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Dedicated to Professor Jean Schmets on the occasion of his 65th birthday

Abstract

A geometric characterization of the extension property is given for Cantor-type sets. The condition can also be done in terms of the rate of growth of certain sequences to the Robin constants of local parts of the set.

1 Introduction

Given a compact set $K \subset \mathbb{R}^n$, $\mathcal{E}(K)$ denotes the space of Whitney jets on K, that is the space of traces on K of C^∞ functions. It is said that K has the extension property if there exists a linear continuous extension operator $L : \mathcal{E}(K) \to C^\infty(\mathbb{R}^n)$. The problem of a geometric characterization of the extension property was raised by Mityagin ([8], Problem 5). Even for the one-dimensional case this problem is still open, in spite of the presence of numerous particular results ([12], [2], [13], [14], [10], [5], [1], [4]). Here we suggest a complete criterion (compare to [14], [5], [1]) of the extension property for Cantor-type sets in certain geometric terms. The condition can be described also in terms of the theory of logarithmic potential.

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2 Sequences of subexponential growth

Let $(\sigma_s)_0^\infty$ be a sequence of positive numbers. We say that $(\sigma_s)_0^\infty$ has a *subexponential growth* if $\sigma_s = exp(o(s))$, that is $\frac{log\,\sigma_s}{s} \to 0$, as $s \to \infty$. Also, the family $(\sigma^{(n)})_{n=1}^\infty = (\sigma_{n,\,s})_{n=1,\,s=0}^{\infty\;\;\infty}$ is of uniform subexponential growth if $\frac{log\,\sigma_{n,\,s}}{s}$ tends to 0, as $s \to \infty$, uniformly with respect to n.

Let $\sigma_s \uparrow \sigma \le \infty$ and $f(s) = log\,\sigma_s$. We are interested in the condition

$$f(s+1) - f(s) \to 0, \qquad \text{as} \quad s \to \infty. \tag{1}$$

Proposition 1. *The condition (1) implies that the sequence $(\sigma_s)_0^\infty$ is of subexponential growth. If the function f is concave or if $\sigma < \infty$, then the subexponential growth of $(\sigma_s)_0^\infty$ is equivalent to (1).*

Proof: Suppose, contrary to our claim, that for some ε_0 and $s_k \uparrow \infty$ we get $f(s_k) \ge 2\varepsilon_0\, s_k$. Then one can find a sequence of disjoint nonempty intervals $(m_k, n_k)_{k=1}^\infty$ with $m_k, n_k \in \mathbb{N}$ such that $f(n_k) - f(m_k) \ge \varepsilon_0\,(n_k - m_k)$. Therefore at least one term $f(j+1) - f(j)$, $j = m_k, \cdots, n_k - 1$ exceeds ε_0, contrary to (1).

If the function f is concave, then $f(s+1) - f(s) \downarrow a$ and the growth condition $f(s)/s \to 0$ implies $a = 0$. In the case $\sigma < \infty$ the result immediately follows from monotonicity of f. ■

An easy example of a sequence $(\sigma_s)_0^\infty$ of subexponential growth without the condition (1) can be done by $f(s) = f(s_k) = f(s_{k-1}) + 1$ for $s_k \le s < s_{k+1}$ provided $s_k/k \to \infty$.

3 Extension property of Cantor-type sets

Given l_1 with $0 < l_1 < 1$ and a sequence $(\alpha_s)_{s=2}^\infty$ with $\alpha_s > 1$ let us denote by $K^{(\alpha_s)}$ the Cantor set associated with the sequence $l_0 = 1, l_1, l_2 = l_1^{\alpha_2}, \cdots, l_s = l_1^{\alpha_2\alpha_3\cdots\alpha_s}, \cdots$, that is $K^{(\alpha_s)} = \bigcap_{s=0}^\infty E_s$, where $E_0 = I_{1,\,0} = [0,1]$, E_s is a union of 2^s closed *basic* intervals $I_{j,\,s}$ of length l_s and E_{s+1} is obtained by deleting of open centric interval of length $l_s - 2\,l_{s+1}$ from each $I_{j,\,s}$, $j = 1, 2, ...2^s$. In the case $\alpha_s = \alpha, s = 2, 3, \cdots$, the compact set $K^{(\alpha)}$ has the extension property if and only if $\alpha \le 2$ ([5], [6]).

Let x be an endpoint of some basic interval. Then there exists the minimal number s (the *type* of x) such that x is the endpoint of some $I_{j,m}$ for every $m \ge s$.

For simplicity, we consider here only the Cantor-type sets such that $\alpha_s \ge 1 + \varepsilon_0, s \ge s_0$ for some positive ε_0 and $l_s \ge 4\,l_{s+1}$ for all s.

We use the notations: $\pi_{n,\,0} := 1$ and $\pi_{n,\,k} = 2^{-k}\,\alpha_{n+1}\alpha_{n+2}\cdots\alpha_{n+k}$ for $n, k \in \mathbb{N}$. Also let $\sigma_{n,\,s} = \sum_{k=0}^s \pi_{n,\,k}$. The condition

$$\sigma_{n,\,s+1} \;/\; \sigma_{n,\,s} \rightrightarrows 1, \qquad \text{as} \quad s \to \infty \tag{2}$$

implies that the the family $(\sigma_{n,\,\cdot}\,)_{n=1}^\infty$ has uniform subexponential growth. Here and in what follows the symbol $\rightrightarrows$ denotes the convergence that is uniform with respect to n. Clearly, (2) is equivalent to

$$\pi_{n,\,s} \Big/ \sum_{k=0}^{s} \pi_{n,\,k} \rightrightarrows 0 \qquad \text{as} \quad s \to \infty \tag{3}$$

and also to

$$\forall v \in \mathbb{N} \qquad \sum_{k=s-v}^{s} \pi_{n,k} \Big/ \sum_{k=0}^{s} \pi_{n,k} \rightrightarrows 0 \quad \text{as} \quad s \to \infty. \tag{4}$$

Therefore the negation of (2) can be written in the following way

$$\exists\, C,\ v : \forall s \quad \text{we get} \quad \sum_{k=0}^{s} \pi_{n,k} \le C \sum_{k=s-v}^{s} \pi_{n,k} \ \text{ for } \ n = n_j \uparrow \infty. \tag{5}$$

In addition we write the condition (2) in geometric terms as

$$\forall M > 0 \quad \exists s_M : \quad l_{n+s}^{M} > l_n^{2^s} l_{n+1}^{2^{s-1}} \cdots l_{n+s}, \quad \forall n \ \ \forall s \ge s_M. \tag{6}$$

We proceed to characterize the extension property of the set $K^{(\alpha_s)}$. The topology of the space $\mathcal{E}(K^{(\alpha_s)})$ is given by the family of norms

$$\| f \|_q = |f|_q + sup \left\{ \frac{|(R_y^q f)^{(k)}(x)|}{|x-y|^{q-k}} : x, y \in K^{(\alpha_s)}, x \ne y, k = 0, 1, ...q \right\},$$

$q = 0, 1, ...,$ where $|\,f|_q = sup\{|\,f^{(k)}(x)| : x \in K^{(\alpha_s)}, k \le q\}$ and $R_y^q f(x) = f(x) - T_y^q f(x)$ is the Taylor remainder.

For an infinitely differentiable function F with compact support, $|\,F|_q^{(\mathbb{R})}$ denotes $sup\{|F^{(k)}(x)| : x \in \mathbb{R}, k \le q\}$.

Theorem 1. *The compact set $K^{(\alpha_s)}$ has the extension property if and only if the condition (2) is fulfilled.*

Proof: Suppose the condition (2) is valid. We can present the extension operator $L : \mathcal{E}(K^{(\alpha_s)}) \to C^\infty(\mathbb{R})$ explicitly. At first we extend properly the basis elements of the space, and then we define the operator L by linearity.

Let us prove that the condition (2) implies boundedness of the sequence (α_s). Suppose, contrary to our claim, that for some subsequence (n_s) we have $\alpha_{n_s} > 2^s$, $s \in \mathbb{N}$. Without loss of generality let $n_s > s$, $s \in \mathbb{N}$. Then for $n = n_s - s$ we get $\sum_{k=0}^{s} \pi_{n,k}/\pi_{n,s} = 1 + \alpha_{n+s}^{-1}\,[2 + \sum_{k=0}^{s-2} \frac{2^{s-k}}{\alpha_{n+k+1} \cdots \alpha_{n+s-1}}] < 1 + \alpha_{n+s}^{-1}\ \sum_{k=0}^{s-1} 2^{s-k}$, since $\alpha_k > 1$, $k \in \mathbb{N}$. Therefore, $\sum_{k=0}^{s} \pi_{n,k}/\pi_{n,s} < 3$, contrary to (3).

Let $A := sup_s\, \alpha_s$.

For a fixed basic interval $I_{j,s} = [a_{j,s}, b_{j,s}]$, let us choose the sequence $(x_{n,j,s})_{n=1}^{\infty}$ of points by including all endpoints of basic subintervals of $I_{j,s}$ in the order of increase of the type. For the points of the same type we first take the endpoints of the largest gaps between the points of this type; here the intervals $(-\infty, x), (x, \infty)$ are considered as gaps. From points adjacent to the equal gaps, we choose the left one x and then $b_{j,s} - x$. Thus, $x_{1,j,s} = a_{j,s}$, $x_{2,j,s} = b_{j,s}$, $x_{3,j,s} = a_{j,s} + l_{s+1}, \cdots, x_{7,j,s} = a_{j,s} + l_{s+1} - l_{s+2}, \cdots$. We follow [7] to define $e_{N,j,s} = \prod_{n=1}^{N}(x - x_{n,j,s})$ if $x \in K^{(\alpha_s)} \cap I_{j,s}$ and $e_{N,j,s} = 0$ on $K^{(\alpha_s)}$ otherwise.

Given a nondecreasing unbounded sequence $(N_s)_{s=0}^{\infty}$ of natural numbers of the form $N_s = 2^{n_s}$, we consider the sequence $\mathcal{B} = (e_{N,j,s})_{s=0,\ j=1,\ N=M_s}^{\infty,\ 2^s,\ N_s}$, where $M_0 = 0$ and for $s \ge 1$ we take $M_s = N_{s-1}/2 + 1$ if the number j is odd, $M_s = N_{s-1}/2$ for even j. By Theorem 2 in [7], the sequence $\mathcal{B}$ forms a basis in the space $\mathcal{E}(K^{(\alpha_s)})$, provided the condition

$$2^{N_s}\, l_s \leq 1 \quad \text{for all} \quad s \geq 1. \tag{7}$$

Given $\delta > 0$, and a compact set E, we take a $C^\infty-$ function $u(\cdot, \delta, E)$ with the properties: $u(\cdot, \delta, E) \equiv 1$ on E, $u(x, \delta, E) = 0$ for $dist(x, E) > \delta$ and $|\,u|_p^{(\mathbb{R})} \leq c_p\, \delta^{-p}$, $p \in \mathbb{N}$, where the constant c_p depends only on p. Let $(c_p) \uparrow$.

Fix $s \in \mathbb{N}$ and N with $M_s \leq N \leq N_s$. Then $2^{m-1} < N \leq 2^m$ for some m from the set $\{n_{s-1}-1, \cdots, n_s\}$. Let $\delta_{N,s} = l_{s+m-1}$. Now for $j = 1, \cdots, 2^s$ we define $L(e_{N,j,s})$ as $\tilde{e}_{N,j,s}\, u(\cdot\,, \delta_{N,s}\,, I_{j,s} \cap K^{(\alpha_s)})$, where $\tilde{e}_{N,j,s}$ denotes the analytic extension of the corresponding polynomial. The operator L is well-defined on the basis elements. For its continuity it is sufficient to show that for any $p \in \mathbb{N}$ there exist $q \in \mathbb{N}$ and $C > 0$ such that

$$|\, \tilde{e}_{N,j,s}\; u(\cdot\,, \delta_{N,s}\,, I_{j,s} \cap K^{(\alpha_s)})|_p^{(\mathbb{R})} \leq C\, ||\, e_{N,j,s}||_q \tag{8}$$

for all admissible values of s, j and N.

Fix $p \in \mathbb{N}$ and q in the form $q = 2^v$ such that for any n we have

$$p\, A^2\, \pi_{n,\, v-3} < \sum_{k=0}^{v-3} \pi_{n,\,k},$$

which is possible due to (3).

Given p, q, using (7), we choose s_0 with $N_{s_0}^p \leq 2^{N_{s_0}}$ and $4\, N_{s_0}^q\, l_{s_0}^{\varepsilon_0} < 1$. In what follows we consider only $s \geq s_0$.

The polynomial $\tilde{e}_{N,j,s}$ has its zeros at all points of the type at most $s+m-2$ and possibly at some points of the type $s+m-1$ on the interval $I_{j,s}$. Let us fix a point z with $dist(z, I_{j,s} \cap K^{(\alpha_s)}) \leq l_{s+m-1}$ and $n \leq p$ such that $|\, \tilde{e}_{N,j,s}\; u|_p^{(\mathbb{R})} = |\,(\tilde{e}_{N,j,s}\; u)^{(n)}(z)|$. For this z we take the point x of the type $\leq s+m-2$ that is the nearest to z. If there are two such points, then we take any of them. Clearly, $|\, x - z\,| \leq 2\, l_{s+m-1}$. By $(\rho_k)_{k=1}^N$ we denote distances from x to the zeros of $\tilde{e}_{N,j,s}$ ordered increasingly. Thus, $\rho_1 = 0$, $l_{s+m-1} \leq \rho_2 \leq l_{s+m-2}$, etcetera.

By the Leibniz Rule,

$$|\,(\tilde{e}_{N,j,s}\; u)^{(n)}(z)| \leq \sum_{i=0}^{n} \binom{n}{i}\, c_{n-i}\, l_{s+m-1}^{i-n}\, |\,(\tilde{e}_{N,j,s})^{(i)}(z)\,|.$$

The derivative $(\tilde{e}_{N,j,s})^{(i)}(z)$ is a sum of $N!/(N-i)!$ terms and every term is a product of $N-i$ factors of type $(z - x_{n,j,s})$. Since $|\, z - x_{n,j,s}\,| \leq |\, x - x_{n,j,s}\,| + 2\, l_{s+m-1}$, we can write $|\,(\tilde{e}_{N,j,s})^{(i)}(z)\,| \leq N^i \prod_{k=i+1}^N (\rho_k + 2\, l_{s+m-1})$ and

$$|\,(\tilde{e}_{N,j,s}\; u)^{(n)}(z)| \leq 2^n\, c_n\, max_{i \leq n}[l_{s+m-1}^{i-n}\, N^i \prod_{k=i+1}^{N} (\rho_k + 2\, l_{s+m-1})].$$

The distance between any two zeros of $\tilde{e}_{N,j,s}$ is not smaller than l_{s+m-1}. It implies that $\rho_k + 2\, l_{s+m-1} \leq \rho_{k+2}$ for $k \leq N-2$. Clearly, $\rho_{N-1} + 2\, l_{s+m-1} \leq \rho_N + 2\, l_{s+m-1} < 2\, l_s$. Therefore,

$$|\,(\tilde{e}_{N,j,s}\; u)^{(n)}(z)| \leq 2^{p+2}\, c_p\, l_s^2\, max_{i \leq n}[\, l_{s+m-1}^{i-n}\, N^i \prod_{k=i+3}^{N} \rho_k].$$

For $i > 0$ the expression in square brackets can be written as $\frac{l_{s+m-1}^{i+1} N^i}{\rho_3 \cdots \rho_{i+2}} \, l_{s+m-1}^{-1-n} \prod_{k=3}^{N} \rho_k$. The fraction here does not exceed 1, because $l_{s+m-1} N^i \leq l_s N_s^p \leq l_s 2^{N_s} \leq 1$, by (7). In the case where $i = 0$ we also have the desired bound. Thus finally,

$$| \tilde{e}_{N,j,s} \, u|_p^{(\mathbb{R})} \leq 2^{p+2} c_p \, l_s^2 \, l_{s+m-1}^{-1-p} \prod_{k=3}^{N} \rho_k. \tag{9}$$

We proceed to get a lower bound of $|| \, e_{N,j,s}||_q$. With the same x as above, we get $|| \, e_{N,j,s}||_q \geq | \, e_{N,j,s}|_q \geq | \, e_{N,j,s}^{(r)}(x) \, |$ for any $r \leq q$.

Any basic subinterval $I_{i,s+m-k}$ contains from 2^{k-1} to 2^k zeros of $\tilde{e}_{N,j,s}$, $k = 1, \cdots, m$. The point x belongs to a certain interval $I_{i,s+m-v}$ that contains r, $q/2 \leq r \leq q$, zeros of $\tilde{e}_{N,j,s}$. Here $\frac{1}{r!} \, e_{N,j,s}^{(r)}(x)$ is a sum of $\binom{N}{r}$ terms and every term is a product of $N - r$ factors of type $(x - x_{n,j,s})$. Only one of these products does not contain $(x - x_{n,j,s})$ for $x_{n,j,s} \in I_{i,s+m-v}$ and the modulus of this product is $\prod_{k=r+1}^{N} \rho_k$. All other products contain terms with $| \, x - x_{n,j,s}| \leq l_{s+m-v}$. Therefore the modulus of any other product does not exceed $\rho_r \prod_{k=r+2}^{N} \rho_k$. The sum of all such products can be estimated from above by $[\binom{N}{r} - 1] \, \rho_r \prod_{k=r+2}^{N} \rho_k$. It follows that

$$| \, e_{N,j,s}^{(r)}(x)| \geq \prod_{k=r+1}^{N} \rho_k - N^r \rho_r \prod_{k=r+2}^{N} \rho_k.$$

Because of the choice of r we get $\rho_r \leq l_{s+m-v}$, $\rho_{r+1} \geq l_{s+m-v-1} - 2 \, l_{s+m-v}$. It is easy to check that $\rho_r / \rho_{r+1} < 2 \, l_s^{\varepsilon_0}$. Therefore, $N^r \rho_r / \rho_{r+1} \leq 1/2$, due to the choice of s_0. It implies that $|| \, e_{N,j,s}||_q \geq \frac{1}{2} \prod_{k=r+1}^{N} \rho_k \geq \frac{1}{2} \prod_{k=q/2+1}^{N} \rho_k$. Comparing this to (9), we see that it is enough to show that the sequence $(l_{s+m-1}^{-p-1} \prod_{k=3}^{q/2} \rho_k)_{s=s_0, \, m \leq n_s}$ is bounded.

In the estimation of the product $\prod_{k=3}^{q/2} \rho_k$ from above we will take into account only the points of the type $\leq s + m - 2$. Clearly, including the points of the type $s + m - 1$ can only decrease the product. Hence, $\rho_3 \leq l_{s+m-3} - l_{s+m-2}$, $\rho_4 \leq l_{s+m-3}, \cdots, \rho_{q/2} \leq l_{s+m-v}$ and

$$\prod_{k=3}^{q/2} \rho_k \leq l_{s+m-3}^2 \, l_{s+m-4}^4 \cdots l_{s+m-v}^{2^{v-2}} = l_{s+m-v}^{\varkappa},$$

where $\varkappa = 2^{v-2} + 2^{v-3} \alpha_{s+m-v+1} + 2^{v-4} \, \alpha_{s+m-v+1} \, \alpha_{s+m-v+2} + \cdots + 2 \, \alpha_{s+m-v+1} \cdots \alpha_{s+m-3} = 2^{v-2} \sum_{k=0}^{v-3} \pi_{s+m-v, \, k}$.

On the other hand, since $\pi_{n,\,k} = \frac{1}{4} \, \alpha_{n+k-1} \, \alpha_{n+k} \, \pi_{n,\,k-2} \leq \frac{1}{4} \, A^2 \, \pi_{n,\,k-2}$, we get $l_{s+m-1} = l_{s+m-v}^{2^{v-1} \pi_{s+m-v,\, v-1}} \geq l_{s+m-v}^{2^{v-3} A^2 \, \pi_{s+m-v,\, v-3}}$. It follows that $l_{s+m-1}^{-p-1} \prod_{k=3}^{q/2} \rho_k \leq l_{s+m-v}^{\varkappa_1}$, where

$$\varkappa_1 = 2^{v-2} \left[\sum_{k=0}^{v-3} \pi_{s+m-v,\,k} - \frac{1}{2}(p+1) A^2 \, \pi_{s+m-v,\,v-3} \right],$$

which is positive by the choice of q. This gives (8) and continuity of the operator L.

We are now in position to show that the condition (5) implies the lack of the extension property for the compact set $K^{(\alpha_s)}$. In [13] Tidten applied Vogt's characterization for splitting of exact sequences of Fréchet spaces and proved that a compact set K has the extension property if and only if the space $\mathcal{E}(K)$ has a dominating norm. Due to Frerick [4], the space of Whitney functions has the property (DN) if and only if for any $\varepsilon > 0$ and for any $q \in \mathbb{N}$ there exist $r \in \mathbb{N}$ and $C > 0$ such that

$$|\cdot|_q^{1+\varepsilon} \le C|\cdot|_0 \;||\cdot||_r^{\varepsilon}.$$

Therefore we need to show that there exists $\varepsilon > 0$ and q such that for any $r \in \mathbb{N}$ one can find a sequence $(f_j) \subset \mathcal{E}(K^{(\alpha_s)})$ with

$$|f_j|_0 \;||f_j||_r^{\varepsilon}\;|f_j|_q^{-1-\varepsilon} \to 0 \quad \text{as} \quad j \to \infty. \tag{10}$$

Given C and v by the condition (5), we take $\varepsilon = C^{-1}$ and $q = 2^v$. Fix any $r = 2^s$. Since the norms $||\cdot||_r$ increase, we can take r in this form. We choose the subsequence (n_j) from the condition (5) and consider $f_j = e_{r,1,n}$ for $n = n_j$.

The zeros of $e_{r,1,n}$ on $I_{1,n}$ are all points of the type $\le n+s-1$. Hence for any $x \in K^{(\alpha_s)} \cap I_{1,n}$ the distance from x to some zero of $e_{r,1,n}$ is not larger than l_{n+s}, the distance from x to other zero of $e_{r,1,n}$ does not exceed l_{n+s-1}. Then we find two other points with $|x - x_{i,1,n}| \le l_{n+s-2}$, etcetera. Therefore,

$$|f_j|_0 \le l_{n+s}\, l_{n+s-1}\, l_{n+s-2}^2\, l_{n+s-3}^4 \cdots l_n^{2^{s-1}}.$$

For the lower bound of $|f_j|_q$ we use the same arguments as above:

$$|f_j|_q \ge |f_j^{(q)}(0)| \ge 1/2\;\, l_{n+s-v-1}^{2^v} \cdots l_n^{2^{s-1}}.$$

Here, instead of the condition $4\, N_s^q\, l_s^{\varepsilon_0} < 1$ we need $4 \cdot 2^{s\,q}\; l_{n_j}^{\varepsilon_0} < 1$, which can be achieved for large enough j.

Since for any $x \in K^{(\alpha_s)} \cap I_{1,n}$ the value $f_j(x)$ is a product of r small terms $(x - x_{i,1,n})$, we get $|f_j|_r = |f_j^{(r)}| = r!$. Also $sup\left\{|(R_y^r f_j)^{(k)}(x)|\,|x-y|^{-r+k}\right\}$ will be realized for $k = r$. Therefore, $||f_j||_r = 2\,r!$.

Thus in order to get (10), it remains to prove that

$$l_{n+s}\, l_{n+s-1}\, l_{n+s-2}^2 \cdots l_{n+s-v}^{2^{v-1}}\;\; (l_{n+s-v-1}^{2^v} \cdots l_n^{2^{s-1}})^{-\varepsilon_0} \to 0, \;\text{ as }\; n = n_j \to \infty.$$

As before, the element of the sequence can be written in the form $l_n^{\varkappa}$, where $\varkappa = \alpha_{n+1}\,\alpha_{n+2}\cdots\alpha_{n+s} + \alpha_{n+1}\cdots\alpha_{n+s-1} + 2\,\alpha_{n+1}\cdots\alpha_{n+s-2} + \cdots + 2^{v-1}\,\alpha_{n+1}\cdots\alpha_{n+s-v} - \varepsilon_0\,[2^v\,\alpha_{n+1}\cdots\alpha_{n+s-v-1} + \cdots + 2^{s-2}\,\alpha_{n+1} + 2^{s-1}] = 2^{s-1}\,[2\,\pi_{n,s} + \sum_{k=s-v}^{s-1}\pi_{n,k} - \varepsilon_0\sum_{k=0}^{s-v-1}\pi_{n,k}]$, which is positive by (5).

This gives (10) and the lack of the dominating norm in the space $\mathcal{E}(K^{(\alpha_s)})$. ■

4 Characterization in potential-theoretic terms

By $Cap\,(K)$ we denote the logarithmic capacity of a compact set $K \subset \mathbb{C}$. We are interested in the minimal value of the logarithmic energy $log(Cap\,(K)^{-1})$, which is also called the Robin constant of K and is denoted by $Rob\,(K)$. Here and subsequently, log denotes the natural logarithm. The Cantor set $K^{(\alpha_s)}$ is polar if and only if $\sum_{k=0}^{\infty} \pi_{1,k} = \infty$ (see e.g.[3]). What is more, Totik in [15] found the bound (in our terms):

$$1/4 \;\; log\, l_1^{-1} \; \sum_{k=0}^{\infty} \pi_{1,k} \le Rob\,(K^{(\alpha_s)}) \le 2 \;\; log\, l_1^{-1} \; \sum_{k=0}^{\infty} \pi_{1,k}.$$

Repeating arguments from [15] for the corresponding part of the set, we obtain for $n \in \mathbb{N}, \; j = 1, \cdots, 2^n$

$$1/4 \;\; log\, l_n^{-1} \; \sum_{k=0}^{\infty} \pi_{n,k} \le Rob\,(K^{(\alpha_s)} \cap I_{j,n}) \le 2 \;\; log\, l_n^{-1} \; \sum_{k=0}^{\infty} \pi_{n,k}.$$

Therefore the condition (3) means a kind of uniform with respect to n regularity of approximation of the sum of the (possibly divergent) series, corresponding to the Robin constant of the set $K^{(\alpha_s)} \cap I_{j,n}$, by its partial sums. Proposition 1 now shows that if the set $K^{(\alpha_s)}$ has the extension property, then the sequences of partial sums, corresponding to the Robin constants of the sets $K^{(\alpha_s)} \cap I_{j,n}$, have uniform with respect to n subexponential growth.

We see that the extension property of the set $K^{(\alpha_s)}$ is not related to the polarity or to the "local" polarity of the set. Neither it is related to the regularity of the Green function $g(\mathbb{C} \setminus K^{(\alpha_s)}, \; z, \infty)$, since by Pleśniak [11], in the case of the Cantor type set, the corresponding Green function is regular if and only if the set is not polar.

Example 1. The set $K^{(2)}$ is polar, but it has the extension property, since here $\pi_{n,k} = 1$ for all n and k. See also [6] for this case.

Example 2. Let us fix an increasing sequence $(k_m)_{m=1}^{\infty}$ of natural numbers and $a \in (1/2, 1)$. We define $\alpha_2 = \cdots = \alpha_{k_1+1} = 2\,a$ and then for $m \in \mathbb{N}$ let $\alpha_{k_1+k_2+\cdots+k_m+m+1} = a^{-k_m}, \; \alpha_{k_1+k_2+\cdots+k_m+j} = 2\,a$ for $j = m+2, m+3, \cdots, k_{m+1}+m+1$. Then $\pi_{1,k} = a^k, \; k = 0, \cdots, k_1$ and for $m \in \mathbb{N}$ we get $\pi_{1,k_1+k_2+\cdots+k_m+m} = 2^{-m}, \; \pi_{1,k_1+k_2+\cdots+k_m+j} = 2^{-m}\,a^{j-m}$ for $j = m+1, m+2, \cdots, k_{m+1}+m$. Therefore, $\sum_{k=0}^{\infty} \pi_{1,k} = \sum_{m=0}^{\infty} 2^{-m}(1+a+\cdots+a^{k_{m+1}})$. Since the series converges, the set $K^{(\alpha_s)}$ is not polar. But it does not have the extension property. For $n = k_1+\cdots+k_m+m+1$ and $s = k_{m+1}+1$ we get $\pi_{n,k} = a^k$ for $k = 0, \cdots, k_{m+1}, \; \pi_{n,s} = 1/2$, contrary to (3). As well the condition (2) can not be fulfilled because the sequence (α_k) is not bounded.

We now turn to the problem of a geometric characterization of the extension property. It is known (see e.g [9]) that there is no general geometric characterization of polarity of compact sets in terms of (Hausdorff) measures. Our condition (2) is more subtle than the statement about the convergence of the series $\sum_{k=0}^{\infty} \pi_{1,k}$. One can conclude that the possibility to find a geometric characterization of the extension property in the general case is rather doubtful.

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