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Quantum canonical transformations in star-product formalism

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Abstract. We study construction of the star-product version of three basic quantum canonical transformations which are known as the generators of the full canonical algebra. By considering the fact that star-product of c-number phase-space functions is in complete isomorphism to Hilbert-space operator algebra, it is shown that while the constructions of gauge and point transformations are immediate, generator of the interchanging transformation deforms this isomorphism. As an alternative approach, we study all of them within the deformed form. How to transform any c-number function under linear-nonlinear transformations and the intertwining method are shown within this argument as the complementary subjects of the text.

1. Introduction

In this work, we construct quantum canonical transformations (QCTs) in phase-space formalism developed by H. Weyl, E. Wigner, H. Groenewold and J. Moyal [1]. We will say Moyal or star-product formalism in short for this phase-space formalism. In doing so, we consider a conjecture stating that any QCT can be written as a composition of three basic CTs. These are gauge, point and interchange CTs [2, 3]. In fact a similar conjecture in classical mechanics has been stated by using only linear and point transformations [4]. The same triplet in quantum mechanics may also be used to generate the classical CTs but this point is out of the scope of this work.

First we give the three basic QCTs in this section. In section 2, we see that if we employ the isomorphism between the Hilbert-space operator algebra and star-product of phase-space functions to get the star-product version of the triplet, the constructions of gauge and point transformations are immediate but the generator of interchange transformation deforms this isomorphism. But still as an alternative way, an ordinary exponential version of it can be constructed. In section 3, as the new approach, all the basic CTs are derived in the ordinary exponential form. Section 4 shows that intertwining transformation is in fact a CT. Finally, section 5 contains a short summary and conclusions including some unanswered matters related with the topic.

So it is natural to start with the definition of a QCT: Given arbitrary complex function (unitary or non-unitary) $\hat{F}(\hat{q},\hat{p})$ which is so called generating function (GF) on quantum phase

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space, QCTs are given by the equations

$$\begin{aligned} \hat{F}(\hat{q},\hat{p})\;\hat{q}\;\hat{F}^{-1}(\hat{q},\hat{p}) &= \hat{Q}(\hat{q},\hat{p}),\\ \hat{F}(\hat{q},\hat{p})\;\hat{p}\;\hat{F}^{-1}(\hat{q},\hat{p}) &= \hat{P}(\hat{q},\hat{p}), \end{aligned} \tag{1}$$

transforming q-numbers into the other variables $\hat{Q}$ and $\hat{P}$, where the new variables must, of course, satisfy the canonicity condition $[\,\hat{Q},\hat{P}\,] = \hat{Q}\hat{P} - \hat{P}\hat{Q} = \mathrm{i}\hbar$ and $\hat{F}^{-1}$ is the algebraic inverse of $\hat{F}$.

Now we list the explicit forms of the three basic QCTs: Th gauge transformation is generated by the function $f(\hat{q})$ such as

$$\begin{aligned} \exp[\lambda f(\hat{q})]\;\hat{q}\;\exp[-\lambda f(\hat{q})] &= \hat{q},\\ \exp[\lambda f(\hat{q})]\;\hat{p}\;\exp[-\lambda f(\hat{q})] &= \hat{p} + \mathrm{i}\hbar\lambda\;\partial_q f, \end{aligned} \tag{2}$$

where λ is a pure imaginary number with a continuous parameter. On the other hand, the explicit form of the point transformation is given by

$$\begin{aligned} \exp[\lambda f(\hat{q})\,\hat{p}]\;\hat{q}\;\exp[-\lambda f(\hat{q})\,\hat{p}] &= A(\hat{q}),\\ \exp[\lambda f(\hat{q})\,\hat{p}]\;\hat{p}\;\exp[-\lambda f(\hat{q})\,\hat{p}] &= (\partial_q A)^{-1}\,\hat{p}, \end{aligned} \tag{3}$$

where the function $A(\hat{q})$ is given by the operation

$$A(\hat{q}) = \exp[-\mathrm{i}\hbar\lambda f(\hat{q})\partial_q]\;\hat{q}. \tag{4}$$

Finally, the interchange transformation

$$\hat{I}\,\hat{q}\,\hat{I}^{-1} = \hat{p}, \qquad \hat{I}\,\hat{p}\,\hat{I}^{-1} = -\hat{q} \tag{5}$$

is achieved by a Fourier transform operator whose definition is given by the action

$$\hat{I}^{\pm} = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \cdot \exp(\pm\mathrm{i}\hat{q}\hat{q}'/\hbar)\;\mathrm{d}q' \tag{6}$$

satisfying

$$\hat{I}\,\hat{q}\,' = -\mathrm{i}\hbar\partial_q\hat{I}, \qquad \hat{I}\,\hat{p}\,' = -\hat{q}\,\hat{I}. \tag{7}$$

The inverse of the transformation means that $\hat{q}$ goes to minus $\hat{p}$ and $\hat{p}$ goes to $\hat{q}$.

Now let us give an immediate example related with the conjecture mentioned above. Our example is the well-known linear CTs:

$$\begin{aligned} \hat{F}\;\hat{q}\;\hat{F}^{-1} &= \hat{Q} = a\,\hat{q} + b\,\hat{p},\\ \hat{F}\;\hat{p}\;\hat{F}^{-1} &= \hat{P} = c\,\hat{q} + d\,\hat{p}, \end{aligned} \tag{8}$$

where a, b, c, d are real constants satisfying $a\,d - b\,c = 1$ and $a + d + 2 \neq 0$. Such a transformation can be decomposed into a five-step sequence:

$$\hat{F}_L = \exp[\,\gamma\hat{q}\,\hat{p}\,]\exp[\,\beta\hat{q}^2\,]\,\hat{I}\exp[\,\alpha\hat{q}^2\,]\,\hat{I}^{-1} = \hat{\mathcal{P}}\,\hat{G}_2\,\hat{I}\,\hat{G}_1\,\hat{I}^{-1} \tag{9}$$

where $\mathcal{G}$ and $\mathcal{P}$ stand for gauge and point respectively and α, β, γ are the pure imaginary numbers compatible with a linear CT [3]. At this stage, it should be note that such a decomposition is not unique, that means of course, one can find many decompositions giving the same transformation. The total transformation after the five steps appears as

$$\begin{aligned} \hat{q} &\to (1 + 4\hbar^2\alpha\beta)\,\hat{q} - 2i\hbar\alpha\,\hat{p},\\ \hat{p} &\to 2i\hbar\beta k\,\hat{q} + \tilde{p}/k \end{aligned} . \tag{10}$$

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2. Canonical transformations in Moyal formalism

In this section, as our main purpose, we will try to construct the three basic CTs in Moyal formalism. Moyal formalism is based on an interesting but very powerful product given by

$$\star \;=\; \exp\left[\,\frac{i\hbar}{2}(\overleftarrow{\partial}_q\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_q)\right] \tag{11}$$

which is so called star-product or Moyal product, where the arrows indicate the direction that the derivatives act. Its explicit action can be seen easily by the general product rule

$$\begin{aligned} F(q,p)\star G(q,p) &= F(q+i\hbar\partial_p/2, p-i\hbar\partial_q/2)G(q,p), \\ &= G\left(q-i\hbar\partial_p/2, p+i\hbar\partial_q/2\right)F(q,p). \end{aligned} \tag{12}$$

Note that, as an advantage, it can be applied to the both sides. On the other hand, by the Groenewold isomorphism given by the integral transform

$$\begin{aligned} \hat{F}(\hat{q},\hat{p})\,\hat{G}(\hat{q},\hat{p}) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} & \mathrm{d}\sigma\,\mathrm{d}\tau\,\mathrm{d}q\,\mathrm{d}p\,[F(q,p)\star G(q,p)] \\ & \times\exp\{\mathrm{i}\,[\sigma(\hat{q}-q)+\tau(\hat{p}-p)]/\hbar\}, \end{aligned} \tag{13}$$

one can make a correspondence between the operator product in quantum phase-space and the Moyal product in c-number phase-space:

$$\hat{F}(\hat{q},\hat{p})\,\hat{G}(\hat{q},\hat{p}) \leftrightarrow F(q,p)\star G(q,p). \tag{14}$$

We should note that after the transformation, the resulting functions $\hat{F}$ and $\hat{G}$ in quantum phase-space become Weyl-ordered or symmetrically ordered. Therefore such a transform provides us a quantization method. Conversely, the inverse transform gives us a de-quantization method. Therefore, by considering this isomorphism, CTs in Moyal formalism can be defined easily by the equations

$$\begin{aligned} F(q,p)\star q\star F^{-1}(q,p) = Q(q,p), \\ F(q,p)\star p\star F^{-1}(q,p) = P(q,p) \end{aligned} \tag{15}$$

satisfying the Moyal bracket

$$\{\,Q\,,P\,\}^M = Q\star P - P\star Q = i\hbar \tag{16}$$

as the canonicity condition. Now we are ready to construct the triplet in terms of star-product. First one is the gauge transformation. Since, by (2) and the correspondence (14), the GF $G=\exp_\star[-\lambda f(q)]$ where the star-exponential is given by the expansion [5]

$$\exp_\star[\lambda\, f(q,p)] = 1 + \lambda\, f(q,p) + \frac{\lambda^2}{2!}\, f(q,p)\star f(q,p) + \cdots\,, \tag{17}$$

contains only q as the variable, the quantum exponential and the star-exponential are equivalent and the new GF generates the same transformation

$$G\star q\star G^{-1} = q, \qquad G\star p\star G^{-1} = p + \mathrm{i}\hbar\lambda\;\partial_q f, \tag{18}$$

just like the previous one (2). When we consider the point transformation

$$\mathcal{P}\star q\star\mathcal{P}^{-1} = A(q), \qquad \mathcal{P}\star p\star\mathcal{P}^{-1} = (\partial_q A)^{-1}\star p \tag{19}$$

generated by the GF $\mathcal{P}=\exp_\star[-\lambda f(q)\star p]$, we see similarly that all hats are gone, and the operator product is replaced by the star-product as expected.

But the matter is the last one, i.e., the interchange transformation. The matter arises from the fact that c-number phase-space functions are not operators taking integral as in (6). But we can find a c-number phase-space function resulting an interchange transformation.

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3. Canonical transformations in Moyal formalism revisited

According to the definition of a CT in the star-product formalism, the interchange transformation must be in the form

$$\mathcal{I} \star q \star \mathcal{I}^{-1} = p, \qquad \mathcal{I} \star p \star \mathcal{I}^{-1} = -q. \tag{20}$$

If we rewrite the transformation in the form

$$\mathcal{I} \star q = p \star \mathcal{I}, \qquad \mathcal{I} \star p = -q \star \mathcal{I}, \tag{21}$$

we get a new definition of the transformation without needing the inverse of the GF $\mathcal{I}$ [6]. This point is important, because getting the inverse of the generator becomes usually a tedious matter. Expansion of the star-products on both sides of (21) gives us the system of partial differential equations

$$(q-p)\,\mathcal{I} = -\frac{i\hbar}{2}(\partial_q - \partial_p)\,\mathcal{I}, \qquad (q+p)\,\mathcal{I} = -\frac{i\hbar}{2}(\partial_q + \partial_p)\,\mathcal{I}. \tag{22}$$

The solution as the generator of the interchange transformation appears in an ordinary exponential form with no $\star$ sign:

$$\mathcal{I} = \exp[\mathrm{i}(q^2 + p^2)/\hbar]. \tag{23}$$

Therefore it is natural to think of getting the other generators in the ordinary exponential form, by converting every transformation into a system of partial differential equations as in the following.

The new GF $\mathcal{G} = \exp[-\lambda f(q)]$ of the gauge transformation is immediate without solving any differential equation due to the single variable dependence of G. Therefore they generate the same transformation as before:

$$\mathcal{G} \star q \star \mathcal{G}^{-1} = q, \qquad \mathcal{G} \star p \star \mathcal{G}^{-1} = p + \mathrm{i}\hbar\lambda\, \partial_q f. \tag{24}$$

On the other hand, getting the generator of the point transformation is not so easy. But similar to the previous one, the converted definition can be given in the form

$$\wp \star q = Q(q) \star \wp, \qquad \wp \star p = \tilde{Q}(q)\, p \star \wp, \qquad \tilde{Q}(q) = (\partial_q Q)^{-1}. \tag{25}$$

This system can be solved by looking for the solutions of type $\wp = \exp\{\lambda[f(q)p + g(q)]\}$. Such a trial function gives the function $f(q)$ as the solution to the algebraic equation

$$Q\,(v) = q - \frac{\mathrm{i}\hbar\lambda}{2}\, f, \qquad v = q + \frac{\mathrm{i}\hbar\lambda}{2}\, f. \tag{26}$$

The function $g(q)$ is determined by the integral

$$g(q) = -\frac{i\hbar}{\lambda}\int \frac{(\partial_q \tilde{Q})(v)}{1+\tilde{Q}(v)}\frac{\partial f}{\partial q}\,\mathrm{dq} \tag{27}$$

resulting from the differential equation (25).

It may be remarkable to point out that this approach, i.e., converting the transformation into a differential equation system, can be used to determine both the generating function and the transformation itself. For example, given transformation $Q = 1/q$, $P = -q^2\, p$, the GF

$$f(q) = \pm\frac{2}{\hbar\lambda}(1-q^2)^{1/2}, \qquad g(q) = -\frac{1}{2\lambda}\ln(q^2 - 1) \tag{28}$$

is obtained by solving the equations in (26) and (27). Conversely, given generating function $f(q) = q$, $g(q) = \text{const.}$, for instance, the result is the scaling transformation

$$Q = k\,q, \quad P = p/k, \quad k = \frac{2 - \mathrm{i}\hbar\lambda}{2 + \mathrm{i}\hbar\lambda}, \quad \lambda \neq 2i/\hbar. \tag{29}$$

After having the construction of the basic transformations in the ordinary exponential form, we are ready to decompose the transformations. For this aim, we consider again the linear CTs

$$Q = a\,q + b\,p, \qquad P = c\,q + d\,p. \tag{30}$$

For the linear CTs we have the system

$$F_L \star q = Q \star F_L, \quad F_L \star p = P \star F_L. \tag{31}$$

This time the solution is given by this ordinary exponential form:

$$F_L(q,p) = \exp\{2\mathrm{i}A[\,bp^2 - cq^2 + (a-d)qp\,]/\hbar\}, \quad A = 1/(a+d+2). \tag{32}$$

On the other hand, if we remember its quantum correspondent (9), we can construct the decomposed form of the compact one in the five-step sequence

$$\begin{aligned} F_L(q,p) &= \exp(\lambda qp) \star \exp(\beta q^2) \star \exp[\mathrm{i}(q^2+p^2)/\hbar] \star \exp(\alpha q^2) \star \Big[\exp[\mathrm{i}(q^2+p^2)/\hbar]\Big]^{-1} \\ &= \wp \star \mathcal{G}_2 \star \mathcal{I} \star \mathcal{G}_1 \star \mathcal{I}^{-1}. \end{aligned} \tag{33}$$

The second example differs from the others, here we consider the transformation of a function. In our example linear potential system $H_0 = p^2 + q$ transforms to the free particle system $H_1 = p^2$ via the transformation rule

$$F \star H_0 \star F^{-1} = H_1. \tag{34}$$

It can be shown that transformations of functions can also be decomposed similarly. The decomposition

$$\begin{aligned} F &= \exp[\mathrm{i}(q^2+p^2)/\hbar] \star \exp[\lambda(fp+q)] \\ &\quad \star \Big[\exp\mathrm{i}(q^2+p^2)/\hbar\Big]^{-1} \star \exp(\mathrm{i}q^3/3\hbar) \star \exp[\mathrm{i}(q^2+p^2)/\hbar] \\ &= \mathcal{I} \star \wp \star \mathcal{I}^{-1} \star \mathcal{G} \star \mathcal{I} \end{aligned} \tag{35}$$

consists of gauge, point and interchange again, where the GF $\wp$ of the point transformation in the forth step is given by the equations

$$\begin{aligned} f(q) &= \frac{\mathrm{i}}{\hbar\lambda}\left[2q + 1 - (1+8q)^{1/2}\right], \\ g(q) &= \frac{1}{2\lambda}\ln\left[\frac{1+(1+8q)^{1/2}}{1+8q}\right]. \end{aligned} \tag{36}$$

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4. Intertwining as a canonical transformation

In this section, we will show that the intertwining method can be treated within this argument. Now, suppose that there exists a c-number phase-space function $L(q,p)$ making a link between two Hamilton functions:

$$L(q,p) \star H_0(q,p) \star L^{-1}(q,p) = H_1(q,p), \tag{37}$$

where the Hamilton functions are in the potential form

$$H_0 = p^2 + V_0(q), \qquad H_1 = p^2 + V_1(q). \tag{38}$$

It is clear that the transformation equation (37) can be rewritten in the form

$$L(q,p) \star H_0(q,p) = H_1(q,p) \star L(q,p). \tag{39}$$

Expansion of the star products on both sides of (39) gives us the differential equation

$$V_1\left(q + \frac{\mathrm{i}\hbar}{2}\partial_p\right) L(q,p) = V_0\left(q - \frac{\mathrm{i}\hbar}{2}\partial_p\right) L(q,p) + 2\mathrm{i}\hbar\, p\, \partial_q L(q,p). \tag{40}$$

relating the two potentials V_0 and V_1. For the solution if we choose the ansatz

$$L(q,p) = p - \mathrm{i}\varphi(q), \tag{41}$$

we get the well known consistency condition

$$V_1(q) + V_0(q) = 2\varphi^2(q), \tag{42}$$

and the Riccati equation

$$V_1(q) = V_0(q) + 2\hbar\partial_q\varphi(q) \tag{43}$$

with the solution $\varphi(q)$. If we look at the ansatz (41) again, we see that the function $L(q,p)$ is in fact a gauge transformation given by

$$L(q,p) = \exp\left[-\int \varphi(q)\mathrm{d}q/\hbar\right] \star p \star \exp\left[\int \varphi(q)\mathrm{d}q/\hbar\right]. \tag{44}$$

As the second step, if we replace the term p in the middle by the decomposition

$$p = \exp\left[\mathrm{i}(q^2+p^2)/\hbar\right] \star \exp(\ln q) \star \exp\left[\mathrm{i}(q^2+p^2)/\hbar\right]^{-1}, \tag{45}$$

we conclude that the function $L(q,p)$ is the sequence

$$L = \mathcal{G}_1^{-1} \star \mathcal{I} \star \mathcal{G}_2 \star \mathcal{I}^{-1} \star \mathcal{G}_1 \tag{46}$$

of the fundamental transformations and this says obviously that the intertwining method is a CT.

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5. Conclusions and summary

Three fundamental CTs, especially the interchange transformation, has been constructed in terms of the ordinary exponential form. It has been shown that the ordinary form may present a powerful tool in determining both the GF and the CT. Within this formalism, transform of functions under the linear (gauge) transformations runs just like in quantum mechanics. The behavior under the nonlinear CTs has been shown and justified by a well-known example.

It should be noted that the GFs in the $\star$-exponential form do not correspond to the quantized versions of those given in the ordinary form. For example, quantized version of the interchange operator $\exp[\mathrm{i}(q^2+p^2)/\hbar]$ does not correspond to $\exp_\star[\mathrm{i}(q^2+p^2)/\hbar] \equiv \exp[\mathrm{i}(\hat{q}^2+\hat{p}^2)/\hbar]$ but to the decomposition $\exp_\star\left(\frac{\mathrm{i}q^2}{2\hbar}\right) \star \exp_\star\left(\frac{\mathrm{i}p^2}{2\hbar}\right) \star \exp_\star\left(\frac{\mathrm{i}q^2}{2\hbar}\right) \equiv \exp\left(\frac{\mathrm{i}\hat{q}^2}{2\hbar}\right) \exp\left(\frac{\mathrm{i}\hat{p}^2}{2\hbar}\right) \exp\left(\frac{\mathrm{i}\hat{q}^2}{2\hbar}\right)$. In general all we can say that $\exp_\star[f(q,p)]$ is always the de-quantized form of $\exp[\hat{f}(\hat{q},\hat{p})]$ provided that $\hat{f}(\hat{q},\hat{p})$ is the Weyl transformed form of $f(q,p)$. Except the some simple cases, getting the quantized version of any ordinary exponentiated function remains an open question not answered yet.

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