LIMITING GIBBS MEASURES OF SOME MODELS OF CLASSICAL STATISTICAL MECHANICS

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By

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

LIMITING GIBBS MEASURES OF SOME MODELS OF CLASSICAL STATISTICAL MECHANICS

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We consider some models of classical statistical mechanics with their random perturbations and investigate the phase diagrams of this models. By using uniqueness theorem we prove the absence of phase transitions in this models.

Keywords: Ground State, Gibbs State, Limiting Gibbs State, Phase Transitions, Hamiltonian.

ÖZET

KLASİK İSTATİSTİKSEL MEKANİĞİN BAZI MODELLERİNDE LİMİT GIBBS ÖLÇÜMLERİ

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Düzensiz bozulmalar ile klasik istatistiksel mekanik modelleri ve bu modellerin faz diyagramlarını araştırıyoruz. Teklik teoremini kullanarak bu modeller arasındaki faz geçişliğinin yokluğunu ispatlıyoruz.

Anahtar kelimeler: Gibbs durumu, Yer durumu, Limit Gibbs durumu, Faz geçişleri, Hamiltonian.

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Contents

Chapter 1

Introduction

The theory of Gibbs Measures is a part of Probability and Measure theory developed with the goal of understanding the cooperative effects in large random systems.This theory is also a rapidly growing branch of Classical Statistical Physics.During the three decades since 1968, this notion has received considerable interest from both mathematical physicists and probabilists. The range of applications also includes various other fields such as biology, medicine chemistry, and economics, but we only concern with the concepts and results which are significant for physics. In probabilistic terms, a Gibbs measure is the distribution of a countably infinite family of random variables which admit some prescribed conditional probabilities.

The notion of a Gibbs measure began in the 1968-1970 with the work of R.L.Dobrushin, O.E.Lanford, and D.Ruelle who introduced the basic concept of a Gibbs measure. This concept combines two elements, (1) the well-known Maxwell-Boltzmann-Gibbs formula for the equilibrium distribution of a physical system with a given energy function, and (2) the familiar probabilistic idea of specifying the interdependence structure of random variables by means of a suitable class of conditional probabilities. An interesting feature of this concept is the fact that a Gibbs measure for a given type of interaction may fail to be unique.This means that, in physical terms, a physical system with this interaction can take several distinct equilibria. This occurrence of non-uniqueness of a Gibbs measure can thus be interpreted as a phase transition.

Chapter 2

Preliminaries

2.1 Basic Notations of Gibbs Fields

In this section, we introduce some basic notations of Gibbs fields, and we consider a simple well-examined example of the so called **Ising Model**.

A set Ω, a σ-algebra Σ of subsets of Ω, and a probability measure μ defined on Σ forms a triple (Ω, Σ, μ), which is called a **probability space.**

Set of configurations of a random field can be denoted by Ω. The σ-algebra generated by open sets in Ω, that is, if Ω is a topological space, Σ denotes its Borel σ-algebra $B(\Omega)$.

μ_0 denotes a free (nonperturbed) measure on Ω (usually independent or Gaussian).

$\Re$ denotes the lattice of all partitions of the set $\aleph = \{1, 2, ..., n\}$.

For any random variable,i.e, a measurable function ξ on a probability space (Ω, Σ, μ), its **mean** (**mathematical expectation**) is denoted by

$$\langle \xi \rangle = \langle \xi \rangle_\mu = \int_\Omega \xi d\mu.$$

$(A_1, ..., A_n)$-an ordered, and $\{A_1, ..., A_n\}$-an unordered collection of sets A_i, $i = 1, ..., n$ (similarly for collections of points).

A partition $\alpha = \{T_1, ..., T_k\}$ of a set A is an unordered collection of nonempty mutually disjoint subsets $T_i \subset A$, $i = 1, ..., k$ whose union is A, $\bigcup_{i=1}^{k} T_i = A$.

U_Λ-a Hamiltonian (energy) in Λ. S-a space of values of a field (a space of "spins" or "charges").Ω_Λ-the space of configurations of a field in Λ ($\Lambda \subset T$ or $\Lambda \subset Q$).

Ising Model :

We consider the lattice Z^ν of points $t = (t^{(1)}, ..., t^{(\nu)}) \in R^\nu$ of the ν-dimensional real space with integer coordinates. Let $\Lambda_N \equiv \Lambda$ be a "cube" in Z^ν centered at the origin, i.e., the set of points in Z^ν whose coordinates have absolute values not greater than N (with an integer $N > 0$). Thus each function $\sigma^\Lambda = \{\sigma_t, t \in \Lambda\}$, defined on the set Λ and taking values $\sigma_t = \pm 1$, is called a **configuration** (in the cube Λ), and the set of all such configurations is denoted by Ω_Λ. The number of configurations in Λ is $2^{|\Lambda|}$, where $|\Lambda|$ is the number of lattice sites in Λ.

Let us consider a function U_Λ on Ω_Λ such that

$$U_\Lambda \equiv U_\Lambda(\sigma^\Lambda) = -(h \sum_{t \in \Lambda} \sigma_t + \beta \sum_{\langle t,t' \rangle} \sigma_t \sigma_{t'}), \tag{2.1}$$

This function is called **energy (hamiltonian)** of the configuration on σ^Λ. The summation in the second part of equation (2.1) is taken over all unordered pairs $\langle t, t' \rangle, t, t' \in \Lambda$, such that $\rho(t, t') = 1$, with

$$\begin{aligned} \rho(t, t') &= \sum_{i=1}^{\nu} |t^{(i)} - t'^{(i)}|, \qquad (2.2) \\ t &= (t^{(1)}, ..., t^{(\nu)}), \quad and \quad t' = (t'^{(1)}, ..., t'^{(\nu)}). \end{aligned}$$

A physical system with the configuration space Ω_Λ of configurations in Λ and a configuration energy of the form (2.1) is usually called the **Ising model**.The real numbers h and β in (2.1) are fixed (parameters of the model).We refer to the case $\beta > 0$, which will be studied here, as the **ferromagnetic** Ising model.

Now, let us introduce a probability distribution on the space Ω_Λ defining the probability of a configuration σ^Λ by

$$P_\Lambda(\sigma^\Lambda) = Z_\Lambda^{-1} \exp\{-U_\Lambda(\sigma^\Lambda)\}. \tag{2.3}$$

The normalization factor Z_Λ is defined by the condition

$$\sum_{\sigma^\Lambda \in \Omega_\Lambda} P_\Lambda(\sigma^\Lambda) = 1,$$

and thus,

$$Z_\Lambda = \sum_{\sigma^\Lambda \in \Omega_\Lambda} \exp\{-U_\Lambda(\sigma^\Lambda)\}. \tag{2.4}$$

The quantity Z_Λ is called **partition function**, and the probability distribution (2.3) is called **Gibbs** probability distribution in Λ corresponding to the Ising model.

The values σ_t of these configurations may be considered as random variables and the formula (2.3) as the joint probability distribution of these random variables.We will denote the mean (value) of an arbitrary function f on the space Ω_Λ under the distribution (2.3) as $\langle f \rangle_\Lambda$.The means $\langle \sigma_t \rangle_\Lambda$ of random variables

$$\sigma_T = \prod_{t \in T} \sigma_t, \qquad \sigma_\emptyset = 1, \tag{2.5}$$

with $T \subset \Lambda$ being an arbitrary subset of Λ, are called **correlation functions** (or **moments**) of the distribution (2.3).

For any $T \subset \Lambda$, $P_\Lambda^{(T)}$ to denote the **joint distribution** of the system of random variables $\{\sigma_t, t \in T\}$, i.e., the collection of probabilities

$$P_\Lambda^{(T)}(\overline{\sigma}_{t_1}, ..., \overline{\sigma}_{t_n}) = Pr(\sigma_{t_1} = \overline{\sigma}_{t_1}, ..., \sigma_{t_n} = \overline{\sigma}_{t_n}), \tag{2.6}$$

with $T = \{t_1, ..., t_n\}$ and $\{\overline{\sigma}_{t_1}, ..., \overline{\sigma}_{t_n}\}$ being an arbitrary collection of values $\overline{\sigma}_{t_i} = \pm 1$, $i = 1, 2, ..., n$. The probabilities (2.6) may be expressed by means of correlation functions $\langle \sigma_T \rangle_\Lambda$.

$$\begin{aligned} P_\Lambda^{(T)}(\overline{\sigma}_{t_1}, ..., \overline{\sigma}_{t_n}) &= (1/2^n)(-1)^k \langle \prod_{i=1}^{n} (\sigma_{t_i} + \overline{\sigma}_{t_i}) \rangle_\Lambda \\ &= (-1)^k / 2^n \sum_{T' \subset T} C_{T'} \langle \sigma_{T'} \rangle_\Lambda, \end{aligned} \tag{2.7}$$

with k being the number of values $\overline{\sigma}_{t_i}$ that equal -1 and

$$C_{T'} = \prod_{t \in T \setminus T'} \overline{\sigma}_t.$$

Thermodynamic Limit :

We fix T and let Λ expand to Z^ν, $\Lambda \nearrow Z^\nu$, i.e., put $N \to \infty$. Now consider,

$$\lim_{\Lambda \nearrow Z^\nu} \langle \sigma_T \rangle_\Lambda \tag{2.8}$$

If we prove the existence of above limit, we may conclude that correlation functions (and finite dimensional distributions) almost do not depend on Λ for sufficiently large Λ in comparison with T. Such a passage to the limit is called the **thermodynamic** limit (the limit of a large number of degrees of freedom σ_t). The limits (2.8) are called **limit correlation functions** and are denoted by $\langle \sigma_t \rangle$. Finite dimensional distributions also have limits (by (2.7)), this limits form a compatible family of finite-dimensional distributions. By the Kolmogorov theorem ([16]), this family defines a system of random variables $\{\sigma_t, t \in Z^\nu\}$, called a (limit) **Gibbs random field** (for the Ising model), their distribution P (a measure) on the space $\Omega = \{-1,1\}^{Z^\nu}$ of infinite configurations in the lattice Z^ν. The existence of the limit distribution P follows from the following theorem.

Theorem 2.1 *The thermodynamic limit (2.8) of correlation functions $\langle \sigma_T \rangle_\Lambda$ exists for $\beta \geq 0$ and every finite T.*

Remark 2.2 *In the case of $\beta = 0$, $\langle \sigma_t \rangle_\Lambda$ can be easily calculated:*

$$\langle \sigma_t \rangle_\Lambda = (\frac{e^h - e^{-h}}{e^h + e^{-h}})^{|T|}. \tag{2.9}$$

Consequently, $\langle \sigma_t \rangle_\Lambda$ does not depend on Λ (for $T \subset \Lambda$). So the thermodynamic limit $\langle \sigma_t \rangle_\Lambda$ exists in this case and equals (2.9). The random variables σ_t are mutually independent, both with respect to the distributions in finite Λ and with respect to the limit distribution.

Proof of Theorem 2.1 It is sufficient to consider the case $h \geq 0$, because of the following property of the Ising model (in the notations introduced below, β and h as subscripts indicate the dependence of Gibbs distributions on these parameters):

$$P_{\Lambda,\beta,h}(\sigma^\Lambda) = P_{\Lambda,\beta,-h}(-\sigma^\Lambda) \tag{2.10}$$

with $-\sigma^\Lambda$ denoting the configuration whose values have an opposite sign to those of the configuration $-\sigma^\Lambda$.

By (2.10)

$$Q(x,y) = \begin{cases} \langle \sigma_t \rangle_{\Lambda,\beta,-h} & |T| \text{ } even, \\ -\langle \sigma_t \rangle_{\Lambda,\beta,-h} & |T| \text{ } odd. \end{cases} \tag{2.11}$$

In particular, for odd $|T|$

$$\langle \sigma_t \rangle_{\Lambda,\beta,0} \tag{2.12}$$

We need some inequalities to prove the theorem and we can consider a general situation. Let Λ be an arbitrary subset of Z^ν, Ω_Λ set of all configurations $\sigma^\Lambda = \{\sigma_t, t \in \Lambda\}$, $\sigma_t = \pm 1$, in Λ, and the energy $U_\Lambda(\sigma^\Lambda)$ of the configuration σ^Λ be of the form

$$U_\Lambda(\sigma^\Lambda) = -(\sum_{t\in\Lambda} h_t\sigma_t + \beta \sum_{\langle t,t'\rangle\in\Lambda} \beta_{t,t'}\sigma_t\sigma_{t'}), \tag{2.13}$$

where $h_t \geq 0$ and $\beta_{t,t'} \geq 0$. The distribution P_Λ on Ω_Λ is given in equation (2.3), and $\langle\ \rangle_\Lambda$ denotes the mean under this distribution.

Lemma 2.3 *The first Griffith inequality*

$$\langle\sigma_T\rangle_\Lambda \geq 0 \tag{2.14}$$

and the second Griffith inequality

$$\langle\sigma_T\sigma_{T'}\rangle_\Lambda - \langle\sigma_T\rangle_\Lambda\langle\sigma_{T'}\rangle_\Lambda \geq 0 \tag{2.15}$$

are valid.

Proof To prove (2.14), let us show that

$$\sum_{\sigma^\Lambda\in\Omega_\Lambda} \sigma_T\ \exp\{-U_\Lambda(\sigma^\Lambda)\} \geq 0. \tag{2.16}$$

Let us first expand the exponential function $\exp\{-U_\Lambda(\sigma^\Lambda)\}$ in the series $\sum_{n=0}^\infty (-U_\Lambda)^n/n!$, by removing the parentheses in each term of this series, and by taking into account that $\sigma_t^2 = 1$, then left side of the inequality (2.16) becomes

$$\sum_{B\subseteq\Lambda} C_B \sum_{\sigma^\Lambda\in\Omega_\Lambda} \sigma_B \tag{2.17}$$

with $C_B \geq 0$. Since for any $t \in \Lambda$

$$\sum_{\sigma_t=\pm1} \sigma_t = 0 \tag{2.18}$$

the sum (2.16) is equal to $C_\emptyset$, which proves (2.14).

Next we investigate two independent samples of the distribution P_Λ to prove (2.15),i.e., a distribution on the space $\Omega_\Lambda \times \Omega_\Lambda$ of pairs $\{\sigma^\Lambda, \widetilde{\sigma}^\Lambda\}$ of configurations of the form

$$\hat{P}_\Lambda(\sigma^\Lambda, \widetilde{\sigma}^\Lambda) = (Z_\Lambda^{-1})^2 \exp\{\sum_{t\in\Lambda} h_t(\sigma_t + \widetilde{\sigma}_t) + \sum_{t,t'\in\Lambda} \beta_{t,t'}(\sigma_t\sigma_{t'} + (\widetilde{\sigma}_t\widetilde{\sigma}_{t'}))\}. \quad (2.19)$$

Let us introduce new variables

$$\xi_t = \sigma_t + \widetilde{\sigma}_t, \qquad \eta_t = \sigma_t - \widetilde{\sigma}_t, \qquad t \in \Lambda,$$

and $(\xi_t, \eta_t) = (2,0), (-2,0), (0,2), (0,-2)$. Taking these variables, the probability (2.19) may be written in the following form

$$Z_\Lambda^{-2}\ exp\ \{\sum_{t\in\Lambda} h_t\ \xi_t +\ \frac{1}{2}\sum_{t,t'\in\Lambda} \beta_{t,t'}\ (\xi_t\xi_{t'} + \eta_t\eta_{t'})\}.$$

Taking

$$\xi_t\ \eta_t = 0 \qquad and \qquad \sum_{\xi_t=-2,0,2} \eta_t^k \geq 0$$

for each integer $k \geq 0$ and each $t \in \Lambda$, and by repeating the proof of (2.14), for all T and $T^{'}$, with ξ_T and η_T, we get

$$\langle \xi_t\ \eta_{t'}\rangle_{\Lambda,\Lambda} \geq 0 \quad (2.20)$$

as defined in (2.5) and the mean $\langle\ \rangle_{\Lambda,\Lambda}$ evaluated by distribution (2.19).

Note that

$$\langle\sigma_T\sigma_{T'}\rangle_\Lambda - \langle\sigma_T\rangle_\Lambda\langle\sigma_{T'}\rangle_\Lambda = \frac{1}{2}\langle(\sigma_T - \widetilde{\sigma}_T)(\sigma_{T'} - \widetilde{\sigma}_{T'})\rangle_{\Lambda,\Lambda}. \quad (2.21)$$

Let us show that

$$(\sigma_T \pm \widetilde{\sigma}_T) = \sum_{A,B\subseteq T} C^{\pm}_{A,B}\ \xi_A\ \eta_B \quad (2.22)$$

with $C^{\pm}_{A,B} \geq 0$. Then we get the inequality (2.16) by using above relations (2.20), (2.21), (2.22), and the above equation (2.22) can be proved by induction on $|T|$ if

$$\sigma_{T\cup\{t\}} + \widetilde{\sigma}_{T\cup\{t\}} = \frac{1}{2}[(\sigma_T + \widetilde{\sigma}_T)\xi_t + (\sigma_T - \widetilde{\sigma}_T)\eta_t],$$
$$\sigma_{T\cup\{t\}} - \widetilde{\sigma}_{T\cup\{t\}} = \frac{1}{2}[(\sigma_T + \widetilde{\sigma}_T)\eta_t + (\sigma_T - \widetilde{\sigma}_T)\xi_t]$$

for $t \notin T \subset \Lambda$. Lemma is proved.

Let us continue the proof of the theorem.

The derivatives are

$$\frac{\partial}{\partial h_t}\langle\sigma_T\rangle_\Lambda = \langle\sigma_T\sigma_t\rangle_\Lambda - \langle\sigma_T\rangle_\Lambda\langle\sigma_t\rangle_\Lambda \geq 0\ ,$$
$$\frac{\partial}{\partial \beta_{t,t'}}\langle\sigma_T\rangle_\Lambda = \langle\sigma_T\sigma_t\sigma_{t'}\rangle_\Lambda - \langle\sigma_T\rangle_\Lambda\langle\sigma_t\sigma_{t'}\rangle_\Lambda \geq 0\ , \tag{2.23}$$

and when increasing the parameters h_t and $\beta_{t,t'}$ the correlation functions increase. In the case of the Ising model, for $T \subset \Lambda_1 \subset \Lambda_2$

$$\langle\sigma_T\rangle_{\Lambda_1} \leq \langle\sigma_T\rangle_{\Lambda_2}. \tag{2.24}$$

By parameters

$$h_t = \begin{cases} h, & t \in \Lambda_1 \\ 0, & t \in \Lambda_2 \setminus \Lambda_1. \end{cases} \tag{2.25}$$

and

$$\beta_{t,t'} = \begin{cases} \beta & \text{if } t, t' \text{ are nearest neighbors in } \Lambda_1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.26}$$

the mean $\langle\sigma_T\rangle_{\Lambda_1}$ coincides with the mean under the distribution of the form (2.13) in Λ_2

We can get (2.24) by using the monotonicity of $\langle\sigma_T\rangle$ with respect to the parameters h_t and $\beta_{t,t'}$. Since $|\langle\sigma_T\rangle| \leq 1$, the statement of the theorem follows from (2.24).

Markov Property:

Let $A \subset Z^{\nu}$ be a set and

$$\partial A = \{t \in Z^{\nu} : \rho(t, A) = 1\}. \tag{2.27}$$

that is boundary ∂A is defined to be the set of all lattice sites of distance 1 from A.

Let $\Lambda \subset Z^{\nu}$ be a cube, and let $A, B \subseteq \Lambda$ be such that $A \cap B = \emptyset$ and $\partial A \subset B$. We use

$$P_{\Lambda}^{(A)}(\overline{\sigma}^{A} / \widetilde{\sigma}^{B}) = Pr\{\sigma_t = \overline{\sigma}_t,\ t \in A \ / \ \sigma_{t'} = \widetilde{\sigma}_{t'}, t' \in B\}$$

to denote the conditional probability that σ^{Λ} equals $\overline{\sigma}^{A} = \{\overline{\sigma}_t,\ t \in A\}$ on the set A under the condition that its values on the set B equal $\widetilde{\sigma}^{B} = \{\widetilde{\sigma}_{t'} : t' \in B\}$.

Lemma 2.4

$$\begin{aligned} P_{\Lambda}^{(A)}(\overline{\sigma}^{A} / \widetilde{\sigma}^{B}) &= P_{\Lambda}^{(A)}(\overline{\sigma}^{A} / \widetilde{\sigma}^{\partial A}) \\ &= Z_A^{-1}(\widetilde{\sigma}^{\partial A}) \ \exp\{-(U_A(\overline{\sigma}^{A}) + U_{A,\partial A}(\overline{\sigma}^{A}, \widetilde{\sigma}^{\partial A}))\}. \end{aligned} \tag{2.28}$$

Above equations hold true, where $U_A(\overline{\sigma}^{A})$ is the energy of the configuration $\overline{\sigma}^{A}$ defined as in (2.1), $U_{A,\partial A}(\overline{\sigma}^{A}, \widetilde{\sigma}^{\partial A})$ is the energy of the interaction between the configurations $\overline{\sigma}^{A}$ and $\widetilde{\sigma}^{\partial A}$:

$$U_{A,\partial A}(\overline{\sigma}^{A}, \widetilde{\sigma}^{\partial A}) = -\beta \sum_{\substack{t \in A,\ t' \in \partial A \\ \rho(t,t')=1}} \overline{\sigma}_t \ \widetilde{\sigma}_{t'}, \tag{2.29}$$

and $Z_A(\widetilde{\sigma}^{\partial A})$ is the conditional partition function

$$Z_A(\widetilde{\sigma}^{\partial A}) = \sum_{\overline{\sigma}^{A}} \exp\{-(U_A(\overline{\sigma}^{A}) + U_{A,\partial A}(\overline{\sigma}^{A}, \widetilde{\sigma}^{\partial A}))\}. \tag{2.30}$$

The first equality in (2.28) is called the **Markov property** of the distribution P_Λ, and the other equality expresses its **Gibbs property**: the conditional distribution $P_\Lambda^{(A)}$ is similar in form to the distribution (2.3), except that the energy $U_{A,\partial A}$ of the interaction with the "boundary" configuration $\widetilde{\sigma}^{\partial A}$ was added to the energy U_A. The distribution given by the formula on the right-hand side of (2.28) is called the **Gibbs distribution in A with the boundary configuration** $\widetilde{\sigma}^{\partial A}$.

Proof By the formula (2.3) we have

$$P_\Lambda^{(A)}(\overline{\sigma}^A/\widetilde{\sigma}^B) = \frac{P_\Lambda^{(A\cup B)}(\overline{\sigma}^A,\widetilde{\sigma}^B)}{P_\Lambda^{(B)}(\widetilde{\sigma}^B)} =$$

$$= \frac{\sum_{\sigma^{\Lambda\setminus\{A\cup B\}}} \exp\,\{-U_\Lambda(\overline{\sigma}^A,\widetilde{\sigma}^B,\sigma^{\Lambda\setminus(A\cup B)})\}}{\sum_{\sigma^{\Lambda\setminus(A\cup B)},\overline{\sigma}^A} \exp\,\{-U_\Lambda(\overline{\sigma}^A,\widetilde{\sigma}^B,\sigma^{\Lambda\setminus(A\cup B)})\}}, \tag{2.31}$$

with $(\overline{\sigma}^A,\widetilde{\sigma}^B,\sigma^{\Lambda\setminus(A\cup B)})$ and $\sigma^{\Lambda\setminus(A\cup B)}$ is a configuration in the set $\Lambda\setminus(A\cup B)$. Also with the energies $U_{A,B}$ and $U_{B,\Lambda\setminus(A\cup B)}$ (similar to (2.29))

$$\begin{aligned} U_\Lambda(\sigma^\Lambda) &= U_A(\overline{\sigma}^A)+U_{A,B}(\overline{\sigma}^\Lambda,\widetilde{\sigma}^B)+U_B(\widetilde{\sigma}^B)+U_{\Lambda\setminus(A\cup B)}(\sigma^{\Lambda\setminus(A\cup B)}) \\ &+ U_{B,\Lambda\setminus(A\cup B)}(\widetilde{\sigma}^B,\sigma^{\Lambda\setminus(A\cup B)}), \end{aligned}$$

Then, the denominator of the right-hand side of the equation (2.31) becomes

$$exp\,\{-U_B(\widetilde{\sigma}^B)\}\; Z_{\Lambda\setminus(A\cup B)}(\widetilde{\sigma}^B)\; Z_A(\widetilde{\sigma}^B),$$

with $Z_{\Lambda\setminus(A\cup B)}(\widetilde{\sigma}^B)$ and $Z_A(\widetilde{\sigma}^B)$ defined in (2.30) and the nominator of the right-hand side of the equation (2.31) becomes

$$exp\,\{-(U_A(\overline{\sigma}^A)+U_{A,B}(\overline{\sigma}^A,\widetilde{\sigma}^B)+U_B(\widetilde{\sigma}^B))\}\; Z_{\Lambda\setminus(A\cup B)}(\widetilde{\sigma}^B),$$

Noticing that $U_{A,B}(\overline{\sigma}^A,\widetilde{\sigma}^B) = U_{A,\partial A}(\overline{\sigma}^A,\widetilde{\sigma}_{\partial A})$ and $Z_A(\widetilde{\sigma}^B) = Z_A(\widetilde{\sigma}^{\partial A})$, inserting above expressions into (2.31), after some cancellations, we get (2.28). The lemma is proved.

Definition 2.5 *A probability distribution P on the space Ω is said to determine a* **Gibbs random field** $\{\sigma_t, t \in Z^\nu\}$ *(for the Ising model) if the conditional distribution $P^{(A)}(\overline{\sigma}^A/\widetilde{\sigma}^B)$, generated by the distribution P, coincides with the Gibbs distribution in A, with the boundary configuration $\widetilde{\sigma}^{\partial A}$ (see the second equality in (2.28)) for arbitrary finite subsets $A, B \subset Z^\nu$ such that $A \cap B = \emptyset$ and $\partial A \subset B$.*

Thus, the limit Gibbs distribution constructed above defines a Gibbs random field in Z^ν. Are there still other Gibbs fields in Z^ν for the Ising model? It turns out that this depends on the dimension of the lattice Z^ν and on the parameters (h, β). The values of parameters (h, β) for which there exists more than one Gibbs field in Z^ν define points of the first order phase transition in the plane (h, β).

Theorem 2.6 *For a ferromagnetic Ising model:*

1) *for $\nu = 1$, there is a unique Gibbs field;*

2) *for $\nu \geq 2$ and $h \neq 0$, or $h = 0$ and β sufficiently small, $0 \leq \beta \leq \beta_0(\nu)$, there is a unique Gibbs field;*

3) *for $\nu \geq 2$, the points $(0, \beta)$ with β sufficiently large, $\beta > \beta_1(\nu)$, are points of the first order phase transition.*

We shall only prove the statements 1) and 3) of this theorem. Let us first investigate the possible ways of construction of Gibbs fields in Z^ν for the Ising model. Let $\Lambda \subset Z^\nu$ be a cube, $\widetilde{\sigma}^{\partial\Lambda}$ be a configuration in the boundary $\partial\Lambda$ of the cube Λ, and let $P_{\Lambda,\widetilde{\sigma}^{\partial\Lambda}}(\sigma^\Lambda)$ denote the Gibbs distribution in Λ (on the space Ω_Λ) with the boundary configuration $\widetilde{\sigma}^{\partial\Lambda}$ (see (2.28)). Let $q^{\partial\Lambda}$ be an arbitrary probability distribution on the set $\Omega_{\partial\Lambda}$ of boundary configurations

$\widetilde{\sigma}^{\partial\Lambda}$. Let us use the $P_{\Lambda,q^{\partial\Lambda}}$ for the distribution

$$P_{\Lambda,q^{\partial\Lambda}}(\sigma^{\Lambda}) = \langle P_{\Lambda,\widetilde{\sigma}^{\partial\Lambda}}(\sigma^{\Lambda})\rangle_{q^{\partial\Lambda}}, \tag{2.32}$$

on the space Ω_{Λ}.This distribution is called **the Gibbs distribution in Λ with a random boundary configuration**. Also the Gibbs distribution P_{Λ}^{per} with the so-called periodic boundary conditions is often considered like $P_{\Lambda,\widetilde{\sigma}^{\partial\Lambda}}$ and $P_{\Lambda,q^{\partial\Lambda}}$. It is defined similarly to the distribution P_{Λ} (see 2.3) except for replacing the "cube" Λ by the "torus" and the energy U_{Λ} in (2.3) by the energy U_{Λ}^{per} of the interaction of the nearest neighbors on this torus. The Gibbs distribution (2.3) is often called the Gibbs distribution in Λ under the "empty boundary conditions". By the proof of lemma (2.4), we can see that the distributions

$$P_{\Lambda,\widetilde{\sigma}^{\partial\Lambda}},\ P_{\Lambda,q^{\partial\Lambda}},\ P_{\Lambda}^{per} \tag{2.33}$$

have the Gibbs property (2.28).

As in the case of Gibbs distributions with the empty boundary conditions, we conclude that the limit $P = \lim_{\Lambda_n\nearrow Z^{\nu}} P_{\Lambda_n}$ of the sequence P_{Λ_n} of distributions of the form (2.33), with Λ_n being an increasing sequence of cubes, $\Lambda_1 \subset \Lambda_2 \subset ... \subset \Lambda_n \subset ... \subset \cup\Lambda_n = Z^{\nu}$, defines a Gibbs field in Z^{ν}.

Lemma 2.7 *Every probability distribution P on the space Ω that is a Gibbs random field in Z^{ν} is the thermodynamic limit of a sequence $P_{\Lambda_n,q_n^{\partial\Lambda_n}}$ for some choice of $q_n^{\partial\Lambda_n}$.*

Proof We choose $q^{\partial\Lambda}$ to be the probability distribution on $\Omega_{\partial\Lambda}$ induced by the distribution P for every cube $\Lambda \subset Z^{\nu}$. $P_{\Lambda,\ q^{\partial\Lambda}}$ coincides in this case with the distribution induced by P on Ω_{Λ}. So $P_{\Lambda,\ q^{\partial\Lambda}} \to P$ (in the sense (2.8)) as $\Lambda \nearrow Z^{\nu}$.

Proof of theorem (2.6)

1) To simplify the formula, we take $h = 0$

Definition (transfer matrix) The 2×2 matrix $J = \|j_{\sigma\sigma'}\|$ with matrix elements $j_{\sigma\sigma'} = e^{\beta\sigma\sigma'}, \ \sigma\sigma' = \pm 1$,

$$J = \begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix} \tag{2.34}$$

is called the **transfer matrix** of the Ising model.

Under the empty boundary conditions P_Λ be the Gibbs distribution in Λ with $\Lambda = [-N, N] \subset Z^1$.

Lemma 2.8

$$P_\Lambda^{\{t_1,\ldots,t_n\}}(\overline{\sigma}_{t_1}, \ldots, \overline{\sigma}_{t_n}) =$$

$$= \frac{(e^{(\overline{\sigma}_{t_1})}, J^{N_1}e)(e^{(\overline{\sigma}_{t_2})}, J^{t_2-t_1}e^{(\overline{\sigma}_{t_1})})\ldots(e, J^{N_2}e^{(\overline{\sigma}_{t_4})})}{(e, J^{2N}e)} \tag{2.35}$$

$$Z_\Lambda = (J^{2N}e, e). \tag{2.36}$$

Above equalities hold with $e = (1,1), \ e^{(1)} = (1,0), \ e^{(-1)} = (0,1), \ N_1 = t_1 + N, \ N_2 = N - t_n, \ -N \le t_1 < t_2 < \ldots < t_n \le N.$

Let $g^{(1)}$ and $g^{(2)}$ be two normalized eigenvectors of the transfer matrix J, with eigenvalues λ_1 and $\lambda_2, \ \lambda_1 > |\lambda_2| \ge 0$. Using the decompositions

$$e = C_1 g^{(1)} + C_2 g^{(2)}, \qquad e^{(\pm 1)} = B_1^{(\pm 1)} g^{(1)} + B_2^{(\pm 1)} g^{(2)},$$

for large N and fixed $\{t_1, \ldots, t_n\}$ we get

$$\begin{aligned} (J^{2N}e, \ e) &\sim C_1^2 \lambda_1^{2N}, \\ (e^{(\overline{\sigma}_{t_1})}, J^{N_1}e) &\sim B_1^{(\overline{\sigma}_{t_1})} C_1 \lambda_1^{N_1}, \\ (e, J^{N_2}e^{(\overline{\sigma}_{t_n})}) &\sim B_1^{(\overline{\sigma}_{t_n})} C_1 \lambda_1^{N_2} \end{aligned}$$

thus

$$\lim_{N\to\infty} P_\Lambda^{\{\overline{\sigma}_{t_1},\dots,\overline{\sigma}_{t_n}\}} = B_1^{(\overline{\sigma}_{t_1})} B_1^{(\overline{\sigma}_{t_n})} \prod_{k=2}^{n} \frac{(e^{(\overline{\sigma}_{t_k})}, J^{t_k-t_{k-1}} e^{(\overline{\sigma}_{t_{k-1}})})}{\lambda_1^{t_k-t_{k-1}}}.$$

Similarly, it can be shown that for any sequence of Gibbs distributions $P_{\Lambda_n, q_n^{\partial\Lambda_n}}, \quad \Lambda_n \nearrow Z^1$ the probabilities $P^{t_1,\dots,t_n}_{\Lambda_n,\ q_n^{\partial\Lambda_n}}$ have the same limit. So the first part is proved.

Remark 2.9 *By our considerations, we may derive that the limit Gibbs field $\{\sigma_t, t\in Z^1\}$ is a stationary Markov chain with the matrix of transition probabilities*

$$P_{\sigma_1,\sigma_2} = \frac{J_{\sigma_1\sigma_2 g^{(1)}_{\sigma_2}}}{\lambda_1 g^{(1)}_{\sigma_1}} \qquad \sigma_1,\sigma_2=\pm 1,$$

and the stationary distribution $\pi_\sigma = (g_\sigma^{(1)})^2,\ \sigma=\pm 1,$ where $g_1^{(1)},\ g_{-1}^{(1)}$ are the components of the eigenvector $g^{(1)}$.

Let us continue the proof of theorem (2.6)

3) We denote the Gibbs distribution in Λ with the boundary configuration $\widetilde{\sigma}_t \equiv +1,\ t\in\partial\Lambda$ ((+)-boundary conditions) by $P_{\Lambda,(+)}$.

Lemma 2.10

$$Pr_{\Lambda,(+)}(\sigma_0=-1) < 1/3 \tag{2.37}$$

the above inequality holds uniformly with respect to all cubes $\Lambda\subset Z^\nu,\ 0\in\Lambda,$ for all sufficiently large $\beta,\ \beta>\beta_1(\nu)$.

Consider the Gibbs distribution $P_{\Lambda,(-)}$ with the boundary configuration $\widetilde{\sigma}_t\equiv -1,\ t\in\partial\Lambda$ ((−)-boundary conditions). For $h=0$, by symmetry we get

$$P_{\Lambda,(+)}(\sigma^\Lambda) \equiv P_{\Lambda,(-)}(-\sigma^\Lambda),$$

then for every Λ

$$Pr_{\Lambda,(-)}(\sigma_0=+1) < 1/3$$

and

$$Pr_{\Lambda,(-)}(\sigma_0 = -1) < 2/3. \tag{2.38}$$

By (2.37) and (2.38) we can see that there are at least two different Gibbs distributions in Z^ν.

Proof To simplify we take the case $\nu = 2$. By shifting the lattice Z^2 by the vector $(1/2,\ 1/2)$, we obtain the dual lattice $\widetilde{Z}^2$. We use $\gamma = \gamma(\sigma^\Lambda)$ for any configuration σ^Λ to denote the collection of those relation of $\widetilde{Z}^2$ that separate two neighboring sites $t, t' \in \Lambda \cup \partial\Lambda$ with $\sigma_t \neq \sigma_{t'}$, ($\sigma_t = 1$ for $t \in \partial\Lambda$). The number of bonds from $\gamma(\sigma^\Lambda)$ attached to a lattice site from $\widetilde{Z}^2$ is always even. Then, the connected components of γ are closed polygons (possibly self-intersecting). Let us call them **contours** and denote them by $\Gamma_1, ..., \Gamma_n$. We shall show that there is a configuration σ^Λ with $\gamma = \gamma(\sigma^\Lambda)$ for each collection $\gamma = \{\Gamma_1, ..., \Gamma_n\}$ of mutually disjoint contours. Let us put $\sigma_t = 1$ for $t \in \Lambda$ that are outside all contours and put $\sigma_t = -1$ for the sites that are inside one contour Γ only, $\sigma_t = 1$ for the sites that are encircled by two contours, and so on. Thus, there is a one-to-one correspondence between the configurations σ^Λ and the collections of contours γ.

Also, for $|\gamma|$ is the number of bonds in γ (the length of γ) and $|\widetilde{\Lambda}|$ is the number of bonds from $\widetilde{Z}^2$ adjacent to at least one site from Λ we get,

$$\begin{aligned} U_{\Lambda,(+)}(\sigma^\Lambda) &= U_\Lambda(\sigma^\Lambda) + U_{\Lambda,\partial\Lambda}(\sigma^\Lambda, \widetilde{\sigma}^{\partial\Lambda} \equiv 1) = 2\beta|\gamma| - \beta|\widetilde{\Lambda}|, \\ Z_{\Lambda,(+)} &= Z_\Lambda(\widetilde{\sigma}^{\partial\Lambda} \equiv 1) = \exp\ \{\beta|\widetilde{\Lambda}|\} \sum_\nu e^{-2\ \beta|\nu|} \end{aligned}$$

Lemma 2.11 *The probability $P_{\Lambda,(+)}(\Gamma)$ of the event that Γ is contained in* *the collection γ can be estimated by*

$$P_{\Lambda,(+)}(\Gamma) \leq e^{-2\beta|\Gamma|}.$$

Proof The probability

$$\begin{aligned} P_{\Lambda,(+)}(\Gamma) &= \sum_{\gamma:\Gamma\in\gamma} P_{\Lambda,(+)}(\gamma) \\ &= \frac{\sum\limits_{\gamma:\Gamma\in\gamma} e^{-2\beta|\gamma|}}{\sum\limits_{\gamma} e^{-2\beta|\gamma|}} = \frac{e^{-2\beta|\Gamma|}\sum\limits_{\gamma}{}' e^{-2\beta|\gamma|}}{\sum\limits_{\gamma} e^{-2\beta|\gamma|}} < e^{-2\beta|\Gamma|}, \end{aligned}$$

where $\sum_{\gamma}{}'$ is taken over all γ not intersecting Γ. So the proof of the lemma is completed.

The number of contours Γ of the length n encircling a given site $t_0 \in Z^2$ is not greater than $n^2 3^n$. Since the event $\sigma_0 = -1$ under the $(+)$-boundary conditions implies the existence of at least one contour Γ encircling the point 0, we have for large enough β

$$Pr_{\Lambda,(+)}(\sigma_0 = -1) \leq \sum_{\Gamma:\ \Gamma encircles\ 0} P_{\Lambda,(+)}(\Gamma) \leq \sum_{n\geq 4} n^2 3^n e^{-2\beta n} < 1/3$$

So the statement **3)** of Theorem (2.6) is proved.

2.2 Gibbs Modifications

2.2.1 Random Fields

We shall concern the following classes of random fields:

1) Random fields in a countable set T with values in a metric (complete and seperable) space S . The probability space $(\Omega,\ \Sigma,\ \mu)$ is represented in this case by the set $S^T = \Omega$ of functions (also called **configurations**) $x = \{x_t,\ t \in T\}$ defined on T, with values in S (S is often called the **set of spins**). The collection of random variables $x_t,\ t \in T$ (i.e. the values of the random configuration x at points $t \in T$) forms a *random field.*

As an example of such a field is the field of independent and identically distributed variables. In this case, the measure μ on $B(S^T)$ is defined to be the product of countably many identical copies of some probability measure λ_0 on the space S.

2) random point fields in a separable metric space Q with values in a space S. The set Ω of all locally finite subsets $x \subset Q$ is considered as probability space. The subset x (at most countable) is called **locally finite** if any bounded set $\Lambda \subset Q$ contains only a finite number of points from x. Every probability measure defined on the Borel σ-algebra $B(\Omega)$ is called a **random point field** in Q.

Let us suppose that a metrizable space S, also called the space of "**charges**" (or "**labels**"), is given. We use Ω^s to denote the space of pairs $\{x, s_x\}$ with $x \in \Omega$ and s_x being a function on x taking values from S. Such pairs will be called **configurations**. In the space Ω^s, as well as in Ω, a metrizable topology can be introduced. Every probability measure on $B(\Omega^s)$ determines a *labelled* random field in Q with values in the space S of charges.

3) Ordinary or generalized fields in R^ν. In this case, the probability space is a topological vector (locally convex) space Ω of functions or distributions defined on R^ν. A random field is given by a definition of a probability measure on the Borel σ-algebra $B(\Omega)$.

2.2.2 Method of Gibbs Modifications :

Gibbs modification is an important device for the construction of new measures from an originally given measure μ_0.

Finite Gibbs Modifications: Let (Ω, Σ, μ_0) be a measurable space with a finite or σ-finite measure μ_0 (called a **"free"** measure), and let $U(x)$, $x \in \Omega$, be a real function on Ω (called **"interaction energy"** or **"hamiltonian"**).

The measure μ with respect to the measure μ_0 with the density $\frac{d\mu}{d\mu_0}(x)$, will be called the Gibbs modification of the measure μ_0 by means of the interaction U, where

$$\frac{d\mu}{d\mu_0}(x) = Z^{-1} \exp \{-U(x)\} \tag{2.39}$$

The normalization factor Z (called the **partition function**) has the *stability condition*

$$Z = \int_\Omega \exp \{-U(x)\} d\mu_0(x) \neq 0, \ \infty. \tag{2.40}$$

The measures absolutely continuous with respect to μ_0 arise using finite Gibbs modifications. Measures which are singular with respect to the original measure μ_0, arise when passing to the weak limit of finite Gibbs modifications.

2.2.3 Weak Convergence of Measures

Let Ω be a topological space, $B = B(\Omega)$ its Borel σ-algebra, and $\Sigma \subset B$ some of its sub-σ-algebras.

Definition 2.12 *Let a directed family $\digamma\{\Lambda\}$ of indices be given. Then the measure μ, defined on the σ-algebra $\Sigma \subset \mathrm{B}$, is called the* **weak limit of the sequence of measures** *μ_Λ, $\Lambda \in \digamma$, defined on Σ if*

$$\int_\Omega f(x)d\mu_\Lambda \to \int_\Omega f(x)d\mu \tag{2.41}$$

for any bounded continuous Σ-measurable function f given on Ω.

For a more general situation, let us consider a complete family $\{\Sigma_\Lambda, \Lambda \in \digamma\}$, $\Sigma_{\Lambda_1} \subset \Sigma_{\Lambda_2}$, $\Lambda_1 \subset \Lambda_2$, of sub-$\sigma$-algebras of the σ-algebra B be given; the σ-algebras Σ_Λ will be called **local** σ-algebras and any function f, defined on Ω and measurable with respect to some of the *local* algebras, will be called a **local** function (function f, measurable with respect to a σ-algebra Σ_A, $A \in \digamma$, will often be denoted by f_A).

Definition 2.13 *Let a finite or σ-finite measure be given on each σ-algebra Σ_Λ. A cylinder measure μ on $\Re$ will be called the* ***weak local limit of the measures*** *μ_Λ if*

$$\lim_\Lambda \int_\Omega f(x)d\mu_\Lambda = \int_\Omega f(x)d\mu \tag{2.42}$$

for any bounded continuous local function f defined on Ω.

A cylinder measure (or its extension to a measure on the σ-algebra B) is the weak local limit of measures $\{\mu_\Lambda,\ \Lambda \in \digamma\}$ if, for each $\Lambda_0 \in \digamma$, the restrictions $\mu_\Lambda|_{\Sigma_{\Lambda_0}} = \mu_\Lambda^{\Lambda_0}$, $\Lambda_0 < \Lambda$, $\Lambda \in \digamma$, of the measures μ_Λ to the σ-algebra Σ_{Λ_0} weakly converge to $\mu|_{\Sigma_{\Lambda_0}} = \mu_{\Lambda_0}$.

Let us consider the case $\Omega = S^T$ (with T is a countable set and S is a metric space; the index Λ runs over finite subsets of T, and $\Sigma_\Lambda = \varphi_\Lambda^{-1}(B(S^\Lambda))$, where $\varphi_\Lambda :\ S^T \to S^\Lambda$ is the restriction mapping, the convergence (2.42) is called the **weak convergence of finite-dimensional distributions** if μ_Λ's are probability measures.

The following proposition gives the relationship between above definitions (2.11) and (2.12).

Proposition 2.14 *Let a family* $\{\Sigma_\Lambda,\ \Lambda \in \mathcal{F}\}$ *of* σ*-algebras be such that the set* $C_0(\Omega)$ *of bounded continuous local functions is dense everywhere in the space* $C(\Omega)$ *of all bounded continuous functions defined on* Ω *(in the uniform metric in* $C(\Omega)$*). Then the necessary and sufficient condition for a measure* μ *on* $\mathrm{B}(\Omega)$ *to be the local limit of probability measures* $\{\mu_\Lambda\}$ *(defined each on the* σ *algebra* Σ_Λ*) is that their arbitrary extensions* $\widetilde{\mu}_\Lambda$ *to a probability measures on the* σ*-algebra* $\mathrm{B}(\Omega)$ *weakly converge to* μ*.*

2.2.4 Limit Gibbs Modifications

Let a free measure μ_Λ^0 and a Hamiltonian U_Λ be defined for each Λ so that the stability condition (2.40) is satisfied, and let $\{\Sigma_\Lambda,\ \Lambda \in \mathcal{F}\}$ be a complete directed family of sub-σ-algebras of the σ-algebra $B(\Omega)$ be given. A cylinder measure μ on the algebra $\Re = \cup\Sigma_\Lambda$ (or its σ-additive extension to the σ-algebra $B(\Omega)$) is called a **limit Gibbs** measure (or a **limit Gibbs modification**) if it is the weak local limit of the Gibbs modifications μ_Λ of the measures μ_Λ^0 (by means of the energies U_Λ).

The theory of Gibbs measures becomes meaningful for a special choice of σ-algebras Σ_Λ, measures μ_Λ^0, and Hamiltonian U_Λ. Let us describe the respective ways of such a choice of Σ_Λ, μ_Λ^0, and U_Λ in connection with the three types of random fields listed above.

1) Gibbs modifications of fields in a countable set T**.** For a finite

$\Lambda \subset T$, we introduce a set of configurations $S^\Lambda = \{x^\Lambda = (x_t,\ t \in \Lambda)\}$. The restriction mapping $\varphi_\Lambda : x \mapsto x^\Lambda = x|_\Lambda$ defines a σ-algebra $\Sigma_\Lambda = \varphi_\Lambda^{-1}(B(S^\Lambda)) \subset B(S^T)$ that will be often identified with $B(S^\Lambda)$. $\{\Sigma_\Lambda,\ \Lambda \in T\}$ is complete in $B(S^T)$.

Remark 2.15 *The set $C_0(S^T) \subset C(S^T)$ of bounded continuous local functions on S^T is dense everywhere in $C(S^T)$, and hence above proposition applies in the case considered.*

Hamiltonians U_Λ are usually defined by potential $\{\Phi_A;\ A \subset T,\ |A| < \infty\}$, i.e., a family of functions Φ_A on Ω that are measurable with respect to σ-algebras Σ_A (i.e., Φ_A can be viewed as a function defined on the space S^A).

Let us put

$$U_\Lambda = \sum_{A \subseteq \Lambda} (\Phi_A) \tag{2.43}$$

for any finite A, often we can use the formal Hamiltonian (formal sum)

$$U = \sum_A \Phi_A. \tag{2.44}$$

Remark 2.16 *In many cases, the free measures μ_Λ^0 are restrictions of some probability measure μ_0 defined on S^T to the respective σ-algebras $\Sigma_\Lambda \subset \mathrm{B}$. In such cases, instead of a Gibbs modification $\hat{\mu}_\Lambda$ given on the σ-algebra $\mathrm{B}(S^T)$ by*

$$\frac{d\hat{\mu}_\Lambda}{d\mu_0}(x) = Z_\Lambda^{-1}\ exp\ \{-U_\Lambda(x)\} \tag{2.45}$$

is investigated. The measure $d\hat{\mu}_\Lambda$ is a "natural" extension of the measure μ_Λ to the whole σ-algebra $\mathrm{B}(S^T)$. This measure is also called a **finite Gibbs modification of the measure** *μ_0. By Remark (2.15), a limit Gibbs measure μ on the space S^T is the weak limit of the measures $d\hat{\mu}$, $\Lambda \nearrow T$.*

2) Gibbs modifications of point fields. Let $\Lambda \subset Q$ be a domain in Q, $\Omega^S(\Lambda,\ n) \subset (\Lambda \times S)^n/\Pi_n$ be the set of sequences of pairs

$$\{(q_1, s_1)\ ,...,\ (q_n, s_n)\},\ q_i \in Q,\ q_i \neq q_j,\ i \neq j,\ s_i \in S, \tag{2.46}$$

factorized with respect to the group Π_n of permutations of n elements (two sequences (2.46)) are considered to be equivalent if one arises from other by means of permutation). In this way, $\Omega^s(\Lambda, n)$ is given with a metrizable topology. Let us use the notation $\Omega^s(\Lambda) = \cup_{n=0}^{\infty}\Omega^s(\Lambda, n),\ \Omega^s(\Lambda,\ 0) = \emptyset$, and let us introduce on $\Omega^s(\Lambda)$ the topology of the direct sum of topological spaces. For Λ is bounded domain in Q and by the restriction mapping

$$\varphi_\Lambda :\ (x, s_x) \mapsto (x \cap \Lambda,\ s_x|_{x\cap\Lambda} \in \Omega^s(\Lambda)), \tag{2.47}$$

the topology on Ω^s is defined as the weakest topology making all mappings φ_Λ continuous. For any bounded domain $\Lambda \subset Q$, the sub-σ-algebra of the Borel σ-algebra $B(\Omega^S)$ is defined as

$$\Sigma_\Lambda = \varphi_\Lambda^{-1}\ [B(\Omega^s(\Lambda))].$$

The family of local σ-algebras Σ_Λ generates the whole Borel σ-algebra $B(\Omega^s)$, and the set $C_0(\Omega^s)$ of bounded continuous local functions is dense everywhere in $C(\Omega^s)$.

Poisson field. For the free measure μ_0 on Ω^s, the distribution of the so-called **labelled Poisson** field in Q is chosen, i.e., let a positive σ-finite (or finite) measure $d\lambda_0$ such that $\lambda_0(\Lambda) < \infty$ for each bounded domain, Λ be given on the space Q, and let a probability measure ds be given on the space S. The measure $(d\lambda_0 \times ds)^n$, defined on the space $(Q \times S)^n$, induces, on the space $\Omega^s(Q, n) \equiv \Omega^s_n$, the factor measure

$$d\nu_n = (d\lambda_0 \times ds)^n/n!,\ n > 0,\ \nu_0(\emptyset) = 1. \tag{2.48}$$

Let us consider a measure ν on the space $\Omega^s_{fin} = \cup_{n\geq 0}\Omega^s_n$ of finite configurations in Q, coinciding on each set Ω^s_n with the measure $\nu_n,\ n = 0,\ 1, ...$

Let $\Lambda \subset Q$ be a bounded domain and μ^0_Λ be a probability measure on $\Omega^s(\Lambda)$ equal to

$$\mu^0_\Lambda = e^{-\lambda_0(\Lambda)}\nu. \tag{2.49}$$

Note that since $\Omega^s(\Lambda) \subset \Omega^s_{fin}$, the measure ν is defined on the space $\Omega^s(\Lambda)$, and $\mu^0_\Lambda(\Omega^s(\Lambda, n))$, i.e., the probability of the occurrence of exactly n points of the labelled field in Λ, equals $\lambda^n_0(\Lambda)\ e^{-\lambda_0(\lambda)}/n!$. Each measure μ^0_Λ can be considered as defined on the σ-algebra Σ_Λ, and we may verify that there is a unique measure μ^0 on the space Ω^s such that its restrictions to sub-σ-algebras Σ_Λ coincide with the measures μ^0_Λ. The labelled point field in Q generated by this measure is called **Poisson field with independent charges**.

Any function $\Phi[(x, s_x)]$ defined on the set Ω^s_{fin} of finite configurations (x, s_x) is called **a potential**. For each bounded domain $\Lambda \subset Q$, we take

$$U_\Lambda[(x, s_x)] = \sum_{y\subseteq x\cap\Lambda} \Phi[(y, s_y)],$$

with $s_y = s_x|_y$ being the restriction of the function s_x to $y \subset x$.

The Gibbs modification μ_Λ of the Poisson field μ^0 is defined with the help of the Lebesgue measure $d\lambda_0 = d^\nu x$ on R^ν, and the energies U_Λ are defined by means of a two-point translation-invariant potential Φ, i.e.,

$$\Phi(x) = \begin{cases} \hat{\mu}, & \text{if } |x| = 1\ , \\ \beta\varphi(q_1 - q_2), & \text{if } x = (q_1, q_2), \\ 0, & \text{if } |x| > 2; \end{cases} \tag{2.50}$$

where $\hat{\mu} \in R^1$ (called chemical potential), φ is an even function defined on the space R^ν, and $\beta > 0$.

Theorem 2.17 *[15] Let φ be a real even upper semi-continuous function on R^ν. Then the followings are equivalent:*

a) *the inequality*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \varphi(q_i - q_j) \geq 0 \tag{2.51}$$

is fulfilled for any n and $q_i \in R^\nu,\ i = 1, ..., n$;

b) *there is a $B \geq 0$ such that*

$$U_\Lambda(x) \geq -\mathrm{B}|x| \tag{2.52}$$

for any $x \in \Omega_{fin}$ and $\Lambda \in R^\nu$.

c) *the partition functions Z_Λ are finite for all bounded domains Λ.*

3) Gibbs modifications of measures on function spaces. Let Ω be some locally convex space of functions $x(t) = \{x_1(t), ..., x_n(t)\},\ t \in R^\nu$, defined on the space R^ν, with values in R^n.

Let us take the topology on Ω is such that the functionals of the form $F_{t_0}(x) = x_k(t_0),\ t_0 \in R^\nu,\ k = 1, 2, ..., n$ are continuous with respect to it (i.e. the convergence of a sequence of functions in Ω implies their pointwise convergence). For each bounded open or closed set $\Lambda \subset R^\nu$, we define the σ-algebra Σ_Λ to be the smallest sub-σ-algebra of the Borel σ-algebra $B(\Omega)$ making all the functionals $\{F_{t_0},\ t_0 \in \Lambda\}$ measurable. Suppose that the family of σ-algebras Σ_Λ is generating for the σ-algebra $B(\Omega)$. Also let us take a probability measure μ_0 (free measure) defined on the Borel σ-algebra $B(\Omega)$, and a functional $U_\Lambda(x)$ is given to each bounded open or closed set

$\Lambda \subset R^\nu$, so that:

1) $U_\Lambda = 0$ if $|\Lambda| = 0$, where $|\Lambda|$ is the Lebesgue measure of Λ;

2) U_Λ is Σ_Λ measurable;

3) $U_{\Lambda_1 \cap \Lambda_2} = U_{\Lambda_1} + U_{\Lambda_2}$if $|\Lambda_1 \cap \Lambda_2| = 0$.

$\{U_\Lambda\}$, a family of functionals satisfying above conditions is called **local additive** functional.

$$0 < \int_\Omega exp\ \{-U_\Lambda(x)\}\ d\mu_0 < \infty \tag{2.53}$$

Suppose that for each bounded domain $\Lambda \subset R^\nu$ satisfies above *stability condition* (2.53) and let us define a Gibbs modification μ_Λ of the measure μ_0 by the formula (2.45). Limit Gibbs modification of the measure μ_0 is defined.

Example In the case when the space Ω contains only smooth locally bounded functions $x(t)$,

$$U_\Lambda(x) = \int_\Lambda \Phi[x_i(t),\ \frac{\partial x_i}{\partial t^{(j)}}]\ d^\nu t, \qquad t = (t^{(1)}, ..., t^{(\nu)}),$$

is a local additive functional with Φ a real function of $n\ (\nu + 1)$ variables that is bounded from below.

Remark 2.18 *Local additive functionals may also be defined on the Schwartz space $D^{'}(R^\nu)$ (space of distributions), in some cases such that they satisfy the stability condition (μ_0 is a probability measure on $D^{'}(R^\nu)$), so the Gibbs modifications μ_Λ and the limit Gibbs modification μ may be defined with the help of them.*

Remark 2.19 *By the additive local functionals we have studied Gibbs modifications of measures on function spaces, and now we may investigate non-local functionals U_ν such that functionals of the following form with Φ being* *a bounded real function of* $2n$ *variables,*

$$U_\Lambda(x) = \int_\Lambda \int_\Lambda \Phi[x(t),\ x(t')]\ d^\nu t d^\nu t'.$$

2.2.5 Weak Compactness of Measures. The concept of Cluster Expansion

A is some collection of measures on whole Borel σ-algebra $B(\Omega)$ of a topological space Ω or on its sub-σ-algebra $\Sigma \subset B(\Omega)$. By the weak compactness of the set A, it is sequentially compact, i.e., there is a weakly converging sequence $\mu_n \to \mu,\ n \to \infty,\ \mu_n \in B$, in any infinite subset $B \subset A$.

Lemma 2.20 *In complete separable metric space* Ω *and* $\Sigma = \mathrm{B}(\Omega)$*, for weak convergence for the set* A*, the below conditions should be satisfied.*
1) *Each* $\mu \in \mathrm{A}$ *is a probability measure, and there is a compact function* $h > 0$ *defined on* Ω *such that*

$$\int_\Omega h(x) d\mu < C$$

for any measure $\mu \in \mathrm{A}$ *where* C *does not depend on* μ*. A function* h *on* Ω *is called compact if the set* $\{x \in \Omega,\ h(x) < a\}$ *is compact for any* $a > 0$*.*
2) *There are a nonnegative measure* μ_0 *on* $\mathrm{B}(\Omega)$ *and a* μ_0*-integrable function* $\varphi(x) \geq 0$ *such that any measure* $\mu \in \mathrm{A}$ *is absolutely continuous with respect to* μ_0 *and*

$$|\frac{d\mu}{d\mu_0}(x)| < \varphi(x), \qquad x \in \Omega.$$

Definition 2.21 *Let* $\{\mu_\Lambda,\ \Lambda \in \digamma\}$ *be a family of measures defined on the* σ*-algebra* Σ_Λ *from a complete family* $\{\Sigma_\Lambda,\ \Lambda \in \digamma\}$ *of sub-*σ*-algebras of the* σ*-algebra* $\mathrm{B}(\Omega)$*. A family* $\{\mu_\Lambda,\ \Lambda \in \digamma\}$ *is called* **weakly locally compact**

if the set $\{\mu_\Lambda^{\Lambda_0},\ \Lambda_0 < \Lambda\}$ *of restrictions of measures* $\{\mu_\Lambda\}$ *to the* σ*-algebra* Σ_{Λ_0} *is weakly compact for any* $\Lambda_0 \in \mathcal{F}$.

Lemma 2.22 *[15] Let* $\{\mu_\Lambda,\ \Lambda \in \mathcal{F}\}$ *be locally compact. Then in any increasing sequence* $\Lambda_1 < \Lambda_2 < \ \ldots\ < \Lambda_n < \ \ldots$ *of indices with the sequence of* σ*-algebras* $\Sigma_{\Lambda_n},\ n = 1, 2, \ldots,$ *is complete, there is a subsequence having the same property and a cylinder measure* μ *on* $\Re = \cup \Sigma_\Lambda$ *such that*

$$\mu = \lim_{h\to\infty} \mu_{i_k} \qquad (\mu_n = \mu_{\Lambda_n}). \tag{2.54}$$

Let $G \subset C_0(S^T)$ be some set of bounded continuous local functions whose linear hull is dense everywhere in the space $C(S^T)$ of all bounded continuous functions. Let the mean $\langle \mathcal{F} \rangle_\mu$ of an arbitrary function $\mathcal{F} \in G$ under a measure μ be expanded in the form

$$\langle \mathcal{F} \rangle_\mu = \sum_{R\subset T,\ |R|<\infty} b_R(\mathcal{F}), \tag{2.55}$$

with $b_R(\mathcal{F})$ being some quantities depending on $\mathcal{F}$ and finite subset $R \subset T$. Such expansions are generally called **cluster expansion** of the measure μ.

Definition 2.23 *Let* $\{\mu_\Lambda,\ \Lambda \subset T\}$ *be a family of measures defined on the* σ*-algebra* $\Sigma_\Lambda = \mathrm{B}(S^\Lambda)$ $(\Lambda \subset T,\ |\Lambda| < \infty)$. *The family* $\{\mu_\Lambda\}$ *is said to admit a* **cluster expansion** *if*

1) *it is weakly locally compact;*

2) *there is a set* $G \subset C_0(S^T)$ *of bounded continuous functions whose linear hull is dense everywhere in the space* $C(S^T)$ *such that the mean* $\langle \mathcal{F} \rangle_{\mu_\Lambda} = \langle \mathcal{F} \rangle_\Lambda$ *of any function* $\mathcal{F} \in G$ *admits an expansion*

$$\langle \mathcal{F} \rangle_\Lambda = \sum_{R\subseteq\Lambda} b_R^{(\Lambda)}(\mathcal{F}) \tag{2.56}$$

with the quantities $b_R^{(\Lambda)}(\mathcal{F})$ satisfying the following conditions:

a) *there is a majorant*

$$|b_R^{(\Lambda)}(\mathcal{F})| < C_R(\mathcal{F}), \qquad \sum_{R \subset T} C_R(\mathcal{F}) < \infty \tag{2.57}$$

b) *there are limits*

$$\lim_{\Lambda \nearrow T} b_R^{(\Lambda)}(\mathcal{F}) = b_R(\mathcal{F}). \tag{2.58}$$

Lemma 2.24 *Let a family $\{\mu_\Lambda\}$ of measures admit a cluster expansion. Then the weak local limit*

$$\mu = \lim_{\Lambda \nearrow T} \mu_\Lambda \tag{2.59}$$

exists and μ admits a cluster expansion.

The cylinder measure μ is probability in the case of probability measures $\{\mu_\Lambda\}$, hence it can be extended to a probability measure on the σ-algebra $B(\Omega)$.

2.3 Gibbs Modifications under Boundary Conditions and Definition of Gibbs Fields by Means of Conditional Distributions

We restrict ourself here only to the case of fields in a countable set T (a metric ρ is given on T) with values in a (metric) space S. We suppose that a finite or σ-finite measure $\mu_\Lambda^0 = \lambda_0^\Lambda$, i.e., the product of $|\Lambda|$ copies of the measure λ_0, as the free measure μ_Λ^0 on the space S^Λ. Also, we suppose that we are given a potential $\{\Phi_A;\ A \subset T,\ |A| < \infty\}$ of a finite range, i.e., $\Phi_A \equiv 0$

if $diamA \equiv \max_{t_1,t_2\in A} \rho(t_1,t_2) > d$ for some constant $d > 0$, and that the Hamiltonian $U_\Lambda = \sum_{A\subseteq\Lambda} \Phi_A$ determined by it satisfies the stability condition

$$0 < \int_{S^\Lambda} \exp\{-U_\Lambda(x)\} d\lambda_0^\Lambda < \infty$$

for any finite $\Lambda \subset T$. Let μ_Λ be a Gibbs modification of the measure λ_0^Λ, and for any $\Lambda_0 \subset \Lambda$, we denote the conditional probability distribution on the set of configurations $x^{\Lambda_0} \in S^{\Lambda_0}$ as $\mu_\Lambda^{\Lambda_0}(.\ /\overline{x}^{\Lambda\setminus\Lambda_0})$ under the condition that a configuration $\overline{x}^{\Lambda\setminus\Lambda_0} \in S^{\Lambda\setminus\Lambda_0}$, in the set $\Lambda \setminus \Lambda_0$, is fixed. And the density of the measure $\mu_\Lambda^{\Lambda_0}(.\ /\overline{x}^{\Lambda\setminus\Lambda_0})$ with respect to the measure $\lambda_0^{\Lambda_0}$ is

$$\frac{d\mu_\Lambda^{\Lambda_0}(x^{\Lambda_0}\ /\overline{x}^{\Lambda\setminus\Lambda_0})}{d\lambda_0^{\Lambda_0}} = Z_{\Lambda_0}^{-1}(\overline{x}^{\Lambda\setminus\Lambda_0})\ \exp\{-U_{\Lambda_0}(x^{\Lambda_0}\ /\overline{x}^{\Lambda\setminus\Lambda_0})\} \tag{2.60}$$

with

$$\begin{aligned} Z_{\Lambda_0}(\overline{x}^{\Lambda\setminus\Lambda_0}) &= \int_{S^{\Lambda_0}} \exp\{-U_{\Lambda_0}(x^{\Lambda_0}\ /\overline{x}^{\Lambda\setminus\Lambda_0})\}\ d\lambda_0^{\Lambda_0}, \\ U_{\Lambda_0}(x^{\Lambda_0}\ /\overline{x}^{\Lambda\setminus\Lambda_0}) &= U_{\Lambda_0}(x^{\Lambda_0}) + \sum_{\substack{A:A\cap\Lambda_0\neq\emptyset \\ A\cap(\Lambda\setminus\Lambda_0)\neq\emptyset}} \Phi_A\ (x^{\Lambda_0}\cup\overline{x}^{\Lambda\setminus\Lambda_0}), \end{aligned} \tag{2.61}$$

where $x^{\Lambda_0}\cup\overline{x}^{\Lambda\setminus\Lambda_0}$ denotes the configuration in Λ whose restrictions to Λ_0 and $\Lambda\setminus\Lambda_0$ are equal to x^{Λ_0} and $\overline{x}^{\Lambda\setminus\Lambda_0}$, respectively. The second expression in (2.61) is called **the energy of the interaction with an external** (boundary) **configuration**. Note that, for a fixed Λ_0 and a sufficiently large $\Lambda \supset \Lambda_0$, the energy $U_{\Lambda_0}\ (x^{\Lambda_0}\ /\overline{x}^{\Lambda\setminus\Lambda_0})\}\ d\lambda_0^{\Lambda_0}$ does not depend on the whole configuration $\overline{x}^{\Lambda\setminus\Lambda_0}$, but only its restriction $\overline{x}^{\partial_d\ \Lambda_0}$ to the d-neighborhood of Λ_0, i.e., $\partial_d\Lambda_0 = \{t \in T \setminus \Lambda_0,\ \rho(t,\ \Lambda_0) \leq\ d\}$. Let us denote this energy by

$$U_{\Lambda_0}\ (x^{\Lambda_0}\ /\overline{x}^{\partial_d\ \Lambda_0}), \tag{2.62}$$

and let us denote the Gibbs modification of the measure $\lambda_0^{\Lambda_0}$ by means of the Hamiltonian (2.62) by $\mu_{\overline{x}^{\partial_d}\ \Lambda_0}^{\Lambda_0}$. This measure $\mu_{\overline{x}^{\partial_d\Lambda_0}}^{\Lambda_0}$ is called **Gibbs distribution on Λ_0 with the boundary configuration** $\overline{x}^{\partial_d\Lambda_0}$ in the neighborhood

$\partial_d \Lambda_0$.

By the formula (2.60) we get the following:

Definition 2.25 *A probability measure μ on the space S^T is called a* **Gibbs distribution in T** *if, for any finite $\Lambda \subset T$ and any configuration $\overline{x} \in S^{T \setminus \Lambda}$, the conditional distribution $\mu(. \ / \ x^{T \setminus \Lambda} = \overline{x}\ )$ on the set S^Λ coincides, under the condition that the external configuration $x^{T \setminus \Lambda}$ is fixed and equal to $\overline{x}$, with the measure $\mu_{\overline{x}^{\partial_d \Lambda_0}}$ given by*

$$\mu(. \ / \ x^{T \setminus \Lambda} = \overline{x}\ ) = \mu_{\overline{x}^{\partial_d \ \Lambda_0}}, \tag{2.63}$$

with $\overline{x}^{\partial_d \ \Lambda_0}$ being a restriction of $\overline{x}$ to $\partial_d \Lambda$.

From (2.63), *d-Markov property* of Gibbs measure μ is formed.

Let $\Lambda \subset T$ be a finite set and let some probability distribution $q = q^{\partial_d \Lambda}$ on the set $S^{\partial_d \Lambda}$ of boundary configurations $\overline{x} = \overline{x}^{\partial_d \Lambda}$ be given, then the measure

$$\mu_q^\Lambda = \int_{S^{\partial_d \Lambda}} \mu_{\overline{x}}^\Lambda dq(\overline{x}) \tag{2.64}$$

on S^Λ is called a Gibbs distribution with a **q-random boundary configuration** in Λ.

Proposition 2.26 *[15] For a measure μ on S^T to be Gibbsian, it is necessary that, for any increasing sequence $\Lambda_n \nearrow T$, $n \to \infty$, of finite sets Λ_n, there is a sequence of distributions $q_n = q^{\partial_d \ \Lambda_n}$ defined each on the set $S^{\partial_d \Lambda_n}$ of boundary configurations, so that the weak local limit of measures $\mu_{q_n}^{\Lambda_n}$ coincides with μ, i.e.,*

$$\lim_{n \to \infty} \mu_{q_n}^{\Lambda_n} = \mu, \tag{2.65}$$

and it is sufficient that the condition (2.65) is satisfied for some increasing sequence $\Lambda_n \nearrow T$.

Corollary 2.27 *Let a family* $\{\mu_{\overline{x}}^{\Lambda}\}$ *of Gibbs modifications be such that there is a unique limit*

$$\mu = \lim_{\Lambda \nearrow T} \mu_{\overline{x}}^{\Lambda}$$

for any sequence $\Lambda \nearrow T$ *and any choice of boundary configurations* $\overline{x} \in S^{\partial_d \Lambda}$*. Then* μ *is the unique Gibbs measure on* S^T*.*

Chapter 3

Markov Fields on the Integers

In this chapter we will study Markov fields on $S = \mathbb{Z}$ under some restrictive assumptions. The state space E will be countable, and we shall look only at Markov specifications γ which are positive and homogeneous. Such specifications γ are always Gibbsian for a suitable shift-invariant nearest-neighbor potential ϕ. We shall pass from ϕ to a closely related positive matrix Q on E, and we shall write γ^Q instead of γ^ϕ (in the terminology of Statistical Physics, Q is called the **transfer matrix**). We shall denote the positive matrix as $Q = (Q(x,y))_{x,y \in E}$ which is defined by

$$Q(\omega_{i-1}, \omega_i) = exp[-\phi_{\{i-1,i\}}(\omega) - (1/2)\phi_{\{i-1\}}(\omega) - (1/2)\phi_{\{i\}}(\omega)]$$

where $\omega \in \Omega$ and $i \in \mathbb{Z}$ (note that the expression on the right depends only on ω_{i-1} and ω_i). Q is often called **the transfer matrix associated with** ϕ. The λ(λ on E is counting measure)-admissibility of ϕ implies that all powers Q^n of Q are well-defined, in that

$$Q^n(x,y) < \infty \quad \textit{for all } x, y \ \in E \ \textit{and } n \geq 1.$$

Indeed, for all $\omega \in \Omega$ and $n \geq 1$ we have

$$Q^n(\omega_0, \omega_n) = Z^{\phi}_{]0,n[}(\omega) \ \exp[-(1/2)\phi_{\{0\}}(\omega) - (1/2)\phi_{\{n\}}(\omega)] < \infty \tag{3.1}$$

Below the set of all Markov fields corresponding to positive matrix Q will be denoted by $G_{\circledast}(Q)$

Theorem 3.1 *[9] Let Q be a positive matrix on E which satisfies equation (3.1). Then either $\mathrm{G}_{\circledast}(Q) = \emptyset$ or $|\mathrm{G}_{\circledast}(Q)| = 1$. The letter case occurs if and only if Q is equivalent to a positive recurrent stochastic matrix P with positive entries. In this case P is unique and $\mathrm{G}_{\circledast}(Q) = \{\mu_p\} \subset exG(Q)$.*

Corollary 3.2 *Let Q be a positive matrix which satisfies equation (3.1).*
(a) *For each $\mu \in \mathrm{G}(Q)$ we have the following alternative: Either μ is shift-invariant, or its translates $\theta_i(\mu)$ $(i \in \mathbb{Z})$ are pairwise distinct.*
(b) *If $Q \sim P$ for some positive recurrent stochastic matrix P with positive entries then either $\mathrm{G}(Q) = \{\mu_p\}$ or $|ex\mathrm{G}(Q)| = \infty$.*
(c) *If Q is not equivalent to any positive recurrent stochastic matrix with positive entries then either $\mathrm{G}(Q) = \emptyset$ or $|ex\mathrm{G}(Q)| = \infty$.*

3.1 Kallikov's example of phase transition

Take $E = \mathbb{Z}_+$, fix the numbers p, q with $0 < q < p < 1$, and we define two numbers $a, b > 0$ by the two requirements

$$a/b = p/q, \qquad a(1-p)^{-1} - b(1-q)^{-1} = 1 \tag{3.2}$$

Thus

$$a = p(1-p)(1-q)(p-q)^{-1} \qquad b = q(1-p)(1-q)(p-q)^{-1}.$$

We also put

$$c = a - b = (1-p)(1-q)$$

Next we introduce a (row) vector $\alpha \in]0,\infty[^E$ by

$$\begin{aligned} \alpha(x) &= ap^x - bq^x \\ &= c(p^{x+1} - q^{x+1})(p-q)^{-1} \\ &= c\sum_{k=0}^{x} p^k q^{x-k} \qquad (x \in E). \end{aligned} \tag{3.3}$$

The equality of the first and second expression on the right comes from the first requirement in (3.2) and the second requirement in (3.2) ensures that α is a probability vector on E. By the second expression for α we realize that α satisfies the recursion relation

$$\alpha(0) = c,\ \alpha(x) = p\alpha(x-1) + cq^x \qquad (x \geq 1) \tag{3.4}$$

Define a positive matrix Q on E by

$$\begin{aligned} Q(x,y) &= p\alpha(x-1)\alpha(x)^{-1}\delta_{x-1}(y) + cq^x\alpha(x)^{-1}\alpha(y) \\ &= [p\delta_{x-1}(y) + cq^x]\alpha(y)/\alpha(x) \end{aligned} \tag{3.5}$$

where $x, y \in E$ and δ_{x-1} is Kronecker's delta. The matrix Q was invented and studied by Kallikov (1977) for a specific choice of p and q.

Since $\delta_{-1}(y) = 0$ for all $y \in E$, we get $Q(0,\cdot) = \alpha$. Moreover, Q is stochastic. Indeed, (3.4) shows that $Q(x,\cdot)$ is a convex combination of the probability vectors δ_{x-1} and α for all $x \geq 1$.

In the case $q = 0$, Q is given by

$$Q(x,y) = \begin{cases} \delta_{x-1}(y) & \text{if } x \geq 1, y \in E, \\ (1-p)p^y & \text{if } x = 0, y \in E \end{cases}$$

and can be thought of as describing the evolution of the number of inhabitants of a fixed territory: The population loses one individual per time unit until the time of extinction, at which time a geometrically distributed number of immigrants enters the territory. In the case $q > 0$, this process is shortened, in that the inhabitants of the territory may be dislodged by an invading population of size distribution α even before extinction (with a positive probability which depends on the number of inhabitants).
It is clear that $\alpha\, Q = \alpha$, to check this we fix some $y \in E$. Then

$$\begin{aligned}\alpha Q(y) &= \alpha(0)\ \alpha(y) + p\ \alpha(y) + \sum_{x \geq 1} c\ q^x \alpha(y) \\ &= [c + p + c\ q(1-q)^{-1}]\ \alpha(y) \\ &= \alpha(y).\end{aligned}$$

According to theorem (2.1), the positive recurrence of Q implies that $G_{\circledast}(Q) = \{\mu_Q\} \subset exG(Q)$. Kallikov's discovery was that Q admits a non-trivial entrance law $\{\alpha_i : i \in \mathbb{Z}\}$ which reaches equilibrium, in that $\alpha_i = \alpha$ for all $i \geq 1$. Let us introduce this entrance law.

We put $s = q/p$. For $i \in \mathbb{Z}$ and $x \in E$ we define

$$\alpha_i(x) = \begin{cases} \alpha(x) & \text{if } i \geq 1 \\ (1 - s^{1-i})\delta_{-i}(x) + s^{1-i}\alpha(x) & \text{if } x \leq 0 \end{cases}$$

Clearly, each α_i is a probability vector on E, and the α_i's with $i \leq 0$ are pairwise distinct. Let us check that $\alpha_i\ Q = \alpha_{i+1}$ for all $i \in \mathbb{Z}$. We already know this when $i \geq 1$. So let $i \leq 0$. For each $y \in E$ we can write

$$\begin{aligned}\alpha_i\ Q(y) &= (1 - s^{1-i})\ Q(-i, y) + s^{1-i}\alpha\ Q(y) \\ &= (1 - s^{1-i})\ p\ \alpha(-i-1)\alpha(-i)^{-1}\delta_{-i-1}(y) + \\ &\quad + [(1 - s^{1-i})\ c\ q^{-i}\alpha(-i)^{-1} + s^{1-i}]\ \alpha(y)\end{aligned}$$

because $\alpha Q = \alpha$. And the second expression on the right of (3.3) shows that

$$\begin{aligned} cq^{-i}\alpha(-i)^{-1} &= q^{-i}(p-q)/p^{1-i}(1-s^{1-i}) \\ &= s^{-i}(1-s)/(1-s^{1-i}). \end{aligned}$$

And using this result, the expression in the square brackets equals

$$s^{-i}(1-s) + s^{1-i} = s^{-i}$$

Thus

$$\alpha_i Q = (1-s^{1-i})p\alpha(-i-1)\alpha(-i)^{-1}\delta_{-i-1} + s^{-i}\alpha$$

Since $\alpha_i Q$ and α are probability vectors, we conclude that

$$\alpha_i Q = (1-s^{-i})\delta_{-i-1} + s^{-i}\alpha = \alpha_{i+1}.$$

So we have proved that $\{\alpha_i : i \in \mathbb{Z}\}$ is an entrance law for Q.

3.2 Spitzer's example of totally broken shift-invariance

In the Kallikov's example of phase transition, Q was a positive recurrent matrix. Then what about if there is any matrix Q which shows a phase transition but is not equivalent to a positive recurrent stochastic matrix. By theorem (2.1), such a Q can never admit a shift-invariant Markov field. Since γ^Q is shift-invariant, one might wonder if such a Q can admit any Markov field. By corollary (2.2) such a Markov field, if it exists, has pairwise distinct translates. Is this case possible? We can answer this question by F.Spitzer's example.

Let us begin by introducing some notations for the binomial and Poisson distributions, respectively

$$b(n,p,k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad (n,\ k \geq 0\ , 0 \leq p \leq 1) \tag{3.6}$$

$$\varrho(q,k) = e^{-q} q^k / k! \qquad (k \geq 0, q > 0) \tag{3.7}$$

Let us introduce the elementary formula

$$\sum_{k \geq 0} b(n,p_1,k) b(k,p_2,\cdot) = b(n,p_1p_2,\cdot) \qquad (n \geq 0,\ 0 \leq p_1, p_2 \leq 1) \tag{3.8}$$

We take $E = \mathbb{Z}_+$. We consider a stochastic matrix P of the form

$$P(x,y) = \begin{cases} b(x,p,y) & \text{if } x \geq 1, y \in E, \\ \alpha(y) & \text{if } x = 0, y \in E. \end{cases} \tag{3.9}$$

Here $o < p < 1$, $b(x,p,\cdot)$ is given by (3.6), and $\alpha > 0$ is a probability vector on E. We consider $Q = P^2 \cdot Q$ is positive since P is not positive and, $Q(x,y) \geq b(x,p,0)\alpha(y) > 0$ for all $x, y \in E$. Also P is irreducible.

As before, let us think P in terms of population dynamics. P describes the evolution of the number of inhabitants of a territory. At each time unit, the inhabitants survive independently of each other with probability p and die with probability $(1-p)$. At the time of extinction, a new population of size distribution α immigrates into the territory.

Now look at the process with the same survival mechanism but without immigration. This process is described by the stochastic matrix

$$\widetilde{P}(x,y) = b(x,p,y) \qquad (x, y \in E). \tag{3.10}$$

According to equation (3.8), the powers of $\widetilde{P}$ are given by

$$\widetilde{P}^n(x,y) = b(x,p^n,y) \qquad (n \geq 1, x, y \in E). \tag{3.11}$$

By an intuitive description of P that P is recurrent. A formal proof is as follows.

For each $x \in E$, we let $\mu_p^x \in P(E^{\mathbb{Z}_+}, \varepsilon^{\mathbb{Z}_+})$ denote the Markov chain with transition matrix P and starting point x. We look at the extinction time $\tau = min\{n \geq 1 : \sigma_n = 0\}$. We obtain

$$\begin{aligned}
\mu_p^0(\tau < \infty) &= \alpha(0) + \Sigma_{x\geq 1}\alpha(x)\mu_p^x(\tau < \infty) \\
&= \alpha(0) + \Sigma_{x\geq 1}\alpha(x) \lim_{n\to\infty} \mu_p^x(\tau \leq n) \\
&= \alpha(0) + \Sigma_{x\geq 1}\alpha(x) \lim_{n\to\infty} \mu_p^x(\sigma_n = 0) \\
&= \alpha(0) + \Sigma_{x\geq 1}\alpha(x) \lim_{n\to\infty} (1-p^n)^x \\
&= 1.
\end{aligned}$$

The next to last equality is a consequence of equation (3.11). Since P is irreducible, the equation $\mu_p^0(\tau < \infty) = 1$ implies that P is recurrent.

Next we can show that α can be chosen in such a way that P is null recurrent. Also, for each $x \geq 1$ we have

$$\mu_p^x(\tau) = \sum_{n\geq 0} \mu_p^x(\tau > n) = \sum_{n\geq 0} (1 - (1-p^n)^x)$$

and therefore, by Fatou's lemma,

$$\liminf_{x\to\infty} \mu_p^x(\tau) = \infty.$$

Consequently, we can find an increasing sequence $(x(k))_{k\geq 1}$ in E such that $\mu_p^{x(k)}(\tau) \geq 2^k$ for all k. Thus, if α is any positive probability vector with $\alpha(x(k)) \geq c\ 2^{-k}$ for some $c > 0$ and all k then

$$\mu_p^0(\tau) = 1 + \sum_{x\geq 1} \alpha(x)\ \mu_p^x(\tau) \geq \sum_{k\geq 1} \alpha(x(k))\ \mu_p^{x(k)}(\tau) = \infty,$$

which means that P is null recurrent.

Finally, we can say that Q is null recurrent whenever P is null recurrent. We have $Q^n(0,0) = P^{2n}(0,0) \geq \alpha(0)P^{2n-1}(0,0)$ *for all* $n \geq 1$ and therefore

$$2\sum_{n\geq 1} Q^n(0,0) \geq \alpha(0)\sum_{k\geq 1} P^k(0,0) = \infty.$$

Thus Q is recurrent. Also, Q is null recurrent because

$$\begin{aligned} \infty = \mu_p^0(\tau) &= \sum_{n\geq 0} \mu_p^0(\tau > n) \leq 2\sum_{k\geq 0} \mu_p^0(\tau > 2k) \\ &\leq 2\sum_{k\geq 0} \mu_Q^0(\tau > k) = 2\mu_Q^0(\tau). \end{aligned}$$

Finally, we can give the Spitzer's result.

Theorem 3.3 *([9]) Let P be given by (3.9), and suppose α is chosen in such a way that P is null recurrent. Define $Q = P^2$. Then $|ex\mathrm{G}(Q)| = \infty$, but $\theta_j(\mu) \neq \mu$ for all $j \in \mathbb{Z}$ and $\mu \in \mathrm{G}(Q)$.*

Chapter 4

Markov Chains And Gibbs States

Let E be a finite state space and S the vertex set of locally finite connected tree, that is there is a distinguished set $B \subset \{b \subset S\colon |b|=2\}$ of "bond" or "edges" b=$\{i,j\}$ between "adjacent" sites $i,j \in S$ which exhibits the three properties below.

1) *local finiteness*

2) *connectedness*

3) *tree property*

- Local Finiteness: For each $i \in S$, the set $\partial i = \{j \in S\colon \{i,j\} \in B\}$ of all neighbors of i is finite. Of course this implies that

  $$\partial\Lambda := \bigcup_i \partial i \setminus \Lambda$$

  is finite for all $\Lambda \in \Im$ where $\Im = \{\Lambda \in S : 0 < |\Lambda| < \infty\}$.

- Connectedness: For any two sites $i, j \in S$ there is a sequence $i = i_0, i_1, ..., i_n = j$ in S such that $\{i_{k-1}, i_k\} \in B$ for all $1 \leq k \leq n$ such a sequence is called a **path** from i to j.

- Tree Property: For each $i,j \in S$ there is only one path from i to j. Consequently, we can introduce a metric d on S by letting $d(i,j)$ be the length n of the unique path from i to j.

Definition 4.1 *Let γ be a specification for E and S. γ is said to be a* ***Markov specification*** *if $\gamma_\Lambda(\sigma_\Lambda = \xi|\cdot)$ is $\mathcal{F}_{\partial\Lambda}$- measurable for all $\xi \in E^\Lambda$ and $\Lambda \in \Im$.*

Clearly, each Gibbs specification for a nearest-neighbor potential is *Markovian.* Also, if γ is Markovian then each $\mu \in G(\gamma)$ is a **Markov field**, in that μ satisfies the local Markov property;

$$\mu(\sigma_\Lambda = \xi|\tau_\Lambda) = \mu(\sigma_\Lambda = \xi|\mathcal{F}_{\partial\Lambda}) \qquad \mu - a.s. \qquad (\xi \in E^\Lambda, \Lambda \in \Im).$$

Notation: For each bond $\{i,j\} \in B$ we let ij denote the associated oriented bond which points from i to j. The symbol $\overrightarrow{B}$ will stand for the set of all oriented bonds. Each site $k \in$S induces a splitting of $\overrightarrow{B}$into the sets,

$$\overrightarrow{B}^k = \{ij \in \overrightarrow{B} : d(k,i) = d(k,j) + 1\}$$

and

$$\overset{k\rightarrow}{B} = \{ij \in \overrightarrow{B} : d(k,j) = d(k,i) + 1\}$$

of oriented bonds that point towards k and away from k, respectively. Similarly, each oriented bond $ij \in \overrightarrow{B}$ defines a splitting of S into the "future interval"

$$]ij, \infty[= \{k \in S : ij \in \overrightarrow{B}^k\}$$

and the "past interval"

$$]-\infty, ij[=\{k \in S : ij \in \overset{k\rightarrow}{B}\}$$

Definition 4.2 *A probability measure μ on $(\Omega,\mathcal{F})$ will be called* ***Markov chain*** *if*

$$\mu(\sigma_j = y|\mathcal{F}_{]-\infty,ij[}) = \mu(\sigma_j = y|\mathcal{F}_{\{i\}}) \qquad \mu - a.s.$$

for all $ij\in\overrightarrow{B}$ and $y\in E$. Any stochastic matrix P_{ij} on E with

$$\mu(\sigma_j = y|\mathcal{F}_{\{i\}}) = P_{ij}(\sigma_i, y) \qquad \mu - a.s.$$

for all $y\in E$ will then be called a ***transition matrix*** *from i to j for μ. A Markov chain μ will be said to be* ***completely homogeneous*** *with transition matrix P if*

$$\mu(\sigma_j = y|\mathcal{F}_{\{i\}}) = P(\sigma_i, y) \qquad \mu - a.s.$$

for all $y\in E$ and all $ij\in\overrightarrow{B}$.

Comments: **(1)** Every Markov chain μ satisfies

$$\mu(A|\mathcal{F}_{]-\infty,ij[}) = \mu(A = y|\mathcal{F}_{\{i\}}) \qquad \mu - a.s.$$

for all $A\in\mathcal{F}_{]-\infty,ij[}$ and all $ij\in\overrightarrow{B}$.

(2) Let μ be a Markov chain with transition matrices $(P_{ij})_{ij\in\overrightarrow{B}}$, and let $\alpha_k = \sigma_k(\mu)$ be the marginal distribution of μ at $k \in S$. Then,

$$\mu(\sigma_\Lambda = \xi) = \alpha_k(\xi_k) \prod_{ij\in \overset{k\rightarrow}{B}:i,j\in\Lambda} P_{ij}(\xi_i, \xi_j) \tag{4.1}$$

for all connected sets $\Lambda \in \Im$ and all $\xi \in E^\Lambda$ and $k \in \Lambda$.

(3) Let μ be a Markov chain, and suppose that V is a copy of $\mathbb{Z}$ which is imbedded (as a graph) into S. Then the marginal distribution $\sigma_v(\mu)$ of μ on V is a Markov chain in the sense of Definition(3.2). This follows from equation (4.1).

(4) Let $(P_{ij})_{ij\in\overrightarrow{B}}$ be a family of stochastic matrices on E. $(P_{ij})_{ij\in\overrightarrow{B}}$ is a family of transition matrices for a Markov chain μ if and only if there exists a family $(\alpha_k)_{k\in S}$ of probability vectors on E such that

$$\alpha_i(x)P_{ij}(x,y) = \alpha_j(y)P_{ji}(y,x) \qquad (ij\in\overrightarrow{B}, x,y\in E) \tag{4.2}$$

This is because (4.2) is equivalent to the statement that the expression on the right of (4.1) is independent of the choice of $k\in\Lambda$ for all connected sets $\xi\in E^\Lambda$ and $\Lambda\in\Im$.

(5) Let P be a positive stochastic matrix on E. P is the transition matrix of a completely homogeneous Markov chain μ if and only if P is reversible, in that there exists a probability vector α on E such that

$$\alpha(x)P(x,y) = \alpha(y)P(y,x) \qquad (x,y\in E).$$

and in this case we have $\alpha = \sigma_k(\mu)$ for all $k\in S$.

(6) Every Markov chain μ is a Markov field. For let $\Lambda\in\Im$ be a connected set with $\Lambda\bigcup\partial\Lambda\subset\Delta$. Equation (4.1) shows that

$$\mu(\sigma_\Delta=\xi\omega\eta)\mu(\sigma_\Delta=\xi'\omega\eta') = \mu(\sigma_\Delta=\xi'\omega\eta)\mu(\sigma_\Delta=\xi\omega\eta')$$

for all $\xi,\xi'\in E^\Lambda$, $\omega\in E^{\partial\Lambda}$and $\eta,\eta'\in E^{\Delta\setminus(\Lambda\bigcup\partial\Lambda)}$. Summing over ξ' and η' we obtain

$$\mu(\sigma_\Delta=\xi\omega\eta)\mu(\sigma_{\partial\Lambda}=\omega) = \mu(\sigma_{\Delta\setminus\Lambda}=\omega\eta)\mu(\sigma_{\Lambda\bigcup\partial\Lambda}=\xi\omega).$$

So, if $\mu(\sigma_{\Delta\setminus\Lambda}=\omega\eta)>0$ then

$$\mu(\sigma_\Lambda=\xi\mid\sigma_{\partial\Lambda}=\omega) = \mu(\sigma_\Lambda=\xi|\sigma_{\Delta\setminus\Lambda}=\omega\eta),$$

and this means that

$$\mu(\sigma_\Lambda = \xi \mid \mathcal{F}_{\partial\Lambda}) = \mu(\sigma_\Lambda = \xi | \mathcal{F}_{\Delta\setminus\Lambda}) \qquad \mu - a.s..$$

Since τ_Λ is generated by the union of all these $\mathcal{F}_{\Delta\setminus\Lambda}$'s, we conclude that

$$\mu(\sigma_\Lambda = \xi \mid \mathcal{F}_{\partial\Lambda}) = \mu(\sigma_\Lambda = \xi | \tau_\Lambda) \qquad \mu - a.s..$$

Hence μ is a Markov field.

Theorem 4.3 *[9] Let γ be a Markov specification. Then each $\mu \in exG(\gamma)$ is a Markov chain.*

We now work towards obtaining characterization of the Markov chains in $G(\gamma)$. For simplicity we shall only consider positive Markov specifications. A positive specification γ is Markovian if and only if $\gamma = \gamma^\phi$ for some nearest-neighbor potential ϕ , [9]. Setting

$$Q_b(\xi) = \exp[-\phi_b(\xi) - |\partial i|^{-1}\phi_{\{i\}}(\xi_i) - |\partial j|^{-1}\phi_{\{j\}}(\xi_j)]$$

when $b = \{i, j\} \in B$ and $\xi \in E^b$, we see that each positive Markov specification γ can be written in the form

$$\gamma_\Lambda(\sigma_\Lambda = \omega_\Lambda | \omega) = Z_\Lambda(\omega^{-1}) \prod_{b \bigcap \Lambda \neq \emptyset} Q_b(\omega_b) \tag{4.3}$$

where $\Lambda \in \Im, \omega \in \Omega$, and $Z_\Lambda(\omega)$ is a normalizing constant. It will often be convenient to think of Q_b as a transfer matrix along the bond b. To emphasize this aspect we introduce a family $\{Q_{ij} : ij \in \overrightarrow{B}\}$ of positive matrices by writing

$$Q_{ij}(x, y) = Q_{ji}(y, x) = Q_b(\xi) \tag{4.4}$$

whenever $b = \{i, j\} \in B,\ \xi \in E^b$, and $x = \xi_i,\ y = \xi_j$.

Definition 4.4 *A family* $\{\ell_{ij} : ij \in \overrightarrow{B}\}$ *of (row) vectors* $\ell_{ij} \in]0, \infty[^E$ *will be called a* ***boundary law*** *for* $\{Q_{ij} : ij \in \overrightarrow{B}\}$ *(or* γ*) if for each* $ij \in \overrightarrow{B}$ *there is a number* $c_{ij} > 0$ *such that*

$$\ell_{ij}(x) = c_{ij} \prod_{k \in \partial i \setminus \{j\}} \ell_{ki} Q_{ki}(x) \qquad \textit{for all} \qquad x \in E$$

Theorem 4.5 *Consider a Markov specification* γ *of the form (4.3) where* $\Lambda \in \Im$, $\omega \in \Omega$, *and* $Z_\Lambda(\omega)$ *is a normalizing constant, and let* $\{Q_{ij} : ij \in \overrightarrow{B}\}$ *be the associated family of transfer matrices.*

(a) *Each boundary law* $\{\ell_{ij} : ij \in \overrightarrow{B}\}$ *for* $\{Q_{ij} : ij \in \overrightarrow{B}\}$ *defines a unique Markov chain* $\mu \in G(\gamma)$ *via the equation.*

$$\mu(\sigma_{\Lambda \bigcap \partial\Lambda} = \xi) = z_\Lambda \prod_{k \in \partial\Lambda} \ell_{kk_\Lambda}(\xi_k) \prod_{b \bigcap \Lambda \neq \emptyset} Q_b(\xi_b) \tag{4.5}$$

Here $\Lambda \in \Im$ *is any connected set,* $\xi \in E^{\Lambda \cap \partial\Lambda}$, *and* $z_\Lambda > 0$ *a suitable normalizing constant.*

(b) *Each Markov chain* $\mu \in G(\gamma)$ *admits a representation of the form (4.5) in terms of boundary law* $\{\ell_{ij} : ij \in \overrightarrow{B}\}$ *which is unique in the sense that each* ℓ_{ij} *is unique up to positive factor.*

Proof : (a) Let us first show that the expressions on the right of (4.5) are consistent. That is whenever $\Lambda, \Delta \in \Im$ are connected sets with $\Lambda \subset \Delta$, $V = (\Delta \cup \partial\Delta) \setminus (\Lambda \cup \partial\Delta)$, and $\xi_{\Lambda \cup \partial\Delta} \in E^{\Lambda \cup \partial\Delta}$ we have

$$\sum_{\xi_\nu \in E^\nu} z_\Delta \prod_{k \in \partial\Delta} \ell_{kk_\Delta}(\xi_k) \prod_{b \cap \Delta \neq \emptyset} Q_b(\xi_b) = z_\Lambda \prod_{k \in \partial\Lambda} \ell_{kk_\Lambda}(\xi_k) \prod_{b \cap \Lambda \neq \emptyset} Q_b(\xi_b) \tag{4.6}$$

It is enough to check this consistency when $\Delta = \Lambda \cup \{i\}$ for some $i \in \partial\Lambda$. Taking $j = i_\Lambda$, we get $V = \partial i \setminus \{j\}$, and the expression on the left side of

(4.6) is equal to

$$z_\Delta \prod_{k\in V} \ell_{kj} Q_{kj}(\xi_j) \prod_{k\in\partial\Lambda\setminus\{i\}} \ell_{kk_\Lambda}(\xi_k) \prod_{b\cap\Lambda\neq\emptyset} Q_b(\xi_b).$$

Since $\{\ell_{ij} : ij \in \overrightarrow{B}\}$ is given as a boundary law, the above expression coincides with the right side of (4.6) up to a factor $z_\Delta/c_{ij}z_\Lambda$. We can see that this factor is 1 by summing over $\xi_{\Lambda\cup\partial\Delta 0}$. This founds (4.6).

Equation (4.5) defines a unique finitely additive measure on the algebra of cylinder events, and thereby a unique probability measure μ on $(\Omega, \mathcal{F})$, as a consequence of (4.6). By definition μ is positive on cylinder events.

Now we should show that μ is a Markov chain, to show this fact we fix any $ij \in \overrightarrow{B}$, $x, y \in E$ and $\omega \in \Omega$, and we let $\Lambda \in \Im$ be a connected set with $i \in \Lambda \subset]-\infty, ij[$. We set $\Delta = \Lambda \cup \partial\Lambda \setminus \{j\}$. Equation (4.5) shows that

$$\mu(\sigma_j = x|\sigma_\Delta = \omega_\Delta)/\mu(\sigma_j = y|\sigma_\Delta = \omega_\Delta) = \ell_{ji}(x)Q_{ji}(x,\omega_i)/\ell_{ji}(y)Q_{ji}(y,\omega_i).$$

We obtain

$$\mu(\sigma_j = y|\sigma_\Delta = \omega_\Delta) = \ell_{ji}(y)Q_{ji}(y,\omega_i)/\ell_{ji}Q_{ji}(\omega_i).$$

by summing over $x \in E$.

The expression on the right depends on ω via ω_i only. We conclude that

$$\mu(\sigma_j = y|\mathcal{F}_{]-\infty,ij[}) = \mu(\sigma_j = y|\mathcal{F}_{\{i\}}) \qquad \mu - a.s.$$

Then we should prove that $\mu \in G(\gamma)$. Let $\Lambda \in \Im$ be given. Take any configurations $\xi, \omega \in \Omega$ with $\xi_{S/\Lambda} = \omega_{S/\Lambda}$. Let $\Delta \in \Im$ be an arbitrary connected set with $\Lambda \in \Delta$. Then we can write

$$\mu(\sigma_\Lambda \;=\; \xi_\Lambda|\sigma_{(\Delta\cup\partial\Delta)\setminus\Lambda} = \omega_{(\Delta\cup\partial\Delta)\setminus\Lambda})/\mu(\sigma_\Lambda = \omega_\Lambda|\sigma_{(\Delta\cup\partial\Delta)\setminus\Lambda} = \omega_{(\Delta\cup\partial\Delta)\setminus\Lambda})$$

$$
\begin{aligned}
&= \mu(\sigma_{\Delta\cup\partial\Delta} = \xi_{\Delta\cup\partial\Delta})/\mu(\sigma_{\Delta\cup\partial\Delta} = \omega_{\Delta\cup\partial\Delta}) \\
&= \prod_{b\cap\Delta\neq\emptyset} Q_b(\xi_b)/Q_b(\omega_b) \\
&= \prod_{b\cap\Lambda\neq\emptyset} Q_b(\xi_b)/Q_b(\omega_b) \\
&= \gamma_\Lambda(\sigma_\Lambda = \xi_\Lambda|\omega)/\gamma_\Lambda(\sigma_\Lambda = \omega_\Lambda|\omega)
\end{aligned}
$$

by using (4.5) and (4.3). Then summing over $\xi_\Lambda \in E^\Lambda$ we see that $\mu \in G(\gamma)$.

(b) We fix any Markov chain $\mu \in G(\gamma)$ to prove part (b). Since γ is positive μ is positive on cylinder events. For $ij \in \overrightarrow{B}$ and $x, y \in E$ we can define $P_{ij}(x,y) = \mu(\sigma_j = y|\sigma_i = x)$. Let $\Lambda \in \Im$ be connected, $\xi \in \Omega$, and $a \in E$ be any fixed reference state. Then for

$$
A = \{\sigma_\Lambda \equiv a\}, \quad B = \{\sigma_{\partial\Lambda} = \xi_{\partial\Lambda}\}, \quad C = \{\sigma_\Lambda = \xi_\Lambda\},
$$

we have

$$
\mu(\sigma_{\Delta\cup\partial\Delta} = \xi_{\Delta\cup\partial\Delta}) = \mu(A)\mu(B \setminus A)\mu(C \setminus B)/\mu(A \setminus B).
$$

Then

$$
\mu(B \setminus A) = \prod_{k\in\partial\Lambda} P_{k_\Lambda k}(a, \xi_k)
$$

by equation (4.1).

Therefore by using (4.3)

$$
\begin{aligned}
\mu(C \setminus B)/\mu(A \setminus B) &= \gamma_\Lambda(C|\xi)/\gamma_\Lambda(A|\xi) \\
&= \prod_{b\cap\Lambda\neq\emptyset} Q_b(\xi_b)/\prod_{b\subset\Lambda} Q_b(aa) \prod_{k\in\partial\Lambda} Q_{k_\Lambda k}(a, \xi_k).
\end{aligned}
$$

We conclude that, equation (4.5) holds with

$$
z_\Lambda = \mu(\sigma_\Lambda \equiv a)/\prod_{b\subset\Lambda} Q_b(aa)
$$

and

$$\ell_{ij}(x) = P_{ji}(a,x)/Q_{ji}(a,x) \qquad (ij \in \overrightarrow{B}, x \in E).$$

Replacing Λ by $\Delta = \Lambda \cup \{i\}$ in (4.5) and comparing the resulting equation with this equation we can obtain equation (4.6). But equation (4.6) implies that $\{\ell_{ij} : ij \in \overrightarrow{B}\}$ is a boundary law, this completes the first part of this proof.

Next, we should prove the uniqueness of ℓ_{ij} up to a factor we assume normalizing constants $z'_\Lambda > 0$ and that μ admits a second representation of the form (4.5) with a boundary law $\{\ell'_{ij} : ij \in \overrightarrow{B}\}$. We obtain the equation

$$\ell'_{ji}(x)/\ell_{ji}(x) = (z\{i\}/z\{i\}') \prod_{k \in \partial i \setminus \{j\}} \ell_{ki}(a)/\ell'_{ki}(a).$$

by applying (4.5) to a singleton $\Lambda = \{i\}$ and a configuration ξ with $\xi_j = x$ for some $j \in \partial i$ and $\xi_k = a$ for all $k \in \partial i \setminus \{j\}$. It follows that ℓ'_{ji} is a positive multiple of ℓ_{ji}. This completes the proof of Theorem (3.5).

Chapter 5

Absence of Phase Transitions in the Long-Range One-Dimensional Antiferromagnetic Models with Random External Field.

Let us consider a classical lattice model of statistical mechanics on a one-dimensional lattice such that the spin variable $\sigma(x)$ at each point x taking the values 0 and 1. The interaction is specified by means of the Hamiltonian:

$$\mathbf{H}(\sigma(x)) = \sum_{x,y\in\mathbf{Z}^1;x>y} U(x-y)\sigma(x)\sigma(y) - \mu \sum_{x\in\mathbf{Z}^1} \sigma(x) \tag{5.1}$$

where μ is the external field. The following conditions are imposed on the potential $U(x)$:

1) $U(x) > 0$ at $x \in \mathbf{Z}^1, x > 0$

2) $\sum\limits_{x \in \mathbf{Z}^1} U(x) < \infty$

3) $U(x+y) + U(x-y) > 2U(x);\ x, y \in \mathbf{Z}^1,\ x > y$

4) The function $U(x)$ can be extended to a twice continuously differentiable function such that $U(x) \sim Ax^{-\gamma},\ U' \sim -A\gamma x^{-\gamma-1},\ U''(x) \sim A\gamma(\gamma+1)\ x^{-\gamma-2}$ at $x \to \infty$; where $\gamma > 1$, and A is a strong positive constant.

(5.1) is antiferromagnetic because of the first condition. For the existence of the thermodynamic limit the second condition is essential. For all further evaluations, the third condition on the convexity of the interaction function $U(x)$ is necessary . The last condition determines a character of potentials decrease at the infinity.

The condition $\sum_{x \in \mathbf{Z}^1, x>0} xU(x) < \infty$ automatically implies the uniqueness of the Gibbs states [5], [6], [18]. Therefore, we investigate the problem of the phase transitions in the model (5.1) for potentials $U(x) \sim Ax^{-\gamma}$, where $\gamma = 1 + \alpha,\ 0 < \alpha < 1$. The ferromagnetic case (when the potential $U(x)$ is negative) was considered by F.Dyson [7], [8]. He considered a model with the following potential (the external field is absent, spin variable $\sigma(x)$ takes the values $(+1, -1)$:

1) $U(x) < 0$

2) $\sum\limits_{x \in \mathbf{Z}^1} |U(x)| < \infty$

3) $U(x+1) > U(x)$

4) $\sum\limits_{x \in \mathbf{Z}^1, x>0} \ln\ln(x+4)^{-1} x^3 U(x)^{-1} < \infty.$

Note that all potentials decreasing as $x^{-1-\alpha},\ 0 < \alpha < 1$ certainly satisfy the stated above conditions.

In the ferromagnetic case we can find β_1 such that if $\beta > \beta_1$ then there exist at least two extremal Gibbs states P^+ and P^- corresponding to the ground states $\sigma(x) = +1$ and $\sigma(x) = -1$. This very important result which is established by F.Dyson is connected with the following fact. Let us consider the boundary conditions $\overline{\sigma}(x) = 1$, the segment $[-n, n]$ and the configuration $\sigma_{-1}(x)$ such that

$$\sigma_{-1}(x) = \begin{cases} -1 & \text{if } x \in [-n, n] \\ 1 & \text{if } x \in Z^1 - [-n.n] \end{cases}$$

Then the difference between the energies of the configurations $\sigma_{-1}(x)$ and $\overline{\sigma}(x)$ has the order $n^{1-\alpha}$. In other words, in the one dimensional case there exists an analog of the notion of the surface tension and this fact leads to the existence of two extremal Gibbs states.

In the antiferromagnetic case, it will be shown that at arbitrary fixed boundary conditions $\overline{\sigma}(x),\ x \in \mathbf{Z}^1 - [-n, n]$ a configuration $\sigma(x),\ x \in [-n, n]$ with maximal weight (or with minimal energy) almost does not differ from the special ground state with the exception of some bounded zone and is stable in the sense of Peierls. This fact has a certain importance in establishing of all further results.

Let Φ^{per} denote the set of all periodic configurations. For every $\sigma \in \Phi^{per}$, we define

$$q = \sum_{y=x+1}^{x+p} \sigma(x)/p,$$

where p is the period of σ, and q does not depend on x. Therefore, the density of each periodic configuration is $\kappa = q/p$. It is more convenient to work with the reciprocal of the density, $\eta(\sigma(x)) = p/q$, which represents the average distance between neighboring points at which $\sigma(x) = 1$. For every configuration $\sigma \in \Phi^{per}$ we define the mean energy $h(\sigma)$ which is independent of x as follows:

$$h(\sigma(x)) = \frac{1}{p} \sum_{y=x+1}^{x+p} \sigma(x) \sum_{z>0} U(z)\sigma(y+z)$$

Let the fixed positive rational number be p/q.

Definition 5.1 *[3]. A configuration $\sigma_0(x) \in \Phi^{per}$ with $\eta(\sigma_0(x)) = p/q$ is called a* **special ground state** *if*

$$h(\sigma(x)) = \inf_{\sigma \in \Phi^{per}, \eta(\sigma)=p/q} h(\sigma)$$

The following proposition readily follows from the convexity of the potential $U(x)$.

Hubbard's criterion [3], [10]. Let $\sigma \in \Phi^{per}$ and $r_i(x;\sigma)$ denotes the distance between $x \in \mathbf{Z}^1$ and i-th particle on the right. If for each x and i

$$[i\eta] \leq r_i(x;\sigma) \leq [i\eta] + 1,$$

(the square brackets denote the integer part of the enclosed number) then σ is a special ground state.

The existence of the configuration satisfying the Hubbard's criterion (the special ground state) is proved in [3]. The remarkable short formula for the special ground states was offered by S.Aubry [4]. Here we give the construction of the special ground state for each fixed rational value of the density κ [3]. Special ground states for irrational densities are investigated in [4],[13].

Every rational number p/q can be uniquely decomposed into a finite continued fraction. We write it as follows :

$p/q = [n_0, n_1, ..., n_s]$, this meaning that:

$$n_0 + \cfrac{1}{n_1 + \frac{1}{n_2 + ... + \frac{1}{n_s}}}$$

The ground state for a configuration with $\kappa = [n_0, n_1, ..., n_s]$ will be constructed by induction.

1. $\kappa = n_0 \geq 1$, n_1 is an integer. It is then obvious that a periodic configuration with equally distant x at which $\sigma(x) = 1$ satisfies Hubbard's criterion, i.e. is a special ground state. In this case $r_i(x;\sigma) = in_0, i > 0$.

2. $\kappa = n_0 + 1/n_1$, where n_0 and n_1 are integers, $n_0 \geq 1, n_1 > 1$. Then the $(n_0 n_1 + 1)-$ periodic configuration

$$\underbrace{0...01}_{n_0+1}\underbrace{\underbrace{0...01}_{n_0}...\underbrace{0...01}_{n_0}}_{n_1-1 times}$$

also satisfies Hubbard's criterion and is a special ground state.

3. $\kappa = [n_0, n_1, ..., n_s]$, where $n_0, n_1, ..., n_s$ are integers, $n_0, n_1, ..., n_s \geq 1$. For $s = 0$ and $s = 1$ the required configurations are already constructed. Suppose we have already constructed a ground state with $s = m$ and $\kappa = [n_0, n_1, ..., n_m]$. Then the following configuration with $s = m + 1$ and $\kappa = [n_0, n_1, ..., n_{m+1}]$ is constructed as:

$$\sigma(n_0, ..., n_{m+1}) = \sigma(n_0, ..., n_{m-1}) \underbrace{\sigma(n_0, ..., n_m)...\sigma(n_0, ..., n_m)}_{n_{m+1} times}$$

Here, $\sigma(n_0, ..., n_j)$, $j = m - 1, m, m + 1$, are the blocks from which the ground states for $\kappa = [n_0, ..., n_j]$ are obtained by periodic continuations.

It can be verified [3] that the constructed configuration satisfies Hubbard's criterion and therefore is a special ground state for $\kappa = [n_0, n_1, ..., n_m, n_{m+1}]$.

Hubbard's criterion allows us to extract an explicit expression for the mean energy of a special ground state [3]:

$$h_\kappa = \kappa \sum_{i=1}^{\infty} U(m_i)\pi_i + U(m_i+1)(1-\pi_i), \tag{5.2}$$

where $m_i = [i\eta],\ \pi_i = 1 + m_i - i\eta$.

By this formula we can see that the function of mean energy is continuous on the set of all rationals and can be extended to a continuous function defined on whole segment $[0, +\infty)$.

Below we formulate theorems describing phase diagrams of our model.

Theorem 5.2 *: [3], [1]*

1) The function h_κ is convex.

2) In each rational point the function h_κ has a left-hand derivative ${\mu_\kappa}^-$ and a right-hand derivative ${\mu_\kappa}^+$, with ${\mu_\kappa}^+ > {\mu_\kappa}^-$.

3) The Lebesgue measure of the complement of the set $\cup_\kappa({\mu_\kappa}^-, {\mu_\kappa}^+)$ in the real line $\mathbf{R}$ is zero.

Theorem 5.3 *: [12]. Suppose that the value of the external field μ of the model (5.1) belongs to the interval $(\mu_\kappa^-\ ,\ \mu_\kappa^+)$ for some number $\kappa = q/p$. Then the special ground state of the model (5.1) is unique within to translation.*

Theorem 5.4 *: Suppose that the value of the external field μ of the model (5.1) belongs to the interval $(\mu_\kappa^-, \mu_\kappa^+)$ for some number $\kappa = q/p$.*

Then the model (5.1) has a unique Gibbs state at all sufficiently small values of the temperature ($\beta^{-1} < const(\mu, U(x))$.

Theorem 5.5 *: Suppose that the value of the external field μ of the model (5.1) belongs to the interval $(\mu_\kappa^-, \mu_\kappa^+)$ for some number $\kappa = q/p$.*

Then there exists constant $c(\mu)$ such that for any $|\mu^{'}| < c$

$$H = H(\sigma(x)) + \mu^{'} \sum_{x \in Z^1}^{k} \sigma(x) \tag{5.3}$$

has a unique limiting Gibbs states of $\beta > \beta_{cr}$ for some β_{cr}.

Proof Let us choose

$$c = \min(|\mu - \mu_k^-|,\ |\mu - \mu_k^+|)\ .$$

Now if

$$-c < \mu^{'} < c\ ,$$

then the external field $\mu + \mu^{'}$ will satisfy

$$\mu_k^- < \mu + \mu^{'} < \mu_k^+$$

and ground states of the model (5.3) will be defined as in [3], and they will be unique up to translations due to [11], we can define contours and interface contours as in [13] and it can be easily seen that contours will satisfy Peierls condition.

Let P_1 and P_2 be two extreme limiting Gibbs states of (5.3) corresponding to the boundary conditions σ_1 and σ_2 . Then by using of Peierls estimation and methods of [13] it can be easily shown that we can find a value β_{cr} such that P_1 and P_2 are absolutely continuous with respect to each other for all $\beta > \beta_{cr}$. Since any two Extreme Limiting Gibbs Measures are singular or coincide [8], we can conclude that P_1 and P_2 coincide. Therefore, Theorem (5.5) is proved.

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