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# WHICH SINGULAR K3 SURFACES COVER AN ENRIQUES SURFACE

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ABSTRACT. We determine the necessary and sufficient conditions on the entries of the intersection matrix of the transcendental lattice of a singular K3 surface for the surface to doubly cover an Enriques surface.

## 1. INTRODUCTION

When  $X$  is a singular K3 surface over the field  $\mathbb{C}$ , the transcendental lattice  $T_X$  of  $X$  is denoted by its intersection matrix

$$(1) \quad \begin{pmatrix} 2a & c \\ c & 2b \end{pmatrix}$$

with respect to some basis  $\{u, v\}$ , where  $a, b > 0$  and  $4ab - c^2 > 0$ . For the definitions and basic facts about K3 surfaces we refer to [1].

Following the works of Horikawa on the period map of Enriques surfaces and work of Nikulin on the embeddings of even lattices, Keum gave an integral lattice theoretical criterion for the existence of a fixed point free involution on a K3 surface, [5, 6, 11, 7]. This criterion is then applied in [7] to show that every algebraic Kummer surface is the double cover of some Enriques surface, in which case the  $a, b, c$  of  $T_X$  are even and  $17 \leq \rho(X) \leq 20$ , see also [10, 8].

If  $U$  denotes the hyperbolic lattice of rank 2 and if  $E_8$  denotes the even unimodular negative definite lattice of rank 8, then a sublattice  $\Lambda^-$  of

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the K3-lattice  $\Lambda$  is defined as

$$\Lambda^- = U \oplus U(2) \oplus E_8(2).$$

A K3 surface with  $12 \leq \rho(X) \leq 20$  covers an Enriques surface if and only if there is a primitive embedding  $\phi : T_X \rightarrow \Lambda^-$  such that the orthogonal complement of the image in  $\Lambda^-$  contains no self intersection -2 vector, and when  $\rho(X) = 10$  or  $11$ , one also needs to have  $\text{length}(T_X) \leq \rho(X) - 2$ , [7, Theorem 1].

We implement this criterion to find explicit necessary and sufficient conditions on the entries of  $T_X$  so that  $X$  covers an Enriques surface when  $\rho(X) = 20$ . In practice, if  $X$  actually covers an Enriques surface it is sometimes, but by no means always, easy to exhibit an embedding  $\phi : T_X \rightarrow \Lambda^-$  such that i) it is possible to demonstrate that  $\phi$  is primitive and that ii) it is possible to show that the existence of a self intersection -2 vector in  $\phi(T_X)^\perp$  leads to a contradiction. However, in case  $X$  does not cover an Enriques surface then it is hard work to demonstrate that for every primitive embedding the orthogonal complement of the image has a self intersection -2 vector. We resolve this difficulty in

**Theorem 1.** *If  $X$  is a singular K3 surface with transcendental lattice given as in (1), then  $X$  covers an Enriques surface if and only if one of the following conditions hold:*

**I**  *$a, b$  and  $c$  are even. (Keum's result, see [7]).*

**II**  *$c$  is odd and  $ab$  is even.*

**III-1**  *$c$  is even.  $a$  or  $b$  is odd. The form  $ax^2 + cxy + by^2$  does not represent 1.*

**III-2**  *$c$  is even.  $a$  or  $b$  is odd. The form  $ax^2 + cxy + by^2$  represents 1, and  $4ab - c^2 \neq 4, 8, 16$ .*

*Equivalently,  $X$  fails to doubly cover an Enriques surface if and only if one of the following conditions hold:*

**III-3**  *$c$  is even.  $a$  or  $b$  is odd. The form  $ax^2 + cxy + by^2$  represents 1, and  $4ab - c^2 = 4, 8, 16$ .*

**IV**  *$abc$  is odd.*

## 2. PARITIES IN TRANSCENDENTAL LATTICE

Before we proceed with the proof we show that the parity properties given in Theorem 1 are well defined.

Let  $\theta = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in SL_2(\mathbb{Z})$ . Then every matrix of the form  ${}^t\theta T_X \theta$  represents the transcendental lattice of  $X$  with respect to some basis. Setting

$$\begin{aligned} {}^t\theta T_X \theta &= \begin{pmatrix} 2(ax^2 + cxz + bz^2) & 2axy + c(xw + yz) + 2bwz \\ 2axy + c(xw + yz) + 2bwz & 2(ay^2 + cyw + bw^2) \end{pmatrix} \\ &= \begin{pmatrix} 2a' & c' \\ c' & 2b' \end{pmatrix}, \end{aligned}$$

we see by inspection that

- I If  $a, b$ , and  $c$  are even, then  $a', b'$  and  $c'$  are even.
- II If  $c$  is odd and  $ab$  is even, then  $c'$  is odd and  $a'b'$  is even.
- III If  $c$  is even with  $a$  or  $b$  odd, then  $c'$  is even with  $a'$  or  $b'$  odd.
- IV If  $abc$  is odd, then  $a'b'c'$  is odd.

### 3. TWO LEMMAS ON INTEGRAL LATTICES

For the fundamental concepts related to integral lattices we refer to [1, 3, 4, 9]. We leave the proofs to the reader.<sup>1</sup>

Let  $M = (\mathbb{Z}^n, A)$  be an integral lattice where  $A = {}^tA$  is the intersection matrix, and let  $\alpha = (\alpha_1, \dots, \alpha_n)$  be a primitive element, i.e.  $\gcd(\alpha_1, \dots, \alpha_n) = 1$ . We denote by  $\langle \alpha, \beta \rangle_M$  the inner product of the vectors  $\alpha$  and  $\beta$  in  $M$ . Denote the orthogonal complement of  $\alpha$  in  $M$  by  $\alpha^\perp$ .

**Lemma 2.** *The index of  $\alpha \oplus \alpha^\perp$  in  $M$  divides  $\langle \alpha, \alpha \rangle_M$ .  $\square$*

Let  $L_1$  and  $L_2$  be two lattices with base elements  $e_1, \dots, e_n$  and  $f_1, \dots, f_m$  respectively where  $m \geq n$ . Assume that we have an embedding of  $L_1$  into  $L_2$  given by

$$\phi(e_i) = a_{i1}f_1 + \dots + a_{im}f_m, \quad i = 1, \dots, n$$

where the  $a_{ij}$ 's are integers. For any choice of integers  $1 \leq t_1, \dots, t_n \leq m$ , define

$$\begin{aligned} \Delta(t_1, \dots, t_n) &= \det(a_{it_j})_{1 \leq i, j \leq n}, \text{ and} \\ d &= \gcd\{\Delta(t_1, \dots, t_n) \mid 1 \leq t_1, \dots, t_n \leq m\}. \end{aligned}$$

<sup>1</sup>The full proofs can be found in arXiv.math.AG/0205282 v1.

**Lemma 3.** *The embedding  $\phi$  is primitive if and only if  $d = 1$ . In other words, a lattice embedding is primitive if and only if the greatest common divisor of the maximal minors of the embedding matrix with respect to any choice of bases is 1.  $\square$*

As an immediate application of this lemma we can indicate that all the mappings in [7, pp106-108] have embedding matrices whose maximal minors have greatest common divisor equal to 1.

#### 4. THE CASE WHEN $c$ IS EVEN WITH $a$ OR $b$ ODD

If  $a$  is even, then set  $\theta = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ . If  $\theta T_X \theta = \begin{pmatrix} 2a' & c' \\ c' & 2b' \end{pmatrix}$ , then  $a'$  and  $b'$  are odd, and  $c'$  is even. If  $b$  is even, then  $\theta = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$  changes  $T_X$  into an equivalent form where again  $a'$  and  $b'$  are odd, and  $c'$  even. So we might assume without loss of generality that  $ab$  is odd, and  $c$  is even.

We will consider a particular embedding of  $T_X$  into  $\Lambda^- = U \oplus U(2) \oplus E_8(2)$ .

Let  $\{u, v\}$  be a basis of  $T_X$ ,  $\{u_1, u_2\}$  be a basis of  $U$  and  $\{v_1, v_2\}$  be a basis of  $U(2)$ .

Define  $\phi : T_X \rightarrow \Lambda^-$  by

$$\begin{aligned} \phi(u) &= u_1 + au_2, \\ \phi(v) &= u_1 + (c - a)u_2 + v_1 + \frac{1}{2}(a + b - c)v_2. \end{aligned}$$

It can be shown by direct computation that this is an embedding and by lemma 3 that this embedding is primitive.

##### 4.1. The form $ax^2 + cxy + by^2$ does not represent 1.

Let  $f = xu_1 + x'u_2 + yv_1 + y'v_2 + e \in \Lambda^-$ , where  $e \in E_8(2)$  with  $e \cdot e = -4k$ ,  $k \geq 0$ . (we will use  $\cdot$  to denote the inner product on  $\Lambda^-$ ).

Impose the condition that  $f$  lies in the orthogonal complement of  $\phi(T_X)$  in  $\Lambda^-$  and that  $f \cdot f = -2$ .

Solving the equations  $f \cdot \phi(u) = 0$ ,  $f \cdot \phi(v) = 0$  for  $x'$ ,  $y'$  and substituting into the equation  $f \cdot f = -2$  gives

$$(2) \quad 1 - (ax^2 + (c - 2a)xy + (a + b - c)y^2) = 2k \geq 0.$$

The binary quadratic form  $ax^2 + (c - 2a)xy + (a + b - c)y^2$  is equivalent to the form  $ax^2 + cxy + by^2$ . Since  $a > 0$  and  $c^2 - 4ab < 0$ , this is a positive definite form. Equation (2) holds if and only if this form represents 1, and then  $k = 0$ . (see [12])

If we assume that the form  $ax^2 + cxy + by^2$  does not represent 1, then equation (2) cannot be solved, so there is no self intersection  $-2$  vector in the orthogonal complement of  $\phi(T_X)$ .

This proves **III-1**.

#### 4.2. The form $ax^2 + cxy + by^2$ does represent 1.

In this case the binary quadratic form  $ax^2 + cxy + by^2$  is equivalent to the form  $x^2 + (ab - c^2/4)y^2$ , see [12, p174]. Then a basis  $\{u, v\}$  of the transcendental lattice exists such that with respect to that basis the matrix

$$T_X = \begin{pmatrix} 2(1) & 0 \\ 0 & 2(\frac{\Delta}{4}) \end{pmatrix}$$

where  $\Delta = 4ab - c^2$ .

Let  $\phi$  be a primitive embedding of  $T_X$  into  $\Lambda^-$  and set  $\phi(u) = \alpha$  with

$$\alpha = a_1u_1 + a_2u_2 + a_3v_1 + a_4v_2 + \omega_1$$

where  $\omega_1 \in E_8(2)$  with  $\omega \cdot \omega = -4k \leq 0$ .

$\alpha \cdot \alpha = 2$  forces  $a_1$  and  $a_2$  to be odd.

If  $\beta = b_1u_1 + b_2u_2 + b_3v_1 + b_4v_2 + \omega_2$  is in the orthogonal complement  $\alpha^\perp$  of  $\alpha$  in  $\Lambda^-$ , then  $\beta \cdot \alpha = 0$  forces  $b_1$  and  $b_2$  to be of the same parity. This in turn implies the following

**Lemma 4.** *If  $\beta, \gamma \in \alpha^\perp$ , then  $\beta \cdot \gamma \equiv 0 \pmod{2}$ .* □

Let  $\beta_1, \dots, \beta_{11}$  be basis elements for  $\alpha^\perp$ , and  $B' = (2b_{ij})$ ,  $2b_{ij} = \beta_i \cdot \beta_j$  the intersection matrix for this basis. Set  $B = (b_{ij})$ .

Let  $C$  be the  $12 \times 12$ -matrix whose rows are the coordinates of  $\alpha, \beta_1, \dots, \beta_{11}$  with respect to the standard basis of  $\Lambda^-$ . And finally let  $A$  denote the

intersection matrix of  $\Lambda^-$  with respect to its standard basis. We have

$$(3) \quad CA^tC = \begin{pmatrix} 2 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & B' & \\ 0 & & & \end{pmatrix}.$$

Since  $\alpha, \beta_1, \dots, \beta_{11}$  is not a basis of  $\Lambda^-$ ,  $|\det C| > 1$ . By lemma 2,  $|\det C|$  divides 2, hence is equal to 2. By interchanging  $\beta_1$  by  $\beta_2$  if necessary, we can assume without loss of generality that  $\det C = 2$ .

It then follows from equation (3) that  $\det B = 1$ .

Define a new lattice  $L = (\mathbb{Z}^{11}, B(-1))$ .  $L$  has signature  $(\tau^+, \tau^-) = (10, 1)$ . Since  $\tau^+ - \tau^- \not\equiv 0 \pmod{8}$ ,  $L$  is odd. Then  $L$  is an indeterminate, odd, unimodular lattice, and as such is isomorphic to  $\langle -1 \rangle^1 \oplus \langle 1 \rangle^{10}$ .

There is an isomorphism  $F : \alpha^\perp \rightarrow L$  which sends  $\beta_i$  to  $e_i = (0, \dots, 1, \dots, 0)$ , where 1 is in the  $i$ -th place. This isomorphism respects inner products in the sense that

$$-2[F(\lambda_1) \cdot F(\lambda_2)] = \lambda_1 \cdot \lambda_2, \text{ for all } \lambda_1, \lambda_2 \in \alpha^\perp.$$

Let  $e'_1, \dots, e'_{11}$  be a basis of  $L$  diagonalizing its intersection matrix. Then the intersection matrix of  $\alpha \oplus \alpha^\perp$  with respect to the basis  $\alpha, F^{-1}(e'_1), \dots, F^{-1}(e'_{11})$  is

$$\begin{pmatrix} 2 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & -2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -2 \end{pmatrix}$$

We are looking for the existence of a primitive embedding

$$\phi : T_X \longrightarrow \alpha \oplus \alpha^\perp \subset \Lambda^-$$

such that with respect to this new basis of  $\alpha \oplus \alpha^\perp$ ,

$$\begin{aligned} \phi(u) &= (1, 0, \dots, 0), \\ \phi(v) &= (0, x_0, \dots, x_{10}) \end{aligned}$$

such that

$$\phi(v) \cdot \phi(v) = 2x_0^2 - 2x_1^2 - \dots - 2x_{10}^2 = 2 \left( \frac{\Delta}{4} \right).$$

Using lemma 3, the problem reduces to a problem in the lattice  $L$ , that of investigating the existence of integers  $x_0, \dots, x_{10}$  such that if  $x = (x_0, \dots, x_{10}) \in L$  then the following conditions are satisfied:

$$\begin{aligned}
 \gcd(x_0, \dots, x_{10}) &= 1, \\
 x \cdot x &= -x_0^2 + x_1^2 + \dots + x_{10}^2 \\
 &= -\left(\frac{\Delta}{4}\right), \text{ and} \\
 (4) \quad y \cdot x = 0 &\implies y \cdot y \neq 1, \text{ for every } y \in L.
 \end{aligned}$$

The existence of such integers is equivalent to  $X$  covering an Enriques surface.

The set of negative self intersection elements of  $L$  span an open convex cone in  $\mathbb{Z}^{11} \otimes \mathbb{R}$ , and we refer to [2] for details. We will utilize the techniques of Vinberg from [15] to investigate the existence of integers as above.

All automorphisms of  $L$  are generated by reflections and a fundamental region for negative self intersecting vectors in  $L$  is bounded by reflecting hyperplanes. Each reflecting hyperplane consists of vectors orthogonal to some vector  $e \in \mathbb{Z}^{11}$  with  $e \cdot e = 1$ . A negative self intersecting vector  $v \in L$  has no vector of self intersection 1 in its orthogonal complement if and only if it is not on one of these reflecting hyperplanes. Such a vector can then be mapped by an automorphism to the interior of the fundamental region. So it suffices to consider only vectors on the interior of the fundamental region. This finally amounts to saying that the conditions in the set of equations (4) holds if and only if (see [15])

$$\begin{aligned}
 \gcd(x_0, \dots, x_{10}) &= 1, \\
 -x_0^2 + x_1^2 + \dots + x_{10}^2 &= -\left(\frac{\Delta}{4}\right), \\
 x_1 &\geq \dots \geq x_{10} > 0, \\
 x_0 &\geq x_1 + x_2 + x_3, \text{ and} \\
 3x_0 &> x_1 + \dots + x_{10}.
 \end{aligned}$$

Let  $P$  denote the set of all  $x \in L$  satisfying the above conditions.

The rest of this case is elementary and we summarize the results in two technical lemmas.

**Lemma 5.** *There is no  $x \in P$  with  $x \cdot x = -1, -2, -4$ .*



*Proof.* Let  $P(m) = \{x \in P \mid x = (m, x_1, \dots, x_{10})\}$ . Then it is easy to show that

$$\begin{aligned} \max_{x \in P(3m)} (x \cdot x) &= 5 - 4m, \quad m > 2, \\ \max_{x \in P(6)} (x \cdot x) &= -5, \\ \max_{x \in P(3m+1)} (x \cdot x) &= 1 - 4m, \quad m \geq 1, \\ \max_{x \in P(3m+2)} (x \cdot x) &= 9 - 8m, \quad m \geq 3, \\ \max_{x \in P(8)} (x \cdot x) &= -12, \\ \max_{x \in P(5)} (x \cdot x) &= -7. \end{aligned}$$

These maximum values are achieved by the vectors

$$\begin{aligned} &[3m, m, \dots, m, m - 2, 1], \\ &[6, 2, \dots, 2, 1, 1, 1], \\ &[3m + 1, m + 1, m, \dots, m, 1], \\ &[3m + 2, m + 2, m, \dots, m, 3], \\ &[8, 4, 2, \dots, 2], \\ &[5, 3, 1, \dots, 1], \text{ respectively.} \end{aligned}$$

It is now clear that  $-1$  and  $-2$  are never achieved. And if  $x \cdot x = -4$ , then  $x \in P(4)$ . But none of the vectors in  $P(4)$  achieve  $-4$ .  $\square$

**Lemma 6.** *For every positive integer  $N$ , other than 1, 2 and 4, there is an  $x \in P$  such that  $x \cdot x = -N$ .*

*Proof.* In the following table each of the given vectors is in  $P$ , and moreover  $X_m(k) \cdot X_m(k) = -(m + 24k)$ ,  $Y_m \cdot Y_m = -m$ ,  $Z(n) \cdot Z(n) = -(4n - 1)$ , and  $W(n) \cdot W(n) = -(4n - 3)$ . This then proves the lemma.

$$\begin{aligned} X_0(k) &= [9k + 4, 3k + 2, 3k + 1, \dots, 3k + 1, 3k - 1, 2], \quad k \geq 1. \\ X_2(k) &= [9k + 4, 3k + 2, 3k + 1, \dots, 3k + 1, 3k, 3k, 2], \quad k \geq 1. \\ X_4(k) &= [12k + 4, 4k + 2, 4k + 1, \dots, 4k + 1, 4k, 1], \quad k \geq 1. \\ X_6(k) &= [9k + 5, 3k + 2, 3k + 2, 3k + 1, \dots, 3k + 1, 2], \quad k \geq 1. \\ X_8(k) &= [9k + 7, 3k + 3, 3k + 2, \dots, 3k + 2, 3k, 2], \quad k \geq 1. \\ X_{10}(k) &= [12k + 7, 4k + 3, 4k + 2, \dots, 4k + 2, 4k + 1, 1], \quad k \geq 0. \\ X_{12}(k) &= [12k + 9, 4k + 3, \dots, 4k + 3, 4k + 2, 4k + 1, 1], \quad k \geq 0. \\ X_{14}(k) &= [9k + 8, 3k + 3, 3k + 3, 3k + 2, \dots, 3k + 2, 2], \quad k \geq 0. \\ X_{16}(k) &= [12k + 10, 4k + 4, 4k + 3, \dots, 4k + 3, 4k + 2, 1], \quad k \geq 0. \\ X_{18}(k) &= [12k + 12, 4k + 4, \dots, 4k + 4, 4k + 3, 4k + 2, 1], \quad k \geq 0. \\ X_{20}(k) &= [6k + 12, 2k + 6, 2k + 3, \dots, 2k + 3, 4], \quad k \geq 1. \\ X_{22}(k) &= [9k + 11, 3k + 4, 3k + 4, 3k + 3, \dots, 3k + 3, 2], \quad k \geq 0. \\ Y_6 &= [4, 1, \dots, 1]. \end{aligned}$$

$$Y_8 = [6, 2, \dots, 2, 1, 1, 1, 1].$$

$$Y_{20} = [6, 2, 2, 1, \dots, 1].$$

$$Z(n) = [3n + 1, n + 1, n, \dots, n, 1], \quad n \geq 1.$$

$$W(n) = [3n, n, \dots, n, n - 1, n - 1, 1], \quad n \geq 2. \quad \square$$

Recalling that  $\Delta = 4ab - c^2$ , these two lemmas complete the proofs of **III-2** and **III-3**.

## 5. THE OTHER CASES

Let  $\{u, v\}$  be a basis of the transcendental lattice giving the matrix representation as in (1), and as before let  $\{u_1, u_2\}$  be the basis of  $U$ , and  $\{v_1, v_2\}$  the basis of  $U(2)$ .

### 5.1. $abc$ is odd.

Consider the mapping  $\phi : T_X \rightarrow \Lambda^-$  defined generically as

$$\begin{aligned} \phi(u) &= a_1u_1 + a_2u_2 + a_3v_1 + a_4v_2 + \omega_1 \\ \phi(v) &= b_1u_1 + b_2u_2 + b_3v_1 + b_4v_2 + \omega_2 \end{aligned}$$

where the  $a_i$ 's and  $b_i$ 's are integers,  $\omega_i \in E_8(2)$ . If  $\phi(u) \cdot \phi(u) = 2a$  and  $\phi(v) \cdot \phi(v) = 2b$ , then  $a_1, a_2, b_1$  and  $b_2$  are odd. But this forces  $\phi(u) \cdot \phi(v)$  to be even. Hence  $T_X$  has no embedding into  $\Lambda^-$ .

This proves **IV**.

### 5.2. $c$ is odd and $ab$ is even.

Consider the mapping  $\phi : T_X \rightarrow \Lambda^-$  defined as

$$\begin{aligned} \phi(u) &= au_1 + u_2 + \frac{1}{2}(c - ab - 1)v_1, \\ \phi(v) &= u_1 + bu_2 + v_2. \end{aligned}$$

This is an embedding and by lemma 3 it is primitive. Let

$$f = c_1u_1 + c_2u_2 + c_3v_1 + c_4v_2 + \omega \in \Lambda^-$$

where  $\omega \in E_8(2)$ .

$f \cdot \phi(u) = 0$  and  $f \cdot \phi(v) = 0$  forces  $c_1c_2$  to be even. Then  $f \cdot f \equiv 0 \pmod{4}$  and hence cannot be  $-2$ .

This proves **II** and completes the proof of theorem 1.

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