Improving AWE Accuracy Using Multipoint Padé Approximation

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Abstract— **A new method is proposed for dominant pole-zero analysis of large linear circuits containing both lumped and distributed elements. This method is based on a multipoint Padé approximation. It finds a reduced order s-domain transfer function using a data set obtained by solving the circuit at only a few frequency points. The proposed method yields more accurate computation of transient and frequency responses with respect to the AWE-type techniques.**

I. Introduction

Asymptotic waveform evaluation (AWE) is an efficient technique that is used for dominant pole-zero approximation [1] [2]. It employs a form of Padé approximation to approximate the behavior of the higher order linear circuit with a reduced order model. The first few terms of power series expansion of the reduced order transfer function are matched to the moments and the Markov parameters of the actual circuit, which can be computed efficiently by a set of simple dc analyses. Actually the moments and the Markov parameters result from a Taylor series expansion of the circuit response about $s = 0$ and $1/s = 0$, respectively. It is then obvious that the moments convey information about the low-frequency characteristics of the circuit, while the Markov parameters can only represent the high-frequency behavior. However, for some applications, specifically in RF and microwave circuits, the mid-frequency range is more important. Hence, we need to make use of the information obtained from the mid-frequency characteristics of the circuit to find a better approximation.

In this paper, we propose a new order reduction technique that gives more accurate results with respect to the conventional AWE technique. In the proposed approach, the circuit matrix is solved a few times in the frequency range under consideration. The derivatives of the network function with respect to complex frequency are obtained efficiently from these the solutions. By using these derivatives at different complex frequency points, a multipoint Padé approximation is performed in order to obtain a reduced order s-domain network function. Poles and zeros (or poles and residues) can be found from this rational network function using standard techniques.

II. The Method

Consider a linear system modeled by a set of linear algebraic equations in the Laplace domain,

$$\mathbf{T}(s)\mathbf{x} = \mathbf{w} \tag{1}$$

where $\mathbf{T}$ is the system matrix, the vector $\mathbf{x}$ is the system response and the vector $\mathbf{w}$ is the system excitation. In general, the system matrix $\mathbf{T}$ is an arbitrary function of complex frequency s. Let the system output be any linear combination of the system response,

$$F = \mathbf{d}^T\mathbf{x}. \tag{2}$$

Using Cramer's rule one can obtain

$$F(s) = \frac{\det\begin{bmatrix} \mathbf{T} & \mathbf{w} \\ -\mathbf{d}^T & 0 \end{bmatrix}}{\det \mathbf{T}}. \tag{3}$$

Our aim is to approximate the network function $F(s)$, regardless whether it is a rational or irrational function of s, with a rational function $\hat{F}(s)$ which has approximately the same frequency characteristics as the original circuit does. Let the approximate function be of the form

$$\hat{F}(s) = \frac{b_0 + b_1 s + \ldots + b_{q-1}s^{q-1}}{1 + a_1 s + \ldots + a_q s^q}. \tag{4}$$

Since there are $2q$ parameters to compute in the reduced model, we need $2q$ constraints from the actual circuit. In the AWE technique $2q$ unknowns are calculated by matching the first r moments and the first $(2q - r)$ Markov parameters of the original circuit to the approximate rational function.

In this work, we propose a method which uses a data set obtained from the circuit to construct the approximate s-domain rational function, $\hat{F}(s)$. This data set contains the *translated moments* obtained at different complex frequency points. In the following, we present the evaluation of the translated moments and how to match them to a rational function.

2.1. Translated moments

The system response $\mathbf{x}(s)$ in (1), can be expanded in Taylor series at $s = s_0$ as:

$$\mathbf{x}(s) = \sum_{k=0}^{\infty} \mathbf{x}_k (s - s_0)^k, \quad (5)$$

provided that $\mathbf{x}(s)$ is analytic at $s = s_0$. The coefficient $\mathbf{x}_k$ in (5) is called the vector of kth *translated moments* and

$$\mathbf{x}_k = \frac{\frac{\partial^k}{\partial s^k}[\mathbf{T}^{-1}] \mid_{s=s_0}}{k!} \mathbf{w}. \quad (6)$$

The first translated moment vector is the solution of the circuit at $s = s_0$,

$$\mathbf{x}_0 = \mathbf{T}^{-1}(s_0)\mathbf{w}. \quad (7)$$

It can be easily shown that the higher order translated moments can be evaluated recursively as,

$$\mathbf{x}_k = -\mathbf{T}^{-1}(s_0) \sum_{r=1}^{k} \frac{\mathbf{T}^{(r)} \mathbf{x}_{k-r}}{r!} \quad (8)$$

where superscript (r) indicates the rth derivative with respect to s evaluated at $s = s_0$. Since the LU factorization of $\mathbf{T}(s_0)$ are known from the solution of the first translated moment vector, each higher order translated moment vector can be obtained only by one forward and back substitution (FBS).

2.2. Multipoint moment matching

The translated moments m_k's of the transfer function are obtained from the translated moment vectors $\mathbf{x}_k$ as,

$$m_k = \mathbf{d}^T \mathbf{x}_k. \quad (9)$$

Note that, the translated moments at $s = s_0^*$ are complex conjugates of the translated moments at $s = s_0$. We match the first q translated moments at $s = s_0$ and their conjugates to a qth order rational function:

$$\begin{aligned} &\frac{b_0 + b_1 s + \ldots + b_{q-1} s^{q-1}}{1 + a_1 s + \ldots + a_q s^q} \\ &= m_0 + m_1(s - s_0) + \ldots + m_{q-1}(s - s_0)^{q-1} + \text{h.o.t.} \\ &= m_0^* + m_1^*(s - s_0^*) + \ldots + m_{q-1}^*(s - s_0^*)^{q-1} + \text{h.o.t.} \end{aligned}$$

One can show that finding the coefficients of the rational function is equivalent to solving the following set of linear equations.

$$\begin{aligned} \hat{b}_0 &= m_0 \hat{a}_0 \\ \hat{b}_0^* &= m_0^* \hat{a}_0^* \\ \hat{b}_1 &= m_0 \hat{a}_1 + m_1 \hat{a}_0 \\ \hat{b}_1^* &= m_0^* \hat{a}_1^* + m_1^* \hat{a}_0^* \\ &\vdots \\ \hat{b}_{q-1} &= m_0 \hat{a}_{q-1} + m_1 \hat{a}_{q-2} + \ldots + m_{q-1} \hat{a}_0 \\ \hat{b}_{q-1}^* &= m_0^* \hat{a}_{q-1}^* + m_1^* \hat{a}_{q-2}^* + \ldots + m_{q-1}^* \hat{a}_0^* \end{aligned} \quad (10)$$

where

$$\begin{aligned} &\hat{a}_0 = 1 + \textstyle\sum_{k=1}^{q} a_k s_0^k \\ &\hat{a}_i = \textstyle\sum_{k=i}^{q} a_k \binom{k}{i} s_0^{k-i}, \qquad i = 1, 2, .., q-1 \\ &\hat{b}_i = \textstyle\sum_{k=i}^{q-1} b_k \binom{k}{i} s_0^{k-i}, \qquad i = 0, 1, .., q-1 \end{aligned}$$

and

$$\begin{aligned} &\hat{a}_0^* = 1 + \textstyle\sum_{k=1}^{q} a_k (s_0^*)^k \\ &\hat{a}_i^* = \textstyle\sum_{k=i}^{q} a_k \binom{k}{i} (s_0^*)^{k-i}, \qquad i = 1, 2, .., q-1 \\ &\hat{b}_i^* = \textstyle\sum_{k=i}^{q-1} b_k \binom{k}{i} (s_0^*)^{k-i}, \qquad i = 0, 1, .., q-1 \end{aligned}$$

Once the coefficients of the rational function are obtained from the above set, the poles and zeros (or poles and residues) can be found from the polynomials of the rational function using standard techniques.

The one-point moment matching technique can be extended to the multipoint case as follows:
i-Select m frequency points on the s-domain.
ii-Calculate p_i moments at the ith frequency point for $i = 1, .., m$, such that $\sum_{i=1}^{m} p_i = q$. With the addition of the conjugates of these q moments, the number of constraints obtained from the circuit equals to the number of unknowns in the reduced (qth) order s-domain network function.
iii-Solve the system of $2q \times 2q$ linear equations which is constructed by taking the first $2p_i$ equations from the ith equation set for $i = 1, .., m$.

As a summary, the coefficients of s-domain rational function are found by solving a $2q \times 2q$ linear equations system. It corresponds to inversion of a real matrix of size $2q \times 2q$ which has a complexity of $O((2q)^3)$. Although there exist some other algorithms of $O((2q)^2)$ to obtain the coefficients of the rational function [3], our experience suggests that the direct inversion of the real matrix yields better results than the methods mentioned in Ref. [3], in terms of numerical accuracy.

2.3. Formulation of the Circuit Equations

Assume that the given circuit contains linear lumped components and linear subcircuits. The subcircuits may contain distributed components. By using Modified Nodal Analysis (MNA) formulation, the frequency domain response of the circuit can be computed from the matrix equation [4],

$$\mathbf{T}(s)\mathbf{x} = \mathbf{w} \tag{11}$$

or explicitly

$$\begin{bmatrix} \mathbf{G}+s\mathbf{C} & \mathbf{D}_1 & \mathbf{D}_2 & \cdots & \mathbf{D}_k \\ \mathbf{Y}_1(s)\mathbf{D}_1^T & \mathbf{I} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \mathbf{Y}_k(s)\mathbf{D}_k^T & 0 & 0 & \cdots & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{V} \\ \mathbf{I}_1 \\ \vdots \\ \mathbf{I}_k \end{bmatrix} = \begin{bmatrix} \mathbf{u} \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where $\mathbf{G}$ and $\mathbf{C}$ are matrices formed by the parameters of the lumped components, $\mathbf{V}$ is a vector containing the node voltages, and the currents of the inductors and the voltage sources, $\mathbf{u}$ is the vector of the values of the independent sources, k is the number of subcircuits, $\mathbf{D}_i$ is the appropriate incidence matrix corresponding to the terminal currents of the ith subcircuit, $\mathbf{I}_i$ is the vector of terminal currents and $\mathbf{Y}_i$ is the admittance matrix of the ith subcircuit.

The derivatives of $\mathbf{T}$ evaluated at $s = s_0$ are

$$\mathbf{T}^{(1)} = \begin{bmatrix} \mathbf{C} & 0 & 0 & \cdots & 0 \\ \mathbf{Y}_1^{(1)}(s_0)\mathbf{D}_1 & 0 & 0 & \cdots & 0 \\ \mathbf{Y}_2^{(1)}(s_0)\mathbf{D}_2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 \\ \mathbf{Y}_k^{(1)}(s_0)\mathbf{D}_k & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{12}$$

and

$$\mathbf{T}^{(r)} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \mathbf{Y}_1^{(r)}(s_0)\mathbf{D}_1 & 0 & 0 & \cdots & 0 \\ \mathbf{Y}_2^{(r)}(s_0)\mathbf{D}_2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 \\ \mathbf{Y}_k^{(r)}(s_0)\mathbf{D}_k & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{13}$$

for $r \geq 2$.

The derivatives of the admittance matrices of the subcircuits are given in Ref. [4].

2.4. Examples

A circuit which contains six of band-pass filters in parallel is considered. It is composed of 20 transmission lines, 23 inductors, 32 capacitors and 26 resistors. The filters are switched by PIN diodes. Because of finite isolation of the PIN switches the active filter is loaded by the other filters, hence the resulted frequency response is very complicated. The magnitude and phase of the frequency response found by the proposed method are shown in Figs. 1- 2 by solid lines. It is a 40th order approximation with three-point moment matching (m=3, $s_1 = 0.65$ GHz, $s_2 = 1.65$ GHz, $s_3 = 2.5$ GHz, $p_1 = 14$, $p_2 = 13$, $p_3 = 13$). The frequency response obtained by solving the circuit at 1000 frequency points, is shown by dots on the same figures. Even 1000 points may not be sufficient, because some sharp peaks in the graph may be missed as can be seen in the Fig. 1. Dot-dashed lines in those figures correspond to a 10th order approximation with one-point moment matching (m=1, $s_1 = 0.65$ GHz, $p_1 = 10$). It approximates the pass-band of the filter sufficiently well, but cannot catch the details for outside the

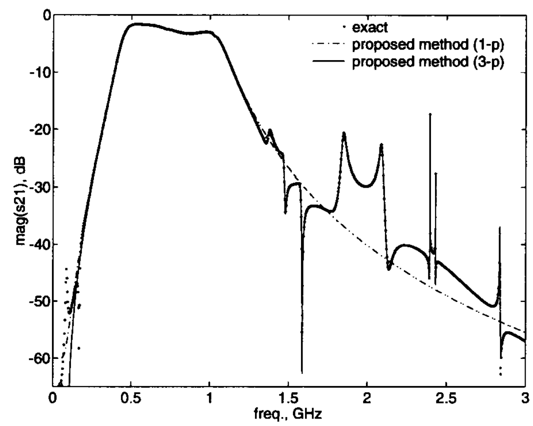

Fig. 1. Frequency response (magnitude) of the band-pass filter circuit.

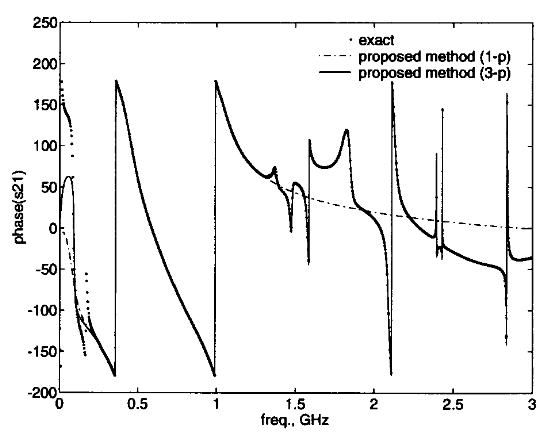

Fig. 2. Frequency response (phase) of the band-pass filter circuit.

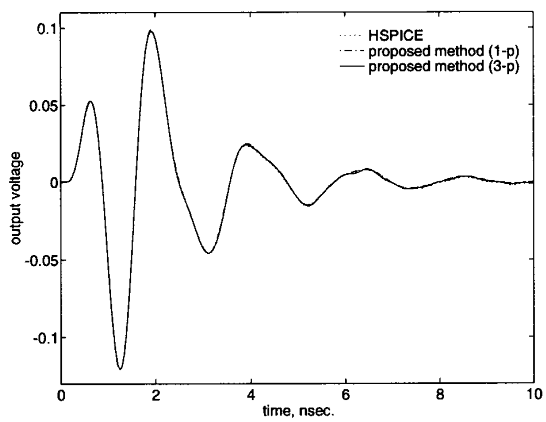

Fig. 3. Step response of the band-pass filter circuit.

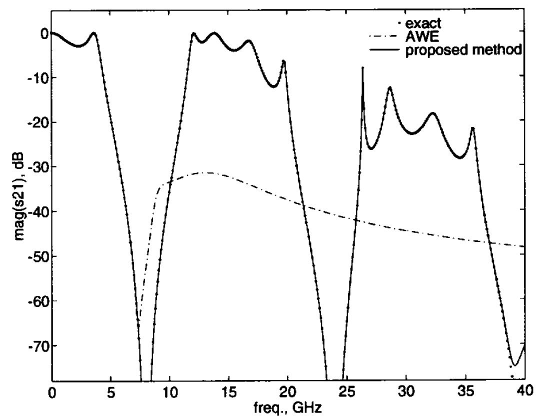

Fig. 4. Frequency response of the low-pass filter circuit.

pass-band region. The step response of the filter circuit is computed by the new method for two different orders of approximations as indicated above. The results are shown in Fig. 3 along with the HSPICE result for comparison. The one-point approximation is very close to the actual response, while the three-point approximation is virtually indistinguishable from it.

The second example is a low-pass filter implemented with transmission lines. The exact frequency response of the filter is shown in Fig. 4. In the same figure, we also give the 10th order AWE approximation. The step response of the circuit corresponding the 10th order AWE approximation is given in Fig. 5 together with the HSPICE response. As it is seen from Figs. 4- 5, the AWE method approximates the low-frequency characteristics of the circuit very well, but it is not able to detect the repetitions of the frequency response and therefore the high frequency transients. However our method, using a three-point moment matching (m=3, s_1 = 4 GHz, s_2 = 16 GHz,s_3 = 32 GHz, $p_1 = 8$, $p_2 = 10$, $p_3 = 12$), finds all the details of the both responses (solid lines in Figs. 4- 5).

III. Conclusions

A new order reduction method for linear circuits has been presented. This method uses a multipoint Padé approximation to find a reduced order s-domain network function. In other words, it computes approximate poles and zeros (or poles and residues) for the given circuit. The obtained poles are not necessarily low-frequency approximations as it is the case in AWE. Consequently, while this method preserves the efficiency of AWE, it

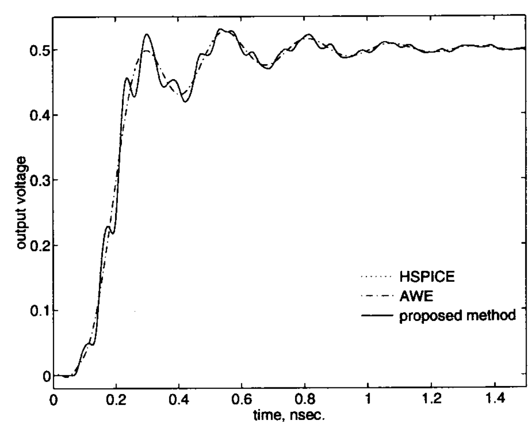

Fig. 5. Step response of the low-pass filter circuit.

improves its accuracy.

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