Proceedings of the 42nd IEEE
Conference on Decision and Control
Maui, Hawaii USA, December 2003

A Stability Result for Delayed Feedback Controllers

Ömer Morgül
Bilkent University
Dept. of Electrical and Electronics Engineering
06800, Bilkent, Ankara, Turkey
morgul@ee.bilkent.edu.tr

***Abstract*— We consider the delayed feedback control (DFC) scheme for one dimensional discrete time systems. To analyze the stability, we construct a map whose fixed points correspond to the periodic orbits of the system to be controlled. Then the stability of the DFC is equivalent to the stability of the corresponding equilibrium point of the constructed map. We obtain a formula for the characteristic polynomial of the Jacobian of this map. Then the Schur stability of this polynomial could be used to analyze the stability of DFC. We also present some simulation results.**

Key words: Chaos control, delayed feedback system, Pyragas controller, stability.

I. INTRODUCTION

In recent years, the analysis and control of chaotic behaviour in dynamical systems has received a great deal of attention among scientists from various disciplines such as engineers, physicists, biologists, etc. Although the chaotic behaviour arising in feedback control systems was known before, [3], the development in the field of chaos control accelerated mainly after the seminal paper [16] where the term "controlling chaos" was introduced. This work had a strong influence, especially, on the approach of the physics community to the problem of controlling chaotic systems and is based on variation of certain parameters which has certain effects on the chaotic behaviour. In such systems usually many unstable periodic orbits are embedded in their chaotic attractors, and as shown in [16], by using small external feedback input, some of these orbits may be stabilized. Therefore, by applying small feedback inputs, it may be possible to force these systems to behave in a regular way.

Following the work of [16], various chaos control techniques have been proposed, see e.g. [4], [7]. Among these, the delayed feedback control (DFC) scheme first proposed in [17] and is also known as Pyragas scheme, has gained considerable attention due to its various attractive features. In this technique the required control input is basically the difference between the current and one period delayed states multiplied by a gain. Hence if the system is already in the periodic orbit, this term vanishes. Also if the trajectories asymptotically approach to the periodic orbit, this term becomes smaller. For more details as well as various applications of DFC, see [8], [20], and the references therein.

DFC has been successfully applied to many systems, including the stabilization of coherent modes of laser [1], [15] ; magnetoelastic systems, [11]; cardiac systems, [2] ; controlling friction, [6]; traffic models, [14]; chaotic electronic oscillators, [9], [18]. For more references on the subject, see e.g. [8].

Despite its simplicity, a detailed stability analysis of DFC is very difficult, [20], [23]. Apparently, DFC has some inherent limitations, [23]. To overcome these limitations, several modifications has been proposed, see e.g. [8], [19], [20], [22], and the references therein.

In this work, we consider the delayed feedback control (DFC) scheme for one dimensional discrete time systems. To analyze the stability, we construct a map whose fixed points correspond to the periodic orbits of the system to be controlled. Then the stability of the DFC is equivalent to the stability of the corresponding equilibrium point of the constructed map. For each periodic orbit, we construct a characteristic polynomial of a related Jacobian matrix. The Schur stability of this polynomial could be used to analyze the stability of DFC. By using Schur-Cohn criterion, we can find bounds on the gain of DFC to ensure stability.

II. STABILITY OF DFC

Let us consider the following one dimensional discrete-time system

$$x(k+1) = f(x(k)) \quad , \tag{1}$$

where $k = 0, 1, \ldots$ is the discrete time index, $f : \mathbf{R} \to \mathbf{R}$ is an appropriate function, which is assumed to be differentiable wherever required. We assume that the system given by (1) possesses a T periodic orbit $x_p^*(\cdot)$ characterized by the set

$$\Sigma_T = \{x_0^*, x_1^*, \ldots, x_{T-1}^*\}$$

i.e. for $x_p^*(0) = x_0^*$, the iterates of (1) yields $x_p^*(1) = x_1^*, \ \ldots, \ x_p^*(T-1) = x_{T-1}^*$, $x_p^*(k) = x_p^*(k-T)$ for $k \geq T$. Let us call this orbit as an uncontrolled periodic orbit (UCPO) for future reference.

Let $x(k)$ be a particular solution of (1) starting with $x(0)$. To characterize the convergence of $x(k)$ to Σ_T, we define a distance measure as follows. For a given k, we define the set Σ_k as follows

$$\Sigma_k = \{x(k), x(k+1), \ldots, x(k+T-1)\} \quad . \tag{2}$$

We also define the following $(j = 0, 1, \ldots, T-1)$:

$$d_k(j) = \sqrt{\sum_{i=0}^{T-1} (x(k+i) - x_p^*(i+j))^2} \quad . \tag{3}$$

0-7803-7924-1/03/$17.00 ©2003 IEEE

Then we define the following distance measure

$$d(\Sigma_k,\Sigma_T) = \min\{d_k(0),\dots,d_k(T-1)\} \quad . \qquad (4)$$

Clearly, if $x(\cdot) = x_p^*(\cdot)$, then we have $d(\Sigma_k,\Sigma_T) = 0$, $\forall k$. Conversely, if $d(\Sigma_k,\Sigma_T) = 0$ for some $k = k_0$, then $d(\Sigma_k,\Sigma_T) = 0$ and $\Sigma_k = \Sigma_T$ for $k \geq 0$.

Let $x(\cdot)$ be the solution of (1) corresponding to a given $x(0)$. We say that the periodic solution $x_p^*(\cdot)$ is asymptotically stable if there exists an $\varepsilon > 0$ such that for any $x(0)$ for which $d(\Sigma_0,\Sigma_T) < \varepsilon$ holds, we have $\lim_{k\to\infty} d(\Sigma_k,\Sigma_T) = 0$. Moreover we say that Σ_T is exponentially stable if this decay is exponential, i.e. the following holds for some $M > 0$ and $\rho \in (0,1)$

$$d(\Sigma_k,\Sigma_T) \leq M\rho^k d(\Sigma_0,\Sigma_T) \quad . \qquad (5)$$

To stabilize periodic orbits of (1), let us apply a control input u as :

$$x(k+1) = f(x(k)) + u(k) \quad . \qquad (6)$$

In DFC, the following simple feedback control input is used to (possibly) stabilize Σ_T :

$$u(k) = K(x(k) - x(k-T)) \quad , \qquad (7)$$

where $K \in \mathbf{R}$ is a constant gain to be determined. Note that if $x(0) \in \Sigma_T$, then $x(k) \in \Sigma_T$ for $k \geq 0$ and $u(k) \equiv 0$. Moreover, if Σ_T is asymptotically stabilized, then $u(k) \to 0$ as $k \to \infty$. In the sequel we will derive some conditions and bounds on K for the stabilization of periodic orbits.

To motivate our analysis, consider the case $T = 1$. In this case we have $\Sigma_1 = \{x_0^*\}$ where $x_0^* = f(x_0^*)$, i.e. period 1 orbits are the same as fixed points of f. By defining $x_1(k) = x(k-1)$, $x_2(k) = x(k)$, we can rewrite (6) and (7) as

$$\begin{aligned} x_1(k+1) &= x_2(k) \quad , \\ x_2(k+1) &= f(x_2(k)) + K(x_2(k) - x_1(k)) \quad . \end{aligned} \qquad (8)$$

Let us define $\hat{x} = (x_1\ x_2)^T \in \mathbf{R}^2$, where here and in the sequel the superscript T denotes transpose, and define $F : \mathbf{R}^2 \to \mathbf{R}^2$ as

$$F(\hat{x}) = (x_2\ \ f(x_2) + K(x_2 - x_1))^T \quad .$$

With this notation, (8) can be written as :

$$\hat{x}(k+1) = F(\hat{x}(k)) \quad . \qquad (9)$$

For $\hat{x}^* = (x_1^*\ x_2^*)^T$, $F(\hat{x}^*) = \hat{x}^*$ holds if and only if $x_1^* = x_2^* = f(x_2^*)$. Hence any fixed point of F corresponds to an UCPO Σ_1 of (1), and vice versa. Hence asymptotic stability of Σ_1 for (6) and (7) can be analyzed by studying the stability of the corresponding fixed point of F for (9). To analyze the latter, let $\Sigma_1 = \{x_0^*\}$ and set $a_1 = Df(x_0^*)$, and

$$J = \frac{\partial F}{\partial x}\Big|_{\Sigma_1} \quad ,$$

where D stands for the derivative and J is the Jacobian of F evaluated at the equilibrium point. Clearly the components of J are given as

$$J(1,1) = 0,\ \ J(1,2) = 1,\ J(2,1) = -K,\ J(2,2) = a_1 + K.$$

The characteristic polynomial $p_1(\lambda)$ of J can easily be found as

$$p_1(\lambda) = \det(\lambda I - J) = \lambda^2 - (a_1 + K)\lambda + K \quad . \qquad (10)$$

We say that a polynomial is Schur stable if all of its eigenvalues are inside the unit disc of the complex plane, i.e. have magnitude less then unity. Hence, the asymptotic stability of the fixed point of F for (9), hence the asymptotic stability of Σ_1 for (6) and (7) could be analyzed by studying the Schur stability of $p_1(\lambda)$ given by (10). Moreover note that the exponential stability of the fixed points of F is equivalent to Schur stability of $p_1(\lambda)$, [13]. Hence we can state the following facts :

Theorem 1 : Let $\Sigma_1 = \{x_0^*\}$ be an UCPO of (1) and set $a_1 = Df(x_0^*)$. Then :

1 : Σ_1 is exponentially stable for (6) and (7) if and only if $p_1(\lambda)$ given by (10) is Schur stable. This condition is only sufficient for asymptotic stability of Σ_1.

2 : If $p_1(\lambda)$ has an unstable root, i.e. outside the unit disc, then Σ_1 cannot be asymptotically stable for (6) and (7). □

Remark 1 : We note that Schur stability of a polynomial can be determined by checking some inequalities in terms of its coefficients; this is known as Schur-Cohn criterion, or equivalently as the Jury test, see [5]. We will apply this test to (10) later. □

To motivate our approach further, let us consider the case $T = 2$. Let the period 2 UCPO of (1) be given as $\Sigma_2 = \{x_0^*, x_1^*\}$ and define $a_1 = Df(x_0^*)$, $a_2 = Df(x_1^*)$. By defining $x_1(k) = x(k-2)$, $x_2(k) = x(k-1)$, $x_3(k) = x(k)$, we can rewrite (6) and (7) as

$$\begin{aligned} x_1(k+1) &= x_2(k) \quad , \quad x_2(k+1) = x_3(k) \\ x_3(k+1) &= f(x_3(k)) + K(x_3(k) - x_1(k)) \quad . \end{aligned} \qquad (11)$$

For $\hat{x} = (x_1\ x_2\ x_3)^T \in \mathbf{R}^3$, let us define $G : \mathbf{R}^3 \to \mathbf{R}^3$ as

$$G(\hat{x}) = (x_2\ x_3\ f(x_3) + K(x_3 - x_1))^T \quad .$$

With this notation, (11) can be written as :

$$\hat{x}(k+1) = G(\hat{x}(k)) \quad . \qquad (12)$$

Note that the fixed points of G do not correspond to the UCPO's of (1), but the fixed points of $F = G^2$ does. To see this, note that

$$F = G^2 = (Y_1\ Y_2\ Y_3)^T \quad , \qquad (13)$$

where

$$Y_1 = x_3,\ Y_2 = f(Y_1) + K(Y_1 - x_1),\ Y_3 = f(Y_2) + K(Y_2 - x_2) \quad . \qquad (14)$$

Now consider the following system

$$\hat{x}(k+1) = F(\hat{x}(k)) \quad . \qquad (15)$$

For $\hat{x}^* = (x_1^*\ x_2^*\ x_3^*)^T$, the fixed points of F, i.e. the solutions of $F(\hat{x}^*) = \hat{x}^*$, are given as $x_1^* = x_3^*$, $x_2^* = f(x_1^*)$, $x_3^* = f(x_2^*) =$

$f^2(x_1^*)$. Hence for any UCPO $\Sigma_2 = \{x_0^*,\ x_1^*\}$ of (1), there corresponds a fixed point $\hat{x}^* = (x_0^*\ x_1^*\ x_0^*)^T$ of F and vice versa. Hence the asymptotic stability of Σ_2 for (6) and (7) is equivalent to the asymptotic stability of the corresponding fixed point of F for the system (15). To analyze the latter, let us define the Jacobian of F at equilibrium as

$$J = \frac{\partial F}{\partial x}|_{\Sigma_2} \quad .$$

The entries of J can be calculated as

$$J(i,j) = \frac{\partial Y_i}{\partial x_j}|_{\Sigma_2} \quad , \quad i,j = 1,2,3 \quad .$$

After straightforward calculations, we obtain

$$\begin{array}{l} J(1,1) = J(1,2) = J(2,2) = 0 \quad , \quad J(1,3) = 1 \\ J(2,1) = -K \quad , \quad J(2,3) = a_1 + K \\ J(3,1) = -K(a_2+K) \quad , \quad J(3,2) = -K \\ J(3,3) = (a_1+K)(a_2+K) \quad . \end{array}$$

The characteristic polynomial $p_2(\lambda)$ of J can be calculated as :

$$\begin{array}{rl} p_2(\lambda) = & \det(\lambda I - J) = \lambda^3 - (a_1+K)(a_2+K)\lambda^2 \\ & + K((a_1+K)+(a_2+K))\lambda - K^2 \quad . \end{array} \qquad (16)$$

Hence for the stability of Σ_2 for (6) and (7), we can study the Schur stability of $p_2(\lambda)$ given above. We will consider the Schur stability of $p_2(\lambda)$ for some cases in the sequel.

Now let us proceed to the general case $T = m$. Assume that (1) has an m periodic UCPO given by $\Sigma_m = \{x_0^*, x_1^*, \ldots, x_{m-1}^*\}$ and define $a_1 = Df(x_0^*)$, $a_2 = Df(x_1^*)$, ..., $a_m = Df(x_{m-1}^*)$. In this case, by defining $x_1(k) = x(k-m)$, $x_2(k) = x(k-m+1)$, ..., $x_m(k) = x(k-1)$, $x_{m+1}(k) = x(k)$, $\hat{x} = (x_1\ x_2\ \ldots\ x_{m+1})^T \in \mathbf{R}^{m+1}$, we can transform (6), (7) into the form

$$\hat{x}(k+1) = G(\hat{x}(k)) \quad , \qquad (17)$$

where $G : \mathbf{R}^{m+1} \to \mathbf{R}^{m+1}$ is defined as

$$G(\hat{x}) = (x_2\ x_3\ \ldots\ x_{m+1}\ f(x_{m+1} + K(x_{m+1} - x_1))^T \quad . \qquad (18)$$

As before, the UCPO Σ_m does not correspond to a fixed point of G, but it corresponds to a fixed point of $F = G^m$. To see this, note that

$$F(\hat{x}) = G^m(\hat{x}) = (Y_1\ Y_2\ \ldots\ Y_{m+1})^T \quad , \qquad (19)$$

where

$$Y_1 = x_{m+1} \quad , \quad Y_{i+1} = f(Y_i) + K(Y_i - x_i) \quad , \quad i = 1,2,\ldots,m \quad . \qquad (20)$$

Now consider the following system

$$\hat{x}(k+1) = F(\hat{x}(k)) \quad . \qquad (21)$$

For $\hat{x}^* = (x_1^*\ x_2*^*\ \ldots\ x_{m+1}^*)^T$, the fixed points of F, i.e. the solutions of $F(\hat{x}^*) = \hat{x}^*$, are given as $x_i^* = Y_i^*$, $i = 1,\ldots,m+1$, which in turn implies $x_1^* = x_{m+1}^*$, $x_2^* = f(x_1^*)$, $x_{j+1}^* = f(x_j^*)$, $j = 1,\ldots,m$. Hence the asymptotic stability of Σ_m for (6) and (7) is equivalent to the asymptotic stability of the corresponding fixed point of F for the system (21). To analyze the latter, let us define the Jacobian of F at the equilibrium as

$$J = \frac{\partial F}{\partial x}|_{\Sigma_m} \quad .$$

The entries of J can be calculated as

$$J(i,j) = \frac{\partial Y_i}{\partial x_j}|_{\Sigma_m} \quad , \quad i,j = 1,\ldots,m+1 \quad .$$

After straightforward calculations, the entries of J are found as follows : For $i = 1,\ldots,m+1$, $j = 1,\ldots,m$ we have

$$J(i,j) = \begin{cases} 0 & i-j<1 \\ -K & i-j=1 \\ -K\prod_{l=j+1}^{i-1}(a_l+K) & i-j>1 \end{cases} \qquad (22)$$

For $j = m+1$, we have

$$\begin{array}{l} J(1,m+1) = 1 \quad , \\ J(i,m+1) = \prod_{l=1}^{i-1}(a_l+K) \quad , \quad i = 2,\ldots,m+1 \end{array} \qquad (23)$$

Clearly the characteristic polynomial $p_m(\lambda)$ of J has the following form :

$$p_m(\lambda) = \lambda^{m+1} + c_m\lambda^m + \ldots + c_1\lambda + c_0 \quad . \qquad (24)$$

Theorem 2 : The coefficients in (24) can be found as follows : (for $1 < l < m$)

$$c_0 = -(-1)^m K^m \quad , \quad c_m = -\prod_{i=1}^{m}(a_i+K) \quad , \qquad (25)$$

$$c_{m-l} = -(-1)^l K^l \sum_{i_1=1}^{m}$$

$$\sum_{i_l=i_{l-1}+1}^{m} \quad \prod_{\substack{i=1 \\ i \neq i_1,\ldots,i_l}}^{m} \quad (a_i+K) \quad . \qquad (26)$$

Proof : By using standard determinant formulas, after lengthy but straightforward calculations, collecting the coefficients of λ^i, we obtain (25), (26). □

Remark 2 : Note that for $m=1$ and $m=2$, $p_m(\lambda)$ given by (24)-(26) reduces to (10) and (16), respectively. □

Now we can state our main results as follows.

Theorem 3 : Let an m period UCPO of (1) be given by $\Sigma_m = \{x_0^*, x_1^*, \ldots, x_{m-1}^*\}$ and define $a_1 = Df(x_0^*)$, $a_2 = Df(x_1^*)$, ..., $a_m = Df(x_{m-1}^*)$. Then :

1 : Σ_m is exponentially stable for (6) and (7) if and only if $p_m(\lambda)$ given by (24)-(26) is Schur stable. This condition is only sufficient for asymptotic stability of Σ_m.

2 : If $p_m(\lambda)$ has at least one unstable root, i.e. magnitude strictly greater than unity, then Σ_m cannot be stabilized by (6) and (7). Hence the proposed method to test stability is not conclusive only if some roots of $p_m(\lambda)$ are on the unit disc, i.e. have unit magnitude, while the rest of the roots are strictly inside the unit disc. □

Remark 3 : We note that the Schur stability of a polynomial can be checked by applying the so called Schur-Cohn criterion, or equivalently the Jury test to the polynomial, see [5]. This test gives some necessary and sufficient conditions on the coefficients of the polynomial. These conditions are in the form of a finite set of inequalities, hence could be checked easily. In our case, once the terms a_i are known, these conditions become some inequalities in terms of some polynomials of K of the following form :

$$q_j(K) = \sum_{i=0}^{n_j} \alpha_i^j K^i > 0 \quad , \quad j = 1,2,\ldots,M \quad , \tag{27}$$

where various constants depend on a_i and m. By finding the roots of these polynomials, we could determine the intervals of K for which Schur stability holds. We will show some examples in the sequel. □

At this point, we can state the following simple necessary condition for the stability of DFC

Theorem 4 : Let an m period UCPO of (1) be given by $\Sigma_m = \{x_0^*, x_1^*, \ldots, x_{m-1}^*\}$ and define $a_1 = Df(x_0^*)$, $a_2 = Df(x_1^*)$, ..., $a_m = Df(x_{m-1}^*)$. If the following holds

$$\prod_{i=1}^{m} a_i \geq 1 \quad . \tag{28}$$

then Σ_m cannot be exponentially stabilized by DFC. If the inequality sign in (28) is strict, i.e. $\geq$ sign is replaced by $>$, then Σ_m cannot be asymptotically stabilized by DFC.

Proof : Note that one necessary condition for Schur stability of $p_m(\lambda)$ for any m is that $p(1) > 0$, see [5]. This results in $p_m(1) = 1 + c_m + \ldots + c_1 + c_0 > 0$. By using (25), (26), this condition reduces to (28). □

Remark 4 : This result indicates an inherent limitation of DFC. We note that similar limitations in terms of some Floquet multipliers have been given in the literature, see [10], [21], [23], [12]. □

III. Applications and Simulations

Now we will consider some special cases. For $m = 1$, $p_1(\lambda)$ given by (10) is Schur stable if and only if

$$\mathbf{i}: \; 1 - a_1 > 0 \quad , \quad \mathbf{ii}: \; 1 + a_1 + 2K > 0 \quad , \quad \mathbf{iii}: \; K < 1 \quad ,$$

see [5]. Clearly these inequalities are satisfied if and only if

$$-3 < a_1 < 1 \quad ,$$

see [23]. If this is the case, then any K satisfying

$$-(1 + a_1)/2 < K < 1$$

will result in the exponential stabilization of the corresponding UCPO. When $K > 1$ or $K < -(1 + a_1)/2$, at least one root of $p_1(\lambda)$ is unstable, hence the corresponding UCPO cannot be stable. For $K = 1$ or $K = -(1 + a_1)/2$, stability cannot be deduced by using our approach.

To elaborate further, let us consider the logistic equation $f(x) = \mu x(1 - x)$. For $\mu = 3.7$, this map has one truly period 2 UCPO $\Sigma_2 = \{x_0^*, x_1^*\}$ given by $x_0^* = 0.390022$, $x_1^* = 0.880248$. The fixed points $x_A = 0$, $x_B = 1 - 1/\mu$ also induce period 2 orbits $\Sigma_{2A} = \{x_A, x_A\}$ and $\Sigma_{2B} = \{x_B, x_B\}$. However, for Σ_{2A} we have $a_1 = a_2 = \mu$, and for Σ_{2B} we have $a_1 = a_2 = 2 - \mu$, and clearly in these cases the necessary condition (28) fails for these orbits, and hence they cannot be stabilized by DFC. For Σ_2, note that $a_1 = \mu - 2\mu x_0^* = 0.8138$, $a_2 = \mu - 2\mu x_1^* = -2.8138$. The coefficients of $p_2(\lambda)$ are given by (16) as : $c_2 = -(a_1 + K)(a_2 + K)$, $c_1 = K((a_1 + K) + (a_2 + K))$ $c_0 = -K^2$.

From the Jury test, $p_2(\lambda)$ is Schur stable if and only if

$$\mathbf{i}: |c_0 + c_2| < 1 + c_1 \; , \; \mathbf{ii}: |c_1 - c_0 c_2| < 1 - c_0^2 \quad ,$$

see [5], p. 180-183. These inequalities are equivalent to the following (see (27)):

$$\mathbf{i}: \quad 1 + 2.29 > 0$$

$$\mathbf{ii}: \quad 4K^2 - 4K - 1.29 > 0$$

$$\mathbf{iii}: \quad 2K^4 - 2K^3 - 4.29K^2 + 2K - 1 < 0$$

$$\mathbf{iv}: \quad 2K^3 + 4.29K^2 - 2K - 1 < 0 \quad .$$

Clearly the sign conditions given above can be converted into some bounds on K once the roots of these polynomials are found. By finding these roots, we conclude that Σ_2 can be exponentially stabilized if and only if $-0.3167 < K < -0.2566$. Note that the precision of these bounds are related to the precision in obtaining the related polynomials and their roots. We performed a numerical simulation for this case. Since the stabilization is only local, the DFC will work when the actual orbit of (1) is sufficiently close to Σ_2. To evaluate the exact domain of attraction for Σ_2 is very difficult, but by extensive numerical simulations we find that when

$$d(i) = d(\Sigma_i, \Sigma_2) < 0.12$$

apparently the orbit is in the domain of attraction, see (2)-(4). By using this idea, we simulated (1) and (7) with the following choice of input :

$$u(k) = \begin{cases} K(x(k) - x(k-2)) & d(i) < 0.12 \\ 0 & d(i) \geq 0.12 \end{cases} \tag{29}$$

Clearly, since the solutions of the logistic equation are chaotic in the uncontrolled case, eventually the control law given above will be effective and the stabilization of Σ_2 will be achieved for any $x(0) \in (0 \; , \; 1)$. We choose $K = -0.2866$, which is the middle of the range given above. The result of this simulation (with $x(0) = 0.6$) is shown in Figures 1 and 2. As can be seen, the decay of solutions to Σ_2 is exponential, and that the required input u is sufficiently small and decays to zero exponentially as well.

A similar analysis shows that for $\mu = 3.75$, the stabilization is possible when $-0.3102 < K < -0.30039$, and for $\mu = 3.76$, the stabilization is possible when $-0.3090 < K < -0.3089$. Similar analysis reveals that the stabilization is not possible for $\mu \geq 3.77$. Hence we conclude that there exists a critical value $3.76 \leq \mu^* < 3.77$ such that DFC can be used for the stabilization of period 2 orbits for $\mu \leq \mu^*$, and cannot be used for $\mu > \mu^*$.

To elaborate further consider the case $m = 3$. Let the UCPO be given as $\Sigma_3 = \{x_0^*, x_1^*, x_2^*\}$, and define $a_i = Df(x_{i-1}^*)$, $i = 1,2,3$. The characteristic polynomial $p_3(\lambda)$ given by (24) has the following coefficients :
$c_3 = -(a_1+K)(a_2+K)(a_3+K)$, $c_2 = K((a_1+K)(a_2+K) + (a_1+K)(a_3+K) + (a_2+K)(a_3+K))$, $c_1 = -K^2((a_1+K) + (a_2+K) + (a_3+K))$, $c_0 = K^3$.

According to the Schur-Cohn criterion, $p_3(\lambda)$ is Schur stable if and only if

$$\mathbf{i}: |c_0| < 1 \quad , \quad \mathbf{ii}: |c_1 + c_3| < 1 + c_0 + c_2 \quad ,$$

$$\mathbf{iii}: |c_2(1-c_0) + c_0(1-c_0^2) + c_3(c_0c_3 - c_1)| < c_0c_2(1-c_0) + (1-c_0^2) + c_1(c_0c_3 - c_1) \quad ,$$

see [5], pp. 180-183. As an example, consider the logistic map with $\mu = 3.85$. In this case, the logistic map has two true period 3 orbits given by $\Sigma_{3+} = \{0.1725, 0.5497, 0.9529\}$ and $\Sigma_{3-} = \{0.4783, 0.9606, 0.1453\}$. The fixed points $x_A = 0$ and $x_B = 1 - 1/\mu$ also induce period 3 orbits in the form $\Sigma_{3A} = \{x_A, x_A, x_A\}$ and $\Sigma_{3B} = \{x_B, x_B, x_B\}$. One can easily show that the necessary condition (28) fails for Σ_{3+} and Σ_{3A}, and hence these orbits cannot be stabilized by DFC. For Σ_{3B}, one can show that the Schur-Cohn criterion, i.e. the inequalities *i-iii* given above, cannot be simultaneously satisfied for any K, hence DFC cannot be used for the stabilization Σ_{3B} as well. For Σ_{3-}, by evaluating these inequalities, one can show that DFC can be used for stabilization when $-0.1041 < K < -0.0315$. We performed a numerical simulation for this case. Since the stabilization is only local, the DFC will work when the actual orbit of (1) is sufficiently close to Σ_{3-}. To evaluate the exact domain of attraction for Σ_{3-} is very difficult, but by extensive numerical simulations we find that when

$$d(i) = d(\Sigma_i, \Sigma_2) < 0.03$$

apparently the orbit is in the domain of attraction, see (2)-(4). By using this idea, we simulated (1) and (7) with the following choice of input :

$$u(k) = \begin{cases} K(x(k) - x(k-3)) & d(i) < 0.03 \\ 0 & d(i) \geq 0.03 \end{cases} \qquad (30)$$

Clearly, since the solutions of the logistic equation are chaotic in the uncontrolled case, eventually the control law given above will be effective and the stabilization of Σ_{3-} will be achieved for any $x(0) \in (0\ ,\ 1)$. We choose $K = -0.0678$, which is the middle of the range given above. The result of this simulation (with $x(0) = 0.6$) is shown in Figures 3 and 4. As can be seen, the decay of solutions to Σ_{3-} is exponential, and that the required input u is sufficiently small and decays to zero exponentially as well.

A similar analysis shows that for $\mu = 3.86$, the stabilization is possible when $-0.1024 < K < -0.0615$, and for $\mu = 3.87$, the stabilization is possible when $-0.1008 < K < -0.087$. Similar analysis reveals that the stabilization is not possible for $\mu \geq 3.88$. Hence we conclude that there exists a critical value $3.87 \leq \mu^* < 3.88$ such that DFC can be used for the stabilization of period 3 orbits for $\mu \leq \mu^*$, and cannot be used for $\mu > \mu^*$.

IV. Conclusion

In conclusion, we analyzed the stability of DFC for a one dimensional discrete-time chaotic system. We first constructed a map whose fixed points correspond to the periodic orbits of the uncontrolled chaotic system. Then the stability of DFC for the original chaotic system is equivalent to the stability of the corresponding fixed point of the constructed map. We derive the form of the characteristic polynomial of the Jacobian matrix of this map at the desired fixed point. Then the stability problem of DFC reduces to determine the Schur stability of the associated characteristic polynomial. By applying the Schur-Cohn criterion, we can determine the bounds on the gain of DFC to ensure the stability. The presented method could be generalized to higher dimensional systems as well. But this requires further research.

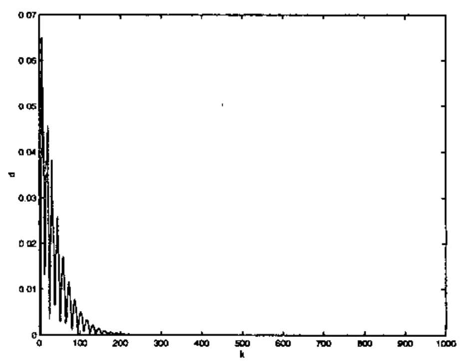

Fig. 1. Stabilization of Σ_2, $d(k)$ vs. k

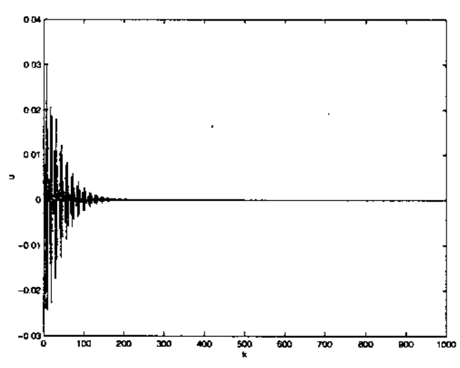

Fig. 2. Stabilization of Σ_2, $u(k)$ vs. k

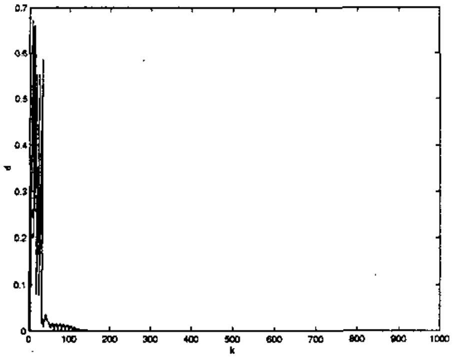

Fig. 3. Stabilization of Σ_{3-}, $d(k)$ vs. k

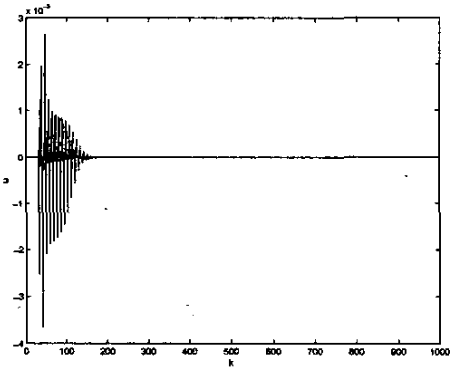

Fig. 4. Stabilization of Σ_{3-}, $u(k)$, vs. k

V. REFERENCES

[1] S. Bielawski, D. Derozier, and P. Glorieux, "Experimental characterization of unstable periodic orbits by controlling chaos," *Physical Review A*, vol. 47, pp. 2492-2495, 1993.

[2] M. E. Brandt, H. T. Shih, and G. Chen, "Linear time delay feedback control of a pathological rhythm in a cardiac conduction model," *Phys. Rev. E.*, vol. 56, pp. 1334-1337, 1997.

[3] R. W. Brockett, "On conditions leading to chaos in feedback systems," *Proc. 21st Conf. Decision and Contr.*, pp. 932-936, 1987.

[4] G. Chen, and X. Dong, *From Chaos to Order : Methodologies, Perspectives and Applications*, World Scientific, Singapore, 1999.

[5] S. N. Elaydi, *An Introduction to Difference Equations*, Springer-Verlag, New York, 1996.

[6] F. J. Elmer, "Controlling friction," *Phys. Rev. E.*, vol. 57, pp. 4903-4906, 1998.

[7] A. L. Fradkov, A. Yu. Pogromsky, *Introduction to Control of Oscillations and Chaos*, World Scientific, Singapore, 1999.

[8] A. L. Fradkov, and R. J. Evans, "Control of Chaos : Survey 1997-2000," *Proceedings of 15th IFAC World Congress*, 21-26 July 2002, Barcelona, Spain, pp. 143-154, 2002.

[9] D. J. Gauthier, D. W. Sukow, H. M. Concannon, and J. E. S. Socolar, "Stabilizing unstable periodic orbits in a fast diode resonator using continuous time-delay autosynchronization," *Phy. Rev. E.*, 50, pp. 2343-2346, 1994.

[10] M. Giona, "Dynamics and relaxation properties of complex systems with memory, *Nonlinearity*, **4**, pp. 911-925, 1991.

[11] W. H. Hai, Y. W. Duan, and L. X. Pan, "An analytical study for controlling unstable periodic motion in magneto-elastic chaos," *Phys. Lett. A.*, vol. 234, pp. 198-204, 1997.

[12] T. Hino, S. Yamamoto, and T. Ushio, "Stabilization of unstable periodic orbits of chaotic discrete-time systems using prediction-based feedback control," *Int. J. Bifurcation Chaos Appl. Sci. Eng.*, **12**, pp. 439-446, 2002.

[13] H. K. Khalil, *Nonlinear Systems*, 3rd. ed, Prentice Hall, Upper Saddle River, 2002.

[14] K. Konishi, H. Kokame, and K. Hirata, "Coupled map car-following model and its delayed feedback control," *Phys. Rev. E.*, vol. 60, pp. 4000-4007, 1999.

[15] N. A. Loiko, A. V. Naumenko, and S. I. Turovets, "Effect of Pyragas feedback on the dynamics of a Q-switched laser," *J. Exper. Theor. Physics*, vol. 85, pp. 827-834, 1997.

[16] E. Ott, C. Grebogi, and J. A. Yorke, "Controlling Chaos," *Phys. Rev. Lett.*, **64**, pp. 1196-1199, 1990.

[17] K. Pyragas, " Continuous control of chaos by self-controlling feedback," *Phys. Lett. A.*, **170**, pp. 421-428, 1992.

[18] K. Pyragas, and A. Tamaševičius, "Experimental control of chaos by delayed self controlling feedback," *Phys. Lett. A*, vol. 170, pp. 99-102, 1993.

[19] K. Pyragas, "Control of chaos via extended delay feedback," *Phys. Lett. A*, 206, pp. 323-330, 1995.

[20] K. Pyragas, "Control of chaos via an unstable delayed feedback controller," *Phys. Rev. Lett.*, **86** pp. 2265-2268, 2001.

[21] H. G. Schuster, and M. B. Stemmler, "Control of chaos by oscillating feedback," *Phys. Rev. E*, **56**, pp. 6410-6417, 1997.

[22] J. E. S. Socolar, D. W. Sukow, and D. J. Gauthier, " Stabilizing unstable periodic orbits in fast dynamical systems," *Phys. Rev. E*, **50** pp. 3245-3248, 1994.

[23] T. Ushio, "Limitation of delayed feedback control in nonlinear discrete-time systems," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, **43** pp. 815-816, 1996.