RECURSION OPERATOR AND DISPERSIONLESS LAX REPRESENTATION

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By
Kostyantyn Zheltukhin
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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Prof. Dr. Metin Gürses(Supervisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Assoc. Prof. Dr. Atalay Karasu

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Assoc. Prof. Dr. Sinan Sertöz

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Prof. Dr. Alexander Shumovsky

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

Prof. Dr. Turgut Önder

Approved for the Institute of Engineering and Science:

Prof. Dr. Mehmet Baray
Director of the Institute

ABSTRACT

RECURSION OPERATOR AND DISPERSIONLESS LAX REPRESENTATION

Kostyantyn Zheltukhin
Ph.D. in Mathematics
Supervisor: Prof. Dr. Metin Gürses
August, 2002

We give a general method for constructing recursion operators for some equations of hydrodynamic type, admitting a dispersionless Lax representation. We consider a polynomial and rational Lax function. We give several examples containing the equations of shallow water waves, polytropic gas dynamics and a degenerate bi-Hamiltonian system with a recursion operator. We also discuss a reduction of $N+1$ systems to N systems of some new integrable equations of hydrodynamic type.

Keywords: Integrable System, Dispersionless Lax Representation, Recursion Operator, Equation of Hydrodynamic type.

ÖZET

SİMETRİ ADIM OPERATÖRLERİ VE DAĞILIMSIZ LAX TEMSİLLERİ

Kostyantyn Zheltukhin
Matematik Bölümü, Doktora
Tez Yöneticisi: Prof. Dr. Metin Gürses
Ağustos, 2002

Dağılımı olmayan Lax temsili kabul eden bazı hidrodinamik tipi denklemlere simetri adım operatörlerinin inşası için genel bir yöntem veriyoruz. Burada polinom ve rasyonel Lax fonksiyonları almaktayız. Sığ su dalga denklemleri, politropik gaz dinamiği denklemlerini de içeren çok çeşitli örnekler ele almaktayız. Bunlar arasında dejenere çift-Hamilton yapı kabul eden bir sistem ve simetri adım operatörünü vermekteyiz. Ayrıca $N+1$ boyutlu bir sistemden N boyutlu bir sisteme indirgeme yaparak integre edilebilir yeni hidrodinamik tipi denklemler ortaya çıkarmaktayız.

Anahtar sözcükler: İntegre Edilebilir Sistemler, Dağılımsız Lax Temsili, Simetri Adım Operatörleri, Hidrodinamik Tipli Denklemler.

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Contents

Chapter 1

Introduction

1.1 Discovery of Soliton Equations

The theory of the nonlinear integrable equations or soliton equations started in 1967 with the paper by Gardner, Greene, Kruskal and Miura [1] on exact solution of the Korteweg-de Vries equation (KdV). The KdV equation

$$u_t = u_{xxx} + uu_x \tag{1.1}$$

was derived by Korteweg and de Vries [2] in 1895 to describe the solitary waves in shallow water.

In 1954, Fermi, Pasta and Ulam [3] while studying numerically a nonlinear oscillator lattice discovered surprisingly slow dissipation of energy in the lattice. To understand this phenomenon Zabusky and Kruskal [4] analyzed numerical solutions of the KdV equation which represents a continuous limit for the Fermi, Pasta and Ulam lattice. They found that the numerical solution with periodic initial conditions developed a train of solitary waves which interacted elastically. When the faster wave catches the slower one they undergo nonlinear interaction but after a long time both waves regain their shape and velocity. Zabusky and Kruskal named such waves solitons.

Definition 1 *A soliton is a solitary wave which asymptotically preserves its shape and velocity upon nonlinear interaction with other solitary waves, or more generally, another (arbitrary) localized disturbance.*

In order to explain such behavior the extensive analytic studies of the KdV equation were undertaking. These studies eventually lead to the development of the inverse scattering transformation method for the exact solutions of the KdV equation [1]. In short time a number of generalizations appeared and many other equations were solved by the inverse scattering transformation method or its generalizations. For the inverse scattering transformation method and its generalizations we refer to [5]–[9] and references therein.

Usually the application of the inverse scattering transformation method to an evolution equation is based on the Lax representation. The evolution part of the Lax equation is written as

$$L_t = [A, L], \tag{1.2}$$

where L and A are differential operators and $[L, A]$ is their commutator (operators L and A are called a Lax pair). In [10], Lax proposed such a representation for the KdV equation.

Example 1 *The Lax pair for the KdV equation is* $L = 4D_x^2 + \frac{3}{2}u$ *and* $A = 4D_x^3 + uD_x + \frac{1}{2}u_x$. *The KdV equation can be written as*

$$L_t - [A, L] = u_t - u_{xxx} - uu_x. \tag{1.3}$$

In some cases it is more convenient to use a zero curvature representation. An evolution equation is written as the compatibility condition of two linear equations

$$v_x = Xv,$$
$$v_t = Yv,$$

where v is an n-dimensional vector and X,Y are $n \times n$ matrices. The Lax equation (1.2) corresponds to the compatibility condition

$$X_t - Y_x + [X, Y] = 0, \tag{1.4}$$

which is equivalent to $v_{xt} = v_{tx}$.

Having Lax representation (1.2) or zero-curvature representation (1.4) for a given system of differential equations in principle leads to the exact solution of a Cauchy problem. However, analytical details are highly nontrivial. On the other hand, the integrable equations have very rich properties. They admit infinitely many conservation laws, multi-Hamiltonian formulation, soliton like solutions, etc. Based on such properties different approaches to study and classify the integrable equations were developed, see for example [11]. A very powerful one turns out to be the algebraic approach. Algebraic aspects of theory of the integrable equations one may find in [12]–[15].

It is worth mentioning that the integrable systems appear in many different areas of physics, biology, engineering. Plasma physics, string theory, non-linear optic, protein dynamic are just a few areas where the integrable systems are widely applicable.

1.2 Some algebraic aspects of theory of the integrable equations

An integrable system of differential equations does not exist separately but rather as a member of an infinite hierarchy. Each equation in the hierarchy generates a flow which commutes with the flow of the original system. Transforming a solution of the system along a commuting flow we again obtain a solution. It means that the equations in the hierarchy are symmetries of the system. Hence, with each integrable system of differential equations we can relate an infinite hierarchy of symmetries.

One of the effective ways to obtain a hierarchy of symmetries is to use a recursion operator which maps a symmetry into a symmetry. Indeed, having a recursion operator for a given system of differential equations we can easily generate an infinite hierarchy of symmetries by applying the recursion operator successively to an initial symmetry. Recursion operators were first presented in their general form by Olver [16] in 1977.

The existence of an infinite hierarchy of symmetries seems to be one of the fundamental properties of the integrable systems. So, in terms of symmetry approach we give the following definition.

Definition 2 *An evolution system of differential equations is called integrable if it possesses a non-constant recursion operator.*

Existence of a hierarchy of symmetries is closely related to a Hamiltonian formulation of a system of differential equations. Actually, if a system admits two distinct but compatible Hamiltonian structures (a bi-Hamiltonian system) then there is an algorithmic way for constructing an infinite hierarchy of symmetries and a recursion operator. All symmetries in the hierarchy are Hamiltonian and generate commuting flows. Hence, multi-Hamiltonian systems are completely integrable systems. The construction of the hierarchy firstly was done by Margi [17] in 1978. Most of the known integrable systems of differential equations posses bi-Hamiltonian formulation.

The multi-Hamiltonian formulation of the Lax equation (1.2) was also studied, see [12]. The Lax equation (1.2) can be considered as an evolution system on an appropriate Lie algebra adjoint to the Lie algebra of Lax operators. Then the multi-Hamiltonian structures appear naturally via R-matrix formalism. This gives direct and systematic means to generate multi-Hamiltonian integrable hierarchies.

Finally, algebraic methods are important for classification of integrable equations. For instance, Mikhailov, Shabat and Sokolov [11] classified all second order integrable evolution systems, using a concept of formal symmetry. Algebraic methods can be also used to obtain special types of solutions such as symmetry invariant solutions and soliton-like solutions see [14],[15].

1.3 Dispersionless Lax equation

Consider an algebra of Laurent series

$$\Lambda = \left\{ \sum_{i=-\infty}^{+\infty} u_i p^i \right\}, \tag{1.5}$$

where u_i are smooth functions on the unit circle S^1, with a Poisson bracket

$$\{f, g\}_k = p^k \left(\frac{\partial f}{\partial p} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial g}{\partial p} \right), \qquad f, g \in \Lambda, \tag{1.6}$$

where k is an integer parameter. The Lax equation on the algebra Λ is

$$L_t = \{A, L\}_k \tag{1.7}$$

with appropriate functions $L, A \in \Lambda$. Such equations are called *dispersionless Lax equations*. They were studied, for example in [21]–[23].

In the present work we discuss the Gel'fand-Dickey [20] integrable hierarchy for polynomial and rational Lax functions. The rational Lax function represents the most general case up to now. Set

$$L = \frac{\Delta_1}{\Delta_2}, \tag{1.8}$$

where Δ_1, Δ_2 are polynomials of degree N and M, respectively, with $N > M$, and let

$$L^{\frac{1}{N-M}+n} = \sum_{j=-\infty}^{\infty} a_{nj} p^j \tag{1.9}$$

be the Laurent series expansion of $L^{\frac{1}{N-M}+n}$ in a neighborhood of $p = \infty$. Then the Gel'fand-Dickey integrable hierarchy is

$$\frac{\partial L}{\partial t_n} = \{A_n, L\} \,, \qquad n = 0, 1, 2, \ldots, \tag{1.10}$$

where

$$A_n = (L^{\frac{1}{N-M}+n})_{\geq k} := \sum_{j=k}^{\infty} a_{nj} p^j. \tag{1.11}$$

This construction appears naturally via R-matrix formalism (see Section 2.5).

Such hierarchies can be found in context of topological field theories (see [24]–[26]). They also correspond to some important equations, for instance, the continuous Toda lattice equation and the polytropic gas dynamics equation (with some modifications of the Lax equation).

We construct a recursion operator associated with the Lax equation (1.10). Then we study integrable reductions of the Lax equation (1.10). The case of the polynomial Lax function was considered by Gürses and Zheltukhin in [28] and the case of the rational Lax function was considered by Zheltukhin in [29]. To construct a recursion operator we use a method recently introduced by Gürses, Karasu and Sokolov in [19] for the standard Lax equation (1.2). Following the method, define operator R_n by the equality

$$A_n = LA_{n-1} + R_n. \tag{1.12}$$

Then we have relations among the symmetries

$$L_{t_n} = LL_{t_{n-1}} + \{R_n, L\}_k. \tag{1.13}$$

The above equation allows us to find R_n in terms of $L_{t_{n-1}}$ and hence L_{t_n}.On the other hand to find L_{t_n} we don't need to know the exact form of A_n. This method was applied for different types of Lax equations, corresponding to field and lattice systems, see [27]-[29]. We give the application of the method in full details in Section 4.2.

The equations in the hierarchy (1.10) are of hydrodynamic type that is of the form

$$u_t^i = \sum_{j=1}^{n} h_j^i(u^1, \ldots, u^n) u_x^j, \qquad i, j = 1, \ldots n. \tag{1.14}$$

There are some other works [33]-[35] which also give recursion operators for some classes of equations of hydrodynamic type, in particular for the diagonal systems. Also, as mentioned before one can obtain a recursion operator using two compatible Hamiltonian operators. Hamiltonian formulation of the equations of hydrodynamic type (sometimes called the dispersionless KdV systems) were studied by Dubrovin and Novikov [36]. See [37] for more details on this subject. For Hamiltonian formulations of the equations admitting a dispersionless Lax representation see [30]–[32].

Chapter 2

Symmetries, recursion operator, Hamiltonian formulation

In this chapter we give basic results of algebraic theory of integrable equations following Olver [15].

Most definitions and theorems are given for a general system of nonlinear differential equations of n-th order

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \ldots, l, \tag{2.1}$$

where $x = (x_1, x_2, \ldots, x_p) \in X$ are independent variables, $u = (u^1, u^2, \ldots, u^q) \in U$ are dependent variables and $\Delta(x, u^{(n)}) = (\Delta_1(x, u^{(n)}), \ldots, \Delta_l(x, u^{(n)}))$ is a smooth function depending on x, u and derivatives of u up to order n with respect to $x_1, x_2, \ldots, x_p$. If we define a jet space $X \times U^{(n)}$ as a space whose coordinates are independent variables, dependent variables and derivatives of dependent variables up to order n then Δ is a smooth mapping

$$\Delta : X \times U^{(n)} \to \mathbb{R}^l.$$

Besides we consider evolution systems of differential equations

$$u_t - K(x, u^{(n)}) = 0, \tag{2.2}$$

where one of the independent variables is distinguished as the time and the other variables $x = (x_1, x_2, \ldots, x_p)$ are spatial.

We omit all proofs in this Chapter. For the proofs the reader may consult [15].

2.1 Lie point symmetries

Definition 3 *A symmetry group of the evolution equation (2.1) is an one-parameter group of transformations G, acting on $X \times U$, such that if $u = f(x)$ is an arbitrary solution of (2.1) and $g_\epsilon \in G$ then $g_\epsilon \cdot f(x)$ is also a solution of (2.1).*

The infinitesimal generator of a symmetry group is called *an infinitesimal symmetry*. To formulate the condition for a group G to be a symmetry group we use infinitesimal generators. It is easier to work with infinitesimal generators (vector fields). First we define a prolongation of a vector field.

Definition 4 *For a vector field*

$$V = \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial}{\partial x_i} + \sum_{i=1}^{q} \eta^i(x, u) \frac{\partial}{\partial u^i} \tag{2.3}$$

the n-th prolongation of V that acts on the jet space $X \times U^{(n)}$ is given by the formula

$$\mathrm{pr}^{(n)} V = V + \sum_{i=1}^{q} \xi^i(x, u) \frac{\partial}{\partial x_i} + \sum_{J} \phi_i^J(x, u^{(n)}) \frac{\partial}{\partial u_J^i}. \tag{2.4}$$

The second summation is over all multi-indices $J = (j_1, \ldots, j_k)$ with $(1 \le j_i \le p)$ and $(1 \le k \le n)$. The coefficient functions ϕ_i^J are given by

$$\phi_\alpha^J(x, u^{(n)}) = D_J\left(\phi_\alpha - \sum_{i=1}^{p} \xi^i \frac{\partial u^\alpha}{\partial x_i}\right) + \sum_{i=1}^{p} \xi^i D_J \frac{\partial u^\alpha}{\partial x_i}, \tag{2.5}$$

where D_J is a total derivative with respect to multi-index J.

The following theorem gives infinitesimal condition for a group G to be a symmetry group.

Theorem 1 *Let*

$$\Delta(x, u^{(n)}) = 0$$

be a system of differential equations. If G is a group of transformations acting on $X \times U$ and

$$\mathrm{pr}^{(n)} V(\Delta(x, u^{(n)})) = 0$$

for every infinitesimal generator V of G, then G is a symmetry group of the system.

This theorem can be used to find all symmetry groups for a given system of equations. For finding symmetry groups we refer to [15]. Note that the set of all infinitesimal symmetries of a system of differential equations forms a Lie algebra of vector fields on $X \times U$.

Let us give some examples of symmetry groups.

Example 2 *The KdV equation (1.1) has the following infinitesimal symmetries:*

$$
\begin{aligned}
v_1 &= \partial_x, \\
v_2 &= \partial_t, \\
v_3 &= -6t\partial_x + \partial_u, \\
v_4 &= x\partial_x + 3t\partial_t - 2u\partial_u,
\end{aligned}
$$

with the corresponding group actions

$$
\begin{aligned}
g_\epsilon \cdot u(x,t) &= u(x+\epsilon, t) && \textit{space translation} \\
g_\epsilon \cdot u(x,t) &= u(x, t+\epsilon) && \textit{time translation} \\
g_\epsilon \cdot u(x,t) &= u(x+6\epsilon t, t) + \epsilon && \textit{Galilean boost} \\
g_\epsilon \cdot u(x,t) &= \mathrm{e}^{2\epsilon} u(\mathrm{e}^{\epsilon} x, \mathrm{e}^{3\epsilon} t) && \textit{scalling.}
\end{aligned}
$$

2.2 Generalized symmetries

The symmetries defined by Definition 3 are called *the Lie point symmetries* or *geometric symmetries* since we can write explicitly corresponding point transformation. The integrable equations have also infinitely many so-called generalized symmetries. Existence of such symmetries is a characteristic feature of the integrable equations. For generalized symmetries we can not write point transformations explicitly. Therefore, working with such symmetries we stay on the level of infinitesimal generators.

Let $\mathcal{A}$ be the space of smooth functions depending on x, u and derivatives of u up to some finite order n where n is not fixed. A function $P(x, u^{(n)}) \in \mathcal{A}$ is a smooth function on some jet space $X \times U^{(n)}$. If it is not important how many derivatives of u that a function P depends on, we write $P(x, u^{(n)}) = P[u]$. Further, define $\mathcal{A}^l$ to be the space of l-tuples $(P_1[u], P_2[u], \ldots, P_l[u])$, where each $P_i \in \mathcal{A}$.

In new notations we can rewrite the equation (2.1) as

$$\Delta_\nu[u] = 0, \qquad \nu = 1, \dots, l. \tag{2.6}$$

Definition 5 *A generalized vector field is an expression of the form*

$$V = \sum_{i=1}^{p} \xi^i[u] \frac{\partial}{\partial x_i} + \sum_{i=1}^{q} \phi^i[u] \frac{\partial}{\partial u^i}. \tag{2.7}$$

So, a generalized vector field may depend not only on x, u but the derivatives of u as well.

Theorem 1 motivates the following definition.

Definition 6 *A generalized vector field is an infinitesimal symmetry of a system of differential equations (2.6) if and only if*

$$\mathrm{pr}V[\Delta_\nu] = 0, \qquad \nu = 1, \dots l, \tag{2.8}$$

for every smooth solution of (2.6).

The prolongation of V is taken up to the order of the system (2.6) and the prolongation formula for a generalized vector field coincides with (2.4).

There is a special class of generalized vector fields, so-called evolutionary vector fields.

Definition 7 *A generalized vector field of the form*

$$V_Q = \sum_{\alpha=1}^{q} Q_\alpha[u] \frac{\partial}{\partial u^\alpha} \tag{2.9}$$

is called an evolutionary vector field and q-tuple $Q[u] = (Q_1[u], \dots, Q_l[u]) \in \mathcal{A}^l$ *is called its characteristic.*

Any generalized vector field (2.7) has the associated evolutionary representative V_Q. The characteristic Q of the evolutionary representative has entries

$$Q_\alpha = \phi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha, \qquad \alpha = 1, 2, \dots q, \tag{2.10}$$

where $u_i^\alpha = \dfrac{\partial u^\alpha}{\partial x_i}$.

Theorem 2 *A generalized vector field is an infinitesimal symmetry generator of a system of differential equations if and only if its evolution representative is.*

A characteristic of an evolution vector field which is an infinitesimal symmetry of a system of differential equations is called *a generalized symmetry of the system.* As in the case of geometrical symmetries we have that the set of all infinitesimal generalized symmetries of a given system of differential equations forms a Lie algebra. Further in our work we deal only with evolution vector fields.

Example 3 *The first three symmetries for the KdV (1.1) equation are*

$$\begin{aligned} Q_0 &= u_x, \\ Q_1 &= u_{xxx} + 6uu_x, \\ Q_3 &= u_{xxxxx} + 10uu_{xxx} + 20u_xu_{xx} + 30u^2u_x, \end{aligned}$$

where Q_3 is the generalized symmetry.

2.3 Recursion operator

To find generalized symmetries one can use Definition 6. In such an approach one has to fix the order of derivatives on which a generalized symmetry may depend. So, one finds symmetry up to some fixed order. Another way to generate a whole infinite hierarchy of symmetries is to use a recursion operator. Although a recursion operator may not give all symmetries.

Definition 8 *A recursion operator for a system (2.1) is a linear operator* $\mathcal{R} : \mathcal{A}^q \to \mathcal{A}^q$ *in the space of q-tuples of differential functions with the property that if* $Q \in \mathcal{A}^q$ *is a generalized symmetry then* $\tilde{Q} = \mathcal{R}Q$ *is also a generalized symmetry.*

Hence, if we know a recursion operator for a system of differential equations we can generate infinitely many symmetries by applying recursion operator $\mathcal{R}$ successively, starting with some symmetry Q_0. The resulting hierarchy of symmetries is

$$Q_n = \mathcal{R}^n Q_0, \qquad n = 1, 2 \ldots. \tag{2.11}$$

To formulate a criteria for a differential operator to be a recursion operator we introduce the Fréchet derivative.

Definition 9 *Let* $P[u] \in \mathcal{A}^r$*. The Fréchet derivative of* P *is a linear differential operator*
$D_P : \mathcal{A}^q \to \mathcal{A}^r$ *defined by*

$$D_P[Q] = \frac{d}{d\epsilon} P[u + \epsilon Q[u]]\bigg|_{\epsilon=0}, \qquad Q[u] \in \mathcal{A}^q. \tag{2.12}$$

It follows that D_P is an $r \times q$ matrix differential operator with entries

$$(D_P)^\alpha_\beta = \sum_J \frac{\partial P^\alpha}{\partial u^\beta_J} D_J, \qquad \alpha = 1, 2, \ldots, r, \quad \beta = 1, 2, \ldots, q, \tag{2.13}$$

where summation is taken over all multi-indices J.

The action of an evolutionary vector filed can be described in terms of Fréchet derivative.

Theorem 3 *If $P[u] \in \mathcal{A}^r$ and $Q[u] \in \mathcal{A}^q$, then*

$$\mathrm{pr}V_Q(P) = D_P(Q). \tag{2.14}$$

The above theorem gives a new characterization for the generalized symmetries. A differential function $Q[u] \in \mathcal{A}^q$ is a symmetry of the equation (2.6) if and only if $D_\Delta(Q) = 0$. This leads to a characterization of a recursion operator.

Theorem 4 *For a system of differential equations $\Delta[u] = 0$ if $\mathcal{R} : \mathcal{A}^q \to \mathcal{A}^q$ is a linear operator such that*

$$D_\Delta \cdot \mathcal{R} = \tilde{\mathcal{R}} \cdot D_\Delta \tag{2.15}$$

for all solutions u of (2.6), where $\tilde{\mathcal{R}} : \mathcal{A}^q \to \mathcal{A}^q$ is a linear differential operator, then $\mathcal{R}$ is a recursion operator for the system.

For an evolution equation (2.2) the condition (2.15) takes the form

$$\mathcal{R}_t = [D_K, \mathcal{R}]. \tag{2.16}$$

Example 4 *The KdV equation (1.1) possesses a recursion operator*

$$\mathcal{R} = D_x^2 + \frac{2}{3}u + \frac{1}{3}D_x^{-1}. \tag{2.17}$$

As the following example shows an evolution equations may have more than one recursion operator.

Example 5 *The Burgers equation*

$$v_t = v_{xx} + vv_x \tag{2.18}$$

possesses two recursion operators

$$\mathcal{R}_1 = D_x + \frac{1}{2}v + \frac{1}{2}v_x D_x^{-1}, \qquad \mathcal{R}_2 = t\mathcal{R}_1 + \frac{1}{2}x + \frac{1}{2}D_x^{-1}. \tag{2.19}$$

The operators in the previous examples are formal pseudo-differential operators. We assume that D_x^{-1} (the formal inverse of D_x) is defined only on functions that are total derivatives. It is possible to show for these operators that if we start with a symmetry u_x then every symmetry in the resulting hierarchy is a total derivative, see [15].

2.4 Conservation Laws

Knowing a recursion operator for a given system one can also generate infinitely many conservation laws for the system. Conservation laws are mathematical formulation of physical laws of conservation. They play an important role in studying properties of solutions.

Definition 10 *For a system (2.1) a conservation law is a divergence expression*

$$\mathrm{Div}\, P[u] = D_{x_1}P_1[u] + D_{x_2}P_2[u] + \ldots D_{x_p}P_p[u] = 0 \tag{2.20}$$

on all solutions of (2.1). Here P is a p-tuple of smooth functions.

In the case of an evolution equation (2.2), the conservation law takes the form

$$D_tT[u] + \mathrm{Div}\, X[u] = 0, \tag{2.21}$$

where $T[u]$ is called a conserved density and $X[u]$ conserved flux.

To give a construction of conservation laws we introduce the formal algebra of pseudo-differential operators.

Definition 11 *A one-dimentional pseudo-differential operator of order n is a formal series*

$$\mathcal{D} = \sum_{i=-\infty}^{n} A_i[u] D_x^i \tag{2.22}$$

where $A_i \in \mathcal{A}^q$. The operator D_x^{-1} is the formal inverse of D_x ($D_x \cdot D_x^{-1} = D_x^{-1} \cdot D_x = 1$).

The multiplication of pseudo-differential operators is described by the formulas

$$D_x^i \cdot D_x^j = D_x^{j+i} \tag{2.23}$$

and

$$D_x Q = Q D_x + Q_x, \qquad Q \in \mathcal{A}, \tag{2.24}$$

which imply the action of D_x^{-1} as

$$D_x^{-1} Q = Q D_x^{-1} + D_x^{-1} Q D_x^{-1} = Q D_x^{-1} - Q_x D_x^{-2} + Q_{xx} D_x^{-3} \dots \tag{2.25}$$

The advantage of introduction of pseudo-differential operators is that now we can take inverse and roots of any pseudo-differential operator.

Lemma 1 *Every nonzero pseudo-differential operator has the inverse.*

Lemma 2 *Every nonzero pseudo-differential operator of order n has the n-th root.*

To find the inverse of a pseudo-differential operator $\mathcal{D}$ we write the inverse operator $\mathcal{D}^{-1}$ with undetermined coefficients and find these coefficients recursively from the equality $\mathcal{D} \cdot \mathcal{D}^{-1} = 1$. In the same way, using the equality $\mathcal{D}^{\frac{1}{n}} \cdot \ldots \cdot \mathcal{D}^{\frac{1}{n}} = \mathcal{D}$ we find roots of a pseudo-differential operator.

Now we can take rational powers of a recursion operator. The coefficients of D_x^{-1} of appropriate powers of a recursion operator will provide conserved densities for a given system.

Definition 12 *The residue of a pseudo-differential operator $\mathcal{D}$ is the coefficient of D_x^{-1}*

$$Res \sum_{i=-\infty}^{n} P_i D_x^i = P_{-1}.$$

Lemma 3 *For any two pseudo-differential operators $\mathcal{D}$ and $\mathcal{B}$ the residue of their commutator is a total x-derivative*

$$\mathrm{Res}[\mathcal{D}, \mathcal{B}] = D_x Q, \tag{2.26}$$

where $Q \in \mathcal{A}^q$.

Let $\mathcal{R}$ be a recursion operator of an evolution equation

$$u_t - K[u] = 0. \tag{2.27}$$

Then it satisfies the equality (2.16)

$$(\mathcal{R})_t + [\mathcal{R}, D_K] = 0. \tag{2.28}$$

It is possible to show that any power of a recursion operator, if exists, satisfies the same equality

$$(\mathcal{R}^{\beta})_t + [\mathcal{R}^{\beta}, D_K] = 0. \tag{2.29}$$

The residue of $((\mathcal{R}^{\beta})_t + [\mathcal{R}^{\beta}, D_K])$ has the form of a conservation law

$$D_t T + D_x X = 0, \tag{2.30}$$

where $T = \mathrm{Res}(\mathcal{R}^{\beta})$ and $X = \mathrm{Res}[\mathcal{R}^{\beta}, \mathcal{D}]$.

Example 6 *Consider the recursion operator (2.17) of the KdV equation. The coefficient* $\frac{1}{3}u_x$ *of* D_x^{-1} *provides us with a conserved density. This density is trivial, since it is a total derivative. The square root of the recursion operator*

$$(\mathcal{R})^{\frac{1}{2}} = D_x + \frac{1}{3}uD_x^{-1} - \frac{1}{18}u^2 D_x^{-3} + \dots \tag{2.31}$$

gives a nontrivial density $\frac{1}{3}u$. *Taking powers* $\mathcal{R}^{\frac{2m+1}{2}}, \quad m \in \mathbb{N}$, *we obtain infinitely many conserved densities.*

2.5 Hamiltonian formalism

The Hamiltonian formalism is very important in the theory of integrability. In particular, bi-Hamiltonian systems, that is systems that admit two Hamiltonian representations on the same set of coordinates, are of great interest. For a bi-Hamiltonian system there is an easy way to construct an infinite hierarchy of conservation laws and recursion operator.

In Hamiltonian formulation of the evolution equations

$$u_t = K[u], \qquad K \in \mathcal{A}^q, \tag{2.32}$$

we use Poisson bracket formalism.

The scalar fields on the space $\mathcal{A}$ are functionals $\mathcal{P} = \int P[u]dx$, $P \in \mathcal{A}$. We denote the space of all such functionals on $\mathcal{A}$ by $\mathfrak{F}$. The derivative of such a scalar field is a variational derivative $\delta\mathcal{P} = \delta \int P[u]dx = (\mathrm{E}_1(P), \ldots \mathrm{E}_q(P)) \in \mathcal{A}^q$, where

$$\mathrm{E}_\nu(P) = \sum_J (-D)_J \frac{\partial P}{\partial u^\nu_J}, \tag{2.33}$$

the sum is extending over all multi-indices $J = (j_1, \ldots, j_k)$ with $1 \leq j_k \leq p$ and $u^\nu_J = D_J u^\nu$. The operator $E = (E_1, \ldots, E_q)$ is called *Euler operator.*

A bracket of two scalar fields is defined by

$$\{\mathcal{P}, \mathcal{Q}\} = \int \delta\mathcal{P}\mathcal{D}\delta\mathcal{Q}dx, \qquad \mathcal{P}, \mathcal{Q} \in \mathfrak{F}, \tag{2.34}$$

where $\mathcal{D} : \mathcal{A}^q \to \mathcal{A}^q$ is a differential operator.

Definition 13 *The bracket defined by (2.34) is a Poison bracket if for all functionals $\mathcal{P}, \mathcal{Q}, \mathcal{O} \in \mathfrak{F}$ the following conditions holds*

1. *skew symmetry*

$$\{\mathcal{P}, \mathcal{Q}\} = -\{\mathcal{Q}, \mathcal{P}\}$$

2. *Jacobi identity.*

$$\{\mathcal{O}, \{\mathcal{P}, \mathcal{Q}\}\} + \{\mathcal{Q}, \{\mathcal{O}, \mathcal{P}\}\} + \{\mathcal{P}, \{\mathcal{Q}, \mathcal{O}\}\} = 0.$$

For a Poisson bracket the corresponding operator $\mathcal{D}$, defined in (2.34), is called a Hamiltonian operator.

For every scalar field $\mathcal{P}$ we can give an associated evolutionary Hamiltonian vector field.

Lemma 4 *Let $\mathcal{D}$ be a Hamiltonian operator with Poisson bracket (2.34). For each functional $\mathcal{P} = \int P dx \in \mathfrak{F}$, there is an evolutionary vector field $\hat{V}_{\mathcal{P}}$, called the Hamiltonian vector field associated with $\mathcal{P}$, such that*

$$\mathrm{pr}\hat{V}_{\mathcal{P}}(\mathcal{O}) = \{\mathcal{O}, \mathcal{P}\} \tag{2.35}$$

for any functional $\mathcal{O} \in \mathfrak{F}$. The characteristic of $\hat{V}_{\mathcal{P}}$ is $\mathcal{D}\delta\mathcal{P}$.

The Hamiltonian flow corresponding to a functional $\mathcal{P} = \int P dx \in \mathfrak{F}$ is determined by Hamiltonian vector field $\hat{V}_{\mathcal{P}}$. Hence, the Hamiltonian system of evolution equations takes the form

$$u_t = \mathcal{D}\delta\mathcal{P}. \tag{2.36}$$

Example 7 *The KdV (1.1) equation admits two Hamiltonian representations*

$$u_t = D_x \delta \int (-\frac{1}{2}u_x^2 + \frac{1}{6}u^3) dx \tag{2.37}$$

with Hamiltonian operator $\mathcal{D}_1 = D_x$ and

$$u_t = (D_x^3 + \frac{1}{3}uD_x + \frac{1}{3}D_x \cdot u)\delta \int \frac{1}{2}u^2 dx \tag{2.38}$$

with Hamiltonian operator $\mathcal{D}_2 = D_x^3 + \frac{1}{3}uD_x + \frac{1}{3}D_x \cdot u$

The characterization of Hamiltonian operators is given by the following theorem.

Theorem 5 *Let $\mathcal{D}$ be a differential operator on space of functionals $\mathfrak{F}$. Then $\mathcal{D}$ is a Hamiltonian operator if the following conditions holds*

1.
$$\int \delta\mathcal{P}\mathcal{D}\delta\mathcal{Q} = -\int \delta\mathcal{Q}\mathcal{D}\delta\mathcal{P}, \qquad \mathcal{P}\mathcal{Q} \in \mathfrak{F}$$

2.
$$\int [P \cdot \mathrm{pr}V_{\mathcal{D}R}Q + R \cdot \mathrm{pr}V_{\mathcal{D}P}P + Q \cdot \mathrm{pr}V_{\mathcal{D}Q}P]dx = 0$$
for any q-tuples $P, Q, R \in A^q$.

For Hamiltonian equations there is a close relation between symmetries and conservation laws. Each generalized symmetry, which corresponds to a Hamiltonian vector field, gives rise to a conserved quantity.

Definition 14 *A scalar field $\mathcal{F}[u]$ is said to be a conserved quantity of evolution equation (2.32) if it is constant along the integral curves, i.e.*

$$\frac{d}{dt}\mathcal{F}[u] = \frac{\partial \mathcal{F}}{\partial t} + \mathrm{pr}V_K\mathcal{F} = 0 \tag{2.39}$$

for any solution u of (2.32).

If V_K is a Hamiltonian vector field, $K = \mathcal{D}\mathcal{H}$, then the condition (2.39) becomes

$$\frac{d}{dt}\mathcal{F}[u] = \frac{\partial \mathcal{F}}{\partial t} + \{\mathcal{F}, \mathcal{H}\} = 0. \tag{2.40}$$

Time independent distinguished functionals are conserved for any Hamiltonian equation. Other conserved quantities can be obtained from Hamiltonian symmetries.

Theorem 6 *Let $\hat{V}_{\mathcal{P}}$ be a Hamiltonian vector field with characteristic $\mathcal{D}\delta\mathcal{P}$. The vector field $\hat{V}_{\mathcal{P}}$ is an infinitesimal generalized symmetry of the system (2.36) if and only if there is an functional $\tilde{\mathcal{P}} = \mathcal{P} - \mathcal{E}$, where $\mathcal{E}$ is a time dependent distinguished functional, such that $\tilde{\mathcal{P}}$ determines a conserved quantity.*

Using the above theorem we can construct a hierarchy of generalized symmetries for bi-Hamiltonian systems. Consider an evolution system that has two Hamiltonian representation

$$u_t = K[u] = \mathcal{D}_1\delta\mathcal{H}_0 = \mathcal{D}_2\delta\mathcal{H}_1. \tag{2.41}$$

The functionals $\mathcal{H}_0$ and $\mathcal{H}_1$ are conserved. So by the Theorem 6, we have that the original vector field $V_K = V_{\mathcal{D}_1\delta\mathcal{H}_0} = V_{\mathcal{D}_2\delta\mathcal{H}_1}$ and two additional vector fields $V_{\mathcal{D}_1\delta\mathcal{H}_1}$ and $V_{\mathcal{D}_2\delta\mathcal{H}_0}$ give generalized symmetries $\mathcal{D}_1\delta\mathcal{H}_0 = \mathcal{D}_2\delta\mathcal{H}_1$ and $\mathcal{D}_1\delta\mathcal{H}_1$, $\mathcal{D}_2\delta\mathcal{H}_0$ respectively. Assume that the vector field $V_{\mathcal{D}_1\delta\mathcal{H}_1}$ is Hamiltonian with respect to the operator $\mathcal{D}_2$, so,

$$\mathcal{D}_1\delta\mathcal{H}_1 = \mathcal{D}_2\delta\mathcal{H}_2 \tag{2.42}$$

for some functional $\mathcal{H}_2$. Again, by Theorem 6 the functional $\mathcal{H}_2$ is conserved and we can obtain a new generalized symmetry $\mathcal{D}_1\delta\mathcal{H}_2$. Next we find $\mathcal{H}_3$ such that $\mathcal{D}_1\delta\mathcal{H}_2 = \mathcal{D}_2\delta\mathcal{H}_3$ and obtain a new generalized symmetry $\mathcal{D}_1\delta\mathcal{H}_3$. We can continue and produce a whole hierarchy of symmetries K_n satisfying the recursion relation

$$K_n = \mathcal{D}_2\delta\mathcal{H}_n = \mathcal{D}_1\delta\mathcal{H}_{n-1}, \qquad n = 1, 2\ldots \tag{2.43}$$

We can define a recursion operator $\mathcal{R} = \mathcal{D}_1\mathcal{D}_2^{-1}$, so, that $\mathcal{R}K_{n-1} = K_n$. To use the above construction of the hierarchy of Hamiltonian symmetries we must put some restrictions on the operators $\mathcal{D}_1$ and $\mathcal{D}_2$, namely, the operators $\mathcal{D}_1$ and $\mathcal{D}_2$ must be compatible, i.e., form a Hamiltonian pair.

Definition 15 *Two Hamiltonian operators $\mathcal{D}_1$ and $\mathcal{D}_2$ are compatible, form a Hamiltonian pair, if any linear combination $a\mathcal{D}_1 + b\mathcal{D}_2, \quad a, b \in \mathbb{R}$, is a Hamiltonian operator.*

Theorem 7 *The Hamiltonian operators $\mathcal{D}_1$ and $\mathcal{D}_2$ are compatible if and only if $\mathcal{D}_1 + \mathcal{D}_2$ is a Hamiltonian operator.*

The importance of compatibility condition is shown by the following lemma.

Lemma 5 *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be compatible Hamiltonian operators. Suppose $P, Q, R \in \mathcal{A}^q$ satisfy*

$$\mathcal{D}_2 P = \mathcal{D}_1 Q, \qquad \mathcal{D}_2 Q = \mathcal{D}_1 R,$$

where $P = \delta\mathcal{P}$ and $Q = \delta\mathcal{Q}$, $\mathcal{P}, \mathcal{Q} \in \mathfrak{F}$. Then R is also a variational derivative of some functional $\mathcal{R} \in \mathfrak{F}$, i.e., $R = \delta\mathcal{R}$.

Now we state the main theorem on bi-Hamiltonian systems.

Theorem 8 *Consider bi-Hamiltonian system (2.41). Assume that the operators $\mathcal{D}_1$ and $\mathcal{D}_2$ are compatible and the operator D_2 is non-degenerate. Define $\mathcal{R} = \mathcal{D}_1\mathcal{D}_2^{-1}$ and put $K_0 = \mathcal{D}_1\delta\mathcal{H}_1$. Assume also that for each $n = 1, 2, \dots$ we can define*

$$K_n = \mathcal{R}K_{n-1} \qquad n = 1, 2, \dots$$

Then there exist a sequence of functionals $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \dots$ such that

1. *the evolution equation*

$$u_t = K_n[u] = \mathcal{D}_1 \delta \mathcal{H}_n = \mathcal{D}_2 \delta \mathcal{H}_{n-1} \tag{2.44}$$

   is bi-Hamiltonian for any $n = 1, 2, \ldots$

2. *the functionals* $\mathcal{H}_n$ *are all in involution with respect to Poisson brackets defined by* $\mathcal{D}_1$ *and* $\mathcal{D}_2$

$$\{\mathcal{H}_m, \mathcal{H}_n\}_{\mathcal{D}_1} = \{\mathcal{H}_m, \mathcal{H}_n\}_{\mathcal{D}_2} = 0. \tag{2.45}$$

Hence the functionals $\mathcal{H}_n$ *provide an infinite hierarchy of generalized symmetries and conservation laws.*

Without the assumption that operators $\mathcal{D}_1$ and $\mathcal{D}_2$ are compatible we have still have the recursion operator $\mathcal{R} = \mathcal{D}_1 \mathcal{D}_2^{-1}$ but the symmetries may not be Hamiltonian.

Chapter 3

Hamiltonian formulation and the Lax Representation

The Lax equation can be treated as an evolution system on the Lie algebra of Lax operators. Such approach allows to construct naturally a multi-Hamiltonian representation for the Lax equation and, moreover, to obtain an infinite hierarchy of such equations. Presenting R-matrix formalism we follow [12].

3.1 Construction of Poisson structure

Definition 16 *Let A be associative algebra with unit I. If there is a Lie bracket on A such that for each element $a \in A$, the operator $ad_a : b \to [a, b]$ is a derivation of the multiplication, then $(A, [.,.])$ is called a Poisson algebra.*

We assume that the algebra A is equipped with a non-degenerate pairing

$$(.,.) : A \times A \to A, \tag{3.1}$$

symmetric with respect to multiplication, $(ab, c) = (a, bc), \quad a, b, c \in A$, and invariant under adjoint map, $([a, b], c) = (a, [b, c]), \quad a, b, c \in A$. We can identify algebra A with its dual algebra A^* via pairing $(.,.)$.

Let $\mathfrak{C}(A)$ be a space of smooth functions on the algebra A. Note that a function $f : A \to \mathbb{R}$ is smooth if there exists a map $df : A \to A$ such that

$$\left.\frac{d}{dt}\right|_{t=0} f(a + ta') = (df(a), a'), \quad a, a' \in A, \tag{3.2}$$

df is a gradient of function $f \in \mathfrak{C}(A)$.

On the space $\mathfrak{C}(A)$ there exists a natural Poisson bracket

$$\{f, g\}(a) = (a, [df, dg]), \qquad a \in A. \tag{3.3}$$

To obtain other Poisson brackets we use R-matrices.

Definition 17 *A linear operator $R \in \mathrm{End}(A)$ is called a classical R-matrix if the R-bracket given by*

$$[a, b]_R = \frac{1}{2}([Ra, b] + [a, Rb]), \quad b, b \in A,$$

is a Lie bracket.

Theorem 9 *A sufficient condition for R to be an R-matrix is*

$$[R(a), R(b)] - R([a, b]_R) = -[a, b]. \tag{3.4}$$

This equation is called *the modified Yang-Baxter equation.*

There is a special class of solutions of Yang-Baxter equation (3.4) important for further applications. Let the algebra A be split into direct sum of two Poisson subalgebras

$$A = A_+ \oplus A_- \tag{3.5}$$

and P_+, P_- are corresponding projections onto this subalgebras. Then the linear operator

$$R = 2(P_+ - P_-) \tag{3.6}$$

satisfies the Yang-Baxter equation (3.4). Hence R is an R-matrix.
Now we introduce brackets on $\mathfrak{C}(A)$

$$\begin{aligned} \{f, g\}_1(L) &= (ad_L df, R(dg)) - (ad_L dg, R(df)) \\ \{f, g\}_2(L) &= (ad_L df, R(Ldg + dgL)) - (ad_L dg, R(Ldf + dfLL)) \\ \{f, g\}_3(L) &= (ad_L df, R(LdgL)) - (ad_L dg, R(LdfL)). \end{aligned} \tag{3.7}$$

Theorem 10

1. *For any R-matrix on A the bracket $\{.,.\}_1$ is a Poisson bracket.*
2. *If R and $\tilde{R} = \frac{1}{2}(R - R^*)$ (where R^* is the adjoint of R with respect to the pairing (3.1)) satisfy equation (3.4) then $\{.,.\}_2$ is a Poisson bracket.*
3. *If R satisfy (3.4) then $\{.,.\}_3$ is a Poisson bracket.*

All three Poisson structures are compatible with each other.

The Hamiltonian operators corresponding to the brackets (3.7) are

$$
\begin{aligned}
\mathcal{D}_1(L) &= ad_L df, R(df) - R^*(ad_L df), \\
\mathcal{D}_2(L) &= -ad_L R(Ldf + dfL) - LR^*(ad_L df) - R^*(ad_L df)L, \qquad (3.8)\\
\mathcal{D}_3(L) &= -ad_L R(LdfL)) - LR^*(ad_L df)L
\end{aligned}
$$

respectively.

In the case of the commutative algebra A we have infinitely many Poisson structures [30].

Theorem 11 *Let A be a commutative Poisson algebra equipped with a non-degenerate, symmetric, ad-invariant pairing (3.1). Assume $R \in End(A)$ is an R-matrix, then for each integer $n \geq -1$, the formula*

$$\{f, h\}_{(n)}(L) = (L, [R(L^{(n+1)}df(L)), dh(L)] + [df(L), R(L^{(n+1)}dh(L))]) \,, \qquad (3.9)$$

where $f, h \in \mathfrak{C}(A)$, defines a Poisson structure on A . All the structures $\{.,.\}_n$, $\quad n = -1, 0, 1, 2 \ldots$, are compatible with each other.

The compatibility of Poisson structures defined by (3.7) and (3.9) shows that they may give a multi-Hamiltonian systems.

Theorem 12 *All the functions $f \in \mathfrak{C}(A)$ satisfying*

$$[df(L), L] = 0, \qquad L \in A, \qquad (3.10)$$

are in involution with respect to all brackets (3.7). The Hamiltonian equations *associated with such functions are Lax equations*

$$\begin{aligned} L_t &= \mathcal{D}_1 df = [R(df), L], \\ L_t &= \mathcal{D}_2 df = [R(2Ldf), L], \\ L_t &= \mathcal{D}_3 df = [R(L^2 df), L]. \end{aligned} \tag{3.11}$$

To apply the above theorem we assume that the pairing (3.1) is given in terms of a non-degenerate trace functional $\mathrm{tr} : A \to \mathbb{R}$

$$(a, b) = \mathrm{tr}(ab), \qquad a, b \in A. \tag{3.12}$$

Corollary 1 *For any function $f_m = \dfrac{1}{m}\mathrm{tr}(L^m)$, $m \in \mathbb{N}$, we have $df_m = L^{m-1}$ and $[df_m(L), L] = 0$. Hence, taking f_m as a Hamiltonian function, we find a hierarchy of tri-Hamiltonian equations*

$$\frac{dL}{dt_m} = [R(L^m), L] = \mathcal{D}_1 df_{m+1} = \frac{1}{2}\mathcal{D}_2 df_m = \mathcal{D}_3 df_{m-1}, \qquad m \in \mathbb{N}. \tag{3.13}$$

All functions f_m are in involution. Therefore, they are conserved quantities for any equation in the hierarchy.

Construction of Poisson structures described in the Theorem 10 and Theorem 11 involves infinitely many fields. Working with Lax equation (1.10) we have a restriction of the original algebra to a subspace involving finite number of fields. If the subspace is not a Poisson sub-manifold for a given Poisson structure then one should apply Dirac reduction to obtain a Poisson structure on the subspace.

Theorem 13 *For two linear subspaces U and V spanned by $u \in U$ and $v \in V$ let*

$$\Theta(u, v) = \begin{pmatrix} \Theta_{uu}(u, v) & \Theta_{uv}(u, v) \\ \Theta_{vu}(u, v) & \Theta_{vv}(u, v) \end{pmatrix}$$

be a Hamiltonian operator on $U \oplus V$. Also let Θ_{vv} be invertible. Then, for arbitrary $c \in V$ the operator

$$\Theta(u, c) = \Theta_{uu}(u, c) - \Theta_{uv}(u, c)[\Theta_{vv}(u, c)]^{-1}\Theta_{vu}(u, c)$$

is a Hamiltonian operator on the affine space $c + U$.

Now we apply the above formalism to some examples of Poisson algebras.

3.2 Algebra of pseudo-differential operators

Let us consider algebra of pseudo-differential operators

$$A = \left\{ L = \sum_{i=-\infty}^{n} u_i(x) D_x^i \right\}, \tag{3.14}$$

where u_i are smooth functions on the unit circle S^1. With the Lie bracket

$$[L_1, L_2] = L_1 \cdot L_2 - L_2 \cdot L_1, \tag{3.15}$$

A is a Poisson algebra. We can equipped A with all the necessary ingredients to construct multi-Hamiltonian systems.

Lemma 6 *The trace form*

$$\operatorname{tr} L = \int_{S^1} \operatorname{Res} L dx, \tag{3.16}$$

where $\operatorname{Res} L$ *is defined by (12), yields a symmetric, ad-invariant and non-degenerate pairing*

$$(L_1, L_2) = \mathrm{tr}(L_1 \cdot L_2). \tag{3.17}$$

We can split A into direct sum of two Poisson subalgebras

$$A = (A)_{\geq 0} \oplus (A)_{<0}, \tag{3.18}$$

where

$$(A)_{\geq 0} = \left\{ L = \sum_{0 \leq i < +\infty} u_i(x) D_x^i \right\},$$

$$(A)_{<0} = \left\{ L = \sum_{i<0} u_i(x) D_x^i \right\},$$

with $k = 0$. Hence, we have R-matrix

$$R = P_{\geq 0} - P_{<0}, \tag{3.19}$$

where $P_{\geq 0}$ and $P_{<0}$ are natural projections on $(A)_{\geq 0}$ and $(A)_{<0}$ respectively.

Now using Corollary 1 we can obtain the integrable multi-Hamiltonian hierarchies.

Example 8 *Let us take $k = 0$ in (3.19) and $L = D_x^2 + u$. Then resulting hierarchy is*

$$L_{t_n} = [(L^{\frac{2n+1}{2}})_{k \geq 0}, L], \qquad n = 0, 1, 2 \ldots . \tag{3.20}$$

To obtain nontrivial equations we take fractional powers of L. Since L is a second order operator we take $L^{\frac{2n+1}{2}}$.

The third equation of this hierarchy is the KdV equation (1.1). The Hamiltonian formulation and recursion operator of this hierarchy was considered in Examples 4 and 7.

3.3 Algebra of Laurent series

Consider the algebra Λ of Laurent series in p

$$\Lambda = \left\{ u(x,p) = \sum_{-\infty}^{+\infty} u_i(x) p^i \right\}, \tag{3.21}$$

where the coefficients u_i are smooth functions on the unit circle S^1. With the Lie bracket defined by

$$\{u, v\} = \frac{\partial u}{\partial p}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial p}, \tag{3.22}$$

$(\Lambda, \{.,.\})$ is a Poisson algebra. The Poisson algebra Λ admits the trace functional

$$\operatorname{tr} u(x,p) = \int_{S^1} \operatorname{Res} u(x,p) dx, \qquad u \in \Lambda, \tag{3.23}$$

where the residue is defined by

$$\operatorname{Res} \sum_{i=-\infty}^{+\infty} u_i(x) p^i = u_{-1}(x). \tag{3.24}$$

Lemma 7 *The trace form*

$$\operatorname{tr} L = \int_{S^1} \operatorname{Res} L dx \tag{3.25}$$

yields a symmetric, ad-invariant and non-degenerate pairing

$$(L_1, L_2) = \operatorname{tr}(L_1 \cdot L_2). \tag{3.26}$$

We introduce the R-matrix associated with the direct sum decomposition

$$A = A_{\geq 1} \oplus A_{\leq 0} \tag{3.27}$$

into subalgebras

$$A_{\geq 1} = \left\{ u \in A : \ u(x, \lambda) = \sum_{i \geq 1} u_i(x) p^i \right\},$$

$$A_{\leq 0} = \left\{ u \in A : \ u(x, \lambda) = \sum_{i \leq 0} u_i(x) p^i \right\}.$$

Since Λ is a commutative algebra we can apply Theorem 11 and obtain an infinite family of Poisson structures $\{.,.\}_n, \quad n = -1, 0, 1 \ldots,$ on A.

Example 9 *The Benny equation in nonlinear waves [38] is given by the quasi-linear system*

$$\begin{aligned} u_{0t} &= u_0 u_{0x} + u_{-1x}, \\ u_{-1t} &= u_{-1} u_{0x} + u_0 u_{-1x}. \end{aligned} \tag{3.28}$$

We consider the case where u_0, u_{-1} are smooth functions on the circle S^1. In [21] , the Benny equation is rewritten as a disspersionless Lax equation on the Poisson algebra Λ. Indeed, the equation (3.28) is equivalent to

$$\frac{dL}{dt} = \left\{ \left(\frac{1}{4} L^2 \right)_{k \geq 1}, L \right\}, \tag{3.29}$$

where the Lax function L is an element of Benny manifold

$$M_{Benny} = \left\{ L \in A : \ L(x,p) = p + u_0(x) + u_{-1} p^{-1} \right\}. \tag{3.30}$$

By Corollary 1 we have a hierarchy of symmetries of (3.28)

$$\frac{dL}{dt_n} = \left\{ \left(\frac{1}{4} L^n \right)_{k \geq 1}, L \right\}. \tag{3.31}$$

The first two Hamiltonian structures associated with the hierarchy are

$$H_{-1}(u) = \begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix} \tag{3.32}$$

and

$$H_0(u) = \begin{pmatrix} D_x & u_0 D_x + u_{0x} \\ u_0 D_x & 2u_{-1} D_x + u_{-1_x} \end{pmatrix}. \tag{3.33}$$

Chapter 4

Polynomial Lax function

The case of the polynomial Lax function was considered by Gürses and Zheltukhin in [28].

4.1 Dispersionless Lax formulation

Consider an algebra of Laurent series introduced in Section 3.3

$$\Lambda = \left\{ \sum_{-\infty}^{+\infty} u_i p^i \ : u_i \in C^\infty(S^1) \right\} \tag{4.1}$$

but with modified bracket

$$\{f,g\}_k = p^k \left(\frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} \right), \qquad f,g \in \Lambda, \tag{4.2}$$

where k is an integer parameter.

Lemma 8 *For any $k \in \mathbb{Z}$ bracket (4.2) is a Poisson bracket.*

Proof. We should check only the Jacobi identity. Other properties of a Poisson bracket are evidently true. Recall that $\{f,g\}$ denotes standard Poisson bracket

$$\{f,g\}=\frac{\partial f}{\partial p}\frac{\partial g}{\partial x}-\frac{\partial f}{\partial x}\frac{\partial g}{\partial g}. \tag{4.3}$$

Let us show that

$$\{\{f,g\}_k,h\}_k+\{\{h,f\}_k,g\}_k+\{\{g,h\}_k,f\}_k=0.$$

First, note that

$$\{\{f,g\}_k,h\}_k=p^k\{p^k\{f,g\},h\}=p^{2k}\{\{f,g\},h\}+kp^{k-1}\{f,g\}h_x.$$

Therefore, we have

$$\{\{f,g\}_k,h\}_k+\{\{h,f\}_k,g\}_k+\{\{g,h\}_k,f\}_k=p^{2k}(\{\{f,g\},h\}+\{\{h,f\},g\}+\{\{g,h\},f\})+$$
$$kp^{k-1}(\{f,g\}h_x+\{h,f\}g_x+\{g,h\}f_x).$$

Equality

$$\{\{f,g\},h\}+\{\{h,f\},g\}+\{\{g,h\},f\}=0$$

holds and it is easy to check that

$$\{f,g\}h_x+\{h,f\}g_x+\{g,h\}f_x=0.$$

Hence, formula (4.2) defines a Poisson bracket. □

The bracket (4.2) is is equivalent to bracket $\{.,.\}$, under $p^k\,\dfrac{d}{dp}=\dfrac{d}{dq}$, where q is the new variable, we shall keep using it. The main reason is technical. There is a nice duality between the systems obtained by polynomial Lax representation, $L=p^N+\cdots$, with Poisson bracket $\{.,.\}_k$ and by Lax representation $L=p^\gamma\left[p^N+\cdots\right]$ with Poisson bracket $\{.,.\}$, which we consider in Chapter 6. For illustration we have examples, equations governing the polytropic gas dynamics, given in Chapter 6.

In the same way as for algebra $(\Lambda, \{.,.\})$ in Section 3.3 we equipped the algebra $(\Lambda, \{.,.\}_k)$ with all the ingredients to construct integrable hierarchies. For algebra $(\Lambda, \{.,.\}_k)$ define the trace functional

$$\operatorname{tr} u(x,p) = \int_{S^1} \operatorname{Res}_k u(x,p)dx, \qquad u \in \Lambda, \tag{4.4}$$

where the residue $\operatorname{Res}_k u(x,p)$ is defined by

$$\operatorname{Res}_k \sum_{i=-\infty}^{+\infty} u_i(x)p^i = u_{k-1}(x). \tag{4.5}$$

Lemma 9 *The trace form*

$$\operatorname{tr}_k L = \int_{S^1} \operatorname{Res}_k L dx \tag{4.6}$$

yields a symmetric, ad-invariant and non-degenerate pairing

$$(L_1, L_2)_k = \operatorname{tr}_k(L_1 \cdot L_2). \tag{4.7}$$

Proof. Since we have

$$\{f,g\}_k = p^k\{f,g\}, \qquad f,g \in \Lambda, \tag{4.8}$$

it is easily follows from Lemma 7 that the pairing

$$(L_1, L_2)_k = \operatorname{tr}_k(L_1 \cdot L_2) \tag{4.9}$$

is symmetric, ad-invariant and non-degenerate. □

We introduce the R-matrix associated with the direct sum decomposition

$$A = A_{\geq k+1} \oplus A_{\leq k} \tag{4.10}$$

into subalgebras

$$A_{\geq k+1} = \left\{ u \in A : \; u(x,\lambda) = \sum_{i \geq k+1} u_i(x) p^i \right\}$$

$$A_{\leq k} = \left\{ u \in A : \; u(x,\lambda) = \sum_{i \leq k} u_i(x) p^i \right\}$$

Now by Corollary 1 for each $k \in \mathbb{Z}$ we can consider multi-Hamiltonian hierarchies of equations of hydrodynamic type, defined in terms of the polynomial Lax function

$$L = p^{N-1} + \sum_{i=-1}^{N-2} p^i S_i(x,t) \tag{4.11}$$

by the Lax equation

$$\frac{\partial L}{\partial t_n} = \left\{ (L^{\frac{n}{N-1}})_{\geq -k+1}, L \right\}_k, \tag{4.12}$$

where $n = j + l(N-1)$ and $j = 1, 2, \ldots, (N-1)$, $l \in \mathbf{N}$. So we have a hierarchy for each k and $j = 1, \ldots, (N-1)$. Also, we require $n \geq -k+1$ to ensure that $(L^{\frac{n}{N-1}})_{\geq -k+1}$ is not zero.

The Lax function (4.11) can also be written in terms of symmetric variables $u_1, \ldots, u_N$

$$L = \frac{1}{p}\prod_{j=1}^{N}(p-u_j), \tag{4.13}$$

that is $u_1, \ldots, u_N$ are zeros of the polynomial

$$p^{N-1} + S_{N-2}p^{N-2} + \ldots + S_{-1}p^{-1}.$$

In new variables the equation (4.12) is invariant under transposition of variables.

4.2 Recursion Operators

For each hierarchy of the equations (4.12), depending on the pair (N,k), we can construct a recursion operator.

Lemma 10 *For any $n \in \mathbb{N}$*

$$L_n = LL_{n-(N-1)} + \{R_n, L\}_k, \tag{4.14}$$

where function R_n has a form

$$R_n = \sum_{i=0}^{N-2} p^{i-k} A_i \left(S_{-1} \ldots S_{N-2}, \frac{\partial S_{-1}}{\partial t_{n-(N-1)}} \ldots \frac{\partial S_{N-2}}{\partial t_{n-(N-1)}} \right). \tag{4.15}$$

Proof.

$$(L^{\frac{n}{N-1}})_{\geq -k+1} = [L(L^{\frac{n}{N-1}-1})_{\geq -k+1} + L(L^{\frac{n}{N-1}-1})_{<-k+1}]_{\geq -k+1}$$

So,

$$(L^{\frac{n}{N-1}})_{\geq -k+1} = L(L^{\frac{n}{N-1}-1})_{\geq -k+1} + (L(L^{\frac{n}{N-1}-1})_{<-k+1})_{\geq -k+1} \\ -(L(L^{\frac{n}{N-1}-1})_{\geq -k+1})_{<-k+1}. \quad (4.16)$$

If we put

$$R_n = (L(L^{\frac{n}{N-1}-1})_{<-k+1})_{\geq -k+1} - (L(L^{\frac{n}{N-1}-1})_{\geq -k+1})_{<-k+1} \quad ,$$

then

$$(L^{\frac{n}{N-1}})_{\geq -k+1} = L(L^{\frac{n}{N-1}-1})_{\geq -k+1} + R_n.$$

Hence,

$$L_n = \left\{(L^{\frac{n}{N-1}})_{\geq -k+1}, L\right\}_k = \left\{L(L^{\frac{n}{N-1}-1})_{\geq -k+1} + R_n, L\right\}_k \\ = LL_{n-(N-1)} + \{R_n, L\}_k, \quad (4.17)$$

and (4.14) is satisfied. Evaluating powers of $(L(L^{\frac{n}{N-1}-1})_{<-k+1})_{\geq -k+1}$ and $-(L(L^{\frac{n}{N-1}-1})_{\geq -k+1})_{<-k+1}$ we get that R_n has form (4.15). $\Box$

Using relation between symmetries (4.14) we construct a recursion operator.

Lemma 11 *A recursion operator for the hierarchy (4.12) is given by equalities, for* $m = N-2, N-3, \ldots, -1$,

$$\begin{aligned}\tfrac{\partial S_m}{\partial t_n} = \textstyle\sum_{j=-1}^{m+1} S_j \tfrac{\partial S_{m-j}}{\partial t_{n-(N-1)}} + \sum_{j=-1}^{m+1}(j+1-k)A_{j+1}S_{m-j,x} - \\ \textstyle\sum_{j=-1}^{m+1}(m-j)A_{j+1,x}S_{m-j},\end{aligned} \tag{4.18}$$

where to simplify the above formula we have defined that $S_{N-1} = 1$ *and* $S_{N-1,x} = 0$, $\dfrac{\partial S_{N-1}}{\partial t_n} = 0$. *Coefficients* $A_{N-2}, A_{N-3}, \ldots, A_0$ *can be found from the recursion relations, for* $m = N-2, \ldots, -1$

$$\begin{aligned}(N-1)A_{m,x} = \sum_{j=m}^{N-1} S_j \frac{\partial S_{N-2+m-j}}{\partial t_{n-(N-1)}} + \sum_{j=m}^{N-2}(j+1-k)A_{j+1}S_{N-2+m-j,x} - \\ \sum_{j=m}^{N-2}(N-2+m-j)A_{j+1,x}S_{N-2+m-j}.\end{aligned} \tag{4.19}$$

Proof. Let us write the equality (4.14), using (4.15) for R_n

$$\begin{aligned}\sum_{i=-1}^{N-2} p^i \frac{\partial S_i}{\partial t_n} = \left(p^{N-1} + \sum_{i=-1}^{N-2} p^i S_i\right)\left(\sum_{i=-1}^{N-2} p^i \frac{\partial S_{(N-2)+m-j}}{\partial t_{n-(N-1)}}\right) + \\ p^k \left(\sum_{j=0}^{N-1}(j-k)p^{j-k-1}A_j\right)\left(\sum_{j=-1}^{N-2} p^j S_{j,x}\right) \\ -p^k\left(\sum_{j=0}^{N-1} p^{j-k}A_{j,x}\right)\left((N-1)p^{N-2} + \sum_{j=-1}^{N-2} jp^{j-1}S_j\right).\end{aligned}$$

To have the equality, the coefficients of $p^{2N-3}, \ldots, p^{N-1}$ and p^{-2} must be zero, it gives recursion relations to find $A_{N-2}, \ldots, A_0$. The coefficients of $p^{N-2}, \ldots, p^{-1}$ give the expressions for $\dfrac{\partial S_{N-2}}{\partial t_n}, \ldots, \dfrac{\partial S_{-1}}{\partial t_n}$. $\square$

Although the recursion operator $\mathcal{R}$, given by (4.18), is a pseudo-differential operator, it gives a hierarchy of local symmetries starting from the equation itself. Indeed, equalities (4.18), (4.19) give expressions $\dfrac{\partial S_{N-2}}{\partial t_n}$, …, $\dfrac{\partial S_{-1}}{\partial t_n}$ in terms of

S_{N-2} ,. . ., S_{-1} and $\dfrac{\partial S_{N-2}}{\partial t_{n-(N-1)}}$, . . . , $\dfrac{\partial S_{-1}}{\partial t_{n-(N-1)}}$. Hence, the recursion operator $\mathcal{R}$ is constructed in such a way that

$$\left\{(L^{\frac{n}{N-1}+1})_{\geq -k+1}; L\right\}_k = \mathcal{R}\left(\left\{(L^{\frac{n}{N-1}})_{\geq -k+1}; L\right\}_k\right) \tag{4.20}$$

Now we apply general formula for recursion operator to particular cases.

4.3 Some Integrable Systems

We shall consider first some examples for $k = 0$, $k = 1$.

4.3.1 Multi-component hierarchy containing also the shallow water wave equations ($k = 0$)

This hierarchy corresponds to the case $k = 0$. Let us give the first equation of hierarchy and a recursion operator for $N = 2, 3$.

Proposition 1 *In the case $N = 2$ one has the Lax function*

$$L = p + S + P\,p^{-1}$$

and the Lax equation for $n = 2$, given by (4.49), when $k = 0$, is

$$\begin{aligned} \tfrac{1}{2}S_t &= SS_x + P_x, \\ \tfrac{1}{2}P_t &= SP_x + PS_x, \end{aligned} \tag{4.21}$$

and the recursion operator, given by (4.50), is

$$\mathcal{R} = \begin{pmatrix} S + S_x D_x^{-1} & 2 \\ 2P + P_x D_x^{-1} & S \end{pmatrix}. \tag{4.22}$$

These equations are known as the shallow water wave equations [38] or as the equations of polytropic gas dynamics for $\gamma = 2$ see Chapter 6.

The first two symmetries of the system (4.21) are given by

$$\begin{aligned} S_{t_1} &= (S^3 + 6SP)_x, \\ P_{t_1} &= (3S^2P + 3P^2)_x, \end{aligned} \tag{4.23}$$

$$\begin{aligned} S_{t_2} &= (S^4 + 12S^2P + 6P^2)_x, \\ P_{t_2} &= (4S^3P + 12SP^2)_x. \end{aligned} \tag{4.24}$$

Remark 1 *In symmetric variables the system (4.21) is written as*

$$\begin{aligned} \tfrac{1}{2}u_t &= (u+v)u_x + uv_x, \\ \tfrac{1}{2}v_t &= vu_x + (u+v)v_x, \end{aligned} \tag{4.25}$$

and the recursion operator (4.22) takes the form

$$\mathcal{R} = \begin{pmatrix} u + v + u_x D_x^{-1} & 2u + u_x D_x^{-1} \\ 2v + v_x D_x^{-1} & u + v + v_x D_x^{-1} \end{pmatrix}. \tag{4.26}$$

Proposition 2 *In the case $N = 3$ one has the Lax function*

$$L = p^2 + pS + P + p^{-1}Q$$

and the Lax equation with $n = 3$ is

$$\begin{aligned}
\tfrac{1}{3}S_t &= (\tfrac{1}{2}P - \tfrac{1}{8}S^2)S_x + \tfrac{1}{2}SP_x + Q_x, \\
\tfrac{1}{3}P_t &= \tfrac{1}{2}QS_x + (\tfrac{1}{8}S^2 + \tfrac{1}{2}P)P_x + SQ_x, \\
\tfrac{1}{3}Q_t &= \tfrac{1}{4}SQS_x + \tfrac{1}{2}QP_x + (\tfrac{1}{8}S^2 + \tfrac{1}{2}P)Q_x.
\end{aligned} \tag{4.27}$$

The recursion operator, corresponding to this equation, is

$$\mathcal{R} = \begin{pmatrix} -\frac{S^2}{4} + P + P_x D_x^{-1} - \frac{S_x}{4} D_x^{-1} \cdot S & \frac{S}{2} + \frac{S_x}{2} D_x^{-1} & 3 \\ \frac{3Q}{2} + (Q_x + \frac{P_x S}{2}) D_x^{-1} - \frac{P_x}{4} D_x^{-1} \cdot S & P + \frac{P_x}{2} D_x^{-1} & 2S \\ \frac{SQ}{4} + (\frac{SQ_x}{2} + \frac{S_x Q}{2}) D_x^{-1} - \frac{Q_x}{4} D_x^{-1} \cdot S & \frac{3Q}{2} + \frac{Q_x}{2} D_x^{-1} & P \end{pmatrix}. \tag{4.28}$$

Proof. Using (4.19) we find the function R_n and using (4.18) we find the recursion operator (4.28). $\square$

Remark 2 *In symmetric variables the equation (4.27) is written as*

$$\begin{aligned}
\tfrac{1}{3}u_t &= (-\tfrac{1}{8}u^2 + \tfrac{1}{2}(uv + uw + vw) + \tfrac{1}{8}(v + w)^2)u_x \\
&\quad + (\tfrac{1}{4}u^2 + \tfrac{1}{4}uv + \tfrac{3}{4}uw)v_x + (\tfrac{1}{4}u^2 + \tfrac{1}{4}uw + \tfrac{3}{4}uv)w_x, \\
\tfrac{1}{3}v_t &= (\tfrac{1}{4}v^2 + \tfrac{1}{4}uv + \tfrac{3}{4}vw)u_x + (\tfrac{1}{4}v^2 + \tfrac{1}{4}vw + \tfrac{3}{4}uv)w_x \\
&\quad + (-\tfrac{1}{8}v^2 + \tfrac{1}{2}(uv + uw + vw) + \tfrac{1}{8}(u + w)^2)v_x, \\
\tfrac{1}{3}w_t &= (\tfrac{1}{4}w^2 + \tfrac{1}{4}uw + \tfrac{3}{4}wv)u_x + (\tfrac{1}{4}w^2 + \tfrac{1}{4}wv + \tfrac{3}{4}uw)v_x \\
&\quad + (-\tfrac{1}{8}w^2 + \tfrac{1}{2}(uv + uw + vw) + \tfrac{1}{8}(v + u)^2)w_x,
\end{aligned} \tag{4.29}$$

and the recursion operator takes the form

$$\mathcal{R} = \begin{pmatrix} \begin{matrix} -\frac{u^2}{4}+\frac{3}{4}(uv+uw)+wv \\ +\frac{u_x}{2}(v+w)D_x^{-1} \\ +\frac{u}{2}(v_x+w_x)D_x^{-1} \\ -\frac{u_x}{4}D_x^{-1}\cdot u+\frac{u_x}{4}D_x^{-1}\cdot v \\ +\frac{u_x}{4}D_x^{-1}\cdot w \end{matrix} & \begin{matrix} \frac{u}{4}(u+v+w)+\frac{3uw}{2} \\ +\frac{u_x}{2}(v+w)D_x^{-1} \\ +\frac{u}{2}(v_x+w_x)D_x^{-1} \\ +\frac{u_x}{4}D_x^{-1}\cdot u-\frac{u_x}{4}D_x^{-1}\cdot v \\ +\frac{u_x}{4}D_x^{-1}\cdot w \end{matrix} & \begin{matrix} \frac{u}{4}(u+v+w)+\frac{3uv}{2} \\ +\frac{u_x}{2}(v+w)D_x^{-1} \\ +\frac{u}{2}(v_x+w_x)D_x^{-1} \\ +\frac{u_x}{4}D_x^{-1}\cdot u+\frac{u_x}{4}D_x^{-1}\cdot v \\ -\frac{u_x}{4}D_x^{-1}\cdot w \end{matrix} \\ \\ \begin{matrix} \frac{v}{4}(u+v+w)+\frac{3vw}{2} \\ +\frac{v_x}{2}(u+w)D_x^{-1} \\ +\frac{v}{2}(u_x+w_x)D_x^{-1} \\ -\frac{v_x}{4}D_x^{-1}\cdot u+\frac{v_x}{4}D_x^{-1}\cdot v \\ +\frac{v_x}{4}D_x^{-1}\cdot w \end{matrix} & \begin{matrix} -\frac{v^2}{4}+\frac{3}{4}(uv+vw)+uw \\ +\frac{v_x}{2}(u+w)D_x^{-1} \\ +\frac{v}{2}(u_x+w_x)D_x^{-1} \\ +\frac{v_x}{4}D_x^{-1}\cdot u-\frac{v_x}{4}D_x^{-1}\cdot v \\ +\frac{v_x}{4}D_x^{-1}\cdot w \end{matrix} & \begin{matrix} \frac{v}{4}(u+v+w)+\frac{3uv}{2} \\ +\frac{v_x}{2}(u+w)D_x^{-1} \\ +\frac{v}{2}(u_x+w_x)D_x^{-1} \\ +\frac{v_x}{4}D_x^{-1}\cdot u+\frac{v_x}{4}D_x^{-1}\cdot v \\ -\frac{v_x}{4}D_x^{-1}\cdot w \end{matrix} \\ \\ \begin{matrix} \frac{w}{4}(u+v+w)+\frac{3vw}{2} \\ +\frac{w_x}{2}(u+v)D_x^{-1} \\ +\frac{w}{2}(u_x+v_x)D_x^{-1} \\ -\frac{w_x}{4}D_x^{-1}\cdot u+\frac{w_x}{4}D_x^{-1}\cdot v \\ +\frac{w_x}{4}D_x^{-1}\cdot w \end{matrix} & \begin{matrix} \frac{w}{4}(u+v+w)+\frac{3uw}{2} \\ +\frac{w_x}{2}(u+v)D_x^{-1} \\ +\frac{w}{2}(u_x+v_x)D_x^{-1} \\ +\frac{w_x}{4}D_x^{-1}\cdot u-\frac{w_x}{4}D_x^{-1}\cdot v \\ +\frac{w_x}{4}D_x^{-1}\cdot w \end{matrix} & \begin{matrix} -\frac{w^2}{4}+\frac{3}{4}(uw+vw)+uv \\ +\frac{w_x}{2}(u+v)D_x^{-1} \\ +\frac{w}{2}(u_x+v_x)D_x^{-1} \\ +\frac{w_x}{4}D_x^{-1}\cdot u+\frac{w_x}{4}D_x^{-1}\cdot v \\ -\frac{w_x}{4}D_x^{-1}\cdot w \end{matrix} \end{pmatrix}. \tag{4.30}$$

4.3.2 Toda hierarchy $(k = 1)$

Toda hierarchy corresponds to the case $k = 1$ (see [32]). Let us give the first equation of hierarchy and a recursion operator for $N = 2$ and $N = 3$.

Proposition 3 *In the case $N = 2$ and $n = 1$ one has the Lax function*

$$L = p + S + P\,p^{-1}$$

and the Lax equation for $n = 1$, given by (4.43),

$$\begin{aligned} S_t &= P_x, \\ P_t &= PS_x, \end{aligned} \tag{4.31}$$

and the recursion operator, given by (4.44),

$$\mathcal{R} = \begin{pmatrix} S & 2 + P_x D_x^{-1} \cdot P^{-1} \\ 2P & S + S_x P D_x^{-1} \cdot P^{-1} \end{pmatrix}. \tag{4.32}$$

The first two symmetries of the equation (4.31) are given by

$$\begin{aligned} S_{t_1} &= (2SP)_x, \\ P_{t_1} &= P(2P + S^2)_x, \end{aligned} \tag{4.33}$$

$$\begin{aligned} S_{t_2} &= (3S^2P + 3P^2)_x, \\ P_{t_2} &= P(6PS + S^3)_x. \end{aligned} \tag{4.34}$$

Remark 3 *In symmetric variables the equation (4.31) is written as*

$$\begin{aligned} u_t &= uv_x, \\ v_t &= vu_x, \end{aligned} \tag{4.35}$$

and the recursion operator (4.32) takes the form

$$\mathcal{R} = \begin{pmatrix} u + v + uv_x D_x^{-1} \cdot u^{-1} & 2u + uv_x D_x^{-1} \cdot v^{-1} \\ 2v + vu_x D_x^{-1} \cdot u^{-1} & u + v + vu_x D_x^{-1} \cdot v^{-1} \end{pmatrix}. \tag{4.36}$$

Proposition 4 *In the case $N = 3$ and $n = 1$ one has the Lax function*

$$L = p^2 + pS_1 + P + p^{-1}Q$$

and the Lax equation with $n = 1$ is

$$\begin{aligned} S_t &= P_x - \tfrac{1}{2} S S_x, \\ P_t &= Q_x, \\ Q_t &= \tfrac{1}{2} Q S_x. \end{aligned} \tag{4.37}$$

The recursion operator, corresponding to this equation, is

$$\mathcal{R} = \begin{pmatrix} P - \frac{1}{4}S^2 + (\frac{1}{2}P_x - \frac{1}{4}SS_x)D_x^{-1} & \frac{1}{2}S & 3 + 2Q_x D_x^{-1} \cdot Q^{-1} \\ \frac{3}{2}Q + \frac{1}{2}Q_x D_x^{-1} & P & 2S + (SQ)_x D_x^{-1} \cdot Q^{-1} \\ \frac{1}{4}SQ + \frac{1}{4}S_x Q D_x^{-1} & \frac{3}{2}Q & P + P_x Q D_x^{-1} \cdot Q^{-1} \end{pmatrix}. \tag{4.38}$$

Proof. Using equalities (4.19) we find the function R_n and using (4.18) we find the recursion operator (4.38). □

Remark 4 *In symmetric variables the equation (4.37) is written as*

$$\begin{aligned} u_t &= \tfrac{1}{2} u(-u_x + v_x + w_x), \\ v_t &= \tfrac{1}{2} v(+u_x - v_x + w_x), \\ w_t &= \tfrac{1}{2} w(+u_x + v_x - w_x), \end{aligned} \tag{4.39}$$

and the recursion operator takes the form

$$\mathcal{R} = \begin{pmatrix} \begin{matrix} -(uv+uw+vw) \\ +\frac{u}{4}(u+v+w) \\ +\frac{u}{4}(u_x-v_x-w_x)D_x^{-1} \\ -u(wv_x+vw_x)D_x^{-1}\cdot u^{-1} \end{matrix} & \begin{matrix} -\frac{u}{4}(u+v+w) \\ -\frac{3uw}{2} \\ +\frac{u}{4}(u_x-v_x-w_x)D_x^{-1} \\ -u(wv_x+vw_x)D_x^{-1}\cdot v^{-1} \end{matrix} & \begin{matrix} -\frac{u}{4}(u+v+w) \\ -\frac{3uv}{2} \\ +\frac{u}{4}(u_x-v_x-w_x)D_x^{-1} \\ -u(wv_x+vw_x)D_x^{-1}\cdot w^{-1} \end{matrix} \\ \\ \begin{matrix} -\frac{v}{4}(u+v+w) \\ -\frac{3vw}{2} \\ +\frac{v}{4}(-u_x+v_x-w_x)D_x^{-1} \\ -v(wu_x+uw_x)D_x^{-1}\cdot u^{-1} \end{matrix} & \begin{matrix} -(uv+uw+vw) \\ +\frac{v}{4}(u+v+w) \\ +\frac{v}{4}(-u_x+v_x-w_x)D_x^{-1} \\ -v(wu_x+uw_x)D_x^{-1}\cdot v^{-1} \end{matrix} & \begin{matrix} -\frac{v}{4}(u+v+w) \\ -\frac{3uv}{2} \\ +\frac{v}{4}(-u_x+v_x-w_x)D_x^{-1} \\ -v(wu_x+uw_x)D_x^{-1}\cdot w^{-1} \end{matrix} \\ \\ \begin{matrix} -\frac{w}{4}(u+v+w) \\ -\frac{3uw}{2} \\ +\frac{w}{4}(-u_x-v_x+w_x)D_x^{-1} \\ -w(uv_x+vu_x)D_x^{-1}\cdot u^{-1} \end{matrix} & \begin{matrix} -\frac{w}{4}(u+v+w) \\ -\frac{3vw}{2} \\ +\frac{w}{4}(-u_x-v_x+w_x)D_x^{-1} \\ -w(uv_x+vu_x)D_x^{-1}\cdot v^{-1} \end{matrix} & \begin{matrix} -(uv+uw+vw) \\ +\frac{w}{4}(u+v+w) \\ +\frac{w}{4}(-u_x-v_x+w_x)D_x^{-1} \\ -w(uv_x+vu_x)D_x^{-1}\cdot w^{-1} \end{matrix} \end{pmatrix}. \tag{4.40}$$

4.4 Lax equation for general k

For general k we shall only consider the case where $N = 2$. We have the Lax function

$$L = p + S + Pp^{-1} \tag{4.41}$$

and the Lax equation

$$\frac{\partial L}{\partial t_n} = \{(L^n)_{\geq -k+1}; L\}_k\,. \tag{4.42}$$

To simplify calculations we deal with two cases $k \geq 1$ and $k \leq 0$.

4.4.1 The first case $k \geq 1$

Proposition 5 *In the case $N = 2$ and $k \geq 1$ one has the Lax equation*

$$
\begin{aligned}
S_t &= kP^{k-1}P_x, \\
P_t &= kP^k S_x.
\end{aligned} \tag{4.43}
$$

and the recursion operator for this equation is

$$
\mathcal{R} = \begin{pmatrix} S + (1-k)S_x D_x^{-1} & 2 + kP^{k-1}P_x D_x^{-1} \cdot P^{-k} \\ 2P + (1-k)P_x D_x^{-1} & S + kS_x P^k D_x^{-1} \cdot P^{-k} \end{pmatrix}. \tag{4.44}
$$

Proof. The smallest power of p in L^n is $-n$. To have powers less than $-k+1$ we must put $n = k$. If there are no such powers then Poisson bracket vanishes, i.e. $\{(L^n); L\}_k = 0$.

Let us calculate the Lax equation

$$
L_t = \left\{(L^k)_{\geq -k+1}; L\right\}_k = -\left\{(L^k)_{\leq -k}; L\right\}_k.
$$

We have $(L^k)_{\leq -k} = [(p + S + Pp^{-1})^k]_{\leq -k} = P^k p^{-k}$, thus

$$
L_t = -\left\{P^k p^{-k}; p + S + Pp^{-1}\right\}_k.
$$

And we get the equation (4.43). Using (4.18), (4.19) we find the recursion operator (4.44). $\square$

The first two symmetries of the equation (4.43) are given as follows

$$\begin{aligned} S_{t_1} &= (k+1)(P^k S)_x, \\ P_{t_1} &= (k+1)P^k \left(P + \frac{k}{2}S^2\right)_x, \end{aligned} \tag{4.45}$$

$$\begin{aligned} S_{t_2} &= (k+1)(k+2)\left(\frac{1}{2}P^k S^2 + \frac{1}{k+1}P^{k+1}\right)_x, \\ P_{t_2} &= (k+1)(k+2)P^k \left(PS + \frac{k}{6}S^3\right)_x. \end{aligned} \tag{4.46}$$

Remark 5 *In symmetric variables the equation (4.43) is written as*

$$\begin{aligned} u_t &= k u^k v^{k-1} v_x, \\ v_t &= k u^{k-1} v^k u_x, \end{aligned} \tag{4.47}$$

and the recursion operator (4.44) takes the form

$$\mathcal{R} = \begin{pmatrix} \begin{array}{c} u + v + (1-k)u_x D_x^{-1} + \\ k u^k v^{k-1} v_x D_x^{-1} \cdot u^{-k} v^{-k+1} \end{array} & \begin{array}{c} 2u + (1-k)u_x D_x^{-1} + \\ k u^k v^{k-1} v_x D_x^{-1} \cdot u^{-k+1} v^{-k} \end{array} \\ \\ \begin{array}{c} 2v + (1-k)v_x D_x^{-1} + \\ k u^{k-1} v^k u_x D_x^{-1} \cdot u^{-k} v^{-k+1} \end{array} & \begin{array}{c} u + v + (1-k)v_x D_x^{-1} + \\ k u^{k-1} v^k u_x D_x^{-1} \cdot u^{-k+1} v^{-k} \end{array} \end{pmatrix}. \tag{4.48}$$

4.4.2 The second case $k \leq 0$

Proposition 6 *In the case $N = 2$ and $k \leq 0$ one has the Lax equation*

$$\begin{aligned} S_t &= (-k+2)(-k+1)SS_x + (-k+2)P_x, \\ P_t &= (-k+2)(-k+1)SP_x + (-k+2)S_x P. \end{aligned} \tag{4.49}$$

and the recursion operator for this equation is

$$\mathcal{R} = \begin{pmatrix} S+(1-k)S_xD_x^{-1} & 2+kP^{k-1}P_xD_x^{-1}\cdot P^{-k} \\ 2P+(1-k)P_xD_x^{-1} & S+kS_xP^kD_x^{-1}\cdot P^{-k} \end{pmatrix}. \tag{4.50}$$

Proof. The largest power of p in L^n is n. To have powers larger than $-k+1$ we must put $n=-k+1$. Then we have

$$(L^{-k+1})_{\geq -k+1} = [(p+S+Pp^{-1})^{-k+1}]_{\geq -k+1} = p^{-k+1},$$

thus

$$L_t = \left\{p^{-k+1}; p+S+Pp^{-1}\right\}_k .$$

Then the Lax equation becomes

$$\begin{aligned} S_t &= S_x, \\ P_t &= P_x. \end{aligned}$$

This is a trivial equation, let us calculate the second symmetry. We have $(L^{-k+2})_{\geq -k+1} = [(p+S+Pp^{-1})^{-k+1}]_{\geq -k+1} = p^{-k+2}+(-k+2)Sp^{-k+1}$, thus

$$L_t = \left\{p^{-k+2}+(-k+2)Sp^{-k+1}; p+S+Pp^{-1}\right\}_k .$$

we get the equation (4.49). Using (4.18), (4.19) we find the recursion operator (4.50). □

The first two symmetries of the equation (4.49) are given as follows

$$
\begin{aligned}
S_{t_1} &= (k-2)(k-3)\left(P\,S+\frac{1}{6}(1-k)S^3\right)_x, \\
P_{t_1} &= (k-2)(k-3)\left(SS_xP+\frac{1}{2}(1-k)S^2\,P_x+PP_x\right),
\end{aligned}
\tag{4.51}
$$

$$
\begin{aligned}
S_{t_2} &= (2-k)(3-k)(4-k)\left(\frac{1}{2}S^2P+\frac{1}{6}S^4+\frac{1}{2(2-k)}P^2\right)_x, \\
P_{t_2} &= (2-k)(3-k)(4-k)\left(\frac{1}{2}S^2S_xP+\frac{1}{6}(1-k)S^3\,P_x\right. \\
&\quad \left.+SPP_x+\frac{1}{(2-k)}P^2S_x\right).
\end{aligned}
\tag{4.52}
$$

Remark 6 *In symmetric variables the equation (4.49) is written as*

$$
\begin{aligned}
u_t &= (-k+2)(1-k)(u+v)u_x+(-k+2)uv_x, \\
v_t &= (-k+2)vu_x+(-k+2)(1-k)(u+v)v_x,
\end{aligned}
\tag{4.53}
$$

and the recursion operator (4.50) takes the form

$$
\mathcal{R}=\begin{pmatrix}
\begin{array}{c} u+v+(1-k)u_xD_x^{-1}+ \\ ku^kv^{k-1}v_xD_x^{-1}\cdot u^{-k}v^{-k+1} \end{array} & \begin{array}{c} 2u+(1-k)u_xD_x^{-1}+ \\ ku^kv^{k-1}v_xD_x^{-1}\cdot u^{-k+1}v^{-k} \end{array} \\
 & \\
\begin{array}{c} 2v+(1-k)v_xD_x^{-1}+ \\ ku^{k-1}v^ku_xD_x^{-1}\cdot u^{-k}v^{-k+1} \end{array} & \begin{array}{c} u+v+(1-k)v_xD_x^{-1}+ \\ ku^{k-1}v^ku_xD_x^{-1}\cdot u^{-k+1}v^{-k} \end{array}
\end{pmatrix}.
\tag{4.54}
$$

In this Section, to obtain the recursion operators we have considered two different cases $k \leq 0$ and $k \geq 1$. At the end we obtained recursion operators having the same forms (4.44) and (4.50). It seems , comparing the results, that the systems of equations in one case are symmetries of the other case. For instance, the system (4.49) is a symmetry of system (4.43). Hence we may consider only one case with recursion operator (4.44) for all integer values of k.

4.5 Reduction

We consider two reductions of the Lax equation (4.12), written in symmetric variables, by setting $u_1 = 0$ and $u_1 = u_N$. These reductions correspond to the Lax equations with two different Lax functions. For reduction $u_1 = 0$ we have a polynomial Lax function with simple roots, $L = (p - u_N)\ldots(p - u_2)$, and for reduction $u_N = u_1$ we have a polynomial Lax function with a multiple root, $L = \frac{1}{p}(p-u_N)^2(p-u_{N-1})\ldots(p-u_2)$. Let us note that one can take a polynomial Lax function with any number of multiple roots and consider the obtained Lax equation as reduction of (4.12) in the same way as reduction $u_N = u_1$.

4.5.1 Reduction $u_1 = 0$

Let us write the equation (4.12) as

$$\Delta(u_N, \ldots, u_1) = 0, \tag{4.55}$$

where Δ is a differential operator. Then

$$\Delta(u_N, \ldots, u_1)|_{u_1=0} = \left(\frac{\tilde{\Delta}(u_N, \ldots, u_2),}{0} \right), \tag{4.56}$$

where $\tilde{\Delta}$ is another differential operator. Indeed, following [31] for the Lax function $L = \frac{1}{p}\prod_{j=1}^{N}(p-u_j)$ we have

$$\frac{\partial L}{\partial t} = L\sum_{j=1}^{N}\frac{u_{j,t}}{p+u_j},$$

$$\frac{\partial L}{\partial x} = L\sum_{j=1}^{N}\frac{u_{j,x}}{p+u_j}$$

and

$$\frac{\partial L}{\partial p} = L\left(-\frac{1}{p}+\sum_{j=1}^{N}\frac{1}{p+u_j}\right).$$

Thus $u_{j,t} = Res_{p=-u_j}\{M, L\}_k$, where $M = (L^{\frac{n}{N-1}})_{\geq -k+1}$. The Lax equation (4.12) can be written as

$$\sum_{j=1}^{N}\frac{u_{j,t}}{p+u_j} = p^k M_p\sum_{j=1}^{N}\frac{u_{j,x}}{p+u_j} - p^k M_x\left(-\frac{1}{p}+\sum_{j=1}^{N}\frac{1}{p+u_j}\right). \tag{4.57}$$

Note, that $p^k M_x$ and $p^k M_p$ are polynomials. So, if we put $u_1 = 0$ and calculate residue of right hand side at $p = 0$ we get (4.56). A new equation

$$\tilde{\Delta}(u_N, \ldots, u_2) = 0 \tag{4.58}$$

is also integrable and a recursion operator of this equation can be obtained as reduction of the recursion operator of the equation (4.55). Let $\mathcal{R}$ be the recursion operator of (4.55) given by lemma 2 , then

$$\mathcal{R}|_{u_1=0} = \left(\begin{array}{c|c} \tilde{R} & * \\ \hline 0\dots 0 & 0 \end{array} \right). \tag{4.59}$$

Indeed, we found the recursion operator using formula (4.14). This formula can be written as

$$\sum_{j=1}^{N} \frac{u_{j,t_n}}{p+u_j} = LL_{n-(N-1)} + p^k R_{n,p} \sum_{j=1}^{N} \frac{u_{j,x}}{p+u_j} - p^k R_{n,x} \left(-\frac{1}{p} + \sum_{j=1}^{N} \frac{1}{p+u_j} \right) \tag{4.60}$$

and in the same way as for reduction of (4.55) we have (4.59). Note that $p^k R_{n,x}$ and $p^k R_{n,p}$ are also polynomials.

Lemma 12 *The operator $\tilde{R}$ is a recursion operator of the equation (4.58).*

Proof. Equation (4.58) is an evolution equation, so, to prove that $\tilde{R}$ is a recursion operator we must prove that for any solution $(u_N, \dots, u_2)$ of (4.58) the following equality holds (see [15])

$$D_{\tilde{\Delta}} \tilde{R} = \tilde{R} D_{\tilde{\Delta}},$$

where $D_{\tilde{\Delta}}$ is a Fréchet derivative of $\tilde{\Delta}$.

If $(u_N, \dots, u_2)$ is a solution of (4.58) then $(u_N, \dots, u_2, u_1 = 0)$ is a solution of (4.55) and for the solution $(u_N, \dots, u_2, u_1 = 0)$ we have

$$D_{\Delta} \mathcal{R} = \mathcal{R} D_{\Delta}. \tag{4.61}$$

$$D_\Delta|_{u_1=0} = \left(\begin{array}{c|c} \tilde{D} & * \\ \hline 0\ldots 0 & * \end{array}\right)$$

and

$$\mathcal{R}|_{u_1=0} = \left(\begin{array}{c|c} \tilde{R} & * \\ \hline 0\ldots 0 & 0 \end{array}\right).$$

Hence by (4.61) we have $\tilde{D}\tilde{R} = \tilde{R}\tilde{D}$. Calculating Fréchet derivative we take derivatives with respect to one variable, considering other variables as constants. Thus, to calculate $\tilde{D}$ we can put $u_1 = 0$ and differentiate with respect to other variables or we can first differentiate and then put $u_1 = 0$. It means that $\tilde{D} = D_{\tilde{\Delta}}$ and

$$D_{\tilde{\Delta}}\tilde{R} = \tilde{R}D_{\tilde{\Delta}}.$$

□

Let us consider reduction of systems, given by Remark 4 and Remark 2 and their recursion operators.

Proposition 7 *Putting $w = 0$ in (4.39) and (4.40) we obtain a new system*

$$\begin{aligned} u_t &= \tfrac{1}{2}u(-u_x + v_x), \\ v_t &= \tfrac{1}{2}v(+u_x - v_x), \end{aligned} \tag{4.62}$$

and its recursion operator

$$\mathcal{R} = \begin{pmatrix} \begin{matrix} -uv + \frac{u}{4}(u+v) \\ +\frac{u}{4}(u_x - v_x)D_x^{-1} \end{matrix} & \begin{matrix} -\frac{u}{4}(u+v) \\ +\frac{u}{4}(u_x - v_x)D_x^{-1} \end{matrix} \\ & \\ \begin{matrix} -\frac{v}{4}(u+v) \\ +\frac{v}{4}(-u_x + v_x)D_x^{-1} \end{matrix} & \begin{matrix} -uv + \frac{v}{4}(u+v) \\ +\frac{v}{4}(-u_x + v_x)D_x^{-1} \end{matrix} \end{pmatrix} \tag{4.63}$$

respectively.

Proposition 8 *Putting $w = 0$ in (4.29) and (4.30) we obtain a new system*

$$\begin{aligned} \tfrac{1}{3}u_t &= (-\tfrac{1}{8}u^2 + \tfrac{1}{2}uv + \tfrac{1}{8}v^2)u_x + (\tfrac{1}{4}u^2 + \tfrac{1}{4}uv)v_x, \\ \tfrac{1}{3}v_t &= (\tfrac{1}{4}v^2 + \tfrac{1}{4}uv)u_x + (-\tfrac{1}{8}v^2 + \tfrac{1}{2}uv + \tfrac{1}{8}u^2)v_x, \end{aligned} \tag{4.64}$$

and its recursion operator

$$\mathcal{R} = \begin{pmatrix} \begin{matrix} -\frac{u^2}{4} + \frac{3uv}{4} + (\frac{u_x v}{2} + \frac{uv_x}{2})D_x^{-1} \\ -\frac{u_x}{4}D_x^{-1}\cdot u + \frac{u_x}{4}D_x^{-1}\cdot v \end{matrix} & \begin{matrix} \frac{u}{4}(u+v) + (\frac{u_x v}{2} + \frac{uv_x}{2})D_x^{-1} \\ +\frac{u_x}{4}D_x^{-1}\cdot u - \frac{u_x}{4}D_x^{-1}\cdot v \end{matrix} \\ & \\ \begin{matrix} \frac{v}{4}(u+v) + (\frac{uv_x}{2} + \frac{u_x v}{2})D_x^{-1} \\ -\frac{v_x}{4}D_x^{-1}\cdot u + \frac{v_x}{4}D_x^{-1}\cdot v \end{matrix} & \begin{matrix} -\frac{v^2}{4} + \frac{3uv}{4} + (\frac{uv_x}{2} + \frac{u_x v}{2})D_x^{-1} \\ +\frac{v_x}{4}D_x^{-1}\cdot u - \frac{v_x}{4}D_x^{-1}\cdot v \end{matrix} \end{pmatrix}, \tag{4.65}$$

respectively.

It is worth to mention that by reduction we obtain a new equation. For example, consider the case $k = 0$. The equation (4.25), corresponding to $N = 2$, and reduction of the equation (4.29), corresponding to $N = 3$, are not related by a linear transformation of variables. Indeed, in the equation (4.25) coefficients of u_x, v_x are linear in u, v but in the equation (4.64) coefficients of u_x, v_x contain quadratic terms. Hence they can not be related by a linear transformation.

4.5.2 Reduction $u_N = u_1$

It follows from (4.57) that the Lax equation (4.12) can be written as

$$u_{i,t} = \sum_{j=1}^{N} h_i^j(u_N, \ldots, u_1) u_{j,x}, \tag{4.66}$$

where $i, j = 1, \ldots, N$ and $h_i^j = h_1(u_i, u_N, \ldots, \hat{u}_i, \ldots, u_1)$ when $i \neq j$ and $h_i^i = h_2(u_i, u_N, \ldots, \hat{u}_i, \ldots, u_1)$, the hat $\wedge$ denotes the absence of the corresponding variable. It also follows from (4.57) that the functions $h_1(x_N, \ldots, x_1)$ and $h_2(x_N, \ldots, x_1)$ are symmetric under permutations of variables $x_{N-1}, \ldots, x_1$.
Reduction $u_N = u_1$ gives us a new integrable equation

$$\begin{aligned} u_{N,t} &= (h_N^N(u_N, u_{N-1}, \ldots, u_2, u_N) + h_N^1(u_N, u_{N-1}, \ldots, u_2, u_N))u_{N,x} + \\ &\quad \sum_{j=2}^{N-1} h_N^j(u_N, u_{N-1}, \ldots, u_2, u_N) u_{j,x}, \\ u_{i,t} &= 2h_i^N(u_N, u_{N-1}, \ldots, u_2, u_N)u_{N,x} + \sum_{j=2}^{N-1} h_i^j(u_N, u_{N-1}, \ldots, u_2, u_N) u_{j,x}, \end{aligned} \tag{4.67}$$

where $i = (N-1), \ldots, 2$.
Fréchet derivative of (4.66), under condition $u_N = u_1$, has the form

$$D_\Delta|_{u_N=u_1} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1(N-1)} & a_{1N} \\ a_{21} & a_{22} & \ldots & a_{2(N-1)} & a_{21} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ a_{(N-1)1} & a_{(N-1)2} & \ldots & a_{(N-1)(N-1)} & a_{(N-1)1} \\ a_{1N} & a_{12} & \ldots & a_{1(N-1)} & a_{11} \end{pmatrix} \tag{4.68}$$

where $a_{ij},\ i,j = 1,\ldots,N$ are differential operators. So, Fréchet derivative of (4.67) can be written as

$$D_{\bar{\Delta}} = \begin{pmatrix} a_{11}+a_{1N} & a_{12} & \ldots & a_{1(N-1)} \\ 2a_{21} & a_{22} & \ldots & a_{2(N-1)} \\ \vdots & \vdots & \ldots & \vdots \\ 2a_{(N-1)1} & a_{(N-1)2} & \ldots & a_{(N-1)(N-1)} \end{pmatrix}. \tag{4.69}$$

Now let us write the recursion operator of (4.66), given by Lemma 11. From (4.60) it follows that, under condition $u_N = u_1$, it has form

$$\mathcal{R}|_{u_N=u_1} = \begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1(N-1)} & b_{1N} \\ b_{21} & b_{22} & \ldots & b_{2(N-1)} & b_{21} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ b_{(N-1)1} & b_{(N-1)2} & \ldots & b_{(N-1)(N-1)} & b_{(N-1)1} \\ b_{1N} & b_{12} & \ldots & b_{1(N-1)} & b_{11} \end{pmatrix}, \tag{4.70}$$

where b_{ij}, $i,j = N,\ldots,1$ are differential operators.

Now we can write a recursion operator for equation (4.67)

$$\bar{\mathcal{R}} = \begin{pmatrix} b_{11}+b_{1N} & b_{12} & \ldots & b_{1(N-1)} \\ 2b_{21} & b_{22} & \ldots & b_{1(N-1)} \\ \vdots & \vdots & \ldots & \vdots \\ 2b_{(N-1)1} & b_{(N-1)2} & \ldots & b_{(N-1)(N-1)} \end{pmatrix}. \tag{4.71}$$

The form of (4.71) can be deduced by applying operator $\mathcal{R}|_{u_N=u_1}$ to a symmetry $\left(\dfrac{\partial u_N}{\partial t_n}, \dfrac{\partial u_{N-1}}{\partial t_n}, \ldots \dfrac{\partial u_2}{\partial t_n}, \dfrac{\partial u_N}{\partial t_n}\right)$.

Lemma 13 *The operator $\bar{R}$ is a recursion operator of the equation (4.67).*

Proof. Equation (4.67) is an evolution equation, so, to prove that $\bar{R}$ is a recursion operator we must prove that for any solution $(u_N, \ldots, u_2)$ of (4.67) the following equality holds (see [15])

$$D_{\bar{\Delta}}\bar{R} = \bar{R}D_{\bar{\Delta}}. \tag{4.72}$$

If $(u_N, \ldots, u_2)$ is a solution of (4.67) then $(u_N, \ldots, u_2, u_1 = u_N)$ is a solution of (4.66) and for the solution $(u_N, \ldots, u_2, u_1 = u_N)$ we have

$$D_{\Delta}\,\mathcal{R} = \mathcal{R}\,D_{\Delta}. \tag{4.73}$$

It is easy to show that (4.72) follows from (4.73). □

Let us consider reduction of systems, given by Remark 4 and Remark 2 and their recursion operators.

Proposition 9 *Putting $w = u$ in (4.39) and (4.40) we obtain a new system*

$$\begin{aligned} u_t &= \tfrac{1}{2}uv_x, \\ v_t &= \tfrac{1}{2}v(2u_x - v_x), \end{aligned} \tag{4.74}$$

and its recursion operator

$$\mathcal{R} = \begin{pmatrix} -(2uv+u^2)-\frac{3}{2}uv & -\frac{1}{4}u(2u+v)-\frac{3}{2}u^2 \\ +\frac{1}{2}uv_x D_x^{-1} & +\frac{1}{4}u(2u_x-v_x)D_x^{-1} \\ -2u(uv)_x D_x^{-1}\cdot u^{-1} & -u(uv)_x D_x^{-1}\cdot v^{-1} \\ & \\ -\frac{1}{2}v(2u+v)-3uv & -(2uv+u^2)+\frac{1}{4}v(2u+v) \\ +\frac{1}{2}v(-2u_x+v_x)D_x^{-1} & +\frac{1}{4}v(2-u_x+v_x)D_x^{-1} \\ -2v(uv)_x D_x^{-1}\cdot u^{-1} & -v(u^2)_x D_x^{-1}\cdot v^{-1} \end{pmatrix} \tag{4.75}$$

respectively.

Proposition 10 *Putting $w = u$ in (4.29) and (4.30) we obtain a new system*

$$\begin{aligned} \tfrac{1}{3}u_t &= (u^2+2uv+\tfrac{1}{8}v^2)u_x+(u^2+\tfrac{1}{4}uv)v_x, \\ \tfrac{1}{3}v_t &= (\tfrac{1}{2}v^2+2uv)u_x+(-\tfrac{1}{8}v^2+uv+u^2)v_x, \end{aligned} \tag{4.76}$$

and its recursion operator

$$\mathcal{R} = \begin{pmatrix} u^2+\frac{7}{2}uv+(u^2+uv)_x D_x^{-1} & 2u^2+\frac{1}{4}uv+\frac{1}{2}(u^2+uv)D_x^{-1} \\ +\frac{1}{2}u_x D_x^{-1}\cdot v & +\frac{1}{2}u_x D_x^{-1}\cdot u-\frac{1}{4}u_x D_x^{-1}\cdot v \\ & \\ 4uv+\frac{1}{2}v^2+2(uv)_x D_x^{-1} & -\frac{1}{4}v^2+\frac{3}{2}uv+u^2+(uv)_x D_x^{-1} \\ +\frac{1}{2}v_x D_x^{-1}\cdot v & +\frac{1}{2}v_x D_x^{-1}\cdot u-\frac{1}{4}v_x D_x^{-1}\cdot v \end{pmatrix}, \tag{4.77}$$

respectively.

We may go on introducing new reductions. For instance a reduction of the type $u_1 = u_2 = u_N$ reduces an N-system to an $(N-2)$-system. One may obtain $(N-2)$-system also from the polynomial Lax function of the form

$$L = p^{-1}(p-u_1)^3(p-u_3)\dots(p-u_{N-1}), \tag{4.78}$$

function L has a zero of multiplicity three. In this way one obtains an infinite number of different classes of integrable systems.

Chapter 5

Rational Lax Function

In this section we discuss the Lax equation (1.10) with a rational Lax function. The general expression for a recursion operator in this case is complicated and difficult to apply. We give several examples in the case $k = 1$.

We have a Lax function

$$L = \frac{\Delta_1}{\Delta_2}, \tag{5.1}$$

where Δ_1, Δ_2 are polynomials of degree N and M, respectively, and $N > M$. The dispersionless Lax equation is

$$\frac{\partial L}{\partial t_n} = \{(L)_{\geq 0}^{\frac{1}{N-M}+n}, L\}_1, \qquad n = 0, 1, 2, \ldots. \tag{5.2}$$

Recall that the bracket $\{.,.\}_1$ is defined by

$$\{f, g\} = p\left(\frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial p}\right), \qquad f, g \in \Lambda. \tag{5.3}$$

First we study symmetry relation for the hierarchy of symmetries (5.2), then we calculate recursion operators for several equations.

Lemma 14 *For any* $n = 0, 1, 2, \ldots,$

$$\frac{\partial L}{\partial t_n} = L\frac{\partial L}{\partial t_{n-1}} + \{R_n, L\}_1. \tag{5.4}$$

Function R_n *has the form*

$$R_n = A + \frac{B}{\Delta_2}, \tag{5.5}$$

where A *is a polynomial of degree* $(N - M)$ *and* B *is a polynomial of degree* $(M - 1)$.

Proof. We have

$$(L^{\frac{1}{N-M}+n})_{\geq 0} = [L(L^{\frac{1}{N-M}+(n-1)})_{\geq 0} + L(L^{\frac{1}{N-M}+(n-1)})_{<0}]_{\geq 0}$$

So,

$$(L^{\frac{1}{N-M}+n})_{\geq 0} = L(L^{\frac{1}{N-M}+(n-1)})_{\geq 0} + (L(L^{\frac{1}{N-M}+(n-1)})_{<0})_{\geq 0} - (L(L^{\frac{1}{N-M}+(n-1)})_{\geq 0})_{<0}.$$

If we take

$$R_n = (L(L^{\frac{1}{N-M}+(n-1)})_{<0})_{\geq 0} - (L(L^{\frac{1}{N-M}+(n-1)})_{\geq 0})_{<0} \quad , \tag{5.6}$$

then

$$(L^{\frac{1}{N-M}+n})_{\geq 0} = L(L^{\frac{1}{N-M}+(n-1)})_{\geq 0} + R_n.$$

Hence,

$$\frac{\partial L}{\partial t_n} = \left\{(L^{\frac{1}{N-M}+n})_{\geq 0}; L\right\}_1 = \left\{L(L^{\frac{1}{N-M}+(n-1)})_{\geq 0} + R_n; L\right\}_1 = L\frac{\partial L}{\partial t_n} + \{R_n; L\}_1,$$

and (5.4) is satisfied. The remainder R_n has the form (5.5). Indeed, in (5.6) we set

$$A = (L(L^{\frac{1}{N-M}+(n-1)})_{<0})_{\geq 0}$$

and

$$B = \Delta_2 \cdot (L(L^{\frac{1}{N-M}+(n-1)})_{\geq 0})_{<0}$$

Then A is a polynomial of degree $(N-M-1)$ and B is a polynomial of degree $(M-1)$. $\square$

Now we can apply this Lemma to construct recursion operators. Let us take the simplest rational Lax function.

Example 10 *Let*

$$L = p + S + \frac{P}{p+Q}. \tag{5.7}$$

Then the Lax equation (5.2) leads to the system (introduced in [32])

$$\begin{aligned} S_t &= P_x, \\ P_t &= PS_x - QP_x - PQ_x, \\ Q_t &= QS_x - QQ_x. \end{aligned} \tag{5.8}$$

The recursion operator of the system is given by

$$\mathcal{R} = \begin{pmatrix} S & 1 & PQ^{-1} + P_x D_x^{-1} \cdot Q \\ 2P & S-Q & -2P + (PS_x - (PQ)_x) D_x^{-1} \cdot Q \\ Q & 1 & PQ^{-1} + S - Q + (QS_x - QQ_x) D_x^{-1} \cdot Q \end{pmatrix}. \tag{5.9}$$

Lemma 15 *A recursion operator for (5.8) is given by (5.9).*

Proof. Using (5.5) for R_n, we have $R_n = A + \dfrac{B}{p+Q}$. So, the symmetry relation (5.4) is

$$\frac{\partial S}{\partial t_n} + \frac{\partial P}{\partial t_n} \cdot \frac{1}{p+Q} + \frac{\partial Q}{\partial t_n} \cdot \frac{P}{(p+Q)^2} =$$

$$\left(p + S + \frac{P}{p+Q}\right)\left(\frac{\partial S}{\partial t_{n-1}} + \frac{\partial P}{\partial t_{n-1}} \cdot \frac{1}{p+Q} + \frac{\partial Q}{\partial t_{n-1}} \cdot \frac{P}{(p+Q)^2}\right) +$$

$$p\left(A_x + \frac{B_x}{p+Q} + \frac{-BQ_x}{(p+Q)^2}\right)\left(1 + \frac{-P}{(p+Q)^2}\right)$$

$$-\frac{pB}{(p+Q)^2}\left(S_x + \frac{P_x}{p+Q} + \frac{-PQ_x}{(p+Q)^2}\right)$$

To have the equality the coefficients of p and $(p+Q)^{-3}$ must be zero. It gives the recursion relations to find A and B. Then the coefficients of p^0, $(p+Q)^{-1}$, $(p+Q)^{-2}$ give expressions for $\dfrac{\partial S}{\partial t_n}$, $\dfrac{\partial P}{\partial t_n}$, $\dfrac{\partial Q}{\partial t_n}$. $\square$

In [32] bi-Hamiltonian representation of this equation was constructed with Hamiltonian operators

$$\mathcal{D}_1 = \begin{pmatrix} 0 & P & Q \\ P & -2PQ & -Q^2 \\ Q & -Q^2 & 0 \end{pmatrix} D_x + \begin{pmatrix} 0 & P_x & Q_x \\ 0 & -(PQ)_x & -QQ_x \\ 0 & -QQ_x & 0 \end{pmatrix} \tag{5.10}$$

and

$$\mathcal{D}_2 = \begin{pmatrix} 2P & P(S-3Q) & Q(S-Q) \\ P(S-3Q) & P(2P-2SQ+4Q^2) & Q(2P-SQ+Q^2) \\ Q(S-Q) & Q(2P-SQ+Q^2) & 2Q^2 \end{pmatrix} D_x +$$

$$\begin{pmatrix} P_x & SP_x - 2(PQ)_x & SQ_x - QQ_x \\ PS_x - (QP)_x & (-SPQ + P^2 + 2PQ^2)_x & Q_x(2P + Q^2 - SQ) \\ QS_x - QQ_x & Q(2P_x + 2QQ_x - S_x - SQQ_x) & 2QQ_x \end{pmatrix}. \tag{5.11}$$

These Hamiltonian operators are degenerate, so, one can not use them to construct a recursion operator. But it turns out that they are related to the recursion operator $\mathcal{R}$ given by (5.9). One can easily check that the following equality holds

$$\mathcal{R}\mathcal{D}_1 = \mathcal{D}_2. \tag{5.12}$$

We observe that the degeneracy in the bi-Hamiltonian operators is due to the following fact. Let $p' = p + F$ then the Lax function becomes

$$L = p' + G + \frac{P}{p'} \ . \tag{5.13}$$

This means that we have two independent variables P and G, where $G = S - F$. The equation corresponding to the Lax function (5.13) has been studied in [28]. To remove degeneracy one can take the Lax function as

$$L = p + S + \frac{P}{p} + \sum_{i=1}^{m} \frac{Q_i}{p + F_i} \ . \tag{5.14}$$

As an example we consider the equation (1.10) with the following Lax function

$$L = p + S + \frac{P}{p} + \frac{Q}{p + F} \ . \tag{5.15}$$

Example 11 *The dispersionless Lax equation (5.2) with the Lax function (5.15), for $n = 1$, gives the following system*

$$
\begin{aligned}
S_t &= P_x + Q_x, \\
P_t &= PS_x, \\
Q_t &= QS_x - FQ_x - QF_x, \\
F_t &= FS_x - FF_x.
\end{aligned}
\tag{5.16}
$$

The recursion operator of the system (5.16) is given by

$$
\begin{pmatrix}
S & 2 + P_x D_x^{-1} \cdot P^{-1} & 1 & QF^{-1} + Q_x D_x^{-1} \cdot F^{-1} \\
 & & & \\
2P & S + QF^{-1} + PS_x D_x^{-1} \cdot P^{-1} & PF^{-1} & -2PQF^{-2} \\
 & PF^{-1}(Q_x - QF^{-1}F_x)D_x^{-1} \cdot P^{-1} & & -PF^{-1}(Q_x - QF^{-1}F_x)D_x^{-1} \cdot F^{-1} \\
 & & & \\
2Q & -QF^{-1} & S - F & -2PQF^{-2} - 2Q \\
 & -PF^{-1}(Q_x - QF^{-1}F_x)D_x^{-1} \cdot P^{-1} & -PF^{-1} & +PF^{-1}(Q_x - QF^{-1}F_x)D_x^{-1} \cdot F^{-1} \\
 & & & +(QS_x - QF_x - FQ_x)D_x^{-1} \cdot F^{-1} \\
 & & & \\
F & 1 + (P_x - PF^{-1}F_x)D_x^{-1} \cdot P^{-1} & -1 & PF^{-1} - F + (FS_x - FF_x)D_x^{-1} \cdot F^{-1} \\
 & & & -(P_x - PF^{-1}F_x)D_x^{-1} \cdot F^{-1}
\end{pmatrix}.
\tag{5.17}
$$

Lemma 16 *A recursion operator for (5.16) is given by (5.17).*

Proof. Using (5.5) for R_n, we have $R_n = C + \dfrac{A}{p} + \dfrac{B}{p+F}$. So, the symmetry relation (5.4) is

$$
\frac{\partial S}{\partial t_n} + \frac{\partial P}{\partial t_n} \cdot \frac{1}{p} + \frac{\partial Q}{\partial t_n} \cdot \frac{1}{(p+F)} + \frac{\partial F}{\partial t_n} \cdot \frac{-Q}{(p+F)^2} =
$$

$$
\left(p + S + \frac{P}{p} + \frac{Q}{p+F}\right)\left(\frac{\partial S}{\partial t_{n-1}} + \frac{\partial P}{\partial t_{n-1}} \cdot \frac{1}{p} + \frac{\partial Q}{\partial t_{n-1}} \cdot \frac{1}{(p+F)} + \frac{\partial F}{\partial t_{n-1}} \cdot \frac{-Q}{(p+F)^2}\right) +
$$

$$
p\left(\frac{-B}{p^2} + \frac{-C}{(p+F)^2}\right)\left(S_x + \frac{P_x}{p} + \frac{Q_x}{(p+F)} + \frac{-QF_x}{(p+F)^2}\right) -
$$

$$
p\left(A_x + \frac{B_x}{p} + \frac{C_x}{(p+F) + \frac{-CF_x}{(p+F)^2}}\right)\left(1 + \frac{P}{p} + \frac{-Q}{(p+F)^2}\right)
$$

Therefore, the coefficients of p, p^{-2}, and $(p+F)^{-3}$ must be zero, it gives the recursion relations to find A, B and C. Then the coefficients of p^0, p^{-1}, $(p+F)^{-1}$ and $(p+F)^{-2}$, give expressions for $\frac{\partial S}{\partial t_n}$, $\frac{\partial P}{\partial t_n}$, $\frac{\partial Q}{\partial t_n}$ and $\frac{\partial F}{\partial t_n}$. □

We also give examples of the Lax function with two poles.

Example 12 *The simplest Lax function with two poles is*

$$L = p + S + \frac{P}{p+F} + \frac{Q}{p+G}. \tag{5.18}$$

Then the Lax equation (5.2) gives us the system

$$\begin{aligned}
S_t &= -P_x - Q_x, \\
P_t &= -PS_x + FP_x + PF_x, \\
Q_t &= -QS_x + GQ_x + QG_x, \\
F_t &= -FS_x + FF_x, \\
G_t &= -GS_x + GG_x.
\end{aligned} \tag{5.19}$$

The recursion operator of the system (5.19) is given by

$$\begin{pmatrix}
S & 1 & 1 & \frac{-P}{F} - P_x D_x^{-1}\cdot F^{-1} & \frac{-Q}{G} - Q_x D_x^{-1}\cdot G^{-1} \\[2ex]
2P & \begin{array}{l} S-F \\ +\frac{Q}{F-G} \end{array} & \frac{P}{F-G} & \begin{array}{l} \frac{-PQ}{F-G} + \frac{-PQ}{(F-G)^2} \\ (-S_xP + FP_x + PF_x)D_x^{-1}\cdot F^{-1} \\ +\frac{F+G}{(F-G)^3}(PQG_x - PQF_x)D_x^{-1}\cdot F^{-1} \\ +\frac{-1}{(F-G)^2}(PGQ_x + FQP_x)D_x^{-1}\cdot F^{-1} \end{array} & \begin{array}{l} \frac{-PQ}{F-G} + \frac{PQ}{(F-G)^2} \\ \\ +\frac{F+G}{(F-G)^3}(PQF_x - PQG_x)D_x^{-1}\cdot G^{-1} \\ +\frac{1}{(F-G)^2}(PGQ_x + FQP_x)D_x^{-1}\cdot G^{-1} \end{array} \\[2ex]
2Q & \frac{Q}{G-F} & \begin{array}{l} S-G \\ +\frac{P}{G-F} \end{array} & \begin{array}{l} \frac{-PQ}{G-F} + \frac{PQ}{(G-F)^2} \\ \\ +\frac{F+G}{(G-F)^3}(PQG_x - PQF_x)D_x^{-1}\cdot F^{-1} \\ +\frac{1}{(G-F)^2}(PGQ_x + FQP_x)D_x^{-1}\cdot F^{-1} \end{array} & \begin{array}{l} \frac{-PQ}{G-F} + \frac{-PQ}{(G-F)^2} \\ (-S_xQ + GQ_x + QG_X)D_x^{-1}\cdot G^{-1} \\ +\frac{F+G}{(G-F)^3}(PQF_x - PQG_x)D_x^{-1}\cdot G^{-1} \\ +\frac{-1}{(G-F)^2}(PGQ_x + FQP_x)D_x^{-1}\cdot G^{-1} \end{array} \\[2ex]
F & -1 & 0 & \begin{array}{l} \frac{-P}{F} - F + S \\ +\frac{-PQ}{(F-G)^2} \\ (FF_x - S_xF)D_x^{-1}\cdot F^{-1} \\ +\frac{1}{(F-G)^2}(FQG_x - FQF_x)D_x^{-1}\cdot F^{-1} \\ \frac{FQ_x}{F-G}D_x^{-1}\cdot F^{-1} \end{array} & \begin{array}{l} \\ +\frac{FQ}{G(F-G)} \\ \\ +\frac{1}{(F-G)^2}(FQF_x - FQG_x)D_x^{-1}\cdot G^{-1} \\ \frac{-FQ_x}{F-G}D_x^{-1}\cdot G^{-1} \end{array} \\[2ex]
G & 0 & -1 & \begin{array}{l} \\ +\frac{GP}{F(G-F)} \\ \\ +\frac{1}{(F-G)^2}(PGG_x - PGF_x)D_x^{-1}\cdot F^{-1} \\ \frac{-GP_x}{G-F}D_x^{-1}\cdot F^{-1} \end{array} & \begin{array}{l} \frac{-Q}{G} - G + S \\ +\frac{-PQ}{(G-F)^2} \\ (GG_x - S_xG)D_x^{-1}\cdot G^{-1} \\ +\frac{1}{(G-F)^2}(PGF_x - PGG_x)D_x^{-1}\cdot G^{-1} \\ \frac{GP_x}{G-F}D_x^{-1}\cdot G^{-1} \end{array}
\end{pmatrix}. \tag{5.20}$$

Lemma 17 *A recursion operator for (5.19) is given by (5.20).*

Proof. Using (5.5) for R_n, we have $R_n = C + \frac{A}{p+F} + \frac{B}{p+G}$. So, the symmetry relation (5.4) is

$$\frac{\partial S}{\partial t_n} + \frac{\partial P}{\partial t_n}\cdot\frac{1}{(p+F)} + \frac{\partial Q}{\partial t_n}\cdot\frac{1}{(p+G)} - P\frac{\partial F}{\partial t_n}\cdot\frac{1}{(p+F)^2} - Q\frac{\partial Q}{\partial t_n}\cdot\frac{1}{(p+G)^2} =$$

$$
\begin{gathered}
\left(p+S+\frac{P}{p+F}+\frac{Q}{p+G}\right)\left(\frac{\partial S}{\partial t_{n-1}}+\frac{\partial P}{\partial t_{n-1}}\cdot\frac{1}{(p+F)}+\frac{\partial Q}{\partial t_{n-1}}\cdot\frac{1}{(p+G)}+\right.\\
\left.-P\frac{\partial F}{\partial t_{n-1}}\cdot\frac{1}{(p+F)^2}+-Q\frac{\partial G}{\partial t_{n-1}}\cdot\frac{1}{(p+G)^2}\right)+\\
p\left(C_x+\frac{A_x}{p+F}+\frac{-AF_x}{(p+F)^2}+\frac{B_x}{p+F}^{2}+\frac{-BF_x}{(p+G)^2}\right)\left(1+\frac{-P}{(p+F)^2}+\frac{-Q}{(p+G)^2}\right)\\
-p\left(\frac{-A}{(p+F)^2}-\frac{-B}{(p+G)^2}\right)\left(S_x+\frac{P_x}{p+F}+\frac{-PF_x}{(p+F)^2}+\frac{Q_x}{p+G}+\frac{-QG_x}{(p+G)^2}\right).
\end{gathered}
$$

To have the equality, the coefficients of p and $(p+F)^{-3}$, $(p+G)^{-3}$ must be zero, it gives the recursion relations to find C, A and B. The coefficients of $p^0, (p+F)^{-1}, (p+F)^{-2}, (p+G)^{-1}, (p+G)^{-2}$ give the expressions for $\frac{\partial S}{\partial t_n}, \frac{\partial P}{\partial t_n}, \frac{\partial Q}{\partial t_n}, \frac{\partial F}{\partial t_n}, \frac{\partial G}{\partial t_n}$. $\square$

Remark that for a general rational Lax function calculation of a recursion operator for the Lax equation (5.2) is similar to calculations in the previous examples.

Chapter 6

Equation of polytropic gas dynamics

In this Chapter we consider the following Lax function

$$L = p^{\gamma-1} + u + \frac{v^{\gamma-1}}{(\gamma-1)^2} p^{-\gamma+1}, \tag{6.1}$$

introduced in [23], and the Lax equation

$$\frac{\partial L}{\partial t} = \frac{\gamma-1}{\gamma} \left\{ (L^{\frac{\gamma}{\gamma-1}})_{\geq 1}, L \right\}_0 . \tag{6.2}$$

The Lax equation (6.2) gives the equations of polytropic gas dynamics

$$\begin{aligned} u_t + uu_x + v^{\gamma-2} v_x &= 0, \\ v_t + (uv)_x &= 0. \end{aligned} \tag{6.3}$$

Proposition 11 *The Lax equation (6.2) corresponds to the equation (6.3).*

Proof. Expanding the function (6.1) around the point $p = \infty$, we have

$$\left(p^{\gamma-1} + u + \frac{v^{\gamma-1}}{(\gamma-1)^2}p^{-\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} = p^\gamma + \frac{\gamma}{\gamma-1}pu + \dots \quad ,$$

all other terms have negative powers of p. Therefore,

$$\left(L^{\frac{\gamma}{\gamma-1}}\right)_{\geq 1} = p^\gamma + \frac{\gamma}{\gamma-1}pu.$$

It easily follows that the Lax equation (6.2) corresponds to the equation (6.3). □

In the same way as for the polynomial Lax function (4.11) we can prove the existence of the hierarchy of symmetries

$$\frac{\partial L}{\partial t_n} = \frac{\gamma-1}{\gamma}\left\{(L^{\frac{\gamma}{\gamma-1}+n})_{\geq 1}, L\right\}_0, \qquad n = 0, 1, 2\dots \quad . \tag{6.4}$$

Let us find the recursion operator for the Lax equation (6.2), corresponding to the hierarchy (6.4).

Proposition 12 *The recursion operator for the equation (6.3) is*

$$\mathcal{R} = \begin{pmatrix} u + \frac{u_x}{\gamma-1}D_x^{-1} & \frac{2v^{\gamma-2}}{\gamma-1} + \frac{(v^{\gamma-2})_x}{\gamma-1}D_x^{-1} \\ \frac{2v}{\gamma-1} + \frac{v_x}{\gamma-1}D_x^{-1} & u + \frac{\gamma-2}{\gamma-1}u_x D_x^{-1} \end{pmatrix}. \tag{6.5}$$

Proof.

First, we note that for any $n \in \mathbb{N}$ a symmetry

$$\left\{\left(L^{\frac{n\gamma+n-1}{\gamma-1}}\right)_{\geq 1}; L\right\}$$

depends only on u, v, u_x, v_x. Hence, the recursion operator must have the form

$$\mathcal{R} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix},$$

where $R_{ij} = R^*_{ij} + R^{**}_{ij} D_x^{-1}$, $i, j = 1, 2$, i.e., it is of zero order. We have the symmetry relation

$$\frac{\partial L}{\partial t_{n+1}} = L \frac{\partial L}{\partial t_n} + \{R_n, L\}. \tag{6.6}$$

The operator R_n must also have the form $R_n = f D_x^{-1} u_n + g D_x^{-1} v_n$, where f, g are functions of u, v, p only. Then,

$$L_{t_{n+1}} = u_{n+1} + v_{n+1} \frac{v^{\gamma-2}}{\gamma-1} p^{-\gamma+1} = R_{11} u_n + R_{12} v_n + \frac{v^{\gamma-2}}{\gamma-1} p^{-\gamma+1} (R_{21} u_n + R_{22} v_n),$$

and the equation (6.6) gives two equations

$$\left\{ \begin{array}{lcl} R^*_{11} + \frac{v^{\gamma-2}}{(\gamma-1)} p^{-\gamma+1} R^*_{21} & = & L + f L_p, \\ R^{**}_{11} + \frac{v^{\gamma-2}}{(\gamma-1)} p^{-\gamma+1} R^{**}_{21} & = & f_x L_P + f_p L_x \end{array} \right. \tag{6.7}$$

and

$$\left\{ \begin{array}{lcl} R^*_{12} + \frac{v^{\gamma-2}}{(\gamma-1)} p^{-\gamma+1} R^*_{22} & = & \frac{v^{\gamma-2}}{(\gamma-1)} p^{-\gamma+1} L + g L_p, \\ R^{**}_{12} + \frac{v^{\gamma-2}}{(\gamma-1)} p^{-\gamma+1} R^{**}_{22} & = & g_x L_P + g_p L_x. \end{array} \right. \tag{6.8}$$

For the equation (6.7) we have

$$L + f L_p = p^{\gamma-1} + u + \frac{v^{\gamma-1}}{(\gamma-1)^2} p^{-\gamma+1} + f \left((\gamma-1) p^{\gamma-2} - \frac{v^{\gamma-1}}{(\gamma-1)} p^{-\gamma} \right).$$

$L + f L_p$ must not contain any power of p different from $p^{\gamma-1}, p^0$. Therefore, the only value for f is

$$f = -\frac{1}{\gamma-1} p.$$

Then, the equation (6.7) takes the form

$$\left\{ \begin{array}{lcl} R^*_{11} + \frac{v^{\gamma-2}}{(\gamma-1)} p^{-\gamma+1} R^*_{21} & = & u + \frac{2v^{\gamma-1}}{(\gamma-1)^2} p^{-\gamma+1}, \\ R^{**}_{11} + \frac{v^{\gamma-2}}{(\gamma-1)} p^{-\gamma+1} R^{**}_{21} & = & \frac{u_x}{\gamma-1} + \frac{v^{\gamma-1} v_x}{(\gamma-1)^2} p^{-\gamma+1}. \end{array} \right.$$

Thus, $R^*_{11} = u$; $R^{**}_{11} = \dfrac{u_x}{\gamma - 1}$; $R^*_{21} = \dfrac{v}{(\gamma-1)}$; $R^{**}_{21} = \dfrac{v_x}{(\gamma-1)}$.

In the same way for the equation (6.8) we have

$$\frac{v^{\gamma-2}}{\gamma-1}p^{-\gamma+1}L+gL_p = \frac{v^{\gamma-2}}{\gamma-1}p^{-\gamma+1}(p^{\gamma-1}+u+\frac{v^{\gamma-1}}{(\gamma-1)^2}p^{-\gamma+1})+g\left((\gamma-1)p^{\gamma-2} - \frac{v^{\gamma-1}}{(\gamma-1)}p^{-\gamma}\right),$$

and

$$g = \frac{v^{\gamma-2}}{(\gamma-1)^2}p^{-\gamma+2}.$$

Then the equation (6.8) takes the form

$$\begin{cases} R^*_{12} + \frac{v^{\gamma-2}}{(\gamma-1)}p^{-\gamma+1}R^*_{22} &= \frac{2v^{\gamma-2}}{(\gamma-1)} + \frac{uv^{\gamma-2}}{(\gamma-1)}p^{-\gamma+2}, \\ R^{**}_{12} + \frac{v^{\gamma-2}}{(\gamma-1)}p^{-\gamma+1}R^{**}_{22} &= \frac{\gamma-2}{\gamma-1}v^{\gamma-3}v_x + \frac{\gamma-2}{\gamma-1}u_xv^{\gamma-3}v_xp^{-\gamma+1}. \end{cases}$$

We have obtained the recursion operator (6.5). $\square$

It is worth mentioning that the equation (4.49) and equations of polytropic gas dynamics (6.3) are related by the following change of variables

$$\begin{aligned} S &= \frac{u}{(-k+2)(-k+1)}, \\ P &= \frac{v^{\frac{1}{-k+1}}}{(-k+2)^2}, \end{aligned} \tag{6.9}$$

where $\gamma = \dfrac{-k+2}{-k+1}$. Note that under this change of variables the recursion operator (4.50) is mapped to the recursion operator (6.5).

Chapter 7

Conclusion

We have constructed the recursion operators of some equations of hydrodynamic type admitting dispersionless Lax representation with a polynomial and rational Lax function. The form of the these operators fall into the class of pseudo differential operators $A + B\,D^{-1}$ where A and B are functions of dynamical variables and their derivatives. The generalized symmetries of these equations are local and all belong to the same class (i.e., they are also equations of hydrodynamic type). We have introduced a method of reduction which leads also to integrable class. Depending upon the type of reductions we may obtain infinitely many different classes of integrable systems.

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