PROCEEDINGS OF SPIE

SPIEDigitalLibrary.org/conference-proceedings-of-spie

# Fast and accurate algorithms for quadratic phase integrals in optics and signal processing

Aykut Koç
Haldun M. Ozaktas
Lambertus Hesselink

SPIE.

# Fast and accurate algorithms for quadratic phase integrals in optics and signal processing

Aykut Koç,[a] Haldun M. Ozaktas[b], and Lambertus Hesselink[a]

[a]Stanford University, Electrical Engineering Department, Stanford, CA
[b]Bilkent University, Electrical Engineering Department, Bilkent, Ankara, Turkey

## ABSTRACT

The class of two-dimensional non-separable linear canonical transforms is the most general family of linear canonical transforms, which are important in both signal/image processing and optics. Application areas include noise filtering, image encryption, design and analysis of ABCD systems, etc. To facilitate these applications, one need to obtain a digital computation method and a fast algorithm to calculate the input-output relationships of these transforms. We derive an algorithm of NlogN time, N being the space-bandwidth product. The algorithm controls the space-bandwidth products, to achieve information theoretically sufficient, but not redundant, sampling required for the reconstruction of the underlying continuous functions.

**Keywords:** Linear Canonical Transforms, Quadratic-Phase Systems, ABCD optics, transforms, fast algorithms

## 1. INTRODUCTION

A quadratic-phase system (QPS) is a unitary system, with parameter matrix $\mathbf{M}$, whose output $f_{\mathbf{M}}(u)$ is related to its input $f(u)$ through a quadratic-phase integral:

$$f_{\mathbf{M}}(u) = \sqrt{\beta}\, e^{-j\pi/4} \int_{-\infty}^{\infty} \exp\left[i\pi(\alpha u^2 - 2\beta u u' + \gamma u'^2)\right] f(u')\, du', \tag{1}$$

where $\alpha$, $\beta$, $\gamma$ are real parameters. This relationship is also known under other names including linear canonical transforms and ABCD-systems.[1–3]

QPSs are identical to the Linear Canonical Transforms (LCTs). LCT is the name given to the same input-output relationships used in signal processing. More importantly, the ABCD systems widely used in optics,,[4] is also represented by linear canonical transforms or quadratic-phase systems.

There are four main classes of LCTs: one dimensional LCTs (1D-LCTs), two dimensional separable LCTs (2D-S-LCTs), two-dimensional non-separable LCTs (2D-NS-LCTs) and complex LCTs (CLCTs).

The class of 1D-LCTs[1,2] is a three-parameter class of linear integral transformations[5] which includes among its many special cases, the one-parameter subclasses of fractional Fourier transforms (FRTs), scaling operations, and chirp multiplication (CM) and chirp convolution (CC) operations, the latter also known as Fresnel transforms.

The class of two-dimensional non-separable linear canonical transforms (2D-NS-LCTs) is the class of linear integral transforms[5] that includes among its several special cases non-separable two-dimensional fractional Fourier transforms (2D-NS-FRTs),[6] two-dimensional versions of chirp multiplication (2D-CM) and chirp convolution (2D-CC) operations, the two-dimensional Fourier transform (2D-FT), and generalized astigmatic scaling (magnification) operations, as well as their separable special cases. The class of non-separable transforms is significantly more general than 2D separable linear canonical transforms (2D-S-LCTs) since it can represent a wide variety of anamorphic/astigmatic/nonorthogonal systems as well.

Two-dimensional *separable* LCTs or symmetrical transforms that do not include the general non-separable case are addressed in .[1,2] The most special case possible are the isotropic 2D-LCTs in which the system is

Further author information: (Send correspondence to Aykut Koç)
Aykut Koç: E-mail: aykutkoc@stanford.edu

Three-Dimensional Imaging, Visualization, and Display 2011, edited by Bahram Javidi, Jung-Young Son,
Proc. of SPIE Vol. 8043, 804304 · © 2011 SPIE · CCC code: 0277-786X/11/$18 · doi: 10.1117/12.884676

fully symmetric, orthogonal and the parameters for both of the dimensions are identical. This case can be represented by only three parameters as in a 1D-LCT.[7] When the system is still orthogonal but the parameters for the orthogonal dimensions differ, the system becomes a 2D-S-LCT, which is represented by six parameters.[7] The separable 2D transforms do not pose much difficulty because the separable transform is essentially two independent one-dimensional transforms along the two dimensions and the dimensions can be treated independently. However, the *non-separable* transform (2D-NS-LCT) is the most general case of this class of integrals where the two dimensions are coupled to each other by four additional cross-parameters, increasing the total number of parameters to ten. This general case is non-separable, non-axially symmetric, non-orthogonal, and anamorphic/astigmatic.[7]

Finally, Bilateral Laplace transforms, Bargmann transforms, Gauss-Weierstrass transforms,,[1] fractional Laplace transforms,,[8] and complex-ordered fractional Fourier transforms[9] are all special cases of complex linear canonical transforms (CLCTs).

These integral transforms are of great importance in electromagnetic, acoustic, and other wave propagation problems since they represent the solution of the wave equation under a variety of circumstances. At optical frequencies, LCTs can model a broad class of optical systems including thin lenses, sections of free space in the Fresnel approximation, sections of quadratic graded-index media, and arbitrary concatenations of any number of these. The efficient and accurate digital computation of LCTs is also of importance in many areas of optical signal processing and general digital image processing. Therefore, given its ubiquitous nature and numerous applications, the discrezitation, sampling and the fast/efficient digital computation of LCTs is of considerable interest. Their fast and accurate digital computation is of vital importance to utilize these tools in applications in a digital domain. Many works have addressed the problem of sampling of real and continuous LCTs and some computation issues, using both decomposition-based and discrete-LCT-based methods.[10–14]

## 2. FAST ALGORITHMS

The proposed algorithm uses matrix factorizations to decompose LCTs into cascade combinations of the elementary LCT blocks discussed above. Since each stage can be computed in $O(N \log N)$ time, the overall LCT can also be. Numerous such decompositions are possible,[2] but they are not equally suited for numerical purposes. For instance, direct naive application of the decomposition of chirp multiplication, Fourier transformation, scaling (magnification), and again chirp multiplication, which suggests itself upon inspection of Eq. (1) will in general lead to very high sampling rates. This is because the early appearance of the chirp multiplication in the cascade and the lack of sampling rate management that controls the first chirp multiplication. If one directly uses this decomposition for every parameter set, there may be two large increases in the number of samples because of the two chirp multiplications. These combined increases in the number of samples results in unnecessary high sampling rates.

### 2.1 1D Algorithm

We have carried out a systematic exhaustive analysis of all possible decompositions of arbitrary 1D-LCTs into the three basic operations of scaling, chirp multiplication (CM), and Fourier transformation (FT). We have considered all possible decompositions with three, four, and five cascade blocks. Every permutation has been checked to see if that decomposition is capable of expressing an LCT with arbitrary parameters. We have also considered the required sampling rates for each decomposition and pick the ones that require the least possible number of samples.

The algorithm can now be outlined as follows:

- If $|\gamma| \leq 1$, use the decomposition:

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \gamma/\beta^2 & 1 \end{bmatrix} \begin{bmatrix} \beta & 0 \\ 0 & 1/\beta \end{bmatrix}.$$

In operator notation:

$$\mathcal{C}_{\mathbf{M}} = \mathcal{Q}_{-\alpha}\, J_{k/2}\, \mathcal{F}_{\mathrm{lc}}\, \mathcal{Q}_{-\gamma/\beta^2}\, J_2\, \mathcal{M}_{\beta}, \tag{2}$$

where $J_x$ represents the $\times x$ oversampling operation. The minimum value of $k$ is:

$$k \geq 1 + |\gamma| + \frac{|\alpha|(1+|\gamma|)^2}{\beta^2}. \tag{3}$$

- If $|\gamma| > 1$, use the decomposition:

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ \alpha - \beta^2/\gamma & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\gamma/\beta^2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -\gamma/\beta & 0 \\ 0 & -\beta/\gamma \end{bmatrix}.$$

  In operator notation:

$$\mathcal{C}_{\mathbf{M}} = \mathcal{Q}_{-\alpha+\beta^2/\gamma}\, J_{k/2}\, \mathcal{F}_{\mathrm{lc}}\, \mathcal{Q}_{\gamma/\beta^2}\, J_2\, \mathcal{F}_{\mathrm{lc}}\, \mathcal{M}_{-\gamma/\beta}, \tag{4}$$

  The minimum value of $k$ is:

$$k \geq 1 + \frac{1}{|\gamma|} + \frac{(1+|\gamma|)^2}{\beta^2}|\alpha - \frac{\beta^2}{\gamma}| \tag{5}$$

We have chosen to avoid unnecessary increases in the time-bandwidth product in the early stages to avoid increasing the number of samples until the last CM stage, where the major and unavoidable increase in sampling rate occurs.

### 2.2 2D Algorithm

For the development of 2D Algorithm, we use the 2D version of the Iwasawa-type decomposition to derive our efficient algorithm. In this decomposition, the first operation is an orthosymplectic system, followed by a scaling (magnification) system, finally followed by a two-dimensional chirp multiplication (2D-CM). (Note that each of the stages of the algorithm are special cases of 2D-NS-LCTs.) The Iwasawa decomposition is:

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{G} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{S^{-1}} \end{bmatrix} \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ -\mathbf{Y} & \mathbf{X} \end{bmatrix} \tag{6}$$

where

$$\mathbf{G} = -(\mathbf{C}\mathbf{A}^{\mathrm{T}} + \mathbf{D}\mathbf{B}^{\mathrm{T}})(\mathbf{A}\mathbf{A}^{\mathrm{T}} + \mathbf{B}\mathbf{B}^{\mathrm{T}})^{\mathbf{-1}} \tag{7}$$

$$\mathbf{S} = (\mathbf{A}\mathbf{A}^{\mathrm{T}} + \mathbf{B}\mathbf{B}^{\mathrm{T}})^{\mathbf{1/2}} \tag{8}$$

$$\mathbf{X} = (\mathbf{A}\mathbf{A}^{\mathrm{T}} + \mathbf{B}\mathbf{B}^{\mathrm{T}})^{\mathbf{-1/2}}\mathbf{A} \tag{9}$$

$$\mathbf{Y} = (\mathbf{A}\mathbf{A}^{\mathrm{T}} + \mathbf{B}\mathbf{B}^{\mathrm{T}})^{\mathbf{-1/2}}\mathbf{B} \tag{10}$$

Given the $4 \times 4$ matrix $\mathbf{M}$, we can determine $2 \times 2$ matrices $\mathbf{G}$, $\mathbf{S}$, $\mathbf{X}$, $\mathbf{Y}$ by using Eqs. 7, 8, 9, 10. We begin with the first and the most sophisticated stage of the decomposition, the orthosymplectic system. This stage of the decomposition can be further decomposed into a two-dimensional *separable* fractional Fourier transform (2D-S-FRT) that is sandwiched between two coordinate rotators. The second stage is the scaling operation and it seems to be the simplest of the three stages. Computationally, such a scaling operation amounts to modifying the information that tells us which coordinates the samples belong to. Since it requires only the reinterpretation of the coordinates of the samples. The last stage of our main Iwasawa decomposition is the 2D-CM operation. The entire algorithm can be summarized stage by stage. The algorithm can be compactly stated in operator notation as follows:

$$\mathcal{C}_{\mathbf{M}} = \mathcal{Q}_{\mathbf{G}}\, K_G\, \mathcal{M}_{\mathbf{S}}\, K_S\, \mathcal{R}_{\mathrm{r}_2}\, \mathcal{F}_{\mathrm{a_x,a_y}}\, J\, \mathcal{R}_{\mathrm{r}_1} \tag{11}$$

where the operators $\mathcal{Q}_{\mathbf{G}}$, $\mathcal{M}_{\mathbf{S}}$, $\mathcal{R}_{\mathrm{r}_2}$, $\mathcal{F}_{\mathrm{a_x,a_y}}$, $\mathcal{R}_{\mathrm{r}_1}$ respectively represent: 2D-CM with parameter matrix $\mathbf{G}$, 2D scaling with parameter matrix $\mathbf{S}$, coordinate rotation with angle $\mathrm{r}_2$, 2D-S-FRT with orders $\mathrm{a_x}$ and $\mathrm{a_y}$, and coordinate rotation with angle $\mathrm{r}_1$. $J$ stands for a simple interpolation without oversampling that is performed to obtain the function on a regular rectangular grid from the rotated samples. $K_S$ and $K_G$ stand for the interpolation operations before the scaling and chirp multiplication operations, respectively.

### 2.3 Complex LCT Algorithm

Extension of RLCTs to complex linear canonical transforms (CLCTs) is rather involved.[1] The extension is very briefly summarized as follows. When we let the entries of the unimodular transform matrices be complex numbers, we obtain the unit determinant matrices

$$\mathbf{M}_{\mathrm{C}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{12}$$

where $a$,$b$,$c$,$d$ are complex parameters. We now show how given $ABCD$ matrices can be decomposed in a manner that leads to a fast algorithm for computation of CLCTs. In the most general case, the matrix $\mathbf{M}_{\mathrm{C}}$ is composed of the four complex parameters $a$,$b$,$c$,$d$, whose real and imaginary parts add up to a total of 8 parameters. These 8 parameters are restricted by the unimodularity condition on $\mathbf{M}_{\mathrm{C}}$, which requires the real part of the determinant to be 1 and the imaginary part to be 0. Because of these two equations, the total number of independent parameters of a general CLCT is 6. These 6 parameters correspond to the 6 parameters of the group $HSp(2,\mathbb{C})$, which is a 6 parameter semigroup of the complex symplectic group. The main decomposition which covers the general case is given as $Sp(2,\mathbb{C})$.

$$\begin{aligned}\mathbf{M}_{\mathrm{C}} = &\begin{bmatrix} 1 & 0 \\ -q_{3r} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -iq_{3c} & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -q_{2r} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -iq_{2c} & 1 \end{bmatrix} \\ &\times \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -q_{1r} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -iq_{1c} & 1 \end{bmatrix}.\end{aligned} \tag{13}$$

This decomposition consists of three imaginary CM and real CM pairs with Fourier/Inverse-Fourier transform operations in between. The imaginary CM and real CM pairs can also be viewed as complex CM (CCM) operations:

$$\begin{aligned}\mathbf{M}_{\mathrm{C}} = &\begin{bmatrix} 1 & 0 \\ -(q_{3r}+iq_{3c}) & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -(q_{2r}+iq_{2c}) & 1 \end{bmatrix} \\ &\times \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -(q_{1r}+iq_{1c}) & 1 \end{bmatrix}.\end{aligned} \tag{14}$$

The three matrices in the center can also be expressed as a CCC operation:

$$\mathbf{M}_{\mathrm{C}} = \begin{bmatrix} 1 & 0 \\ -(q_{3r}+iq_{3c}) & 1 \end{bmatrix} \begin{bmatrix} 1 & (q_{2r}+iq_{2c}) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -(q_{1r}+iq_{1c}) & 1 \end{bmatrix} \tag{15}$$

which is nothing but the complex version of the well-known CM-CC-CM decomposition.[2]

## 3. CONCLUSIONS

In this paper, fast and accurate algorithms for the computation of linear canonical transforms (LCTs) from the $N$ samples of the input signal in $O(N \log N)$ time are discussed. Our approach is based on concepts from signal analysis and processing rather than conventional numerical analysis. With careful consideration of sampling issues, $N$ can be chosen very close to the time-bandwidth product of the signals, and need not be much larger. The transform output may have a higher space-bandwidth product due to the nature of the transform family.

All algorithms relate the samples of the input function to the samples of the continuous LCT of this function in the same sense that the fast Fourier transform (FFT) implementation of the discrete Fourier transform (DFT) computes the samples of the continuous FT of a function. Since the sampling rates are carefully controlled, the output samples obtained are accurate approximations to the true ones and the continuous LCT can be recovered via interpolation of these samples. The only inevitable source of deviation from exactness arises from the fundamental fact that a signal and its transform cannot both be of finite extent. This is the same source of deviation encountered when using the DFT/FFT to compute the continuous FT. Thus the algorithms compute LCTs with a performance similar to the DFT/FFT in computing the Fourier transform, both in terms of speed and accuracy.

We have considered several examples to illustrate and compare the presented methods. Because of paper-length constraints, we only present here the results for the 1D algorithm. However, the corresponding tests to prove the accuracy of the other algorithms have also similar results. We consider the chirped pulse function $\exp(-\pi u^2 - i\pi u^2)$, denoted F1, and the trapezoidal function $1.5\text{tri}(u/3) - 0.5\text{tri}(u)$, denoted F2 ($\text{tri}(u) = \text{rect}(u) * \text{rect}(u)$). Since these two functions are well confined to a circle with diameter $\Delta u = 8$ we take $N = 8^2$. We also consider the binary sequence 01101010 occupying $[-8, 8]$ with each bit 2 units in length, so that $N = 16^2$. This binary sequence is denoted by F3. We consider two transforms, the first (T1) with parameters $(\alpha, \beta, \gamma) = (-3, -2, -1)$, and the second (T2) with parameters $(-4/5, 1, 2)$. The LCTs T1 and T2 of the functions F1, F2, F3, F4 have been computed both by the presented fast method (A) and by a highly inefficient brute force numerical approach based on composite Simpson's rule with extensive number of intervals that can handle highly oscillatory functions which is here taken as a reference. The results for all functions (F1, F2, F3, F4) and our LCT algorithm are tabulated in Table 1 for both transforms (T1, T2). Also shown are the errors that arise when using the DFT in approximating the FT of the same functions, which serves as a reference. (The error is defined as the energy of the difference normalized by the energy of the reference, expressed as a percentage.)

| | A T1 | A T2 | DFT |
|---|---|---|---|
| F1 | $\mathbf{3.2 \times 10^{-22}}$ | $\mathbf{9.5 \times 10^{-22}}$ | $\mathbf{2.0 \times 10^{-21}}$ |
| F2 | $\mathbf{7.8 \times 10^{-4}}$ | $\mathbf{8.1 \times 10^{-4}}$ | $\mathbf{6.2 \times 10^{-4}}$ |
| F3 | **1.5** | **1.6** | **1.2** |
| F4 | $\mathbf{9.7 \times 10^{-2}}$ | $\mathbf{11 \times 10^{-2}}$ | $\mathbf{8.3 \times 10^{-2}}$ |

Table 1. Percentage errors for different functions F, transforms T, and algorithms A.

## REFERENCES


[1] Wolf, K. B., [*Integral Transforms in Science and Engineering (Chapter 9: Construction and properties of canonical transforms)*], New York: Plenum Press (1979).

[2] Ozaktas, H. M., Zalevsky, Z., and Kutay, M. A., [*The Fractional Fourier Transform with Applications in Optics and Signal Processing*], New York: Wiley (2001).

[3] Abe, S. and Sheridan, J. T., "Optical operations on wavefunctions as the abelian subgroups of the special affine fourier transformation," *Opt. Lett.* **19**, 1801–1803 (1994).

[4] Hecht, E., [*Optics, 4th Ed.*], Addison Wesley (2001).

[5] Bastiaans, M. J., "The wigner distribution function applied to optical signals and systems," *Optics Communications* **25**(1), 26 – 30 (1978).

[6] Sahin, A., Kutay, M. A., and Ozaktas, H. M., "Nonseparable two-dimensional fractional fourier transform," *Appl. Opt.* **37**(23), 5444–5453 (1998).

[7] Alieva, T. and Bastiaans, M. J., "Properties of the canonical integral transformation," *J. Opt. Soc. Am. A* **24**, 3658–3665 (2007).

[8] Torre, A., "Linear and radial canonical transforms of fractional order," *J. Compt. and Appl. Math.* **153**, 477–486 (2003).

[9] Shih, C. C., "Optical interpretation of a complex-order fourier transform," *Opt. Lett.* **20**(10), 1178–1180 (1995).

[10] Hennelly, B. M. and Sheridan, J. T., "Fast numerical algorithm for the linear canonical transform," *J. Opt. Soc. Am. A* **22**, 928–937 (2005).

[11] Hennelly, B. M. and Sheridan, J. T., "Generalizing, optimizing, and inventing numerical algorithms for the fractional fourier, fresnel, and linear canonical transforms," *J. Opt. Soc. Am. A* **22**, 917–927 (2005).

[12] Ozaktas, H. M., Arıkan, O., Kutay, M. A., and Bozdağı, G., "Digital computation of the fractional fourier transform," *IEEE Transactions on Signal Processing* **44**, 2141–2150 (1996).

[13] Oktem, F. S. and Ozaktas, H. M., "Exact relation between continuous and discrete linear canonical transforms," *Signal Processing Letters, IEEE* **16**(8), 727 –730 (2009).

[14] Healy, J. J. and Sheridan, J. T., "Sampling and discretization of the linear canonical transform," *Signal Process.* **89**(4), 641–648 (2009).