Uncertainty and dissipation

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Abstract

This paper discusses the question of the energy confinement in mechanical structures in the light of the uncertainties affecting the natural frequencies of the system. More precisely, recent studies have shown that energy can be introduced to a linear system with near irreversibility, or energy within a system can migrate to a subsystem nearly irreversibly, even in the absence of dissipation, provided that the system has a particular natural frequency distribution. In this paper, the case of uncertainties in the system's natural frequencies is discussed and a remarkable statistical property of the natural frequency is derived for permanent energy confinement within a part of the system. The results demonstrate the existence of a special class of linear non-dissipative dynamic systems that exhibit nearly-irreversible energy confinement (IEC) if they satisfy a minimum-variance-response (MIVAR) property. In this case, if the probability density function of the natural frequencies has a special distribution, the conservative system shows an unexpected decaying impulse response.

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1 Average impulse response: population vs. single sample

In complex built-up structures, comprised of many individual structural components and modes, the spatial redistribution of vibratory energy throughout the structure, in many ways, seems to be similar to reduction of vibration due to classical dissipation mechanisms and "appears" as damping, but is distinct from dissipation of vibratory energy as heat. In fact, this phenomenon consists of a fast energy transfer from the part of the structure directly excited to a subsystem that receives and stores this energy permanently. This phenomenon has been reported in references [1-8], which illustrate and, in some cases, provide theoretical fundamentals of this phenomenon.

The present work shows that in order to observe such an unusual effect in a system with random natural frequencies, the properties the pdf of the resonant frequencies must satisfy certain conditions. In particular, it is shown that energy confinement effect is related to a requirement for a minimum variance response. The main statements of the theory are formulated in this and in the next section, while in section 3 an application of the theory is discussed.

In many problems of mechanics, a combination of harmonic functions can br used to represent the impulse response *h(t)* of a conservative linear dynamical system, which commonly yields a discrete eigenfrequency spectrum that can be expressed as:

$$h(t) = \sum_{i=1}^{N} \alpha_i G(\omega_i) \sin \omega_i t \tag{1}$$

where $\alpha_i G(\omega_i), \omega_i, N$ represent mode shape dependent factors, system natural frequencies and the number of the modes involved in the system response.

Except for the special and simple cases where frequencies ω_i are integer multiples of a fundamental frequency, the properties of a harmonic series such as $h(t)$ can be, in general, quite complicated. However, for conservative systems, expression (1) does not vanish asymptotically but behaves as an *almost periodic function*.

The theory outlined in this paper investigates special occurrences of random natural frequencies ω_i in equation (1). In this case, the function $\sigma = G(\omega)\sin\omega t$, appearing in each term of equation (1), is also random. Assumption that each natural frequency ω_i has the same probability density function p_ω, physically corresponds to systems with a cluster of modes with the same average natural frequency spread randomly over a given frequency range. Such a class of systems reveals surprising theoretical and technical attributes, as shown in the examples later in this paper.

Expected value $E\{\sigma\}$ of σ is:

$$E\{\sigma\} = \int_{-\infty}^{+\infty} p_\omega(\omega) G(\omega) \sin\omega t \, d\omega \tag{2}$$

Equivalently, in terms of the probability density function of σ, p_σ, the same expectation has the form:

$$E\{\sigma\} = \int_{-\infty}^{+\infty} p_\sigma(\sigma)\, \sigma \, d\sigma \tag{3}$$

where:

$$p_\sigma(\sigma) \frac{d\sigma}{d\omega} = p_\omega(\omega) \tag{4}$$

The expected value of the impulse response *h(t)* is therefore:

$$\begin{gathered} E\{h\} = \sum_{i=1}^{N} \alpha_i \int_{-\infty}^{+\infty} p_\omega(\omega) G(\omega) \sin\omega t \, d\omega, \qquad \alpha = \sum_{i=1}^{N} \alpha_i \\ E\{h\} = \alpha E\{\sigma\} \end{gathered} \tag{5}$$

Asymptotic expansion of the integral (2) shows that $E\{h\}$ obeys the time asymptotic property

$$\lim_{t \to \infty} E\{h\} = 0 \tag{6}$$

A comparison between h and $E\{h\}$, from equation (1) and (2), respectively, reveals that the two functions do not share the fundamental asymptotic property (6): for a conservative system, the single sample h is almost periodic and does not exhibit a vanishing impulse response, which is indeed the case for $E\{h\}$.

Property (6) has a deep implication in the energy behavior of the system considered here. The vanishing impulse response in the absence of any dissipation implies that the energy injected by the impulse is irreversibly stored in some part of the system far away from the point of energy injection. Therefore, we can conclude that property (6) implies an **irreversible energy confinement (IEC)** away from the point at which h is computed. This property, besides its theoretical importance, also offers a potential in the design of new shock absorbers, so that it can be regarded, under certain circumstances, as a desirable property.

In this light, we can ask if there are particular populations of systems, i.e. some probability density functions p_ω, that can make the expected value $E\{h\}$ as close as possible to the samples *h(t)*. This requirement has its strict statistical translation: find the function p_ω that minimizes the variance of the impulse response h. If this property, called here **minimum-variance-response (MIVAR),** holds, we expect that each sample *h(t)* of the population will exhibit a property close to the one expressed by equation (6).

Accordingly, this paper investigates, for linear and conservative systems, the link between irreversible energy confinement and minimum-variance-response.

Note that the resonant frequencies of a single vibrating structure with N modes is also a collection of N samples of natural frequencies ω_i, *i=1,2,...N,* of the random variable ω. Thus, its modal density $n(\omega)$ is directly related to the pdf p_ω, namely $n(\omega) = N\, p_\omega$. This point is of practical interest; it shows how the result of statistical analysis related to a population can be applied even to a single sample.

2 MIVAR: Minimum-Varaince-Response

Although our aim is to find p_ω leading to the minimum-variance-impulse-response-MIVAR, the problem is more conveniently formulated in terms of p_σ from which p_ω is obtained through equation (3), with some care. Before embarking on the analysis of MIVAR some preliminary definitions and results are helpful.

Def 1) The set of functions $\sigma_i = G(\omega_i)\sin\omega_i t,\;\; i = 1,2,\ldots,N$, defines a random point (or a random vector) $\boldsymbol{\sigma} = [\sigma_1, \sigma_2, \ldots, \sigma_N]^T$ in a space Σ and $\boldsymbol{\sigma}$ exists within the hypercube $C \equiv \{E \text{x}\, E \text{x} \ldots \text{x}\, E\}$, with:

$$E \equiv [-G_{\max}, G_{\max}] \quad \text{and} \quad G_{\max} = \max\{G_1, G_2, \ldots, G_N\}$$

With this definition the impulse response is a linear function of the random vector $\boldsymbol{\sigma}$:

$$h(\boldsymbol{\sigma}) = \boldsymbol{\alpha}^T \cdot \boldsymbol{\sigma} = \sum_{i=1}^{N} \alpha_i\, \sigma_i\, .$$

Def 2) The joint probability of $\boldsymbol{\sigma}$ is $P(\boldsymbol{\sigma}) = \prod_{k=1}^{N} p_\sigma(\sigma_k)$, with:

$$\int_C P(\boldsymbol{\sigma})\, dC = 1\,, \quad E\{h\} = \int_C h(\boldsymbol{\sigma}) P(\boldsymbol{\sigma})\, dC \qquad (7)$$

Def 3) The variance D^2 of *h* is $D^2 = \int_C \ [h(\boldsymbol{\sigma}) - E\{\ h(\boldsymbol{\sigma})\}\ ]^2 P(\boldsymbol{\sigma})\, dC$

Lemma 1) With $E\{*\} = \overline{*}$, $\overline{h} = \alpha\,\overline{\sigma}$, and from equation (7):

$$\int_C \frac{\partial}{\partial \overline{h}} P(\boldsymbol{\sigma})\ dC = 0 \ \rightarrow \ \int_C \left[\frac{1}{P(\boldsymbol{\sigma})}\frac{\partial}{\partial \overline{h}} P(\boldsymbol{\sigma})\right] P(\boldsymbol{\sigma})\ dC = \int_C \frac{\partial}{\partial \overline{h}}\left[\log P(\boldsymbol{\sigma})\right] P(\boldsymbol{\sigma})\ dC = 0$$

i.e.:

$$\overline{\frac{\partial}{\partial \overline{h}} \log P} = 0$$

The last equation is equivalent also to:

$$\overline{\overline{h}\,\frac{\partial}{\partial \overline{h}} \log P} = 0 \tag{8}$$

Lemma 2) From equation (7):

$$\int_C h\frac{\partial}{\partial \overline{h}} P(\boldsymbol{\sigma})\ dC = 1 \quad \rightarrow \quad \int_C h\left[\frac{1}{P(\boldsymbol{\sigma})}\frac{\partial}{\partial \overline{h}} P(\boldsymbol{\sigma})\right] P(\mathbf{s}, I)\ dC = 1$$

i.e.:

$$\overline{h\,\frac{\partial}{\partial \overline{h}} \log P} = 1 \tag{9}$$

To minimize the distance *D* between *h* and $\overline{h}$ in the space Σ, consider the difference between equations (8) and (9) produces a condition as a scalar product between the functions as:

$$\overline{(h - \overline{h}) \cdot \left(\frac{\partial}{\partial \overline{h}} \log P\right)} = 1 \tag{10}$$

.

The Schwartz inequality applied to equation (10) leads to:

$$\overline{(h - \overline{h}) \cdot \left(\frac{\partial}{\partial \overline{h}} \log P\right)}^{\,2} \le \overline{(h - \overline{h})^2}\ \ \overline{\left(\frac{\partial}{\partial \overline{h}} \log P\right)^2} \tag{11}$$

which, using equation (10) and the Def 3 for *D*, becomes:

$$D^2 \geq \frac{1}{\overline{\left(\frac{\partial}{\partial \bar{h}} \log P\right)^2}} \tag{12}$$

The right-hand side of (12) presents a lower bound for D, depending on the probability function *P* or, equivalently, depending on p_σ and through equation (3) on p_ω. Among the possible choices for *P*, those providing a minimum for the variance *D* match the lower bound exactly:

$$D^2 = \frac{1}{\overline{\left(\frac{\partial}{\partial \bar{h}} \log P\right)^2}} \tag{13}$$

As the minimum for *D* is reached through equation (13), the Schwartz inequality (11) becomes:

$$\overline{(h-\bar{h}) \cdot \left(\frac{\partial}{\partial \bar{h}} \log P\right)}^2 = 1$$

i.e. the scalar product between $(h-\bar{h})$ and $\left(\frac{\partial}{\partial \bar{h}} \log P\right)$ becomes unity; this implies that the two functions are proportional (i.e. they are parallel functions, since their scalar product is 1:

$$\frac{\partial}{\partial \bar{h}} \log P = a(\bar{h})(h-\bar{h}) \tag{14}$$

where $a(\bar{h})$ is a proportionality constant**.**

Condition (14) represents a differential equation in terms of *P*, and its solution leads to a family of exponential functions $P(\boldsymbol{\sigma}) = \prod_{k=1}^{N} p_\sigma(\sigma_k)$. The solution to Eq. (14), originally obtained by Pitman and Koopman in the context of the theory of estimators, is given as:

$$p_\sigma(\sigma) = \exp\left\{ \alpha(\bar{h})\varepsilon(\sigma) + \zeta(\sigma) + \beta(\bar{h}) \right\}$$

where $\alpha(\bar{h})$, $\beta(\bar{h})$, $\varepsilon(\sigma)$, and $\zeta(\sigma)$ are arbitrary functions of their respective arguments. Gauss function also belongs to this family of solutions:

$$p_\sigma(\sigma) = \frac{1}{r\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\frac{\left(\sigma - \bar{\sigma}\right)^2}{r^2}\right\} \tag{15}$$

$p_\sigma(\sigma)$ has a distribution that depends on the functions $\sigma(G,\omega)$, $\bar{\sigma}$ and on the parameter *r*. Together with equation (15), equation (3) provides the probability density function $p_\omega(\omega)$ of the natural frequencies that minimizes *D*, i.e. that provides a MIVAR:

$$p_\omega(\omega) = \frac{1}{r\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\frac{\left(\sigma - \bar{\sigma}\right)^2}{r^2}\right\} \frac{d\sigma}{d\omega} \tag{16}$$

Equation (16) shows that time appears as a parameter in the pdf that minimizes *D*. The consideration here is limited to time-invariant pdf*s*, thus equation (16) is is considered at a particular time t_0. The choice for t_0 and selection of the frequency interval $[\omega_{\min}, \omega_{\max}]$ for $p_\omega(\omega)$, which also depends the choice of t_0, deserve special care. In fact, in equation (3) both $p_\omega(\omega)$ and $p_\sigma(\sigma)$ must be positive implying that the derivative $d\sigma/d\omega$ must also be positive within $[\omega_{\min}, \omega_{\max}]$. Moreover, $p_\omega(\omega)$ through equation (3), must satisfy conditions (7), used in deriving lemmas 1 and 2. Since integration domain of our integrals is finite $E \equiv [-G_{\max}, G_{\max}]$, *r* and $\bar{\sigma}(t_0)$ must satisfy the constraints:

$$r << G_{\max}, \quad \bar{\sigma}(t_0) \in E \tag{17}$$

These guarantee that $p_\sigma(\sigma)$ has its main peak within the interval *E* and therefore, at least approximately, satisfies equations (7) . Therefore, the explicit expression for $p_\omega(\omega)$ becomes

$$p_\omega(\omega) = \frac{1}{r\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\frac{\left(G(\omega)\sin\omega t_0 - \bar{\sigma}(t_0)\right)^2}{r^2}\right\} \left[\frac{dG(\omega)}{d\omega}\sin\omega t_0 + t_0 G(\omega)\cos\omega t_0\right] \tag{18}$$

where $r, \bar{\sigma}(t_0), t_0$ can be regarded as arbitrary parameters but verifying restraints (17). This is the fundamental result of this paper: **the pdf from equation (18) leads to a MIVAR (minimum-variance-response) and therefore produces an IEC (irreversible energy confinement)**.

The previous theory applies to the impulse response of a random system, and, furthermore, in general, to any physical quantity represented by *h(t)* that is a suitable linear combination of harmonic terms with random frequencies as in equation (1).

3 Application of the theory

The examples given in this section illustrate an application for the theory described above.

The system under consideration consists of a set of resonators with random natural frequencies $\omega_i\ (i = 1,2..N)$, connected in parallel to a common principal structure of natural frequency ω_M, see Figure 1.

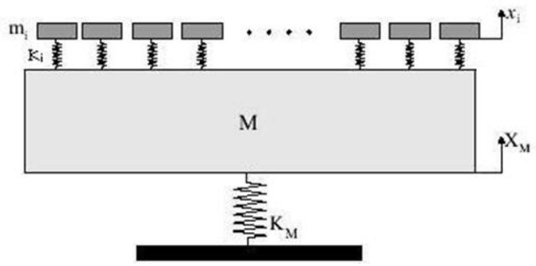

Figure 1

The system does not possess any means of energy dissipation. The coupled equation of master-cluster system are:

$$\begin{cases} M\,\ddot{x}_M(t) + K_M x_M(t) + \sum_{i=1}^{N} k_i\left[x_M(t) - x_i(t)\right] = 0 \\ m\,\ddot{x}_i(t) - k_i\left[x_M(t) - x_i(t)\right] = 0 \end{cases} \tag{19}$$

where M, K_M, x_M are the mass, stiffness and displacement of the master structure, respectively; m, k_i, x_i represent the same quantities for the oscillators in the attached set. The solution of the second equation provides:

$$x_i(t) = \frac{k_i}{m\,\omega_i}\sin\omega_i\,t * x_M(t) = \omega_i \sin\omega_i\,t * x_M(t)$$

which when introduced into the first produces:

$$M\,\ddot{x}_M(t) + \left(K_M + \sum_{i=1}^{N} k_i\right) x_M(t) + m\,h(t) * x_M(t) = 0 \tag{20}$$

where the coupling effect of the cluster on the master is represented in the last term, characterized by the kernel $h(t) = \sum_{i=1}^{N} \omega_i^3 \sin\omega_i\,t$, again of the form (1) with $G(\omega) \equiv \omega^3, \alpha_i = 1$.

Let us consider together with equation (20) a companion equation obtained by replacing h with $\bar{h}$:

$$M\,\ddot{x}_M(t)+\left(K_M+\sum_{i=1}^{N}k_i\right)x_M(t)+m\,\bar{h}(t)*x_M(t)=0 \tag{21}$$

$$\bar{h}(t)=N\int_0^{\omega_{max}} p_\omega(\omega)\,\omega^3\,\sin\omega\,t\;d\omega \tag{22}$$

How the cluster affects the master motion in equation (21) can be investigated examining the Fourier transform $\bar{H}(\Omega)$ of $\bar{h}(t)$:

$$\bar{H}(\Omega)=-j\,\Omega\left[\frac{\pi}{4}\Omega^2\,p_\omega(\Omega)\right]+\int_{-\infty}^{+\infty}\frac{\pi}{2}\frac{p_\omega(\zeta)\zeta^3}{(\zeta-\Omega)}\,d\zeta,\quad \text{for}\quad \Omega\in[0,\omega_{max}]$$

$$\bar{H}(\Omega)=0 \qquad \text{elsewhere} \tag{23}$$

Equation (23) shows the coupling term between the master and the cluster in equation (21) that acts as a damper, because of the imaginary part of $\bar{H}(\Omega)$. Namely, an equivalent viscous damping $C_{eq}=\frac{\pi}{4}\Omega^2\,p_\omega(\Omega)$ appears as a function of the natural frequency pdf. This phenomenon amounts to an irreversible energy confinement (IEC) within the cluster of resonators. It is important to note that a similar effect for the companion equation (20) is not seen. Therefore, the IEC effect appears only if in equation (20) h is replaced by its average $\bar{h}$. Therefore the conclusion can be stated as: if $p_\omega(\omega)$ is of the form given by equation (18) with $G(\omega)\equiv\omega^3$, then the population of h exhibits a minimum deviation from its average $\bar{h}$ and the IEC effect is close to be satisfied also for equation (20). As already mentioned, this conclusion has a practical implication even for the case of a single sample: if the modal density within the cluster is $n(\omega)=N\,p_\omega(\omega)$, then it approaches, and in the closest possible manner to an IEC.

As an example, consider a master structure, with an uncoupled natural frequency $\omega_M=1$, and with N=100 attached oscillators. Assumption of $t_0=\frac{\pi}{4}$ follows that, within the frequency interval $\omega\in[0,2]$, $\sigma\in[0,8]$, the derivative $d\sigma/d\omega$ is positive. The values of r and $\bar{\sigma}(t_0)$ [r = 0.4 and $\bar{\sigma}(t_0)$= 0.6] are selected to be consistent with inequalities (17), and to assure that the function represented by equation (18) has its peak around $\omega=\omega_M=1$.

Figure 2 shows the modal density of the attached oscillators from equation (18).

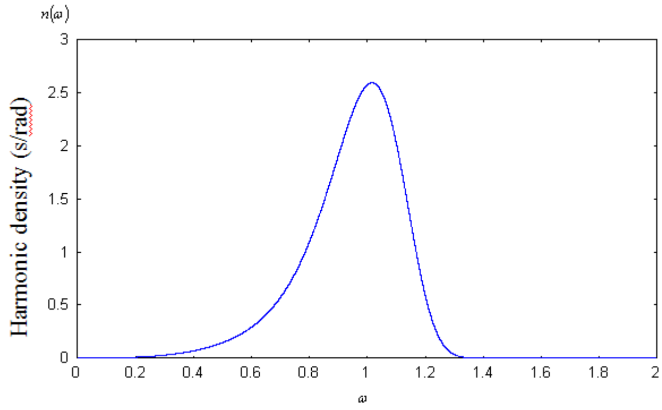

Figure 2

Figure 3 shows the master response following an impulse applied at t=0, which illustrates how a significant part of its energy is transferred to the set of oscillators and remains there without returning back to the master, producing an irreversible energy confinement.

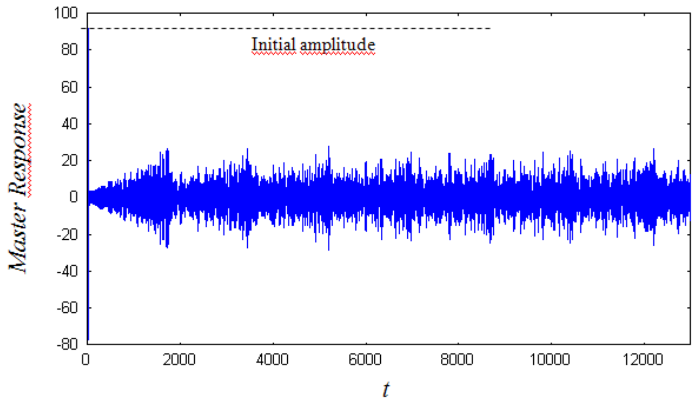

Figure 3

We conclude the paper with an example of a built-up aerospace structure equipped with a special patented vibration absorber [9] developed on the basis of the present theory and consisting of a cluster of beam shaped resonators all clamped to the same structural base. The lengths of the beams, i.e. the natural frequencies of the set, obey the rules of selection discussed in the previous sections.

Based on these ideas, a device was designed for use on board of UNISAT (UNIversity SATellite), see Figs. 4-6, which is a permanent space project developed at the University of Rome La Sapienza by the Gauss Group. UNISAT is a small scientific satellite (14 kg- 20 kg depending on the payload), launched periodically starting in 2000. The latest version of UNISAT will be equipped with the vibration suppressor described in this manuscript. Severe vibrations affect the electronic instrumentation of the satellite during lift-off of (by a DNEPR rocket launched from Russia) and the present device designed to reduce the shock and vibration the structure that carries the electronic package. Moreover, due to the safety requirements imposed, all electronic equipment in the satellites on board the launcher must be switched-off during the lift-off, making it unfeasible to use any active vibration suppressor.

The material used for the absorber is steel ($\rho = 7780\,kg/m^3$, $E = 187.5\,GPa$,) and the cluster of resonators is actually made of beams manufactured by milling them from a stack of steel sheets, each with a thickness of h=0.6mm. The maximum allowed space for the device is 90 mm x 90 mm x 40 mm, with a maximum allowed mass of 0.15 kg.

Measurements are made to validate the estimated equivalent damping given by equation (23) and effectiveness of the absorber are determined following the procedure.

As the first step, the best location for attachment point P is identified considering both the response and the geometric constraints (Figs. 5 and 6). An electro-dynamic shaker was used to excite the structure employing a spectrum similar to that under operating conditions while monitoring several sensitive positions of interest. The point where the maximum amplitude response develops is identified as P (Fig. 6). The peak frequency is identified from the drive point frequency response function (FRF) at P (Fig. 7) for determining the tuning frequency and the bandwidth of the damper.

To verify the effectiveness of the built-up device shown in figure 8, it was installed on the satellite platform (see Fig. 9). Figure 10 presents a comparison of the attenuated drive point FRF with attached absorber as well as the one without the absorber. The results, given for the frequency range covered by the set of first modes of the beams in the cluster, show significant reduction in the FRF amplitude, and are expected to provide the necessary protection.

Finally, figure 11 shows the same comparison but for the frequency bandwidth 2200-3000 Hz. This bandwidth covers the set of natural frequencies of the second modes of the beams producing an additional attenuation of the amplitude of vibration.

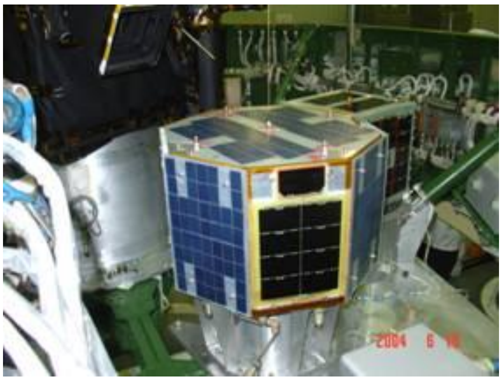

Figure 4: UNISAT (version n. 3 – launched in 2004). View of the satellites room in the rocket before the launch

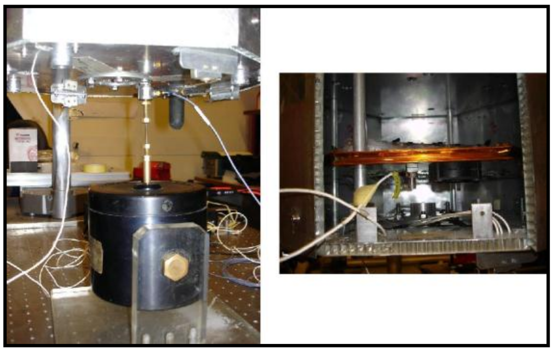

Figure 5: Left: electrodynamic exciter acting on the bottom panel of the satellite

Right: inner view of the satellite structure

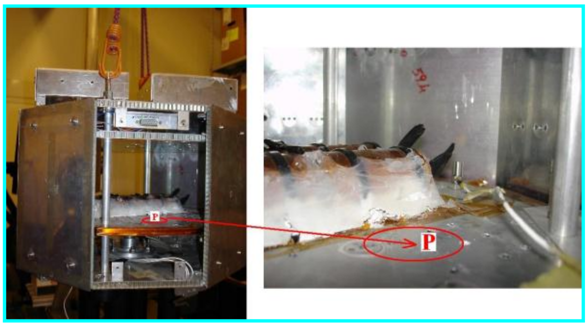

Figure 6: Selection of the “point P” at which the device is applied

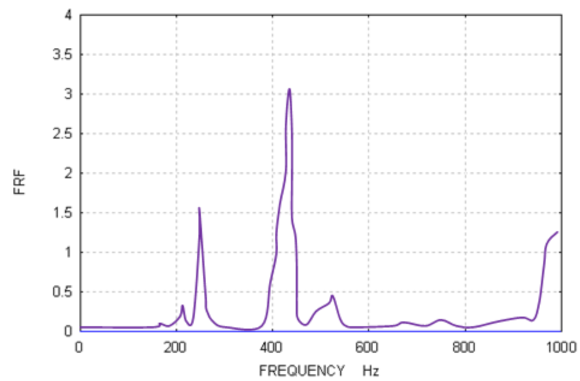

Figure 7: Experimental drive-point FRF

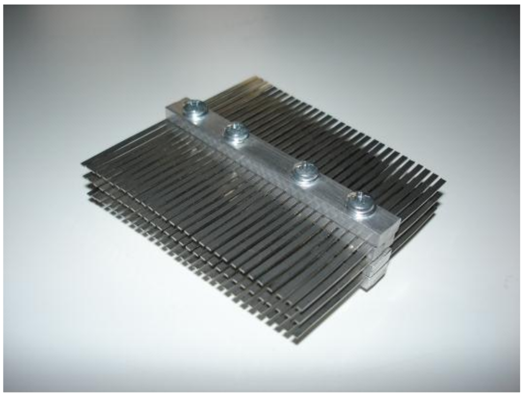

Figure 8: View of the built-up device – tuning frequency 440 Hz, total weight 130 g

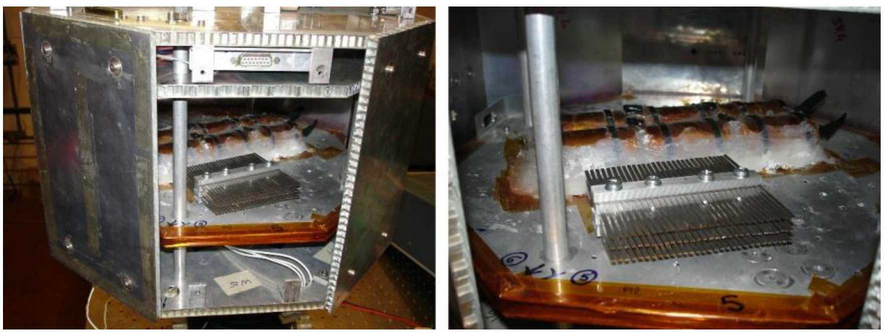

Figure 9: View of the final installation of the device on board of UNISAT

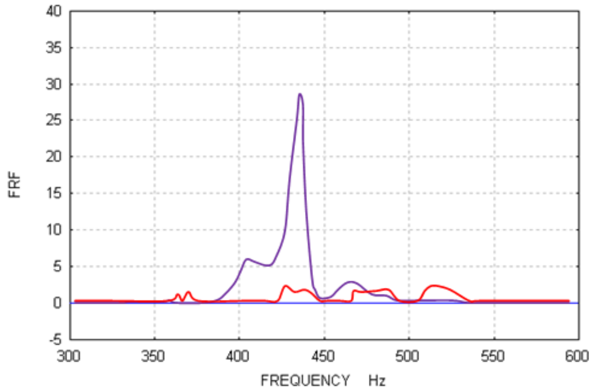

Figure 10: Comparison of the FRFs at “point P” with (red curve) and without (blue curve) the vibration suppression device.

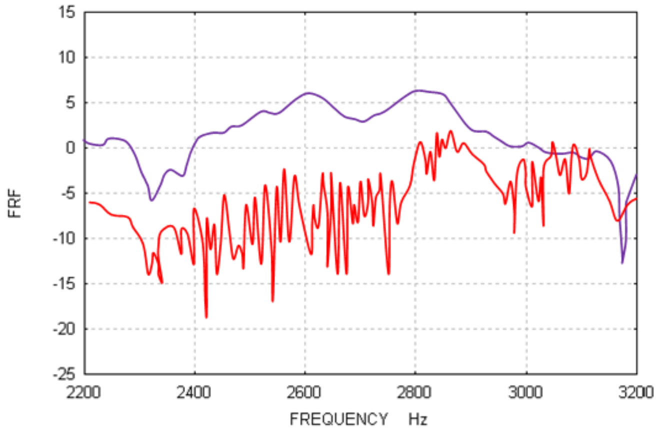

Figure 11: Evidence of the response reduction related to the second mode

4 Final remarks

This paper presents an investigation aimed at correlating uncertainty in complex vibrating systems to their energy storage capability. It appears that the pdf of the system’s natural frequencies is strictly related to the damping exhibited by the system, and namely to an irreversible storage of energy into some of the degrees of freedom of the same system. The question has an interest both on the theoretical ground as well as for applications.

The problem considered here is related to the development of a thermodynamic approach to the study of energy sharing between mechanical resonators.

An attempt to formulate a new thermodynamic theory bridging the gap between the classical thermodynamic and the theory of dynamical systems is contained in [10]. In this context, some relevant concepts, originally related to thermodynamics, arise naturally also in the analysis of dynamical systems: the energy equipartition and the existence of an entropy-like quantity. The first, indeed one of the cardinal element of statistical thermodynamics, has been investigated in [11,12,16] and has also points of contact with the topic of the present paper, namely the irreversible energy storgae. The concept of entropy has been indeed proved to be consistent even in the frame of structural vibrations and acoustics [13,14,15]. Finally, partly related to the entropy concept, the question of irreversibility in structural dynamics, even in the absence of dissipation effects, has been considered in [7,16,17,19], with significant implications also on the practical design of special devices characterized by an energy trapping capability and energy sink storage. The problem of energy distribution along real structures has been also approached attacking the problem by dimensional analysis as shown in [20,21].

An additional area of investigation in this context is represented by the energy control and confinement through nonlinearities, the most active group in the field names this phenomenon energy pumping. A series of papers [22-25] are devoted to this topic and a general overview of the problem can be found in the book [26]. The authors clearly show how, especially non-smooth nonlinearities, can be used to design special nonlinear attachments to be coupled to a primary linear structure to damp its motion, subtracting a significant amount of vibration energy form the primary. This process again shows how the shock vibration energy can be confined into a subpart of the excited system.

An interesting and different point of view on the energy storage and energy re-distribution in complex systems, that belongs to a different context and research field, is indeed contained in [27-31]. Suitable structural modifications are there conceived coupling a mechanical structure to an electrical system. Under the dynamic point of view, the investigated electro-mechanical coupling is characterized by a single state vector collecting both mechanical and electrical degrees of freedom. The analysis developed by the authors, shows how it is possible to properly select some characteristics of the coupled dynamic matrix in a way most of the energy can be re-directed into the electrical degrees of freedom. In this case, an energy storage is physically confined into the electrical part of the system, but the essence of the process has the same aims of the present analysis: try to determine a permanent energy storage within some selected degrees of freedom of a complex dynamic system. In this sense, the results here presented can have also potential applications in the context of design of an electrical shunt piezo-electrically coupled to a vibrating structures.

Finally, a special mention deserves the formation of regions of energy accumulation in continuous media, as a consequence of non–classical strain-stress relationships. In this context, deformation or kinetic energy can be concentrated in small regions through the addition of parts constituted by materials exhibiting a state of deformation effectively described by suitable tridimensional higher gradient continua as in [32,33] or two-dimensional structured continua as, e.g. in [34,35].

From the previous considerations it appears that the problem of energy storage into systems of different physical nature (mechanical and/or electromechanical or even purely electrical) is far to being solved, and the questions that emerge have also a deep engineering interest. This paper proposes an additional contribution in this general context, providing additional insights into probabilistic features of the natural frequency distribution that determines the ability of energy storage into a confined part of a dynamical system.

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