THE DISCRETE FRACTIONAL FOURIER TRANSFORMATION

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ABSTRACT

Based on the fractional Fourier transformation of sampled periodic functions, the discrete form of the fractional Fourier transformation is obtained. It is found that only for a certain dense set of fractional orders such a discrete transformation is possible to define. Also, for its efficient computation a fast algorithm, which has the same complexity as the FFT, is given.

1. INTRODUCTION

The fractional Fourier transform [1, 2, 3, 4] has found many applications in the solution of differential equations [2, 3], quantum mechanics and quantum optics [5, 6], optical diffraction theory and optical beam propagation (including lasers), and optical systems and optical signal processing [1, 8, 9, 10, 11, 12], swept-frequency filters [4], time-variant filtering and multiplexing [1], pattern recognition and study of time-frequency distributions [13]. The recently studied Radon transformation of the Wigner spectrum [14] is also known to be the magnitude square of the fractional Fourier transform [1, 15]. The fractional Fourier transform has been related to wavelet transforms [1, 16], neural networks [16], and is also related to various chirp-related operations [1, 17, 18, 19]. It can be optically realized much like the usual Fourier transform [1, 8, 9, 10].Other applications which are currently under study or which have been suggested include phase retrieval, signal detection, radar, tomography, and data compression.

In this paper, the discrete form of the fractional Fourier transformation is given. It is obtained as a result of the investigation on the fractional Fourier transformation of sampled periodic signals. It is shown that, the obtained discrete transformation is valid for a discrete set of transformation orders. However, since this set is dense in the set of all possible orders, it does not impose tight restrictions. Also, an $O(N \log N)$ algorithm for the computation of the N-point discrete transformation is provided. Hence, in many of the above mentioned applications it is now possible to use the efficient discrete algorithm for effective use of the fractional Fourier transformation.

2. THE FRACTIONAL FOURIER TRANSFORMATION

The ath order fractional Fourier transform $\mathcal{F}^a[f(x)]$ of the function $f(x)$ may be defined for $0 < |a| < 2$ as

$$\begin{aligned} \mathcal{F}^a[f(x)] &= \int_{-\infty}^{\infty} B_a(x,x')f(x')\,dx', \\ B_a(x,x') &= A_\phi \exp\left[j\pi(x^2\cot\phi - 2xx'\csc\phi + x'^2\cot\phi)\right], \\ A_\phi &= \frac{\exp(-j\pi \mathrm{sgn}(\sin\phi)/4 + j\phi/2)}{|\sin\phi|^{1/2}} \end{aligned} \quad (2.1)$$

where $\phi \equiv \frac{a\pi}{2}$. The kernel approaches $B_0(x,x') \equiv \delta(x-x')$ and $B_{\pm 2}(x,x') \equiv \delta(x+x')$ for $a=0$ and $a=\pm 2$ respectively. The definition is easily extended outside the interval $[-2,2]$ by remembering that $\mathcal{F}^{4m}$ is the identity operator for any integer m and that the fractional Fourier transform operator is additive in index, that is, $\mathcal{F}^{a_1}\mathcal{F}^{a_2} = \mathcal{F}^{a_1+a_2}$. The ath fractional Fourier transform $\mathcal{F}^a[f(x)]$ of the function $f(x)$ will be abbreviated by $f_a(x)$.

3. THE DISCRETE FRACTIONAL FOURIER TRANSFORMATION

In this section, the discrete form of the fractional Fourier transformation is obtained. For this purpose, fractional Fourier transform of a sampled periodic signal is investigated. This approach can be motivated by the similar analysis that can be carried out for the Fourier transformation leading to the definition of discrete Fourier transformation (DFT) [24]. Let $f(x)$ be a sampled periodic signal with a period Δ_0:

$$f(x) = \sum_{k=-N/2}^{N/2-1} f(k\frac{\Delta_0}{N}) \sum_{n=-\infty}^{\infty} \delta(x-(n+\frac{k}{N})\Delta_0) \quad , \quad (3.2)$$

where N, number of samples in a period is taken to be even. Its order-a fractional Fourier transform, f_a, can be obtained by using Eqn. 2.1, giving

$$\begin{aligned} f_a(x) &= \int_{-\infty}^{\infty} B_a(x,x')f(x')\,dx', \\ &= \sum_{k=-N/2}^{N/2-1} f(k\frac{\Delta_0}{N}) \sum_{n=-\infty}^{\infty} B_a(x,(n+\frac{k}{N})\Delta_0) \end{aligned} \quad (3.3)$$

For the transformed function in Eqn. 3.3 to be periodic with a period Δ_a, $f_a(x)$ should be equal to $f_a(x + l\Delta_a)$ for all x and l. It can be shown with a little algebra that, this condition of periodicity is satisfied if and only if:

$$\Delta_a \cos\phi = p\Delta_0 \quad (3.4)$$
$$\Delta_a \sin\phi = q\frac{N}{\Delta_0} \quad (3.5)$$

where for integer p and q such that pqN is even, which is satisfied since N is even. The ratio of Eqn. 3.5 and Eqn. 3.4 gives the condition on the transformation order:

$$\tan\phi = \frac{q}{p}\frac{N}{\Delta_0^2} \quad (3.6)$$

which implies that the $f_a(x)$ is periodic for only a set of orders which satisfies Eqn. 3.6 and hence have a cardinality equal to the rational numbers. Since the set of rationals are dense in the set of real numbers, this set of orders also forms a dense set in the set of all possible orders. The conditions in Eqn. 3.4 and Eqn. 3.5 can be easily interpreted by the identification of $\frac{N}{\Delta_0}$ as the period of $f(x)$ in the Fourier transform domain [24]. The first condition in Eqn. 3.4 implies that the projection of the period on the transform domain of order-a onto the original domain should be a multiple of the period in the original domain. Likewise, the second condition in Eqn. 3.4 implies that the projection of the period on the transform domain of order-a onto the Fourier transform domain should be a multiple of the period in the Fourier transform domain. By choosing $N = \Delta_0^2$, the periods in the original and the Fourier transform domain can be made equal to each other, resulting a square tiling pattern in the time-frequency domain [21]. In this case, the conditions becomes:

$$\Delta_a \cos\phi = p\Delta_0 \quad (3.7)$$
$$\Delta_a \sin\phi = q\Delta_0 \quad (3.8)$$

which are satisfied by $\tan\phi = q/p$ and $\Delta_a = \sqrt{p^2+q^2}\Delta_0$. In the rest of the paper, we assume that ϕ satisfies these conditions. By using the shift property of the fractional Fourier transformation, Eqn. 3.3 can be written as:

$$f_a(x) = \sum_{k=-N/2}^{N/2-1} f(k\frac{\Delta_0}{N}) \exp(j\pi(\sin\phi\cos\phi(\frac{k\Delta_0}{N})^2 - 2\sin\phi x\frac{k\Delta_0}{N})) \sum_{n=-\infty}^{\infty} B_a(x - (\frac{k}{N})\Delta_0\cos\phi, n\Delta_0) \ . \quad (3.9)$$

The inner summation in Eqn. 3.9 is the shifted fractional Fourier transformation of a uniform impulse train with period Δ_0, which is denoted by $\bar{\delta}_a(x)$:

$$\bar{\delta}_a(x) = \int_{-\infty}^{\infty} B_a(x,x') \sum_{n=-\infty}^{\infty} \delta(x' - n\Delta_0) \ . \quad (3.10)$$

By using the relation between the projections of the Wigner distribution and the magnitude square of the fractional Fourier transformation, it can be shown that $\bar{\delta}_a(x)$ is also a uniform impulse train with a quadratic phase term:

$$\bar{\delta}_a(x) = A_\phi\sqrt{\frac{2r^2}{pqN}} \sum_{n=-\infty}^{\infty} \exp(j\pi\cot\phi x^2)\delta(x - \frac{nr}{\Delta_a}) \ , \quad (3.11)$$

where r is the greatest common divisor of q and N. If q and N are relatively prime, then we get the following transform relation for $f_a(x)$ in Eqn. 3.9:

$$f_a(x) = \sqrt{\frac{2}{pqN}}A_\phi \sum_{k=-N/2}^{N/2-1} f(\frac{k\Delta_0}{N}) \exp(j\pi(x^2\cot\phi - 2x\frac{k\Delta_0}{N}\csc\phi + (\frac{k\Delta_0}{N})^2\cot\phi)) \cdot \sum_{n=-\infty}^{\infty} \delta(x - \frac{n}{\Delta_a}) \ . \quad (3.12)$$

Note that to get Eqn. 3.12 we made use of the fact that $(\frac{k}{N})\Delta_0\cos\phi = kp/\Delta_a$. In this form of the transform relation, we see that the transform of the sampled periodic signal is also a sampled and periodic signal with a sampling interval of $1/\Delta_a$. This is the important property that allows us to claim the discrete fractional Fourier transform as being the relation between the magnitudes of the impulsive sequences in respective domains. This leads us to the following form for the discrete definition of the fractional Fourier transformation (DFFT):

$$f_a(\frac{n}{\Delta_a}) = \sum_{k=-N/2}^{N/2-1} T_\phi(\frac{n}{\Delta_a}, \frac{k}{\Delta_0}) f(\frac{k}{\Delta_0}) \ , \quad (3.13)$$

where we made use of $N = \Delta_0^2$ to get the above symmetric form of the transform, and the discrete transformation kernel T_ϕ is given by:

$$T_\phi(\frac{n}{\Delta_a}, \frac{k}{\Delta_0}) = \sqrt{\frac{2}{pqN}}A_\phi \exp(j\pi((\frac{n}{\Delta_a})^2\cot\phi - 2\frac{n}{\Delta_a}\frac{k}{\Delta_0}\csc\phi + (\frac{k}{\Delta_0})^2\cot\phi)) \ . \quad (3.14)$$

A fast algorithm for the efficient computation of the DFFT can be obtained as in the following steps:

$$h_a(k) = \exp(-j\pi(\cot\phi - \frac{\Delta_0}{\Delta_a}\csc\phi)(\frac{k}{\Delta_0})^2) f(\frac{k}{\Delta_0})$$
$$g_a(n) = \sqrt{\frac{2}{pqN}}A_\phi \sum_{k=-N/2}^{N/2-1} \exp(j\pi\frac{(n-k)^2}{\Delta_a\Delta_0}\csc\phi) h_a(k)$$
$$f_a(n) = \exp(-j\pi(\cot\phi - \frac{\Delta_a}{\Delta_0}\csc\phi)(\frac{n}{\Delta_a})^2) g_a(n)$$

where in the first step $h_a(k)$ can be obtained by computing N multiplications, in the second step the required convolution can be performed by using FFTs with a requirement of $O(N\log N)$ multiplications, and in the final step $f_a(n)$ for $-N/2 \leq n \leq N/2 - 1$ can be computed by using N multiplications. Hence, the total number of multiplications required by the above algorithm is $O(N\log N)$ which the same as that of FFT.

4. CONCLUSIONS

Based on the investigation of the fractional Fourier transformation of sampled periodic signals, the discrete fractional Fourier transformation is obtained. Similar to the generalization of the Fourier transformation by the fractional Fourier transformation, the discrete fractional Fourier transformation generalizes the DFT. A fast algorithm, which has the same complexity as FFT, is given for the efficient computation of the discrete transformation. Hence, it is expected that in many of the applications where fractional Fourier transformation has been found useful, the discrete fractional Fourier transformation will be used to increase efficiency.

5. REFERENCES

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