partial X^l}du^k\wedge\frac{\delta X^l}{F}=$$
$$\frac{1}{2}R_j{}^i{}_{kl}du^k\wedge du^l+P_j{}^i{}_{kl}du^k\wedge\frac{\delta X^l}{F}.\tag{3.161}$$

Comparing the coefficients of $du^k\wedge du^l$ and $du^k\wedge\frac{\delta X^l}{F}$ in (3.161), we obtain formulas for $R_j{}^i{}_{kl}$ and $P_j{}^i{}_{kl}$ in terms of natural coordinates, respectively as

$$R_j{}^i{}_{kl}=\frac{\delta\Gamma^i{}_{jl}}{\delta X^k}-\frac{\delta\Gamma^i{}_{jk}}{\delta X^l}+\Gamma^i{}_{hk}\Gamma^h{}_{jl}-\Gamma^i{}_{hl}\Gamma^h{}_{jk},\tag{3.162}$$

and

$$P_j{}^i{}_{kl}=-F\frac{\partial\Gamma^i{}_{jk}}{\partial X^l}.\tag{3.163}$$

Let us introduce the following quantities which will simplify some calculations to obtain the Ricci curvature.

$$R\,^i{}_k\equiv e_m^{\ j}R_j{}^i{}_{kl}e_m^{\ l},\tag{3.164}$$

$$R_{ik}\equiv e_m^{\ j}R_{jikl}e_m^{\ l},\tag{3.165}$$

$$R\,^i{}_{kl}\equiv e_m^{\ j}R_j{}^i{}_{kl},\tag{3.166}$$

and

$$P\,^i{}_{kl}\equiv e_m^{\ j}P_j{}^i{}_{kl}.\tag{3.167}$$

3.4.2 Relations for Ω, R and P in Natural Coordinates and Arbitrary Orthogonal Basis

We have made use of the natural coordinates for the curvature two-form and its R, P parts. Let us denote the curvature two-form of the Chern connection for the arbitrary orthogonal basis sections as $\Omega_\beta{}^\alpha$ such that

$$\Omega_\beta{}^\alpha = dw_\beta{}^\alpha - w_\beta{}^\gamma \wedge w_\gamma{}^\alpha, \tag{3.168}$$

where $w_\beta{}^\alpha$ are components of the Chern connection. It is torsion-free such that

$$dw^\alpha = w^\beta \wedge w_\beta{}^\alpha. \tag{3.169}$$

By taking exterior derivative of (5.2), we have

$$0 = w^\gamma \wedge (dw_\gamma{}^\alpha - w_\gamma{}^\beta \wedge w_\beta{}^\alpha), \tag{3.170}$$

or equivalently

$$w^\gamma \wedge \Omega_\gamma{}^\alpha = 0. \tag{3.171}$$

$\Omega_\beta{}^\alpha$ are two-form on PTM, so we can express it as linear combinations of $w^\rho \wedge w^\sigma$, $w^\rho \wedge w_m^{\bar{\sigma}}$ and $w_m^{\bar{\rho}} \wedge w_m^{\bar{\sigma}}$ as

$$\Omega_\beta{}^\alpha = \frac{1}{2} R_\beta{}^\alpha{}_{\rho\sigma} w^\rho \wedge w^\sigma + P_\beta{}^\alpha{}_{\rho\bar{\sigma}} w^\rho \wedge w_m^{\bar{\sigma}} + \frac{1}{2} Q_\beta{}^\alpha{}_{\bar{\rho}\bar{\sigma}} w_m^{\bar{\rho}} \wedge w_m^{\bar{\sigma}}. \tag{3.172}$$

By using (4.106) which is consequence of torsion-freeness of the Chern connection and as in natural coordinates, we get

$$Q_\beta{}^\alpha{}_{\bar{\rho}\bar{\sigma}} = 0. \tag{3.173}$$

Thus $\Omega_\beta{}^\alpha$ take the form

$$\Omega_\beta{}^\alpha = \frac{1}{2} R_\beta{}^\alpha{}_{\rho\sigma} w^\rho \wedge w^\sigma + P_\beta{}^\alpha{}_{\rho\bar{\sigma}} w^\rho \wedge w_m^{\bar{\sigma}}. \tag{3.174}$$

We now obtain formulas for $\Omega_\beta{}^\alpha$, $R_\beta{}^\alpha{}_{\rho\sigma}$ and $P_\beta{}^\alpha{}_{\rho\bar{\sigma}}$ in terms of the $\Omega_j{}^i$, $R_j{}^i{}_{kl}$ and $P_j{}^i{}_{kl}$. Consider $\Omega_j{}^i$ as

$$\Omega_j{}^i = \frac{1}{2} R_j{}^i{}_{kl} du^k \wedge du^l + P_j{}^i{}_{kl} du^k \wedge \frac{\delta X^l}{F}. \tag{3.175}$$

By using $du^i = e_\alpha{}^i w^\alpha$ and $\frac{\delta X^l}{F} e_{\bar{\alpha}}{}^i w_m^{\bar{\alpha}}$ in (5.5) and contracting the result equation with $e_\beta{}^j e^\alpha{}_i$, we get

$$e_\beta{}^j \Omega_j{}^i e^\alpha{}_i = \frac{1}{2} e_\beta{}^j e_\rho{}^k e_\sigma{}^l R_j{}^i{}_{kl} e^\alpha{}_i w^\rho \wedge w^\sigma + e_\beta{}^j e_\rho{}^k e_{\bar{\sigma}}{}^l P_j{}^i{}_{kl} e^\alpha{}_i w^\rho \wedge w^{\bar{\sigma}}. \tag{3.176}$$

Considering (5.4) with (5.6), we can write the following relations

$$\Omega_\beta{}^\alpha = e_\beta{}^j \Omega_j{}^i e^\alpha{}_i, \tag{3.177}$$

$$R_\beta{}^\alpha{}_{\rho\sigma} = e_\beta{}^j e_\rho{}^k e_\sigma{}^l R_j{}^i{}_{kl} e^\alpha{}_i, \tag{3.178}$$

$$P_\beta{}^\alpha{}_{\rho\bar{\sigma}} = e_\beta{}^j e_\rho{}^k e_{\bar{\sigma}}{}^l P_j{}^i{}_{kl} e_i{}^\alpha. \tag{3.179}$$

3.4.3 Some Computations

In this section we obtain some relations which will be utilized later. Almost metric-compatibility condition of the Chern connection in natural coordinates is

$$dg_{ij} - g_{kj}\theta_i{}^k - g_{ik}\theta_j{}^k = 2A_{ijs}\frac{\delta X^s}{F}. \tag{3.180}$$

Exterior derivative of Cartan tensor $A = A_{ijk}du^i \otimes du^j \otimes du^k$ is

$$\begin{aligned} DA &= \Big(dA_{ijl} - A_{kjl}\theta_i{}^k - A_{ikl}\theta_j{}^k - A_{ijk}\theta_l{}^k\Big)du^i \otimes du^j \otimes du^l \\ &\equiv (DA)_{ijl}du^i \otimes du^j \otimes du^l. \end{aligned} \tag{3.181}$$

Here $(DA)_{ijl}$ are one-forms on $TM \setminus 0$ and can be expressed in terms of the bases du^s and $\frac{\delta X^s}{F}$ as

$$(DA)_{ijl} = A_{ijk|s}du^s + A_{ijk;s}\frac{\delta X^s}{F}. \tag{3.182}$$

Using (3.116) and $dA_{ijl} = \frac{\delta A_{ijl}}{\delta u^s} + F\frac{\partial A_{ijl}}{\partial X^s}\frac{\delta X^s}{F}$ in (3.181) and considering the du^s and $\frac{\delta X^s}{F}$ parenthesis, we get $A_{ijk|s}$ and $A_{ijk;s}$, respectively as

$$A_{ijk|s} = \frac{\delta A_{ijk}}{\delta u^s} - A_{ljk}\Gamma^l{}_{is} - A_{ikl}\Gamma^l{}_{js} - A_{ijl}\Gamma^l{}_{ks}, \tag{3.183}$$

and

$$A_{ijk;s} = F\frac{\partial A_{ijk}}{\partial X^s}. \tag{3.184}$$

Here $A_{ijk|s}$ and $A_{ijk;s}$ denote the horizontal and vertical covariant derivatives of A_{ijk}. We obtain some facts which will be useful in the next section. Consider

$$\begin{aligned} d(\log F) &= \frac{dF}{F} = \frac{1}{F}(F_{X^i}dX^i + F_{u^i}du^i) \\ &= \frac{F_{X^i}}{F}\Big(\delta X^i - N_j{}^i du^j\Big) + \frac{F_{u^i}}{F}du^i. \end{aligned} \tag{3.185}$$

Here we used $dX^i = \delta X^i - N_j{}^i du^j$. By using (3.136), (3.9), (3.31), (3.32), $e^m_{\ i}g^{is} = e_m^{\ s}$, $e_m^{\ i}A_{ijk} = 0$, $X^sX^kg_{ks} = F^2$, we can write (3.185) as

$$\begin{aligned} d(\log F) &= \frac{e^m_{\ i}}{F}\delta X^i - \frac{e^m_{\ i}}{F}\gamma^i{}_{jk}X^k du^j - \frac{e^m_{\ i}}{F}A^i{}_{jk}\gamma^k{}_{rs}X^rX^s du^j + \frac{F_{u^i}}{F}du^i \\ &= \frac{e^m_{\ i}}{F}\delta X^i - \frac{X^sX^k}{2F^2}\Big(\frac{\partial g_{sj}}{\partial u^k} - \frac{\partial g_{jk}}{\partial u^s} + \frac{\partial g_{ks}}{\partial u^j}\Big)du^j - \frac{e_m^{\ i}}{F}A_{ijk}\gamma^k{}_{rs}X^rX^s du^j + \frac{F_{u^i}}{F}du^i \\ &= \frac{e^m_{\ i}}{F}\delta X^i - \frac{1}{F^2}\frac{\partial(X^sX^kg_{ks})}{\partial u^j}du^j + \frac{F_{u^i}}{F}du^i \\ &= \frac{e^m_{\ i}}{F}\delta X^i - \frac{F_{u^j}}{F}du^j + \frac{F_{u^i}}{F}du^i = \frac{e^m_{\ i}}{F}\delta X^i. \end{aligned} \tag{3.186}$$

Thus we obtain

$$d(\log F) = \frac{F_{X^i}}{F}\delta X^i. \tag{3.187}$$

By using (3.135), $X^j A_{ijk} = 0$, symmetry of Cartan tensor and (3.136), we have

$$\begin{aligned}
\Gamma^l{}_{jk}\frac{X^j}{F} &= \gamma^l{}_{jk}\frac{x^j}{F} - \frac{X^j}{F}g^{li}\Big(A_{ijs}\frac{N_k{}^s}{F} - A_{jks}\frac{N_i{}^s}{F} + A_{kis}\frac{N_j{}^s}{F}\Big) \\
&= \gamma^l{}_{jk}\frac{x^j}{F} - g^{li}\frac{X^j}{F^2}A_{kis}N_j{}^s \\
&= \gamma^l{}_{jk}\frac{x^j}{F} - g^{li}\frac{X^j}{F^2}A_{kis}\Big(\gamma^s{}_{jt}X^t - \frac{1}{F}A^s{}_{jt}\gamma^t{}_{rs}X^rX^s\Big) \\
&= \gamma^l{}_{jk}\frac{x^j}{F} - \frac{A^l{}_{ks}}{F^2}\gamma^s{}_{jt}X^jX^t = \frac{1}{F}\Big(\gamma^l{}_{jk}X^j - \frac{A^l{}_{ks}}{F}\gamma^s{}_{jt}X^jX^t\Big) \\
&= \frac{N_k{}^l}{F}.
\end{aligned} \tag{3.188}$$

Thus we obtain

$$\Gamma^l{}_{jk}\frac{X^j}{F} = \frac{N_k{}^l}{F}. \tag{3.189}$$

By using (3.183), (3.83), (3.107) and (3.189), we have

$$\begin{aligned}
A_{ijk|s}e_m{}^k &= \Big(\frac{\delta A_{ijk}}{\delta u^s} - A_{ljk}\Gamma^l{}_{is} - A_{ilk}\Gamma^l{}_{js} - A_{ijl}\Gamma^l{}_{ks}\Big)e_m{}^k \\
&= \frac{\delta A_{ijk}}{\delta u^s}e_m{}^k - A_{ijl}\Gamma^l{}_{ks}e_m{}^k \\
&= e_m{}^k\frac{\partial A_{ijk}}{\partial u^s} - N_s{}^r\frac{\partial A_{ijk}}{X^r}e_m{}^k - A_{ijl}\frac{N_s{}^l}{F} \\
&= \frac{1}{F}\frac{\partial(A_{ijk}X^k)}{\partial u^s} - \frac{N_s{}^r}{F}\Big(\frac{\partial(A_{ijk}X^k)}{X^r} - A_{ijk}\delta^k_r\Big) - A_{ijl}\frac{N_s{}^l}{F} \\
&= \frac{N_s{}^r}{F}A_{ijr} - \frac{N_s{}^l}{F}A_{ijl} = 0.
\end{aligned} \tag{3.190}$$

Thus we get

$$A_{ijk|s}e_m{}^k = 0. \tag{3.191}$$

Let us denote $A_{ijk|s}e_m{}^s$ as

$$\dot{A}_{ijk} \equiv A_{ijk|s}e_m{}^s. \tag{3.192}$$

By using (3.191), we can wtite

$$\dot{A}_{ijk}e^k_m = 0. \tag{3.193}$$

By using (3.184) and (3.83), we have

$$\begin{aligned} A_{ijk;s}e_m^{\ k} &= F\frac{\partial A_{ijk}}{\partial X^s}\frac{X^k}{F} \\ &= \frac{\partial(A_{ijk}X^k)}{\partial X^s} - A_{ijk}\delta_s^k. \end{aligned} \tag{3.194}$$

Hence we get

$$A_{ijk;s}e_m^{\ k} = -A_{ijs}. \tag{3.195}$$

Exterior derivative of e_m is formed

$$\begin{aligned} De_m &= \Big(de_m^{\ i} + e_m^{\ j}\theta_j^{\ i}\Big)\frac{\partial}{\partial u^i} \\ &= \Big(d(\frac{X^i}{F}) + \frac{X^j}{F}\theta_j^{\ i}\Big)\frac{\partial}{\partial u^i} \\ &\equiv (De_m)^i\frac{\partial}{\partial u^i}. \end{aligned} \tag{3.196}$$

By using (3.189) and (3.105), $(De_m)^i$ can be written as

$$\begin{aligned} (De_m)^i &= \frac{dX^i}{F} - \frac{X^i dF}{F^2} + \frac{X^j}{F}\theta_j^{\ i} = \frac{dX^i}{F} - \frac{X^i dF}{F^2} + \frac{X^j}{F}\Gamma^i_{\ jk}du^k \\ &= \frac{dX^i}{F} - \frac{X^i dF}{F^2} + \frac{N_k^{\ i}}{F}du^k \\ &= \frac{\delta X^i}{F} - \frac{X^i}{F}d(\log F). \end{aligned} \tag{3.197}$$

By (3.196) and (3.197), we can write

$$\frac{\delta X^i}{F} = de_m^{\ i} + e_m^{\ j}\theta_j^{\ i} + e_m^{\ i}d(\log F). \tag{3.198}$$

Taking exterior derivative of (3.198) and using (3.139), (3.198), we get

$$\begin{aligned} d\Big(\frac{\delta X^i}{F}\Big) &= d(e_m^{\ j})\theta_j^{\ i} + e_m^{\ j}d(\theta_j^{\ i}) + d(e_m^{\ i})d(\log F) \\ &= e_m^{\ j}\Omega_j^{\ i} + \frac{\delta X^j}{F}\Big(\theta_j^{\ i} - e^m_{\ j}\frac{\delta X^i}{F}\Big). \end{aligned} \tag{3.199}$$

3.4.4 Some Relations for R and P from Almost Metric-Compatibility in Natural Coordinates

Taking exterior derivative of (3.180) and using (3.139) for $\Omega_i^{\ j}$, $g_{kj}\Omega_i^{\ j} = \Omega_{ik}$ and after some simplification we have

$$\Omega_{ij} + \Omega_{ji} = -2\Big[dA_{ijl} - A_{kjl}\theta_i^{\ k} - A_{ikl}\theta_j^{\ k}\Big] \wedge \frac{\delta X^l}{F} - 2A_{ijl}d\Big(\frac{\delta X^l}{F}\Big) \tag{3.200}$$

By using (3.181) in the above expression, we get

$$\begin{aligned}\Omega_{ij}+\Omega_{ji} &= -2\Big[(DA)_{ijl}+A_{ijk}\theta_l{}^k\Big]\wedge\frac{\delta X^l}{F}-2A_{ijl}d\Big(\frac{\delta X^l}{F}\Big) \\ &= -2(DA)_{ijk}\wedge\frac{\delta X^k}{F}-2A_{ijk}\Big[d(\frac{\delta X^k}{F})+\theta_l{}^k\wedge\frac{\delta X^l}{F}\Big].\end{aligned} \tag{3.201}$$

By using (3.140), left hand side of (3.201) can be written as

$$\begin{aligned}\Omega_{ij}+\Omega_{ji} &= \Omega_i{}^k\delta_{kj}+\Omega_j{}^k\delta_{ki} \\ &= \delta_{kj}\Big(\frac{1}{2}R_i{}^k{}_{rs}du^r\wedge du^s+P_i{}^k{}_{rs}du^r\wedge\frac{\delta X^r}{F}\Big)+ \\ &\quad \delta_{ki}\Big(\frac{1}{2}R_j{}^k{}_{rs}du^r\wedge du^s+P_j{}^k{}_{rs}du^r\wedge\frac{\delta X^r}{F}\Big) \\ &= \frac{1}{2}\Big(R_{ijrs}+R_{jirs}\Big)du^r\wedge du^s+\Big(P_{ijrs}+P_{jirs}\Big)du^r\wedge\frac{\delta X^s}{F}.\end{aligned} \tag{3.202}$$

By using(3.182), (3.199), (3.140), $e_m^r R_r{}^k{}_{st}=R{}^k{}_{st}$ and $e_m^r P_r{}^k{}_{st}=P{}^k{}_{st}$ left hand side of (3.201) can be written as

$$\begin{aligned}&-2(DA)_{ijk}\wedge\frac{\delta X^k}{F}-2A_{ijk}\Big[d(\frac{\delta X^k}{F})+\theta_l{}^k\wedge\frac{\delta X^l}{F}\Big] \\ =\;& -2\Big(A_{ijk|s}du^s+A_{ijk;s}\frac{\delta X^s}{F}\Big)\wedge\frac{\delta X^s}{F}-2A_{ijk}e_m^r\Omega_r{}^k-2A_{ijk}\frac{\delta X^r}{F}\wedge\theta_r{}^k+ \\ & 2A_{ijk}e^m_r\frac{\delta X^r}{F}\wedge\frac{\delta X^k}{F}-2A_{ijk}\theta_l{}^k\wedge\frac{\delta X^l}{F} \\ =\;& -\Big(A_{iju}R{}^u{}_{kl}\Big)du^k\wedge du^l-2\Big(A_{iju}P{}^u{}_{kl}+A_{ijl|k}\Big)du^k\wedge\frac{\delta X^l}{F}+ \\ & 2\Big(A_{ijk;l}-A_{ijl}e^m_l\Big)\frac{\delta X^k}{F}\wedge\frac{\delta X^l}{F}.\end{aligned} \tag{3.203}$$

Inserting (3.202) and (3.203) into (3.201), we get

$$\begin{aligned}&\frac{1}{2}\Big(R_{ijrs}+R_{jirs}\Big)du^r\wedge du^s+\Big(P_{ijrs}+P_{jirs}\Big)du^r\wedge\frac{\delta X^s}{F} \\ =\;& -\Big(A_{iju}R{}^u{}_{kl}\Big)du^k\wedge du^l-2\Big(A_{iju}P{}^u{}_{kl}+A_{ijl|k}\Big)du^k\wedge\frac{\delta X^l}{F}+ \\ & 2\Big(A_{ijk;l}-A_{ijl}e_l{}^m\Big)\frac{\delta X^k}{F}\wedge\frac{\delta X^l}{F}.\end{aligned} \tag{3.204}$$

Comparing the coefficients of $du^k\wedge du^l$ in (3.204), we get the relation

$$R_{ijkl}+R_{jikl}=2\Big(-A_{iju}R{}^u{}_{kl}\Big)\equiv 2B_{ijkl}. \tag{3.205}$$

By using (3.205), we calculate the following combination

$$
\begin{aligned}
& \Big(B_{ijkl} - B_{klij}\Big) + \Big(B_{iljk} + B_{jkil}\Big) + \Big(B_{ljki} + B_{kilj}\Big) - R_{ijkl} + R_{klij} \\
= & \frac{R_{ijkl} + R_{jikl}}{2} - \frac{R_{klij} + R_{lkij}}{2} + \frac{R_{iljk} + R_{lijk}}{2} + \frac{R_{jkil} + R_{kjil}}{2} + \\
& \frac{R_{jkil} + R_{jlki}}{2} + \frac{R_{kilj} + R_{iklj}}{2} - R_{ijkl} + R_{klij} \\
= & -\frac{1}{2}\Big(R_{ijkl} + R_{kjli} + R_{ljik}\Big) + \frac{1}{2}\Big(R_{klij} + R_{iljk} + R_{jlki}\Big) + \\
& \frac{1}{2}\Big(R_{lijk} + R_{jikl} + R_{kilj}\Big) + \frac{1}{2}\Big(R_{iklj} + R_{lkji} + R_{jkil}\Big) = 0.
\end{aligned}
\tag{3.206}
$$

Here we used the Bianchi identities of the form $R_{kijl} + R_{jikl} + R_{likj} = 0$. Thus we get another Bianchi identity as

$$
R_{ijkl} - R_{klij} = \Big(B_{ijkl} - B_{klij}\Big) + \Big(B_{iljk} + B_{jkil}\Big) + \Big(B_{ljki} + B_{kilj}\Big). \tag{3.207}
$$

Comparing the coefficients of $du^k \wedge \frac{\delta X^l}{F}$ in (3.204), we get

$$
P_{ijkl} + P_{jikl} = -2A_{iju}P^{\ \ u}_{\ kl} - 2A_{ijl|k} \equiv 2E_{ijkl} - 2A_{ijl|k}. \tag{3.208}
$$

Consider the following combination

$$
\Big(P_{ijkl} + P_{jikl}\Big) - \Big(P_{jkil} + P_{kjil}\Big) + \Big(P_{kijl} + P_{ikjl}\Big). \tag{3.209}
$$

By using the symmetry of P, (3.209) takes the form

$$
\Big(P_{ijkl} + P_{jikl}\Big) - \Big(P_{jkil} + P_{kjil}\Big) + \Big(P_{kijl} + P_{ikjl}\Big) = 2P_{jikl}. \tag{3.210}
$$

By using (3.208), (3.209) takes the form

$$
\begin{aligned}
& \Big(P_{ijkl} + P_{jikl}\Big) - \Big(P_{jkil} + P_{kjil}\Big) + \Big(P_{kijl} + P_{ikjl}\Big) \\
= & \; 2E_{ijkl} - 2E_{jkil} + 2E_{kijl} - 2A_{ijl|k} + 2A_{jkl|i} - 2A_{kil|j}.
\end{aligned}
\tag{3.211}
$$

Considering (3.210) with (3.211), we get the following relation

$$
P_{jikl} = \Big(E_{ijkl} - E_{jkil} + E_{kijl}\Big) - \Big(A_{ijl|k} - A_{jkl|i} + A_{kil|j}\Big). \tag{3.212}
$$

By using (3.163) and Euler's theorem, we have

$$
P_{j\ kl}^{\ i}e_m^{\ l} = -F\frac{\partial \Gamma^i_{\ jk}}{\partial X^l}\frac{X^l}{F} = -X^l\frac{\partial \Gamma^i_{\ jk}}{\partial X^l} = 0, \tag{3.213}
$$

where ${\Gamma^i}_{jk}$ is homogeneous of degree zero in X^i. By (3.208) and (3.83), we have

$$E_{ijkl}e_m^{\ j} = -A_{iju}e_m^{\ j}P^u_{\ kl} = 0. \tag{3.214}$$

Contracting (3.212) with $e_m^{\ j}$ and using (3.192), (3.193), (3.191), (3.195), we get

$$P_{jikl}e_m^{\ j} \equiv P_{ikl} = -A_{kil|j}e_m^{\ j} = -\dot{A}_{kil}. \tag{3.215}$$

And from this

$$P_{jikl}e_m^{\ j}e_m^{\ k} = -\dot{A}_{kil}e_m^{\ k} = 0. \tag{3.216}$$

By using (3.215) and symmetry of Cartan tensor, we get

$$E_{ijkl} = -A_{iju}P^u_{\ kl} = -{A^u}_{ij}P_{ukl} = -{A^u}_{ij}(-\dot{A}_{ukl}) = {A^u}_{ij}\dot{A}_{ukl}. \tag{3.217}$$

Thus, we get the second Chern curvature tensor P in terms of the Cartan tensor and its horizontal covariant derivative $A_{ijk|s}$ as

$$P_{jikl} = -\Big(A_{ijl|k} + A_{jkl|i} + A_{kil|j}\Big) + {A^u}_{ij}\dot{A}_{ukl} - {A^u}_{jk}\dot{A}_{uil} + {A^u}_{ki}\dot{A}_{ujl}. \tag{3.218}$$

similarly we obtain

$$P_{ijkl} + P_{jikl} = -2A_{iju}P^u_{\ kl} - 2A_{ijl|k} = -2{A^u}_{ij}\dot{A}_{ukl} - 2A_{ijl|k}, \tag{3.219}$$

or equivalently

$$A_{ijl|k} = {A^u}_{ij}\dot{A}_{ukl} - \frac{1}{2}\Big(P_{ijkl} + P_{jikl}\Big). \tag{3.220}$$

From there we can say P_{jikl} vanishes if and only if $A_{ijk|l}$ vanishes. We need the following facts to obtain ${R^{\ i}}_k$ in terms of ${N_i}^j$ and F.

$$e_m^{\ r}\frac{\delta {\Gamma^i}_{jk}}{\delta u^s} = \frac{\delta\big(e_m^{\ r}{\Gamma^i}_{jk}\big)}{\delta u^s} + {\Gamma^i}_{jk}\frac{{N_s}^r}{F}, \tag{3.221}$$

which comes directly from definition of $\frac{\delta}{\delta u^i}$ and (3.189). By contracting (3.162) with $e_m^{\ j}e_m^{\ l}$ and using (3.164), (3.221) and (3.189) we get ${R^{\ i}}_k$ as

$${R^{\ i}}_k = e_m^{\ j}\left(\frac{\delta}{\delta u^k}\left(\frac{{N_j}^i}{F}\right) - \frac{\delta}{\delta u^j}\left(\frac{{N_k}^i}{F}\right)\right). \tag{3.222}$$

By multiplying the above expression with g_{il} we get the Ricci curvature in terms of ${N_i}^j$ as

$$R_{lk} = g_{il}e_m^{\ j}\left(\frac{\delta}{\delta u^k}\left(\frac{{N_j}^i}{F}\right) - \frac{\delta}{\delta u^j}\left(\frac{{N_k}^i}{F}\right)\right). \tag{3.223}$$

We can state a theorem to summarize the symmetry properties of Chern curvature tensors R_{jikl} and P_{jikl}.

Theorem 3.4.1 *First and second curvature tensors of Chern connection have the following properties:*

1) $R_{jikl} = -R_{jilk}$,

2) $R_{jikl} + R_{kilj} + R_{lijk} = 0$,

3) $R_{jikl} + R_{ijkl} = 2(-A_{iju}R^{u}{}_{kl}) \equiv 2B_{ijkl}$,

4) $R_{ijkl} - R_{klij} = \Big(B_{ijkl} - B_{klij}\Big) + \Big(B_{iljk} + B_{jkil}\Big) + \Big(B_{ljki} + B_{kilj}\Big)$,

5) $P_{kijl} = P_{jikl}$,

6) $P_{ijkl} + P_{jikl} = -2A_{iju}P\,{}^{u}{}_{kl} - 2A_{ijl|k} \equiv 2E_{ijkl} - 2A_{ijl|k}$.

where $A_{ijl|k} = \frac{\delta A_{ijl}}{\delta u^k} - A_{rjl}\Gamma^{r}{}_{ik} - A_{irl}\Gamma^{r}{}_{jk} - A_{ijr}\Gamma^{r}{}_{lk}$.

3.4.5 Formulas for Horizontal and Vertical Covariant Derivative of R and P

In this part, we will obtain some formulas for horizontal and vertical covariant derivatives of R and P. We have the curvature two-form of the Chern connection as

$$\Omega_j{}^i = d\theta_j{}^i - \theta_j{}^k \wedge \theta_k{}^i. \tag{3.224}$$

By taking exterior derivative of (3.224), we have

$$\begin{aligned} d\Omega_j{}^i &= -d\theta_j{}^k \wedge \theta_k{}^i - \theta_j{}^k \wedge d\theta_k{}^i \\ &= \theta_j{}^k \wedge \Omega_k{}^i - \theta_k{}^i \wedge \Omega_j{}^k. \end{aligned} \tag{3.225}$$

Thus we get the second Bianchi identity as

$$d\Omega_j{}^i - \theta_j{}^k \wedge \Omega_k{}^i + \theta_k{}^i \wedge \Omega_j{}^k = 0. \tag{3.226}$$

By using (3.140) for $\Omega_j{}^i$, (3.199), (3.215), we can write (3.226) as

$$\frac{1}{2}d(R_j{}^i{}_{kl})du^k \wedge du^l + d(P_j{}^i{}_{kl})du^k \wedge \frac{\delta X^l}{F} - P_j{}^i{}_{kl}du^k \wedge d(\frac{\delta X^l}{F})$$

$$
\begin{aligned}
&- \frac{1}{2}\theta_j{}^k R_k{}^i{}_{rs} du^r \wedge du^s - \theta_j{}^k P_k{}^i{}_{rs} du^r \wedge \frac{\delta X^s}{F} \\
&+ \frac{1}{2}\theta_k{}^i R_j{}^k{}_{rs} du^r \wedge du^s - \theta_k{}^i P_j{}^i{}_{rs} du^r \wedge \frac{\delta X^s}{F} \\
&= \frac{1}{2}\Big(R_j{}^i{}_{kl|t} du^t + R_j{}^i{}_{kl;t}\frac{\delta X^t}{F}\Big) du^k \wedge du^l + \Big(P_j{}^i{}_{kl|t} du^t + P_j{}^i{}_{kl;s}\frac{\delta X^t}{F}\Big) du^k \wedge \frac{\delta X^l}{F} \\
&- P_j{}^i{}_{kl} du^k \wedge \Big[e_m{}^t\Big(\frac{1}{2}R_t{}^l{}_{rs} du^r \wedge du^s + P_t{}^l{}_{rs} du^r \wedge \frac{\delta X^s}{F}\Big) + \frac{\delta X^t}{F}\Big(\theta_t{}^l - e_t{}^m\frac{\delta X^l}{F}\Big)\Big] \\
&- \frac{1}{2}\theta_j{}^k R_k{}^i{}_{rs} du^r \wedge du^s - \theta_j{}^k P_k{}^i{}_{rs} du^r \wedge \frac{\delta X^s}{F} \\
&+ \frac{1}{2}\theta_k{}^i R_j{}^k{}_{rs} du^r \wedge du^s - \theta_k{}^i P_j{}^i{}_{rs} du^r \wedge \frac{\delta X^s}{F} \\
&= \frac{1}{2}\Big(R_j{}^i{}_{kl|t} - P_j{}^i{}_{ku} R^{\ u}{}_{lt}\Big) du^k \wedge du^l \wedge du^t + \frac{1}{2}\Big(R_j{}^i{}_{kl;t} - 2P_j{}^i{}_{kt|l}\dot{A}^u{}_{lt}\Big) du^k \wedge du^l \wedge \frac{\delta X^t}{F} \\
&+ \Big(P_j{}^i{}_{kl;t} - P_j{}^i{}_{kl} e_t{}^m\Big) du^k \wedge \frac{\delta X^l}{F} \wedge \frac{\delta X^t}{F} = 0. \qquad (3.227)
\end{aligned}
$$

From this equation and changing order of wedge product and indices, we get the following three equations

$$R_j{}^i{}_{kl|t} = P_j{}^i{}_{ku} R^{\ u}{}_{lt}, \tag{3.228}$$

$$R_j{}^i{}_{kl;t} = P_j{}^i{}_{kt|l} - P_j{}^i{}_{lt|k} - \Big(P_j{}^i{}_{kl}\dot{A}^u{}_{lt} - P_k{}^i{}_{lu}\dot{A}^u{}_{kt}\Big), \tag{3.229}$$

$$P_j{}^i{}_{kl;t} = P_j{}^i{}_{kl} e^m_{\ t}. \tag{3.230}$$

Similarly by using (3.228), we get the following identity

$$R_j{}^i{}_{kl|t} + R_j{}^i{}_{lt|k} + R_j{}^i{}_{tk|l} = P_j{}^i{}_{ku} R^{\ u}{}_{lt} + P_j{}^i{}_{lu} R^{\ u}{}_{tk} + P_j{}^i{}_{tu} R^{\ u}{}_{kl}. \tag{3.231}$$

By using (3.230), we get the following identity

$$P_j{}^i{}_{kl;t} - P_j{}^i{}_{kt;l} = P_j{}^i{}_{kl} e^m_{\ t} - P_j{}^i{}_{kt} e^m_{\ l}. \tag{3.232}$$

By contracting (3.231) with e^j_m and using (3.215), we get

$$R_j{}^i{}_{kl|t} + R_j{}^i{}_{lt|k} + R_j{}^i{}_{tk|l} = -\dot{A}^i{}_{ku} R^{\ u}{}_{lt} - \dot{A}^i{}_{lu} R^{\ u}{}_{tk} - \dot{A}^i{}_{tu} R^{\ u}{}_{kl}, \tag{3.233}$$

similarly by contracting (3.232) with e^j_m and using (3.215), we have

$$P_j{}^i{}_{kl} - P_l{}^i{}_{kj} = \dot{A}^i{}_{kj;l} - \dot{A}^i{}_{kl;j}. \tag{3.234}$$

By using (3.218), the above expression can be written as

$$
\begin{aligned}
\dot{A}^i{}_{kj;l} - \dot{A}^i{}_{kl;j} &= A^i{}_{kj|l} - A^i_{kl|j} \qquad (3.235) \\
&+ \Big(A^i{}_{ju}\dot{A}^u{}_{kl} + \dot{A}^i{}_{ju} A^u{}_{kl}\Big) - \Big(A^i{}_{lu}\dot{A}^u{}_{kj} + \dot{A}^i{}_{lu} A^u{}_{kj}\Big).
\end{aligned}
$$

Similarly by taking the vertical derivative of $e_m^{\ j} R_j{}^i{}_{kl} = R^{\ i}{}_{kl}$ with respect to X^t, we get

$$e_m^{\ j} R_j{}^i{}_{kl;t} = R^{\ i}{}_{kl;t} - R_t{}^i{}_{kl} + e^m_{\ t} R^{\ i}{}_{kl}. \tag{3.236}$$

By contracting (3.229) with $e_m^{\ j}$ and using (3.236), (3.215), we have

$$\begin{aligned} R_j{}^i{}_{kl;t} e_m^{\ j} &= e_m^{\ j} P_j{}^i{}_{kt|l} - e_m^{\ j} P_j{}^i{}_{lt|k} - \Big(e_m^{\ j} P_j{}^i{}_{ku} \dot{A}^u{}_{lt} - e_m^{\ j} P_k{}^i{}_{lu} \dot{A}^u{}_{kt} \Big) \qquad (3.237)\\ R^{\ i}{}_{kl;j} - R_j{}^i{}_{kl} + e^m_{\ j} R^{\ i}{}_{kl} &= -\Big(\dot{A}^i{}_{lj|k} - \dot{A}^i{}_{kj|l} + \dot{A}^i{}_{ku} \dot{A}^u{}_{lj} - \dot{A}^i{}_{lu} \dot{A}^u{}_{kj} \Big)\\ R_j{}^i{}_{kl} &= R^{\ i}{}_{kl;j} + e^m_{\ j} R^{\ i}{}_{kl} - \Big(\dot{A}^i{}_{lj|k} - \dot{A}^i{}_{kj|l} + \dot{A}^i{}_{ku} \dot{A}^u{}_{lj} - \dot{A}^i{}_{lu} \dot{A}^u{}_{kj} \Big). \end{aligned}$$

By taking the vertical derivative of $e_m^{\ u} R^{\ i}{}_{ku} = R^{\ i}{}_k$ with respect to X^t, we get

$$e_m^{\ u} R^{\ i}_{\ ku;l} = R^{\ i}_{\ k;l} + R^{\ i}_{\ k} e^m_{\ l} - R^{\ i}_{\ kl}. \tag{3.238}$$

Contracting (3.237) with $e_m^{\ l}$ and using (3.191), (3.193), we get

$$R_j{}^i{}_{kl} e_m^{\ l} = R^{\ i}_{\ k;j} + 2R^{\ i}_{\ k} e^m_{\ j} + R_j{}^i{}_k + \dot{A}^{\ i}_{\ jk|l} e_m^{\ l}. \tag{3.239}$$

Consider the Bianchi identity of the form

$$R_j{}^i{}_{kl} + R_k{}^i{}_{lj} + R_l{}^i{}_{jk} = 0. \tag{3.240}$$

Contracting (3.240) with $e_m^{\ l}$, we get

$$R_j{}^i{}_{kl} e_m^{\ l} + R_k{}^i{}_{lj} e_m^{\ l} + R_j{}^i{}_k = 0. \tag{3.241}$$

Inserting (3.239) into (3.241), we obtain

$$R^{\ i}{}_{kl} = \frac{1}{2}\Big(R^{\ i}{}_{k;l} - R^{\ i}{}_{l;k} \Big) + \frac{2}{3}\Big(R^{\ i}_{\ k} e^m_{\ l} - R^{\ i}_{\ l} e^m_{\ k} \Big). \tag{3.242}$$

By using (3.237), (3.242), we obtain an expression for the curvature tensor

$$\begin{aligned} R_j{}^i{}_{kl} &= \frac{1}{3}\Big(R^{\ i}_{\ k;l;j} - R^{\ i}_{\ l;k;j} + e^m_{\ j} R^{\ i}_{\ k;l} - e^m_{\ j} R^{\ i}_{\ l;k} \Big) \qquad (3.243)\\ +\ & \frac{2}{3}\Big(R^{\ i}_{\ k;j} e^m_{\ l} - R^{\ i}_{\ l;j} e^m_{\ k} + R^{\ i}_{\ k} g_{jl} - R^{\ i}_{\ l} g_{jk} \Big) - \Big(\dot{A}^i{}_{jl|k} - \dot{A}^i{}_{uk} + \dot{A}^i{}_{uk} \dot{A}^u{}_{jl} - \dot{A}^i{}_{ul} \dot{A}^u{}_{jk} \Big). \end{aligned}$$

3.5 Geodesic Spray

Let us introduce

$$G^i \equiv \gamma^i{}_{jk} X^j X^k, \tag{3.244}$$

where $\gamma^i{}_{jk}$ are the Christoffel symbols. By taking the partial derivative of (3.244) with respect to X^j and using (3.79), (3.104) and (3.9) we obtain

$$\begin{aligned}
\frac{\partial G^i}{\partial X^j} &= \frac{\partial}{\partial X^j}\Big(\gamma^i{}_{rk} X^r X^k\Big) = \frac{\partial}{\partial X^j}\Big(\gamma^i{}_{rk}\Big) X^r X^k + 2\gamma^i{}_{jk} X^k \\
&= \frac{\partial (g^{is})}{\partial X^j}\frac{1}{2}\Big(\frac{\partial g_{sr}}{\partial u^k} - \frac{\partial g_{rk}}{\partial u^s} + \frac{\partial g_{ks}}{\partial u^r}\Big) + 2\gamma^i{}_{jk} X^k \\
&= -\frac{2}{F} A^{is}{}_j \gamma_{sjk} X^j X^k + 2\gamma^i{}_{jk} X^k \\
&= 2\Big(\gamma^i{}_{jk} X^k - \frac{A^i{}_{sj}}{F}\gamma^s{}_{jk} X^j X^k\Big) = 2N_j{}^i.
\end{aligned} \tag{3.245}$$

Thus from the first partial derivative of $\frac{1}{2}G^i$ with respect to X^j, we obtain the nonlinear connection as

$$\frac{1}{2}\frac{\partial G^i}{\partial X^j} = N_j{}^i. \tag{3.246}$$

Consider the second derivative by using (3.189), symmetry of P, (3.163) and (3.215) as

$$\begin{aligned}
\frac{1}{2}\Big(G^i\Big)_{X^j X^k} &= (N_j{}^i)_{X^k} = \Big(\Gamma^i{}_{jl} X^l\Big)_{X^k} \\
&= \frac{\partial \Gamma^i{}_{jl}}{\partial X^k} X^l + \Gamma^i{}_{jl}\delta^l_k = \Gamma^i{}_{jk} - P_j{}^i{}_{lk}\frac{X^l}{F} \\
&= \Gamma^i{}_{jk} - P_l{}^i{}_{jk} e_m{}^l = \Gamma^i{}_{jk} + \dot{A}^i{}_{jk} \equiv {}^b\Gamma^i{}_{jk}.
\end{aligned} \tag{3.247}$$

Thus, by taking the second partial derivative of $\frac{1}{2}G^i$ with respect to X^k, we get the Berwald connection. The differences between Chern connection and Berwald connection is the $\dot{A}^i{}_{jk}$ term.

By using (3.107), (3.246) and Euler's theorem, consider the vector field

$$\begin{aligned}
X^k\frac{\delta}{\delta u^k} &= X^k\Big(\frac{\partial}{\partial u^k} - N_k{}^i\frac{\partial}{\partial X^i}\Big) = X^k\frac{\partial}{\partial u^k} - \frac{1}{2}X^k\frac{\partial G^i}{\partial X^k}\frac{\partial}{\partial X^k} \\
&= X^k\frac{\partial}{\partial u^k} - \frac{1}{2}2G^i\frac{\partial}{\partial X^i},
\end{aligned} \tag{3.248}$$

where G^i is homogeneous of degree two in X^i. The above vector field is called the **geodesic spray**. We will obtain the curvature tensors ${}^bR$ and ${}^bP$ for the

Berwald connection. Consider the following expression

$$d\theta_j^{\ i} - \theta_j^{\ h} \wedge \theta_h^{\ i} = \frac{1}{2} {}^b R_j^{\ i}{}_{kl} du^k \wedge du^l +^b P_j^{\ i}{}_{kl} du^k \wedge du^l. \tag{3.249}$$

Now, we will calculate $d\theta_j^{\ i}$ and $-\theta_j^{\ h} \wedge \theta_h^{\ i}$ with respect to the Berwald connection. By using (3.116), (3.247), we have

$$\begin{aligned} d\theta_j^{\ i} &= d\Big({}^b\Gamma^i{}_{jl} du^l \Big) = \Big(d(\Gamma^i{}_{jl}) + d(\dot{A}^i{}_{jl}) \Big) \wedge du^l \\ &= \frac{\delta \Gamma^i{}_{jl}}{\delta u^k} du^k \wedge du^l + F \frac{\partial \Gamma^i{}_{jl}}{\partial X^k} \frac{\delta X^k}{F} \wedge du^l \\ &+ \dot{A}^i{}_{jl;k} \frac{\delta X^k}{F} \wedge du^l + \dot{A}^i{}_{jl|k} du^k \wedge \frac{\delta X^l}{F}, \end{aligned} \tag{3.250}$$

and by using (3.116), (3.247), we have

$$\begin{aligned} -\theta_j^{\ h} \wedge \theta_h^{\ i} &= \theta_h^{\ i} \wedge \theta_j^{\ h} = \Big(\Gamma^i{}_{hk} + \dot{A}^i{}_{hk} \Big) du^k \wedge \Big(\Gamma^h{}_{jl} + \dot{A}^h{}_{jl} \Big) du^l \\ &= \Big(\Gamma^i{}_{hk} \Gamma^h{}_{jl} + \dot{A}^i{}_{hk} \dot{A}^h{}_{jl} \Big) du^k \wedge du^l. \end{aligned} \tag{3.251}$$

Writing (3.250) and (3.251) into (3.249), we have

$$\begin{aligned} & \frac{1}{2} \Big[\frac{\delta \Gamma^i{}_{jl}}{\delta u^k} - \frac{\delta \Gamma^i{}_{jk}}{\delta u^l} + \Gamma^i{}_{hk} \Gamma^h{}_{jl} - \Gamma^i{}_{hl} \Gamma^h{}_{jk} \\ + & \dot{A}^i{}_{jl|k} - \dot{A}^i{}_{jk|l} + \dot{A}^i{}_{hk} \dot{A}^h{}_{jl} - \dot{A}^i{}_{hl} \dot{A}^h{}_{jk} \Big] du^k \wedge du^l - \Big(F \frac{\partial \Gamma^i{}_{jk}}{\partial X^l} + \dot{A}^i{}_{jk;l} \Big) du^k \wedge \frac{\delta X^l}{F} \\ = & \frac{1}{2} {}^b R_j^{\ i}{}_{kl} du^k \wedge du^l +^b P_j^{\ i}{}_{kl} du^k \wedge \frac{\delta X^l}{F}. \end{aligned} \tag{3.252}$$

Comparing the coefficients of $du^k \wedge du^l$ and $du^k \wedge \frac{\delta X^l}{F}$ in (3.252) and using (3.162), (3.163), we get the expressions relating the Berwald and the Chern curvatures

$${}^b R_j^{\ i}{}_{kl} = R^i_{j\ kl} + \Big[\dot{A}^i{}_{jl|k} - \dot{A}^i{}_{jk|l} + \dot{A}^i{}_{hk} \dot{A}^h{}_{jl} - \dot{A}^i{}_{hl} \dot{A}^h{}_{jk} \Big], \tag{3.253}$$

and

$${}^b P_j^{\ i}{}_{kl} = P_j^{\ i}{}_{kl} - \dot{A}^i{}_{jk;l}. \tag{3.254}$$

3.6 Flag Curvature and Ricci Curvature

In this section, we will define the flag curvature which is the generalization of the sectional curvature in Riemannian geometry and obtain a formula for Ricci curvature in terms of Flag curvature. We can define the curvature tensor as

$$R(X,Y) : \Gamma(p^*TM) \to \Gamma(p^*TM).$$

As in Riemannian geometry it is given by

$$R(Z,W)X = R_{j\ kl}^{\ i} X^j Z^k W^l \frac{\partial}{\partial u^i}, \tag{3.255}$$

and

$$R(X,Y,Z,W) = R_{jikl} X^j Y^i Z^k W^l, \tag{3.256}$$

or equivalently

$$R(X,Y,Z,W) \equiv G(R(Z,W)X,Y), \tag{3.257}$$

where X, Y, Z, W are local sections of p^*TM over $TM \setminus 0$.

By Theorem 3.4.1, $R(X,Y,Z,W)$ have the following properties:

1) $R(X,Y,Z,W) = -R(X,Y,W,Z)$,

2) $R(X,Y,Z,W) + R(Z,Y,W,X) + R(W,Y,X,Z) = 0$,

3) $R(X,Y,Z,W) + R(Y,X,Z,W) = -2A(X,Y,R(Z,W)e_m) \equiv 2B(XYZW)$,

4) $R(X,Y,Z,W) - R(Z,W,X,Y) = [B(XYZW) - B(ZWXY)]$
$+ [B(XWYZ) + B(YZXW)] + [B(WYZX) + B(ZXWY)]$.

Flag curvature is one of the invariants which is the generalization of the sectional curvature of Riemannian geometry. Flag is an object which is based at $p \in M$ with flagpole $X \in T_u(M)$ and transverse edge $V = V^i \frac{\partial}{\partial u^i}$. Denote the flag as

$$K(X,V) = \frac{R(V,X,X,V)}{G(X,X)G(V,V) - G(X,V)^2}, \tag{3.258}$$

or equivalently

$$K(X,V) = \frac{V^i (X^j R_{jikl} X^l) V^k}{G(X,X)G(V,V) - G(X,V)^2}, \tag{3.259}$$

where $(u,X) \in TM \setminus 0$ and X, V are section of pulled-back bundle π^*TM.
By using corollary of Euler's theorem, we have

$$\begin{aligned} G(e_m, e_m) &= g_{ij} du^i du^j \Big(\frac{X^k}{F} \frac{\partial}{\partial u^k}, \frac{X^l}{F} \frac{\partial}{\partial u^l} \Big) \\ &= (F F_{X^i X^j} + F_{X^i} F_{X^j}) \frac{X^i}{F} \frac{X^j}{F} = 1. \end{aligned} \tag{3.260}$$

Taking $X = e_m$ in (3.259) and using (3.260), right hand side does not change so we have

$$K(e_m, V) = \frac{V^i(e_m^{\ j} R_{jikl} e_m^{\ l})V^k}{G(e_m, e_m)G(V,V) - G(e_m,V)^2} = \frac{V^i(e_m^{\ j} R_{jikl} e_m^{\ l})V^k}{G(V,V) - G(e_m,V)^2}, \tag{3.261}$$

which is called **flag curvature** of the flag $e_m \wedge V$. To obtain the Ricci curvature in terms of K, let us define the following object

$$K(e_m, V, W) = \frac{V^i\big(e_m^{\ j} R_{jikl} e_m^{\ l}\big)W^k}{G(V,W) - G(e_m,V)G(e_m,W)} = \frac{V^i R_{ik} W^k}{G(V,W) - G(e_m,V)G(e_m,W)}, \tag{3.262}$$

which is algebraically a predecessor of the Flag curvature. In (3.262) by taking $V = \frac{\partial}{\partial u^i}$, $W = \frac{\partial}{\partial u^j}$, we get

$$K\left(e_m, \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right) = \frac{R_{ik}}{G\big(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\big) - G\big(e_m, \frac{\partial}{\partial u^i}\big)G\big(e_m, \frac{\partial}{\partial u^j}\big)}. \tag{3.263}$$

Let us obtain the quantities $G\big(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\big)$, $G\big(e_m, \frac{\partial}{\partial u^i}\big)$ and $G\big(e_m, \frac{\partial}{\partial u^j}\big)$ to determine Ricci curvature in a closed form.

$$G\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right) = g_{kl} du^k du^l \left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right) = g_{ij}, \tag{3.264}$$

$$G\left(e_m, \frac{\partial}{\partial u^i}\right) = g_{kl} du^k du^l \left(e_m^{\ j}\frac{\partial}{\partial u^j}, \frac{\partial}{\partial u^i}\right) = g_{ij} e_m^{\ j} = e_i^{\ m}, \tag{3.265}$$

$$G\left(e_m, \frac{\partial}{\partial u^i}\right) = g_{kl} du^k du^l \left(e_m^{\ i}\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right) = g_{ij} e_m^{\ i} = e_j^{\ m}. \tag{3.266}$$

By inserting (3.264), (3.265) and (3.266) into (3.262) we get

$$K\left(e_m, \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right) = \frac{R_{ik}}{g_{ij} - e_i^{\ m} e_j^{\ m}}. \tag{3.267}$$

Thus we obtain the Ricci curvature in terms of K as

$$R_{ij} = \big(g_{ij} - e_i^{\ m} e_j^{\ m}\big) K\left(e_m, \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right). \tag{3.268}$$

3.7 Schur's Lemma

By rotating the transverse edge V around the flagpole X, if the flag curvature does not change then we call it a **scalar flag curvature**. Denote that scalar as $\lambda = \lambda(u, X)$. For ease of calculation, we use the following notations

$$h_{ij} \equiv g_{ij} - e^m_{\ i} e^m_{\ j}, \tag{3.269}$$

$$h_{ijk} \equiv g_{ij} e^m_{\ k} - g_{ik} e^m_{\ j}. \tag{3.270}$$

We obtain some useful relations.

$$\lambda_{;s} e_m^{\ s} = F \frac{\partial \lambda}{\partial X^s} \frac{X^s}{F} = X^s \frac{\partial \lambda}{\partial X^s} = 0, \tag{3.271}$$

since λ is homogeneous of degree zero in X^i.

$$\begin{aligned} \lambda_{;r;s} e_m^{\ s} &= F \frac{\partial \lambda_{;r}}{\partial X^s} \frac{X^s}{F} = X^s \frac{\partial F}{\partial X^s} \frac{\partial \lambda}{\partial X^r} + F X^s \frac{\partial (\frac{\partial \lambda}{\partial X^r})}{\partial X^s} \\ &= F \frac{\partial \lambda}{\partial X^r} - F \frac{\partial \lambda}{\partial X^r} = 0, \end{aligned} \tag{3.272}$$

since F and $\frac{\partial \lambda}{\partial X^r}$ are homogeneous of degree 1 and -1, respectively.

$$h_{is} e_m^{\ s} = (g_{is} - e^m_{\ i} e^m_{\ s}) e_m^{\ s} = e^m_{\ i} - e^m_{\ i} = 0, \tag{3.273}$$

$$h_{ijs} e_m^{\ s} = (g_{ij} e^m_{\ s} - g_{is} e^m_{\ j}) e_m^{\ s} = g_{ij} - e^m_{\ i} e^m_{\ j} = h_{ij}, \tag{3.274}$$

$$g^{ij} h_{ik} = g^{ij} (g_{ik} - e^m_{\ i} e^m_{\ k}) = \delta^j_k - e_m^{\ j} e^m_{\ k}, \tag{3.275}$$

$$g^{ij} h_{ij} = g^{ij} (g_{ij} - e^m_{\ i} e^m_{\ j}) = n - 1, \tag{3.276}$$

$$g^{ij} h_{ijk} = g^{ij} (g_{ij} e^m_{\ k} - g_{ik} e^m_{\ j}) = n e^m_{\ k} - \delta^j_k e^m_{\ j} = (n-1) e^m_{\ k}, \tag{3.277}$$

$$h_{ij|k} = (g_{ij} - e^m_{\ i} e^m_{\ j})_{|k} = g_{ij|k} - e^m_{\ i|k} e^m_{\ j} - e^m_{\ i} e^m_{\ j|k} = 0, \tag{3.278}$$

since vertical derivative of g_{ij} and $e^m_{\ i}$ are zero.

$$h_{ijk|l} = g_{ij|l} e^m_{\ k} - g_{ij} e^m_{\ k|l} - g_{ik|l} e^m_{\ j} - g_{ik} e^m_{\ j|l} = 0. \tag{3.279}$$

We will now state a proposition which gives some relations with first curvature tensor R and flag curvature scalar λ.

Proposition 1 . *For the Finsler manifold M, the followings are equivalent:*

1) $R_{ik} = \lambda(u, X)h_{ik}$,

2) $R_{ikl} = \lambda(u, X)h_{ikl} + \frac{1}{3}(h_{ik}\lambda_{;l} - h_{il}\lambda_{;k})$,

3) $R_{ijkl} = \lambda(u, X)(g_{ik}g_{jl} - g_{il}g_{jk}) - \Big(\dot{A}_{ijl|k} - \dot{A}_{ijk|l} + \dot{A}_{isk}\dot{A}^{s}{}_{jl} - \dot{A}_{isl}\dot{A}^{s}{}_{jk}\Big) + \lambda_{;j}h_{ikl} + \frac{1}{3}\Big[\lambda_{;k}(e^{m}_{i}h_{jl} + h_{ijl}) - \lambda_{;l}(e^{m}_{i}h_{jk} + h_{ijk})\Big] + \frac{1}{3}(h_{ik}\lambda_{;l;j} - h_{il}\lambda_{;k;j})$.

Proof: 1) $\Rightarrow$ 2) By using (3.242) with lowering index i, vertical derivative of (3.269) with respect to X^l, (3.270) and after some cancellation, we have

$$\begin{aligned}
R_{ikl} &= \frac{1}{3}\Big(R_{ik;l} - R_{il;k}\Big) + \frac{2}{3}\Big(R_{ik}e^{m}_{l} - R_{il}e^{m}_{k}\Big) \\
&= \frac{1}{3}\Big(\lambda_{;l}h_{ik} + \lambda h_{ik;l} - \lambda_{;k}h_{il} - \lambda h_{il;k}\Big) + \frac{2}{3}\Big(\lambda h_{ik}e^{m}_{l} - \lambda h_{il}e^{m}_{k}\Big) \\
&= \frac{1}{3}\Big(\lambda_{;l}h_{ik} - \lambda_{;k}h_{il}\Big) + \frac{\lambda}{3}\Big(g_{ik}e^{m}_{l} - g_{il}e^{m}_{k}\Big) + \frac{2\lambda}{3}\Big(g_{ik}e^{m}_{l} - g_{il}e^{m}_{k}\Big) \\
&= \lambda h_{ikl} + \frac{1}{3}\Big(\lambda_{;l}h_{ik} - \lambda_{;k}h_{il}\Big).
\end{aligned}$$

2) $\Rightarrow$ 1) Contracting 3) with $e_{m}^{\ l}$ and using (3.271), (3.273), (3.274), we have

$$\begin{aligned}
&R_{ikl}e_{m}^{\ \ l} = \lambda h_{ikl}e^{\ l}_{m} + \frac{1}{3}\Big(h_{ik}\lambda_{;l}e^{\ l}_{m} - h_{;l}e^{\ l}_{m}\lambda_{;k}\Big) \\
&R_{ik} = \lambda h_{ik}.
\end{aligned}$$

2) $\Rightarrow$ 3) Consider (3.243) with lowering index i as

$$\begin{aligned}
&R_{jikl} = \frac{1}{3}\Big(R_{ik;l;j} - R_{il;k;j} + e^{m}_{j}R_{ik;l} - e^{m}_{j}R_{il;k}\Big) \\
+ \ & \frac{2}{3}\Big(R_{ik;j}e^{m}_{l} - R_{il;j}e^{m}_{k} + R_{ik}g_{jl} - R_{il}g_{jk}\Big) \\
- \ & \Big(\dot{A}_{ijl|k} - \dot{A}_{iuk} + \dot{A}_{iuk}\dot{A}^{u}{}_{jl} - \dot{A}_{iul}\dot{A}^{u}{}_{jk}\Big).
\end{aligned}$$

By using (3.269) and 1), we can get an expression for the $R_{ik;l}$. By using that and similar terms in the above equation and after some simplification we have

$$\begin{aligned}
R_{jikl} &= \lambda(g_{ik}g_{jl} - g_{il}g_{jk}) + \Big(\dot{A}_{ijl|k} - \dot{A}_{iuk} + \dot{A}_{iuk}\dot{A}^{u}{}_{jl} - \dot{A}_{iul}\dot{A}^{u}{}_{jk}\Big) \\
&+ \frac{\lambda_{;j}}{3}(g_{ik}e^{m}_{l} - g_{il}e^{m}_{k}) - \frac{\lambda_{;l}}{3}(g_{ij}e^{m}_{k} - g_{ik}e^{m}_{j}) - \frac{\lambda_{;l}}{3}e^{m}_{i}(g_{kj} - e^{m}_{k}e^{m}_{j}) \\
&+ \frac{\lambda_{;k}}{3}(g_{ij}e^{m}_{l} - g_{il}e^{m}_{j}) + \frac{\lambda_{;k}}{3}e^{m}_{i}(g_{lj} - e^{m}_{l}e^{m}_{j}) \\
&+ \frac{1}{3}(h_{ik}\lambda_{;l;j} - h_{il}\lambda_{;k;j})
\end{aligned}$$

$$\begin{aligned}
&= \lambda(g_{ik}g_{jl} - g_{il}g_{jk}) - \Big(\dot{A}_{ijl|k} - \dot{A}_{ijk|l} + \dot{A}_{isk}\dot{A}^{s}{}_{jl} - \dot{A}_{isl}\dot{A}^{s}{}_{jk}\Big) \\
&+ \lambda_{;j}h_{ikl} + \frac{1}{3}\Big[\lambda_{;k}(e^{m}_{\ i}h_{jl} + h_{ijl}) - \lambda_{;l}(e^{m}_{\ i}h_{jk} + h_{ijk})\Big] + \frac{1}{3}(h_{ik}\lambda_{;l;j} - h_{il}\lambda_{;k;j}).
\end{aligned}$$

3) $\Rightarrow$ 2) Contracting 3) with $e_{m}^{\ j}$ and using (3.191), (3.193), (3.273) and (3.272), we have

$$\begin{aligned}
R_{ijkl} &= \lambda(g_{ik}e_{m}^{\ j}g_{jl} - g_{il}e_{m}^{\ j}g_{jk}) - e_{m}^{\ j}\Big(\dot{A}_{ijl|k} - \dot{A}_{ijk|l} + \dot{A}_{isk}\dot{A}^{s}{}_{jl} - \dot{A}_{isl}\dot{A}^{s}{}_{jk}\Big) \\
&+ e_{m}^{\ j}\lambda_{;j}h_{ikl} + \frac{1}{3}\Big[\lambda_{;k}(e^{m}_{\ i}e_{m}^{\ j}h_{jl} + h_{ijl}) - \lambda_{;l}(e^{m}_{\ i}e_{m}^{\ j}h_{jk} + h_{ijk})\Big] \\
&+ \frac{1}{3}(h_{ik}e_{m}^{\ j}\lambda_{;l;j} - h_{il}e_{m}^{\ j}\lambda_{;k;j}) \\
&= \lambda(g_{ik}e^{m}_{\ l} - g_{il}e^{m}_{\ k}) + \frac{1}{3}\Big(h_{ik}\lambda_{;l} - h_{il}\lambda_{;k}\Big) \\
&= \lambda h_{ikl} + \frac{1}{3}\Big(h_{ik}\lambda_{;l} - h_{il}\lambda_{;k}\Big).
\end{aligned}$$

$\boxdot$

By lowering the indices i of (3.231) and contracting it with $e_{m}^{\ j}$, $e_{m}^{\ l}$ and using (3.213), (3.215) we have

$$R_{\ ikl|t}e_{m}^{\ t} + R_{\ il|k} - R_{\ ik|l} = -\dot{A}^{s}{}_{ik}R_{sl} + \dot{A}^{s}{}_{il}R_{sk}. \tag{3.280}$$

By using 1) and 2) of Proposition, (3.278), (3.279) in the above equation, we get

$$\begin{aligned}
&\lambda_{|t}e_{m}^{\ t}h_{ikl} + \frac{1}{3}(h_{ik}\lambda_{;l|t}e_{m}^{\ t} - h_{il}\lambda_{;k|t}e_{m}^{\ t}) + \lambda_{|k}h_{il} - \lambda_{|l}h_{ik} \\
= \ &-\dot{A}^{s}{}_{ik}\lambda h_{sl} + \dot{A}^{s}{}_{il}\lambda h_{sk}.
\end{aligned} \tag{3.281}$$

Contracting the above equation with g^{ik} and using (3.272), (3.275), (3.277) and denoting $\lambda_{|t}e_{m}^{\ t} \equiv \dot{\lambda}$, we have

$$\begin{aligned}
&\dot{\lambda}h_{ikl}g^{ik} + \frac{1}{3}(g^{ik}h_{ik}\lambda_{;l|t}e_{m}^{\ t} - g^{ik}h_{il}\lambda_{;k|t}e_{m}^{\ t}) + \lambda_{|k}g^{ik}h_{il} + \\
- \ &\lambda_{|l}g^{ik}h_{ik} = -\dot{A}^{s}{}_{ik}g^{ik}\lambda h_{sl} + \dot{A}^{s}{}_{il}g^{ik}\lambda h_{sk} \\
&\dot{\lambda}(m-1)e^{m}_{\ l} + \frac{1}{3}\Big((m-1)\lambda_{;l|t}e_{m}^{\ t} - (\delta^{k}_{l} - e_{m}^{\ k}e^{m}_{\ l})\lambda_{;k|t}e_{m}^{\ t}\Big) + \\
+ \ &\lambda_{|k}(\delta^{k}_{l} - e_{m}^{\ k}e^{m}_{\ l}) - (m-1)\lambda_{|l} = 0 \\
&(m-2)\dot{\lambda}e^{m}_{\ l} - (m-2)\lambda_{|l} + \frac{(m-2)}{3}\lambda_{;l|t}e_{m}^{\ t} = 0.
\end{aligned} \tag{3.282}$$

We will now state the **Schur's lemma** when the scalar of the scalar flag curvature depends on only u .i.e. $\lambda(u)$.

Lemma 3 (Schur's lemma) . *Suppose M is Finsler manifold with dimension $m \geq 3$ and scalar of the scalar flag curvature deponds on only u .i.e. $\lambda = \lambda(u)$ then λ is constant.*

Proof: We have

$$\lambda_{;i} = F\frac{\partial \lambda}{\partial X^i} = 0, \tag{3.283}$$

and

$$\lambda_{|i} = \frac{\delta \lambda}{\delta u^i} = \frac{\partial \lambda}{\partial u^i} - N_i{}^j\frac{\partial \lambda}{\partial X^j} = \frac{\partial \lambda}{\partial u^i}, \tag{3.284}$$

since λ does not depend on X. From there (3.282) takes the form

$$\lambda_{|i} = \dot{\lambda} e^m_{\ i}. \tag{3.285}$$

By taking vertical derivative of the above equation with respect to j, we get

$$\lambda_{|i;j} = \dot{\lambda}_{;j} e^m_{\ i} + \dot{\lambda} h_{ij}. \tag{3.286}$$

λ is a function of only u and by (3.284) $\lambda_{|i}$ is also a function of u only, so $\lambda_{|i;j} = F\frac{\partial \lambda|i}{\partial X^j} = 0$. (3.286) takes the form

$$\dot{\lambda}_{;j} e^m_{\ i} + \dot{\lambda} h_{ij} = 0. \tag{3.287}$$

Dimension of M is greater than 1 so there is at least one nonzero arbitrary basis section U orthogonal to $e^m_{\ i}$. Contracting (3.287) with U^i, we have

$$\dot{\lambda}_{;j} e^m_{\ i}(U^i) + \dot{\lambda} h_{ij}(U^i) = \dot{\lambda}\Big(g_{ij} - e^m_{\ i} e^m_{\ j}\Big)(U^i) = \dot{\lambda} g_{ij} U^i = \dot{\lambda} U_j = 0, \tag{3.288}$$

where U^i is orthogonal to $e^m_{\ i}$. From (3.288), we have $\dot{\lambda} U = 0$ and also $\dot{\lambda} = 0$. By using (3.285) and (3.284), we have $\lambda_{|i} = \frac{\partial \lambda}{\partial u^i} = 0$. From there, we can say λ is constant on the connected manifold M.

$\boxdot$

3.8 Special Finsler Spaces

A) Riemannian Spaces

C^∞ manifold M with metric $G = g_{ij}(u)du^i du^j$ is called Riemannian manifold. That G defines a Finsler function F on $TM \setminus 0$ by $F(u, X) = \sqrt{g_{ij}(u)X^i X^j}$. From there it is clear that, every Riemannian manifold is Finslerian. Finsler manifold is Riemannian if the components of Finsler metric g_{ij} is independent of X. In Riemannian manifold, Cartan tensor and second curvature tensor of Chern connection vanish in natural coordinates. Chern connection coefficients ${\Gamma^i}_{jk}$ take the form of ${\gamma^i}_{jk}$ which is the second kind Christoffel symbols in natural coordinates. First curvature tensor of the Chern connection ${R_j}^i{}_{kl}$ is of the form

$$ {R_j}^i{}_{kl} = \frac{\partial {\gamma^i}_{jl}}{\partial u^k} - \frac{\partial {\gamma^i}_{jk}}{\partial u^l} + {\gamma^i}_{hk}{\gamma^h}_{jl} - {\gamma^i}_{hl}{\gamma^h}_{jk}. $$

B) Berwald Spaces

A Finsler manifold M is a **Berwald space** if the Chern connection coefficients ${\Gamma^i}_{jk}$ in natural coordinates are independent of X. As a consequence of this condition, we can say that the second curvature tensor of Chern connection P vanishes. In Berwald space, first curvature tensor ${R_j}^i{}_{kl}$ in natural coordinates take the form

$$ {R_j}^i{}_{kl} = \frac{\partial {\Gamma^i}_{jl}}{\partial u^k} - \frac{\partial {\Gamma^i}_{jk}}{\partial u^l} + {\Gamma^i}_{hk}{\Gamma^h}_{jl} - {\Gamma^i}_{hl}{\Gamma^h}_{jk}. $$

C) Randers space

Suppose M is an $m-$dimensional manifold. A **Randers metric** is a Finsler function on $TM \setminus 0$ of the form

$$ F(u, X) \equiv \alpha(u, X) + \beta(u, X), \tag{3.289} $$

where $\alpha(u, X) \equiv \sqrt{\tilde{a}_{ij}(u)X^i X^j}$ and $\beta(u, X) \equiv \tilde{b}_i(u)X^i$.
Here $\tilde{a}_{ij}$ are components of a Riemannian metric on M with inverse $\tilde{a}^{ij}$ and also we lower and raise the indices by $\tilde{a}_{ij}$. $\tilde{b}_i$ are one-forms on M. $m-$dimensional manifold M with Finsler function of the form (3.289) is called **Randers space**.

D) Locally Minkowskian Spaces

A Finsler manifold M is **locally Minkowskian** if the first and second curvature tensors of the Chern connection $R_j{}^i{}_{kl}$, $P_j{}^i{}_{kl}$ in natural coordinates vanish. We will state a lemma without proof. One can find the proof in [17].

Lemma 4 . *Suppose M is a finite dimensional manifold with torsion-free connection D. Let p be any point in M. If the curvature of D vanishes in a neighbourhood of p, then there exist a local coordinate system u^i such that all the connection coefficients $\Gamma^i{}_{jk}$ are zero.*

That lemma helps us to prove the lemma given below which puts the restriction on the Finsler function F to make the manifold locally Minkovskian.

Lemma 5 . *A Finsler manifold M is locally Minkowskian if and only if, for every point p of M and local coordinates u^i around p, there exist local coordinates (u^i, X^i) on TM which the Finsler function F is independent of u^i.*

Proof: ($\Leftarrow$) Suppose there exist local coordinate system (u^i, X^i) and $F_{u^i} = 0$. $g_{ij} = \left(FF_{X^iX^j} + F_{X^i}F_{X^j}\right)$, so $\frac{\partial g_{ij}}{\partial u^k} = 0$. Consequence of that $\gamma^i{}_{jk} = \frac{g^{jl}}{s}\left(\frac{\partial g_{jl}}{\partial u^k} - \frac{\partial g_{kj}}{\partial u^k} + \frac{\partial g_{lk}}{\partial u^j}\right) = 0$. By (3.136) $N_j{}^i$ is directly depends on $\gamma^i{}_{jk}$ so $N_j{}^i = 0$. By using that and $\frac{\partial g_{ij}}{\partial u^k} = 0$, $\Gamma^i{}_{jk}$ in (3.135) vanishes. By using that fact $R_j{}^i{}_{kl}$ and $P_j{}^i{}_{kl}$ vanish since they have the form $R_j{}^i{}_{kl} = \frac{\delta\Gamma^i{}_{jl}}{\delta X^k} - \frac{\delta\Gamma^i{}_{jk}}{\delta X^l} + \Gamma^i{}_{hk}\Gamma^h{}_{jl} - \Gamma^i{}_{hl}\Gamma^h{}_{jk}$ and $P_j{}^i{}_{kl} = -F\frac{\partial\Gamma^i{}_{jk}}{\partial X^l}$. Thus $R = P = 0$ which is definition of locally Minkovskian.
($\Rightarrow$) Suppose M is locally Minkovskian such that first and second Chern curvature tensors vanish. From the vanishing of second curvature tensor we have

$$\frac{\partial\Gamma^i{}_{jk}}{\partial X^l} = 0, \tag{3.290}$$

By using that fact and vanishing of first curvature tensor we have

$$R_j{}^i{}_{kl} = \frac{\partial\Gamma^i{}_{jl}}{\partial u^k} - \frac{\partial\Gamma^i{}_{jk}}{\partial u^l} + \Gamma^i{}_{hk}\Gamma^h{}_{jl} - \Gamma^i{}_{hl}\Gamma^h{}_{jk} = 0, \tag{3.291}$$

By (3.290) and (3.291) we have a torsion-free connection and curvature of this connection vanishes. By applying Lemma 4, we have a local coordinate system

u^i which all the connection coefficients ${\Gamma^i}_{jk}$ vanishes. By using that fact and (3.137) we have ${N_j}^i = 0$. By using that ${N_j}^i = 0$, ${\Gamma^i}_{jk} = 0$ and (3.127), we get $\frac{\partial g_{ij}}{\partial u^k} = 0$ which means g_{ij} is independent of u^i. Consider $g_{ij} = (FF_{X^iX^j} + F_{X^i}F_{X^j})$. Contracting that with X^iX^j and using Euler's theorem we get $X^iX^jg_{ij} = F^2$. Thus F is independent of u^i since g_{ij} is independent of u^i.

$\boxdot$

3.9 Summary

F is positively homogeneous with degree one in X^i: $X^i \frac{\partial F}{\partial X^i} = F$,
and have identities

1) $X^j F_{X^iX^j} = 0$,

2) $X^k F_{X^iX^jX^k} = -F_{X^iX^j}$,

3) $X^l F_{X^iX^jX^kX^k} = -2F_{X^iX^jX^k}$,

4) $X^i F_{X^iu^i} = F_{u^i}$.

Finsler metric: $G = g_{ij}(u, X)du^idu^j = \left(\frac{1}{2}F^2\right)_{X^iX^j} du^idu^j$,
Components of Finsler metric: $g_{ij} = FF_{X^iX^j} + F_{X^i}F_{X^j}$,
with $g_{ij}X^iX^j = F^2$
Orthonormal frames: $e_\alpha = {e_\alpha}^i \frac{\partial}{\partial u^i}$ and its dual $w^\alpha = e^\alpha_{\ i} du^i$,
$(e_\alpha, e_\beta) = {e_\alpha}^k g_{ki} {e_\beta}^i \equiv \delta_{\alpha\beta}$,
Global sections: $e_m = \frac{X^i}{F}\frac{\partial}{\partial u^i} = {e_m}^i \frac{\partial}{\partial u^i}$, and $w = w^m = F_{X^i}du^i = e^m_{\ i}du^i$,
Chern connection: $De_\alpha = {w_\alpha}^\beta e_\beta$,
Torsion-free condition: $dw^\alpha = w^\beta \wedge {w_\beta}^\alpha$,

1) $dw^m = w^{\bar\alpha} \wedge {w_{\bar\alpha}}^m$,

2) $dw^{\bar\alpha} = w^{\bar\beta} \wedge {w_{\bar\beta}}^{\bar\alpha} + w^m \wedge {w_m}^{\bar\alpha}$,

where

$w_{\bar\alpha}{}^m = -e_{\bar\alpha}{}^i F_{X^iX^j} dX^j + \frac{e_{\bar\alpha}{}^i}{F}(F_{u^i} - X^j F_{X^iu^j})w^m + e_{\bar\alpha}{}^i e_{\bar\beta}{}^j F_{u^iX^j} w^{\bar\beta} + \lambda_{\bar\alpha\bar\beta} w^{\bar\beta}$,

$\lambda_{\bar\rho\bar\sigma} = -\frac{1}{2} e_{\bar\rho}{}^i e_{\bar\sigma}{}^j \Big(G_{ijm} + F_{X^ju^i} + F_{X^iu^j} \Big)$,

$G_{ijm} = g^{ks} F F_{X^kX^iX^j} \Big(F_{u^s} - X^r F_{X^su^r} \Big) + X^s F_{u^sX^iX^j} + \frac{X^s}{F} F_{u^s} F_{X^iX^j}$,

$w_{\bar\beta}{}^{\bar\alpha} = e^{\bar\alpha}{}_k de_{\bar\beta}{}^k + \xi_{\bar\beta}{}^{\bar\alpha} w^m + \mu_{\bar\beta\bar\gamma}{}^{\bar\alpha} w^{\bar\gamma}$,

$w_m{}^{\bar\alpha} = \frac{1}{F} e^{\bar\alpha}{}_k dX^k + \xi_\nu{}^{\bar\alpha} w^\nu$,

$\xi_m{}^{\bar\alpha} = -\frac{\delta^{\bar\alpha\bar\sigma} e_{\bar\sigma}{}^i}{F} \Big(F_{u^i} - X^j F_{X^iu^j} \Big)$,

$\xi_{\bar\beta}{}^{\bar\alpha} = -\delta^{\bar\alpha\bar\sigma} \Big(e_{\bar\sigma}{}^i e_{\bar\beta}{}^j F_{X^ju^i} + \lambda_{\bar\sigma\bar\beta} \Big)$,

$\mu_{\bar\rho\bar\sigma}{}^{\bar\alpha} = \frac{1}{2} \delta^{\bar\alpha\bar\beta} \Big(e_{\bar\beta}{}^i e_{\bar\rho}{}^j G_{ij\bar\sigma} - e_{\bar\rho}{}^i e_{\bar\sigma}{}^j G_{ij\bar\beta} + e_{\bar\sigma}{}^i e_{\bar\beta}{}^j G_{ij\bar\rho} \Big)$,

$G_{ij\bar\beta} = e_{\bar\beta}{}^s \Big\{ \frac{1}{2} \Big(F_{u^rX^s} - F_{X^ru^s} - G_{rsm} \Big) \Big(F^2 g^{rk} F_{X^iX^jX^k} + X^r F_{X^iX^j} \Big) + F_{u^s} F_{X^iX^j} +$
$F F_{u^sX^iX^j} \Big\}$,

Components of Cartan tensor in natural basis: $A_{ijk} = \frac{F}{2} \frac{\partial g_{ij}}{\partial X^k}$,

with $X^i A_{ijk} = X^j A_{ijk} = X^k A_{ijk} = 0$,

Components of Cartan tensor in frame w^α: $A_{\rho\sigma\alpha} = \frac{F}{2} \frac{\partial g_{ij}}{\partial X^k} e_\sigma{}^j e_\rho{}^i e_\alpha{}^k$,

Almost metric-compatibility condition: $w_{\gamma\beta} + w_{\beta\gamma} = -2 A_{\gamma\beta\alpha} w_m{}^\alpha$,

$\mathcal{G} \equiv \frac{1}{2} F^2$,

$\mathcal{G}_l \equiv \frac{1}{2} \Big(X^s \frac{\partial^2 \mathcal{G}}{\partial u^s \partial X^l} - \frac{\partial \mathcal{G}}{\partial u^l} \Big) = \frac{1}{2} \Big(X^s F_{u^s} F_{X^l} + X^s F F_{X^lu^s} - F F_{u^l} \Big)$,

$\mathcal{G}^i \equiv g^{il} \mathcal{G}_l$,

Basis of $T^*(PTM)$: $w_m{}^{\bar\alpha} = e^{\bar\alpha}{}_j \delta X^j$ and $w^\alpha = e^\alpha{}_i du^i$, where

$\delta X^j \equiv dX^j F + N_k{}^j du^k$ and $N_j{}^i \equiv \frac{1}{F} \frac{\partial \mathcal{G}^i}{\partial X^j}$,

Basis of $T(PTM)$: $\hat e_\alpha = e_\alpha{}^i \frac{\delta}{\delta u^i}$ and $\hat e_{m+\bar\alpha} = e_{\bar\alpha}{}^j \frac{\delta}{\delta X^j}$, where

$\frac{\delta}{\delta u^i} \equiv \frac{\partial}{\partial u^i} - F N_i{}^j \frac{\partial}{\partial X^j}$ and $\frac{\delta}{\delta X^i} \equiv F \frac{\partial}{\partial X^i}$,

Chern connection in natural basis: $D \frac{\partial}{\partial u^i} = \theta_i{}^j \frac{\partial}{\partial u^j}$, $D du^i = -\theta_j{}^i du^j$,

or $D \frac{\partial}{\partial u^i} = \Gamma^j{}_{il} du^l \otimes \frac{\partial}{\partial u^j}$, where $\theta_i{}^j = \Gamma^j{}_{il} du^l$,

Connection coefficients:

$\Gamma_{ijk} = \gamma_{ijk} - \frac{F}{2} \Big(\frac{\partial g_{ij}}{\partial X^l} N_k{}^l + \frac{\partial g_{ki}}{\partial X^l} N_j{}^l - \frac{\partial g_{jk}}{\partial X^l} N_i{}^l \Big)$,

$\Gamma^r{}_{jk} = \gamma^r{}_{jk} - \frac{F}{2} g^{ir} \Big(\frac{\partial g_{ij}}{\partial X^l} N_k{}^l + \frac{\partial g_{ki}}{\partial X^l} N_j{}^l - \frac{\partial g_{jk}}{\partial X^l} N_i{}^l \Big)$,

$\gamma_{ijk} \equiv \frac{1}{2} \Big(\frac{\partial g_{ij}}{\partial u^k} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{jk}}{\partial u^i} \Big)$,

$\gamma^l{}_{jk} \equiv \frac{g^{il}}{2} \Big(\frac{\partial g_{ij}}{\partial u^k} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{jk}}{\partial u^i} \Big)$,

$N_j{}^i$ in terms of Cartan tensor and Christoffel symbols:

$N_j{}^i = \gamma^i{}_{jk}\frac{X^k}{F} - A^i{}_{jk}\gamma^k{}_{rs}\frac{X^rX^s}{F^2}$,
$\Gamma^i{}_{jk}\frac{X^j}{F} = N_k{}^i$,
Curvature two-form in natural basis: $\Omega_j{}^i = d\theta_j{}^i - \theta_j{}^k \wedge \theta_k{}^i$, and
$\Omega_j{}^i = \frac{1}{2}R_j{}^i{}_{kl}du^k \wedge du^l + P_j{}^i{}_{kl}du^k \wedge \frac{\delta X^k}{F}$, where
$R_j{}^i{}_{kl} = \frac{\delta\Gamma^i{}_{jl}}{\delta X^k} - \frac{\delta\Gamma^i{}_{jk}}{\delta X^l} + \Gamma^i{}_{hk}\Gamma^h{}_{jl} - \Gamma^i{}_{hl}\Gamma^h{}_{jk}$,
$P_j{}^i{}_{kl} = -F\frac{\partial\Gamma^i{}_{jk}}{\partial X^l}$,
$R^{\ i}{}_k \equiv e_m^{\ j}R_j{}^i{}_{kl}e_m^{\ l}$,
$R_{ik} \equiv e_m^{\ j}R_{jikl}e_m^{\ l}$, $R^{\ i}{}_{kl} \equiv e_m^{\ j}R_j{}^i{}_{kl}$,
$P^{\ i}{}_{kl} \equiv e_m^{\ j}P_j{}^i{}_{kl}$,
Curvature matrix in basis w^α and $w_m^{\ \bar{\alpha}}$:
$\Omega_\beta^{\ \alpha} = \frac{1}{2}R_\beta{}^\alpha{}_{\rho\sigma}w^\rho \wedge w^\sigma + P_\beta{}^\alpha{}_{\rho\bar{\sigma}}w^\rho \wedge w_m^{\ \bar{\sigma}}$,
Relations of $\Omega_i^{\ j}$ and $\Omega_\alpha^{\ \beta}$:
$\Omega_\beta^{\ \alpha} = e_\beta^{\ j}\Omega_j{}^i e^\alpha{}_i$,
$R_\beta{}^\alpha{}_{\rho\sigma} = e_\beta^{\ j}e_\rho^{\ k}e_\sigma^{\ l}R_j{}^i{}_{kl}e^\alpha{}_i$,
$P_\beta{}^\alpha{}_{\rho\bar{\sigma}} = e_\beta^{\ j}e_\rho^{\ k}e_{\bar{\sigma}}^{\ l}P_j{}^i{}_{kl}e_i{}^\alpha$,
Propeties of first and second curvature tensors R, P and their horizontal and vertical derivatives, respectively:

1) $R_{jikl} = -R_{jilk}$,

2) $R_{jikl} + R_{kilj} + R_{lijk} = 0$,

3) $R_{jikl} + R_{ijkl} = 2(-A_{iju}R^u{}_{kl}) \equiv 2B_{ijkl}$,

4) $R_{ijkl} - R_{klij} = \Big(B_{ijkl} - B_{klij}\Big) + \Big(B_{iljk} + B_{jkil}\Big) + \Big(B_{ljki} + B_{kilj}\Big)$,

5) $P_{kijl} = P_{jikl}$,

6) $P_{ijkl} + P_{jikl} = -2A_{iju}P^{\ u}{}_{kl} - 2A_{ijl|k} \equiv 2E_{ijkl} - 2A_{ijl|k}$,

7) $R_j{}^i{}_{kl|t} + R_j{}^i{}_{lt|k} + R_j{}^i{}_{tk|l} = -\dot{A}^i{}_{ku}R^{\ u}{}_{lt} - \dot{A}^i{}_{lu}R^{\ u}{}_{tk} - \dot{A}^i{}_{tu}R^{\ u}{}_{kl}$,

8) $P_j{}^i{}_{kl} - P_l{}^i{}_{kj} = \dot{A}^i{}_{kj;l} - \dot{A}^i{}_{kl;j}$,

9) $R_j{}^i{}_{kl} = \frac{1}{3}\Big(R^{\ i}{}_{k;l;j} - R^{\ i}{}_{l;k;j} + e^m_{\ j}R^{\ i}{}_{k;l} - e^m_{\ j}R^{\ i}{}_{l;k}\Big) +$
$\frac{2}{3}\Big(R^{\ i}{}_{k;j}e^m_{\ l} - R^{\ i}{}_{l;j}e^m_{\ k} + R^{\ i}{}_k g_{jl} - R^{\ i}{}_l g_{jk}\Big) - \Big(\dot{A}^i{}_{jl|k} - \dot{A}^i{}_{uk} + \dot{A}^i{}_{uk}\dot{A}^u{}_{jl} - \dot{A}^i{}_{ul}\dot{A}^u{}_{jk}\Big)$

Geodesic sparay: $X^k \frac{\delta}{\delta u^k} = X^k \frac{\partial}{\partial u^k} - \frac{1}{2} 2G^i \frac{\partial}{\partial X^k}$,
Flag curvature: $K(e_m, V) = \frac{V^i \left(e_m{}^j R_{jikl} e_m{}^l \right) V^k}{G(V,V) - G(e_m,V)^2}$.

Chapter 4

The Geometry of Tangent Bundle

In this chapter we study the geometry of the tangent bundle [18]. We first define a metric on that tangent bundle and an almost complex structure, then for the ease of calculations we define special frame which makes the metric and almost complex structure simpler. By using this special frame we define an anholonomic basis. We calculate Levi-Civita connection and curvature tensor on tangent bundle. Finally we obtain some relations among the connection coefficients and components of Riemannian curvature tensor.

4.1 Defining Metric on T(M)

Suppose M is an m-dimensional C^∞ manifold with local coordinates u^i, $1 \leq i \leq m$. $T(M)$ is $2m$-dimensional tangent bundle of M with local coordinates $Y^I = \left(y^i, y^{i^*}\right)$, $1 \leq i \leq m$, $m+1 \leq i^* \leq 2m$. Here y^i and y^{i^*} are respectively the base manifold and fiber coordinates. It is convenient to write

$$Y^I = \left(y^i, y^{i^*}\right) \equiv \left(u^i, X^i\right) \tag{4.1}$$

for the consistency of our notations. Except when otherwise stated; upper case Latin indices and Greek indices denote the natural basis and anholonomic basis, respectively. First eight letters of Latin alphabet and the remainings are used

to expand the Greek indices and upper case Latin indices, respectively.i.e. $\alpha = (a, a^*)$, $1 \le a \le m$, $m+1 \le a^* \le 2m$ and $I = (i, i^*)$, $1 \le i \le m$, $m+1 \le i^* \le 2m$. The indices without star and the indices with star denote the components of base manifold (except the indices of X) and the components of fiber, respectively.
$F(u^i, X^i)$ is a function which is defined on $T(M)$ and satisfies the properties of Finsler function. By using F, the components of the metric tensor has the form

$$g_{ij}(u, X) = \left(\frac{1}{2}F^2\right)_{X^i X^j}, \tag{4.2}$$

which is homogeneous in X with degree 0. As in previous chapter let w be a globally defined one-form or Hilbert form as

$$w = e^m_{\ i} du^i. \tag{4.3}$$

(Here m is not an index. It is the last values of every index) By using $e_m^{\ j} g_{ij} = e^m_{\ i}$ as in previous chapter, we can write (4.3) as

$$w = g_{ij} e_m^{\ j} du^i. \tag{4.4}$$

Here we take $e_m^{\ j} = X^j$ which is different from the previous chapter. By taking exterior derivative of (4.4) and using the fact (3.79), we have

$$\begin{aligned} dw &= dg_{ij} e_m^{\ j} du^i + g_{ij} d\left(e_m^{\ j}\right) du^i \\ &= \frac{\partial g_{ij}}{\partial u^k} du^k X^j du^i + \frac{\partial g_{ij}}{\partial X^k} dX^k X^j du^i + g_{ij} dX^j \wedge du^i \\ &= \frac{1}{2}\frac{\partial g_{ij}}{\partial u^k} X^j du^k \wedge du^i - \frac{1}{2}\frac{\partial g_{kj}}{\partial u^i} X^j du^k \wedge du^i + \frac{1}{2} g_{ij} dX^j \wedge du^i - \frac{1}{2} g_{ij} du^j \wedge dX^i \\ &= \frac{1}{2}\left[\frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ik}}{\partial u^j}\right] X^k du^i \wedge du^j + \frac{1}{2} g_{ij} dX^i \wedge du^j - \frac{1}{2} g_{ij} du^i \wedge dX^j. \end{aligned} \tag{4.5}$$

We can write dw in a closed form as

$$dw = \frac{1}{2} L_{IJ} dY^I \wedge dY^J, \tag{4.6}$$

where $Y^I = \left(y^i, y^{i^*}\right) \equiv (u^i, X^i)$ and $L_{ij} = \left(\frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ik}}{\partial u^j}\right) X^k$, $L_{ij^*} = -g_{ij}$, $L_{i^*j} = g_{ij}$ and $L_{i^*j^*} = 0$ or we can write L_{IJ} in a matrix form as

$$L_{IJ} = \begin{pmatrix} \left(\frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ik}}{\partial u^j}\right) X^k & g_{ij} \\ -g_{ij} & 0 \end{pmatrix}. \tag{4.7}$$

Here L is skew-symmetric in lower-indices and has maximum rank $2m$. Let G_{IJ} be a positive definite metric on $T(M)$ such that

$$L_I{}^K L_J{}^L G_{KL} = G_{IJ}, \tag{4.8}$$

where

$$L_I{}^K = L_{IJ} G^{JK}, \tag{4.9}$$

and

$$L_I{}^K L_K^J = -\delta_I^J. \tag{4.10}$$

From these equations we can say that $L_I{}^J$ denote an almost complex structure compatible with metric G. Later we will obtain G and L in terms of F. Now, by using (4.7) and (4.9) we obtain the components of $L_I{}^J$ as

$$\begin{aligned} L_i{}^j &= L_{iK} G^{Kj} = L_{ik} G^{kj} + L_{ik^*} G^{k^* j} \\ &= \left(\frac{\partial g_{kj}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k} \right) X^j G^{kj} - g_{ik} G^{k^* j}, \end{aligned} \tag{4.11}$$

$$\begin{aligned} L_{i^*}^j &= L_{i^* K} G^{Kj} = L_{i^* k} G^{kj} + L_{i^* k^*} G^{k^* j} \\ &= g_{ik} G^{kj}, \end{aligned} \tag{4.12}$$

$$\begin{aligned} L_i{}^{j^*} &= L_{iK} G^{Kj^*} = L_{ik} G^{kj^*} + L_{ik^*} G^{k^* j^*} \\ &= \left(\frac{\partial g_{kj}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k} \right) X^j G^{kj^*} - g_{ik} G^{k^* j^*}, \end{aligned} \tag{4.13}$$

$$\begin{aligned} L_{i^*}^{j^*} &= L_{i^* K} G^{Kj^*} = L_{i^* k} G^{kj^*} + L_{i^* k^*} G^{k^* j^*} \\ &= g_{ik} G^{kj^*}. \end{aligned} \tag{4.14}$$

We can write $L_I{}^J$ in a matrix form as

$$L_I{}^J = \begin{pmatrix} L_{ik} G^{kj} - g_{ik} G^{k^* j} & g_{ik} G^{kj} \\ L_{ik} G^{kj^*} - g_{ik} G^{k^* j^*} & g_{ik} G^{kj^*} \end{pmatrix}. \tag{4.15}$$

By using (4.9) and (4.10), we can write the following relation

$$L_{IK} L_{JL} G^{KL} = G_{IJ}. \tag{4.16}$$

By using (4.7) and (4.16), we obtain some relations for G_{IJ} as

$$\begin{aligned} G_{ij} &= L_{iK}L_{jL}G^{KL} \\ &= L_{ik}L_{jl}G^{kl} - L_{ik}g_{jl}G^{kl^*} - g_{ik}L_{jl}G^{k^*l} + g_{ik}g_{jl}G^{k^*l^*}, \end{aligned} \quad (4.17)$$

$$\begin{aligned} G_{i^*j} &= L_{i^*K}L_{jL}G^{KL} \\ &= g_{ik}L_{jl}G^{kl} - g_{ik}g_{jl}G^{kl^*}, \end{aligned} \quad (4.18)$$

$$\begin{aligned} G_{ij^*} &= L_{iK}L_{j^*L}G^{KL} \\ &= g_{jl}L_{ik}G^{kl} - g_{ik}g_{jl}G^{k^*l}, \end{aligned} \quad (4.19)$$

$$\begin{aligned} G_{i^*j^*} &= L_{i^*K}L_{j^*L}G^{KL} \\ &= g_{ik}g_{jl}G^{kl}. \end{aligned} \quad (4.20)$$

Consider the fibre of $T(M)$ which is obtained from $y^{i^*} = X^i$ and also it is n-dimensional submanifold of $T(M)$. The vectors which are tangent to these fiber, are

$$C_{i^*}{}^J = \delta_{i^*}^J = \left(0, \delta_i^j\right), \quad (4.21)$$

and by using L, we get its orthogonal which has the form

$$B_j{}^I = L_K^I C_{j^*}{}^K = L_{j^*}^I. \quad (4.22)$$

By using (4.15) and (4.22), we can write the components of $B_j{}^I$ as

$$B_j{}^i = L_{j^*}^i = g_{jk}G^{ki}, \qquad B_j{}^{i^*} = L_{j^*}^{i^*} = g_{jk}G^{ki^*}. \quad (4.23)$$

$C_{i^*}^I$ are tangent to fiber of $T(M)$ so it is an element of $T\left(T\left(M\right)\right)$ and its orthogonal $B_j{}^I$ are an element of $T^*\left(T\left(M\right)\right)$.

4.2 Determination of Metric G and Almost Complex Structure L

Let us denote

$$G_{KL}B_j{}^K B_i{}^L = G_{KL}L_{j^*}^K L_{i^*}^L \equiv g_{ij}. \quad (4.24)$$

By (4.8), left hand side of the above equation is $G_{j^*i^*}$. Thus, we get the one of the components of G in terms of F as

$$G_{j^*i^*} = g_{ji}. \tag{4.25}$$

By using (4.21) and (4.25) we can write

$$G_{KL} C_{j^*}^{\ K} C_{i^*}^{\ L} = G_{j^*i^*} = g_{ji}. \tag{4.26}$$

Thus we get the relations for $C_{i^*}^{\ K}$ and $B_i^{\ K}$ in terms of g_{ij}. By using (4.20) and (4.25), we have

$$g_{ik} g_{jl} G^{kl} = g_{ij}. \tag{4.27}$$

That equality holds if $G^{kl} = g^{kl}$ with $g^{kl} g_{lj} = \delta^k_j$. Here we get another relation for G as

$$G^{ij} = g^{ij}, \tag{4.28}$$

where g_{ij} and g^{ij} help us to lower and raise the indices of objects. Let us introduce

$$- G^{ij^*} = -G^{j^*i} \equiv N^{ij}. \tag{4.29}$$

By using (4.20), (4.28) and (4.29), we have

$$G_{ij^*} = g_{jl} L_{ik} G^{kl} - g_{ik} g_{jl} G^{k^*l} = L_{ij} + N_{ji}. \tag{4.30}$$

Denoting $G_{ij^*} \equiv N_{ij}$ and using the above relation, we get

$$L_{ij} = N_{ij} - N_{ji}. \tag{4.31}$$

Let us introduce

$$G^{k^*l^*} \equiv g^{kl} + g^{rs} N_r^{\ k} N_s^{\ l}. \tag{4.32}$$

By using (4.17), (4.31), (4.32) and after some simplification, we get another relation for G as

$$\begin{aligned} G_{ij} &= (N_{ik} - N_{ki})(N_{jl} - N_{lj}) g^{kl} + g_{jl}(N_{ik} - N_{ki}) N^{kl} \\ &+ g_{ik}(N_{jl} - N_{lj}) N^{lk} + g_{ik} g_{jl} \left(g^{kl} + g^{rs} N_r^{\ k} N_s^{\ l}\right) \\ &= g_{ij} + g_{kl} N_i^{\ l} N_j^{\ k}. \end{aligned} \tag{4.33}$$

By using (4.19) and (4.31), we have

$$G_{i^*j} = g_{ik} L_{jl} G^{kl} - g_{ik} g_{jl} G^{kl^*} = N_{ji}. \tag{4.34}$$

Thus we can write G_{IJ} and G^{IJ} in terms of g_{ij} and $N_i{}^j$ in the matrix form as

$$G_{IJ} = \begin{pmatrix} g_{ij} + g_{lk} N_i{}^l N_j{}^k & N_{ji} \\ N_{ij} & g_{ij} \end{pmatrix}, \tag{4.35}$$

and

$$G^{IJ} = \begin{pmatrix} g^{ij} & -N^{ji} \\ -N^{ij} & g^{ij} + g^{kl} N_k{}^i N_l{}^j \end{pmatrix}. \tag{4.36}$$

By using (4.35) and (4.36), the components of $L_I{}^J$ can be rewritten as

$$L_i{}^j = \left(N_{ik} - N_{ki}\right) g^{kj} + g_{ik} N^{jk} = N_i{}^j, \tag{4.37}$$

$$L_{i^*}^{\ j} = g_{ik} g^{kj} = \delta_i^j, \tag{4.38}$$

$$\begin{aligned} L_i^{j^*} &= \left(N_{ik} - N_{ki}\right)\left(-N^{kj}\right) - g_{ik}\left(g^{kj} + g^{rs} N_r{}^k N_s{}^j\right) \\ &= -\delta_i^j - N_{ik} N^{kj}, \end{aligned} \tag{4.39}$$

$$L_{i^*}^{j^*} = g_{ik}\left(-N^{kj}\right) = -N_i{}^j. \tag{4.40}$$

Thus $L_I{}^J$ in terms of $N_i{}^j$ in the matrix form is

$$L_I{}^J = \begin{pmatrix} N_i{}^j & \delta_i^j \\ -\delta_i^j - N_i{}^k N_k{}^j & -N_i{}^j \end{pmatrix}. \tag{4.41}$$

4.3 A Special Frame and Anholonomic Basis

In the previous section we obtained the metric G which has all components G_{ij}, G_{i^*j}, G_{ij^*}, $G_{i^*j^*}$. To obtain G which has components no mixed parts, we define special frame as

$$E_\alpha{}^I = \left(B_a{}^I, C_{a^*}^{\ I}\right), \qquad \text{with inverse} \qquad E^\alpha{}_I = \left(B^a{}_I, C^{a^*}{}_I\right). \tag{4.42}$$

By using (4.21), (4.22), (4.23) and (4.41), we can write

$$B_a{}^J = L_K^{\ J} C_{a^*}^{\ K} = L_{a^*}^{\ J} = \left(\delta_a^j, -N_a{}^j\right), \qquad C_{a^*}^{\ J} = \left(0, \delta_a^j\right), \tag{4.43}$$

and

$$B^a{}_I = \left(\delta_i^a, 0\right), \qquad C^{a^*}{}_J = \left(N_j{}^a, \delta_j^a\right). \tag{4.44}$$

By using this special frame we can write the components of G and L as

$$G_{\alpha\beta} = G_{KL}E_{\alpha}{}^{K}E_{\beta}{}^{L}, \tag{4.45}$$

and

$$L_{\alpha\beta} = L_{KL}E_{\alpha}{}^{K}E_{\beta}{}^{L}. \tag{4.46}$$

By using (4.35), (4.41), (4.42) and after some simplification, we obtain components of G and L in this special frame as

$$\begin{aligned} G_{ab} &= G_{KL}E_{a}{}^{K}E_{b}{}^{L} \\ &= G_{kl}B_{a}{}^{k}B_{b}{}^{l} + G_{k^*l}B_{a}{}^{k^*}B_{b}{}^{l} + G_{kl^*}B_{a}{}^{k}B_{b}{}^{l^*} + G_{k^*l^*}B_{a}{}^{k^*}B_{b}{}^{l^*} \\ &= g_{ab}, \end{aligned} \tag{4.47}$$

$$\begin{aligned} G_{a^*b} &= G_{KL}E_{a^*}{}^{K}E_{b}{}^{L} \\ &= G_{kl}C_{a^*}{}^{k}B_{b}{}^{l} + G_{k^*l}C_{a^*}{}^{k^*}B_{b}{}^{l} + G_{kl^*}C_{a^*}{}^{k}B_{b}{}^{l^*} + G_{k^*l^*}C_{a^*}{}^{k^*}B_{b}{}^{l^*} \\ &= N_{ba} - N_{ba} = 0, \end{aligned} \tag{4.48}$$

$$\begin{aligned} G_{ab^*} &= G_{KL}E_{a}{}^{K}E_{b^*}{}^{L} \\ &= G_{kl}B_{a}{}^{k}C_{b^*}{}^{l} + G_{k^*l}B_{a}{}^{k^*}C_{b^*}{}^{l} + G_{kl^*}B_{a}{}^{k}C_{b^*}{}^{l^*} + G_{k^*l^*}B_{a}{}^{k^*}C_{b^*}{}^{l^*} \\ &= N_{ab} - N_{ab} = 0, \end{aligned} \tag{4.49}$$

$$\begin{aligned} G_{a^*b^*} &= G_{KL}E_{a^*}{}^{K}E_{b^*}{}^{L} \\ &= G_{kl}C_{a^*}{}^{k}C_{b^*}{}^{l} + G_{k^*l}C_{a^*}{}^{k^*}C_{b^*}{}^{l} + G_{kl^*}C_{a^*}{}^{k}C_{b^*}{}^{l^*} + G_{k^*l^*}C_{a^*}{}^{k^*}C_{b^*}{}^{l^*} \\ &= g_{ab}. \end{aligned} \tag{4.50}$$

We can write $G_{\alpha\beta}$ and $G^{\beta\gamma}$ in matrix form as

$$G_{\alpha\beta} = \begin{pmatrix} g_{ab} & 0 \\ 0 & g_{ab} \end{pmatrix}, \qquad G^{\beta\gamma} = \begin{pmatrix} g^{bc} & 0 \\ 0 & g^{bc} \end{pmatrix}. \tag{4.51}$$

Thus we obtained metric in a simpler form by defining special frame.

$$L_{ab} = L_{KL}E_{a}{}^{K}E_{b}{}^{L} = L_{ij} - (N_{ij} - N_{ji}) = 0, \tag{4.52}$$

$$L_{a^*b} = L_{KL}E_{a^*}{}^{K}E_{b}{}^{L} = g_{kl}\delta_{a}^{k}\delta_{b}^{l} = g_{ab}, \tag{4.53}$$

$$L_{ab^*} = L_{KL} E_a{}^K E_{b^*}{}^L = -g_{kl}\delta_a^k \delta_b^l = -g_{ab}, \tag{4.54}$$

$$L_{a^*b^*} = L_{KL} E_{a^*}^{\ K} E_{b^*}^{\ L} = 0. \tag{4.55}$$

We can write $L_{\alpha\beta}$ and $L_\beta{}^\gamma$ in matrix form as

$$L_{\alpha\beta} = \begin{pmatrix} 0 & g_{ab} \\ -g_{ab} & 0 \end{pmatrix}, \qquad L_\beta{}^\gamma = \begin{pmatrix} 0 & \delta_b^c \\ -\delta_b^c & 0 \end{pmatrix}, \tag{4.56}$$

where we obtained components of $L_\beta{}^\gamma$ by multiplying the components of $L_{\alpha\beta}$ with g^{bk}. Now we introduce the anholonomic basis vectors by using the special frame as

$$E_\alpha \equiv E_\alpha{}^K \frac{\partial}{\partial Y^K}, \tag{4.57}$$

and its dual as

$$W^\alpha \equiv E^\alpha_{\ K} dY^K. \tag{4.58}$$

where $E_\alpha = (E_a, E_{a^*})$, $W^\alpha = \left(W^a, W^{a^*}\right)$ and $E_\alpha\left(W^\beta\right) = \delta_\alpha^\beta$.
By using (4.1), (4.42), (4.43), (4.44) and expanding the indices as $K = (k, k^*)$, we obtain

$$\begin{aligned} E_a = \frac{\delta}{\delta y^a} &= \frac{\delta}{\delta u^a} = E_a{}^K \frac{\partial}{\partial Y^K} = E_a{}^k \frac{\partial}{\partial y^k} + E_a{}^{k^*} \frac{\partial}{\partial y^{k^*}} \\ &= \frac{\partial}{\partial u^a} - N_a{}^k \frac{\partial}{\partial X^k}, \end{aligned} \tag{4.59}$$

$$\begin{aligned} E_{a^*} = \frac{\delta}{\delta y^{a^*}} &= \frac{\delta}{\delta X^a} = E_{a^*}^{\ K} \frac{\partial}{\partial Y^K} = E_{a^*}^{\ k} \frac{\partial}{\partial y^k} + E_{a^*}^{\ k^*} \frac{\partial}{\partial y^{k^*}} \\ &= \frac{\partial}{\partial X^a}, \end{aligned} \tag{4.60}$$

$$W^a = \delta y^a = \delta u^a = E^a_{\ K} dY^K = E^a_{\ k} dy^k + E^a_{\ k^*} dy^{k^*} = du^a, \tag{4.61}$$

$$\begin{aligned} W^{a^*} = \delta y^{a^*} &= \delta X^a = E^{a^*}_{\ K} dY^K = E^{a^*}_{\ k} dy^k + E^{a^*}_{\ k^*} dy^{k^*} \\ &= dX^a + N_k{}^a du^k. \end{aligned} \tag{4.62}$$

Thus we obtained an anholonomic basis as

$$\begin{aligned} E_\alpha &= (E_a, E_{a^*}) = \left(\frac{\partial}{\partial u^a} - N_a{}^k \frac{\partial}{\partial X^k}, \frac{\partial}{\partial X^a}\right), \\ W^\alpha &= \left(W^a, W^{a^*}\right) = \left(du^a, dX^a + N_k{}^a du^k\right). \end{aligned} \tag{4.63}$$

So that G has no mixed components. We can write the line element of the tangent bundle in the anholonomic basis as

$$d\sigma^2 = G_{\alpha\beta} W^\alpha W^\beta \tag{4.64}$$

where

$$G_{\alpha\beta} = \begin{pmatrix} g_{ab} & 0 \\ 0 & g_{ab} \end{pmatrix}, \qquad W^\alpha = \left(W^a, W^{a^*}\right) = \left(du^a, dX^a + N_k{}^a du^k\right). \tag{4.65}$$

Our special frame (4.42) is not a natural frame, so we have anholonomic objects. We now obtain the components of the anholonomic objects by using the Lie brackets of (4.59), (4.60), (4.61) and (4.62). By definition of Lie brackets we have

$$[E_\alpha, E_\beta] = C_{\alpha\beta}{}^\mu E_\mu, \tag{4.66}$$

where $C_{\alpha\beta}{}^\mu$ are structure constants or components of the anholonomic objects and skew-symmetric in lower indices. By using (4.66) for E_a and E_b, we have

$$[E_a, E_b] = C_{ab}{}^\alpha E_\alpha = C_{ab}{}^c E_c + C_{ab}{}^{c^*} E_{c^*}, \tag{4.67}$$

By using the formula (4.59) for E_a, direct calculation of Lie bracket of E_a and E_b is

$$\begin{aligned}
[E_a, E_b] &= \left(\frac{\partial}{\partial u^a} - N_a{}^k \frac{\partial}{\partial X^k}\right)\left(\frac{\partial}{\partial u^b} - N_b{}^k \frac{\partial}{\partial X^k}\right) \\
&- \left(\frac{\partial}{\partial u^b} - N_b{}^k \frac{\partial}{\partial X^k}\right)\left(\frac{\partial}{\partial u^a} - N_a{}^k \frac{\partial}{\partial X^k}\right) \\
&= \left[\frac{\partial N_a{}^k}{\partial u^b} - N_b{}^l \frac{\partial N_a{}^k}{\partial X^l}\right]\frac{\partial}{\partial X^k} - \left[\frac{\partial N_b{}^l}{\partial u^a} - N_a{}^k \frac{\partial N_b{}^l}{\partial X^k}\right]\frac{\partial}{\partial X^l} \\
&= \left[\frac{\delta N_a{}^k}{\delta u^b} - \frac{\delta N_b{}^k}{\delta u^a}\right]\frac{\partial}{\partial X^k} = \left[\frac{\delta N_a{}^{c^*}}{\delta u^b} - \frac{\delta N_b{}^{c^*}}{\delta u^a}\right] E_{c^*}.
\end{aligned} \tag{4.68}$$

Comparing (4.67) and (4.68) we get

$$C_{ab}{}^c = 0, \qquad C_{ab}{}^{c^*} = \frac{\delta N_a{}^{c^*}}{\delta u^b} - \frac{\delta N_b{}^{c^*}}{\delta u^a} \equiv F^{c^*}_{\ ab}. \tag{4.69}$$

At the end of the chapter we will get an explicit formula for $F^\gamma_{\ \alpha\beta}$ on the whole tangent bundle. By using (4.66) for E_a and E_{b^*}, we have

$$[E_a, E_{b^*}] = C_{ab^*}{}^\alpha E_\alpha = C_{ab^*}{}^c E_c + C_{ab^*}{}^{c^*} E_{c^*}, \tag{4.70}$$

By using the formula (4.59) and (4.60) for E_a and E_{b^*}, direct calculation of Lie bracket of E_a and E_{b^*} is

$$\begin{aligned}[E_a, E_{b^*}] &= \left(\frac{\partial}{\partial u^a} - N_a{}^k \frac{\partial}{\partial X^k}\right)\left(\frac{\partial}{\partial X^b}\right) - \left(\frac{\partial}{\partial X^b}\right)\left(\frac{\partial}{\partial u^a} - N_a{}^k \frac{\partial}{\partial X^k}\right) \\ &= \frac{\partial N_a{}^k}{\partial X^b}\frac{\partial}{\partial X^k} = \frac{\partial N_a{}^{c^*}}{\partial y^{b^*}} E_{c^*}. \end{aligned} \tag{4.71}$$

Comparing (4.70) and (4.71) we get

$$C_{ab^*}{}^c = 0, \qquad C_{ab^*}{}^{c^*} = \frac{\partial N_a{}^{c^*}}{\partial y^{b^*}} \equiv \phi^{c^*}{}_{ab^*}. \tag{4.72}$$

Using similar method we get

$$C_{a^*b}{}^c = 0, \qquad C_{a^*b}{}^{c^*} = -\frac{\partial N_b{}^{c^*}}{\partial y^{a^*}} \equiv -\phi^{c^*}{}_{ba^*}. \tag{4.73}$$

By using (4.66) for E_{a^*} and E_{b^*}, we have

$$[E_{a^*}, E_{b^*}] = C_{ab^*}{}^{\alpha} E_\alpha = C_{a^*b^*}{}^{c} E_c + C_{a^*b^*}{}^{c^*} E_{c^*}. \tag{4.74}$$

By using the formula (4.60) for E_{a^*} and E_{b^*}, direct calculation of Lie bracket of E_{a^*} and E_{b^*} given

$$[E_{a^*}, E_{b^*}] = \left(\frac{\partial}{\partial X^a}\right)\left(\frac{\partial}{\partial X^b}\right) - \left(\frac{\partial}{\partial X^b}\right)\left(\frac{\partial}{\partial X^a}\right) = 0. \tag{4.75}$$

Comparing (4.70) and (4.71) we get

$$C_{a^*b^*}{}^{c} = 0, \qquad C_{a^*b^*}{}^{c^*} = 0. \tag{4.76}$$

4.4 Determination of Connection Coefficients Γ and Curvature Tensor R

Levi-Civita connection on the tangent bundle in anholonomic basis defined as

$$D(E_\beta) = \Gamma^\lambda{}_{\beta\alpha} W^\alpha \otimes E_\lambda, \tag{4.77}$$

where $\Gamma^{\lambda}{}_{\beta\alpha}$ is Levi-Civita connection coefficients. Now we obtain a formula for $\Gamma^{\mu}{}_{\alpha\beta}$ in terms of metric of tangent bundle $G_{\alpha\beta}$ and components of anholonomic objects $C_{\alpha\beta}{}^{\mu}$. From the definition of Lie bracket we have

$$\begin{aligned} [E_\alpha, E_\beta] &= D_{E_\alpha} E_\beta - D_{E_\beta} E_\alpha \\ &= \left[\Gamma^{\mu}{}_{\beta\alpha} - \Gamma^{\mu}{}_{\alpha\beta}\right] E_\mu. \end{aligned} \tag{4.78}$$

Comparing that with (4.66), we get the relation of connection coefficients and components of anholonomic objects as

$$-C_{\alpha\beta}{}^{\mu} = \Gamma^{\mu}{}_{\alpha\beta} - \Gamma^{\mu}{}_{\beta\alpha}. \tag{4.79}$$

From the covariant derivative of $G_{\alpha\beta}$ we have

$$D_\mu G_{\alpha\beta} = \frac{\delta G_{\alpha\beta}}{\delta Y^\mu} - \Gamma^{\gamma}{}_{\alpha\mu} G_{\gamma\beta} - \Gamma^{\gamma}{}_{\beta\mu} G_{\gamma\alpha} = 0, \tag{4.80}$$

which is the consequence of metric-compatibility of the Levi-Civita connection. From there we have

$$\frac{\delta G_{\alpha\beta}}{\delta Y^\mu} = \Gamma^{\gamma}{}_{\alpha\mu} G_{\gamma\beta} + \Gamma^{\gamma}{}_{\beta\mu} G_{\gamma\alpha}. \tag{4.81}$$

By changing the indices α, β, μ and adding and subtracting $\Gamma^{\gamma}{}_{\alpha\beta} G_{\gamma\mu}$, we have

$$\begin{aligned} \frac{\delta G_{\mu\alpha}}{\delta Y^\beta} + \frac{\delta G_{\mu\beta}}{\delta Y^\alpha} - \frac{\delta G_{\alpha\beta}}{\delta Y^\mu} &= 2\Gamma^{\gamma}{}_{\alpha\beta} G_{\gamma\mu} + \left(\Gamma^{\gamma}{}_{\mu\alpha} - \Gamma^{\gamma}{}_{\alpha\mu}\right) G_{\gamma\beta} + \left(\Gamma^{\gamma}{}_{\mu\beta} - \Gamma^{\gamma}{}_{\beta\mu}\right) G_{\gamma\alpha} \\ &- \left(\Gamma^{\gamma}{}_{\alpha\beta} - \Gamma^{\gamma}{}_{\beta\alpha}\right) G_{\gamma\mu}. \end{aligned} \tag{4.82}$$

Using (4.79) and arranging the above equality and multiplying both sides by $G^{\mu\lambda}$ and lowering the indices with $G_{\gamma\beta}$, we get the components of Levi-Civita connection on the bundle manifold as

$$\Gamma^{\lambda}{}_{\alpha\beta} = \frac{1}{2} G^{\mu\lambda} \left[\frac{\delta G_{\mu\beta}}{\delta Y^\alpha} + \frac{\delta G_{\mu\alpha}}{\delta Y^\beta} - \frac{\delta G_{\alpha\beta}}{\delta Y^\mu} + C_{\mu\alpha\beta} + C_{\mu\beta\alpha} - C_{\alpha\beta\mu}\right]. \tag{4.83}$$

Let us set

$$G_{\alpha\beta} = \begin{pmatrix} g_{ab} & 0 \\ 0 & g_{ab} \end{pmatrix} \equiv \begin{pmatrix} g_{ab} & 0 \\ 0 & g_{a^*b^*} \end{pmatrix}, \tag{4.84}$$

and

$$G^{\alpha\beta} = \begin{pmatrix} g^{ab} & 0 \\ 0 & g^{ab} \end{pmatrix} \equiv \begin{pmatrix} g^{ab} & 0 \\ 0 & g^{a^*b^*} \end{pmatrix}. \tag{4.85}$$

Let us introduce

$$\bar{\bar{\gamma}}^{\alpha}{}_{\beta\gamma} \equiv \frac{1}{2} G^{\lambda\alpha} \left(\frac{\delta G_{\lambda\beta}}{\delta Y^{\gamma}} + \frac{\delta G_{\lambda\gamma}}{\delta Y^{\beta}} - \frac{\delta G_{\beta\gamma}}{\delta Y^{\lambda}} \right). \tag{4.86}$$

Now we obtain Γ for different indices by using the formula (4.83). These components of the connection Γ will help us to calculate Ricci curvature and scalar curvature tensors of the tangent bundle in the next chapter. Scalar curvature will play a central role to get the Schwarzchild solution and to compute the gravitational red shift.

By using (4.59), (4.69), (4.84), (4.85) and (4.86) we have

$$\begin{aligned} \Gamma^{a}{}_{bc} &= \frac{1}{2} G^{\mu a} \left[\frac{\delta G_{\mu b}}{\delta Y^{c}} + \frac{\delta G_{\mu c}}{\delta Y^{b}} - \frac{\delta G_{cb}}{\delta Y^{\mu}} + C_{\mu cb} + C_{\mu bc} - C_{bc\mu} \right] \\ &= \frac{1}{2} g^{da} \left[\frac{\delta g_{db}}{\delta u^{c}} + \frac{\delta g_{dc}}{\delta u^{b}} - \frac{\delta g_{cb}}{\delta u^{d}} \right] = \bar{\bar{\gamma}}^{a}{}_{bc}. \end{aligned} \tag{4.87}$$

By denoting

$$\gamma^{a}{}_{bc} \equiv \frac{1}{2} g^{da} \left(\frac{\partial g_{db}}{\partial u^{c}} + \frac{\partial g_{dc}}{\partial u^{b}} - \frac{\partial g_{cb}}{\partial u^{d}} \right) \tag{4.88}$$

and using (4.59) for $\frac{\delta}{\delta u^a}$, we can obtain

$$\begin{aligned} \Gamma^{a}{}_{bc} &= \frac{1 g^{da}}{2} \left[\frac{\partial g_{db}}{\partial u^{c}} + \frac{\partial g_{dk}}{\partial u^{b}} - \frac{\partial g_{kb}}{\partial u^{d}} \right] - \frac{1}{2} g^{da} \left[N_{c}{}^{r} \frac{\partial g_{db}}{\partial X^{r}} + N_{b}{}^{r} \frac{\partial g_{dc}}{\partial X^{r}} - N_{d}{}^{r} \frac{\partial g_{cb}}{\partial X^{r}} \right] \\ &= \gamma^{a}{}_{bc} - \frac{1}{2} g^{da} \left[N_{c}{}^{r} \frac{\partial g_{db}}{\partial X^{r}} + N_{b}{}^{r} \frac{\partial g_{dc}}{\partial X^{r}} - N_{d}{}^{r} \frac{\partial g_{cb}}{\partial X^{r}} \right]. \end{aligned} \tag{4.89}$$

Here $\gamma^{a}{}_{bc}$ are the Christoffel symbols of the base manifold.

By using (4.60), (4.69), (4.72), (4.73), (4.84) and (4.85) we have

$$\begin{aligned} \Gamma^{a}{}_{bc^*} &= \frac{1}{2} G^{\mu a} \left[\frac{\delta G_{\mu b}}{\delta y^{c^*}} + \frac{\delta G_{\mu c^*}}{\delta y^{b}} - \frac{\delta G_{c^* b}}{\delta Y^{\mu}} + C_{\mu c^* b} + C_{\mu b c^*} - C_{b c^* \mu} \right] \\ &= \frac{1}{2} g^{da} \frac{\partial g_{db}}{\partial X^{c}} - \frac{1}{2} g^{da} F_{c^* db}. \end{aligned} \tag{4.90}$$

Let us denote

$$\bar{\gamma}^{a}{}_{bc} \equiv \frac{1}{2} g^{da} \left(\frac{\partial g_{db}}{\partial X^{c}} + \frac{\partial g_{dc}}{\partial X^{b}} - \frac{\partial g_{cb}}{\partial X^{d}} \right), \tag{4.91}$$

By using (4.91), we get

$$\begin{aligned} \bar{\gamma}_{dcb} + \bar{\gamma}_{bcd} &= \frac{1}{2} \left(\frac{\partial g_{dc}}{\partial X^{b}} + \frac{\partial g_{db}}{\partial X^{c}} - \frac{\partial g_{bc}}{\partial X^{d}} + \frac{\partial g_{bd}}{\partial X^{c}} + \frac{\partial g_{bc}}{\partial X^{d}} - \frac{\partial g_{cd}}{\partial X^{b}} \right) \\ &= \frac{\partial g_{db}}{\partial X^{c}}. \end{aligned} \tag{4.92}$$

Using the identity

$$\bar{\gamma}_{dc^*b} + \bar{\gamma}_{bc^*d} = \frac{\delta g_{db}}{\delta y^{c^*}}, \tag{4.93}$$

and $F_{c^*db} = -F_{c^*bd}$ we can write (4.90) as

$$\Gamma^a{}_{bc^*} = \frac{1}{2}\left(\bar{\gamma}^a{}_{c^*b} + \bar{\gamma}_{bc^*}{}^a + F_{c^*b}{}^a\right). \tag{4.94}$$

Similarly we can obtain

$$\Gamma^a{}_{c^*b} = \frac{1}{2}\left(\bar{\gamma}^a{}_{c^*b} + \bar{\gamma}_{bc^*}{}^a + F_{c^*b}{}^a\right). \tag{4.95}$$

By using (4.59), (4.72), (4.76), (4.84), (4.85) we have

$$\begin{aligned}\Gamma^a{}_{b^*c^*} &= \frac{1}{2}G^{\mu a}\left[\frac{\delta G_{\mu b^*}}{\delta y^{c^*}} + \frac{\delta G_{\mu c^*}}{\delta y^{b^*}} - \frac{\delta G_{c^*b^*}}{\delta Y^\mu} + C_{\mu c^*b^*} + C_{\mu b^*c^*} - C_{b^*c^*\mu}\right]\\ &= \frac{1}{2}g^{da}\frac{\delta g_{c^*b^*}}{\delta u^d} + \frac{1}{2}g^{da}\phi_{b^*dc^*} + \frac{1}{2}g^{da}\phi_{c^*db^*}.\end{aligned} \tag{4.96}$$

By using (4.84), (4.85) and (4.86) we get the following relation

$$\bar{\bar{\gamma}}_{c^*db^*} + \bar{\bar{\gamma}}_{b^*dc^*} = \frac{\delta g_{b^*c^*}}{\delta u^d}, \tag{4.97}$$

which leads to

$$\begin{aligned}\Gamma^a{}_{b^*c^*} &= -\frac{1}{2}\left[(\bar{\bar{\gamma}}_{c^*}{}^a{}_{b^*} - \phi_{c^*}{}^a{}_{b^*}) + (\bar{\bar{\gamma}}_{b^*}{}^a{}_{c^*} - \phi_{b^*}{}^a{}_{c^*})\right]\\ &= -\frac{1}{2}\left(T_{c^*}{}^a{}_{b^*} + T_{b^*}{}^a{}_{c^*}\right).\end{aligned} \tag{4.98}$$

where

$$T^\alpha_{\ \beta\gamma} \equiv \bar{\bar{\gamma}}^\alpha{}_{\beta\gamma} - \phi^\alpha{}_{\beta\gamma}. \tag{4.99}$$

By using (4.60), (4.69), (4.72), (4.84), (4.85) and the fact (4.93) we have

$$\begin{aligned}\Gamma^{a^*}_{bc} &= \frac{1}{2}G^{\mu a^*}\left[\frac{\delta G_{\mu b}}{\delta y^c} + \frac{\delta G_{\mu c}}{\delta y^b} - \frac{\delta G_{cb}}{\delta Y^\mu} + C_{\mu cb} + C_{\mu bc} - C_{bc\mu}\right]\\ &= -\frac{1}{2}g^{d^*a^*}\frac{\partial g_{cb}}{\partial X^d} + \frac{1}{2}g^{d^*a^*}F_{d^*bc} = \frac{1}{2}\left[F^{a^*}{}_{bc} - \bar{\gamma}_c{}^{a^*}{}_b - \bar{\gamma}_b{}^{a^*}{}_c\right].\end{aligned} \tag{4.100}$$

By using (4.59), (4.72), (4.73), (4.84), (4.85), (4.97) and (4.99) we obtain

$$\begin{aligned}\Gamma^{a^*}_{bc^*} &= \frac{1}{2}G^{\mu a^*}\left[\frac{\delta G_{\mu b}}{\delta y^{c^*}} + \frac{\delta G_{\mu c^*}}{\delta y^b} - \frac{\delta G_{c^*b}}{\delta Y^\mu} + C_{\mu c^*b} + C_{\mu bc^*} - C_{bc^*\mu}\right]\\ &= -\frac{1}{2}g^{d^*a^*}\frac{\delta g_{d^*c^*}}{\delta u^b} - \frac{1}{2}g^{d^*a^*}\phi_{c^*bd^*} - \frac{1}{2}g^{d^*a^*}\phi_{d^*bc^*}\\ &= \frac{1}{2}\left[\bar{\bar{\gamma}}^{a^*}_{bc^*} - \phi^{a^*}_{bc^*} + \bar{\bar{\gamma}}_{c^*b}{}^{a^*} - \phi_{c^*b}{}^{a^*}\right]\\ &= \frac{1}{2}\left[T^{a^*}_{bc^*} + T_{c^*b}{}^{a^*}\right]\end{aligned} \tag{4.101}$$

By using (4.59), (4.73), (4.84), (4.85), (4.97), (4.99), adding and subtracting $\bar{\bar{\gamma}}^{a^*}_{bc^*}$ we obtain

$$\begin{aligned}
\Gamma^{a^*}_{c^*b} &= \frac{1}{2}G^{\mu a^*}\left[\frac{\delta G_{\mu b}}{\delta y^{c^*}} + \frac{\delta G_{\mu c^*}}{\delta y^b} - \frac{\delta G_{c^*b}}{\delta Y^\mu} + C_{\mu c^* b} + C_{\mu b c^*} - C_{c^* b\mu}\right] \\
&= -\frac{1}{2}g^{d^*a^*}\frac{\delta g_{d^*c^*}}{\delta u^b} - \frac{1}{2}g^{d^*a^*}\phi_{c^*bd^*} + \frac{1}{2}g^{d^*a^*}\phi_{d^*bc^*} \\
&= \bar{\bar{\gamma}}^{a^*}_{bc^*} + \frac{1}{2}\left[\bar{\bar{\gamma}}_{c^*b}{}^{a^*} - \phi_{c^*b}{}^{a^*} - \left(\bar{\bar{\gamma}}^{a^*}_{bc^*} - \phi^{a^*}_{bc^*}\right)\right] \\
&= \bar{\bar{\gamma}}^{a^*}_{bc^*} + \frac{1}{2}\left[T_{c^*b}{}^{a^*} - T^{a^*}_{bc^*}\right] \qquad (4.102)
\end{aligned}$$

By using (4.60), (4.76), (4.84), (4.85) and (4.86) we obtain

$$\begin{aligned}
\Gamma^{a^*}_{b^*c^*} &= \frac{1}{2}G^{\mu a^*}\left[\frac{\delta G_{\mu b^*}}{\delta y^{c^*}} + \frac{\delta G_{\mu c^*}}{\delta y^{b^*}} - \frac{\delta G_{b^*c^*}}{\delta Y^\mu} + C_{\mu c^* b^*} + C_{\mu b^* c^*} - C_{b^* c^*\mu}\right] \\
&= \frac{1}{2}g^{d^*a^*}\left(\frac{\partial g_{d^*b^*}}{\partial X^c} + \frac{\partial g_{d^*c^*}}{\partial X^b} - \frac{\partial g_{b^*c^*}}{\partial X^d}\right) = \bar{\bar{\gamma}}^{a^*}{}_{b^*c^*}. \qquad (4.103)
\end{aligned}$$

Curvature tensor of the Levi-Civita connection D for the anholonomic basis is defined as

$$R\left(E_\alpha, E_\beta\right) = D_{E_\alpha}D_{E_\beta} - D_{E_\beta}D_{E_\alpha} - D_{\left[E_\alpha,E_\beta\right]}. \qquad (4.104)$$

where componentwise form of R is

$$R = R_{\alpha\ \gamma\lambda}^{\ \beta}E_\beta \otimes W^\alpha \otimes W^\gamma \otimes W^\lambda. \qquad (4.105)$$

We can write the components of the curvature tensor in the anholonomic basis as

$$R_{\alpha\ \gamma\lambda}^{\ \beta} = R\left(E_\alpha, W^\beta, E_\gamma, E_\lambda\right) = W^\beta\left(R\left(E_\gamma, E_\lambda\right)\cdot E_\alpha\right). \qquad (4.106)$$

By using (4.57), (4.58), (4.66), (4.78), (4.106) and (4.104) we get the components of the curvature tensor as

$$\begin{aligned}
R_{\alpha\ \gamma\lambda}^{\ \beta} &= W^\beta\left[\left(D_{E_\gamma}D_{E_\lambda} - D_{E_\lambda}D_{E_\gamma} - D_{[E_\gamma,E_\lambda]}\right)E_\alpha\right] \\
&= W^\beta\Big[E_\gamma\left(\Gamma^\delta{}_{\alpha\lambda}E_\delta\right) + \Gamma^\delta{}_{\alpha\lambda}\Gamma^\rho{}_{\delta\gamma}E_\rho - E_\lambda\left(\Gamma^\delta{}_{\alpha\gamma}\right)E_\delta \\
&- \Gamma^\delta{}_{\alpha\gamma}\Gamma^\rho{}_{\delta\lambda}E_\rho - C_{\gamma\lambda}{}^\delta\Gamma^\rho{}_{\alpha\delta}E_\rho\Big] \\
&= E_\gamma(\Gamma^\beta{}_{\alpha\lambda}) - E_\lambda(\Gamma^\beta{}_{\alpha\gamma}) + \Gamma^\delta{}_{\alpha\lambda}\Gamma^\beta{}_{\delta\gamma} - \Gamma^\delta{}_{\alpha\gamma}\Gamma^\beta{}_{\delta\lambda} - C_{\gamma\lambda}{}^\delta\Gamma^\beta{}_{\alpha\delta} \\
&= \frac{\delta\Gamma^\beta{}_{\alpha\lambda}}{\delta Y^\gamma} - \frac{\delta\Gamma^\beta{}_{\alpha\gamma}}{\delta Y^\lambda} + \Gamma^\beta{}_{\delta\gamma}\Gamma^\delta{}_{\alpha\lambda} - \Gamma^\beta{}_{\delta\lambda}\Gamma^\delta{}_{\alpha\gamma} - C_{\gamma\lambda}{}^\delta\Gamma^\beta{}_{\alpha\delta}. \qquad (4.107)
\end{aligned}$$

From the previous chapter, Equation (3.189) gives the following relation

$$X^b \bar{\bar{\gamma}}^{a^*}{}_{bc} = N_c{}^{a^*}. \tag{4.108}$$

By using that fact and (4.69) we obtain a formula for $C_{ab}{}^{c^*}$ in terms of R as

$$\begin{aligned}
C_{ab}{}^{c^*} &= \frac{\partial\left(X^d \bar{\bar{\gamma}}^{c^*}{}_{da}\right)}{\partial u^b} - X^r \bar{\bar{\gamma}}^d{}_{rb} \frac{\partial\left(X^s \bar{\bar{\gamma}}^{c^*}{}_{sa}\right)}{\partial X^d} - \frac{\partial\left(X^d \bar{\bar{\gamma}}^{c^*}{}_{db}\right)}{\partial u^a} + X^r \bar{\bar{\gamma}}^d{}_{ra} \frac{\partial\left(X^s \bar{\bar{\gamma}}^{c^*}{}_{sb}\right)}{\partial X^d} \\
&= -X^d \left[\frac{\partial \bar{\bar{\gamma}}^{c^*}{}_{db}}{\partial u^a} - N_a{}^s \frac{\partial \bar{\bar{\gamma}}^{c^*}{}_{db}}{\partial X^s} - \left(\frac{\partial \bar{\bar{\gamma}}^{c^*}{}_{da}}{\partial u^b} - N_b{}^s \frac{\partial \bar{\bar{\gamma}}^{c^*}{}_{da}}{\partial X^s}\right) + \bar{\bar{\gamma}}^r{}_{db} \bar{\bar{\gamma}}^{c^*}{}_{ra} - \bar{\bar{\gamma}}^r{}_{da} \bar{\bar{\gamma}}^{c^*}{}_{rb}\right] \\
&= -X^d \left[\frac{\delta \bar{\bar{\gamma}}^{c^*}{}_{db}}{\delta u^a} - \frac{\delta \bar{\bar{\gamma}}^{c^*}{}_{da}}{\delta u^b} + \bar{\bar{\gamma}}^r{}_{db} \bar{\bar{\gamma}}^{c^*}{}_{ra} - \bar{\bar{\gamma}}^r{}_{da} \bar{\bar{\gamma}}^{c^*}{}_{rb}\right] \equiv -X^d \bar{R}_d{}^{c^*}{}_{ab}.
\end{aligned} \tag{4.109}$$

Comparing that with (4.69) we get

$$F^{c^*}{}_{ab} = X^d \bar{R}_d{}^{c^*}{}_{ab}, \tag{4.110}$$

Here F is called the gauge curvature field and on the whole tangent bundle it has the form

$$\begin{aligned}
F^\gamma{}_{\alpha\beta} &\equiv Y^\delta \bar{R}_\delta{}^\gamma{}_{\alpha\beta} \\
&\equiv Y^\delta \left[\frac{\delta \bar{\bar{\gamma}}^\gamma{}_{\delta\beta}}{\delta Y^\alpha} - \frac{\delta \bar{\bar{\gamma}}^\gamma{}_{\delta\alpha}}{\delta Y^\beta} + \bar{\bar{\gamma}}^\lambda{}_{\delta\beta} \bar{\bar{\gamma}}^\gamma{}_{\lambda\alpha} - \bar{\bar{\gamma}}^\lambda{}_{\delta\alpha} \bar{\bar{\gamma}}^\gamma{}_{\lambda\beta}\right],
\end{aligned} \tag{4.111}$$

where gauge curvature field is skew-symmetric in the lower indices.i.e. $F^\gamma{}_{\alpha\beta} = -F^\gamma_{\beta\alpha}$.

4.5 Summary

Metric on the tangent bundle in natural basis:
$G_{IJ} = \begin{pmatrix} g_{ij} + g_{lk} N_i{}^l N_j{}^k & N_{ji} \\ N_{ij} & g_{ij} \end{pmatrix},$
Almost complex structure in natural basis:
$L_I{}^J = \begin{pmatrix} N_i{}^j & \delta_i^j \\ -\delta_i^j - N_i{}^k N_j{}^l & -N_i{}^j \end{pmatrix},$
Special frame: $E_\alpha{}^I = \left(B_a{}^I, C_{a^*}{}^I\right)$, with inverse $E^\alpha{}_I = \left(B^a{}_I, C^{a^*}{}_I\right)$, where
$B_a{}^I = (\delta_a^i, -N_a{}^i)$, $C_{a^*}{}^I = (0, \delta_a^i)$ and $B^a{}_I = (\delta_i^a, 0)$, $C^{a^*}{}_I = (N_i{}^a, \delta_i^a)$.

Anholonomic basis:
$E_\alpha = (E_a, E_{a^*}) = \left(\frac{\partial}{\partial u^a} - N_a{}^k \frac{\partial}{\partial X^k}, \frac{\partial}{\partial X^a}\right),$
and its dual
$W^\alpha = \left(W^a, W^{a^*}\right) = \left(du^a, dX^a + N_k{}^a du^k\right).$
Components of the bundle metric in special frame:
$G_{\alpha\beta} = \begin{pmatrix} g_{ab} & 0 \\ 0 & g_{a^*b^*} \end{pmatrix}.$
Alomost comlex structure in special frame:
$L_\alpha{}^\beta = \begin{pmatrix} 0 & \delta_a^b \\ -\delta_a^b & 0 \end{pmatrix}.$
The line element of the tangent bundle: $d\sigma^2 = G_{\alpha\beta} W^\alpha W^\beta$.
Components of the anholonomic objects such that $[E_\alpha, E_\beta] = C_{\alpha\beta}{}^\mu E_\mu$ where
$C_{ab}{}^c = C_{ab^*}{}^c = C_{a^*b}{}^c = C_{a^*b^*}{}^c = C_{a^*b^*}{}^{c^*} = 0,$
$C_{ab}{}^{c^*} = \frac{\delta N_a{}^{c^*}}{\delta u^b} - \frac{\delta N_b{}^{c^*}}{\delta u^a} \equiv F^{c^*}{}_{ab},$
$C_{ab^*}{}^{c^*} = \frac{\partial N_a{}^{c^*}}{\partial y^{b^*}} \equiv \phi^{c^*}{}_{ab^*},$
$C_{a^*b}{}^{c^*} = -\frac{\partial N_b{}^{c^*}}{\partial y^{a^*}} \equiv -\phi^{c^*}{}_{ba^*}.$
Levi-Civita connection on the tangent bundle:
$D(E_\beta) = \Gamma^\lambda{}_{\beta\alpha} W^\alpha \otimes E_\lambda,$
where the components of the connection coefficients are
$\Gamma^\lambda{}_{\alpha\beta} = \frac{1}{2} G^{\mu\lambda} \left[\frac{\delta G_{\mu\beta}}{\delta Y^\alpha} + \frac{\delta G_{\mu\alpha}}{\delta Y^\beta} - \frac{\delta G_{\alpha\beta}}{\delta Y^\mu} + C_{\mu\alpha\beta} + C_{\mu\beta\alpha} - C_{\alpha\beta\mu}\right].$
The components of the connection coefficients for different indices which belong to base and fiber sectors:
$\Gamma^a{}_{bc} = \bar{\bar{\gamma}}^a{}_{bc},$
$\Gamma^a{}_{bc^*} = \Gamma^a{}_{c^*b} = \frac{1}{2}\left(\bar{\gamma}^a{}_{c^*b} + \bar{\gamma}_{bc^*}{}^a + F_{c^*b}{}^a\right),$
$\Gamma^a{}_{b^*c^*} = -\frac{1}{2}\left(T_{b^*}{}^a{}_{c^*} + T_{c^*}{}^a{}_{b^*}\right),$
$\Gamma^a{}_{b^*c^*} = -\frac{1}{2}\left(T_{b^*}{}^a{}_{c^*} + T_{c^*}{}^a{}_{b^*}\right),$
$\Gamma^{a^*}{}_{bc} = \frac{1}{2}\left(F^{a^*}{}_{bc} - \bar{\gamma}_b{}^{a^*}{}_c - \bar{\gamma}_c{}^{a^*}{}_b\right),$
$\Gamma^{a^*}{}_{bc^*} = \frac{1}{2}\left(T^{a^*}{}_{bc^*} + T_{c^*b}{}^{a^*}\right),$
$\Gamma^{a^*}{}_{c^*b} = \bar{\bar{\gamma}}^{a^*}{}_{bc^*} + \frac{1}{2}\left(T_{c^*b}{}^{a^*} - T^{a^*}{}_{bc^*}\right),$
$\Gamma^{a^*}{}_{b^*c^*} = \bar{\bar{\gamma}}^{a^*}{}_{b^*c^*},$
where
$\bar{\bar{\gamma}}^\alpha{}_{\beta\gamma} = \frac{1}{2} G^{\lambda\alpha} \left(\frac{\delta G_{\lambda\beta}}{\delta Y^\gamma} + \frac{\delta G_{\lambda\gamma}}{\delta Y^\beta} - \frac{\delta G_{\beta\gamma}}{\delta Y^\lambda}\right),$
$\bar{\gamma}^a{}_{bc} = \frac{1}{2} g^{da} \left(\frac{\partial g_{db}}{\partial X^c} + \frac{\partial g_{dc}}{\partial X^b} - \frac{\partial g_{bc}}{\partial X^d}\right),$

$\gamma^a_{\ bc} = \frac{1}{2} g^{da} \left(\frac{\partial g_{db}}{\partial u^c} + \frac{\partial g_{dc}}{\partial u^b} - \frac{\partial g_{bc}}{\partial u^d} \right),$
$\phi^\alpha_{\ \beta\gamma} = \frac{\partial N_\beta^{\ \alpha}}{\partial Y^\gamma},$
$T^\alpha_{\ \beta\gamma} = \bar{\bar{\gamma}}^\alpha_{\ \beta\gamma} - \phi^\alpha_{\ \beta\gamma}.$
Components of the curvature tensor without the components of the anholonomic objects:
$\bar{R}_{\delta\ \beta\gamma}^{\ \alpha} = Y^\delta \left[\frac{\delta \bar{\bar{\gamma}}^\alpha_{\ \delta\gamma}}{\delta Y^\beta} - \frac{\delta \bar{\bar{\gamma}}^\alpha_{\ \delta\beta}}{\delta Y^\gamma} + \bar{\bar{\gamma}}^\alpha_{\ \lambda\beta} \bar{\bar{\gamma}}^\lambda_{\ \delta\gamma} - \bar{\bar{\gamma}}^\alpha_{\ \lambda\gamma} \bar{\bar{\gamma}}^\lambda_{\ \delta\beta} \right].$
The gauge curvature field:
$F^\alpha_{\ \beta\gamma} = Y^\delta \bar{R}_{\delta\ \beta\gamma}^{\ \alpha}.$
Components of curvature tensor of the tangent bundle:
$R_{\alpha\ \gamma\lambda}^{\ \beta} = \frac{\delta \Gamma^\beta_{\ \alpha\lambda}}{\delta Y^\gamma} - \frac{\delta \Gamma^\beta_{\ \alpha\gamma}}{\delta Y^\lambda} + \Gamma^\beta_{\ \delta\gamma} \Gamma^\delta_{\ \alpha\lambda} - \Gamma^\beta_{\ \delta\lambda} \Gamma^\delta_{\ \alpha\gamma} - \Gamma^\beta_{\ \alpha\delta} C_{\gamma\lambda}^{\ \ \delta}.$

Chapter 5

Structure of Spacetime Tangent Bundle

In this chapter make applications of the geometry of tangent bundle [19], [20], [21], [22]. We construct the spacetime tangent bundle by using spacetime and four-velocity space. We first define a metric on spacetime tangent bundle by using the proper acceleration a, and then we define connection and obtain the connection coefficients, curvature tensor, Ricci curvature and scalar curvature. Then we define an action on the spacetime tangent bundle and considering the Schwarzschild-like spacetime we obtain the Euler-Lagrange equations of motion coming from the action and obtain a Schwarzschild-like solution. Finally we obtain the modified red shift formula on the Schwarzschild-like spacetime.

5.1 Metric, Connection and Curvature of Spacetime Tangent Bundle

In Riemannian geometry, we have a positive definite quadratic differential two-form $ds^2 = g_{ij}du^i du^j$ and is called Riemannian metric. u^i are local coordinates and g_{ij} are symmetric smooth functions on Riemannian manifold. Here we obtain a positive definite quadratic differential form by using the proper acceleration a, of an object in curved spacetime, which is given by Einstein's theory of general relativity as

$$a^2 = -c^4 g_{ij} \frac{DX^i}{ds} \frac{DX^j}{ds}, \tag{5.1}$$

where c is velocity of light in vacuum, g_{ij} are components of spacetime metric tensor. u^i and $X^i = \frac{du^i}{ds}$, $(i = 0, 1, 2, 3)$ are spacetime coordinates of an object and four-velocity coordinates, respectively. Covariant derivative of X^i is

$$\frac{DX^i}{ds} = \frac{dX^i}{ds} + X^j \Gamma^i{}_{jk} \frac{du^k}{ds}, \tag{5.2}$$

where $\Gamma^i{}_{jk}$ are affine connection coefficients of spacetime. Let a_0 be the maximal proper acceleration. We can state the following condition

$$a^2 \leq {a_0}^2, \tag{5.3}$$

since the proper acceleration can not reach to the maximal acceleration. By using (5.1) and (5.3) we have

$$\begin{aligned}
-c^4 g_{ij} \frac{DX^i}{ds} \frac{DX^j}{ds} &\leq a_0^2 \\
-\left(\frac{c^2}{a_0}\right)^2 g_{ij} \frac{DX^i}{ds} \frac{DX^j}{ds} &\leq 1 \\
{\rho_0}^2 g_{ij} \frac{DX^i}{ds} \frac{DX^j}{ds} &\geq 1 \geq 0,
\end{aligned} \tag{5.4}$$

where $\rho_0 = \frac{c^2}{a_0}$. Spacetime and four-velocity spaces determine an $8-$dimensional tangent bundle with local coordinates $Y^I = (y^i, y^{i^*}) \equiv (u^i, \rho_0 X^i)$. First three components of spacetime are spatial coordinates and the fourth component is the time component. Let us define the positive definite quadratic differential form

$$d\sigma^2 \equiv G_{IJ} DY^I DY^J = g_{ij} du^i du^j + {\rho_0}^2 g_{ij} DX^i DX^j. \tag{5.5}$$

By using (5.2) and (5.4) we can write (5.5) as

$$\begin{aligned}
d\sigma^2 &= g_{ij} du^i du^j + {\rho_0}^2 g_{ij} \left(dX^i + X^k \Gamma^i{}_{kl} du^l\right) \left(dX^j + X^m \Gamma^j{}_{mn} du^n\right) \\
&= g_{ij} du^i du^j + {\rho_0}^2 g_{ij} dX^i \wedge dX^j + {\rho_0}^2 g_{ij} X^m \Gamma^j{}_{mn} dX^i \wedge du^n \\
&+ {\rho_0}^2 g_{ij} X^k \Gamma^i{}_{kl} du^l \wedge dX^j + {\rho_0}^2 g_{ij} X^k X^m \Gamma^i{}_{kl} \Gamma^j{}_{mn} du^l \wedge du^n \\
&= \left[g_{ij} + \rho_0^2 g_{ln} X^k \Gamma^l{}_{ki} X^m \Gamma^n{}_{mj}\right] du^i \wedge du^j + \rho_0^2 g_{nj} X^m \Gamma^n{}_{mi} du^i \wedge dX^j \\
&+ \rho_0^2 g_{in} X^m \Gamma^n{}_{mj} dX^i \wedge du^j + \rho_0^2 g_{ij} dX^i \wedge dX^j \geq 0.
\end{aligned} \tag{5.6}$$

Let us write the above expressions as

$$d\sigma^2 = G_{IJ} dY^I dY^J, \tag{5.7}$$

which is the line element with positive definite metric G_{IJ} on the spacetime tangent bundle in the natural coordinates. G_{IJ} has the form

$$G_{IJ} = \begin{pmatrix} g_{ij} + \rho_0^2 g_{ln} X^k \Gamma^l{}_{ki} X^m \Gamma^n{}_{mj} & \rho_0^2 g_{nj} X^m \Gamma^n{}_{mi} \\ \rho_0^2 g_{in} X^m \Gamma^n{}_{mj} & \rho_0^2 g_{ij} \end{pmatrix}, \tag{5.8}$$

or by using the fact

$$X^k \Gamma^l{}_{ki} = N_i{}^l, \tag{5.9}$$

and lower the indices by g_{ij} we can write G_{IJ} as

$$G_{IJ} = \begin{pmatrix} g_{ij} + \rho_0^2 g_{ln} N_i{}^l N_j{}^n & \rho_0^2 N_{ij} \\ \rho_0^2 N_{ji} & \rho_0^2 g_{ij} \end{pmatrix}. \tag{5.10}$$

Thus we get the metric on spacetime tangent bundle which is the same form of the previous chapter. Again as in previous chapter for the ease of calculations let us introduce the special frame as

$$E_\alpha{}^I = \left(B_a{}^I, C_{a^*}{}^I\right) \qquad \text{with inverse} \qquad E^\alpha{}_I = \left(B^a{}_I, C^{a^*}{}_I\right), \tag{5.11}$$

where

$$B_a{}^I = \left(\delta_a^i, -N_a{}^i\right), \qquad C_{a^*}{}^I = \left(0, \delta_a^i\right), \tag{5.12}$$

and

$$B^a{}_I = \left(\delta_i^a, 0\right), \qquad C^{a^*}{}_I = \left(N_i{}^a, \delta_i^a\right). \tag{5.13}$$

The bundle metric in the special frame has the form

$$G_{\alpha\beta} = G_{IJ} E_\alpha{}^I E_\beta{}^J, \tag{5.14}$$

or in matrix form as

$$G_{\alpha\beta} = \begin{pmatrix} g_{ab} & 0 \\ 0 & g_{a^*b^*} \end{pmatrix}. \tag{5.15}$$

The anholonomic basis have the form

$$E_\alpha = (E_a, E_{a^*}) = \left(\frac{\partial}{\partial u^a} - \rho_0^{-1} N_a{}^k \frac{\partial}{\partial X^k}, \rho_0^{-1} \frac{\partial}{\partial X^a}\right), \tag{5.16}$$

and its dual has the form

$$W^{\alpha} = \left(W^{a}, W^{a^*}\right) = \left(du^{a}, \rho_0 dX^{a} + N_k{}^{a} du^{k}\right). \tag{5.17}$$

This anholonomic basis also satisfy the duality condition

$$E_{\alpha}\left(W^{\beta}\right) = \delta^{\beta}_{\alpha}. \tag{5.18}$$

The line element of the spacetime tangent bundle in the anholonomic basis has the form

$$d\sigma^2 = G_{\alpha\beta} W^{\alpha} W^{\beta}. \tag{5.19}$$

We define the Levi-Civita connection on spacetime tangent bundle in the anholonomic basis as

$$DE_{\alpha} = \Gamma^{\lambda}{}_{\alpha\beta} W^{\beta} \otimes E_{\lambda},$$

where the connection coefficients have been found in previous chapter as

$$\Gamma^{\lambda}{}_{\alpha\beta} = \frac{1}{2} G^{\mu\lambda}\left[\frac{\delta G_{\mu\beta}}{\delta Y^{\alpha}} + \frac{\delta G_{\mu\alpha}}{\delta Y^{\beta}} - \frac{\delta G_{\alpha\beta}}{\delta Y^{\mu}} + C_{\mu\alpha\beta} + C_{\mu\beta\alpha} - C_{\alpha\beta\mu}\right], \tag{5.20}$$

where $\frac{\delta}{\delta Y^{\alpha}} = \left(\frac{\delta}{\delta u^{a}}, \rho_0^{-1}\frac{\partial}{\partial X^{a}}\right) = \left(\frac{\partial}{\partial u^{a}} - \rho_0^{-1} N_a{}^{b}\frac{\partial}{\partial X^{b}}, \rho_0^{-1}\frac{\partial}{\partial X^{a}}\right)$. Here $C_{\alpha\beta}{}^{\gamma}$ are components of the anholonomic objects and defined as $[E_{\alpha}, E_{\beta}] = C_{\alpha\beta}{}^{\mu} E_{\mu}$. In previous chapter, $C_{\alpha\beta}{}^{\mu}$ have been found for different indices as

$$\begin{aligned}
C_{ab}{}^{c} &= C_{a^*b}{}^{c} = C_{ab^*}{}^{c} = C_{a^*b^*}{}^{c} = C_{a^*b^*}{}^{c^*} = 0,\\
C_{ab}{}^{c^*} &= \frac{\delta N_a{}^{c^*}}{\delta u^{b}} - \frac{\delta N_b{}^{c^*}}{\delta u^{a}} \equiv F^{c^*}{}_{ab}\\
C_{ab^*}{}^{c^*} &= \frac{\partial N_a{}^{c^*}}{\partial y^{b^*}} \equiv \phi^{c^*}{}_{ab^*}\\
C_{a^*b}{}^{c^*} &= \frac{\partial N_b{}^{c^*}}{\partial y^{a^*}} \equiv -\phi^{c^*}{}_{ba^*}.
\end{aligned} \tag{5.21}$$

The connection coefficients Γ of the tangent bundle for different indices have the form

$$\Gamma^{a}{}_{bc} = \bar{\bar{\gamma}}^{a}{}_{bc}, \tag{5.22}$$

$$\Gamma^{a}{}_{bc^*} = \Gamma^{a}{}_{c^*b} = \frac{1}{2}\left(\bar{\gamma}^{a}{}_{c^*b} + \bar{\gamma}_{bc^*}{}^{a} + F_{c^*b}{}^{a}\right), \tag{5.23}$$

$$\Gamma^{a}{}_{b^*c^*} = -\frac{1}{2}\left(T_{b^*}{}^{a}{}_{c^*} + T_{c^*}{}^{a}{}_{b^*}\right), \tag{5.24}$$

$$\Gamma^{a^*}{}_{bc} = \frac{1}{2}\left(F^{a^*}{}_{bc} - \bar{\gamma}_b{}^{a^*}{}_c - \bar{\gamma}_c{}^{a^*}{}_b\right), \tag{5.25}$$

$$\Gamma^{a^*}{}_{bc^*} = \frac{1}{2}\left(T^{a^*}{}_{bc^*} + T_{c^*b}{}^{a^*}\right), \tag{5.26}$$

$$\Gamma^{a^*}{}_{c^*b} = \bar{\bar{\gamma}}^{a^*}{}_{bc^*} + \frac{1}{2}\left(T_{c^*b}{}^{a^*} - T^{a^*}{}_{bc^*}\right), \tag{5.27}$$

$$\Gamma^{a^*}{}_{b^*c^*} = \bar{\bar{\gamma}}^{a^*}{}_{b^*c^*}. \tag{5.28}$$

Here $\bar{\bar{\gamma}}^{\alpha}{}_{\beta\gamma}$, $\bar{\gamma}^{a}{}_{bc}$, $\gamma^{a}{}_{bc}$, $\phi^{\alpha}{}_{\beta\gamma}$, $T^{\alpha}{}_{\beta\gamma}$ have the forms as

$$\bar{\bar{\gamma}}^{\alpha}{}_{\beta\gamma} = \frac{1}{2}G^{\lambda\alpha}\left(\frac{\delta G_{\lambda\beta}}{\delta Y^{\gamma}} + \frac{\delta G_{\lambda\gamma}}{\delta Y^{\beta}} - \frac{\delta G_{\beta\gamma}}{\delta Y^{\lambda}}\right), \tag{5.29}$$

$$\bar{\gamma}^{a}{}_{bc} = \frac{1}{2}g^{da}\left(\frac{\partial g_{db}}{\partial X^{c}} + \frac{\partial g_{dc}}{\partial X^{b}} - \frac{\partial g_{bc}}{\partial X^{d}}\right), \tag{5.30}$$

$$\gamma^{a}{}_{bc} = \frac{1}{2}g^{da}\left(\frac{\partial g_{db}}{\partial u^{c}} + \frac{\partial g_{dc}}{\partial u^{b}} - \frac{\partial g_{bc}}{\partial u^{d}}\right), \tag{5.31}$$

$$\phi^{\alpha}{}_{\beta\gamma} = \frac{\partial N_{\beta}{}^{\alpha}}{\partial Y^{\gamma}}, \tag{5.32}$$

$$T^{\alpha}{}_{\beta\gamma} = \bar{\bar{\gamma}}^{\alpha}{}_{\beta\gamma} - \phi^{\alpha}{}_{\beta\gamma}. \tag{5.33}$$

The gauge curvature field $F^{\alpha}{}_{\beta\gamma}$ has the form

$$F^{\alpha}{}_{\beta\gamma} = Y^{\delta}\bar{R}_{\delta}{}^{\alpha}{}_{\beta\gamma}, \tag{5.34}$$

where

$$\bar{R}_{\delta}{}^{\alpha}{}_{\beta\gamma} = Y^{\delta}\left[\frac{\delta\bar{\bar{\gamma}}^{\alpha}{}_{\delta\gamma}}{\delta Y^{\beta}} - \frac{\delta\bar{\bar{\gamma}}^{\alpha}{}_{\delta\beta}}{\delta Y^{\gamma}} + \bar{\bar{\gamma}}^{\alpha}{}_{\lambda\beta}\bar{\bar{\gamma}}^{\lambda}{}_{\delta\gamma} - \bar{\bar{\gamma}}^{\alpha}{}_{\lambda\gamma}\bar{\bar{\gamma}}^{\lambda}{}_{\delta\beta}\right]. \tag{5.35}$$

Curvature tensor R of the bundle manifold in the anholonomic basis has the form

$$R_{\alpha}{}^{\beta}{}_{\gamma\lambda} = \frac{\delta\Gamma^{\beta}{}_{\alpha\lambda}}{\delta Y^{\gamma}} - \frac{\delta\Gamma^{\beta}{}_{\alpha\gamma}}{\delta Y^{\lambda}} + \Gamma^{\beta}{}_{\delta\gamma}\Gamma^{\delta}{}_{\alpha\lambda} - \Gamma^{\beta}{}_{\delta\lambda}\Gamma^{\delta}{}_{\alpha\gamma} - \Gamma^{\beta}{}_{\alpha\delta}C_{\gamma\lambda}{}^{\delta}. \tag{5.36}$$

5.2 Ricci Curvature And Scalar Curvature of Bundle Manifold

The Ricci and scalar curvature tensor of the tangent bundle have the form

$$R_{\alpha\lambda} = R_{\alpha}{}^{\beta}{}_{\beta\lambda}, \tag{5.37}$$

$$R = R_{\alpha\lambda}G^{\alpha\lambda} = R_{ac}g^{ac} + R_{a^*c^*}g^{a^*c^*}. \tag{5.38}$$

We obtain the scalar curvature tensor on tangent bundle in terms of Γ's. But firstly we have to obtain Ricci vurvature. The Ricci curvature tensor of spacetime is

$$R_{ac} = R_{a\ \beta c}^{\ \beta} = R_{a\ bc}^{\ b} + R_{a\ b^*c}^{\ b^*}. \tag{5.39}$$

By using (5.36) we can write (5.39) as

$$\begin{aligned} R_{ac} &= \frac{\delta\Gamma^b{}_{ac}}{\delta Y^b} - \frac{\delta\Gamma^b{}_{ab}}{\delta Y^c} + \Gamma^b{}_{db}\Gamma^d{}_{ac} + \Gamma^b{}_{d^*b}\Gamma^{d^*}{}_{ac} \\ &- \Gamma^b{}_{dc}\Gamma^d{}_{ab} - \Gamma^b{}_{d^*c}\Gamma^{d^*}{}_{ab} - \Gamma^b{}_{db}C_{bc}{}^d - \Gamma^b{}_{ad^*}C_{bc}{}^{d^*} \\ &+ \frac{\delta\Gamma^{b^*}{}_{ac}}{\delta Y^{b^*}} - \frac{\delta\Gamma^{b^*}{}_{ab^*}}{\delta Y^c} + \Gamma^{b^*}{}_{db^*}\Gamma^d{}_{ac} + \Gamma^{b^*}{}_{d^*b^*}\Gamma^{d^*}{}_{ac} \\ &- \Gamma^{b^*}{}_{dc}\Gamma^d{}_{ab^*} - \Gamma^{b^*}{}_{d^*c}\Gamma^{d^*}{}_{ab^*} - \Gamma^{b^*}{}_{db^*}C_{b^*c}{}^d - \Gamma^{b^*}{}_{ad^*}C_{b^*c}{}^{d^*}. \end{aligned} \tag{5.40}$$

By contracting (5.40) with g^{ac} and using (5.21), formulas (5.22)-(5.28) for Γ's, (5.35) for $\bar{R}_{\delta\ \beta\gamma}^{\ \alpha}$ and (5.33) for $T^{\alpha}_{\beta\gamma}$ we get

$$\begin{aligned} R_{ac}g^{ac} &= g^{ac}\left[\frac{\delta\bar{\bar{\gamma}}^b{}_{ac}}{\delta Y^b} - \frac{\delta\bar{\bar{\gamma}}^b{}_{ab}}{\delta Y^c} + \bar{\bar{\gamma}}^b{}_{db}\bar{\bar{\gamma}}^d{}_{ac} - \bar{\bar{\gamma}}^b{}_{dc}\bar{\bar{\gamma}}^d{}_{ab}\right] \\ &+ \frac{g^{ac}}{4}\left(F_{d^*b}{}^b + \bar{\gamma}^b{}_{d^*b} + \bar{\gamma}_{bd^*}{}^b\right)\left(F^{d^*}{}_{ac} - \bar{\gamma}_a{}^{d^*}{}_c - \bar{\gamma}_c{}^{d^*}{}_a\right) \\ &- \frac{g^{ac}}{4}\left(F_{d^*c}{}^b + \bar{\gamma}^b{}_{d^*c} + \bar{\gamma}_{cd^*}{}^b\right)\left(F^{d^*}{}_{ab} - \bar{\gamma}_a{}^{d^*}{}_b - \bar{\gamma}_b{}^{d^*}{}_a\right) \\ &- \frac{g^{ac}}{2}\left(F_{d^*a}{}^b + \bar{\gamma}^b{}_{d^*a} + \bar{\gamma}_{ad^*}{}^b\right)\left(F^{d^*}{}_{bc}\right) \\ &+ \frac{g^{ac}}{2}\frac{\partial}{\partial y^{b^*}}\left(F^{b^*}{}_{ac} - \bar{\gamma}_a{}^{b^*}{}_c - \bar{\gamma}_c{}^{b^*}{}_a\right) - \frac{g^{ac}}{2}\frac{\delta}{\delta u^c}\left(T^{b^*}{}_{ab^*} + T_{b^*a}{}^{b^*}\right) \\ &+ \frac{g^{ac}}{2}\left(T^{b^*}{}_{db^*} + T_{b^*d}{}^{b^*}\right)\bar{\bar{\gamma}}^d{}_{ac} + \frac{g^{ac}}{2}\bar{\bar{\gamma}}^{a^*}{}_{b^*c^*}\left(F^{d^*}{}_{ac} - \bar{\gamma}_a{}^{d^*}{}_c - \bar{\gamma}_c{}^{d^*}{}_a\right) \\ &- \frac{g^{ac}}{4}\left(F^{b^*}{}_{dc} - \bar{\gamma}_d{}^{b^*}{}_c - \bar{\gamma}_c{}^{b^*}{}_d\right)\left(F_{b^*a}{}^d + \bar{\gamma}^d{}_{b^*a} + \bar{\gamma}_{ab^*}{}^d\right) \\ &- \frac{g^{ac}}{2}\left[\bar{\bar{\gamma}}^{b^*}{}_{cd^*} + \frac{1}{2}\left(T_{d^*c}{}^{b^*} - T^{b^*}{}_{cd^*}\right)\right]\left(T^{d^*}{}_{ab^*} + T_{b^*a}{}^{d^*}\right) \\ &+ \frac{g^{ac}}{2}\left(T^{b^*}{}_{ad^*} + T_{d^*a}{}^{b^*}\right)\left(T^{d^*}{}_{cb^*} - \bar{\bar{\gamma}}^{d^*}{}_{cb^*}\right). \end{aligned} \tag{5.41}$$

In the above equation expanding the parenthesis, arranging the indices and terms, using the skew-symmetry of $F^{\alpha}_{\beta\gamma}$ in the lower indices and after complicated simplification we get

$$R_{ac}g^{ac} = g^{ac}\bar{R}_{ac} - \frac{1}{2}F^{bdc}F_{bdc} - 2\bar{\gamma}^{bd^*}{}_b\bar{\gamma}^c{}_{d^*c}$$

$$\begin{aligned}
&+ \quad \frac{1}{2}\left[\bar{\gamma}^{cd^*b}\bar{\gamma}_{cd^*b} + \bar{\gamma}^{bd^*c}\bar{\gamma}_{cd^*b} + \bar{\gamma}^{cd^*b}\bar{\gamma}_{bd^*c} + \bar{\gamma}^{bd^*c}\bar{\gamma}_{bd^*c}\right] \\
&- \quad \frac{1}{4}\left[T^{d^*cb^*}T_{d^*cb^*} + T^{b^*cd^*}T_{d^*cb^*} + T^{d^*cb^*}T_{b^*cd^*} + T^{b^*cd^*}T_{b^*cd^*}\right] \\
&+ \quad T^{b^*}{}_{db^*}\bar{\bar{\gamma}}_a^{da} - g^{ac}\frac{\delta T^{b^*}{}_{ab^*}}{\delta u^c} - g^{ac}\frac{\partial \bar{\gamma}_a{}^{b^*}{}_c}{\partial y^{b^*}}. \qquad (5.42)
\end{aligned}$$

We obtain some useful facts to write the above expression in a more compact form. Consider

$$\bar{\gamma}_{d^*a^*c^*} + \bar{\gamma}_{c^*a^*d^*} = \frac{\partial g_{d^*c^*}}{\partial y^{a^*}}, \qquad (5.43)$$

Contracting that equation with $g^{c^*a^*}$ and using $g^{ac}g_{cd} = \delta^a_d$ we get

$$\begin{aligned}
\bar{\gamma}_{d^*a^*}{}^{a^*} + \bar{\gamma}^{a^*}{}_{a^*d^*} &= g^{c^*a^*}\frac{\partial g_{d^*c^*}}{\partial y^{a^*}} = \frac{\partial \delta^{a^*}_{d^*}}{\partial y^{a^*}} - g_{d^*c^*}\frac{\partial g^{c^*a^*}}{\partial y^{a^*}} \\
&= -g_{d^*c^*}\frac{\partial g^{c^*a^*}}{\partial y^{a^*}}. \qquad (5.44)
\end{aligned}$$

Contracting that expression with $g^{d^*b^*}$ we get

$$\bar{\gamma}^{b^*}{}_{a^*}{}^{a^*} + \bar{\gamma}^{a^*}{}_{a^*}{}^{b^*} = -\frac{\partial g^{a^*b^*}}{\partial y^{a^*}}. \qquad (5.45)$$

Consider

$$\bar{\gamma}_a{}^{b^*}{}_c = \frac{g^{b^*e^*}}{2}\frac{\partial g_{ac}}{\partial y^{e^*}}. \qquad (5.46)$$

By taking the partial derivative of the above expression with respect to y^{b^*}, contracting the resulting equation with g^{ac}, using $\bar{\gamma}_{da^*c} + \bar{\gamma}_{ca^*d} = \frac{\partial g_{dc}}{\partial y^{a^*}}$ and (5.44) we get

$$\begin{aligned}
& g^{ac}\frac{\partial \bar{\gamma}_a{}^{b^*}{}_c}{\partial y^{b^*}} = \frac{g^{ac}}{2}\frac{\partial g^{b^*e^*}}{\partial y^{b^*}}\frac{g_{ac}}{\partial y^{e^*}} + \frac{g^{ac}g^{b^*e^*}}{2}\frac{\partial}{y^{b^*}}\left(\frac{\partial g_{ac}}{\partial y^{e^*}}\right) \\
= \quad & -\frac{g^{ac}}{2}\left(\bar{\gamma}^{e^*}{}_{b^*}{}^{b^*} + \bar{\gamma}^{b^*}{}_{b^*}{}^{e^*}\right)\left(\bar{\gamma}_{ae^*c} + \bar{\gamma}_{ce^*a}\right) + \frac{g^{ac}g^{b^*e^*}}{2}\frac{\partial\left(\bar{\gamma}_{ae^*c} + \bar{\gamma}_{ce^*a}\right)}{\partial y^{b^*}} \\
= \quad & -\frac{1}{2}\left[g^{ac}\bar{\gamma}^{e^*}{}_{b^*}{}^{b^*}\bar{\gamma}_{ae^*c} + g^{ac}\bar{\gamma}^{e^*}{}_{b^*}{}^{b^*}\bar{\gamma}_{ce^*a} + g^{ac}\bar{\gamma}^{b^*}{}_{b^*}{}^{e^*}\bar{\gamma}_{ae^*c} + g^{ac}\bar{\gamma}^{b^*}{}_{b^*}{}^{e^*}\bar{\gamma}_{ce^*a}\right] \\
+ \quad & \frac{g^{b^*e^*}}{2}\left[\frac{\partial\left(g^{ac}\bar{\gamma}_{ae^*c}\right)}{\partial y^{b^*}} - \bar{\gamma}_{ae^*c}\frac{\partial g^{ac}}{\partial y^{b^*}} + \frac{\partial\left(g^{ac}\bar{\gamma}_{ce^*a}\right)}{\partial y^{b^*}} - \bar{\gamma}_{ce^*a}\frac{\partial g^{ac}}{\partial y^{b^*}}\right]. \qquad (5.47)
\end{aligned}$$

By arranging the indices and terms and using the similar fact of (5.45) for $\frac{\partial g^{ac}}{\partial y^{b^*}}$ we can write the above expression as

$$\begin{aligned}
g^{ac}\frac{\partial \bar{\gamma}_a{}^{b^*}{}_c}{\partial y^{b^*}} &= \frac{g^{e^*b^*}}{2}\frac{\partial \bar{\gamma}^c{}_{e^*c}}{\partial y^{b^*}} - \bar{\gamma}^{e^*b^*}{}_{b^*}\bar{\gamma}^c{}_{e^*c} - \bar{\gamma}^{b^*e^*}{}_{b^*}\bar{\gamma}^c{}_{e^*c} \\
&+ \quad \frac{1}{2}\left[\bar{\gamma}_{ae^*c}\bar{\gamma}^{ae^*c} + \bar{\gamma}_{ae^*c}\bar{\gamma}^{ce^*a} + \bar{\gamma}_{ce^*a}\bar{\gamma}^{ae^*c} + \bar{\gamma}_{ce^*a}\bar{\gamma}^{ce^*a}\right]. \qquad (5.48)
\end{aligned}$$

Since $\bar{\gamma}_{ae^*c} = \frac{1}{2}\frac{\partial g_{ac}}{\partial y^{e^*}}$ or we can write it as

$$\bar{\gamma}_{aec} = \frac{1}{2}\frac{\partial g_{ac}}{\partial X^e}. \tag{5.49}$$

By using that fact we can write (5.48) as

$$\begin{aligned} g^{ac}\frac{\partial \bar{\gamma}_a{}^{b^*}{}_c}{\partial y^{b^*}} &= \frac{g^{e^*b^*}}{2}\frac{\partial \bar{\gamma}^c{}_{e^*c}}{\partial y^{b^*}} - \bar{\gamma}^{eb^*}{}_{b^*}\bar{\gamma}^c{}_{ec} - \bar{\gamma}^{b^*e}{}_{b^*}\bar{\gamma}^c{}_{ec} \\ &+ \frac{1}{2}\left[\bar{\gamma}_{aec}\bar{\gamma}^{aec} + \bar{\gamma}_{aec}\bar{\gamma}^{cea} + \bar{\gamma}_{cea}\bar{\gamma}^{aec} + \bar{\gamma}_{cea}\bar{\gamma}^{cea}\right]. \end{aligned} \tag{5.50}$$

Substituting that expression into (5.42), making simplification in the resulting equation, arranging terms and indices and using the fact (5.49) we get

$$\begin{aligned} R_{ac}g^{ac} &= {}^s\bar{R} - \frac{1}{2}F^{bdc}F_{bdc} - \bar{\gamma}^{be}{}_b\bar{\gamma}^c{}_{ec} \\ &- \frac{1}{4}\left[T^{bdc}T_{bdc} + T^{cdb}T_{bdc} + T^{bdc}T_{cdb} + T^{cdb}T_{cdb}\right] \\ &- \left[g^{e^*b^*}\frac{\partial \bar{\gamma}^c{}_{e^*c}}{\partial y^{b^*}} - \bar{\gamma}^c{}_{ec}\bar{\gamma}^{be^*}{}_{b^*}\right] - \left[g^{ac}\frac{\delta T^b{}_{ab}}{\delta u^c} - \bar{\bar{\gamma}}^{da}{}_a T^b{}_{db}\right]. \end{aligned} \tag{5.51}$$

Let us introduce

$$\frac{\bar{D}T^b{}_{ab}}{Du^b} \equiv \frac{\delta T^b{}_{ab}}{\delta u^b} - \bar{\bar{\gamma}}^c{}_{ab}T^b{}_{cb}, \tag{5.52}$$

and

$$\frac{D\bar{\gamma}^c{}_{e^*c}}{Dy^{b^*}} \equiv \frac{\partial \bar{\gamma}^c{}_{e^*c}}{\partial y^{b^*}} - \bar{\gamma}^d{}_{e^*b^*}\bar{\gamma}^c{}_{dc}. \tag{5.53}$$

We can write the last two terms of (5.51) in terms of these new notations. Consider

$$\begin{aligned} g^{e^*b^*}\frac{\partial \bar{\gamma}^c{}_{e^*c}}{\partial y^{b^*}} - \bar{\gamma}^c{}_{ec}\bar{\gamma}^{be^*}{}_{b^*} &= g^{e^*b^*}\left[\frac{\partial \bar{\gamma}^c{}_{e^*c}}{\partial y^{b^*}} - g_{e^*b^*}\bar{\gamma}^c{}_{ec}\bar{\gamma}^{be^*}{}_{b^*}\right] \\ &= g^{e^*b^*}\left[\frac{\partial \bar{\gamma}^c{}_{e^*c}}{\partial y^{b^*}} - \bar{\gamma}^c{}_{ec}\bar{\gamma}^e{}_{e^*b^*}\right] \\ &= g^{e^*b^*}\frac{D\bar{\gamma}^c{}_{e^*c}}{Dy^{b^*}} \equiv \frac{D\bar{\gamma}^c{}_{e^*c}}{Dy_{e^*}}. \end{aligned} \tag{5.54}$$

Here we used the fact

$$\begin{aligned} \bar{\gamma}^{ce}{}_c\bar{\gamma}^{b^*}{}_{eb^*} &= \bar{\gamma}^{ce}{}_c\bar{\gamma}_{f^*eb^*}g^{f^*b^*} = \bar{\gamma}^{ce}{}_c\bar{\gamma}_{ef^*b^*}g^{f^*b^*} \\ &= \bar{\gamma}^c{}_{ec}\bar{\gamma}^{eb^*}{}_{b^*}, \end{aligned} \tag{5.55}$$

and $g_{e^*b^*}\bar{\gamma}^{ce}{}_c\bar{\gamma}^{b^*}{}_{eb^*} = g_{e^*b^*}\bar{\gamma}^c{}_{ec}\bar{\gamma}^{eb^*}{}_{b^*} = \bar{\gamma}^c{}_{ec}\bar{\gamma}^e{}_{e^*b^*}$. Consider

$$\begin{aligned} g^{ac}\frac{\delta T^b{}_{ab}}{\delta u^c} - \bar{\gamma}^{da}{}_a T^b{}_{db} &= g^{ac}\left[\frac{\delta T^b{}_{ab}}{\delta u^c} - g_{ac}\bar{\bar{\gamma}}^{da}{}_a T^b{}_{db}\right] \\ &= g^{ac}\left[\frac{\delta T^b{}_{ab}}{\delta u^c} - \bar{\bar{\gamma}}^d{}_{ca} T^b{}_{db}\right] \\ &= g^{ac}\frac{\bar{D}T^b{}_{ab}}{Du^c} \equiv \frac{\bar{D}T^b{}_{ab}}{Du_a}. \end{aligned} \tag{5.56}$$

Inserting (5.54) and (5.56) into (5.51) we get the first term of the scalar curvature tensor of the tangent bundle as

$$\begin{aligned} R_{ac}g^{ac} &= {}^s\bar{R} - \frac{1}{2}F^{cdb}F_{cdb} - \bar{\gamma}^{be}{}_b\bar{\gamma}^c{}_{ec} \\ &- \frac{1}{4}\left[T^{cdb}T_{cdb} + T^{bdc}T_{cdb} + T^{cdb}T_{bdc} + T^{bdc}T_{bdc}\right] \\ &- \frac{D\bar{\gamma}^c{}_{ec}}{D\rho_0 X_e} - \frac{\bar{D}T^b{}_{ab}}{Du_a}, \end{aligned} \tag{5.57}$$

where ${}^s\bar{R}$ is the spacetime sector of $\bar{R}$. Now we obtain the second term of the scalar curvature of tangent bundle. Let us write Ricci curvature of the fiber as

$$R_{a^*c^*} = R_{a^*}{}^{\beta}{}_{\beta c^*} = R_{a^*}{}^{b}{}_{bc^*} + R_{a^*}{}^{b^*}{}_{b^*c^*}. \tag{5.58}$$

By using (5.36) we can write (5.58) as

$$\begin{aligned} R_{a^*c^*} &= \frac{\delta\Gamma^b{}_{a^*c^*}}{\delta Y^b} - \frac{\delta\Gamma^b{}_{a^*b}}{\delta Y^{c^*}} + \Gamma^b{}_{db}\Gamma^d{}_{a^*c^*} + \Gamma^b{}_{d^*b}\Gamma^{d^*}{}_{a^*c^*} \\ &- \Gamma^b{}_{dc^*}\Gamma^d{}_{a^*b} - \Gamma^b{}_{d^*c^*}\Gamma^{d^*}{}_{a^*b} - \Gamma^b{}_{db}C_{bc^*}{}^d - \Gamma^b{}_{a^*d^*}C_{bc^*}{}^{d^*} \\ &+ \frac{\delta\Gamma^{b^*}{}_{a^*c^*}}{\delta Y^{b^*}} - \frac{\delta\Gamma^{b^*}{}_{a^*b^*}}{\delta Y^{c^*}} + \Gamma^{b^*}{}_{db^*}\Gamma^d{}_{a^*c^*} + \Gamma^{b^*}{}_{d^*b^*}\Gamma^{d^*}{}_{a^*c^*} \\ &- \Gamma^{b^*}{}_{dc^*}\Gamma^d{}_{a^*b^*} - \Gamma^{b^*}{}_{d^*c^*}\Gamma^{d^*}{}_{a^*b^*} - \Gamma^{b^*}{}_{db^*}C_{b^*c^*}{}^d - \Gamma^{b^*}{}_{a^*d^*}C_{b^*c^*}{}^{d^*}. \end{aligned} \tag{5.59}$$

By contracting (5.59) with $g^{a^*c^*}$ and using (5.21), formulas (5.22)-(5.28) for Γ's, (5.35) for $\bar{R}_\delta{}^\alpha{}_{\beta\gamma}$ and (5.33) for $T^\alpha{}_{\beta\gamma}$ we get

$$\begin{aligned} R_{a^*c^*}g^{a^*c^*} &= g^{a^*c^*}\left[\frac{\delta\bar{\bar{\gamma}}^{b^*}{}_{a^*c^*}}{\delta Y^{b^*}} - \frac{\delta\bar{\bar{\gamma}}^{b^*}{}_{a^*b^*}}{\delta Y^{c^*}} + \bar{\bar{\gamma}}^{b^*}{}_{d^*b^*}\bar{\bar{\gamma}}^{d^*}{}_{a^*c^*} - \bar{\bar{\gamma}}^{b^*}{}_{d^*c^*}\bar{\bar{\gamma}}^{d^*}{}_{a^*b^*}\right] \\ &- \frac{g^{a^*c^*}}{4}\left(T^{b^*}{}_{db^*} + T_{b^*d}{}^{b^*}\right)\left(T_{a^*}{}^{d}{}_{c^*} + T_{c^*}{}^{d}{}_{a^*}\right) \\ &+ \frac{g^{a^*c^*}}{4}\left(T^{b^*}{}_{dc^*} + T_{c^*d}{}^{b^*}\right)\left(T_{a^*}{}^{d}{}_{b^*} + T_{b^*}{}^{d}{}_{a^*}\right) \end{aligned}$$

$$
\begin{aligned}
&- \frac{g^{a^*c^*}}{2}\frac{\delta}{\delta u^b}\left(T_{a^*}{}^b{}_{c^*} + T_{c^*}{}^b{}_{a^*}\right)\\
&- \frac{g^{a^*c^*}}{2}\frac{\delta}{\delta Y^{c^*}}\left(F_{a^*b}{}^b + \bar{\gamma}^b{}_{a^*b} + \bar{\gamma}_{ba^*}{}^b\right)\\
&- \frac{g^{a^*c^*}}{2}\bar{\bar{\gamma}}^b{}_{db}\left(T_{a^*}{}^d{}_{c^*} + T_{c^*}{}^d{}_{a^*}\right)\\
&+ \frac{g^{a^*c^*}}{2}\left(F_{d^*b}{}^b + \bar{\gamma}^b{}_{d^*b} + \bar{\gamma}_{bd^*}{}^b\right)\bar{\bar{\gamma}}^{d^*}{}_{a^*c^*}\\
&- \frac{g^{a^*c^*}}{4}\left(F_{c^*d}{}^b + \bar{\gamma}^b{}_{c^*d} + \bar{\gamma}_{dc^*}{}^b\right)\left(F_{a^*b}{}^d + \bar{\gamma}^d{}_{a^*b} + \bar{\gamma}_{ba^*}{}^d\right)\\
&+ \frac{g^{a^*c^*}}{2}\left[\bar{\bar{\gamma}}^{d^*}{}_{ba^*} + \frac{1}{2}\left(T_{a^*b}{}^{d^*} - T^{d^*}{}_{ba^*}\right)\right]\left(T_{d^*}{}^b{}_{c^*} + T_{c^*}{}^b{}_{d^*}\right)\\
&+ \frac{g^{a^*c^*}}{2}\left(T_{a^*}{}^b{}_{d^*} + T_{d^*}{}^b{}_{a^*}\right)\left(T^{d^*}{}_{bc^*} - \bar{\bar{\gamma}}^{d^*}{}_{bc^*}\right). \qquad (5.60)
\end{aligned}
$$

In the above expression expanding the parenthesis, arranging the indices and terms, using skew-symmetry of $F^\alpha_{\beta\gamma}$ in lower indices, $\bar{\bar{\gamma}}^{a^*}{}_{b^*c^*} = \bar{\gamma}^{a^*}{}_{b^*c^*}$, (5.49), (5.54) and (5.56) we get

$$
\begin{aligned}
R_{a^*c^*}g^{a^*c^*} &= g^{a^*c^*}\bar{R}_{a^*c^*} + \frac{1}{4}F^{abd}F_{abd}\\
&- \frac{1}{4}\left[\bar{\gamma}_{abd}\bar{\gamma}^{abd} + \bar{\gamma}_{dba}\bar{\gamma}^{abd} + \bar{\gamma}_{abd}\bar{\gamma}^{dba} + \bar{\gamma}_{dba}\bar{\gamma}^{dba}\right]\\
&- T^{bd}{}_bT^a{}_{da} - g^{a^*c^*}\left(\frac{\delta\bar{\gamma}^b{}_{a^*b}}{\delta Y^{c^*}} - g_{a^*c^*}\bar{\gamma}^{bd^*}{}_b\bar{\gamma}_{d^*}{}^{a^*}{}_{a^*}\right)\\
&- g^{db}\left(\frac{\delta T^{a^*}{}_{da^*}}{\delta u^b} - g_{db}\bar{\gamma}^{bd}{}_bT^{a^*}{}_{da^*}\right)\\
&= {}^v\bar{R} + \frac{1}{4}F^{abd}F_{abd} - \frac{D\bar{\gamma}^b{}_{ab}}{D\rho_0X_a} - \frac{\bar{D}T^a{}_{da}}{Du_d} - T^{bd}{}_bT^a{}_{da}\\
&- \frac{1}{4}\left[\bar{\gamma}_{abd}\bar{\gamma}^{abd} + \bar{\gamma}_{dba}\bar{\gamma}^{abd} + \bar{\gamma}_{abd}\bar{\gamma}^{dba} + \bar{\gamma}_{dba}\bar{\gamma}^{dba}\right], \qquad (5.61)
\end{aligned}
$$

where ${}^v\bar{R}$ is the four-velocity sector of $\bar{R}$. Writing the expressions (5.57) and (5.61) into (5.38) and arranging the indices we get the scalar curvature of bundle manifold as

$$
\begin{aligned}
R &= {}^s\bar{R} + {}^v\bar{R} - \frac{1}{4}F^{abc}F_{abc} - T^{ab}{}_aT^c{}_{bc} - 2\frac{\bar{D}T^a{}_{ba}}{Du_b}\\
&- \frac{1}{4}\left[T^{abc}T_{abc} + T^{cba}T_{abc} + T^{abc}T_{cba} + T^{cba}T_{cba}\right]\\
&- \frac{1}{4}\left[\bar{\gamma}^{abc}\bar{\gamma}_{abc} + \bar{\gamma}^{cba}\bar{\gamma}_{abc} + \bar{\gamma}^{abc}\bar{\gamma}_{cba} + \bar{\gamma}^{cba}\bar{\gamma}_{cba}\right]\\
&- \bar{\gamma}^{ab}{}_a\bar{\gamma}^c{}_{bc} - 2\frac{D\bar{\gamma}^a{}_{ba}}{D\rho_0X_e}. \qquad (5.62)
\end{aligned}
$$

5.3 Scalar Curvature of Riemannian Spacetime Tangent Bundle

In the Riemannian case, spacetime metric g_{ab} is independent of X^c.i.e.

$$\frac{\partial g_{ab}}{\partial X^c} = 0. \tag{5.63}$$

Consequence of that

$$\bar{\bar{\gamma}}^a{}_{bc} = \gamma^a{}_{bc}, \tag{5.64}$$

$$\bar{\gamma}^a{}_{bc} = 0, \tag{5.65}$$

and

$${}^s\bar{R} = g^{ac}\left(\frac{\partial \gamma^b{}_{ac}}{\partial u^b} - \frac{\partial \gamma^b{}_{ab}}{\partial u^c} + \gamma^b{}_{db}\gamma^d{}_{ac} - \gamma^b{}_{dc}\gamma^d{}_{ab}\right) \equiv {}^sR, \tag{5.66}$$

which is the ordinary spacetime scalar curvature. By using (5.63), (5.64) and the formula for ${}^vR$, we can say ${}^vR$ vanishes i.e. ${}^vR = 0$. By using (5.34) and the fact $\bar{R}_a{}^b{}_{cd} = {}^sR_a{}^b{}_{cd}$ we get

$$F^b{}_{cd} = \rho_0(X^a)^s R_a{}^b{}_{cd}, \tag{5.67}$$

where

$${}^sR_a{}^b{}_{cd} = \frac{\partial \gamma^b{}_{ad}}{\partial u^c} - \frac{\partial \gamma^b{}_{ac}}{\partial u^d} + \gamma^b{}_{ec}\gamma^e{}_{ad} - \gamma^b{}_{ed}\gamma^e{}_{ac}, \tag{5.68}$$

which is the spacetime curvature. By using (5.9), (5.32), (5.63) and (5.64) we get

$$T^a{}_{bc} = 0. \tag{5.69}$$

In Riemannian spacetime tangent bundle, by using the above facts (5.62) takes the form

$$R = {}^sR - \frac{1}{4}F^{abc}F_{abc}, \tag{5.70}$$

which is the Riemannian scalar curvature of the Riemannian spacetime tangent bundle.

5.4 Action for the Spacetime Tangent Bundle

We can define an action with Lagrangian $\sqrt{G}R$ on maximal-acceleration invariant spacetime tangent bundle as

$$\mathcal{L} = \int D^8Y\sqrt{G}R = \int d^4u\sqrt{-g}\rho_0^4 d^4X\sqrt{-g}R, \tag{5.71}$$

where G is the determinant of the bundle metric (5.15), $D^8Y\sqrt{G}$ is the volume measure of the spacetime tangent bundle and R is the scalar curvature of the bundle manifold. In the case of Finslerian spacetime manifold, the components of the spacetime metric $g_{ab}(u, X)$ are homogeneous in the fiber coordinates X^c with degree zero. We can write $g_{ab}(u, X)$ in terms of the Finsler function F as

$$g_{ab}(u, X) = \left(\frac{1}{2}F^2(u, X)\right)_{X^aX^b}. \tag{5.72}$$

If spacetime manifold is Finslerian and the spacetime connection is the Levi-Civita connection then the connection coefficients which are given (5.22)-(5.28), are Levi-Civita connection coefficients of the tangent bundle of the Finsler manifold. To have the consistency of the connection coefficients with Cartan's theory of Finsler space, the gauge curvature field $F^a_{\ bc}$ must be vanish [18].
In fact, the maximal acceleration fibre bundle is defined on a general spacetime manifold. A Finslerian spacetime geometry and Riemannian spacetime tangent bundle are special spacetime manifolds. But needed general spacetime manifold did not find yet. Here, we consider one of the special case, the Riemannian spacetime manifold. We consider Schwarzschild-like solution for the Riemannian spacetime metric. Spacetime manifold is Riemannian so we have

$$\frac{\partial g_{ab}}{\partial X^c} = 0, \tag{5.73}$$

which leads to

$$\Gamma^a_{\ bc} = \gamma^a_{\ bc}. \tag{5.74}$$

Let us consider a body with mass m which is stationary and spatially symmetric gravitating and an observer on the four-velocity space X defined by

$$X^a = \left(\frac{du^0}{ds}, \frac{du^{\bar{k}}}{ds}\right) = \left(g_{00}^{-\frac{1}{2}}, 0, 0, 0\right), \qquad \bar{k} = 1, 2, 3 \ a = 0, 1, 2, 3. \tag{5.75}$$

Here we can say, the action (5.71) is restricted to a distribution in the spacetime tangent bundle. That distribution is determined by the singular distribution in terms of the Dirac delta function as

$$\frac{1}{\sqrt{-g}}\delta\left(\rho_0 X^0 - \rho_0 \frac{1}{\sqrt{g_{00}}}\right)\delta^3\left(\rho_0 \bar{X}\right),\ X = \left(X^0, X^1, X^2, X^3\right),\ \bar{X} = \left(X^1, X^2, X^3\right). \tag{5.76}$$

We can insert that distribution into the integration measure. By using that fact and (5.70) for R we can write (5.71) as

$$\mathcal{L} = \int \rho_0^4 d^4u\sqrt{-g}d^4X\sqrt{-g}\frac{1}{\sqrt{-g}}\delta\left(\rho_0 X^0 - \rho_0 \frac{1}{\sqrt{g_{00}}}\right)\delta^3\left(\rho_0 \bar{X}\right)\left[{}^sR - \frac{1}{4}F^{abc}F_{abc}\right]. \tag{5.77}$$

By taking integral on tangent space, using (5.67) and properties of Dirac delta function we get a reduced action defined on the base manifold

$$\mathcal{L} = \int \rho_0^4 d^4u\sqrt{-g}\frac{1}{\rho_0^4}\left({}^sR - \frac{\rho_0^2}{4}\left(g_{00}\right)^{\frac{1}{2}}\left(g^{00}\right)^{\frac{1}{2}}\left({}^sR^{a0bc}\right)\left({}^sR_{a0bc}\right)\right),$$

$$\mathcal{L} = \int d^4u\sqrt{-g}\left({}^sR - \frac{\rho_0^2}{4}\left({}^sR^{a0bc}\right)\left({}^sR_{a0bc}\right)\right). \tag{5.78}$$

5.5 Schwarzschild-like Spacetime

Now we consider a Schwarzschild-like metric on the spacetime manifold as

$$ds^2 = e^{\nu}d(ct)^2 - e^{\lambda}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2, \tag{5.79}$$

where ν and λ depend on r.i.e. $\nu(r)$,$\lambda(r)$. The metric on the spacetime tangent bundle has the form

$$d\sigma^2 = ds^2 + D\left(\rho_0 X\right)^2 = \left(1 - \frac{a^2}{a_0^2}\right)ds^2, \tag{5.80}$$

where the covariant acceleration of the spacetime curve is constant and given [20]

$$\frac{DX^a}{ds} = \left(\frac{a}{c^2}\right)n^a, \qquad n^2 = -1 \text{ and } \rho_0 = \frac{c^2}{a_0}. \tag{5.81}$$

Let us denote the Lagrangian of (5.78) as

$$L\left(r;\lambda,\nu\right) \equiv \sqrt{-g}\left({}^sR - \frac{\rho_0^2}{4}\left({}^sR^{a0bc}\right)\left({}^sR_{a0bc}\right)\right). \tag{5.82}$$

We can write the Euler-Lagrange equations of (5.78) as

$$\frac{\partial L}{\partial \lambda} - \frac{\partial}{\partial r}\left(\frac{\partial L}{\partial \lambda'}\right) + \frac{\partial^2}{\partial r^2}\left(\frac{\partial L}{\partial \lambda''}\right) = 0, \tag{5.83}$$

and

$$\frac{\partial L}{\partial \nu} - \frac{\partial}{\partial r}\left(\frac{\partial L}{\partial \nu'}\right) + \frac{\partial^2}{\partial r^2}\left(\frac{\partial L}{\partial \nu''}\right) = 0. \tag{5.84}$$

From now on λ', λ'' and ν', ν'' denote first and second derivative of λ and ν with respect to r, respectively. In Schwarzschild-like metric (5.79) let us use the following notations

$$g_{00} = e^{\nu},\ g_{11} = -e^{\lambda},\ g_{22} = -r^2,\ g_{33} = -r^2 \sin^2\theta, \tag{5.85}$$

with inverses

$$g^{00} = e^{-\nu},\ g^{11} = -e^{-\lambda},\ g^{22} = -r^{-2},\ g^{33} = \frac{-1}{r^2 \sin^2\theta}, \tag{5.86}$$

and

$$u^0 = t,\ u^1 = r,\ u^2 = \theta,\ u^3 = \phi. \tag{5.87}$$

We obtain the Lagrangian (5.82) by using Schwarzschild-like metric and (5.68). By expanding the summation indices in (5.82), using (5.68) and lowering the indices with g_{00}.i.e. ${}^sR^{100}{}_1 = (g^{00})^2\,({}^sR^1{}_{001})$ we can write the Lagrangian (5.82) with the nonvanishing terms as

$$L = \sqrt{-g}\left\{{}^sR - \frac{\rho_0^2}{4}\left(g^{00}\right)^2\left[\left({}^sR^1{}_{001}\right)^2 + \left({}^sR^2{}_{002}\right)^2 + \left({}^sR^3{}_{003}\right)^2\right]\right\}. \tag{5.88}$$

The nonvanishing curvature tensors and determinant of the metric for the Schwarzschild-like metric are found as

$$\left({}^sR^1{}_{001}\right)^2 = \frac{c^2}{2}e^{\nu-\lambda}\left(\frac{\lambda'\nu'}{2} - \frac{(\nu')^2}{2} - \nu''\right), \tag{5.89}$$

$$\left({}^sR^1{}_{001}\right)^2 = -\frac{c^2}{2}e^{\nu-\lambda}\nu', \tag{5.90}$$

$$\left({}^sR^1{}_{001}\right)^2 = -\frac{c^2}{2}e^{\nu-\lambda}\frac{\nu'}{r}, \tag{5.91}$$

$$g = g_{00}g_{11}g_{22}g_{33} \text{ and } \sqrt{-g} = r^2 e^{(\nu+\lambda)/2}\sin\theta. \tag{5.92}$$

To find scalar curvature ${}^sR$ we need the Ricci curvature. By using (5.68) and Schwarzschild-like metric we obtain the components of Ricci curvature as

$$ {}^sR_{00} = \frac{c^2}{2} e^{\nu-\lambda} \left(\frac{4\lambda'}{r} - \lambda'\nu' + (\nu')^2 + 2\nu'' \right), \tag{5.93} $$

$$ {}^sR_{11} = \frac{\lambda'}{r} + \frac{\lambda'\nu'}{r} - \frac{(\nu')^2}{4} - \frac{\nu''}{2}, \tag{5.94} $$

$$ {}^sR_{22} = 1 - e^{-\lambda} \left[\frac{r}{2} (\nu' - \lambda') + 1 \right] \tag{5.95} $$

$$ {}^sR_{33} = \sin^2\theta \left\{ e^{-\lambda} \left[\frac{r}{2} (\lambda' - \nu') - 1 \right] + 1 \right\}. \tag{5.96} $$

By using these components and (5.86) we obtain the scalar curvature as

$$ \begin{aligned} {}^sR &= {}^sR_{00}g^{00} + {}^sR_{11}g^{11} + {}^sR_{22}g^{22} + {}^sR_{33}g^{33} \\ &= e^{-\lambda} \left(\frac{2\nu'}{r} - \frac{2\lambda'}{r} - \frac{\nu'\lambda'}{2} + \frac{(\nu')^2}{2} + \nu'' + \frac{2}{r^2} \right) - \frac{2}{r^2}. \end{aligned} \tag{5.97} $$

By using (5.89), (5.90) and (5.91) we get

$$ \begin{aligned} \left({}^sR^1{}_{001}\right)^2 + \left({}^sR^2{}_{002}\right)^2 + \left({}^sR^3{}_{003}\right)^2 &= c^4 e^{2(\nu-\lambda)} \left(\frac{(\nu')^2}{2r^2} + \frac{(\nu')^4}{16} \right. \\ &+ \left. \frac{(\nu')^2(\lambda')^2}{16} - \frac{(\nu')^3\lambda'}{8} - \frac{\lambda'\nu'\nu''}{4} + \frac{(\nu')^2\nu''}{4} + \frac{(\nu'')^2}{4} \right). \end{aligned} \tag{5.98} $$

By using (5.92), (5.97) and (5.98) we get the Lagrangian as

$$ \begin{aligned} L &= r^2 \sin\theta \left[e^{(\nu-\lambda)/2} \left(\nu'' - \frac{\nu'\lambda'}{2} + \frac{(\nu')^2}{2} + \frac{2\nu'}{r} - \frac{2\lambda'}{r} + \frac{2}{r} \right) \right. \\ &- \left. \frac{2e^{(\nu+\lambda)/2}}{r^2} \right] - \frac{\rho_0^2}{8} r^2 \sin\theta e^{(\nu-3\lambda)/2} \left[\frac{(\nu')^4}{4} - \frac{\lambda'(\nu')^3}{2} + (\nu')^2\nu'' \right. \\ &+ \left. \frac{(\lambda')^2(\nu')^2}{4} - \lambda'\nu'\nu'' + (\nu'')^2 \frac{2(\nu')^2}{r^2} \right]. \end{aligned} \tag{5.99} $$

Using this Lagrangian in the Euler-Lagrangian equations (5.83) and (5.84) and in the resulting equation arranging the terms and after complicated simplification we get

$$ \begin{aligned} \frac{\nu'}{r} + \frac{1}{r^2} - \frac{e^\lambda}{r^2} &= -\frac{\rho_0^2}{16} e^{-\lambda} \left[\frac{(\nu')^4}{4} + \frac{\lambda'(\nu')^3}{2} - (\nu')^2\nu'' \right. \\ &- \frac{3(\lambda')^2(\nu')^2}{4} + 2\lambda'\nu'\nu''(\nu'')^2 + \frac{6(\nu')^2}{r^2} - \frac{2(\nu')^3}{r} \\ &+ \left. \frac{2\lambda'(\nu')^2}{r} - \frac{4\nu'\nu''}{r} + \lambda''(\nu'')^2 - 2\nu''\nu''' \right], \end{aligned} \tag{5.100} $$

and

$$\begin{aligned}
\frac{\lambda'}{r}-\frac{1}{r^2}+\frac{e^{\lambda}}{r^2} &= -\frac{\rho_0^2}{8}e^{-\lambda}\Bigg[-\frac{(\nu')^4}{8}+\frac{\lambda'(\nu')^3}{4}-\frac{(\nu')^2\nu''}{2}+\frac{11(\lambda')^2(\nu')^2}{8} \\
&- \frac{9\lambda'\nu'\nu''}{2}+\frac{3(\nu'')^2}{2}+\frac{(\nu')^2}{r^2}-\frac{5\lambda'(\nu')^2}{r}+\frac{8\nu'\nu''}{r} \\
&+ \frac{5(\lambda')^2\nu'}{r}-\frac{14\lambda'\nu''}{r}-\frac{3(\lambda')^3\nu'}{2}+\frac{11(\lambda')^2\nu''}{2} \\
&+ \frac{4\lambda'\nu'}{r^2}+2\nu'\nu'''+\frac{7\nu'\lambda'\lambda''}{2}-6\lambda'\nu'''-\frac{4\lambda''\nu'}{r} \\
&+ \frac{8\nu'''}{r}-\lambda''(\nu')^2-4\lambda''\nu''-\lambda'''\nu'+2\nu''''\Bigg]. \qquad (5.101)
\end{aligned}$$

Here we take perturbative solutions for $r \gg \rho_0$ as (perturbation about the classical Schwarzschild solution)

$$e^{\nu}=1-\frac{2Gm}{c^2r}+\varepsilon(r), \qquad (5.102)$$

and

$$e^{\lambda}=\left(1-\frac{2Gm}{c^2r}\right)^{-1}+\delta(r), \qquad (5.103)$$

which are given by [20]. Here $\varepsilon(r)$ and $\delta(r)$ are small perturbations about the classical Schwarzschild solutions. By taking derivative of (5.102) with respect to r, from now on denoting $\frac{2Gm}{c^2}\equiv A$ and arranging the resulting equation we get

$$\begin{aligned}
\nu'e^{\nu} &= \frac{A}{r^2}+\varepsilon' \\
\nu' &= \frac{1}{1-A/r+\varepsilon}\left(\frac{A}{r^2}+\varepsilon'\right) \\
&= \left(\frac{1}{1-A/r}\right)\frac{1}{1+\frac{\varepsilon}{1-A/r}}\left(\frac{A}{r^2}+\varepsilon'\right). \qquad (5.104)
\end{aligned}$$

By using the series expansion of $1/\left(1+\frac{\varepsilon}{1-A/r}\right)$ and taking the first two terms (because of $\varepsilon \ll 1$) we get

$$\begin{aligned}
\nu' &= \frac{1}{1-A/r}\left(1-\frac{\varepsilon}{1-A/r}\right)\left(\frac{A}{r^2}+\varepsilon'\right) \\
&= \frac{1}{1-A/r}\left(\frac{A}{r^2}+\varepsilon'-\left(\frac{A}{r^2}\right)\frac{\varepsilon}{1-A/r}\right) \\
&= \frac{A}{r-A}+\frac{r}{r-A}\varepsilon'-\frac{A}{(r-A)^2}\varepsilon, \qquad (5.105)
\end{aligned}$$

where $\varepsilon\varepsilon' \to 0$ since $\varepsilon \ll 1$. By taking derivative of (5.103) with respect to r, arranging the terms and using the similar facts which we used above we get

$$\lambda' = \delta'\left(\frac{r-A}{r}\right) - \frac{A}{r\,(r-A)} + \frac{A}{r^2}\delta. \tag{5.106}$$

Inserting (5.103), (5.105) and (5.103), (5.106) into the left hand sides of (5.100) and (5.101), respectively we get

$$\frac{\nu'}{r} + \frac{1}{r^2} - \frac{e^\lambda}{r^2} = \frac{1}{r-A}\left(\varepsilon' - \frac{A}{r\,(r-A)}\varepsilon - \frac{(r-A)}{r^2}\delta\right), \tag{5.107}$$

$$\frac{\lambda'}{r} - \frac{1}{r^2} + \frac{e^\lambda}{r^2} = \frac{(r-A)}{r^2}\left(\delta' + \frac{(r+A)}{r\,(r-A)}\delta\right). \tag{5.108}$$

On right hand sides of (5.100) and (5.101) all the terms are products of powers of the derivatives of λ and ν. So we do not consider the perturbative terms ε and δ in (5.102) and (5.103). Therefore we have

$$e^\nu = 1 - \frac{A}{r}, \tag{5.109}$$

and

$$e^\lambda = \left(1 - \frac{A}{r}\right)^{-1}. \tag{5.110}$$

By using (5.109) and (5.110) all the needed derivatives of λ and ν with respect to r in the right hand sides of (5.100) and (5.101) have the forms

$$\begin{aligned} \nu' &= \frac{A}{r^2 - rA},\ \nu'' = -\frac{A\,(2r-A)}{(r^2-rA)^2},\ \nu''' = -\frac{2A}{(r^2-rA)^2} + \frac{2A\,(2r-A)^2}{(r^2-rA)^3} \\ \nu'''' &= \frac{4A\,(2r-A)}{(r^2-rA)^2} + \frac{8A\,(2r-A)}{(r^2-rA)^3} - \frac{6A\,(2r-A)^3}{(r^2-rA)^4}, \end{aligned} \tag{5.111}$$

and

$$\begin{aligned} \lambda' &= -\frac{A}{r^2-rA},\ \lambda'' = \frac{A\,(2r-A)}{(r^2-rA)^2}, \\ \lambda''' &= \frac{2A}{(r^2-rA)^2} - \frac{2A\,(2r-A)^2}{(r^2-rA)^3}. \end{aligned} \tag{5.112}$$

By inserting (5.111) and (5.112) into the right hand sides of (5.100) and (5.101), respectively and after complicated calculation we get

$$-\frac{3\rho_0^2A^2}{8r^5\,(r-A)}, \tag{5.113}$$

and

$$\frac{3\rho_0^2 A^2}{8r^5\left(r-A\right)}. \qquad (5.114)$$

Considering (5.107), (5.108) and (5.113), (5.114), respectively we get the following differential equations

$$\varepsilon' - \frac{A}{r\left(r-A\right)}\varepsilon - \frac{\left(r-A\right)}{r}\delta + \frac{3\rho_0^2 A^2}{8r^5} = 0, \qquad (5.115)$$

and

$$\delta' + \frac{\left(r+A\right)}{r\left(r-A\right)}\delta - \frac{3\rho_0^2 A^2}{8r^5\left(r-A\right)^2} = 0. \qquad (5.116)$$

First let us solve (5.116). Solution of the homogeneous part of (5.116) has the form

$$\delta = c_1 \frac{r}{\left(r-A\right)^2}, \ c_1 \text{ is an arbitrary constant.} \qquad (5.117)$$

To obtain the solution of (5.116) we parameterize the constant of the homogeneous part solution as

$$\delta = c_1(r)\frac{r}{\left(r-A\right)^2}. \qquad (5.118)$$

We need to determine $c_1(r)$. From the derivative of δ in (5.118) we have

$$\delta' = \left(\frac{1}{\left(r-A\right)^2} - \frac{2r}{\left(r-A\right)^3}\right)c_1(r) + \frac{r}{\left(r-A\right)^2}c_1'. \qquad (5.119)$$

Substituting (5.118) and (5.119) into (5.116) we get $c_1(r)$ as

$$c_1(r) = -\frac{\rho_0^2 A^2}{2r^3}. \qquad (5.120)$$

Thus the solution of (5.116) is

$$\delta = c_1(r)\frac{r}{\left(r-A\right)^2} = -\frac{\rho_0^2 A^2}{2r^2\left(r-A\right)^2}. \qquad (5.121)$$

By substituting that δ into (5.115) and applying the similar procedure which we used to solve (5.116) we get the solution of (5.115) as

$$\varepsilon = \frac{\rho_0^2 A^2}{8r^4}. \qquad (5.122)$$

By writing (5.121) and (5.122) into (5.102) and (5.103), respectively and writing the expression for A we obtain

$$e^{\nu} = 1 - \frac{2Gm}{c^2 r} + \frac{\rho_0^2 G^2 m^2}{2c^4 r^4}, \qquad (5.123)$$

and

$$e^{\lambda} = \frac{1}{1 - 2Gm/c^2r} - \frac{\rho_0^2 G^2 m^2}{2c^4 r^4 \left(1 - 2Gm/c^2r\right)^2}. \tag{5.124}$$

Thus we get the nontrivial components of the Schwarzschild-like metric as

$$g_{00} = e^{\nu} = 1 - \frac{2Gm}{c^2 r} + \frac{\rho_0^2 G^2 m^2}{2c^4 r^4}, \tag{5.125}$$

and

$$g_{11} = -e^{\lambda} = \frac{1}{1 - 2Gm/c^2r} - \frac{\rho_0^2 G^2 m^2}{2c^4 r^4 \left(1 - 2Gm/c^2r\right)^2}. \tag{5.126}$$

5.6 Red Shift For Static Emitter and Observer

Let an emitter emit wave with frequency $f(r)$ and number of wave crest $n(r)$ in a proper time $\Delta\tau(r)$ at radius r. So the number of crests $n(r)$ at radius r is

$$n(r) = f(r)\Delta\tau(r), \tag{5.127}$$

and the number of wave crests $n(r)$ at large r is

$$\lim_{r\to\infty} n(r) = \lim_{r\to\infty} \left(F(r)\Delta\tau(r)\right) = \left(\lim_{r\to\infty} f(r)\right) \Delta t \equiv f(\infty)\Delta t. \tag{5.128}$$

They are equal so we have

$$n(r) = f(r)\Delta\tau(r) = f(\infty)\Delta t. \tag{5.129}$$

If the maximal acceleration invariance is require in the frame of the emitter then the infinitesimal path $d\sigma(r)$ which is traversed by the emitter in the bundle, must have the same value at the infinitesimal path at $a = 0$ as

$$d\sigma(r) = d\sigma(r)|_{a=0}, \tag{5.130}$$

equivalently by using

$$d\sigma^2 = ds^2 + \rho_0^2 DX^2 = \left(1 - \frac{a^2}{a_o^2}\right) ds^2, \tag{5.131}$$

we have

$$d\sigma(r) = d\left(c\tau\left(r\right)\right) = d\sigma(r)|_{a=0} = \left(1 - \frac{a^2}{a_o^2}\right)^{1/2} {g_{00}}^{1/2} c dt, \tag{5.132}$$

this leads to

$$\Delta\tau(r) = \left(1 - \frac{a^2}{a_o^2}\right)^{1/2} {g_{00}}^{1/2}\Delta t. \tag{5.133}$$

Substituting (5.133) into (5.129) we get

$$f(r)\left(1 - \frac{a^2}{a_o^2}\right)^{1/2} {g_{00}}^{1/2}\Delta t = f(\infty)\Delta t, \tag{5.134}$$

equivalently

$$f(\infty) = f(r)\left(1 - \frac{a^2}{a_o^2}\right)^{1/2} {g_{00}}^{1/2}. \tag{5.135}$$

From the begining of the chapter we know that the proper acceleration a in a curved spacetime is

$$a^2 = -c^2 g_{ab}\frac{DX^a}{ds}\frac{DX^b}{ds}, \tag{5.136}$$

and $\frac{DX^a}{ds}$ was given as

$$\frac{DX^a}{ds} = \frac{dX^a}{ds} + \gamma^a_{\ bc}X^bX^c. \tag{5.137}$$

And from the previous section we have $X^a = \left(g_{00}^{-1/2}, 0, 0, 0\right)$. Using this and expanding the summation indices in (5.137) we get

$$\frac{DX^a}{ds} = \frac{dg_{00}^{-1/2}}{ds} + \gamma^a_{\ 00}g_{00}^{-1/2}g_{00}^{-1/2} = \frac{1}{g_{00}}\gamma^a_{\ 00}. \tag{5.138}$$

By using that fact we can write (5.136) as

$$a^2 = \frac{-c^4 g_{ab}\gamma^a_{\ 00}\gamma^b_{\ 00}}{g_{00}^2}. \tag{5.139}$$

Expanding the summation indices of $g_{ab}\gamma^a_{\ 00}\gamma^b_{\ 00}$ we see that the only nonvanishing term is $g_{11}\gamma^1_{\ 00}\gamma^1_{\ 00}$. Let us compute $\gamma^1_{\ 00}$ which has form

$$\gamma^1_{\ 00} = \frac{g^{11}}{2}\left(\frac{\partial g_{01}}{\partial u^0} + \frac{\partial g_{10}}{\partial u^0} - \frac{g_{00}}{u^1}\right). \tag{5.140}$$

In the above expression, by using $g_{10} = 0$, $u^1 = r$, $g_{00} = 1 - A/r + \rho_0^2A^2/8r^4$, $g^{11} = 1/\left\{\rho_0^2A^2/8r^4\,(1 - A/r) - 1/\left[1 - A/r\right]\right\}$ and applying the procedure similar to (5.105) we get

$$\gamma^1_{\ 00} = \frac{1}{2}\left(\frac{1}{\frac{1}{1-A/r} - \frac{\rho_0^2A^2}{8r^4(1-A/r)}}\right)\left(\frac{A}{r^2} - \frac{\rho_0^2A^2}{2r^5}\right)$$

$$\begin{aligned}
&= \frac{1}{2}\frac{1}{\frac{1}{1-A/r}}\left(1-\frac{\rho_0^2A^2}{8r^4\left(1-A/r\right)^2}\left(1-A/r\right)\right)\left(\frac{A}{r^2}-\frac{\rho_0^2A^2}{2r^5}\right)\\
&= \frac{r-A}{2r}\left(1+\frac{\rho_0^2A^2}{8r^4\left(1-A/r\right)}\right)\left(\frac{A}{r^2}-\frac{\rho_0^2A^2}{2r^5}\right)\\
&= \frac{r-A}{2r}\left(\frac{A}{r^2}-\frac{\rho_0^2A^2}{2r^5}+\frac{\rho_0^2A^3}{8r^5\left(r-A\right)}\right), \qquad (5.141)
\end{aligned}$$

which leads to

$$g_{11}\gamma^1{}_{00}\gamma^1{}_{00} = -\frac{1}{4}g_{11}g^{11}\left(\frac{A}{r^2}-\frac{\rho_0^2A^2}{2r^5}\right)\frac{(r-A)}{r}\left(\frac{A}{r^2}-\frac{\rho_0^2A^2}{2r^5}+\frac{\rho_0^2A^3}{8r^5\left(r-A\right)}\right).$$

By using the fact $g_{11}g^{11}=1$ and r is large enough so that the terms $\frac{1}{r^6}, \frac{1}{r^7}, \ldots$ are ignored and the above equation takes the form

$$g_{11}\gamma^1{}_{00}\gamma^1{}_{00} = -\frac{1}{4}\frac{(r-A)\,A^2}{r^5}, \qquad (5.142)$$

and using the facts in (5.141) we can write g_{00}^2 as

$$g_{00}^2 = \frac{(r-A)^2}{r^2}\left(1+\frac{\rho_0^2A^2}{4r^3\left(r-A\right)}\right). \qquad (5.143)$$

By substituting (5.142) and (5.143) into (5.139) we get

$$\begin{aligned}
a^2 &= -c^4\frac{1}{\frac{(r-A)^2}{r^2}\left(1+\frac{\rho_0^2A^2}{4r^3(r-A)}\right)}\left(-\frac{1}{4}\frac{(r-A)\,A^2}{r^5}\right)\\
&= \frac{c^4}{4}\frac{r}{r-A}\left(1-\frac{\rho_0^2A^2}{4r^3\left(r-A\right)}\right)\left(\frac{A^2}{r^4}\right), \qquad (5.144)
\end{aligned}$$

Considering the above equation with r large enough, we get

$$a^2 = \frac{c^4}{4}\frac{A^2}{r^4}. \qquad (5.145)$$

By using (5.125) for g_{00}, $a_0=\frac{c^2}{\rho_0}$, (5.135) for $f(\infty)$ and (5.145) for a^2 we get the modified red shift as

$$\begin{aligned}
\frac{\delta f}{f} \equiv \frac{f(\infty)-f(r)}{f(r)} &= \frac{f(\infty)}{f(r)}-1 = \left(1-\frac{a^2}{a_0^2}\right)^{1/2} g_{00}^{1/2}-1\\
&= \left(1-\frac{1}{2}\frac{a^2}{a_0^2}\right)\left(1-\frac{A}{r}+\frac{\rho_0^2A^2}{8r^4}\right)^{1/2}-1\\
&= \left(1-\frac{1}{2}\frac{c^4}{4}\frac{A^2r}{r^4\left(r-A\right)}\frac{\rho_0^2}{c^4}\right) \qquad (5.146)\\
&= \left[\left(1-\frac{A}{r}\right)^{1/2}-1\right]-\frac{\rho_0^2A^2}{8r^4\left(1-A/r\right)^{1/2}}.
\end{aligned}$$

And writing the expression for A we get the modified red shift as

$$\frac{\delta f}{f}=\left[\left(1-\frac{2Gm}{c^2r}\right)^{1/2}-1\right]-\frac{\rho_0^2G^2m^2}{2c^4r^4\left(1-2Gm/c^2r\right)^{1/2}}. \tag{5.147}$$

Hence we obtain the red shift formula of the Schwarzschild geometry modified by the tangent bundle metric. Contribution of the tangent bundle metric is the order of $\frac{\rho_0^2G^2m^2}{c^4}$ which is extremely smaller than the contribution of the pure Schwarzschild metric.

5.7 Summary

Proper acceleration of an object: a,

Proper acceleration which is given by Einstein's theory of general relativity:
$a^2=-c^4g_{ij}\frac{DX^i}{ds}\frac{DX^j}{ds}$,

Metric on spacetime tangent bundle in natural basis:
$G_{IJ}=\begin{pmatrix} g_{ij}+g_{lk}N_i{}^lN_j{}^k & N_{ji}\\ N_{ij} & g_{ij}\end{pmatrix}$,

Special frame: $E_\alpha{}^I=\left(B_a{}^I,C_{a^*}{}^I\right)$, with inverse $E^\alpha{}_I=\left(B^a{}_I,C^{a^*}{}_I\right)$, where $B_a{}^I=\left(\delta_a^i,-N_a{}^i\right)$, $C_{a^*}{}^I=\left(0,\delta_a^i\right)$ and $B^a{}_I=\left(\delta_i^a,0\right)$, $C^{a^*}{}_I=\left(N_i{}^a,\delta_i^a\right)$.

Anholonomic basis:
$E_\alpha=\left(E_a,E_{a^*}\right)=\left(\frac{\partial}{\partial u^a}-\rho_0^{-1}N_a{}^k\frac{\partial}{\partial X^k},\rho_0^{-1}\frac{\partial}{\partial X^a}\right)$,

and its dual
$W^\alpha=\left(W^a,W^{a^*}\right)=\left(du^a,\rho_0dX^a+N_k{}^adu^k\right)$.

Components of the bundle metric in special frame:
$G_{\alpha\beta}=\begin{pmatrix} g_{ab} & 0\\ 0 & g_{a^*b^*}\end{pmatrix}$.

The line element of the spacetime tangent bundle: $d\sigma^2=G_{\alpha\beta}W^\alpha W^\beta$.

Components of the anholonomic objects such that $[E_\alpha,E_\beta]=C_{\alpha\beta}{}^\mu E_\mu$:
$C_{ab}{}^c=C_{ab^*}{}^c=C_{a^*b}{}^c=C_{a^*b^*}{}^c=C_{a^*b^*}{}^{c^*}=0$,
$C_{ab}{}^{c^*}=\frac{\delta N_a{}^{c^*}}{\delta u^b}-\frac{\delta N_b{}^{c^*}}{\delta u^a}\equiv F^{c^*}{}_{ab}$,
$C_{ab^*}{}^{c^*}=\frac{\partial N_a{}^{c^*}}{\partial y^{b^*}}\equiv\phi^{c^*}{}_{ab^*}$,
$C_{a^*b}{}^{c^*}=-\frac{\partial N_b{}^{c^*}}{\partial y^{a^*}}\equiv-\phi^{c^*}{}_{ba^*}$.

Levi-civita connection on the spacetime tangent bundle:

$D(E_\beta) = \Gamma^\lambda{}_{\beta\alpha} W^\alpha \otimes E_\lambda$,

where the components of the connection coefficients are

$\Gamma^\lambda{}_{\alpha\beta} = \frac{1}{2} G^{\mu\lambda} \left[\frac{\delta G_{\mu\beta}}{\delta Y^\alpha} + \frac{\delta G_{\mu\alpha}}{\delta Y^\beta} - \frac{\delta G_{\alpha\beta}}{\delta Y^\mu} + C_{\mu\alpha\beta} + C_{\mu\beta\alpha} - C_{\alpha\beta\mu} \right]$.

The components of the connection coefficients for different indices which belong to spacetime and fiber sectors:

$\Gamma^a{}_{bc} = \bar{\bar{\gamma}}^a{}_{bc}$,

$\Gamma^a{}_{bc^*} = \Gamma^a{}_{c^*b} = \frac{1}{2} \left(\bar{\gamma}^a{}_{c^*b} + \bar{\gamma}_{bc^*}{}^a + F_{c^*b}{}^a \right)$,

$\Gamma^a{}_{b^*c^*} = -\frac{1}{2} \left(T_{b^*}{}^a{}_{c^*} + T_{c^*}{}^a{}_{b^*} \right)$,

$\Gamma^a{}_{b^*c^*} = -\frac{1}{2} \left(T_{b^*}{}^a{}_{c^*} + T_{c^*}{}^a{}_{b^*} \right)$,

$\Gamma^{a^*}{}_{bc} = \frac{1}{2} \left(F^{a^*}{}_{bc} - \bar{\gamma}_b{}^{a^*}{}_c - \bar{\gamma}_c{}^{a^*}{}_b \right)$,

$\Gamma^{a^*}{}_{bc^*} = \frac{1}{2} \left(T^{a^*}{}_{bc^*} + T_{c^*b}{}^{a^*} \right)$,

$\Gamma^{a^*}{}_{c^*b} = \bar{\bar{\gamma}}^{a^*}{}_{bc^*} + \frac{1}{2} \left(T_{c^*b}{}^{a^*} - T^{a^*}{}_{bc^*} \right)$,

$\Gamma^{a^*}{}_{b^*c^*} = \bar{\bar{\gamma}}^{a^*}{}_{b^*c^*}$,

where

$\bar{\bar{\gamma}}^\alpha{}_{\beta\gamma} = \frac{1}{2} G^{\lambda\alpha} \left(\frac{\delta G_{\lambda\beta}}{\delta Y^\gamma} + \frac{\delta G_{\lambda\gamma}}{\delta Y^\beta} - \frac{\delta G_{\beta\gamma}}{\delta Y^\lambda} \right)$,

$\bar{\gamma}^a{}_{bc} = \frac{1}{2} g^{da} \left(\frac{\partial g_{db}}{\partial X^c} + \frac{\partial g_{dc}}{\partial X^b} - \frac{\partial g_{bc}}{\partial X^d} \right)$,

$\gamma^a{}_{bc} = \frac{1}{2} g^{da} \left(\frac{\partial g_{db}}{\partial u^c} + \frac{\partial g_{dc}}{\partial u^b} - \frac{\partial g_{bc}}{\partial u^d} \right)$,

$\phi^\alpha{}_{\beta\gamma} = \frac{\partial N_\beta{}^\alpha}{\partial Y^\gamma}$,

$T^\alpha{}_{\beta\gamma} = \bar{\bar{\gamma}}^\alpha{}_{\beta\gamma} - \phi^\alpha{}_{\beta\gamma}$,

Components of the curvature tensor without the components of the anholonomic objects:

$\bar{R}_\delta{}^\alpha{}_{\beta\gamma} = Y^\delta \left[\frac{\delta \bar{\bar{\gamma}}^\alpha{}_{\delta\gamma}}{\delta Y^\beta} - \frac{\delta \bar{\bar{\gamma}}^\alpha{}_{\delta\beta}}{\delta Y^\gamma} + \bar{\bar{\gamma}}^\alpha{}_{\lambda\beta} \bar{\bar{\gamma}}^\lambda{}_{\delta\gamma} - \bar{\bar{\gamma}}^\alpha{}_{\lambda\gamma} \bar{\bar{\gamma}}^\lambda{}_{\delta\beta} \right]$.

The gauge curvature field:

$F^\alpha{}_{\beta\gamma} = Y^\delta \bar{R}_\delta{}^\alpha{}_{\beta\gamma}$.

Components of curvature tensor of the spacetime tangent bundle:

$R_\alpha{}^\beta{}_{\gamma\lambda} = \frac{\delta \Gamma^\beta{}_{\alpha\lambda}}{\delta Y^\gamma} - \frac{\delta \Gamma^\beta{}_{\alpha\gamma}}{\delta Y^\lambda} + \Gamma^\beta{}_{\delta\gamma} \Gamma^\delta{}_{\alpha\lambda} - \Gamma^\beta{}_{\delta\lambda} \Gamma^\delta{}_{\alpha\gamma} - \Gamma^\beta{}_{\alpha\delta} C_{\gamma\lambda}{}^\delta$.

Ricci curvature of the spacetime tangent bundle: $R_{\alpha\lambda} = R_\alpha{}^\beta{}_{\beta\lambda}$,

Scalar curvature of the spacetime tangent bundle:

$R = R_{\alpha\lambda} G^{\alpha\lambda} = R_{ac} g^{ac} + R_{a^*c^*} g^{a^*c^*} =$

$= {}^s\bar{R} + {}^v\bar{R} - \frac{1}{4} F^{abc} F_{abc} - T^{ab}{}_a T^c{}_{bc} - 2 \frac{\bar{D} T^a{}_{ba}}{D u_b}$

$- \frac{1}{4} \left[T^{abc} T_{abc} + T^{cba} T_{abc} + T^{abc} T_{cba} + T^{cba} T_{cba} \right]$

$- \frac{1}{4} \left[\bar{\gamma}^{abc} \bar{\gamma}_{abc} + \bar{\gamma}^{cba} \bar{\gamma}_{abc} + \bar{\gamma}^{abc} \bar{\gamma}_{cba} + \bar{\gamma}^{cba} \bar{\gamma}_{cba} \right]$

$- \bar{\gamma}^{ab}{}_a \bar{\gamma}^c{}_{bc} - 2 \frac{D \bar{\gamma}^a{}_{ba}}{D \rho_0 X_e}$.

Chapter 6

Conclusion

In this thesis we have studied the Finsler geometry and the geometry of the tangent bundle. In the last chapter we have considered an application of the geometry of the tangent bundle to physics. We defined the maximal acceleration invariant fiber bundle on a Riemannian spacetime and an action on the bundle manifold. Then we considered the Riemannian Schwarzschild-like spacetime. We obtained the Euler-Lagrange equations which yields two nonlinear differential equations. We solved these partial differential equations using a perturbative expansions around the Schwarzschild solutions. The solutions of these equations define the components of the Schwarzschild-like metric. By using this Schwarzschild-like solution we obtained the modified red shift formula for a static emitter and observer on a Schwarzschild-like spacetime. We have observed that the contribution coming from the acceleration is extremely small compared with the standard red shift expression.

In general the maximal acceleration invariant fiber bundle is defined on a general spacetime manifold which does not need to be Riemannian, Finslerian or Kählerian spacetime. All these are special cases. Applications of general spacetime tangent bundles have not been done yet and remain to be explored.

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