A BOUND ON THE ZERO-ERROR LIST CODING CAPACITY*

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Abstract

We present a new bound on the zero-error list coding capacity, and using which, show that the list-of-3 capacity of the 4/3 channel is at most 6/19 bits, improving the best previously known bound of 3/8. The relation of the bound to the graph-entropy bound of Körner and Marton is also discussed.

The Bound

Consider a discrete memoryless channel $K = (\mathcal{I}, \mathcal{J}, P)$ where $\mathcal{I}$ denotes the input alphabet, $\mathcal{J}$ the output alphabet, and $P(j|i)$ the probability that $j \in \mathcal{J}$ is received given that $i \in \mathcal{I}$ is transmitted. A set $S \subset \mathcal{I}^N$ is called *independent* if for every $y \in \mathcal{J}^N$

$$\prod_{x \in S} \prod_{n=1}^{N} P(y_n|x_n) = 0.$$

A set $\mathcal{C} \subset \mathcal{I}^N$ is called a zero-error list-of-L code, $L \geq 1$, if every $S \subset \mathcal{C}$ with $|S| = L + 1$ is an independent set. Zero-error list-of-L capacity is defined by

$$C_L = \limsup_{N \to \infty} \frac{1}{N} \log M(N, L)$$

where $M(N, L)$ is the maximum possible size for a list-of-L code of length N. (All logarithms are to to base 2.)

We call a channel k-uniform if k is the smallest integer for which $C_k > 0$. The new bound is as follows.

Theorem 1 *The rate R of any list-of-k code $\mathcal{C}$ on a k-uniform channel K satisfies*

$$R - \epsilon \leq \min_{1 \leq m \leq k} \min_{x_{m+1}, \ldots, x_k} \min_{P'} \frac{1}{mN} \sum_{n=1}^{N} I(X_{1n}, \ldots, X_{mn}; Y_n | x_{(m+1)n}, \ldots, x_{kn})$$

where P' ranges through all conditional probability assignments such that whenever $\{i_1, \ldots, i_m, i'_1, \ldots, i'_m, i_{m+1}, \ldots, i_k\}$ is independent in K

$$P'(j|i_1, \ldots, i_m, i_{m+1}, \ldots, i_k) P'(j|i'_1, \ldots, i'_m, i_{m+1}, \ldots, i_k) = 0$$

for all j. The mutual information term is computed using the probability assignment

$$Pr\{X_{1n} = x_{1n}, \ldots, X_{mn} = x_{mn}, Y_n = y_n\} = Q_n(x_{1n}) \cdots Q_n(x_{mn}) P'(y_n | x_{1n}, \ldots, x_{kn})$$

where Q_n is the empirical distribution of the nth coordinate of the codewords in $\mathcal{C}$, i.e., $Q_n(i)$ equals the fraction of codewords $x \in \mathcal{C}$ with $x_n = i$, $i \in \mathcal{I}$. The number ϵ goes to zero as N increases for any fixed $R \geq 0$.

For comparison, the Körner-Marton graph-entropy bound [3] states (in the above notation) that

$$R - \epsilon \leq \min_{m, P'} \frac{|\mathcal{C}|^{-(k-m)}}{mN} \sum_{x_{m+1}, \ldots, x_k} \sum_{n=1}^{N} I(X_{1n}, \ldots, X_{mn}; Y_n | x_{(m+1)n}, \ldots, x_{kn})$$

where the outer summation is over all possible choices of distinct codewords $x_{m+1}, \ldots, x_k \in \mathcal{C}$. Thus, the Körner-Marton bound upperbounds the rate R by (essentially) the average of the quantity $\sum_{n=1}^{N} I(X_{1n}, \ldots, X_{mn}; Y_n | x_{(m+1)n}, \ldots, x_{kn})$, whereas here R is bounded by the minimum of the same quantity.

The bound here may also be seen as a generalization of the Shannon bound on zero-error capacity [1], [2]. Shannon's bound is obtained by looking at the zero-error code through a single user channel; here we look at the code through a multiaccess channel.

The 4/3 Channel

The 4/3 channel has a four letter input and output alphabet $A = \{0, 1, 2, 3\}$, and the transition probabilities $P(j|i) = 1/3$ for all $i, j \in A$, $i \neq j$. The bound $C_3 \leq 6/19$ is obtained (after some manipulation) by applying the above theorem using the following P'. (i) For any $i, i_1, j \in A$, $P'(j|i_1, i, i) = \delta_{ij}$. (ii) For any $i_1, i_2, i_3, j \in A$ with $i_2 \neq i_3$,

$$P'(j|i_1, i_2, i_3) = \begin{cases} 0 & \text{if } j \in \{i_1, i_2, i_3\}; \\ (4 - |\{i_1, i_2, i_3\}|)^{-1} & \text{otherwise.} \end{cases}$$

References

[1] C.E. Shannon, 'The zero error capacity of a noisy channel,' *IEEE Trans. Inform. Theory*, vol. IT-2, no. 3, pp. 8-19, 1956.

[2] P. Elias, 'Zero error capacity under list decoding,' *IEEE Trans. Inform. Theory*, vol. IT-34, No. 5, pp. 1070-1074,ept. 1988.

[3] J. Körner and K. Marton, 'On the capacity of uniform hypergraphs,' *IEEE Trans. Inform. Theory*, vol. IT-36, No.1, pp. 153-156, Jan. 1990.

*This work has been supported by TÜBİTAK under project TBAG 1053.