LINEAR TOPOLOGICAL STRUCTURE OF SPACES OF WHITNEY FUNCTIONS DEFINED ON SEQUENCES OF POINTS

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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In this work we consider the spaces of Whitney functions defined on convergent sequences of points.By means of linear topological invariants we analyze linear topological structure of these spaces .Using diametral dimension we found a continuum of pairwise non-isomorphic spaces for so called regular type and proved that more refined invariant compound invariants are not stronger than diametral dimension in this case .

On the other hand, we get the same diametral dimension for the spaces of Whitney functions defined on irregular compact sets.

Keywords: Linear Topological Invariants,Whitney Functions,Diametral Dimension.

ÖZET

DİZİ NOKTALARI ÜZERİNDE TANIMLI WHITNEY FONKSİYON UZAYLARININ TOPOLOJİK YAPISI

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Bu çalışmada yakınsak dizi noktaları üzerinde tanmlanmış Whitney fonksiyon uzaylarını ele aldık. Lineer topolojik invariantlar vasıtası ile bu uzayların topolojik yapısını inceledik. Diametral dimensionı kullanarak düzgün türdeki dizi noktaları üzerinde tanımlı sonsuz çoklukta karşılıklı izomorfik olmayan uzaylar bulduk ve bu durum için bileşik invariantların daha kuvvetli olmadığını gösterdik.

Bununla beraber, düzgün olmayan kompakt kümeler üzerinde tanımlı Whitney fonksiyon uzayları içinde aynı diametral dimensionu elde ettik.

Anahtar kelimeler: Lineer topolojik invariantlar, Whitney fonksiyonları, Diametral dimension.

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Contents

Chapter 1

Introduction

1.1 Linear Topological Invariants

We begin with a short summary of invariants which are distinct characteristics of linear topological spaces; that is in order to show two linear topological spaces are not isomorphic, it is enough to prove that these characteristics of spaces differs from each other. More precisely,

If Φ is a class of linear topological spaces, φ is a set with a equivalence relation $\sim$ and $\tau : \Phi \to \varphi$ is a mapping, such that

$$X \simeq Y \Rightarrow \tau(X) \sim \tau(Y)$$

then τ is called linear topological invariant and τ is said to be complete invariant on the class Φ if for any $X, Y \in \Phi$

$$\tau(X) \sim \tau(Y) \Rightarrow X \simeq Y.$$

We restrict our attention to Fréchet spaces.

Definition 1.1 *A* $\mathbf{K}$*-vector space* F*, equipped with a metric, is called a metric linear space, if in* E *addition is uniformly continuous and scalar multiplication is continuous. A metric linear space* E *is said to be locally convex, if for each zero neighborhood* V *there exists a convex zero neighborhood* U *with* $U \subset V$*.*

A complete metric locally convex space is called a Fréchet space.

Definition 1.2 *Let* E *be a locally convex space. A collection* $\mathcal{U}$ *of zero neighborhoods in* E *is called a* fundamental system of zero neighborhoods, *if for every zero neighborhood* U *there exists a* $\mathrm{V} \in \mathcal{U}$ *and* $\epsilon > 0$ *with* $\epsilon \mathrm{V} \subset \mathrm{U}$*.*

A family $(\|.\|_\alpha)_{\alpha \in A}$ of continuous seminorms on E is called a *fundamental system of seminorms,* if the sets

$$U_\alpha := \{x \in E : \|x\|_\alpha \leq 1\}, \quad \alpha \in A,$$

form a fundamental system of neighborhoods.

Let E be a locally convex space which has countable fundamental system of neighborhoods $(U_n)_{n \in \mathbf{N}}$. Without lose of generality one can assume that

$$U_{n+1} \subset U_n, \quad \forall n \in \mathbf{N}.$$

1.1.1 Counting invariants.

First of this kind of invariants, Approximative dimension, was introduced by A.N. Kolmogorov [13] and A.Pelczynski [19] and they proved $A(D) \ncong A(G)$ if the domains $D \subset \mathbb{C}^n, G \subset \mathbb{C}^m, \quad n \neq m$ and $A(\mathbb{D}^n) \ncong A(\mathbb{C}^n)$ where $\mathbb{D}^n$ is the unit polydisc in $\mathbb{C}^n$ and $A(\mathbb{D})$ is the space of all analytic functions on $\mathbb{D}$.

Later on the so called diametral dimension $\Gamma(X)$ and dual diametral dimension $\Gamma'(X)$ were introduced (definition see below) by C. Bessaga, A. Pelczynsky and S. Rolewicz [2]. These kind of invariants turn to be more strong then approximative dimensions (see [16]).

Characterization of nuclear spaces in terms of diametral dimension was given by Mitiagin (see e.g. [16]).

In [5] Dragilev has shown that the invariants $\Gamma(X), \Gamma'(X)$ are very useful for distinguishing some special classes of spaces with regular absolute basis.

Moreover Crone and Robinson [4], Kondakov [14] proved that the invariant $\Gamma'(X)$ is complete on the class of all nuclear spaces with regular basis.

It must be remarked here that $\Gamma(X), \Gamma'(X)$ are not effective invariants for consideration of distinguishing spaces without regular absolute basis as it can be seen by the following proposition (see ([6], [17], [20])).

Proposition 1.1 *The spaces A(U) and $A(U) \times A(\mathbb{C})$ are not isomorphic, although $\Gamma'(A(U)) = \Gamma'(A(U) \times A(\mathbb{C}))$.*

Definition 1.3 *Let U be an absolutely convex absorbent set and V be any set in the locally convex space X. Then n^{th} Kolmogorov diameters of V with respect to U is defined as*

$$d_n(V, U) = \inf_{L \in \mathcal{L}_n} \sup_{x \in V} \inf_{y \in L} \|x - y\|_U$$

where infimum is taken over the collection $\mathcal{L}_n$ of all subspaces of X of dimension $\leq n$. Here $\|.\|_U$ is the gauge functional of the set U (see definition (1.6)).

It is easy to see that definition of $d_n(V,U)$ can also be given as:

$$d_n(V,U) = \inf_{L\in\mathcal{L}_n} \inf\{\delta : V \subset \delta U + L\}.$$

The diametral dimension is given as follows,

$$\Gamma(X) = \{\gamma = (\gamma_n) : \forall U \quad \exists V \quad ; \gamma_n d_n(V,U) \to 0 \text{ as } n \to \infty\}$$

and

$$\Gamma'(X) = \{\gamma = (\gamma_n) : \exists V \quad \forall U \quad ; \frac{\gamma_n}{d_n(V,U)} \to 0 \text{ as } n \to \infty\}$$

We consider the counting function corresponding to the diametral dimension $\Gamma(X)$,

$$\beta(t) := \beta(U_q, U_p, t) = \min\{\dim L : tU_q \subset U_p + L\}, \qquad t > 0.$$

One can show that

$$\beta(t) = |\{n : d_n(U_q, U_p) > \frac{1}{t}\}|,$$

where $|K|$ denotes the cardinality of the set K, and $(U_k)_{k=1}^{\infty}$ is the basis of neighborhoods of X.

If X is a Schwartz space (that is $\forall p \quad \exists q$ such that U_q is precompact in $X_p := X/Z_p$, $Z_p = \{x \in X : \|x\|_p = 0\}$) and p,q are sufficiently apart from each other, then $\beta(U_q, U_p, t)$ takes finite values.

The following well-known propositions express the direct relation between $\Gamma(X)$ and $\beta(t)$:

Proposition 1.2 $(\gamma_n) \in \Gamma(X) \Longleftrightarrow \forall p \quad \exists q \quad ; \forall C \quad \exists n_0$*:*

$$\beta(U_q, Up, C\gamma_n) \leq n \qquad for \qquad n \geq n_0.$$

Proposition 1.3 *If Fréchet spaces X and Y are isomorphic, then*

$$\forall p_1 \exists p \quad \forall q \exists q_1, C :$$

$$\beta^Y(V_{q_1}, V_{p_1}, t) \leq \beta^X(U_q, U_p, Ct), \quad t > 0,$$

and vice-versa.

Here $(V_p)_{p=1}^{\infty}, (U_q)_{p=1}^{\infty}$ are bases of neighborhoods of spaces X and Y respectively.

1.1.2 Interpolating Invariants.

There are certain interpolating properties of seminorms which are invariant under isomorphisms. Various forms of such invariants were introduced by

Dragilev [5], Zahariuta [26], Vogt [24] and others. As an example we can present here only *Dominating Norm (DN)* property which will be mentioned in the sequel.

A Fréchet space X with a fundamental system of seminorms $(\|.\|_q)_{q=o}^{\infty}$ *is said to have the DN property [24](also* D_1 *in [25]), if*

$$\exists p \forall q \exists r, C > 0 : \|.\|_q \leq t\|.\|_p + \frac{C}{t}\|.\|_r, \qquad t > o$$

with $p, q, r \in \mathbb{N}_0 = 0, 1, 2, \ldots$.

1.1.3 Compound Invariants.

In [27] Zahariuta suggested a method of combining possibilities of both counting and interpolating invariants, to produce new characteristics which are considered as the invariants based on the asymptotic behavior of classical n-diameters of pairs of "synthetic" neighborhoods of zero, built in an invariant way for a given pair, triple and so on of neighborhoods.

We give here two of these, namely β_1 and β_{II}, in what follows $t \to \infty$ and $\tau = \tau(t) \to 0$; for $0 \leq p < q < r$ let $U = \tau U_p \cap tU_r$

$$\beta_1(U, U_q) = \beta_1(\tau, t, U_p, U_q, U_r) = \inf\{\dim L : U \subset U_q + L\} \tag{1.1}$$

$$= |\{n : d_n(U, U_q) > 1\}|, \tag{1.2}$$

$$\beta_{\mathrm{II}}(U_q, V) = \beta_{\mathrm{II}}(\tau, t, U_p, U_q, U_r) = \inf\{\dim L : U_q \subset V + L\}, \tag{1.3}$$

where $V = conv(\tau U_p \cup tU_r)$ and infumum is taken over all finite dimensional subspaces of X; (here conv(K) denotes the convex hull of the set K).

Proposition 1.4 *Let the spaces X and Y be isomorphic Fréchet spaces with fundamental systems of neigborhoods* $(U_p)_1^\infty and (V_p)_1^\infty$ *respectively. Then*

$$\forall p \exists p_1 \forall q_1 \exists q \forall r \exists r_1, \exists C$$

such that

$$\beta_1^Y(\tau, t, V_{p_1}, V_{q_1}, V_{r_1}) \leq \beta_1^X(C\tau, Ct, U_p, U_q, U_r), \qquad \forall t > 0, \forall \tau > 0 \tag{1.4}$$

and vice-versa.

Proposition 1.5 *Let the spaces X and Y be isomorphic Fréchet spaces with fundamental systems of neigborhoods $(U_p)_1^\infty and (V_p)_1^\infty$ respectively. Then*

$$\forall p_1 \exists p \forall q \exists q_1 \forall r_1 \exists r, \exists \epsilon$$

such that

$$\beta_{\mathrm{II}}^Y(\tau, t, V_{p_1}, V_{q_1}, V_{r_1}) \leq \beta_{\mathrm{II}}^X(\epsilon\tau, \epsilon t, U_p, U_q, U_r), \qquad \forall t > 0, \forall \tau > 0 \qquad (1.5)$$

and vice-versa.

Proofs are similar, so we give the proof of the second one.

Proof: Assume $\tau : X \to Y$ is an isomorphism. Then according to the above order of quantifiers, for some $C \geq 1$ we have

$$U_p \subset C\tau^{-1}(V_{p_1}) \quad \Rightarrow \frac{1}{C}\tau(U_p) \subset V_{p_1}$$
$$V_{q_1} \subset C\tau(U_q)$$
$$U_r \subset C\tau^{-1}(V_{r_1}) \quad \Rightarrow \frac{1}{C}\tau(U_r) \subset V_{r_1}$$

then according to the definition of β_{II}, it follows that,

$$\beta_{\mathrm{II}}^Y(\tau, t, V_{p_1}, V_{q_1}, V_{r_1}) \leq \beta_{\mathrm{II}}^X(\tau, t, \frac{1}{C}\tau(U_p), C\tau(U_q), \frac{1}{C}\tau(U_r))$$
$$\leq \beta_{\mathrm{II}}^X(\frac{1}{C^2}\tau, \frac{1}{C^2}t, U_p, U_q, U_r) \qquad \square$$

1.2 Continuous Norm and Tikhomirov's Theorem for Spaces Without Continuous Norm

In this section we will consider the continuous norm property of spaces and Tikhomirov's theorem which is used for finding lower bound for the counting functions β, β_1 and so on.

Definition 1.4 *A Fréchet space X is said to have continuous norm, if one of its seminorms is a norm. Similarly, X has no continuous norm if every neighborhood contains a line.*

THEOREM 1.1 *(Tikhomirov [16,Prop.6]) Let X be linear space with continuous norm. If U is an absolutely convex set in X then for any set $V \in X$, if*

$$\alpha U \cap L_{n+1} \subset V \cap L_{n+1}$$

is satisfied for some n+1 dimensional subspace L_{n+1} of X and for $\alpha > 0$, then

$$d_n(V, U) \geq \alpha.$$

We remark here that, Tikhomirov's theorem which is given in [16] can not be applied to the spaces without continuous norm.

For example let us consider ω, the space of all sequences with the topology given by the seminorms

$$|x|_p = \sup_{n \leq p} |x_n| \qquad with \qquad x = (x_n) \in \omega.$$

It is clear that for $L = span(e_n)_{n=p+1}^{\infty}$ we get $L \subset U_p \quad \forall p$ and ω has no continuous norm.

Thus, $\forall n \in \mathbf{N}$ we have

$$U_p \cap L_{n+1} \subset U_q \cap L_{n+1}$$

for some n+1 dimensional subspace in ω;

In fact we can choose $L = span(e_k)_{k=r}^{r+n}, \quad r > q > p$. But that would mean to get

$$d_n(U_q, U_p) \geq 1, \quad \forall n,$$

which is impossible, as for $q > p$ we get trivially

$$d_n(U_q, U_p) = 1 \qquad for \quad n < p$$

and

$$d_n(U_q, U_p) = 0 \qquad for \quad n \geq p.$$

We continue with the following definitions which are necessary for this subject.

Definition 1.5 *If F is a subspace of K-vector space E, then the set E/F of all so-called cosets $[x]_F := x + F$, $x \in E$, becomes a linear space with respect to the addition and the scalar multiplication defined by $(x+F)+(y+F) := x+y+F$ and $k(x+F) := kx+F, \quad \forall x, y \in E, \quad \forall k \in K$. This is the quotient vector space of E modulo F. The map $\tau : E \to E/F, \tau(x) := x+F$, is called the quotient map and it is linear.*

Definition 1.6 *Let X be a locally convex space and U be absolutely convex absorbent set in X, define the* gauge (or Minkowski) functional of the set U $\|x\|_U : X \to \mathbf{R}$ *by*

$$\|x\|_U = \inf\{\delta > 0 : x \in \delta U\}.$$

It is clear that the kernel of $\|.\|_U$, $Z_U := \{x \in X : \|x\|_U = 0\}$, is a closed subspace of X. Let X_U be the completion of X/Z_U with respect to the norm $\|.\|_U$.

After small modification we present here the following version of Tikhomirov's theorem which is valid for any locally convex space X.

THEOREM 1.2 *Let U be absolutely convex absorbent set and V be any set in X; if for some $\alpha > 0$ and for (n+1)-dimensional subspace L_{n+1} in X_U*

$$\alpha U/Z_U \cap L_{n+1} \subset V/Z_U \cap L_{n+1} \tag{1.6}$$

then

$$d_n(V, U) \geq \alpha. \tag{1.7}$$

Proof: It is clear that, if the space has continuous norm, then

$$\alpha U/Z_U \cap L_{n+1} \subset V/Z_U \cap L_{n+1}$$

implies

$$\alpha U \cap L'_{m+1} \subset V \cap L'_{m+1}$$

for some (m+1) dimensional subspace L'_{m+1} in X with $m \geq n$, it follows that

$$d_m(V, U) \geq \alpha$$

by the previous theorem. Since Kolmogorov diameters are decreasing, we obtain

$$d_n(V, U) \geq \alpha.$$

If the space has no continuous norm, then for the Banach space X_U with the norm $\|.\|_U$

$$\alpha U/Z_U \cap L_{n+1} \subset V/Z_U \cap L_{n+1}$$

implies

$$d_n(V/Z_U, U/Z_U) \geq \alpha$$

by Theorem 1.1.

Then it is enough to show

$$d_n(V, U) \geq d_n(V/Z_U, U/Z_U)$$

which will imply the result.

Let

$$\delta_0 := d_n(V, U) = \inf_{L \in \mathcal{L}_n} \inf\{\delta : V \subset \delta U + L\}.$$

Then

$$\forall \epsilon > 0, \qquad \exists L \in \mathcal{L}_n \qquad \text{and} \qquad \exists \beta \in (\delta_0, \delta_0 + \epsilon)$$

such that

$$V \subset \beta U + L \Rightarrow V \subset (\delta_0 + \epsilon)U + L$$

Then $\qquad \tau(V) \subset (\delta_0 + \epsilon)\tau(U) + \tau(L)$.

But as $\dim(\tau(L)) \leq \dim L = n$ and say $\dim(\tau(L)) = m \leq n$

Then

$$V/Z_U \subset (\delta_0 + \epsilon)U/Z_U + \tau(L) \qquad \forall \epsilon$$

So according to the definition of m^{th} Kolmogorov diameter

$$d_m(V/Z_U, U/Z_U) \leq \delta_0 + \epsilon \qquad \forall \epsilon.$$

Since $m \leq n$, we get

$$d_n(V/Z_U, U/Z_U) \leq d_m(V/Z_U, U/Z_U) \leq \delta_0 + \epsilon \qquad \forall \epsilon.$$

That is $\quad d_n(V/Z_U, U/Z_U) \le d_n(V,U) \quad$, which gives us the result that

$$d_n(V,U) \ge \alpha \qquad\qquad \square$$

1.3 Whitney Functions and Whitney Jets

Let K be a perfect (that is without isolated points) compact set on the line. By $\mathcal{E}(K)$ we denote the space of Whitney functions on K; that is functions $f : K \to \mathbb{R}$ which are extendable to a C^∞-function $\tilde{f}$ on$\mathbb{R}$.

$\mathcal{E}(K)$ is a Fréchet space with the topology defined by the family of seminorms

$$\|f\|_p = \sup_{0\le i\le p} |f^{(i)}(x)| + \sup \frac{|(R_y^p f)^{(i)}(x)|}{|x-y|^{p-i}} \qquad \forall x, y \in K, \quad x \neq y, \tag{1.8}$$

where

$$R_y^p f(x) = f(x) - \sum_{k=0}^{p} f^{(k)}(y) \frac{(x-y)^k}{k!}$$

is the p^{th} Taylor remainder, $p \in \mathbf{N}_0$.

Here, given values of the function f on K, using perfectness of the compact set K we can define the values of all its derivatives on K. In other words the compact set K is C^∞-determining in this case.

Definition 1.7 *$K \subset \mathbb{R}^m$ is C^∞-determining if for any extendable function f on K with $f|_K = 0$ we obtain $f^{(j)}|_K = 0, \quad \forall j \in \mathbb{N}^m$.*

On the other hand, suppose that a compact set K contains an isolated point, let it be 0. Then in order to define an extendable function f completely with all derivatives at 0, we have to give not only the values of f at 0, but also the values of all its derivatives $a_j = f^{(j)}(0), j \in \mathbb{N}_0$.

Moreover, since the Borel problem (given sequence (a_j) construct a function $f \in C^\infty[-1,1]$ such that $f^{(j)}(0) = a_j, \forall j \in \mathbf{N}_0$) has a solution for any sequence (a_j), we have no restriction on the growth of "derivatives" of f. That is if $K = K_1 \cup \{0\}$ then $\mathcal{E}(K) \simeq \mathcal{E}(K_1) \bigoplus \omega$. In particular $\mathcal{E}(\{0\}) \simeq \omega$.

But since the space ω has no continuous norm, we get the following trivial proposition in accordance.

Proposition 1.6 *Let $K \subset \mathbb{R}$ be a compact set; then $\mathcal{E}(K)$ has no continuous norm if and only if K has an isolated point.*

So in general, for a compact set $K \subset \mathbb{R}$, we will define the space of Whitney jets $\mathcal{E}(K)$ to be the space of all infinite sequences $f = (f^{(i)}(x))_{i \in \mathbb{N}_0}, \quad x \in K$, for which there exists an extension $F \in C^\infty(\mathbb{R})$ such that $f^{(i)}(x) = F^{(i)}(x) \quad \forall x \in K, \forall i \in \mathbf{N}_0$.

$\mathcal{E}(K)$ is Fréchet space with the topology defined by the seminorm family $\|.\|_p, \quad p \in \mathbb{N}_0$, as defined in (1.8).

It is clear that a compact set K with isolated point does not have the Extension property.

Definition 1.8 For $K \subset \mathbf{R}^n$, K has the Extension property if there exists a linear continuous extension operator $L : \mathcal{E}(K) \to \mathbf{C}^\infty(\mathbf{R}^n)$.

Mitiagin [16] proved that $K = [-1, 1] \subset \mathbb{R}$ has the Extension property whereas $K = 0$ does not.

In [21] Tidten has shown that the property DN of the space $\mathcal{E}(K)$ is equivalent to the Extension property for the compact set K. We give the following trivial proof that the singleton has no Extension property .

Assume there exists an extension operator $L : \mathcal{E}(\{0\}) \to \mathbf{C}^\infty(\mathbf{R})$. Then $\forall p, \exists C, q$ such that

$$\|Lf\|_p \leq C\|f\|_q \text{ for all } f \in \mathcal{E}(\{0\}).$$

Then for $p = 0$ there exists q_0, C_0 such that, $\|Lf\|_0 \leq C_0\|f\|_{q_0}$ for all $f \in \mathcal{E}(\{0\})$. Consider $f = (f^{(j)})_{j=0}^\infty = 1$ for $j = q + 1$, and zero otherwise.

Clearly $\|f\|_q = 0$ and we get

$$\|Lf\|_0 \leq C_0\|f\|_{q_0} = 0.$$

This is contradiction as $Lf \neq 0$.

Clear that it is not possible to use the interpolating invariants for the spaces $\mathcal{E}(K)$, if K has an isolated point.

The problem of isomorphic classification of spaces of C^∞ and Whitney functions was considered in several cases. As a result the families having the cardinality of the continuum of pairwise non-isomorphic spaces were given. In [7] and [22] it was done for the spaces of $C^\infty-$functions on the sharp cusp, in [10] for the spaces of Whitney functions given on so-called "running duck" set and in [11] and [1] for the spaces of Whitney functions defined on Cantor-type sets by using counting, interpolating and compound invariants.

It must be remarked here that the diametral dimension can not be applied to distinguish the spaces of the type C^∞ or $\mathcal{E}(K)$ with $K^0 \neq 0$. In fact, these spaces contain a subspace which is isomorphic to the space s of rapidly decreasing sequences. Since for a subspace Y in X we have $\Gamma(Y) \supset \Gamma(X)$ [16-prop 7], and all these spaces contain a subspace isomorphic to the space s, we get that their diametral dimension is not larger than $\Gamma(s)$.

On the other hand, the space s has the minimal possible diametral dimension in the class of nuclear spaces [see 16], thus we obtain $\Gamma(\mathcal{E}(K)) = \Gamma(s)$.

Here we restrict our attention to the following model case of compact sets

$$K = \{0\} \cup \cup_{n=1}^{\infty} \{a_n\} \text{ with } a_n \to 0.$$

Chapter 2

Regular Case

Let $\mathcal{E}(K)$ be a space of Whitney jets, defined on the set $K = \{0\} \cup \cup_{n=1}^{\infty}\{a_n\}$ such that $a_n \to 0$ monotonically.

Definition 2.1 *We say that a compact set* $K = \{0\} \cup \cup_{n=1}^{\infty}\{a_n\}$ *is of regular type if*

$$\exists Q \geq 1 \quad \textit{such that} \quad |a_{n-1} - a_n| \geq a_n^Q \quad , n \geq n_0 \textit{ for some } n_0 \in \mathbb{N}. \quad (2.1)$$

2.1 Counting Function $\beta(t)$

THEOREM 2.1 *Let* $K = \{0\} \cup \cup_{n=1}^{\infty} a_n$ *be of regular type with the corresponding constant* Q_a. *Then for the counting function corresponding to the diametral dimension of the space* $\mathcal{E}(K)$ *and for* $q > p > 0$ *with* $q - pQ_a > 0$, *we get*

$$N_2 \leq \beta(t, U_p, U_q) \leq (q+1)N_1, \qquad t \geq 4$$

where

$$N_1 = \min\{n : a_n \leq (\frac{1}{2et})^{\frac{1}{q-pQ}}\} \tag{2.2}$$

and

$$N_2 = \max\{n : |a_n - a_{n+1}| \geq (\frac{8}{t})^{\frac{1}{q-p}}\} \quad . \tag{2.3}$$

Proof: *Upper bound for* β

From definition of β we see that $\beta(t) \leq \dim L$ for any subspace L satisfying

$$tU_q \subset U_p + L.$$

Let us consider the following functions

$$H_{0j} = \begin{cases} \frac{x^j}{j!} & \text{if } x \in [0, a_{N_1}] \cap K \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

and

$$h_{nj} = \begin{cases} \frac{(x-a_k)^j}{j!} & \text{if } x = a_k \in K \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

and define

$$L = \text{span}\{H_{0j} \cup h_{nj} : n = 1....N_1;\ j = 0....q\}$$

then $\dim L = N_1(q+1)$.

For any $f \in U_q$ choose $g \in L$ such that

$$g(x) = \sum_{j=0}^{q} f^{(j)}(0)\frac{x^j}{j!} + \sum_{k=1}^{N_1-1}\sum_{j=0}^{q} f^{(j)}(a_k)\frac{(x-a_k)^j}{j!}$$

Now let us show that with above choice of the subspace L, $U_q \subset \frac{1}{t}U_p + L$ is satisfied; that is to show that $\|f-g\|_p \leq \frac{1}{t}$.

a. *Upper bound for* $|f-g|_p = \sup_{x\in K}|f^{(i)}(x) - g^{(i)}(x)| \quad i = 0, ..., p.$

a.1 $\quad x \leq a_{N_1}$

$$\Rightarrow \qquad g(x) = \sum_{j=0}^{q} f^{(j)}(0)\frac{x^j}{j!} \qquad \text{then} \qquad |f(x) - g(x)| = R_0^q f(x)$$

$$\Rightarrow \qquad |f^i(x) - g^i(x)| = |(R_0^q f)^i(x)| \leq \|f\|_q |x|^{q-i} \Rightarrow a_{N_1}^{q-p} \leq \frac{1}{2t} \quad by \quad (2.1).$$

a.2 $\quad x > a_{N_1}$ and let $x := a_l$

$$\Rightarrow \qquad g(x) = \sum_{j=1}^{q} f^{(i)}(a_l)\frac{(x-a_l)^j}{j!} \qquad \text{then} \qquad |f^{(i)}(x) - g^{(i)}(x)| = 0 < \frac{1}{2t}.$$

b *Upper bound for $b_{i,p}$.*

Here $\quad b_{i,p} = \frac{|(R_y^p(f-g))^{(i)}(x)|}{|x-y|^{p-i}} \qquad \forall x, y \in K, \quad x \neq y \quad and \quad i = 0, 1...p.$

b.1 $\quad x > a_{N_1}, \quad y > a_{N_1}.$

$$\Rightarrow (R_y^p(f-g))^{(i)}(x) = f^{(i)}(x) - g^{(i)}(x) - \sum_{j=1}^{p}(f^{(j)}(y) - g^{(j)}(y))\frac{(x-y)^{j-i}}{(j-i)!} = 0$$

$$\Rightarrow \qquad b_{i,p} = 0 < \tfrac{1}{2t}$$

b.2 $\quad x \leq a_{N_1}$ and $y \leq a_{N_1}$.

$$\Rightarrow \qquad g(x) = \sum_{j=0}^{q} f^{(j)}(0)\frac{x^j}{j!} \quad \Rightarrow \quad |f(x) - g(x)| = R_o^q f(x)$$

then

$$R_y^p(f-g)(x) = R_o^q f(x) - \sum_{j=0}^{p}(R_0^q f)^{(j)}(y)\frac{(x-y)^j}{j!}$$

$$\Rightarrow \qquad b_{i,p} \leq \frac{|(R_0^q(f))^{(i)}(x)|}{|x-y|^{p-i}} + \frac{\sum_{j=i}^{p}(R_0^q f)^{(j)}(y)\frac{|x-y|^{j-i}}{(j-i)!}}{|x-y|^{p-i}}$$

$$\Rightarrow b_{i,p} \leq \|f\|_q |x|^{q-i}|x-y|^{i-p} + \sum_{j=i}^{p}\|f\|_q |y|^{q-i}\frac{|x-y|^{j-p}}{(j-i)!}$$

$$\leq \quad a_{N_1}^{q-p} + a_{N_1}^{q-p} \sum_{j=i}^{p} \frac{1}{(j-i)!}$$

$$\leq \quad a_{N_1}^{q-p}(e+1) \leq (\frac{1}{2et})^{\frac{q-p}{q-pQ}}.e \leq \frac{1}{2t} \quad \text{by} \quad (2.2).$$

b.3 $\quad x > a_{N_1}$ and $y \leq a_{N_1}$.

Then $\quad f^{(i)}(x) - g^{(i)}(x) = 0$ and $f^{(j)}(y) - g^{(j)}(y) = (R_0^q f)^{(j)}(y)$

$$\Rightarrow \quad |R_y^p(f-g)^{(i)}(x)| \leq \sum_{j=i}^{p} |(R_0^q f)^{(j)}(y) \frac{|x-y|^{j-i}}{(j-i)!}|$$

$$\leq \quad \|f\|_q \sum_{j=i}^{p} |y|^{q-j} \frac{|x-y|^{j-i}}{(j-i)!}$$

$$\Rightarrow \quad b_{i,p} \leq \sum_{j=i}^{p} |y|^{q-j} \frac{|x-y|^{j-i}}{(j-i)!} |x-y|^{i-p} = \sum_{j=i}^{p} |y|^{q-j} \frac{|x-y|^{j-p}}{(j-i)!}$$

$$\leq \sum_{j=i}^{p} |a_{N_1}|^{q-j} \frac{|a_{N_1-1} - a_{N_1}|^{j-p}}{(j-i)!} \leq \sum_{j=i}^{p} |a_{N_1}|^{q-j} \frac{|a_{N_1}|^{Qj-Qp}}{(j-i)!} \qquad by \quad (2.1)$$

$$\leq \sum_{j=i}^{p} \frac{|a_{N_1}|^{q-Qp}.|a_{N_1}|^{Qj-j}}{(j-i)!} \qquad , \text{ since} \quad Q \geq 1,$$

$$\Rightarrow \qquad b_{i,p} \leq |a_{N_1}|^{q-pQ} e \leq \frac{1}{2t} \qquad \text{by } (2.2).$$

b.4 $\quad x \leq a_{N_1}$ and $y > a_{N_1}$.

$$\Rightarrow \quad |f^{(i)}(y) - g^{(i)}(y)| = 0 \qquad \text{then}$$

$$|(R_y^p(f-g))^{(i)}(x)| \leq |f^{(i)}(x) - g^{(i)}(x)| = |(R_0^q f)^{(i)}(x)|$$

$$\Rightarrow \quad b_{i,p} \leq |(R_0^q f)^{(i)}(x)|.|x-y|^{i-p} \leq \|f\|_q |x|^{q-i} |x-y|^{i-p}$$

similar to $(b.3)$,

$$b_{i,p} \leq |a_{N_1}|^{q-i} |a_{N_1-1} - a_{N_1}|^{i-p} \leq \frac{1}{2t}.$$

Therefore $\|f-g\|_p \leq \frac{1}{t}$ and $U_q \subset \frac{1}{t}U_p + L$, that is

$$\beta(t, U_p, U_q) \leq \dim L = (q+1)N_1.$$

Lower bound for β.

Here we are going to use Tikhomirov's theorem (see thm (1.2)) and the second definition of β as a tool. That is

$$\alpha U_p/Z_p \cap L_{n+1} \subset U_q/Z_p \qquad \text{with } \dim L_{n+1} = n+1$$

implies $d_n(U_q, U_p) \geq \alpha$. Then

$$\beta(t, U_p, U_q) \geq \sup\{\dim L : 2U_p/Z_p \cap L \subset tU_q/Z_p\},$$

where supremum is taken over all finite dimensional subspaces L of $\mathcal{E}^p(K)$, which is the space of Whitney jets of order p,

$$\mathcal{E}^p(K) = \{f \in C^p(K) : \exists F \in C^p(R) \text{ such that } F^{(i)}|_K = f^{(i)}, \quad i \leq p\}$$

and Z_p is defined as

$$Z_p = \{f \in \mathcal{E}(K) : \|f\|_p = 0\}.$$

Define $L =$ span $\{[h_{np}]_p : n = 1, ..., N_2\}$.

Let us show with the above choice of subspace L, the following embedding is satisfied:

$$2U_p/Z_p \cap L \subset tU_q/Z_p.$$

Take $f \in 2U_p/Z_p \cap L$, then

$$f(x) = \sum_{k=1}^{N_2} \alpha_k \frac{(x-a_k)^p}{p!} + Z_p = \tilde{f} + Z_p$$

where

$$\tilde{f} = \sum_{k=1}^{N_2} \alpha_k \frac{(x - a_k)^p}{p!}$$

then $\tilde{f} \in 2U_p$

$$\Rightarrow \qquad 2 \geq \|\tilde{f}\|_p > |\tilde{f}^p(x)| \geq |\alpha_k|$$

$$\Rightarrow \quad |\alpha_k| \leq 2 \qquad \forall k = 1, ..., N_2.$$

Clearly, in order to show $f \in tU_q/Z_p$ it is sufficient to show $\tilde{f} \in tU_q$, that is $\|\tilde{f}\|_q \leq t, \qquad t > 0$.

a. *Upper bound for* $\quad |\tilde{f}|_q = \sup |\tilde{f}^{(i)}(x)| \quad , i \leq q \quad x \in K.$

a.1 $\quad x < a_{N_2}$.

$$\text{Then} \quad \tilde{f}^{(i)}(x) = 0 < \frac{t}{2} \quad \forall i \leq q.$$

a.2 $\quad x \geq a_{N_2}$ and let $x := a_l$.

$$\text{Then} \quad \tilde{f}(x) = \alpha_l \frac{(x - a_l)^p}{p!}$$

$$\Rightarrow \quad |\tilde{f}^{(i)}(x)| = |\alpha_l \frac{(x - a_l)^{p-i}}{(p-i)!}| \leq 2 \leq \frac{t}{2}$$

$$\text{as} \qquad \frac{(x - a_k)^{p-i}}{(p-i)!} \neq 0 \qquad \text{only for} \quad i = p.$$

b. *Upper bound for* b_{iq} .

$$\text{Here} \quad b_{iq} = \frac{|(R_y^q \tilde{f})^{(i)}(x)|}{|x - y|^{q-i}} \ , \quad i = 0, 1, ..., q, \quad x, y \in K \quad , x \neq y.$$

Remark. For $p < i \leq q$

$$\tilde{f}^{(i)}(x) = 0 \quad \forall x \in K \Rightarrow b_{i,q} = 0$$

then, without lose of generality it is enough to take $i = 0, 1, ..., p$.

b.1 $x < a_{N_2}$ and $y \geq a_{N_2}$.Let $y := a_s$

$$\Rightarrow \quad \tilde{f}(x) = 0 \text{ and } \tilde{f}(y) = \alpha_s \frac{(y-a_s)^{p-i}}{(p-i)!}$$

$$\text{then} \quad |(R_y^q \tilde{f})^{(i)}(x)| \leq |\sum_{k=i}^{q} \tilde{f}^{(k)}(y) \frac{(x-y)^{k-i}}{(k-i)!}|$$

$$\leq \alpha_s \frac{|x-y|^{p-i}}{(p-i)!}$$

since $\tilde{f}^{(k)}(y) \neq 0$ only for $k = p$

$$\Rightarrow \quad b_{i,q} \leq \frac{2}{(p-i)!}|x-y|^{p-i}|x-y|^{i-q} = \frac{2}{(p-i)!}|x-y|^{p-q}$$

$$\leq 2|a_{N_2} - a_{N_2+1}|^{p-q} \leq \frac{t}{2} \quad \text{by} \quad (2.3).$$

b.2 $x \geq a_{N_2}$ and $y \geq a_{N_2}$. Let $x := a_l$and $y := a_s$

$$\Rightarrow |(R_y^q \tilde{f})^{(i)}(x)| = |\alpha_l \frac{(x-a_l)^{p-i}}{(p-i)!} - \sum_{k=i}^{q} \alpha_s \frac{(y-a_s)^{p-k}}{(p-k)!} \frac{(x-y)^{k-i}}{(k-i)!}|$$

where $\frac{(y-a_s)^{(p-k)}}{(p-k)!} \neq 0$ only for $p = k$.

$$\Rightarrow \quad |(R_y^q \tilde{f})^{(i)}(x)| \leq |\alpha_l \frac{(x-a_l)^{p-i}}{(p-i)!} - \alpha_s \frac{(x-y)^{p-i}}{(p-i)!}|$$

$$\leq 2\frac{|x-a_l|^{p-i}}{(p-i)!} + 2\frac{|x-y|^{p-i}}{(p-i)!}$$

$$\Rightarrow b_{i,q} \leq \frac{2}{(p-i)!}|x-a_l|^{p-i}|x-y|^{i-q} + \frac{2}{(p-i)!}|x-y|^{p-i}|x-y|^{i-q}; \ (2.6)$$

- i. If $i < p$, then first term of (2.6) is "0"

$$\Rightarrow b_{i,q} \leq \frac{2}{(p-i)!}|x-y|^{p-q} \leq 2|x-y|^{p-q} \leq 2|a_{N_2} - a_{N_2+1}|^{p-q} \leq \frac{t}{2} \quad \text{by} \quad (2.3).$$

- ii. If $i = p$,

then $2|x-y|^{p-q} + 2|x-y|^{p-q} \leq 4|a_{N_2} - a_{N_2+1}|^{p-q} \leq \frac{t}{2}$ by (2.3).

b.3 $x \geq a_{N_2}$ and $y < a_{N_2}$. Let $x := a_l$

$$\text{then} \qquad \tilde{f}(y) = 0 \quad \text{and} \quad \tilde{f}(x) = \alpha_l \frac{(x-a_l)^p}{p!}$$

$$\Rightarrow \quad |(R_y^q \tilde{f})^{(i)}(x)| = |\alpha_l \frac{|x-a_l|^{(p-i)}}{(p-i)!}|$$

$$\Rightarrow \quad b_{i,q} \leq |\alpha_l \frac{|x-a_l|^{p-i}}{(p-i)!} |x-y|^{i-q} \leq 2|x-y|^{p-q}$$

$$\leq 2|a_{N_2} - a_{N_2}|^{p-q} \leq \frac{t}{2}.$$

b.4 $x < a_{N_2}$ and $y < a_{N_2}$

$$\text{then} \qquad \tilde{f}^{(i)}(x) = \tilde{f}^{(i)}(y) = 0$$

$$\Rightarrow \quad |(R_y^q \tilde{f})^{(i)}(x)| = |\tilde{f}^{(i)}(x) - \sum_{k=i}^{q} f^{(k)}(y) \frac{(x-y)^{k-i}}{(k-i)!}| = 0$$

$$\Rightarrow b_{i,q} = 0 < \frac{t}{2}.$$

Thus $\|\bar{f}\|_q \leq t$, which implies $f \in tU_q/Z_p$.

$$\Rightarrow \beta(t, U_p, U_q) \geq N_2 \qquad \square$$

2.1.1 Geometric Criterion.

Here we give geometric condition of being isomorphic for the spaces $X := \mathcal{E}(K_a)$ and $Y := \mathcal{E}(K_b)$, K_a and K_b are of regular type, in terms of the elements of sequences and by means of Proposition 1.3, where

$$K_a = \{0\} \cup \cup_{n=1}^{\infty}\{a_n\} \text{ and } K_b = \{0\} \cup \cup_{n=1}^{\infty}\{b_n\}$$

and define $f(n) := a_n$ and $g(n) := b_n$. Suppose without loss of generality that the functions f, g are monotonic and g is differentiable.

Proposition 2.1 *If $X \simeq Y$ then;*

$$\forall p_1 \quad \exists p \quad \forall q \quad \exists q_1, \quad \exists C \textit{such that}$$

$$(|g'|^{-1}((\frac{8}{t})^{\frac{1}{q_1-p_1}}) - 1) \leq (q+1)(f^{-1}((\frac{1}{2et})^{\frac{1}{q-pQ_a}}) + 1) \tag{2.7}$$

where Q_a is the constant from Definition 2.1.

Proof: We will first estimate counting function corresponding to an ordinary space $\mathcal{E}(K_d)$ in terms of the general term of the sequence d_n and use the Proposition 1.3 to get above result.

Let us given $Z = \mathcal{E}(K_d)$ such that $K_d := \{0\} \cup \cup_{n=1}^{\infty}\{d_n\}$ which is of regular type. Then,

$$N_2 \leq \beta(t, U_p, U_q) \leq (q+1).N_1, \quad \text{such that} \quad N_1 \text{ and } N_2$$

are defined as in Theorem 2.1 .

Let h be monotone function with $h(n) := d_n$, then N_1 can also be given as:

$$N_1 = \max\{n : d_{n-1} > (\frac{1}{2et})^{\frac{1}{q-pQ_d}}\}$$

$$= \max\{n : h(n-1) > (\frac{1}{2et})^{\frac{1}{q-pQ_d}}\} = \max\{n : (n-1) < h^{-1}((\frac{1}{2et})^{\frac{1}{q-pQ_d}})\}$$

$$\Rightarrow N_1 < h^{-1}((\frac{1}{2et})^{\frac{1}{q-pQ_d}}) + 1$$

Thus

$$\beta_d(t, U_p, U_q) < (h^{-1}((\frac{1}{2et})^{\frac{1}{q-pQ_d}}) + 1)q$$

Now we find the lower bound for $\beta_d(t, U_p, U_q)$ in terms of the function $h(n)$.

N_2 can be given as :

$$N_2 + 1 = \min\{n : |a_n - a_{n+1}| < (\frac{8}{t})^{\frac{1}{q-p}}\}$$

that is

$$N_2 + 1 = \min\{n : |h(n) - h(n+1)| < (\frac{8}{t})^{\frac{1}{q-p}}\}.$$

On the other hand by the mean value theorem we have

$\exists x \in (n, n+1)$ such that $|h'(x)| = |h(n) - h(n+1)|$ and we obtain

$$N_2 + 1 = \min\{[x] : |h'(x)| < (\frac{8}{t})^{\frac{1}{q-p}}\}$$

where $[x]$ is the greatest integer at x.

$$\Rightarrow N_2 + 1 = \min\{int(x) : x > |h'|^{-1}(\frac{8}{t})^{\frac{1}{q-p}}\}$$

then,

$$N_2 > |h'|^{-1}(\frac{8}{t})^{\frac{1}{q-p}} - 1$$

that is

$$\beta_d(t, U_p, U_q) > |h'|^{-1}(\frac{8}{t})^{\frac{1}{q-p}} - 1.$$

Thus for the space $\mathcal{E}(K_d)$ we get,

$$(|h'|^{-1}(\frac{8}{t})^{\frac{1}{q-p}} - 1) < \beta_d(t, U_p, U_q) < (q+1)(f^{-1}((\frac{1}{2et})^{\frac{1}{q-pQ_d}}) + 1).$$

Now we can combine this inequality with the Proposition 1.3 to obtain the criterion in terms of the general terms of the sequences.

If $X \simeq Y$ then

$$\forall p_1 \quad \exists p \quad \forall q \quad \exists q_1, \quad \exists C \text{such that}$$

$$(|g'|^{-1}(\frac{8}{t})^{\frac{1}{q_1-p_1}} - 1) < (q+1)(f^{-1}((\frac{1}{2et})^{\frac{1}{q-pQ_a}}) + 1)$$

and vice-versa. $\square$

2.1.2 Example of a Continuum of Pairwise Non-Isomorphic Spaces

Here we consider the spaces $X_\alpha := \mathcal{E}(K_\alpha)$, $\alpha > 1$. Such that,

$$K_\alpha = \{0\} \cup \cup_{n=1}^{\infty} \{a_n^\alpha\} \quad , \qquad a_n^\alpha = \exp(-\ln^\alpha n)$$

and define

$$f_\alpha(n) := a_n^\alpha.$$

First let us show show that K_α is of regular type.

$$|a_{n-1}^\alpha - a_n^\alpha| = |\frac{1}{\exp(\ln^\alpha(n-1))} - \frac{1}{\exp(\ln^\alpha n)}| = |\frac{\exp(\ln^\alpha n) - \exp(\ln^\alpha(n-1))}{\exp(\ln^\alpha(n-1))\exp(\ln^\alpha n)}|$$

But if we choose $Q = 2$

$$|\frac{\exp(\ln^\alpha n) - \exp(\ln^\alpha(n-1))}{\exp(\ln^\alpha(n-1))}| \geq (\frac{1}{\exp^{\ln^\alpha n}})^2,$$

since $|\exp(\ln^\alpha n) - \exp(\ln^\alpha(n-1))| \geq 1$ for large enough n, the inequality (2.1) is realized .

Now let us find the upper and the lower bounds of $\beta_\alpha(t)$, $\forall \alpha > 1$. Such that $\beta_\alpha(t)$ is the counting function corresponding to the space X_α.

That is, according to the previous proposition we need to estimate f_α^{-1} and $|f'_\alpha|^{-1}$ from below and from above respectively for arbitrary $\alpha > 1$.

$$f_\alpha(n) = \exp(-\ln^\alpha n) = m$$

$$\Rightarrow \ln^\alpha(n) = \ln(\frac{1}{m}) \Rightarrow \ln n = \ln^{\frac{1}{\alpha}}(\frac{1}{m})$$

$$\Rightarrow n = \exp(\ln^{\frac{1}{\alpha}}(\frac{1}{m})), \text{ then } \qquad f^{-1}(m) = \exp(\ln^{\frac{1}{\alpha}}(\frac{1}{m})).$$

On the other hand,

$$f'_\alpha(n) = -\alpha\exp(-\ln^\alpha n)\ln^{\alpha-1} n\frac{1}{n} \qquad n = 1, 2, ...$$

$$\Rightarrow |f'_\alpha(n)| = \alpha\exp(-\ln^\alpha n)\ln^{\alpha-1} n\frac{1}{n} > \alpha\frac{\exp(-\ln^\alpha n)}{\exp\ln^\alpha n}$$

$$= \alpha\exp(-2\ln^\alpha n). \text{ Define } k(n) := \alpha\exp(-2\ln^\alpha n)$$

$$\Rightarrow k^{-1}(n) - 1 < |f'_\alpha|^{-1} - 1 < \beta_\alpha$$

then

$$k^{-1}(n) = \exp 2^{(-\frac{1}{\alpha})}\ln^{\frac{1}{\alpha}}(\frac{\alpha}{n})$$

Thus for any α we obtain,

$$\exp 2^{(-\frac{1}{\alpha})}\ln^{\frac{1}{\alpha}}(\alpha(\frac{t}{8})^{\frac{1}{q-p}}) - 1 < \beta_\alpha(t, U_p, U_q) < (q+1)(\exp(\ln^{\frac{1}{\alpha}}((2et)^{\frac{1}{q-2p}})) - 1).$$

Now we apply criterion to the spaces X_α and X_γ, $\forall\alpha, \gamma > 1, \alpha \neq \gamma$ are fixed constants;

$$if \qquad X_\alpha \simeq X_\gamma \qquad \text{then,}$$

$$\forall p_1 \quad \exists p \quad \forall q \quad \exists q_1, \quad \exists C\text{such that}$$

$$\exp 2^{(-\frac{1}{\gamma})}\ln^{\frac{1}{\gamma}}(\gamma(\frac{t}{8})^{\frac{1}{q_1-p_1}}) - 1 < (q+1)(\exp(\ln^{\frac{1}{\alpha}}((2et)^{\frac{1}{q-2p}})) - 1).$$

But for $p_1 = 0$, $q = 1 + 2p$, $\alpha > \gamma > 1$ and for large t, this inequality is impossible .

2.2 Compound Invariant Over $\mathcal{E}(K)$

In this part we will consider the invariant effect of $\beta_1(t,\tau)$ which is known to be more refined invariant then counting function $\beta(t)$.

In what follows we will focus on the question that whether $\beta_1(t,\tau)$ is strictly more refined then the invariant $\beta(t)$ for spaces $\mathcal{E}(K)$ where the set K is of regular type.

THEOREM 2.2 *Let $K = \{0\} \cup \cup_{n=1}^{\infty}\{a_n\}$ be of the regular type. Then the invariant $\beta_1(t,\tau)$ is not strictly more refined than the invariant $\beta(t)$ for the spaces $\mathcal{E}(K)$.*

Proof: Let us show that $\beta_1(t,\tau)$ and $\beta(t)$ have the same upper and the lower bounds asymptotically.

For upper bound we remark the following. Consider

$$\tilde{\beta}(U,V) = \min\{\dim L : U \subset V + L\}$$

It is clear that if $U_1 \subset U_2$ and $V_1 \supset V_2$, then $\tilde{\beta}(U_1,V_1) \leq \tilde{\beta}(U_2,V_2)$.

It turns out that,

$$\beta_1(\tau,t,U_p,U_q,U_r) = \tilde{\beta}(\tau U_p \cap tU_r, U_q)$$

$$\beta(t,U_r,U_q) = \tilde{\beta}(tU_r,U_q)$$

then

$$\beta_1(\tau,t,U_p,U_q,U_r) \leq \beta(t,U_r,U_q)$$

and by Theorem 2.1 we have

$$\beta(t, U_r, U_q) < (r+1)N_1 \quad \text{with} \quad N_1 = \min\{n : a_n \leq (\frac{1}{2et})^{\frac{1}{r-qQ}}\}$$

thus

$$\beta_1(\tau, t, U_p, U_q, U_r) < (r+1)N_1.$$

Lower bound for $\beta_1(t,\tau)$

We use theorem (1.2) for lower bound which implies,

$$\beta_1(\tau, t, U_p, U_q, U_r) \geq \sup\{\dim L : 2U_q/Z_q \cap L \subset (\tau U_p \cap tU_r)/Z_q\}$$

where supremum is taken over all finite dimensional subspaces of $\mathcal{E}^q(K)$. We define

$$L = span\{[h_{nq}]_q\}_{n=1}^{N_2}$$

where

$$N_2 = \max\{n : |a_n - a_{n+1}| \geq (\frac{8}{t})^{\frac{1}{r-q}}\} \tag{2.8}$$

then $dimL = N_2$. Now let us show with the above choice of subspace L the following embedding is satisfied:

$$2U_q/Z_q \cap L \subset (\tau U_p \cap tU_r)/Z_q.$$

Let $f \in 2U_q/Z_q \cap L$ be arbitrary. Then

$$f = \sum_{k=1}^{N_2} \alpha_k [h_{kq}]_q = \sum_{k=1}^{N_2} \alpha_k h_{kq} + Z_q = \tilde{f} + Z_q$$

where

$$\tilde{f} := \sum_{k=1}^{N_2} \alpha_k h_{kq},$$

that is

$$f(x) = \begin{cases} \tilde{f}(x) + Z_q = \alpha_k \frac{(x-a_k)^q}{q!} + Z_q & \text{if } x = a_k \geq a_{N_2} \\ 0 & \text{otherwise} \end{cases}$$

Since $f \in 2U_q/Z_q$

$$2 \geq \|\tilde{f}\|_q \geq |\alpha_k| \Rightarrow |\alpha_k| \leq 2.$$

Now let's show

$$f \in (\tau U_p \cap tU_r)/Z_q. \tag{2.9}$$

On the other hand, it is clear that to show $\tilde{f} \in (\tau U_p \cap tU_r)$ is sufficient for (2.9). That is to show

$$\|\tilde{f}\|_p \leq \tau \text{ and } \|\tilde{f}\|_r \leq t.$$

Bound for $\|\tilde{f}\|_p \leq$. Here $b_{i,p} = \frac{|(R_y^p\tilde{f})^{(i)}(x)|}{|x-y|^{p-i}} \quad i \leq p$.

a *Upper bound for* $|\tilde{f}^{(i)}(x)|$, $\quad i \leq p$.

a.1 $\quad x < a_{N_2}$. $\quad$ Then $\quad \tilde{f}^{(i)}(x) = 0 < \frac{\tau}{2}$.

a.2 $\quad x \geq a_{N_2} \quad x := a_l$.

$$\text{Then} \quad \tilde{f}(x) = \alpha_l \frac{(x-a_l)^q}{q!}$$

$$\Rightarrow |\tilde{f}^{(i)}(x)| = |\alpha_l \frac{(x-a_l)^{q-i}}{(q-i)!}| = 0 < \frac{\tau}{2} \quad \text{as } i \leq p.$$

b *Upper bound for* $b_{i,p}$

b.1 $\quad x < a_{N_2}$ and $y < a_{N_2}$.

$$\Rightarrow (R_y^p\tilde{f})^{(i)}(x) = 0 \Rightarrow b_{i,p} = 0 < \frac{\tau}{2}$$

b.2 $x < a_{N_2}$ and $y \geq a_{N_2}$. Let $x := a_l$ and $y := a_s$.

$$|(R_y^p \tilde{f})(x)| = |\tilde{f}(x) - \sum_{k=0}^{p} \tilde{f}^{(k)}(y) \frac{(x-y)^k}{k!}|$$

here $\tilde{f}^{(i)}(x) = 0 \quad \forall i$, since $x < a_{N_2}$.

$$\Rightarrow |(R_y^p \tilde{f})^{(i)}(x)| \leq |\sum_{k=i}^{p} \alpha_s \frac{(y-a_s)^{q-k}}{(q-k)!} \frac{(x-y)^{(k-i)}}{(k-i)!}| = 0$$

since $(y - a_s)^{q-k} = 0$ as $k \leq p < q$,

$$\Rightarrow \qquad b_{ip} < \frac{\tau}{2}.$$

b.3 $x \geq a_{N_2}$ and $y \geq a_{N_2}$. Let $x := a_l$ and $y := a_s$.

$$\text{Then} \qquad \tilde{f}(x) = |\alpha_l \frac{(x-a_l)^q}{q!}| \text{ and } \tilde{f}(y) = |\alpha_s \frac{(x-a_s)^q}{q!}|$$

$$\Rightarrow \quad |(R_y^p \tilde{f})^{(i)}(x)| \leq |\alpha_l \frac{(x-a_l)^{q-i}}{(q-i)!} - \sum_{k=0}^{p} \alpha_s \frac{(x-a_s)^{q-k}}{(q-k)!} \frac{(x-y)^{k-i}}{(k-i)!}| = 0$$

$$\text{since}(x-a_l)^{q-i} = (y-a_s)^{q-k} = 0 \qquad \text{as} \quad i,k \leq p < q \quad \forall i,k \leq p.$$

$$\Rightarrow \qquad b_{ip} < \frac{\tau}{2}$$

b.4 $x \geq a_{N_2}$ and $y < a_{N_2}$Let $x := a_l$ and $y := a_s$.

$$\Rightarrow \qquad \tilde{f}(y) = 0 \text{ as } y < a_{N_2} \quad \text{and} \quad \tilde{f}(x) = \alpha_l \frac{(x-a_l)^q}{q!}$$

Since $\tilde{f}(y) = 0$ for $y < a_{N_2}$,

$$\Rightarrow |(R_y^p \tilde{f})^{(i)}(x)| = |\alpha_l \frac{(x-a_l)^{q-i}}{(q-i)!}| = 0 \quad \text{as} \quad i \leq p < q,$$

$$\Rightarrow \qquad b_{i,p} < \frac{\tau}{2}$$

Bound for $\|f\|_r$. Here $b_{ir} = \frac{|(R_y^r\tilde{f})^{(i)}(x)|}{|(x-y)^{r-i}|}$, $i \leq r$.

a *Upper bound for* $|\tilde{f}^{(i)}(x)|$,$i \leq r$.

a.1 $x < a_{N_2}$ $\Rightarrow \tilde{f}^{(i)}(x) = 0 \quad \forall i \leq r \Rightarrow |\tilde{f}^{(i)}(x)| < \frac{t}{2}$.

a.2 $x \geq a_{N_2}$.Let $x := a_l$

$$\Rightarrow \quad |\tilde{f}^{(i)}(x)| = |\alpha_l \frac{(x-a_l)^{q-i}}{(q-i)!}| \neq 0 \text{ only for i=q },$$

$$\text{then for} \quad i = q, \quad |\tilde{f}^{(i)}(x)| \leq |\alpha_l| \leq 2 \leq \frac{t}{2}.$$

b *Upper bound for* b_{ir}, $i \leq r$.

b.1 $x < a_{N_2}$ and $y < a_{N_2}$.

$$\Rightarrow (R_y^r\tilde{f})^{(i)}(x) = 0 \quad \forall i, \text{ then } b_{ir} = 0 < \frac{t}{2}.$$

b.2 $x < a_{N_2}$ and $y \geq a_{N_2}$. Let $x := a_l$ and$y := a_s$.

$$\Rightarrow \qquad \tilde{f}^{(i)}(x) = 0 \quad \forall i \text{ and } \tilde{f}^{(i)}(y) = \alpha_s \frac{(y-a_s)^q}{q!}$$

$$\Rightarrow b_{i,r} \leq |\sum_{k=i}^{r} \alpha_s \frac{(y-a_s)^{q-k}}{(q-k)!}\frac{(x-y)^{k-i}}{(k-i)!}||x-y|^{i-r}$$

$$\text{here} \quad (y-a_s)^{q-k} \neq 0 \quad \text{only for} \quad k = q$$

$$\Rightarrow b_{i,r} \leq |\alpha_s||x-y|^{q-r} \leq 2|x-y|^{q-r} < |a_{N_2} - a_{N_2+1}|^{q-r} \leq \frac{t}{2} \quad \text{by (2.8).}$$

b.3 $x \geq a_{N_2}$ and $y \geq a_{N_2}$. Let $x := a_l$ and $y := a_s$.

$$\Rightarrow \quad |(R_y^r\tilde{f})^{(i)}(x)| = |\alpha_l \frac{(x-a_l)^{q-i}}{(q-i)!} - \sum_{k=i}^{r} \alpha_s \frac{(y-a_s)^{q-k}}{(q-k)!}\frac{(x-y)^{k-i}}{(k-i)!}|$$

similar to previous case

$$\leq |\alpha_l \frac{(x-a_l)^{q-i}}{(q-i)!} - \alpha_s \frac{(x-y)^{q-i}}{(q-i)!}|$$

$$\leq 2|x-a_l|^{q-i} + 2|x-y|^{q-i}$$

$$\Rightarrow b_{i,r} \leq 2|x-a_l|^{q-i}|x-y|^{i-r} + 2|x-y|^{q-i}|x-y|^{i-r}$$

$$= 2|x-a_l|^{q-i}|x-y|^{i-r} + 2|x-y|^{q-r}$$

here the first term ($\neq 0$) only for $i = q$

$$\Rightarrow \quad b_{ir} \leq 4|x-y|^{q-r} \leq 4|a_{N_2} - a_{N_2+1}|^{q-r} \leq \frac{t}{2} \quad \text{by (2.8).}$$

b.4 $\quad x \geq a_{N_2}$ and $y < a_{N_2}$.Let $x := a_l$ and $y := a_s$.

$$\Rightarrow \tilde{f}^{(i)}(y) = 0 \quad \forall i.$$

$$\Rightarrow \quad |(R^r_y \tilde{f})^{(i)}(x)| = |\tilde{f}^{(i)}(x)| \leq 2\frac{|x-a_l|^{q-i}}{(q-i)!} \neq 0 \text{ only for } i{=}q,$$

$$\Rightarrow \quad b_{ir} \leq \frac{2}{|x-y|^{r-i}} \leq 2|a_{N_2} - a_{N_2+1}|^{q-r} \leq \frac{t}{2}.$$

Thus

$$N_2 < \beta_1(\tau, t, U_p, U_q, U_r) < (r+1)N_1.$$

That is $\beta(t, U_q, U_r)$ and $\beta_1(\tau, t, U_p, U_q, U_r)$ have the same upper and the lower bounds.

Chapter 3

Irregular Case

For irregular case we restrict ourselves to the case

$K_a = \{0\} \cup \cup_{n=1}^{\infty} \{a_n\}$ such that $\forall Q, \quad \exists n_0 : \quad |a_n - a_{n+1}| \leq a_n^Q \quad \forall n \geq n_0.$

And we see with the following theorem that the spaces $\mathcal{E}(K)$, where K is of irregular type, are not distinguishable by means of the function $\beta(t)$.

THEOREM 3.1 *Given the space $X = \mathcal{E}(K_a)$, K_a of irregular type, we have*

$$\beta_X(t, U_p, U_q) \sim \beta_s(t, V_p, V_q)$$

Where s is the space of rapidly decreasing sequences and $(U_n)_{n=1}^{\infty}, (V_k)_{k=1}^{\infty}$ are the bases of neighborhoods of the spaces X and s respectively.

Proof. We know what β_s is and we will just show here that it is the same as β_X asymptotically.

For the space s we have $\beta_s(t, V_p, V_q) \sim t^{\frac{1}{q-p}}$ and β_s is maximal among all nuclear Fréchet spaces (see [16]).

Thus we naturally obtain the upper bound for β_X.That is

$$\beta_X(t, U_p, U_q) < t^{\frac{1}{q-p}}.$$

On the other hand, lower bound is done exactly the same way with regular case (see Theorem 2.1), since estimating lower bound has nothing to do with regularity of the set K_a. Thus

$$\beta_X(t, U_p, U_q) > N_2 \quad \text{where } N_2 \text{ is defined as in (2.3).}$$

And using (2.7) we obtain

$$(|f'|^{-1}((\frac{8}{t})^{\frac{1}{q_1-p_1}}) - 1) < \beta_X(t, U_p, U_q) < t^{\frac{1}{q-p}},$$

where $f(n) := a_n$.

To obtain asymptotic equivalence we use the irregularity of the set K_a. That is

$$\forall \quad Q > 1 \quad f^Q(n) > |f(n) - f(n+1)| \quad \text{for } n \geq n_0$$

and it follows from here that

$$f(n) > \frac{1}{n^\epsilon} \quad \forall \quad \epsilon > 0,$$

it is because $\forall\epsilon \quad \exists Q = Q(\epsilon)$ such that $|\frac{1}{n^\epsilon} - \frac{1}{(n+1)^\epsilon}| > (\frac{1}{n^\epsilon})^Q$.

$$\Rightarrow \qquad |f'|(n) > (|\frac{1}{n^\epsilon}|)' = \frac{\epsilon}{n^{\epsilon+1}} \quad \forall\epsilon \quad \Rightarrow \qquad |f'|^{-1}(m) > (\frac{\epsilon+1}{m})^{\frac{1}{\epsilon+1}} \quad \forall\epsilon$$

$$\Rightarrow \qquad \beta_X(t, U_p, U_q) > |f'|^{-1}((\frac{8}{t})^{\frac{1}{q-p}}) > ((\epsilon+1)(\frac{8}{t})^{\frac{1}{q-p}\frac{1}{\epsilon+1}})$$

so we get as a result

$$((\epsilon+1)(\frac{8}{t})^{\frac{1}{q-p}\frac{1}{\epsilon+1}}) < \beta_X(t, U_p, U_q) < t^{\frac{1}{q-p}} \qquad \forall \epsilon$$

and the desired result follows from this inequality $\square$.

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