Stability of delayed feedback controllers for discrete time systems

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Abstract

We consider the delayed feedback control (DFC) scheme for one dimensional discrete time systems. To analyze the stability, we construct a map whose fixed points correspond to the periodic orbits of the system to be controlled. Then the stability of the DFC is equivalent to the stability of the corresponding equilibrium point of the constructed map.

Key words: Chaos control, delayed feedback system, Pyragas controller, stability.

1 Introduction

In recent years, the analysis and control of chaotic behaviour in dynamical systems has received a great deal of attention among scientists from various disciplines such as engineers, physicists, biologists, etc. The development in the field of chaos control accelerated mainly after the seminal paper [1] where the term "controlling chaos" was introduced. This work had a strong influence, especially, on the approach of the physics community to the problem of controlling chaotic systems and is based on variation of certain parameters which has certain effects on the chaotic behaviour. In such systems usually many unstable periodic orbits are embedded in their chaotic attractors, and as shown in [1], by using small external feedback input, some of these orbits may be stabilized. Therefore, by applying small feedback inputs, it may be possible to force these systems to behave in a regular way. Following the work of [1], various chaos control techniques have been proposed, [2], [3]. Among these, the delayed feedback control (DFC) scheme first proposed in [4] and is also known as Pyragas scheme, has gained considerable attention due to its various attractive features. In this technique the required control input is basically the difference between the current and one period delayed states multiplied by a gain. Hence if the system is already in the periodic orbit, this term vanishes. Also if the trajectories asymptotically approach to the periodic orbit, this term becomes smaller. For more details as well as various applications of DFC, see [5], [6] and the references therein.

Despite its simplicity, a detailed stability analysis of DFC is very difficult, [5], [7]. Apparently, DFC has some inherent limitations, [7]. To overcome these limitations, several modifications has been proposed, see e.g. [8], [5], [6] and the references therein.

In this work, we consider the delayed feedback control (DFC) scheme for one dimensional discrete time systems. To analyze the stability, we construct a map whose fixed points correspond to the periodic orbits of the system to be controlled. Then the stability of the DFC is equivalent to the stability of the corresponding equilibrium point of the constructed map. For each periodic orbit, we construct a characteristic polynomial of a related Jacobian matrix. The Schur stability of this polynomial could be used to analyze the stability of DFC. By using Schur-Cohn criterion, we can find bounds on the gain of DFC to ensure stability.

2 Stability of DFC

Let us consider the following one dimensional discrete-time system

$$x(k+1) = f(x(k)) \quad , \qquad (1)$$

where $k = 0, 1, \ldots$ is the discrete time index, $f : \mathbf{R} \to \mathbf{R}$ is an appropriate function, which is assumed to be differentiable wherever required. We assume that the system given by (1) possesses a T periodic orbit characterized by the set $\Sigma_T = \{x_0^*, x_1^*, \ldots, x_{T-1}^*\}$, i.e. for $x(0) = x_0^*$, the iterates of (1) yields $x(1) = x_1^*, \ldots, x(T-1) = x_{T-1}^*$, $x(k) = x(k-T)$ for $k \geq T$.

Let us call this orbit as an uncontrolled periodic orbit (UCPO) for future reference.

Let $\mathcal{S} \subset \mathbf{R}$ be a set, and $y \in \mathbf{R}$. We define the distance $d(y, \mathcal{S})$ between y and $\mathcal{S}$ as

$$d(y, \mathcal{S}) = \min_{z \in \mathcal{S}} | y - z | \quad . \tag{2}$$

We say that Σ_T is asymptotically stable if for some $\epsilon > 0$. for any $y \in \mathbf{R}$ satisfying $d(y, \Sigma_T) < \epsilon$, the iterates of (1) with $x(0) = y$ yields $\lim_{k \to \infty} d(x(k), \Sigma_T) = 0$. Moreover we say that Σ_T is exponentially stable if this decay is exponential, i.e. the following holds for some $M > 0$ and $\rho \in (0, 1)$

$$d(x(k), \Sigma_T) \leq M \rho^k d(y, \Sigma_T) \quad . \tag{3}$$

To stabilize periodic orbits of (1), let us apply a control input u as :

$$x(k+1) = f(x(k)) + u(k) \tag{4}$$

In DFC, the following simple feedback control input is used to (possibly) stabilize Σ_T :

$$u(k) = K(x(k) - x(k-T)) \quad , \tag{5}$$

where $K \in \mathbf{R}$ is a constant gain to be determined. Note that if $x(0) \in \Sigma_T$, then $x(k) \in \Sigma_T$ for $k \geq 0$ and $u(k) \equiv 0$. Moreover, if Σ_T is asymptotically stabilized, then $u(k) \to 0$ as $k \to \infty$. In the sequel we will derive some conditions and bounds on K for the stabilization of periodic orbits. To motivate our analysis, consider the case $T = 1$. In this case we have $\Sigma_1 = \{x_0^*\}$ where $x_0^* = f(x_0^*)$, i.e. period 1 orbits are the same as fixed points of f. By defining $x_1(k) = x(k-1)$, $x_2(k) = x(k)$, we can rewrite (4) and (5) as

$$\begin{aligned} x_1(k+1) &= x_2(k) \quad , \\ x_2(k+1) &= f(x_2(k)) + K(x_2(k) - x_1(k)) \quad . \end{aligned} \tag{6}$$

Let us define $\hat{x} = (x_1 \ x_2)^T \in \mathbf{R}^2$, where here and in the sequel the superscript T denotes transpose, and define $F : \mathbf{R}^2 \to \mathbf{R}^2$ as $F(\hat{x}) = (x_2 \ \ Y_2)^T$ where $Y_2 = f(x_2) + K(x_2 - x_1)$. For $\hat{x}^* = (x_1^* \ x_2^*)^T$, $F(\hat{x}^*) = \hat{x}^*$ holds if and only if $x_1^* = x_2^* = f(x_2^*)$. Hence any fixed point of F corresponds to an UCPO Σ_1 of (1), and vice versa. Hence asymptotic stability of Σ_1 for (4) and (5) can be analyzed by studying the stability of the corresponding fixed point of F for (6). To analyze the latter, let $\Sigma_1 = \{x_0^*\}$ and set $a_1 = Df(x_0^*)$, and

$$J = \frac{\partial F}{\partial x} |_{\Sigma_1} \quad ,$$

where D stands for the derivative and J is the Jacobian of F evaluated at the equilibrium point. Clearly the components of J are given as

$J(1,1) = 0$, $J(1,2) = 1$, $J(2,1) = -k$, $J(2,2) = a_1 + k$

The characteristic polynomial $p_1(\lambda)$ of J can easily be found as

$$p_1(\lambda) = \det(\lambda I - J) = \lambda^2 - (a_1 + k)\lambda + k \quad . \tag{7}$$

We say that a polynomial is Schur stable if all of its eigenvalues are inside the unit disc of the complex plane, i.e. have magnitude less then unity. Hence, the asymptotic stability of the fixed point of F for (6), hence the asymptotic stability of Σ_1 for (4) and (5) could be analyzed by studying the Schur stability of $p_1(\lambda)$ given by (7). Moreover note that the exponential stability of the fixed points of F is equivalent to Schur stability of $p_1(\lambda)$, [9]. Hence we can state the following facts :

Theorem 1 : Let $\Sigma_1 = \{x_0^*\}$ be an UCPO of (1) and set $a_1 = Df(x_0^*)$. Then :

1 : Σ_1 is exponentially stable for (4) and (5) if and only if $p_1(\lambda)$ given by (7) is Schur stable. This condition is only sufficient for asymptotic stability of Σ_1.

2 : If $p_1(\lambda)$ has an unstable root, i.e. outside the unit disc, then Σ_1 cannot be asymptotically stable for (4) and (5). □

Remark 1 : We note that Schur stability of a polynomial can be determined by checking some inequalities in terms of its coefficients; this is known as the Jury test, see [10]. We will apply this test to (7) later. □

To motivate our approach further, let us consider the case $T = 2$. Let the period 2 UCPO of (1) be given as $\Sigma_2 = \{x_0^*, x_1^*\}$ and define $a_1 = Df(x_0^*)$, $a_2 = Df(x_1^*)$. By defining $x_1(k) = x(k-2)$, $x_2(k) = x(k-1)$, $x_3(k) = x(k)$, we can rewrite (4) and (5) as

$$\begin{aligned} x_1(k+1) &= x_2(k) \quad , \quad x_2(k+1) = x_3(k) \\ x_3(k+1) &= f(x_3(k)) + K(x_3(k) - x_1(k)) \quad . \end{aligned} \tag{8}$$

For $\hat{x} = (x_1 \ x_2 \ x_3)^T \in \mathbf{R}^3$, let us define $G : \mathbf{R}^3 \to \mathbf{R}^3$ as $G(\hat{x}) = (x_2 \ Y_1 \ Y_2)^T$ where $Y_1 = x_3$, $Y_2 = f(Y_1) + K(Y_1 - x_1)$. Note that the fixed points of G do not correspond to the UCPO's of (1), but the fixed points of $F = G^2$ does. To see this, note that $F = (Y_1 \ Y_2 \ Y_3)^T$ where $Y_3 = f(Y_2) + K(Y_2 - x_2)$. For $\hat{x}^* = (x_1^* \ x_2^* \ x_3^*)^T$, the fixed points of F, i.e. the solutions of $F(\hat{x}^*) = \hat{x}^*$, are given as $x_1^* = x_3^*$, $x_2^* = f(x_1^*)$, $x_3^* = f(x_2^*) = f^2(x_1^*)$. Hence for any UCPO $\Sigma_2 = \{x_0^*, \ x_1^*\}$ of (1), there corresponds a fixed point $\hat{x}^* = (x_0^* \ x_1^* \ x_0^*)^T$ of F and vice versa. Hence the asymptotic stability of Σ_2 for (4) and (5) is equivalent to the asymptotic stability of the corresponding fixed point of F for the system $\hat{x}(k+1) = F(\hat{x}(k))$. To analyze the latter, let us define the Jacobian of F at equilibrium as

$$J = \frac{\partial F}{\partial x} |_{\Sigma_2} \quad .$$

The entries of J can be calculated as

$$J(i,j) = \frac{\partial Y_i}{\partial x_j} |_{\Sigma_2} \quad , \quad i,j = 1,2,3 \quad .$$

After straightforward calculations, we obtain

$$\begin{array}{l} J(1,1) = J(1,2) = J(2,2) = 0 \quad , \quad J(1,3) = 1 \\ J(2,1) = -K \quad , \quad J(2,3) = a_1 + K \\ J(3,1) = -K(a_2+K) \quad , \quad J(3,2) = -K \\ J(3,3) = (a_1+K)(a_2+K) \quad . \end{array}$$

The characteristic polynomial $p_2(\lambda)$ of J can be calculated as :

$$\begin{aligned} p_2(\lambda) = \ & \det(\lambda I - J) = \lambda^3 - (a_1+K)(a_2+K)\lambda^2 \\ & + K((a_1+K)+(a_2+K))\lambda - K^2 \quad . \end{aligned} \tag{9}$$

Hence for the stability of Σ_2 for (4) and (5), we can study the Schur stability of $p_2(\lambda)$ given above. We will consider the Schur stability of $p_2(\lambda)$ for some cases in the sequel.

Now let us proceed to the general case $T = m$. Assume that (1) has an m periodic UCPO given by $\Sigma_m = \{x_0^*, x_1^*, \ldots, x_{m-1}^*\}$ and define $a_1 = Df(x_0^*)$, $a_2 = Df(x_1^*)$, ..., $a_m = Df(x_{m-1}^*)$. In this case, by defining $x_1(k) = x(k-m)$, $x_2(k) = x(k-m+1)$, ..., $x_m(k) = x(k-1)$, $x_{m+1}(k) = x(k)$, $\hat{x} = (x_1 \ x_2 \ \ldots \ x_{m+1})^T \in \mathbf{R}^{m+1}$, and $Y_2 = f(x_{m+1}) + K(x_{m+1} - x_1)$, we can transform (4), (5) into the form $\hat{x}(k+1) = G(\hat{x}(k))$ where $G : \mathbf{R}^{m+1} \to \mathbf{R}^{m+1}$ is defined as $G(\hat{x}) = (x_2 \ x_3 \ \ldots \ x_{m+1} \ Y_2)^T$. As before, the UCPO Σ_m does not correspond to a fixed point of G, but it corresponds to a fixed point of $F = G^m$. To see this, note that

$$F(\hat{x}) = (Y_1 \ Y_2 \ \ldots \ Y_{m+1})^T$$

where $Y_1 = x_{m+1}$, $Y_{i+1} = f(Y_i) + K(Y_i - x_i)$, $i = 1, 2, \ldots, m$. For $\hat{x}^* = (x_1^* \ x_2 *^* \ \ldots \ x_{m+1}^*)^T$, the fixed points of F, i.e. the solutions of $F(\hat{x}^*) = \hat{x}^*$, are given as $x_i^* = Y_i^*$, $i = 1, \ldots, m+1$, which in turn implies $x_1^* = x_{m+1}^*$, $x_2^* = f(x_1^*)$, $x_{j+1}^* = f(x_j^*)$, $j = 1, \ldots, m$. Hence the asymptotic stability of Σ_m for (4) and (5) is equivalent to the asymptotic stability of the corresponding fixed point of F for the system $\hat{x}(k+1) = F(\hat{x}(k))$. To analyze the latter, let us define the Jacobian of F at the equilibrium as

$$J = \frac{\partial F}{\partial x} |_{\Sigma_m} \quad .$$

The entries of J can be calculated as

$$J(i,j) = \frac{\partial Y_i}{\partial x_j} |_{\Sigma_m} \quad , \quad i,j = 1, \ldots, m+1 \quad .$$

After straightforward calculations, the entries of J are found as follows : For $i = 1, \ldots, m+1$, $j = 1, \ldots, m$ we have

$$J(i,j) = \begin{cases} 0 & i-j<1 \\ -K & i-j=1 \\ -K\prod_{l=j+1}^{i-1}(a_l+K) & i-j>1 \end{cases} \tag{10}$$

For $j = m+1$, we have

$$\begin{array}{l} J(1,m+1) = 1 \quad , \\ J(i,m+1) = \prod_{l=1}^{i-1}(a_l+K) \quad , \quad i = 2, \ldots, m+1 \end{array} \tag{11}$$

Clearly the characteristic polynomial $p_m(\lambda)$ of J has the following form :

$$p_m(\lambda) = \lambda^{m+1} + c_m\lambda^m + \ldots + c_1\lambda + c_0 \quad . \tag{12}$$

Theorem 2 : The coefficients in (12) can be found as follows : (for $1 < l < m$)

$$c_0 = -(-1)^m K^m \quad , \quad c_m = -\prod_{i=1}^{m}(a_i+K) \quad , \tag{13}$$

$$c_{m-l} = -(-1)^l K^l \sum_{i_1=1}^{m} \sum_{i_2=i_1+1}^{m} \cdots \sum_{i_l=i_{l-1}+1}^{m} \prod_{\substack{i=1 \\ i \neq i_1, \ldots, i_l}}^{m} (a_i+K) \quad . \tag{14}$$

Proof : By using standard determinant formulas, after lengthy but straightforward calculations, collecting the coefficients of λ^i, we obtain (13), (14). □

Remark 2 : Note that for $m = 1$ and $m = 2$, $p_m(\lambda)$ given by (12)-(14) reduces to (7) and (9), respectively. □

Now we can state our main results as follows.

Theorem 3 : Let an m period UCPO of (1) be given by $\Sigma_m = \{x_0^*, x_1^*, \ldots, x_{m-1}^*\}$ and define $a_1 = Df(x_0^*)$, $a_2 = Df(x_1^*)$, ..., $a_m = Df(x_{m-1}^*)$. Then :

1 : Σ_m is exponentially stable for (4) and (5) if and only if $p_m(\lambda)$ given by (12)-(14) is Schur stable. This condition is only sufficient for asymptotic stability of Σ_m.

2 : If $p_m(\lambda)$ has at least one unstable root, i.e. magnitude strictly greater than unity, then Σ_m cannot be stabilized by (4) and (5). Hence the proposed method to test stability is not conclusive only if some roots of $p_m(\lambda)$ are on the unit disc, i.e. have unit magnitude, while the rest of the roots are strictly inside the unit disc. □

Remark 3 : We note that the Schur stability of a polynomial can be checked by applying the so called Schur-Cohn criterion, or equivalently the Jury test to the polynomial, see [10]. This test gives some necessary and sufficient conditions on the coefficients of the polynomial. These conditions are in the form of a finite set of inequalities, hence could be checked easily. In our case, once the terms a_i are known, these conditions become some inequalities in terms of some polynomials of K. By finding the roots of these polynomials, we could determine the intervals of K for which Schur stability holds. We will show some examples in the sequel. □

At this point, we can state the following simple necessary condition for the stability of DFC

Theorem 4 : Let an m period UCPO of (1) be given by $\Sigma_m = \{x_0^*, x_1^*, \ldots, x_{m-1}^*\}$ and define $a_1 = Df(x_0^*)$, $a_2 = Df(x_1^*)$, ..., $a_m = Df(x_{m-1}^*)$. If the following holds

$$\prod_{i=1}^{m} a_i > 1 \quad . \tag{15}$$

then Σ_m cannot be stabilized by DFC.

Proof : Note that one necessary condition for Schur stability of $p_m(\lambda)$ for any m is that $p(1) > 0$, see [10]. This results in $p_m(1) = 1 + c_m + \ldots + c_1 + c_0 > 0$. By using (13), (14), this condition reduces to (15). □

Remark 4 : This condition gives an inherent limitation of DFC in the sense that when it holds, DFC cannot stabilize the corresponding Σ_m. We note that similar limitations in terms of some Floquet multipliers have been given in the literature, see [11], [7], [13], [12]. □

3 Applications and Simulations

Now we will consider some special cases. For $m = 1$, $p_1(\lambda)$ given by (7) is Schur stable if and only if

$$\textbf{i}: 1 - a_1 > 0 \ , \ \textbf{ii}: 1 + a_1 + 2K > 0 \ , \ \textbf{iii}: K < 1 \ ,$$

see [10]. Clearly these inequalities are satisfied if and only if

$$-3 < a_1 < 1 \ ,$$

see [7]. If this is the case, then any K satisfying

$$-(1 + a_1)/2 < K < 1$$

will result in the exponential stabilization of the corresponding UCPO. When $K > 1$ or $K < -(1 + a_1)/2$, at least one root of $p_1(\lambda)$ is unstable, hence the corresponding UCPO cannot be stable. For $K = 1$ or $K = -(1 + a_1)/2$, stability cannot be deduced by using our approach.

To elaborate further, let us consider the logistic equation

$$f(x) = \mu x(1 - x) \ .$$

For $\mu = 3.75$, this map has one truly period 2 UCPO $\Sigma_2 = \{x_0^*, x_1^*\}$ given by $x_0^* = 0.884994$, $x_1^* = 0.381672$. The fixed points $x_A = 0$, $x_B = 1 - 1/\mu$ also induce period 2 orbits $\Sigma_{2A} = \{x_A, x_A\}$ and $\Sigma_{2B} = \{x_B, x_B\}$. However, one can easily show that the condition (15) holds for these orbits, and hence they cannot be stabilized by DFC. For Σ_2, note that $a_1 = \mu - 2\mu x_0^* = -2.8874$, $a_2 = \mu - 2\mu x_1^* = 0.8874$. The coefficients of $p_2(\lambda)$ are given by (9) as $c_2 = -(a_1 + K)(a_2 + K)$, $c_1 = K((a_1 + K) + (a_2 + K))$, $c_0 = -K^2$. From the Jury test, $p_2(\lambda)$ is Schur stable if and only if

$$\textbf{i}: \ | c_0 + c_2 | < 1 + c_1 \ , \ \textbf{ii}: \ | c_1 - c_0 c_2 | < 1 - c_0^2 \ ,$$

see [10], p. 180-183. These inequalities are equivalent to the following :

$$\begin{gathered} 1 + 2.5625 > 0 \\ 4K^2 - 4K - 1.5625 > 0 \\ 2K^4 - 2K^3 - 4.5625K^2 + 2K - 1 < 0 \\ 2K^3 + 4.5625K^2 - 2K - 1 < 0 \ . \end{gathered}$$

Clearly the sign conditions given above can be converted into some bounds on K once the roots of these polynomials are found. By finding these roots, we conclude that Σ_2 can be exponentially stabilized if and only if $-0.3102 < K < -0.30039$. Note that the precision of these bounds are related to the precision in obtaining the related polynomials and their roots. We performed a numerical simulation for this case with $K = -0.305$. Since the stabilization is only local, the DFC will work when the actual orbit of (1) is sufficiently close to Σ_2. To evaluate the exact domain of attraction for Σ_2 is very difficult, but by extensive numerical simulations we find that when

$$d(i) = \sqrt{\sum_{j=0}^{2} d(x(i-j), \Sigma_2)^2} < 0.09 \ ,$$

apparently the orbit is in the domain of attraction (note that the system is actually has dimension 3, see (8)). By using this idea, we simulated (4) and (5) with the following choice of input :

$$u(k) = \begin{cases} K(x(k) - x(k-2)) & d(i) < 0.09 \\ 0 & d(i) \geq 0.09 \end{cases} \tag{16}$$

The results of the simulation (with $\mu = 3.75$, $K = -0.305$, $x(0) = 0.7$) are shown in Figures 1 and 2. For this particular simulation, the trajectories entered into the domain of attraction of Σ_2 at the iteration $k = 36$, and we plotted $u(k)$ and $d(x(k), \Sigma_2)$ versus k for $k \geq 36$ in Figures 1 and 2, respectively. As can be seen, the decay of solutions to Σ_2 is exponential, and that the required input u is sufficiently small and decays to zero exponentially as well. A similar analysis shows that for $\mu = 3.76$, the stabilization is possible when $-0.3090 < K < -0.3089$, and is not possible for $\mu \geq 3.77$. Hence we conclude that there exists a critical value $3.76 \leq \mu^* < 3.77$ such that DFC can be used for the stabilization of period 2 orbits for $\mu \leq \mu^*$, and cannot be used for $\mu > \mu^*$.

To elaborate further consider the case $m = 3$. Let the UCPO be given as $\Sigma_3 = \{x_0^*, x_1^*, x_2^*\}$, and define $a_i = Df(x_{i-1}^*)$, $i = 1, 2, 3$. The characteristic polynomial $p_3(\lambda)$ given by (12) has the coefficients $c_3 = -(a_1 + K)(a_2+K)(a_3+K)$, $c_2 = K((a_1+K)(a_2+K)+(a_1+K)(a_3+K)+(a_2+K)(a_3+K))$, $c_1 = -K^2((a_1+K)+(a_2+K)+(a_3+K))$, $c_0 = K^3$. According to the Jury test, $p_3(\lambda)$ is Schur stable if and only if

$$\mathbf{i} : | c_0 | < 1 \quad , \quad \mathbf{ii} : | c_1 + c_3 | < 1 + c_0 + c_2 \quad ,$$

$$iii : | c_2(1 - c_0) + c_0(1 - c_0^2) + c_3 \ (c_0c_3 - c_1) | < c_0c_2(1 - c_0) + (1 - c_0^2) + c_1(c_0c_3 - c_1)$$

see [10], pp. 180-183. As an example, consider the logistic map with $\mu = 3.87$. In this case, the logistic map has two true period 3 orbits given by $\Sigma_{3+} = \{0.1768, 0.5632, 0.9520\}$ and $\Sigma_{3-} = \{0.4643, 0.9625, 0.1394\}$. The fixed points $x_A = 0$ and $x_B = 1 - 1/\mu$ also induce period 3 orbits in the form $\Sigma_{3A} = \{x_A, x_A, x_A\}$ and $\Sigma_{3B} = \{x_B, x_B, x_B\}$. One can easily show that the condition (15) holds for Σ_{3+} and Σ_{3A}, and hence these orbits cannot be stabilized by DFC. For Σ_{3B}, one can show that the Jury test, i.e. the inequalities *i-iii* given above, cannot be simultaneously satisfied for any K, hence DFC cannot be used for the stabilization Σ_{3B} as well. For Σ_{3-}, by evaluating these inequalities, one can show that DFC can be used for stabilization when $-0.1008 < K < -0.087$. We performed a numerical simulation for this case with $K = -0.095$. To evaluate the domain of attraction for Σ_{3-}, we performed various simulations, and it appears that when

$$d(i) = \sqrt{\sum_{j=0}^{3} d(x(i-j), \Sigma_{3-})^2} < 0.03 \quad ,$$

apparently the orbit is in the domain of attraction (note that the system is actually has dimension 4). By using this idea, as in the previous simulation, we simulated (4) and (5) with the following choice of input

$$u(k) = \begin{cases} K(x(k) - x(k-3)) & d(i) < 0.03 \\ 0 & d(i) \geq 0.03 \end{cases} \tag{17}$$

The results of the simulation (with $\mu = 3.87$, $K = -0.095$, $x(0) = 0.7$) are shown in Figures 3 and 4. For this particular simulation, the trajectories entered into the domain of attraction of Σ_{3-} at the iteration $k = 531$, and we plotted $u(k)$ and $d(x(k), \Sigma_{3-})$ versus k for $k \geq 531$ in Figures 3 and 4, respectively. As can be seen, the decay of solutions to Σ_{3-} is exponential, and that the required input u is sufficiently small and decays to zero exponentially as well. A similar analysis shows that DFC cannot be used for stabilization of period 3 orbits for $\mu \geq 3.88$. Hence we conclude that there exists a critical value $3.87 \leq \mu^* < 3.88$ such that DFC can be used for the stabilization of period 3 orbits for $\mu \leq \mu^*$, and cannot be used for $\mu > \mu^*$. Clearly, this procedure can be extended to arbitrary period m.

4 Conclusion

In conclusion, we analyzed the stability of DFC for a chaotic system. We first constructed a map whose fixed points correspond to the periodic orbits of the uncontrolled chaotic system. Then the stability of DFC for the original chaotic system is equivalent to the stability of the corresponding fixed point of the constructed map. We derive the form of the characteristic polynomial of the Jacobian matrix of this map at the desired fixed point. Then the stability problem of DFC reduces to determine the Schur stability of the associated characteristic polynomial. By applying Jury test, we can determine the bounds on the gain of DFC to ensure the stability. The presented method could be generalized to higher dimensional systems as well. But this requires further research.

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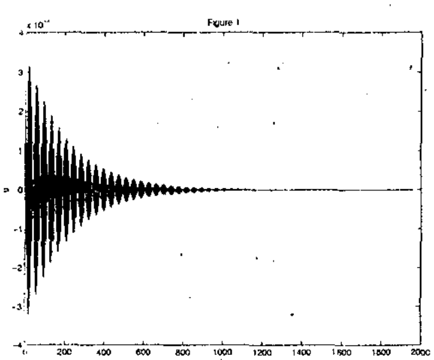

Figure 1: Stabilization of Σ_2, $u(k)$, plotted for $k > 36$

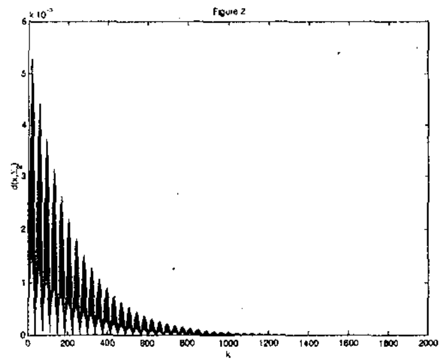

Figure 2: Stabilization of Σ_2, $d(x(k), \Sigma_2)$, plotted for $k > 36$

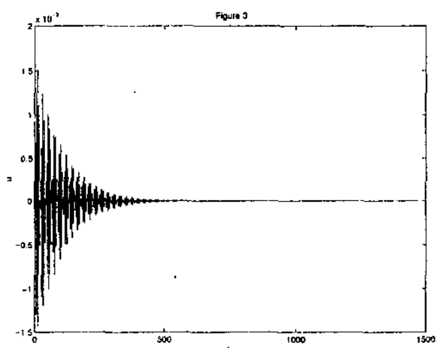

Figure 3: Stabilization of Σ_3, $u(k)$, plotted for $k > 531$

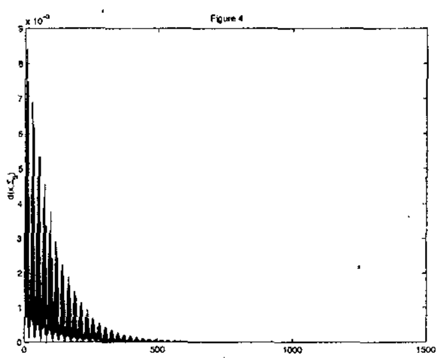

Figure 4: Stabilization of Σ_3, $d(x(k), \Sigma_3)$, plotted for $k > 531$