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On Switching $\mathcal{H}^\infty$ Controllers for a Class of LPV Systems*

Peng Yan
Dept. of Electrical Engineering
The Ohio State University
2015 Neil Ave. Columbus, OH 43210
yanp@ee.eng.ohio-state.edu

Hitay Özbay**
Dept. of Electrical & Electronics Engineering
Bilkent University
Bilkent, Ankara, Turkey TR-06800
ozbay@ee.eng.ohio-state.edu

***Abstract*— We consider switching $\mathcal{H}^\infty$ controllers for a class of LPV systems scheduled along a measurable parameter trajectory. The candidate controllers are selected from a given controller set according to the switching rules based on the scheduling variable. We provide sufficient conditions to guarantee the stability of the switching LPV systems in terms of the dwell time and the average dwell time. Our results are illustrated with an example, where switching between two robust controllers is performed for an LPV system.**

I. Introduction

This paper addresses a switching $\mathcal{H}^\infty$ control strategy for a class of linear parameter varying (LPV) systems scheduled along a measurable parameter trajectory. LPV systems are ubiquitous in chemical processes, robotics systems, automative systems and many manufacturing processes. Meanwhile Jacobian linearization of nonlinear systems also results in LPV models, where gain scheduled controllers can be developed for the nonlinear plants. The analysis and control of LPV systems has been studied widely [1], [2], [12], [17], [18], [9], [16]. A systematic gain scheduling method was developed in [1], [2] based on LMI (Linear Matrix Inequality) algorithms; [18] provided sufficient conditions for the stability of LPV systems with parameter-varying time delays, where gain scheduled controller was designed based on LMIs. Fast gain scheduling was considered in [9], where derivative information on the scheduling variable was utilized in a new control law. In a very recent publication [17], an improved stability analysis for LPV systems was given and the robust gain-scheduled controller was constructed in terms of LMIs. We refer to [13] for a general review on gain scheduling methods.

An alternative method is switching control where a family of controllers are designed at different operating points and the system performs controller switching based on the switching logic. As stated in [3], a challenging point of switching control is its hybrid nature of the continuous and discrete-valued signals. Stability analysis and the design methodology have been investigated recently in the literature of hybrid dynamical systems [6], [11], [10], [14], [15]. For LTI systems, [15] provided sufficient conditions on the stability of the switching control systems based on Filippov solutions to discontinuous differential equations and Lyapunov functionals; [11] proposed a dwell-time based switching control, where a sufficiently large dwell-time can guarantee the system stability. A more flexible result was obtained in [6], where the average dwell-time was introduced for switching control. Besides stability analysis, a number of results have been published on related topics, such as optimal control [14] and tracking problem [7].

Due to the time-varying and the hybrid natures of the switching LPV systems, it is challenging to explore the stability conditions and switching schemes similarly to those for LTI systems. Theoretical and practical results have been presented in recent publications [3], [10], [13]. In particular, [3] analyzed the bounded amplitude performance and derived the conditions related to dwell time, and [10] proposed switching $\mathcal{H}^\infty$ controllers for nonlinear systems which exhibits LPV nature after linearization. In the present paper, we discuss the switching $\mathcal{H}^\infty$ control methodology for a class of LPV systems, where each candidate $\mathcal{H}^\infty$ controller guarantees robust property at the selected operating condition and the switching rules are developed to cover a large operating range. By constructing Lyapunov functionals for time-varying systems as [4], [8], this paper extends the stability results of [6], [11] to LPV systems.

The paper is organized as follows. The problem definition is stated in Section 2, where the structure of the candidate $\mathcal{H}^\infty$ controllers is described and the switching control architecture is proposed. In section 3, the main results on the stability of the switching systems are presented in terms of the dwell time and the average dwell time. An illustrative example is given in Section 4, followed by concluding remarks in Section 5.

II. Problem Definition

The general structure of the switching control scheme that we consider in this paper is depicted in Figure 1, where $w_p \in \mathbb{R}^{n_w}$ is the exogenous input, $u \in \mathbb{R}^{n_u}$ is the control input, $z_p \in \mathbb{R}^{n_z}$ is the regulated output and $y \in \mathbb{R}^{n_y}$ is the measured output. The LPV system depends on a parameter $\theta(t)$, where $\theta(t) \in \mathbb{R}$ is assumed to be continuously differentiable and $\theta \in \Theta$ where Θ is a compact set.

Under further assumptions of $|\theta(t)| < \bar{\theta}$ and $|\dot{\theta}(t)| < \bar{\rho}$, the stability and performance analysis of LPV system (1) can be formulated in terms of LMIs, where gain scheduling $\mathcal{H}^\infty$ controllers can be derived based on convex optimization using LMIs [13], [1], [2], [12], [17]. In the present paper, we

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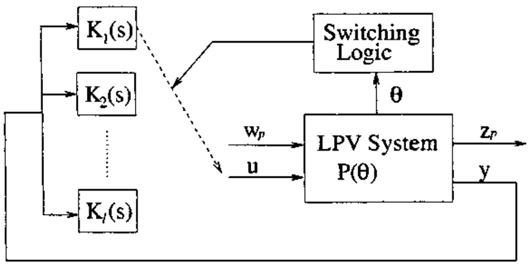

Fig. 1. The switching control system

propose to construct a family of $\mathcal{H}^\infty$ controllers designed at selected operating points $\theta = \theta_i$, $i = 1,2,...,l$, and perform controller switching for the above LPV system, which allows for more freedom in the controller design and has the advantage of simplicity.

The candidate controllers are chosen from a controller set $\mathcal{K} := \{K_i(s) : i = 1,2,...,l\}$, where $K_i(s)$ is an LTI $\mathcal{H}^\infty$ controller designed at $\theta = \theta_i$. Consider an operating range Θ_i, $\theta_i \in \Theta_i$, the LPV system in Figure 1 can be represented as $\mathcal{F}_u(G_{\theta_i}, \Delta_{\theta_i})$, where Δ_{θ_i} is the time varying portion of the LPV system, G_{θ_i} is the LTI portion with nominal value θ_i and $\mathcal{F}_u$ denotes the upper LFT (Linear Fractional Transformation). The closed loop system is depicted in Figure 2, where G_{θ_i} is the nominal transfer function at a specified θ_i:

$$G_{\theta_i}(s) = \left[\begin{array}{c|cc} A(\theta_i) & B_1(\theta_i) & B_2(\theta_i) \\ \hline C_1(\theta_i) & D_{11}(\theta_i) & D_{12}(\theta_i) \\ C_2(\theta_i) & D_{21}(\theta_i) & D_{22}(\theta_i) \end{array}\right], \ i = 1,2,...l, \quad (1)$$

and an $\mathcal{H}^\infty$ optimization problem is defined as finding $K_i(s)$ for the LTI plant G_{θ_i} such that

(i). The closed loop system is asymptotically stable for $\theta \in \Theta_i$;

(ii). $\inf\{\sup_{w\neq 0} \frac{\|z\|_2}{\|w\|_2} : K_i(s)\text{satisfies (i)}\} \leq \gamma$ for the smallest possible γ, where

$$z = \begin{bmatrix} z_r \\ z_p \end{bmatrix}, \ w = \begin{bmatrix} w_r \\ w_p \end{bmatrix}.$$

Denote $\|\cdot\|_{i,2}$ to be the $\mathcal{L}^2$ induced norm and let M_{θ_i} to be the transfer function from w_r to z_r. A sufficient condition on robust stability satisfying (i) is $\|M_{\theta_i}\|_\infty < 1$ and $\|\Delta_{\theta_i}\|_{i,2} < 1$, which can be obtained by applying small gain analysis [1], [13], [19]. The above treatment results in the $\mathcal{H}^\infty$ controller design for the LTI system, where standard $\mathcal{H}^\infty$ optimization methods can be employed [5]. The state space expression of each candidate controller $K_i(s)$ is given by

$$K_i(s) = \left[\begin{array}{c|c} A_{K_i} & B_{K_i} \\ \hline C_{K_i} & D_{K_i} \end{array}\right], \ i = 1,2,...,l. \quad (2)$$

Note that $K_i(s)$ robustly stabilizes the LPV system for $\|\Delta_{\theta_i}\|_{i,2} < 1$, which can be guaranteed by properly choosing

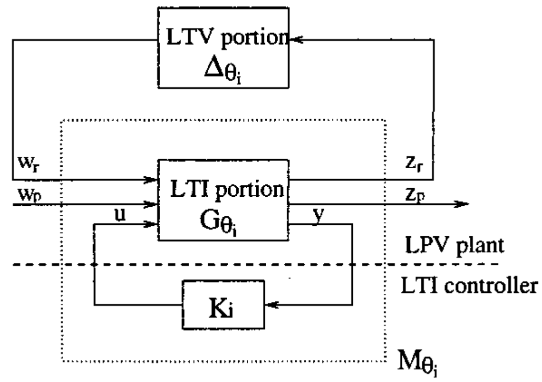

Fig. 2. LPV plant and the controller

θ_i^-, θ_i^+ and $\beta_i > 0$, such that

$$\theta \in \Theta_i := [\theta_i^-, \theta_i^+], \ |\dot{\theta}(t)| < \beta_i. \quad (3)$$

Accordingly, the robust bound Θ_i can be determined for each candidate controller $K_i(s)$.

In order to cover a large operating range Θ, we need to develop stable switching schemes over $\mathcal{K}$. Obviously, a necessary condition for stable switching is:

$$\Theta \subseteq \bigcup_{i=1}^{l} \Theta_i. \quad (4)$$

III. Main Results

Applying the switching rules over $\mathcal{K}$ and invoking (1) and (2), we obtain the closed loop A-matrix $A_{cl} \in \{A_i(\theta), i = 1,2,...,l\}$, where

$$A_i(\theta) = \begin{bmatrix} A + B_2 D_{K_i}(I - D_{22}D_{K_i})^{-1}C_2 & B_2(I - D_{K_i}D_{22})^{-1}C_{K_i} \\ B_{K_i}(I - D_{22}D_{K_i})^{-1}C_2 & A_{K_i} + B_{K_i}(I - D_{22}D_{K_i})^{-1}D_{22}C_{K_i} \end{bmatrix}$$

For switching LTI systems, it has been shown in [11] that a sufficiently large dwell time can guarantee stability; and [6] provided a more flexible and stronger result based on the average dwell time. We claim that similar results can be obtained for switching LPV systems.

Consider the following switching LPV system:

$$\dot{\xi}(t) = A_q(\theta(t))\xi(t), \ t \geq 0 \quad (5)$$

where q is a piecewise constant signal taken values on the set $\mathcal{F} := \{1,2,...,l\}$, i.e. $q(t) = i$, $i \in \mathcal{F}$, for $\forall t \in [t_j, t_{j+1})$, where t_j, $j \in \mathbb{Z}^+ \cup \{0\}$, is the j^{th} switching time instant. Here $A_i \in \mathcal{A} := \{A_i(\theta(t)) : i \in \mathcal{F}, \theta(t) \in \Theta\}$, which is a family of parameter varying matrices. We further assume that:

H1. There is a $\lambda_i > 0$, such that for any $\theta \in \Theta$, the eigenvalues of $A_i(\theta)$ have real parts no greater than $-2\lambda_i$, $\forall i \in \mathcal{F}$;

H2. $\exists K_A^i > 0$, $\|A_i(\theta(t))\| \leq K_A^i$, $\forall i \in \mathcal{F}$;

H3. $\exists K_D^i > 0$, $\|\frac{\partial A_i(\theta)}{\partial \theta}\| \leq K_D^i$, $\forall i \in \mathcal{F}$;
where $\|\cdot\|$ denotes the (pointwise in time) Euclidean norm of a time-varying vector and the corresponding induced norm on matrices.

The dwell time based switching rule set is denoted by $S[\tau_D]$, where τ_D is a constant such that for any $q \in S[\tau_D]$, the distance between any consecutive discontinuities of $q(t)$, $t_{j+1} - t_j$, $j \in \mathbb{Z}^+ \cup \{0\}$, is larger than τ_D [6], [11]. Clearly

$$S[\tau_{D1}] \subset S[\tau_{D2}], \ \forall \tau_{D1} > \tau_{D2} > 0. \tag{6}$$

A sufficient condition on the minimum dwell time to guarantee the stable switching can now be given using Lyapunov stability analysis (a similar result is obtained in [10], using the same technique, for switched gain scheduling controllers in uncertain nonlinear systems).

Theorem 3.1: Assume (H1-H3). Then there exist finite constants $\bar{\beta} > 0$, $\tau_D > 0$, such that the switching LPV system (5) is stable in the sense of Lyapunov for any switching rule $q \in S[\tau_D]$ if $|\dot{\theta}(t)| < \bar{\beta}$.

Proof. First we notice that

$$\hat{A}_i(\theta(t)) := A_i(\theta(t)) + \lambda_i I, \ \forall i \in \mathcal{F} \tag{7}$$

is Hurwitz, which is straightforward from (H1). Let

$$Q_i(t) = \int_0^\infty e^{\hat{A}_i^T(\theta(t))\zeta} e^{\hat{A}_i(\theta(t))\zeta} d\zeta, \ \forall i \in \mathcal{F}. \tag{8}$$

Note that $Q_i(t)$ is well defined, continuously differentiable, and is the unique positive-definite solution of

$$\hat{A}_i^T(\theta(t))Q_i(t) + Q_i(t)\hat{A}_i(\theta(t)) = -I, \tag{9}$$

i.e.

$$A_i^T(\theta(t))Q_i(t) + Q_i(t)A_i(\theta(t)) = -2\lambda_i Q_i(t) - I. \tag{10}$$

Define a family of Lyapunov functions

$$\mathcal{V} := \{V_i : V_i(t,\xi(t)) := \xi^T(t)Q_i(t)\xi(t), i \in \mathcal{F}\} \tag{11}$$

for the following LPV systems respectively

$$\dot{\xi}(t) = A_i(\theta(t))\xi(t), \ \forall i \in \mathcal{F}. \tag{12}$$

Recall that there exist positive constants $M_i \geq \mu_i > 0$, $i \in \mathcal{F}$, depending only on λ_i and K_A^i, such that

$$\mu_i \|\xi(t)\|^2 \leq V_i(t,\xi(t)) \leq M_i \|\xi(t)\|^2, \ t \geq 0. \tag{13}$$

We refer to [4], [8] for details.

Consider an arbitrary switching interval $[t_j, t_{j+1})$, where $q(t) = i$, $i \in \mathcal{F}$, for $\forall t \in [t_j, t_{j+1})$. Using the quadratic form of V_i as shown in (11), a straightforward calculation gives the time derivative of $V_i(t,\xi(t))$ along the trajectory of (12)

$$\begin{aligned} \frac{d}{dt}V_i(t,\xi(t)) &= -\xi^T(t)\xi(t) - 2\lambda_i \xi^T(t)Q_i(t)\xi(t) \\ &\quad + \xi^T(t)\dot{Q}_i(t)\xi(t), \ t \in [t_j, t_{j+1}). \end{aligned} \tag{14}$$

Note that differentiating (8) with respect to t gives

$$\begin{aligned} \dot{Q}_i(t) &= \int_0^\infty e^{\hat{A}_i^T(\theta(t))\zeta}[\dot{\hat{A}}_i^T(\theta(t))Q_i(t) \\ &\quad + Q_i(t)\dot{\hat{A}}_i(\theta(t))]e^{\hat{A}_i(\theta(t))\zeta} d\zeta, \end{aligned} \tag{15}$$

where

$$\dot{\hat{A}}_i(\theta(t)) = \frac{\partial}{\partial \theta}\hat{A}_i(\theta(t))\dot{\theta}(t), \ t \geq 0. \tag{16}$$

Invoking (H3) and Lemma 3 of [8] we have

$$\begin{aligned} \|\dot{\hat{A}}_i(\theta(t))\| &\leq K_D^i |\dot{\theta}(t)| \\ \|\dot{Q}_i(t)\| &\leq K_Q^i |\dot{\theta}(t)| \end{aligned} \tag{17}$$

where $K_Q^i > 0$ is a constant depending only on λ_i, K_A^i and K_D^i.

Define

$$\bar{\beta} := \min_{i \in \mathcal{F}} \{\frac{1 + 2\lambda_i \mu_i}{K_Q^i}\}. \tag{18}$$

Since $|\dot{\theta}(t)| < \bar{\beta}$, we can pick up $0 < \beta < \bar{\beta}$ such that $|\dot{\theta}(t)| \leq \beta$. Thus

$$\begin{aligned} \frac{d}{dt}V_i(t,\xi(t)) &\leq -(1 + 2\lambda_i \mu_i - K_Q^i \beta)\|\xi(t)\|^2 \\ &= -b_i \|\xi(t)\|^2, \ t \in [t_j, t_{j+1}). \end{aligned} \tag{19}$$

where

$$b_i := (1 + 2\lambda_i \mu_i - K_Q^i \beta) > 0. \tag{20}$$

Recall (13) and (19), we have

$$\frac{\dot{V}_i(t,\xi(t))}{V_i(t,\xi(t))} \leq \frac{-b_i \|\xi(t)\|^2}{M_i \|\xi(t)\|^2} = -\frac{b_i}{M_i}. \tag{21}$$

Thus

$$\|\xi(t)\| \leq \|\xi(t_j)\| \sqrt{\frac{M_i}{\mu_i}} e^{\frac{-b_i}{2M_i}(t - t_j)}, \ t \in [t_j, t_{j+1}). \tag{22}$$

Choosing the minimum dwell time as follows:

$$\tau_D = \max_{i \in \mathcal{F}} \{\frac{2M_i \ln \sqrt{\frac{M_i}{\mu_i}}}{b_i}\} \tag{23}$$

we claim that any switching rule $q \in S[\tau_D]$ is stable in the sense of Lyapunov.

In fact, recall the definition of the dwell time τ_D, we have $t_{j+1} - t_j > \tau_D$, $j \in \mathbb{Z}^+ \cup \{0\}$. Thus

$$\begin{aligned} \|\xi(t_{j+1})\| &= \lim_{t \uparrow t_{j+1}} \|\xi(t)\| \leq \lim_{t \uparrow t_{j+1}} \|\xi(t_j)\| \sqrt{\frac{M_i}{\mu_i}} e^{\frac{-b_i}{2M_i}(t - t_j)} \\ &= \|\xi(t_j)\| \sqrt{\frac{M_i}{\mu_i}} e^{\frac{-b_i}{2M_i}(t_{j+1} - t_j)} \\ &< \|\xi(t_j)\| \sqrt{\frac{M_i}{\mu_i}} e^{\frac{-b_i}{2M_i}(\tau_D)} \leq \|\xi(t_j)\| \end{aligned}$$

Thus we have a decreasing sequence $\{\|\xi(t_j)\|, j = 0, 1, 2, \ldots\}$ with upper bound $\|\xi(t_0)\| = \|\xi(0)\|$, which indicates the stability of the overall switching system. Thus the proof is complete. ■

Note that for switching LTI systems, we can set

$$\mu_i = \sigma_{min}[Q_i], \; M_i = \sigma_{max}[Q_i], \; \forall i \in \mathcal{F} \tag{24}$$

where $\sigma_{min}[Q_i]$ denotes the smallest singular value of Q_i and $\sigma_{max}[Q_i]$ the largest singular value of Q_i.

The dwell time condition in Theorem 3.1 can be applied to the switching $\mathcal{H}^\infty$ control problem discussed in Section 2. As depicted in Figure 3, two possible switching schemes [10] are (a) critical-point switching, (b) hysteresis switching. For the critical-point switching, the stability of the closed-loop system cannot be guaranteed. In fact, in the worst case where $\theta(t)$ oscillates within a neighborhood of $c_{i,i+1}$, fast switching or chattering will happen, which may violate the dwell time requirement. The following corollary addresses a sufficient condition for the hysteresis switching scheme over $\mathcal{H}^\infty$ controller set $\mathcal{K}$.

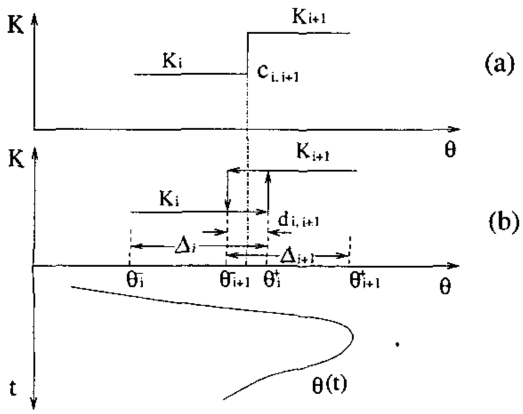

Fig. 3. Switching logic

Corollary 3.1: For the hysteresis switching over robust controller set $\mathcal{K}$ with operating range Θ_i obeying (4), a sufficient condition to guarantee Lyapunov stability is

$$|\dot{\theta}(t)| < \min\{\min_{i \in \mathcal{F}}\{\frac{|d_{i,i+1}|}{\tau_D}, \beta_i\}, \bar{\beta}\} \tag{25}$$

where $d_{i,i+1} = \Theta_i \cap \Theta_{i+1}$ is the i^{th} hysteresis interval as shown in Figure 3.

Proof. For simplicity, we consider only two neighboring controllers, i.e. $K_i(s)$ and $K_{i+1}(s)$ in switching time interval $[t_j, t_{j+1})$, $j \in \mathbb{Z}^+ \cup \{0\}$. As discussed in Theorem 3.1, $t_{j+1} - t_j > \tau_D$ should be satisfied to guarantee stability of the switching system, which requires the currently working controller $K_i(s)$ to hold on at least τ_D. In the worst case of switching where $\theta(t)$ oscillates around the center of the interval $d_{i,i+1}$, with amplitude $|d_{i,i+1}|/2$, the condition $|\dot{\theta}(t)| < d_{i,i+1}/\tau_D$ is sufficient to guarantee stable switching. Taking all the possible controllers into consideration and invoking (3) and $|\dot{\theta}(t)| < \bar{\beta}$, we come up with (25) and complete the proof. ■

Note that Theorem 3.1, as well as Corollary 3.1, is in fact conservative in the sense that the minimum dwell time should be satisfied, which does not allow for fast switching. In the following, we present another result based on the average dwell time for switching LPV systems, which can guarantee exponential stability of switching LPV systems in the more general sense.

Similar to [6], we define the average dwell time τ_D^* and the corresponding switching rule set $S_{ave}[\tau_D^*, N_0]$ as follows. For $t > \tau \geq 0$, let $N(t,\tau) \in \mathbb{Z}^+ \cup \{0\}$ denote the number of discontinuities (switching number) of a switching signal q in the time interval (τ, t); $S_{ave}[\tau_D^*, N_0]$ is defined as the set of all switching rules, q, that satisfy:

$$N(t,\tau) \leq N_0 + \frac{t-\tau}{\tau_D^*} \tag{26}$$

where τ_D^* is called the average dwell time and N_0 the chatter bound. Obviously

$$S[\tau_D^*] \subset S_{ave}[\tau_D^*, 1]$$

In the rest of this section, a sufficient condition on the exponential stability is given in term of the average dwell time, which is an extension of theorem 1 of [6] to the switching LPV systems.

Theorem 3.2: Define $\bar{\lambda} > 0$ as

$$\bar{\lambda} = \min_{i \in \mathcal{F}}\{\frac{b_i}{2M_i}\} \tag{27}$$

where b_i and M_i can be obtained from (20) and (13) respectively. For $\forall \lambda \in (0, \bar{\lambda})$, there exists $\tau_D^* > 0$ such that the switching LPV system (5) is exponentially stable with decay rate no slower than λ for all the switching rules over $S_{ave}[\tau_D^*, N_0]$, where $N_0 \geq 0$ is any finite chatter bound.

Proof. Given time interval $[t_0, \bar{t}]$, where $\bar{t} > t_0 = 0$, denote $t_1 < t_2 < \cdots < t_{N(\bar{t},t_0)}$ to be the switching time instants of q in $(t_0, \bar{t})$. Let

$$\rho := \max_{i \in \mathcal{F}}\{\sqrt{\frac{M_i}{\mu_i}}\}. \tag{28}$$

Recall (22), we have

$$\begin{aligned} \|\xi(t_1)\| &= \lim_{t \uparrow t_1} \|\xi(t)\| \leq \|\xi(t_0)\| \sqrt{\frac{M_i}{\mu_i}} e^{\frac{-b_i}{2M_i}(t_1 - t_0)} \\ &\leq \rho \|\xi(t_0)\| e^{-\bar{\lambda}(t_1 - t_0)}. \end{aligned} \tag{29}$$

Iterating the above inequality from 0 to $N(\bar{t}, t_0) - 1$ yields

$$\begin{aligned} \|\xi(t_{N(\bar{t},t_0)})\| &\leq \rho \|\xi(t_{N(\bar{t},t_0)-1})\| e^{-\bar{\lambda}(t_{N(\bar{t},t_0)} - t_{N(\bar{t},t_0)-1})} \leq \dots \\ &\leq \rho^{N(\bar{t},t_0)} \|\xi(t_0)\| e^{-\bar{\lambda}(t_{N(\bar{t},t_0)} - t_0)}. \end{aligned} \tag{30}$$

Based on (22), (30),

$$\begin{aligned} \|\xi(\bar{t})\| &\leq \rho^{N(\bar{t},t_0)+1} \|\xi(t_0)\| e^{-\bar{\lambda}(\bar{t} - t_0)} \\ &\leq \|\xi(t_0)\| e^{-\bar{\lambda}(\bar{t} - t_0) + (N(\bar{t},t_0)+1)\ln\rho}. \end{aligned} \tag{31}$$

For any $\lambda \in (0, \bar{\lambda})$, we define

$$N_0 := \frac{k}{\ln \rho} \text{ and } \tau_D^* := \frac{\ln \rho}{\bar{\lambda} - \lambda} \tag{32}$$

where $k > 0$ is a constant. Based on the definition of $S_{ave}[\tau_D^*, N_0]$, we come up with

$$N(\bar{t}, t_0) \leq N_0 + \frac{\bar{t} - t_0}{\tau_D^*}$$

which is equivalent to

$$-\bar{\lambda}(\bar{t} - t_0) + N(\bar{t}, t_0) \ln \rho \leq k - \lambda(\bar{t} - t_0). \tag{33}$$

Thus

$$\|\xi(\bar{t})\| \leq \rho \|\xi(t_0)\| e^{-\bar{\lambda}(\bar{t}-t_0) + N(\bar{t}, t_0) \ln \rho} \leq \rho \|\xi(t_0)\| e^{k - \lambda(\bar{t} - t_0)}. \tag{34}$$

We conclude from (34) that the switching LPV system (5) is exponentially stable for all switching rules over $S_{ave}[\tau_D^*, N_0]$ with decay rate no slower than λ. ■

Recall (32), (27) and (28), we have

$$\tau_D^* > \bar{\tau}_D^* := \frac{\ln \rho}{\bar{\lambda}} = \frac{\ln\left(\max_{i \in \mathcal{F}}\{\sqrt{\frac{M_i}{\mu_i}}\}\right)}{\min_{i \in \mathcal{F}}\{\frac{b_i}{2M_i}\}} \geq \tau_D \tag{35}$$

Thus the average dwell time τ_D^* derived in Theorem 3.2 is larger than the minimum dwell time τ_D in Theorem 3.1. However, the former doesn't require any minimum dwell time for switching, which could allow for some fast switchings.

Note that we assume $\theta(t)$ is a scalar function of time t in this paper. For the scenario $\theta(t) \in \mathbb{R}^{n_\theta}$ being a vector, similar results can be easily obtained without further complication.

IV. Numerical Example

In this section, we apply the above switching $\mathcal{H}^\infty$ control method to the following LPV system shown in Figure 4. We employ $\mathcal{L}\{f(t,\theta)|_{\theta=\theta_0}\} = f_{\theta_0}(s)$ to describe the LPV dynamic equations in Laplace domain at fixed parameter values, by which the LPV plant P_θ can be written as:

$$P_\theta(s) = \frac{(1 - \tau s)(1 + \alpha s)}{(1 + \tau s)(s^2 + 2\xi_0(\theta)\omega_0 s + \omega_0^2)(1 - \alpha s)} \tag{36}$$

where $\tau = 0.1$, $\omega_0 = 10$, $\alpha = 15$, $\xi_0(\theta) = 0.075\theta + 0.085$, and $\theta(t) = \theta(t + T)$ is periodical obeying

$$\begin{aligned} \theta(t) &= (3 + 2\sin(\frac{4\pi t}{T}))(\mathbb{U}(t) - \mathbb{U}(t - \frac{3}{8}T)) + \mathbb{U}(t - \frac{3}{8}T) \\ &\quad - \mathbb{U}(t - \frac{7}{8}T) + (3 + 2\sin(\frac{4\pi t}{T}))\mathbb{U}(t - \frac{7}{8}T), \ t \leq T \end{aligned}$$

where $T = 3.6 \times 10^4$ and $\mathbb{U}(t)$ is the unit step function. Thus $\theta \in \Theta := [1, 5]$ and $\xi_0(\theta) \in [0.16, 0.46]$.

We would like to design $\mathcal{H}^\infty$ controllers to stabilize the system and minimize $\sup_{w \neq 0}\{\frac{\|z\|_2}{\|w\|_2}\}$, where the regulated output z and the exogenous input w are defined as:

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \ w = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}.$$

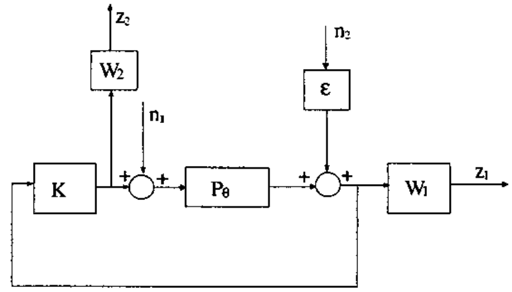

Fig. 4. Block diagram

Note that n_2 is a fictitious noise that we added so that the rank conditions of standard four block $\mathcal{H}^\infty$ design can be satisfied [5]. The weighting functions W_1 and W_2 are chosen as $W_1 = (s + 100)/(4s + 4)$ and $W_2 = 2$ respectively.

We consider switching $\mathcal{H}^\infty$ control scheme as discussed in the above sections, where we design two $\mathcal{H}^\infty$ controllers K_1 and K_2 at the selected operating points $\theta = \theta_1 = 2.2$ and $\theta = \theta_2 = 3.8$ respectively, and employ controller switching between K_1 and K_2. The operating range is chosen as $\Theta_1 = [\theta_1^-, \theta_1^+] = [1, 3.4]$ for controller K_1, and $\Theta_2 = [\theta_2^-, \theta_2^+] = [2.6, 5]$ for K_2. The two candidate $\mathcal{H}^\infty$ controllers K_1 and K_2 can be constructed using standard $\mathcal{H}^\infty$ optimization methods [5], [19]:

$$\left[\begin{array}{c|c} A_{K_1} & B_{K_1} \\ \hline C_{K_1} & D_{K_1} \end{array}\right] \sim K_1(s) =$$

$$\frac{9654s^4 + 1.54 \times 10^5 s^3 + 1.59 \times 10^6 s^2 + 1.11 \times 10^7 s + 9.64 \times 10^6}{s^5 + 88.48s^4 + 1710s^3 + 2.93 \times 10^4 s^2 + 2.96 \times 10^4 s + 1840}$$

and

$$\left[\begin{array}{c|c} A_{K_2} & B_{K_2} \\ \hline C_{K_2} & D_{K_2} \end{array}\right] \sim K_2(s) =$$

$$\frac{9661s^4 + 1.78 \times 10^5 s^3 + 1.85 \times 10^6 s^2 + 1.13 \times 10^7 s + 9.64 \times 10^6}{s^5 + 89.83s^4 + 1815s^3 + 3.04 \times 10^4 s^2 + 3.05 \times 10^4 s + 1901}$$

The following analysis shows that K_1 and K_2 can robustly stabilize the LPV system within the operating range Θ_1 and Θ_2 respectively.

Define

$$P_e^i(s) = P_\theta(s) - P_{\theta_i}(s), \ i = 1, 2,$$

and assume

$$|P_e^i(j\omega)| < |W_e^i|, i = 1, 2. \tag{37}$$

A sufficient condition to guarantee robust stability is given by [19]:

$$\|W_e^i K_i (1 + P_{\theta_i} K_i)^{-1}\|_\infty \leq 1, \ i = 1, 2. \tag{38}$$

As depicted in Figure 5, (37) can be satisfied by choosing

$$\begin{aligned} W_e^1(s) &= \frac{55(s+2)^2}{(s+7)^2(s+8)(s+9)(s+12)} \\ W_e^2(s) &= \frac{30(s+2)^2}{(s+7)^2(s+8)(s+9)(s+12)}. \end{aligned}$$

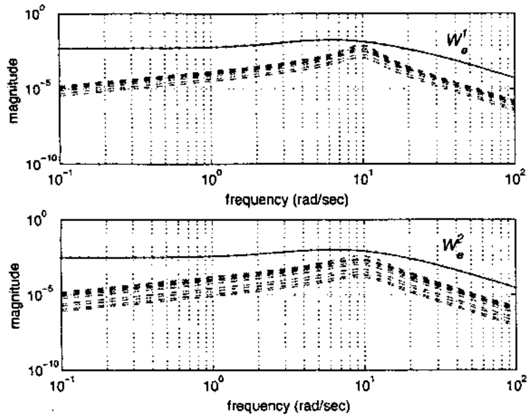

Fig. 5. Weights for uncertainties with respect to θ_1 and θ_2

Figure 6 shows that the robust stability condition (38) is satisfied for K_1 and K_2 respectively.

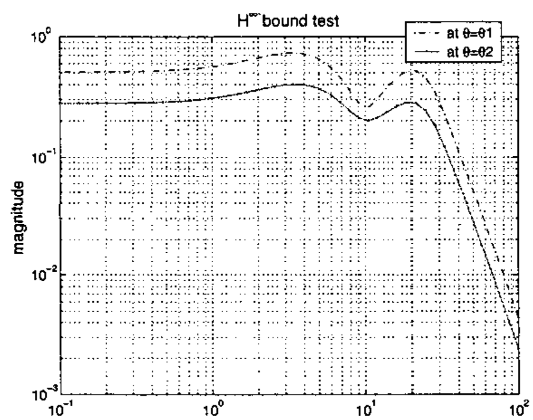

Fig. 6. Robustness test

Thus K_1 and K_2 can robustly stabilize the LPV system with respect to Θ_1 and Θ_2. Consider the closed-loop A-matrices $A_1(\theta), \theta \in \Theta_1$ and $A_2(\theta), \theta \in \Theta_2$, we can numerically obtain the following parameters:

TABLE I

PARAMETERS λ_i, μ_i, M_i AND K_Q^i

	λ_i	μ_i	M_i	K_Q^i
$i=1$	0.8	1.0	261.59	3028.2
$i=2$	1.4	1.0	261.49	1190.9

Recall (18), we have $\bar{\beta} = 8.6 \times 10^{-4}$. Also notice that

$$|\dot{\theta}(t)| \leq |\frac{4\pi}{18000}| \approx 7 \times 10^{-4}.$$

Choosing $\beta = 8.1 \times 10^{-4} < \bar{\beta}$ and invoking (20), we have $b_1 = 0.15$ and $b_2 = 2.84$. Furthermore, we can pick $\lambda = 1.02 \times 10^{-4}$ such that $\tau_D^* = 1.5 \times 10^4$, which is straightforward from (27), (28) and (32). Thus, the switching scheme for $\theta(t)$ belongs to $S_{ave}[\tau_D^*, 1]$, which is due to the fact that there are only 2 switchings per period T (Fig.8). Based on Theorem 3.2, we conclude that the switching LPV system with K_1 and K_2 are stable.

The closed loop system with the determined switching $\mathcal{H}^\infty$ control scheme is simulated using MATLAB. For the purpose of comparison, we also provide an $\mathcal{H}^\infty$ controller $\bar{K}$ designed at $\theta = \frac{\theta_1^- + \theta_2^+}{2} = 3$ for the LPV system, by which the performance of a single $\mathcal{H}^\infty$ controller can be simulated. The disturbance n_1 is set to be $n_1(t) = \sin(2\pi/6000) + \frac{1}{2}\sin(2\pi/3000) + \delta(t)$, where $\delta(t)$ is a Gaussian distributed signal of mean 0 and variance 0.2.

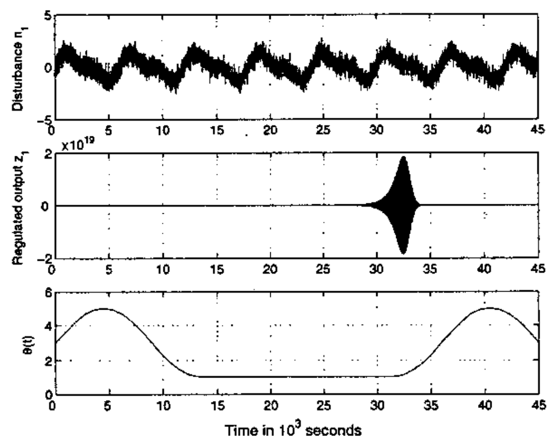

Fig. 7. The case of a single $\mathcal{H}^\infty$ controller

First, we give the simulation result for the case of single $\mathcal{H}^\infty$ controller $\bar{K}$ (for comparison purposes) in Figure 7, where the divergence of the output signal is observed because $\bar{K}$ itself can not robustly stabilize the LPV system for the whole operating range Θ. The simulation results of the switching $\mathcal{H}^\infty$ control method are depicted in Figure 8. Note that the system remains stable and the magnitude of the regulated output z_1 is much smaller than the magnitude of the disturbance n_1 for all $\theta \in \Theta$.

Note that for the proposed switching control scheme, Theorem 3.1 is not valid. In fact, the minimum dwell time τ_D to guarantee stability in Theorem 3.1 is given by

$$\tau_D = \max_{i=1,2}\{\frac{2M_i \ln\sqrt{\frac{M_i}{\mu_i}}}{b_i}\} = 9.71 \times 10^3 > \tau_{min},$$

where $\tau_{min} = T/4 = 9000$ is the minimum distance between two consecutive switchings in our design, which is depicted in Fig.8. Meanwhile, Corollary 3.1 also turns out to be too

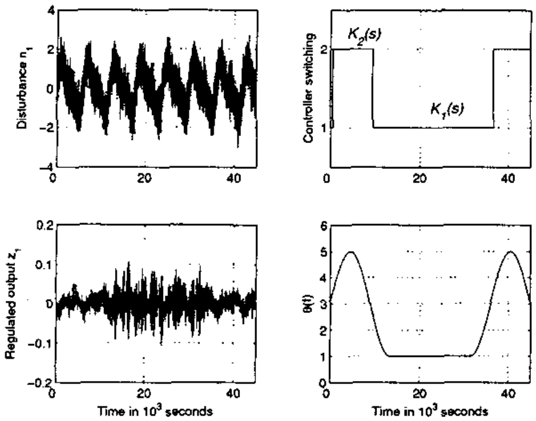

Fig. 8. The switching $\mathcal{H}^\infty$control method

conservative for this design due to the fact that

$$\max\{\dot{\theta}(t)\} = \frac{4\pi}{18000} > \frac{|d_{1,2}|}{\tau_D} = \frac{0.8}{9710} = 8.2\times10^{-5},$$

which violates (25). The analysis of this numerical example affirms a good coincidence with the discussion of Section 3. It suggests that Theorem 3.2 is a less conservative result allowing faster switching.

V. Concluding Remarks

Switching $\mathcal{H}^\infty$ controllers are proposed for a class of LPV systems with slow parameter variations. Controller robustness is combined with the switching policy, which results in the hysteresis switching over a set of $\mathcal{H}^\infty$ controllers designed at selected operating points. The stability analysis is provided in terms of the dwell time and the average dwell time. The proposed switching $\mathcal{H}^\infty$ control method is illustrated by a numerical example, where the comparison between the single $\mathcal{H}^\infty$ controller and our design is also given. A further extension of this work would be switching control for LPV systems with fast parameter variations.

VI. References

[1] P. Apkarian and P. Gahinet, "A convex characterization of gain-scheduled $\mathcal{H}^\infty$ controllers", *IEEE Trans. Automatic Control*, vol. 40, pp. 853-864, 1995.

[2] P. Apkarian, P. Gahinet and G. Becker, " Self-scheduled $\mathcal{H}^\infty$ control of linear parameter-varying systems: a design example", *Automatica*, vol. 31, pp. 1251-1261, 1995.

[3] C. Bett and M. Lemmon, "Bounded amplitude performance of switched LPV systems with applications to hybrid systems", *Automatica*, vol. 35, pp. 491-503, 1999.

[4] C. Desoer, "Slowly varying system $\dot{x} = A(t)x$", *IEEE Trans. Automatic Control*, vol. 14, pp. 780-781, 1969.

[5] J. Doyle, K. Glover, P. Khargonekar and B. Francis, "State-space solutions to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems", *IEEE Trans. Automatic Control*, vol. 34, pp. 831-847, 1989.

[6] J. Hespanha and S. Morse, "Stability of switched systems with average dwell-time", *Proc. of the 38th IEEE Conference on Decision and Control*, 1999.

[7] J. Hochcerman-Frommer, S. Kulkarni and P. Ramadge, "Controller switching based on output prediction errors", *IEEE Trans. Automatic Control*, vol. 43, pp. 596-607, 1998.

[8] D. Lawrence and W. Rugh, "On a stability theorem for nonlinear systems with slowly varying inputs", *IEEE Trans. Automatic Control*, vol. 35, pp. 860-864, 1990.

[9] S.-H. Lee and J.-T. Lim, "Fast gain scheduling on tracking problems using derivative information", *Automatica*, vol. 33, pp. 2265-2268, 1997.

[10] S.-H. Lee and J.T. Lim, "Switching control of $\mathcal{H}^\infty$ gain scheduled controllers in uncertain nonlinear systems", *Automatica*, vol. 36, pp. 1067-1074, 2000.

[11] S. Morse, "Supervisory control of families of linear setpoint controllers: part 1: exact matching", *IEEE Trans. Automatic Control*, vol. 41, pp. 1413-1431, 1996.

[12] A. Packard, "Gain-scheduling via linear fractional transformations", *System and Control Letters*, vol. 22, pp. 79-92, 1994.

[13] W. Rugh and J. Shamma, "Research on gain scheduling", *Automatica*, vol. 36, pp. 1401-1425, 2000.

[14] A. Savkin, E. Sakfidas and R. Evans, "Robust output feedback stabilizability via controller switching", *System and Control Letters*, vol. 29, pp. 81-90, 1996.

[15] E. Skafidas, R. Evans, A. Savkin and I. Peterson, "Stability results for switched controller systems", *Automatica*, vol. 35, pp. 553-564, 1999.

[16] D. Stilwell and W. Rugh, "Stability and $\mathcal{L}_2$ gain properties of LPV systems", *Proc. of the American Control Conference*, 1999.

[17] F. Wang and V. Balakrishnan, "Improved stability analysis and gain-scheduled controller synthesis for parameter-dependent systems", *IEEE Trans. Automatic Control*, vol. 47, pp. 720-734, 2002.

[18] F. Wu and K. Grigoriadis, "LPV systems with parameter-varying time delays: analysis and control", *Automatica*, vol. 37, pp. 221-229, 2001.

[19] K. Zhou, J. Doyle and K. Glover, "Robust and Optimal Control", *Prentice-Hall*, New Jersey, 1996.