Observer Based Control of Chaos

Ercan Solak, Ömer Morgül and Umut Ersoy
Dept. of Electrical Engineering and Electronics
Bilkent University, 06533, Ankara, Turkey
fax : +90-312-266 41 26
e-mail : ercan@ee.bilkent.edu.tr

Abstract

In this work we consider the control of forced chaotic oscillators. To obtain any desirable behavior, the system parameters are effectively modified using state feedback. The system states used in the feedback are estimated through a nonlinear observer. The application of the proposed method is illustrated for Duffing and Van der Pol oscillators.

Key Words : Chaotic systems, State observers, Feedback Systems.

1 INTRODUCTION

Many different aspects of chaotic dynamics have attracted extensive interest from different disciplines in recent years. An interesting and challenging research subject in the field is the control of chaotic systems. However, there is neither a general method, nor a common framework for the control problem. Many publications on the subject [5, 6, 7] focus on driving a system from a chaotic regime to periodic orbits and from periodic orbits to chaotic trajectories.

Main control strategies that have been studied are open-loop methods [9], OGY technique [6, 10] and control engineering tools [5, 11]. There are yet other approaches that are more complex and difficult to apply in many cases [12, 13].

In this paper we present a different approach to the control of a class of chaotic systems. The class we consider covers the forced oscillators such as Duffing equation or the Van der Pol oscillator. We consider the Brunowsky canonical form state space realizations of these forced oscillators and apply our control strategy to this form. We note that systems other than forced oscillators can also be considered in our framework provided that a transformation to Brunowsky form can be found. Our aim is to change the dynamics of the given chaotic system by using state feedback so that desirable behavior are obtained, i.e. to change from chaos to periodic regime, or vice versa. If the behavior of the system to be controlled depends on some parameters in the dynamics, and if the bifurcation structure with respect to these parameters is known, then by applying input term appropriately, these coefficients could be changed to obtain desirable dynamics. We use state observers to estimate the states of the chaotic system. In constructing the input function we use those estimated states instead of the original system states which may not be available through direct measurements. Under relatively mild conditions, any control scheme involving asymptotically converging state estimates performs as well as the one that uses the original system states.

In the next section we present a brief reminder of full order observers. Then we consider the observer based control of chaotic systems. Finally the results are applied to Duffing and Van der Pol Oscillators.

2 FULL ORDER OBSERVER

We begin with the notion of observer for a nonlinear dynamical system. Consider the following time-invariant, single-input, single-output system

$$\begin{aligned} \dot{x} &= Ax + g(x) + h(t), \ x(0) = x_0, && (1) \\ y &= Cx, && (2) \end{aligned}$$

where, $x \in \mathbf{R}^n$ denotes the state vector, $g : \mathbf{R}^n \to \mathbf{R}^n$, $A \in \mathbf{R}^{n \times n}$, $h : \mathbf{R}_+ \to \mathbf{R}^n$, $C \in \mathbf{R}^{1 \times n}$ and $y \in \mathbf{R}$ is the system output. Here $h(t)$ can be viewed as a forcing term. A full order observer is another dynamical system which, by using the output and the input of (1),(2), estimates the states of (1),(2), see e.g. [1]. The most common observer construction scheme involves replicating the system dynamics together with an additive output error injection term. Namely,

$$\dot{\hat{x}} = A\hat{x} + g(\hat{x}) + h(t) + K(y - C\hat{x}), \qquad (3)$$

where $K \in \mathbf{R}^n$ is the gain vector. With this choice of observer, we define the estimation error to be $\varepsilon = x - \hat{x}$. Then the error dynamics is given by

$$\dot{\varepsilon} = (A - KC)\varepsilon + g(x) - g(\hat{x}). \qquad (4)$$

Thus the estimation problem is equivalent to the stability of (4).

We note that if the pair (C, A) is observable, then by an appropriate choice of the feedback gain K, the eigenvalues of $A_c = A - KC$ can be assigned arbitrarily, see e.g. [14].

Further, we impose the following two assumptions on the system dynamics.

1. The system is in the form

$$\dot{x} = Ax + Bf(x) + h(t), \quad (5)$$
$$y = Cx, \quad (6)$$

where $f : \mathbf{R}^n \rightarrow \mathbf{R}$ and A, B, C are given by the Brunowsky canonical form,

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, C = [1\ 0\ \dots\ 0]. \quad (7)$$

2. $f(\cdot)$ satisfies the following global Lipschitz condition

$$\|f(x_1) - f(x_2)\| \leq \gamma \|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathbf{R}^n \quad (8)$$

for some $\gamma > 0$.

In [2], it has been shown that there exists a gain vector K such that the system

$$\dot{\hat{x}} = A\hat{x} + Bf(\hat{x}) + h(t) + K(y - C\hat{x}) \quad (9)$$

is an exponential observer for (5),(6). For details and the procedure to determine the gain vector see [2],[3],[4].

Remark 1 *The condition (8) may seem too restrictive. However for chaotic systems, the trajectories always remain in a bounded region of the state space. Hence whenever $f(\cdot)$ differentiable, a Lipschitz constant can be found in that bounded region.*

Remark 2 *The above observer construction scheme is still valid when the system is not in the form (5),(6) but can be transformed into this form by a diffeomorphic coordinate transformation.*

Corollary 1 *If the nonlinearity in (1) can be expressed as a function of the output, i.e. $g(x) = \hat{g}(y)$, and if the pair (C, A) is observable then the observer (3) can be modified as*

$$\dot{\hat{x}} = A\hat{x} + \hat{g}(y) + h(t) + K(y - C\hat{x}), \quad (10)$$

yielding a linear and stable error system. Note that by an appropriate choice of K, the eigenvalues of $A - KC$ can be placed at arbitrary location on the left half plane. In this case the nonlinearity need not be continuous (e.g. hysteresis or signum type nonlinearity).

3 OBSERVER BASED CONTROL

In this section we consider the application of observer theory given in the previous section to the control of forced chaotic oscillators. We consider the systems described by the following differential equation.

$$x^{(n)} + F(x,\ \dot{x}, \dots, x^{(n-1)}) = h(t) + r(t) \quad (11)$$

where $x^{(i)}$ denotes the i^{th} time derivative of x for $i = 1, 2, \dots, n-1$, $h(t)$ is a known forcing function and $r(t)$ is the control input to be determined. We assume that F is differentiable with respect to its arguments. This class of systems cover a wide range of chaotic oscillators, e.g. Duffing equation, Van der Pol oscillator. We first explain our methodology.

Assume that the signal x as well as its time derivatives $x^{(i)}, i = 1, 2, \dots, n-1$ are available. Then by choosing the control input r as

$$r(t) = \hat{h}(t) - h(t) + F(x, \dots, x^{(n-1)}) - \hat{F}(x, \dots, x^{(n-1)}) \ . \quad (12)$$

we rewrite (11) as

$$x^{(n)} + \hat{F}(x,\ \dot{x}, \dots, x^{(n-1)}) = \hat{h}(t) \ . \quad (13)$$

Hence we can effectively change the dynamics of the forced oscillator (11) to any desired dynamics given by (13) by the choice of the feedback input (12). In particular, assume that F in (11) has the following form:

$$F(x, \dots, x^{(n-1)}) = \sum_{i=1}^{p} \alpha_i F_i(x, \dots, x^{(n-1)}) + F_r(x, \dots, x^{(n-1)}), \quad (14)$$

where F_i for $i = 1, 2, \dots, p$ are known and differentiable functions, and α_i are real and constant parameters. Assume that the dynamical behavior of (11) depends on the parameters α_i, (e.g. a bifurcation diagram in terms of parameters α_i is known). Then, by choosing the control input $r(t)$ as

$$r(t) = \sum_{i}^{p} (\alpha_i - \hat{\alpha}_i) F_i + \hat{h}(t) - h(t) \ . \quad (15)$$

Then (11) reduces to

$$x^{(n)} + \sum_{i}^{p} \hat{\alpha}_i F_i + F_r = \hat{h}(t). \quad (16)$$

Hence any behavior in the bifurcation diagram of (11) can be obtained with appropriate choice of the parameters $\hat{\alpha}_i$.

The basic problem in the scheme presented above is the availability of the signals $x^{(i)}(t)$, for $i = 1, \dots, n-1$. We assume that $x(t)$ is available through measurements, however the implementation of derivative in physical settings is adversely corrupted by the presence of noise. Alternatively we could use an observer to estimate the states, hence the derivatives. Using these estimated states in the control input (12) would inevitably bring a perturbation term $\epsilon(t)$ in the right hand sides of (13),(16). However if the observer states converge exponentially to those of the forced oscillator, asymptotically we can neglect the perturbation term, [1]. Assuming that the different regimes of (11) are structurally stable, we could expect to observe the same control effect even when we use the state estimates in (12).

For the observer part of the control scheme we first transform (11) to state space coordinates by assigning the state variables as

$$x_1 = x, \quad x_{i+1} = x^{(i)}, \quad i = 1, 2, \ldots, n-1. \tag{17}$$

Then, in state space (11) becomes

$$\begin{aligned} \dot{x}_1 &= x_2, && (18)\\ \dot{x}_2 &= x_3, && (19)\\ &\vdots && (20)\\ \dot{x}_{n-1} &= x_n, && (21)\\ \dot{x}_n &= -F(x_1, x_2, \ldots, x_n) + h(t), && (22)\\ y &= x, && (23) \end{aligned}$$

which is in the form described by (5),(6). Since F is differentiable, we can find a Lipschitz bound on F in the compact region bounding the system trajectories. Hence an exponential observer can be readily constructed to estimate the states of (20). As a special case, assume that F in (11) has the following special form

$$F(x, \dot{x}, \ldots, x^{(n-1)}) = a_0 x + a_1 \dot{x} + \ldots + a_{n-1} x^{(n-1)} + f_r(x), \tag{24}$$

where $a_0, \ldots, a_{n-1}$ are arbitrary real constants and f_r is an arbitrary function. Then, without the requirement of being Lipschitz we can directly apply Corollary 1 to construct an exponential observer.

Example 1 (Duffing System) We consider the following system :

$$\ddot{x} + a_0 x + a_1 \dot{x} + a_2 x^3 = q \cos \omega t + r(t) \quad . \tag{25}$$

For the uncontrolled case (i.e $r = 0$), the bifurcation structure of the Duffing equation given above with respect to parameters a_0, a_1, a_2, q and ω could be found in many sources, see e.g. [8]. By writing (25) in state space and defining the state vector $u = (x_1\ x_2)$ we have

$$\begin{aligned} \dot{u} &= Au + Bf(y) + B(q \cos wt + r(t)), && (26)\\ y &= Cu, && (27) \end{aligned}$$

where

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ C = [1\ 0],\\ f(y) &= -a_2 y^3. && (28) \end{aligned}$$

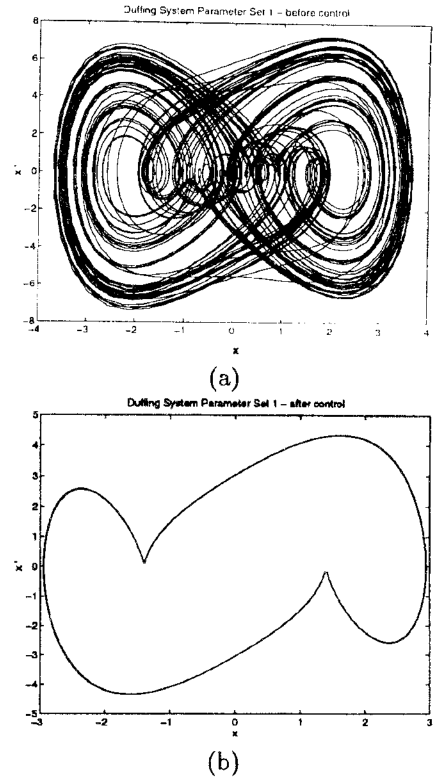

Figure 1: State trajectories of the Duffing Equation for Set 1, (a) before control (b) after control

Note that the nonlinearity $f(y)$ is a function of the output only. Hence applying Corollary 1 we construct the observer as

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + k_1(x - \hat{x}_1), && (29)\\ \dot{\hat{x}}_2 &= -a_0 \hat{x}_1 - a_1 \hat{x}_2 + k_2(x - \hat{x}_1) - a_2 y^3 \\ &\quad + q \cos \omega t + r(t). && (30) \end{aligned}$$

We consider two parameter sets for the Duffing system. Set 1 : $(a_0 = 0,\ a_1 = 0.25,\ a_2 = 1,\ q = 11,\ \omega = 1)$, Set 2 : $(a_0 = 0,\ a_1 = 1.45,\ a_2 = 1,\ q = 11,\ \omega = 1)$, The first set corresponds to chaotic behavior and the second set corresponds to a limit cycle, [8]. By applying the control law $r(t) = -1.2\hat{x}_2$ the behavior corresponding Set 1 can be effectively changed to that of Set 2. Figure 1 shows the system behavior before and after the application of control. Figure 1(b) is plotted after the transients have died out. Observer gains for Set 1 are chosen to be $k_1 = 2.75$ and $k_2 = 1.3125$. It is also possible to drive the system behavior from that of Set 2(limit cycle) to that of Set 1(chaos) using the control input $r(t) = 1.2\hat{x}_2$.

Example 2 (Van der Pol Oscillator) As a second example, consider the following forced Van der Pol oscillator :

$$\ddot{x} + d(x^2 - 1)\dot{x} + x = a \cos \omega t + r(t). \tag{31}$$

For various values of d, ω, a this oscillator exhibits a variety of nonlinear phenomena, including chaos, [15]. Transforming to state space we obtain

$$\dot{u} = Au + Bf(u) + Bh(t), \quad (32)$$
$$y = Cu, \quad (33)$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C = [1\ 0],$$
$$f(u) = -x_1 - d(x_1^2 - 1)x_2, \; h(t) = a\cos wt + r(t). \quad (34)$$

Obviously the system (32),(33) is in the Brunowsky canonical form given in (5),(6). Although $f(u)$ is not globally Lipschitz, we can still find a Lipschitz bound on $f(u)$ since the trajectories which are of interest to us remain in a bounded convex region of the state space. Hence there exist gains k_1 and k_2 such that

$$\dot{\hat{x}}_1 = \hat{x}_2 + k_1(x - \hat{x}_1), \quad (35)$$
$$\dot{\hat{x}}_2 = -d(\hat{x}_1^2 - 1)\hat{x}_2 - \hat{x}_1 + k_2(x - \hat{x}_1) + a\cos\omega t + r(t), \quad (36)$$

is an exponential observer for (32),(33).
We again select two sets of parameters of the uncontrolled system from the bifurcation diagrams.
Set 1 : $(a = 2.5,\ d = 6,\ \omega = 3)$, chaos,
Set 2 : $(a = 2.5,\ d = 0.5,\ \omega = 3)$, limit cycle.
For the first parameter set we choose the control law $r(t) = 5.5(\hat{x}_1^2 - 1)\hat{x}_2$, driving the system from chaos (Set 1) to limit cycle (Set 2). We assigned the observer gains as $k_1 = 12$ and $k_2 = 35$. The simulation results are shown in Figure 2. It is also possible to drive the system behavior from that of Set 2(limit cycle) to that of Set 1(chaos) using the control input $r(t) = -5.5(\hat{x}_1^2 - 1)\hat{x}_2$.

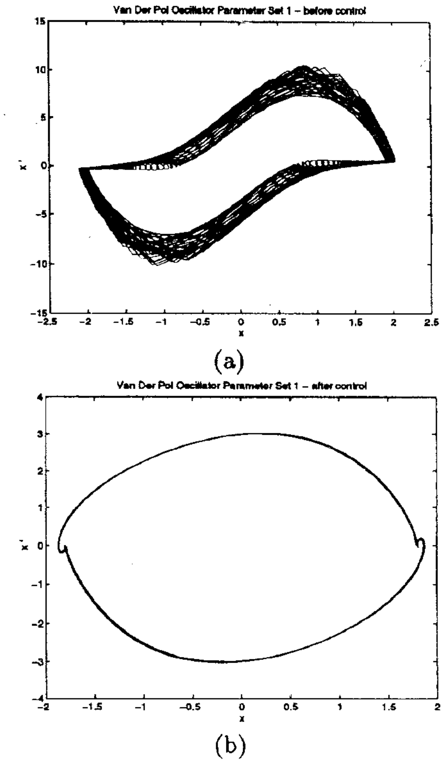

Figure 2: State trajectories of the Van der Pol Oscillator for Set 1, (a) before control (b) after control

References

[1] M. Vidyasagar, *Nonlinear Systems Analysis.* Prentice Hall, New Jersey, second edition, 1993.

[2] G. Ciccarella, M. Dalla Mora, and A. Germani, "A Luenberger-like observer for nonlinear systems," *International Journal of Control*, vol. 57, pp. 537–556, 1993.

[3] Ö. Morgül and E. Solak, "On the observer based synchronization of chaotic systems," *Physical Review E*, vol. 54, no. 5, pp. 4803–4811, 1996

[4] Ö. Morgül and E. Solak, "On the synchronization of chaotic systems by using state observers," to appear in *International J. of Bifurcation and Chaos.*

[5] G. Chen, X. Dong, "On feedback control of chaotic continuous-time systems," *IEEE Trans. on Circuits and Systems*, vol. 40, no. 9, 1993.

[6] E. Ott, C. Grebogi, and J. A. Yorke, "Controlling chaos," *Physical Review Letters*, vol. 64, no. 11, pp. 1196–1199, 1990.

[7] A. Y. Loskutov and A. I. Shismarev, "Control of dynamical systems behavior by parametric perturbations: An analytic approach," *Chaos*, vol. 4, no. 2, pp. 391–395, 1994.

[8] V. Chadran, S. Elgar and C. Pezeshki, "Bispectral and trispectral characterization of transition to chaos in the Duffing oscillator," *Int. J. of Bifurcation and Chaos* vol. 3, no. 3, pp. 551–557, 1993.

[9] A. Hübler, "Adaptive Control of Chaotic Systems," *Helv. Phys. Acta*, vol. 62, 1989.

[10] U. Dressler and G. Nitsche, "Controlling chaos using time delay coordinates," *Physical Review Letters*, vol. 68, no. 1, pp. 1–4, 1992.

[11] S. Bhajekar, E. Jonckeere and A. Hammad. "$\mathcal{H}^\infty$ control of chaos", in *Proceedings of the 33rd Conference on Decision and Control, Lake Buena Vista, 1994*, pp. 3285–3286.

[12] P. M. Alsing, A. Gavrielides and V. Kovanis, "Using neural networks for controlling chaos," *Physical Review E*, vol. 49, no. 2, pp. 1225–1231, 1994.

[13] J. K. John and R. E. Amritkar, "Synchronization by feedback and adaptive control," *Int. J. of Bifurcation and Chaos*, vol. 4, pp. 1687–1695, 1994.

[14] T. Kailath, *Linear Systems* (Prentice-Hall, 1980).

[15] R. Mettin, U. Parlitz, and W. Lauterborn "Bifurcation structure of the driven Van der Pol Oscillator," *Int. J. of Bifurcation and Chaos*, vol. 3, pp. 1529–1555, 1993.