Linear Algebraic Theory of Partial Coherence: Discrete Fields and Measures of Partial Coherence

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ABSTRACT

We present a linear algebraic theory of partial coherence which allows precise mathematical definitions of concepts such as coherence and incoherence. This not only provides new perspectives and insights, but also allows us to employ the tools of linear algebra in applications. We define a scalar measure of the degree of partial coherence of an optical field which is zero for complete incoherence and unity for full coherence.

1. INTRODUCTION

The theory of partial coherence is a well established area of optics.[1,2] In this work, we formulate the theory in terms of the standard concepts of linear algebra, leading to a number of new perspectives. While not containing any new physics, this approach offers new insights, understanding, and operationality and has the potential to facilitate applications, especially in optical information processing. In this paper we consider the case of discrete light fields, which lead to a particularly simple matrix-algebraic formulation. Once the framework is established, it is not difficult to translate the matrix formalism for discrete fields to a continuous formalism. We restrict our attention to quasi-monochromatic conditions. One-dimensional notation is employed for simplicity.

2. CORRELATION MATRICES AND THEIR PROPERTIES

Let $\mathbf{f} = [f(1), f(2), \ldots, f(N)]^{\mathrm{T}}$ be a vector representing the samples of the continuous optical field $f(x)$. Here N is the space-bandwidth product of $f(x)$. We define the *mutual intensity matrix* $\mathbf{J}$ as $\mathbf{J} = \langle \mathbf{f}\mathbf{f}^{\mathrm{H}} \rangle$, where the angle brackets denote ensemble averaging. The diagonal elements of $\mathbf{J}$ correspond to the intensity of the field. $J(m,n)$ will represent the elements of $\mathbf{J}$. The following properties of $\mathbf{J}$ hold for any field $\mathbf{f}$: (i) $\mathbf{J}$ is Hermitian symmetric and positive semi-definite, and thus all eigenvalues of $\mathbf{J}$ are real and non-negative. (ii) A complete set of orthogonal eigenvectors can always be found and $\mathbf{J}$ can be diagonalized by a unitary matrix $\mathbf{U}$ whose columns are the orthonormal eigenvectors of $\mathbf{J}$; mathematically $\mathbf{J} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\mathrm{H}}$, where $\mathbf{\Lambda}$ is a diagonal matrix of the eigenvalues of $\mathbf{J}$. This can also be written as

$$\mathbf{J} = \sum_{k=1}^{R} \lambda_k \, \mathbf{u}_k \mathbf{u}_k^{\mathrm{H}}, \tag{1}$$

where $\mathbf{u}_k$ is the kth column of $\mathbf{U}$, λ_k is the corresponding eigenvalue, and R is the number of non-zero eigenvalues.[3] This expression is also known as the *coherent-mode representation.*[4] (iii) If $\mathbf{J}$ can be expressed as the outer product of two vectors $\mathbf{u}'$ and $\mathbf{u}''$ in the form $\mathbf{u}'\mathbf{u}''^{\mathrm{H}}$, then by appropriate scaling $\mathbf{J}$ can also be expressed in self-outer-product form $\mathbf{u}\mathbf{u}^{\mathrm{H}}$. (iv) The rank of $\mathbf{J}$ is equal to the number of non-zero eigenvalues.[3]

Sometimes it is more convenient to work with the normalized version $\mathbf{L}$ of $\mathbf{J}$:

$$L(m,n) = \frac{\langle f(m) f^*(n) \rangle}{\sqrt{\langle |f(m)|^2 \rangle \langle |f(n)|^2 \rangle}} = \frac{J(m,n)}{\sqrt{J(m,m) J(n,n)}}, \tag{2}$$

where $L(m,n)$ are the elements of the matrix $\mathbf{L}$. $\mathbf{L}$ is referred to as the complex coherence matrix.[5] Since $\mathbf{L}$ is obtained by normalizing $\mathbf{J}$, the properties given above also hold for $\mathbf{L}$. In addition to these, the following are also true: The diagonal entries of $\mathbf{L}$ are all equal to 1, $|L(m,n)| \leq 1$, and if $\mathbf{J}$ has unit rank, then all elements of $\mathbf{L}$ have unit magnitude. Conversely, if all elements of $\mathbf{L}$ have unit magnitude, positive semi-definiteness implies that $\mathbf{J}$ has unit rank.[5]

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3. THE DEGREE OF PARTIAL COHERENCE

A field is considered coherent if any two samples of the field are fully correlated; that is, if they are just as correlated with each other as they are with themselves. A field is considered incoherent if any two distinct samples are fully uncorrelated; that is, if the magnitude of their normalized correlation is zero.

We first consider fully coherent fields. Since any two samples of such a field must be fully correlated, the magnitude of their normalized correlation must be unity. This means that all of the elements of matrix $\mathbf{L}$ must have unit magnitude. In this case the matrices $\mathbf{J}$ and $\mathbf{L}$ both have unit rank, are of outer-product form, and consequently have only one non-zero eigenvalue.[5] The sole non-zero eigenvalue of $\mathbf{L}$ is equal to N. We can thus say that a discrete optical field is fully coherent if any of the following alternative conditions are satisfied: (i) All elements of the associated $\mathbf{L}$ matrix have unit magnitude: $|L(m,n)| = 1$; (ii) The associated mutual intensity matrix $\mathbf{J}$ has unit rank; (iii) $\mathbf{J}$ (or $\mathbf{L}$) has only one non-zero eigenvalue; (iv) $\mathbf{J}$ (or $\mathbf{L}$) is of outer-product form.

Next we consider fully incoherent fields. Since any two distinct samples of such a field must be uncorrelated, the mutual intensity matrix $\mathbf{J}$ and its normalized version $\mathbf{L}$ must be diagonal. In fact, $\mathbf{L}$ is the identity matrix. Therefore we can say that a discrete optical field is fully incoherent if the following alternative conditions are satisfied: (i) The associated normalized mutual intensity matrix $\mathbf{L}$ is the identity matrix: $\mathbf{L} = \mathbf{I}$; (ii) The associated mutual intensity matrix $\mathbf{J}$ is diagonal. Though trivial, we also note that for the fully incoherent case, the matrix $\mathbf{L}$ is of full rank ($R = N$), and all of its eigenvalues are equal to unity.

Based on the definitions of full coherence and full incoherence in terms of correlation matrices, we can define a scalar measure of the degree of partial coherence of a field. This can be accomplished by interpolating any of the characteristics of the matrices in question.[5] There are many ways of constructing such interpolation functions, leading to several definitions of such a measure. Here we will present one possible definition, referring the reader elsewhere[5] for other possible definitions and further discussion.

As already stated, incoherent light is characterized by all unity eigenvalues and coherent light by one non-zero eigenvalue of the matrix $\mathbf{L}$. We also note that the eigenvalues of $\mathbf{L}$ are all non-negative and their sum is equal to N. Based on these observations, we can state that the more concentrated the eigenvalues are, the more coherent the light, and the more uniformly spread they are, the more incoherent the light. One measure of this is the variance of the eigenvalues:

$$c' = \frac{1}{N}\sum_{n=1}^{N}(\lambda_n - 1)^2, \tag{3}$$

where the 1 subtracted from λ_n is the average value of the eigenvalues. When all eigenvalues are unity we have $c' = 0$, and when only one eigenvalue is non-zero we have $c' = N - 1$. For convenience, we define $c = c'/(N-1)$, so that $c = 0$ corresponds to full incoherence and $c = 1$ corresponds to full coherence.

4. CONCLUSION

In this paper, we present certain elements of a linear algebraic theory of partial coherence. While containing no new physics, the presented formulation allows precise definitions of concepts such as coherence and incoherence, offers new insights, and allows us to make the most of the conceptual and algebraic tools of linear algebra. We offered a definition for the degree of partial coherence of a light field. Various alternatives and their continuous versions are discussed elsewhere.[5] The formulation presented should be especially useful in optical information processing applications, allowing the precise analytical and numerical formulation of such problems. For instance, this formalism can be applied to the problem of synthesizing light with desired mutual intensity.[6]

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