Proceedings of the 40th IEEE
Conference on Decision and Control
Orlando, Florida USA, December 2001

Stabilization and Disturbance Rejection for the Beam Equation

Ömer Morgül
Bilkent University, Dept. of Electrical and Electronics Engineering
06533, Bilkent, Ankara, Turkey

Abstract

We consider a system described by the Euler-Bernoulli beam equation in a bounded domain with appropriate boundary conditions. To stabilize the system, we propose a dynamic boundary controller applied at the free end of the system. We show that with the proposed controller, the closed-loop system is asymptotically stable. Moreover, we consider the case in which the output of the controller is corrupted by disturbance.

Key Words : Distributed Parameter Systems, Stability, Disturbance Rejection.

1 Introduction

In this note, we consider a linear time-invariant system which is represented by one dimensional Euler-Bernoulli beam equation in a bounded domain. We assume that the system is clamped at one end and the boundary control input is applied at the other end. For this system, we propose a finite dimensional *dynamic* boundary controller. The transfer function of the controller is a proper rational function of the complex variable s, and may contain a single pole at $s = 0$ and another one $s = j\omega_1$, $\omega_1 \neq 0$, provided that the residues corresponding to these poles are nonnegative; the rest of the transfer function is required to be a strictly positive real function. We then show that if ω_1 does not belong to a countable set then the closed loop system is asymptotically stable. We also consider the case where the output of the controller is corrupted by disturbance. We show that if the structure of the disturbance is known (i.e. the frequency spectrum), then it may be possible to choose the controller accordingly to attenuate the effect of the disturbance at the system output.

2 Problem Statement

We consider a flexible beam clamped at one end and is free at the other end. Without loss of generality, we assume that the beam length, mass density and the flexural rigidity have unit values. We denote the displacement of the beam by $u(x,t)$ at $x \in (0,1)$ and $t \geq 0$. The beam is clamped at one end and is controlled by a boundary control force at the other end. The equations are given as (see [2])

$$u_{tt} + u_{xxxx} = 0 \quad , \quad u(0,t) = u_x(0,t) = 0 \qquad (1)$$

$$u_{xx}(1,t) = 0 \quad , \quad u_{xxx}(1,t) = f(t) \qquad (2)$$

where a subscript, as in u_t denotes a partial differential with respect to the corresponding variable, and $f(t)$ is the boundary control force applied at the free end of the beam.

We assume that $f(t)$ is generated by the following controller

$$\dot{z}_1 = Az_1 + bu_t(1,t), \dot{x}_2 = -\omega_1 x_1 + u_t(1,t) \qquad (3)$$

$$\dot{x}_1 = \omega_1 x_2, f(t) = c^T z_1 + du_t(1,t) + ku(1,t) + k_1 x_2 \qquad (4)$$

where $z_1 \in \mathbf{R}^n$ is the controller state, $A \in \mathbf{R}^{n\times n}$ is a constant matrix, $b, c \in \mathbf{R}^n$ are constant column vectors, d, k, k_1 are positive constants, and the superscript T denotes transpose. If we take the Laplace transform, then the controller transfer function $g(s)$ between its input $u_t(1,t)$ and output $f(t)$ may be found as

$$g(s) = g_1(s) + \frac{k}{s} + \frac{k_1 s}{s^2 + \omega_1^2} \quad , \qquad (5)$$

where $g_1(s) = c^T(sI - A)^{-1}b + d$. We make the following assumptions concerning the actuator given by (3)-(4) thoroughout this work :

Assumption 1 : A is Hurwitz stable, and the triple (A, b, c) is minimal.

Assumption 2 : $d \geq 0, k \geq 0, k_1 \geq 0$; moreover there exists a constant γ, $d \geq \gamma \geq 0$, such that the following holds :

$$d + \mathcal{R}e\{c^T(j\omega I - A)^{-1}b\} > \gamma \quad , \quad \omega \in \mathbf{R} \quad , \qquad (6)$$

Moreover for $d > 0$, we assume $\gamma > 0$ as well.

3 Stability Results

To analyze the system given by (1)-(4), we define the function space $\mathcal{H}$ as follows : $z = (u\ v\ z_1\ x_1\ x_2)^T$

$$\mathcal{H} := \{z | u \in \mathbf{H}_0^2, v \in \mathbf{H}_0^1, z_1 \in \mathbf{R}^n,\ x_1, x_2 \in \mathbf{R},\} \quad , (7)$$

for the definition of various spaces, see e.g. [2]-[3]. Then, the system given by (1)-(4) can be written in the following abstract form :

$$\dot{z} = Lz \quad , \quad z(0) \in \mathcal{H} \quad , \qquad (8)$$

where $z = (u\ u_t\ z_1\ x_1\ x_2)^T \in \mathcal{H}$, and the operator $L : \mathcal{H} \to \mathcal{H}$ is a linear unbounded operator. Our main result is the following :

Theorem 1 : Consider the system given by (8).

i : The operator L generates a C_0-semigroup of contractions $T(t)$ in $\mathcal{H}$, (for the terminology of semigroup theory, the reader is referred to e.g. [1]).

ii : If $\omega = \omega_1$ is not one of the roots of the following transcendental equation

$$\cosh\tau \sin\tau - \sinh\tau \cos\tau = 0 \quad , \quad \tau = \sqrt{\omega} \quad , \qquad (9)$$

then the semigroup $T(t)$ generated by L is asymptotically stable, that is all solutions of (8) asymptotically converge to zero.

iii : If $\gamma > 0$, (see (6)), then the semigroup $T(t)$ is exponentially stable.

Proof : The proof of this theorem requires some lenghty calculations and is omitted here due to space limitations. □

4 Disturbance Rejection

In this section we show the effect of the proposed control law given by (3)-(4) on the solutions of the system given by (1)-(2), when the output of the controller is corrupted by a disturbance $d(t)$, that is (4) has the following form :

$$f(t) = c^T z_1 + du_t(1,t) + ku(1,t) + k_1 x_2 + d(t) \quad (10)$$

or equivalently we have $\hat{f}(s) = g(s)\hat{y}_t(1,s) + \hat{d}(s)$. By taking the Laplace transform of (1)-(2) and using zero initial conditions, after some straightforward calculations we obtain $\hat{u}_t(1,s) = -G_1(s)\hat{f}(s)$ where

$$G_1(s) = \frac{s(\cosh\tau \sin\tau - \sinh\tau \cos\tau)}{\tau^3(1 + \cosh\tau \cos\tau)} \quad , \qquad (11)$$

and $s^2 = -\tau^4$. By using (11) we obtain :

$$\hat{u}_t(1,s) = -\frac{G_1(s)}{1 + G_1(s)g(s)}\hat{d}(s) \quad . \qquad (12)$$

From (12) we can also derive a procedure to design $g(s)$ if we know the structure of $d(t)$. For example if $d(t)$ has a band-limited frequency spectrum, (i.e. has frequency components in an interval of frequencies $[\Omega_1, \Omega_2]$), then we can choose $g(s)$ to minimize

$$c(\omega) = \left| \frac{G_1(j\omega)}{1 + G_1(j\omega)g(j\omega)} \right| \quad \omega \in [\Omega_1, \Omega_2] \ . \qquad (13)$$

Note that to ensure the stability of the closed-loop system, $g_1(s)$ should be a strictly positive real function as well, (see (5)). As a simple example, assume that $d(t) = a\cos\omega_0(t)$. Then we may choose $g(s)$ in the form (5) with $\omega_1 = \omega_0$. Provided that the assumptions 1-2 are satisfied and that $j\omega_0$ is not a zero of $G_1(s)$, the closed-loop system is asymptotically stable, (see Theorem 1). Moreover, if $k_2 > 0$, then $c(\omega)$ given above satisfies $c(\omega_0) = 0$. From (13) we may conclude that this eliminates the effect of the disturbance at the output $u_t(1,t)$. For an application of the ideas presented here to the wave equation, see [3], [4].

5 Conclusion

In this note, we considered a linear time-invariant system which is represented by one dimensional Euler-Bernoulli beam equation in a bounded domain. We assumed that the system is clamped at one end and the boundary control force input is applied at the other end. For this system, we proposed a finite dimensional *dynamic* boundary controller. We then proved that the closed-loop system is asymptotically stable under certain conditions. We also studied the case where the output of the controller is corrupted by a disturbance. We showed that, if the frequency spectrum of the disturbance is known, then by choosing the controller appropriately we can obtain better disturbance rejection.

References

[1] Z. H. Luo, B. Z. Guo, and Ö. Morgül, *Stability and Stabilization of Infinite Dimensional Systems with Applications*, Springer Verlag, series in Communications and Control Eng., London, 1999.

[2] Ö. Morgül, "Dynamic boundary control of a Euler-Bernoulli beam," *IEEE Trans. on Auto. Contr.*, vol. 37, No. 5, 1992, pp. 639-642.

[3] Ö. Morgül, "Stabilization and disturbance rejection for the wave equation," *IEEE Trans. on Auto. Contr.*, vol. 43, no. 1, 1998, pp. 89-95.

[4] Ö. Morgül, "A dynamic control law for the wave equation," *Automatica*, vol. 30, no. 11, pp. 1785-1792, 1994.