OPTIMAL SAMPLING OF MULTIDIMENSIONAL PERIODIC BAND-LIMITED SIGNALS

Sayit Korkmaz

Dept. of Electrical and Electronics Engineering, Bilkent University, Ankara TR-06800
phone: +(90) 312-290-1456, fax: +(90) 312-266-4192
email: sayit@ee.bilkent.edu.tr, web: www.ee.bilkent.edu.tr/ ~sayit

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In this paper, we present an algebraic description of the aliasing phenomena evident in the linear sampling process of multidimensional periodic band limited signals. Opposed to the classical Shannon sampling, periodic band limited signals underlie a different aliasing structure providing further freedom in the sampling strategy due to the discreteness of the spectrum. An algebraic formulation of the optimal sampling problem is also presented.

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If we consider the widespread use of FFT algorithms in signal processing, it would not be wrong to say that harmonic functions are the most important functions in DSP. This evidently makes the sampling and quantization of them an important problem from a practical and theoretical standpoint. In spite of their simplicity in comparison with general signals faced in engineering applications, the aliasing effects are strikingly different than usual aliasing effects in non-periodic signals.

In this paper, we will mainly focus on sampling theorems and aliasing effects in multidimensional periodic band-limited signals. Opposed to the classical Shannon sampling theorem for strictly bandlimited signals, sampling periodic bandlimited signals is relatively different. This stems from the discreteness of the spectrum. The interpolation of such signals has been studied in [1, 2, 3] however the authors do not discuss the aliasing structure. The interpolating function of periodic band-limited signals is simply a periodic-sinc function also known as the Dirichlet kernel [1, 4]. The Dirichlet kernel is used to define half-integer representations for finite operators [4]. As a complementary to the previous work, we will focus on the aliasing phenomena in a very general multidimensional case.

It must be noted that discreteness of the spectrum gives us further freedom for choosing a suitable sampling rate which can easily be below the Nyquist rate [4]. Despite it is easy to decrease the sampling rates by using Papoulis extensions, periodic band limited signals do not need a modification in the sampling structure [4]. If restated in an information theory context, all of the information is contained in one period. In order to represent the signal from its samples the same information can also be collected from the other periods. This is closely related with the discreteness of the spectrum. It will be shown that the no-aliasing condition can be stated compactly as:

$$\mathbf{r}_1 \neq \mathbf{r}_2 \mod \mathbf{N}^T \tag{1}$$

which reveals the algebraic nature of the aliasing phenomena and is a compact mathematical description of the general sense that the pulses in the spectrum should not overlap.

The main motivation for studying the sampling and aliasing structure for periodic band-limited signals is the fact that we face a relatively different aliasing structure. In other words, by applying the ubiquitous theory of sampling to these very simple harmonic functions, new aliasing structures are faced and hence all the relevant theories has to carefully be adapted to the structure of the discrete spectrum.

Another important issue for both practice and theory is the optimal sampling problem. For the continuous case, it is very difficult to classify the optimal sampling architecture for a given class of closed region in the plane or in a higher dimensional space since this requires classification of surfaces in multi-dimensional vector spaces which makes the problem extensively difficult. Even a precise mathematical formulation does not exist to the best of the author's knowledge. However, for a small class of regions eg. -bounded in a sphere in the dimension of the space-, it has been shown that the optimal sampling geometry is hexagonal [5]. We will show that in the discrete case it is possible to formulate the optimal sampling problem in the most general case. This will be achieved by formulating the aliasing phenomena in an algebraic manner.

We will consider only linear sampling and the nonlinear sampling case will be discussed elsewhere.

2 [illegible]

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In order to obtain a sampling theorem and conditions for aliasing, we shall follow the well known results from multidimensional linear sampling theory [5]. However, the aliasing structure will be treated differently than the usual case where signals are assumed to be only band-limited.

Bold lower case letters denote vectors and bold upper case letter denote matrices throughout this paper. Let f_a be a strictly band limited multidimensional periodic signal with fundamental period $\mathbf{A}$.

$$f_a(\mathbf{t}) = f_a(\mathbf{t} + \mathbf{A}\ ') \tag{2}$$

In the above representation, $\mathbf{r}'$ denotes an integer valued vector. Since the signal is periodic, it can be represented with Fourier series as:

$$f_a(\mathbf{t}) = f_a(\mathbf{t}+\mathbf{A}\mathbf{r}') = \sum_{\mathbf{r}\in N_a} c_{\mathbf{r}} e^{j2\pi(\mathbf{A}^{-T}\mathbf{r})^T\mathbf{t}} \tag{3}$$

In the above expression $\mathbf{r}$ is an integer valued vector belonging to a finite set denoted by N_a and $c_{\mathbf{r}}$ denotes Fourier series coefficients. N_a is the set of $\mathbf{r}$ vectors for which the Fourier series coefficients $c_{\mathbf{r}}$ are nonzero.

The samples of f_a are given as $f(\mathbf{n}) = f_a(\mathbf{V}\mathbf{n})$ where $\mathbf{V}$ is the sampling matrix. The discrete signal $f(\mathbf{n})$ and the sampled signal $f_s(\mathbf{t})$ are respectively,

$$f(\mathbf{n}) = \sum_{\mathbf{r}\in N_a} c_{\mathbf{r}} e^{j2\pi(\mathbf{A}^{-T}\mathbf{r})^T\mathbf{V}\mathbf{n}}, \tag{4}$$

$$f_s(\mathbf{t}) = \sum_{\mathbf{n}} f_a(\mathbf{V}\mathbf{n})\delta(\mathbf{t}-\mathbf{V}\mathbf{n}). \tag{5}$$

If we write these signals in transform domain,

$$\begin{aligned}\hat{f}_a(\mathbf{\Omega}) &= \int f_a(\mathbf{t})e^{-j\,\mathbf{\Omega}^T\mathbf{t}}d\mathbf{t} \\ &= \sum_{\mathbf{r}\in N_a} c_{\mathbf{r}} 2\pi\delta(\mathbf{\Omega}-2\pi\mathbf{A}^{-T}\mathbf{r}).\end{aligned} \tag{6}$$

In the above representations $\mathbf{\Omega}$ denotes the radial frequency. f_s in transform domain is:

$$\hat{f}_s(\mathbf{\Omega}) = \int f_s(\mathbf{t})e^{-j\,\mathbf{\Omega}^T\mathbf{t}}d\mathbf{t}. \tag{7}$$

The last equation can further be simplified by using the well known results from multidimensional sampling theory [5]:

$$\begin{aligned}\hat{f}_s(\mathbf{\Omega}) &= \sum_{\mathbf{n}} f_a(\mathbf{V}\mathbf{n})e^{-j\mathbf{\Omega}^T\mathbf{V}\mathbf{n}} \\ &= \frac{1}{|\det(\mathbf{V})|}\sum_{\mathbf{k}} \hat{f}_a(\mathbf{\Omega}-2\pi\mathbf{V}^{-T}\mathbf{k}) \\ &= \frac{2\pi}{|\det(\mathbf{V})|}\sum_{\mathbf{k}}\sum_{\mathbf{r}\in N_a} c_{\mathbf{r}}\delta(\mathbf{\Omega}-2\pi\mathbf{A}^{-T}\mathbf{r}-2\pi\mathbf{V}^{-T}\mathbf{k})\end{aligned} \tag{8}$$

where the transition from the second line to the third line is done by replacing $\hat{f}_a$ with (6).

Up to now, we followed directly the well known results from the multidimensional sampling theory [5]. However, the conditions for aliasing has to be carefully justified since that the last equation is simply a summation of delta functions. Hence, for perfect reconstruction, there must be no overlapping of the pulses. This guarantees to avoid loss of information. However opposed to the non-periodic case, we have further freedom in choosing the sampling rates since there are empty spaces between the pulses. On the other hand, we must be careful on the intersections.

In order to avoid aliasing, there must be no overlapping of the pulses in (8). This can be achieved if and only if for all combinations of $\mathbf{r}_1 \in N_a$ and $\mathbf{r}_2 \in N_a$ and $\mathbf{k}$ such that $\mathbf{r}_1 \neq \mathbf{r}_2$ the following inequality is satisfied

$$2\pi\mathbf{A}^{-T}(\mathbf{r}_1-\mathbf{r}_2) \neq 2\pi\mathbf{V}^{-T}\mathbf{k} \tag{9}$$

It must be noted that $\mathbf{r}_1 \in N_a$ and $\mathbf{r}_2 \in N_a$ take finite values opposed to $\mathbf{k}$ which can take all the nonzero integer valued elements. It is easy that this statement is necessary and sufficient for no-aliasing.

Note that in general while sampling harmonic functions, the condition for no-aliasing (9) does not guarantee the periodicity of the samples $f(\mathbf{n})$. We can obtain non-periodic samples from a strictly periodic signal [4]. Indeed, if the sampling matrix $\mathbf{V}$ is chosen such that $\mathbf{A}^{-1}\mathbf{V}$ has at least one element being irrational, the no-aliasing condition (9) will always be satisfied. This also gives non-periodic samples from a periodic signal. Since this case is not very significant for the practical cases, we shall prefer the samples

$$f(\mathbf{n}) = \sum_{\mathbf{r}\in N_a} c_{\mathbf{r}} e^{j2\pi(\mathbf{A}^{-T}\mathbf{r})^T\mathbf{V}\mathbf{n}} \tag{10}$$

to be also periodic. This can be achieved if and only if we choose $\mathbf{V}$ such that

$$\mathbf{A}^{-1}\mathbf{V} = \mathbf{N}^{-1} \tag{11}$$

is satisfied where $\mathbf{N}$ is an integer valued invertible matrix and denotes the periodicity of $f(\mathbf{n})$. For notational brevity periodic $f(\mathbf{n})$ will be replaced with $f[\mathbf{n}]$.

$$f[\mathbf{n}] = f[\mathbf{n}+\mathbf{N}] = \sum_{\mathbf{r}\in N_a} c_{\mathbf{r}} e^{j2\pi\mathbf{r}^T\mathbf{N}^{-1}\mathbf{n}} \tag{12}$$

The last equation is the discrete Fourier series representation of the periodic samples. If the sampling rate $\mathbf{V}$ is chosen such that equation (11) is satisfied, the condition for no-aliasing in equation (9) can be written as:

$$2\pi\mathbf{A}^{-T}(\mathbf{r}_1-\mathbf{r}_2) \neq 2\pi\mathbf{A}^{-T}\mathbf{N}^T\mathbf{k} \tag{13}$$

which can further be simplified as:

$$(\mathbf{r}_1-\mathbf{r}_2) \neq \mathbf{N}^T\mathbf{k}, \qquad \forall\mathbf{r}_1,\mathbf{r}_2 \in N_a, \mathbf{r}_1 \neq \mathbf{r}_2, \forall\mathbf{k}, \tag{14}$$

Finally, this statement can be formulated simply as:

$$\mathbf{r}_1 \neq \mathbf{r}_2 \mod \mathbf{N}^T \qquad \forall\mathbf{r}_1,\mathbf{r}_2 \in N_a, \mathbf{r}_1 \neq \mathbf{r}_2 \tag{15}$$

This result, as expected, is brief however, for the multidimensional case it is difficult to test whether for a given $\mathbf{r}_1, \mathbf{r}_2$ and $\mathbf{N}$ the condition $\mathbf{r}_1 \neq \mathbf{r}_2 \mod \mathbf{N}^T$ is satisfied. Nevertheless, the no-aliasing expression in equation (15) is a compact mathematical description of the general sense that the pulses in the spectrum should not overlap.

3 [redacted]

The number of samples in one period and and the sampling density can be related to each other by using equation (11):

$$|\det\mathbf{N}| = \frac{|\det\mathbf{A}|}{|\det\mathbf{V}|} \tag{16}$$

Hence, minimizing the sampling density and minimizing the number of samples in one period are equivalent problems.

A necessary condition for no aliasing is:

$$\frac{|\det(\mathbf{A})|}{|\det(\mathbf{V})|} \geq \text{number of elemnts of the set } N_a \tag{17}$$

which is a statement of the minimum number of degrees of freedom. It is easy to see that there exists also an upper bound for the determinant of $\mathbf{N}$. However since the number of matrices having a fixed determinant N are infinitely many, we can not directly adopt a brute force approach for the optimization.

In general, the optimal sampling problem is reduced to integer optimization:

$$\min \ |\det \mathbf{N}| \tag{18}$$

$$\mathbf{r}_1 \neq \mathbf{r}_2 \mod \mathbf{N}^T, \quad \mathbf{r}_1, \mathbf{r}_2 \in N_a \tag{19}$$

This optimization is desirable since we want to represent the signal with as few as possible samples and minimum sampling density. Although we do not yet provide a solution to this optimization problem, note that in the non-periodic case it is extremely difficult even to state the optimal sampling problem. Furthermore this optimization only minimizes the sampling density, however this might be achieved by a very dense sampling in one dimension and very coarse sampling in other dimensions. Such solutions are optimal however it does not seem very practical to use very dense and coarse sampling rates together. Nevertheless it must be noted that the stated optimization problem is not standard and is with great possibility very difficult to solve.

4 [illegible]

The main motivation of this paper was to reveal the algebraic structure in the sampling process of periodic bandlimited signals. Reconstruction and interpolation from the samples will be published elsewhere.

It is very interesting from both theoretical and practical perspective whether the the presented formulation of the optimal sampling problem can be extended to a restricted class of non-periodic band-limited signals or periodic but non band-limited signals.

The algebraic structure of the optimal sampling problem and its connection to modules will be published elsewhere.

5 [illegible]

In this paper, we studied the aliasing phenomena in the linear sampling process of multidimensional periodic band-limited signals. We provided an algebraic formulation in order to handle the discreteness of the spectrum. Based on this formulation it was shown that the optimal sampling can be formulated as nonlinear integer programming.

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[1] T. Schanze, "Sinc Interpolation of Discrete Periodic Signals," *IEEE Transactions on Signal Processing*, vol. 43, no:6, pp. 1502-1503, June 1995.

[2] F. Candocia and J. C. Principe, "Comments on Sinc Interpolation of Discrete Periodic Signals",*IEEE Transactions on Signal Processing*, vol. 46, no. 7, pp. 2044-2047, July 1998.

[3] S.R. Dooley and A.K. Nandi, "Notes on the interpolation of discrete periodic signals using sinc function related approaches",*IEEE Transactions on Signal Processing*, vol. 48, no. 4, pp. 1201-1203, April 2000.

[4] B. Gulbahar, Discrete Wigner Distribution and Half integer representations of discrete-time signals and systems, M.Sc. Thesis, Electrical Engineering Department, Bilkent University, 2002.

[5] D. E. Dudgeon and R. M. Mersereau, Multidimensional Digital Signal Processing, Prentice Hall Signal Processing Series, 1984.