Optimal Hybrid Control of a Two-Stage Manufacturing System

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***Abstract*— We consider a two-stage serial hybrid system for which the arrival times are known and the service times are controllable. We derive some optimal sample path characteristics, in particular, we show that no buffering is observed between stages. The original non-smooth optimal control problem is first transformed into a convex optimization problem which is then simplified by the no buffer property. Further simplifications are possible for the bulk arrival case.**

I. INTRODUCTION

The term "hybrid" is used to characterize systems that include time-driven and event-driven dynamics. The former are represented by differential (difference) equations, while the latter may be described through various frameworks used for Discrete Event Systems (DES), such as timed automata, max-plus equations, queueing networks, or Petri nets (see [1]). Broadly speaking, two categories of modeling frameworks have been proposed to study hybrid systems: Those that extend event-driven models to include time-driven dynamics; and those that extend the traditional time-driven models to include event-driven dynamics (for an overview, see [2], [3], [4], [5])

The hybrid system modeling framework used in this paper falls into the first category above and is motivated by the structure of many manufacturing systems. In these systems, discrete entities (referred to as jobs) move through a network of work-centers which process the jobs so as to change their physical characteristics according to certain specifications. Associated with each job are a physical state and a temporal state. The physical state of job C_i at stage j denoted by $z_{i,j}$, which, depending on the particular problem being studied, describes quantities such as the temperature, size, weight, chemical composition, bacteria level, or some other measure of the "quality" of the job, evolves according to time-driven dynamics described by the differential equations

$$\dot{z}_{i,j}(t) = f_j(z_{i,j}(t), u_{i,j}(t)) \quad (1)$$

$$z_{i,j}(\tau_{i,j}) = \zeta^0_{i,j} \quad z_{i,j}(\tau_{i,j}+s_{i,j}) = \zeta^d_{i,j} \quad (2)$$

Applying the input $u_{i,j}(t)$ between the times $\tau_{i,j}$, when the service starts, and $\tau_{i,j}+s_{i,j}$, when the service ends, the physical state is brought from the initial value $\zeta^0_{i,j}$ to a desired final value $\zeta^d_{i,j}$. The length of service $s_{i,j}$ depends on the input $u_{i,j}(t)$ as well as the initial $\zeta^0_{i,j}$ and the desired $\zeta^d_{i,j}$ states. In this work we will assume identical jobs, i.e., $\zeta^0_{i,j} = \zeta^0_j$ and $\zeta^d_{i,j} = \zeta^d_j$ are given; therefore, a change in service time $s_{i,j}$ can only be achieved by adjusting the controllable input $u_{i,j}(t)$. The temporal state $x_{i,j}$, on the other hand, keeps the time information; the departure time for job C_i from stage j, in particular. It evolves according to event-driven dynamics given by the Lindley Equation (see in [1])

$$x_{i,1} = \max(a_i, x_{i-1,1}) + s_{i,1}(u_{i,1}) \qquad x_{0,1} = -\infty \quad (3)$$

$$x_{i,2} = \max(x_{i,1}, x_{i-1,2}) + s_{i,2}(u_{i,2}) \qquad x_{0,2} = -\infty \quad (4)$$

where a_i denotes the arrival time of job C_i to the system. Due to the existence of $s_{i,j}$ and $u_{i,j}$ in both time-driven dynamics and event-driven dynamics, an interaction is observed, which leads to a natural trade-off between temporal requirements on job completion times and physical requirements on the quality of the completed jobs: In order to meet job completion deadlines and to decrease inventory costs, one may set the processing times as small as possible; however, this usually comes at the expense of more resources, e.g., in a turning operation a faster process will increase tooling costs and will require extra supervision. Our objective, therefore, is to formulate and solve optimal control problems associated with such trade-offs.

In [6], [7], [8], and [9], the hybrid system framework is adopted to analyze a single-stage manufacturing process operating under a deterministic setting, i.e., with a known job arrival schedule and controllable service times. For the hybrid systems with a certain separable cost structure, a hierarchical method is proposed in [10] and [11] to decompose the original hybrid control problem into several lower-level continuous-time optimal control problems with well-established solution methods, and a challenging higher-level discrete-event control problem of determining the optimal service times. An efficient algorithm to solve this discrete-event control problem for single-stage systems is presented in [8]. Approximate solutions for two-stage systems are obtained in [12] using the Bezier approximation method to smooth out the $\max$ functions in the event-driven dynamics. [13] considers a multistage model with constrained service times, and presents some optimal sample path characteristics. In this paper, we consider two-stage manufacturing systems, and identify some new optimal sample path characteristics to simplify the discrete-event control problem. In particular, we show that no buffering is observed between stages on the optimal sample path, which leads to the transformation of what is otherwise a non-smooth optimal control problem into an equivalent convex programming problem involving only smooth differentiable functions that can be efficiently solved using standard calculus techniques.

II. Problem Formulation

Let us consider a two stage serial manufacturing system. A sequence of N identical jobs arrive to the system from the first stage at known times $0 \leq a_1 \leq a_2 \leq ... \leq a_N$ and are processed in the first stage and the second stage consecutively. We denote these jobs by C_i, $i = 1, 2, ..., N$. Servers process one job at a time on a first-come first-served non-preemptive basis (i.e. a job in service can not be interrupted until its service completion).

We consider the optimal service time control problem

$$\min_{\substack{s_{i,j} \\ s_{i,j} \geq 0 \\ i=1,\dots,N \quad j=1,2}} \quad J = \sum_{i=1}^{N} [\theta_1(s_{i,1}) + \theta_2(s_{i,2}) + \phi_i(x_{i,2})] \tag{5}$$

subject to

$$x_{i,1} = \max(a_i, x_{i-1,1}) + s_{i,1} \quad i = 1, ..., N \tag{6}$$

$$x_{i,2} = \max(x_{i,1}, x_{i-1,2}) + s_{i,2} \quad i = 1, ..., N \tag{7}$$

where $x_{0,1} = x_{0,2} = -\infty$. In this formulation, $\theta_j(s_{i,j})$ denotes the process cost at stage j resulting from applying the optimal control $u^*_{i,j}(s_{i,j})$ (see in [10] and [11]), and $\phi_i(x_{i,2})$ denotes the departure time cost for job C_i. The optimal service times are denoted by $s^*_{i,j}$ and the optimal departure times are denoted by $x^*_{i,j}$ for jobs C_i, where $i = 1, ..., N$, at stage j, where $j = 1, 2$. Moreover, the optimal cost is denoted by J^*. This optimization problem is non-convex and non-differentiable over the service times space due to the max function. In Section IV, we will formulate an equivalent convex and differentiable optimization problem over a larger space with a unique solution.

In this setup, following assumptions are necessary to make the problem somewhat more tractable while preserving the originality of the problem.

Assumption 1: $\theta_j(s)$, for $j = 1, ..., M$ is continuously differentiable, monotonically decreasing, i.e.,$\frac{d\theta_j(s)}{s} < 0$, and strictly convex, i.e., $\frac{d\theta_j(s)}{s}$ is monotonically increasing in s.

Assumption 2: $\phi_i(x)$ for $i = 1, ..., N$ is continuously differentiable, monotonically increasing, and strictly convex.

An example set of costs that will satisfy these assumptions would be

$$\theta_j(s_{i,j}) = \frac{\beta_j}{s_{i,j}} \tag{8}$$

and

$$\phi_i(x_{i,2}) = \alpha(x_{i,2} - a_i)^2 \tag{9}$$

Note that for this example set of costs, longer services are cheaper, however; there is a quadratic cost on the system time, which will increase by the longer service times.

III. Characteristics of the Optimal Control

We begin the development of the optimal sample path characteristics of this system with the following definitions:

Definition 1: A job C_i is *critical* at stage j if it departs at the arrival time of the next job, i.e. $x_{i,j} = x_{i+1,j-1}$.

Definition 2: A contiguous set of jobs $\{C_k, ..., C_n\}$ is said to form a *block* at stage j if

1) $x_{k-1,j} \leq x_{k,j-1}$ and $x_{n,j} \leq x_{n+1,j-1}$.
2) $x_{i-1,j} > x_{i,j-1}$ for $i = k+1, ..., n$.

Definition 3: A contiguous set of jobs $\{C_k, ..., C_n\}$ is said to form a *busy period* at stage j if

1) $x_{k-1,j} < x_{k,j-1}$ and $x_{n,j} < x_{n+1,j-1}$.
2) $x_{i-1,j} \geq x_{i,j-1}$ for $i = k+1, ..., n$.

where $x_{k,0} = a_k$. Note that busy periods are formed of blocks that are separated from each other by the critical jobs.

Applying calculus of variations techniques (see in [14]) on the optimal control problem, we obtain a set of necessary conditions for optimality.

Lemma 1: The optimal solution $\{s^*_{i,j}\}$ must satisfy the following conditions:

For $i = 1, ..., N$,

$$\theta'_1(s^*_{i,1}) + \lambda^*_{i,1} = \theta'_2(s^*_{i,2}) + \lambda^*_{i,2} = 0 \tag{10}$$

$$x^*_{i,1} = \max(a_i, x^*_{i-1,1}) + s^*_{i,1} \tag{11}$$

$$x^*_{i,2} = \max(x^*_{i,1}, x^*_{i-1,2}) + s^*_{i,2} \tag{12}$$

For $i = 1, ..., N-1$,

$$\begin{aligned} \lambda^*_{i,1} &= \lambda^*_{i+1,1} \left. \frac{d\max(a_{i+1}, x_{i,1})}{dx_{i,1}} \right|_{x_{i,1}=x^*_{i,1}} \\ &\quad + \lambda^*_{i,2} \left. \frac{d\max(x_{i,1}, x^*_{i-1,2})}{dx_{i,1}} \right|_{x_{i,1}=x^*_{i,1}} \end{aligned} \tag{13}$$

$$\lambda^*_{i,2} = \phi'_i(x^*_{i,2}) + \lambda^*_{i+1,2} \left. \frac{d\max(x^*_{i+1,1}, x_{i,2})}{dx_{i,2}} \right|_{x_{i,2}=x^*_{i,2}} \tag{14}$$

$$\lambda^*_{N,1} = \phi'_N(x^*_{N,2}) \left. \frac{d\max(x_{N,1}, x^*_{N-1,2})}{dx_{N,1}} \right|_{x_{N,1}=x^*_{N,1}} \tag{15}$$

$$\lambda^*_{N,2} = \phi'_N(x^*_{N,2}) \tag{16}$$

Proof: We first form the augmented cost as

$$\begin{aligned} \overline{J} = \sum_{i=1}^{N} \{ & \theta_1(s_{i,1}) + \theta_2(s_{i,2}) + \phi_i(x_{i,2}) \\ & + \lambda_{i,1}[\max(a_i, x_{i-1,1}) + s_{i,1} - x_{i,1}] \\ & + \lambda_{i,2}[\max(x_{i,1}, x_{i-1,2}) + s_{i,2} - x_{i,2}] \} \end{aligned} \tag{17}$$

Then, we invoke basic variational calculus techniques to obtain the necessary conditions for an optimal solution. For all $i = 1, ..., N$ and $j = 1, 2$, by differentiating (17) with respect to $s_{i,j}$'s we get the optimality equations (10), by differentiating with respect to $x_{i,j}$'s, we get the co-state equations (13)-(14) and the boundary conditions (15)-(16), and finally by differentiating with respect to $\lambda_{i,j}$'s, we obtain the state equations (11)-(12). ■

Using the optimality equations (10) and the co-state equations (13)-(16), we can show the following monotonicity properties of the optimal service times.

Lemma 2: (Monotonicity Properties) If jobs C_i and C_{i+1} are in the same block of the first stage on the optimal sample path, then their service times satisfy

$$s^*_{i,1} \leq s^*_{i+1,1}$$

If these jobs are in the same block of the second stage on the optimal sample path, then their service times satisfy

$$s^*_{i,2} < s^*_{i+1,2}$$

Proof: If we consider equations (14) and (16), since $\phi'_i(x^*_{i,2}) > 0$ by Assumption 2 and by the fact that the derivative of the $\max$ function is non-negative, we can conclude that $\lambda^*_{i,2} > 0$ for all $i = 1, ..., N$.

If jobs C_i and C_{i+1} are in the same block of the first stage on the optimal sample path, then $a_{i+1} < x^*_{i,1}$. Therefore, from (13) we have

$$\lambda^*_{i,1} = \lambda^*_{i+1,1} + \lambda^*_{i,2} \left. \frac{d \max(x_{i,1}, x^*_{i-1,2})}{dx_{i,1}} \right|_{x_{i,1}=x^*_{i,1}} \quad (18)$$

It follows from (10) and (18) that

$$\begin{aligned} \theta'_1(s^*_{i+1,1}) - \theta'_1(s^*_{i,1}) &= \lambda^*_{i,1} - \lambda^*_{i+1,1} \\ &= \lambda^*_{i,2} \left. \frac{d \max(x_{i,1}, x^*_{i-1,2})}{dx_{i,1}} \right|_{x_{i,1}=x^*_{i,1}} \\ &\geq 0. \end{aligned}$$

Since $\theta'_1(s)$ is monotonically increasing, $s^*_{i,1} \leq s^*_{i+1,1}$.

If jobs C_i and C_{i+1} are in the same block of the second stage on the optimal sample path, then $x^*_{i+1,1} < x^*_{i,2}$. Therefore, from (14) we have

$$\lambda^*_{i,2} = \phi'_i(x^*_{i,2}) + \lambda^*_{i+1,2} \quad (19)$$

It follows from (10), (19) and Assumption 2 that

$$\begin{aligned} \theta'_2(s^*_{i+1,2}) - \theta'_2(s^*_{i,2}) &= \lambda^*_{i,2} - \lambda^*_{i+1,2} \\ &= \phi'_i(x^*_{i,2}) > 0. \end{aligned}$$

Since $\theta'_2(\cdot)$ is monotonically increasing, $s^*_{i,2} < s^*_{i+1,2}$. ■

The following lemma establishes that, on the optimal sample path, no job leaves the first stage idle and arrives at a busy second stage.

Lemma 3: The inequality

$$x^*_{k,1} \geq \min(a_{k+1}, x^*_{k-1,2})$$

always holds for all $k = 1, 2, ..., N$ on the optimal sample path.

Proof: Assume that $x^*_{k,1} < \min(a_{k+1}, x^*_{k-1,2})$ for some arbitrary $k \in \{1, ..., N\}$, then

$$\begin{aligned} x^*_{k+1,1} &= \max(a_{k+1}, x^*_{k,1}) + s^*_{k+1,1} \quad (20) \\ &= a_{k+1} + s^*_{k+1,1} \end{aligned}$$

and

$$\begin{aligned} x^*_{k,2} &= \max(x^*_{k,1}, x^*_{k-1,2}) + s^*_{k,2} \quad (21) \\ &= x^*_{k-1,2} + s^*_{k,2} \end{aligned}$$

Let us define Δ to be

$$\Delta = \min(a_{k+1}, x^*_{k-1,2}) - x^*_{k,1}$$

Note that $\Delta > 0$. Also let us define service times $s_{i,1}$ and $s_{i,2}$ for $i = 1, ..., N$ to be

$$s_{i,1} = \begin{cases} s^*_{i,1} + \Delta & i = k \\ s^*_{i,1} & o.w. \end{cases}$$

and

$$s_{i,2} = s^*_{i,2}$$

for all $i = 1, ..., N$ and let J be the cost of the applying service times $s_{i,1}$ and $s_{i,2}$ for $i = 1, ..., N$. Then, we can write

$$x_{i,1} = x^*_{i,1} \text{ for } i = 1, 2, ..., (k-1)$$

and

$$\begin{aligned} x_{k,1} &= \max(a_k, x_{k-1,1}) + s_{k,1} \\ &= \max(a_k, x^*_{k-1,1}) + s^*_{k,1} + \Delta \\ &= x^*_{k,1} + \min(a_{k+1}, x^*_{k-1,2}) - x^*_{k,1} \\ &= \min(a_{k+1}, x^*_{k-1,2}) \end{aligned}$$

Also from (20),

$$\begin{aligned} x_{k+1,1} &= \max(a_{k+1}, x_{k,1}) + s_{k+1,1} \\ &= \max(a_{k+1}, \min(a_{k+1}, x^*_{k-1,2})) + s_{k+1,1} \\ &= a_{k+1} + s_{k+1,1} \\ &= a_{k+1} + s^*_{k+1,1} \\ &= x^*_{k+1,1} \end{aligned}$$

therefore,

$$x_{i,1} = x^*_{i,1} \text{ for } i = k+2, ..., N$$

The respective departure times $x_{i,2}$ for the second stage are

$$x_{i,2} = x^*_{i,2} \quad \text{for } i = 1, 2, ..., (k-1)$$

and from (21) we have

$$\begin{aligned} x_{k,2} &= \max(x_{k,1}, x_{k-1,2}) + s_{k,2} \\ &= \max(\min(a_{k+1}, x^*_{k-1,2}), x^*_{k-1,2}) + s^*_{k,2} \\ &= x^*_{k-1,2} + s^*_{k,2} \\ &= x^*_{k,2} \end{aligned}$$

$$x_{i,2} = x^*_{i,2} \quad \text{for } i = k+1, ..., N$$

Hence, by Assumption 1

$$\begin{aligned} J - J^* &= \sum_{i=1}^{N} \{\theta_1(s_{i,1}) + \theta_2(s_{i,2}) + \phi_i(x_{i,2})\} \\ &\quad - \sum_{i=1}^{N} \{\theta_1(s^*_{i,1}) + \theta_2(s^*_{i,2}) + \phi_i(x^*_{i,2})\} \\ &= \theta_1(s_{k,1}) - \theta_1(s^*_{k,1}) \\ &= \theta_1(s^*_{k,1} + \Delta) - \theta_1(s^*_{k,1}) < 0 \end{aligned}$$

which is a contradiction. Therefore, $x^*_{k,1} \geq \min(a_{k+1}, x^*_{k-1,2})$ for all $k = 1, ..., N$. ■

The following lemma will become useful while proving the main result of this paper, which is presented next.

Lemma 4: Consider the job sequence $\{C_k, ..., C_n\}$ that constitutes a busy period for the first stage on the optimal sample path. If for some i, $k \leq i < n$,

$$x^*_{i+1,1} \geq x^*_{i,2}$$

and

$$x^*_{i,1} < x^*_{i-1,2}$$

are satisfied then

$$x^*_{l,1} < x^*_{l-1,2} \quad \text{for } l = k, ..., i$$

Proof: (By Induction) It is already given that

$$a_{i+1} \leq x^*_{i,1} < x^*_{i-1,2}$$

We also have

$$\begin{aligned} x^*_{i+1,1} &\geq x^*_{i,2} \\ \max(a_{i+1}, x^*_{i,1}) + s^*_{i+1,1} &\geq \max(x^*_{i,1}, x^*_{i-1,2}) + s^*_{i,2} \end{aligned}$$

hence

$$s^*_{i+1,1} > s^*_{i,2} \tag{22}$$

Let us assume that

$$x^*_{l,1} < x^*_{l-1,2} \quad \text{for } l = r, ..., i$$

Since all these jobs $\{C_{r-1}, ..., C_i\}$ are in the same block for the second stage, we have from Lemma 2

$$s^*_{i,2} > s^*_{i-1,2} > ... > s^*_{r-1,2} \tag{23}$$

In order to show a contradiction let us assume that

$$x^*_{r-1,1} \geq x^*_{r-2,2} \tag{24}$$

In that case

$$\begin{aligned} x^*_{r,1} &< x^*_{r-1,2} \\ \max(a_r, x^*_{r-1,1}) + s^*_{r,1} &< \max(x^*_{r-1,1}, x^*_{r-2,2}) + s^*_{r-1,2} \end{aligned}$$

It follows from (24) that

$$s^*_{r,1} < s^*_{r-1,2} \tag{25}$$

From (23), (25), and (22)

$$s^*_{i+1,1} > s^*_{r,1}$$

Let us pick a positive Δ such that

$$\Delta < \min\left(\frac{s^*_{i+1,1} - s^*_{r,1}}{2}, \min_{l \in \{r,...,i\}} (x^*_{l-1,2} - x^*_{l,1})\right)$$

and define

$$s_{l,1} = \begin{cases} s^*_{l,1} + \Delta & l = r \\ s^*_{l,1} - \Delta & l = i+1 \\ s^*_{l,1} & o.w. \end{cases}$$

and

$$s_{l,2} = s^*_{l,2} \quad l = 1, ..., N$$

Under these service times, the departure times will be

$$x_{l,1} = \begin{cases} x^*_{l,1} + \Delta & l = r, ..., i \\ x^*_{l,1} & o.w. \end{cases}$$

and since $\Delta < \min_{l \in \{r,...,i\}} (x^*_{l-1,2} - x^*_{l,1})$

$$x_{l,2} = x^*_{l,2} \quad l = 1, ..., N$$

Hence, the change in cost due to applying these non-optimal service times $s_{l,1}$ and $s_{l,2}$ will be

$$J - J^* = \left\{ \begin{array}{c} \left(\theta_1(s^*_{r,1} + \Delta) - \theta_1(s^*_{r,1})\right) \\ - \left(\theta_1(s^*_{i+1,1}) - \theta_1(s^*_{i+1,1} - \Delta)\right) \end{array} \right\} < 0$$

because $\theta_1^{'}$ is monotonically increasing and $s^*_{i+1,1} > s^*_{r,1}$. Since the cost J is lower than the optimal cost J^*, a contradiction is observed implying that

$$x^*_{r-1,1} < x^*_{r-2,2}$$

which concludes the induction proof. ■

We present next the main result of this paper, which is shown for all busy periods of the first stage, hence for all jobs, that no buffering between stages is observed.

Theorem 1 (No buffer property): Consider the job sequence $\{C_k, ..., C_n\}$ that constitutes a busy period of the first stage on the optimal sample path. Then,

$$x^*_{i,1} \geq x^*_{i-1,2}$$

for all $i = k, ..., n$.

Proof: (By Induction) Let us start with $i = n$. From Lemma 3, we have

$$x^*_{n,1} \geq \min(a_{n+1}, x^*_{n-1,2}) \tag{26}$$

Since C_n is the last job of the busy period of the first stage on the optimal sample path

$$x^*_{n,1} < a_{n+1} \tag{27}$$

Combining (26) and (27), we obtain

$$x^*_{n,1} \geq x^*_{n-1,2}$$

Next, let us assume that

$$x^*_{l,1} \geq x^*_{l-1,2} \text{ for all } l = i+1, ..., n$$

We need to show that $x^*_{i,1} \geq x^*_{i-1,2}$ holds. In order to prove by contradiction, let us assume that

$$x^*_{i,1} < x^*_{i-1,2} \tag{28}$$

From $x^*_{i+1,1} \geq x^*_{i,2}$, we have

$$\max(x^*_{i,1}, a_{i+1}) + s^*_{i+1,1} \geq \max(x^*_{i,1}, x^*_{i-1,2}) + s^*_{i,2}$$

Since C_i is not the last job of the busy period of the first stage on the optimal sample path, $x^*_{i,1} \geq a_{i+1}$ and from (28)

$$x^*_{i,1} + s^*_{i+1,1} \geq x^*_{i-1,2} + s^*_{i,2}$$

Hence,

$$s^*_{i+1,1} > s^*_{i,2} \tag{29}$$

By Lemma 4, (28) implies that

$$x_{l,1}^* < x_{l-1,2}^* \quad l = k, ..., i \tag{30}$$

Since all these jobs $\{C_{k-1}, ..., C_i\}$ are in the same block for the second stage, we have from Lemma 2

$$s_{i,2}^* > s_{i-1,2}^* > ... > s_{k-1,2}^* \tag{31}$$

Since job C_k starts the busy period, $a_k > x_{k-1,1}^*$ and from Lemma 3,

$$x_{k-1,1}^* \geq \min(a_k, x_{k-2,2}^*) = x_{k-2,2}^* \tag{32}$$

It follows from (30) that

$$\max(x_{k-1,1}^*, a_k) + s_{k,1}^* < \max(x_{k-1,1}^*, x_{k-2,2}^*) + s_{k-1,2}^*$$

Since C_k is the first job of the busy period of the first stage on the optimal sample path, $x_{k-1,1}^* < a_k$ and from (32), we have

$$a_k + s_{k,1}^* < x_{k-1,1}^* + s_{k-1,2}^*$$

Hence,

$$s_{k,1}^* < s_{k-1,2}^* \tag{33}$$

From (29), (31) and (33)

$$s_{k,1}^* < s_{i+1,1}^*$$

Let us analyze the cost for the following service times:

$$s_{l,1} = \begin{cases} s_{l,1}^* + \Delta & l = k \\ s_{l,1}^* - \Delta & l = i+1 \\ s_{l,1}^* & o.w. \end{cases}$$

and

$$s_{l,2} = s_{l,2}^* \quad \text{for all} \quad l = 1, 2, ..., N$$

where a positive Δ is picked such that

$$\Delta < \min\left(\frac{s_{i+1,1}^* - s_{k,1}^*}{2}, \min_{l \in \{k,...,i\}} (x_{l-1,2}^* - x_{l,1}^*)\right)$$

Under these service times, the departure times will be

$$x_{l,1} = \begin{cases} x_{l,1}^* + \Delta & l = k, ..., i \\ x_{l,1}^* & o.w. \end{cases}$$

and

$$x_{l,2} = x_{l,2}^* \quad l = 1, ..., N$$

Hence, the change in cost due to applying these non-optimal service times $s_{l,1}$ and $s_{l,2}$ will be

$$J - J^* = \left\{ \begin{array}{c} \left(\theta_1(s_{k,1}^* + \Delta) - \theta_1(s_{k,1}^*)\right) \\ -\left(\theta_1(s_{i+1,1}^*) - \theta_1(s_{i+1,1}^* - \Delta)\right) \end{array} \right\} < 0$$

because θ_1' is monotonically increasing and $s_{i+1,1}^* > s_{k,1}^*$. Since the cost J is lower than the optimal cost J^*, a contradiction is observed implying that

$$x_{i,1}^* \geq x_{i-1,2}^*$$

which concludes the induction proof. ■

Note that this theorem presents a result stronger than the one in Lemma 3

$$x_{i,1}^* \geq x_{i-1,2}^* \geq \min(a_{i+1}, x_{i-1,2}^*) \tag{34}$$

IV. Convex Programming Problem

In this section, we will create a convex programming problem that is equivalent to the original optimal control problem given in (5)-(7). Then, we will utilize the optimal control characteristics to simplify this convex programming problem.

Recall the optimization problem (5)-(7), and replace the constraint (6) by the constraints

$$\begin{aligned} x_{i,1} &\geq a_i + s_{i,1} & i = 1, ..., N \\ x_{i,1} &\geq x_{i-1,1} + s_{i,1} & i = 1, ..., N \end{aligned}$$

and the constraint (7) by the constraints

$$\begin{aligned} x_{i,2} &\geq x_{i,1} + s_{i,2} & i = 1, ..., N \\ x_{i,2} &\geq x_{i-1,2} + s_{i,2} & i = 1, ..., N \end{aligned}$$

i.e., let us define a surrogate convex optimization problem

$$\min_{\substack{s_{i,j} \geq 0, x_{i,j} \\ i=1,2,...,N \ \& \ j=1,2}} \overline{J} = \sum_{i=1}^{N} \{\theta_1(s_{i,1}) + \theta_2(s_{i,2}) + \phi_i(x_{i,2})\} \tag{35}$$

subject to

$$\begin{aligned} x_{i,1} &\geq a_i + s_{i,1} \\ x_{i,1} &\geq x_{i-1,1} + s_{i,1} \\ x_{i,2} &\geq x_{i,1} + s_{i,2} \\ x_{i,2} &\geq x_{i-1,2} + s_{i,2} \end{aligned}$$

for all $i = 1, ..., N$. Note that since the optimization in the surrogate problem is over a larger set, $\overline{J}^* \leq J^*$.

The following theorem establishes that the original and the surrogate problems have the same unique solution.

Theorem 2: The unique optimal solution of the surrogate problem satisfies

$$x_{i,1}^* = \max(a_i^*, x_{i-1,1}^*) + s_{i,1}^*$$

$$x_{i,2}^* = \max(x_{i,1}^*, x_{i-1,2}^*) + s_{i,2}^*$$

for all $i = 1, ..., N$, therefore $\overline{J}^* = J^*$.

Proof: Assume that the optimal solution satisfies

$$\begin{aligned} x_{i,1}^* &> a_i + s_{i,1}^* \\ x_{i,1}^* &> x_{i-1,1}^* + s_{i,1}^* \end{aligned}$$

for some i and define $\Delta_1 = x_{i,1}^* - \max(a_i^*, x_{i-1,1}^*) - s_{i,1}^* > 0$. If we perturb the optimal solution so that $s_{i,1}^*$ is replaced by $s_{i,1} = s_{i,1}^* + \Delta_1$, the cost $\overline{J}$ will be increased by

$$\Delta\overline{J} = \theta_1(s_{i,1}^* + \Delta_1) - \theta_1(s_{i,1}^*)$$

The cost of service at the first stage, $\theta_1(\cdot)$, is assumed to be monotonically decreasing, therefore, $\Delta\overline{J} < 0$, which contradicts the optimality assumption. Hence

$$x_{i,1}^* = \max(a_i^*, x_{i-1,1}^*) + s_{i,1}^*$$

Now, assume that the optimal solution satisfies

$$\begin{aligned} x_{i,2}^* &> x_{i,1}^* + s_{i,2}^* \\ x_{i,2}^* &> x_{i-1,2}^* + s_{i,2}^* \end{aligned}$$

for some i and define $\Delta_2 = x_{i,2}^* - \max(x_{i,1}^*, x_{i-1,2}^*) - s_{i,2}^* > 0$. If we perturb the optimal solution so that $s_{i,2}^*$ is replaced by $s_{i,2} = s_{i,2}^* + \Delta_2$, the cost $\overline{J}$ will be increased by

$$\Delta\overline{J} = \theta_2(s_{i,2}^* + \Delta_2) - \theta_2(s_{i,2}^*)$$

The cost of service at the second stage, $\theta_2(\cdot)$, is assumed to be monotonically decreasing, therefore, $\Delta\overline{J} < 0$, which contradicts the optimality assumption. Hence

$$x_{i,2}^* = \max(x_{i,1}^*, x_{i-1,2}^*) + s_{i,2}^*$$

Since the optimal solution of the surrogate problem is feasible in the region defined by the constraints (6) and (7), the costs J^* and $\overline{J}^*$ are equal. Note that perturbing the service times with Δ_1 and Δ_2 values defined above, we did not lose feasibility. ■

The convexity and differentiability in the surrogate problem are gained at the expense of increased the number of decision variables and constraints, each from $2N$ to $4N$ excluding the non-negativity constraints. We will next employ optimal solution characteristics to simplify the surrogate problem.

From Theorem 1, we know that $x_{i,1}^* \geq x_{i-1,2}^*$ for all $i = 2, ..., N$. We also know that $x_{1,1}^* = a_1 + s_{1,1}^*$. These characteristics of the optimal solution allow us to simplify the surrogate problem to

$$\min_{\substack{s_{i,j} \geq 0, x_{i,1} \\ i=1,2,\ldots,N \ \& \ j=1,2}} \tilde{J} = \sum_{i=1}^{N} \{\theta_1(s_{i,1}) + \theta_2(s_{i,2}) + \phi_i(x_{i,1} + s_{i,2})\}$$

subject to

$$\begin{aligned} x_{1,1} &= a_1 + s_{1,1} \\ x_{i,1} &\geq a_i + s_{i,1} \\ x_{i,1} &\geq x_{i-1,1} + s_{i,1} \\ x_{i,1} &\geq x_{i-1,1} + s_{i-1,2} \end{aligned}$$

for all $i = 2, ..., N$. The number of decision variables for this equivalent convex optimization problem is $3N$ and the number of constraints (excluding the non-negativity constraints) is $3N - 2$.

A. Bulk Arrivals

For the case of bulk arrivals where $a_i = 0$ for all $i = 1, ..., N$, further simplifications are possible. In particular, the surrogate problem can be simplified to

$$\min_{\substack{s_{i,j} \geq 0 \\ i=1,2,\ldots,N \ j=1,2}} \tilde{J} = \sum_{i=1}^{N} \left\{ \begin{array}{c} \theta_1(s_{i,1}) + \theta_2(s_{i,2}) \\ +\phi_i \left(\sum_{k=1}^{i} s_{k,1} + s_{i,2} \right) \end{array} \right\}$$

subject to

$$s_{i,1} \geq s_{i-1,2}$$

for all $i = 2, ..., N$. The number of decision variables in this case is only $2N$ and the number of constraints (excluding the non-negativity constraints) are further reduced to only $N - 1$.

V. Conclusion

In this paper, we considered a two-stage serial manufacturing system where all the arrival times are known. Our control variables were the deterministic service times for both stages. We derived some characteristics of the optimal control and showed that no buffering between stages is observed on the optimal sample path. The original non-smooth optimization problem is transformed into a convex optimization problem over a larger set, which is then simplified by the no buffer property to have fewer constraints and variables. Further simplifications are shown to be possible for the bulk arrivals case. The resulting convex optimization problem can be efficiently solved using standard calculus techniques.

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