BESSEL FUNCTIONS - BASED RECONSTRUCTION OF NON-UNIFORMLY SAMPLED DIFFRACTION FIELDS

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ABSTRACT

A discrete computational model for the diffraction process is essential in forward problems related to holographic TV. The model must be as general as possible, since the shape of the displayed objects does not bear any restrictions. We derive a discrete diffraction model which suits the problem of reconstruction of diffraction fields from a set of non-uniformly distributed samples. The only restriction of the model is the wave nature of the field. The derivation takes advantage of changing the spatial and frequency coordinates to polar form and ends up with a model stated in terms of Bessel functions. The model proves to be a separable orthogonal basis. It shows rapid convergence when evaluated in the framework of the non-uniform sampling problem.

Index Terms— diffraction, non-uniform sampling, polar coordinates, Bessel functions

1. INTRODUCTION

The computation of the light field distribution which arises over the entire three-dimensional (3D) space from an abstract representation of a 3D scene is known as the *forward problem* in holography [1]. This problem is important in holographic three-dimensional television (3DTV), because the display device has to be controlled by signals which depend on the optical field of the scene. In order to solve forward problems, it is relevant to have an accurate and general model that simulates numerically the diffraction process. Since numerical calculations usually are carried out by a computer, the model has to be also discrete, i.e. it must be dependent on finite number of parameters. In addition, the model must be as general as possible, because the nature of the displayed scene is usually not known in advance.

In our earlier work [2, 3] the forward problem is stated as reconstruction of the diffraction field from a set of given samples, non-uniformly distributed in the space. This statement is general enough, because it does not assume any particular shape for the object. However, the model used to calculate the diffraction bears certain limitations. The plane wave decomposition integral [4] is discretized in order to achieve discrete diffraction model. The discretization is done by sampling the Fourier transform of first plane of the field. Since this sampling is equivalent to periodization of the function on the first plane, this function must have finite spatial extend. It is also assumed to be essentially band-limited in order to obtain a model which depends on a finite number of Fourier coefficients. Moreover, the diffraction field can be considered up to a limited distance along the propagation direction z. The discretization of the spectra on the first line assumes periodization along the transverse directions. On the other hand, a space-limited pattern on the first plane spreads when propagated along the z-axis. The spread is proportional to the distance from the initial plane. After certain distance the adjacent periodic replicas on the same plane start overlapping [5] and the model starts producing erroneous results. One way to overcome this problem is to assume larger periods in the transverse directions when discretizing the Fourier spectra of the initial plane, so that there is more distance between the adjacent replicas. However, this will increase the dimensionality of the model, because larger period in spatial domain corresponds to denser sampling in frequency domain and the essential frequency band is represented by more coefficients. Therefore the model in [2, 3] can be used to calculate accurately the diffraction field up to some distance, which depends on the support of the diffraction pattern and the chosen dimensionality of the model.

Our motivation in this paper is to search for a model which can be used to calculate diffraction patterns with less restrictions. Yet the model must be discrete and finite-dimensional. In order to do so, we start from the wave nature of the diffraction field. The Helmholtz wave equation puts the only restriction on our model - the Fourier transform of the wave is nonzero only on a sphere. We arrive at an elegant model involving Bessel functions of the first kind [6] by writing the spatial and frequency coordinates in a polar form. There exist accurate numerical methods to calculate the Bessel functions [7] which make our model practically feasible.

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2. DIFFRACTION MODEL IN POLAR COORDINATES

We consider optical fields generated by monochromatic (single wavelength) light waves, propagating in free space [4]. For simplicity, the discussions are restricted to one transverse dimension only. Extension to two transverse dimensions is straightforward. The light field $u(x,z)$ can be expressed in terms of its two-dimensional Fourier transform $A(k_x,k_z)$ as

$$u(x,z)=\iint_{-\infty}^{\infty} A(k_x,k_z)e^{j(k_x x+k_z z)}dk_x dk_z. \qquad (1)$$

The x axis is the transverse axis and the z axis is the optical axis along which the field propagates. The variables k_x and k_z are the spatial frequencies for x and z respectively. They can be written in polar form as $k_x=k\sin\theta$ and $k_z=k\cos\theta$, where $k\in[0,\infty)$ is the radius and $\theta\in[0,2\pi)$ is the angle. After the change of the variables the 2D Fourier transform integral of Eq.1 becomes

$$u(x,z)=\int_0^{\infty}\int_0^{2\pi} kA(k\sin\theta,k\cos\theta)e^{j(kx\sin\theta+kz\cos\theta)}dkd\theta. \qquad (2)$$

The light field $u(x,z)$ satisfies the Helmholtz wave equation $\nabla^2u(x,z)+k^2u(x,z)=0$. Therefore the Fourier transform $A(k_x,k_z)$ is nonzero only on a circle, centered at the origin with radius $k_0=\frac{2\pi}{\lambda}$, where λ is the wavelength of the monochromatic light. This implies that the spectra $A(k_x,k_z)$ written in polar coordinates depends only on one variable:

$$A(k\sin\theta,k\cos\theta)=A(k_0\sin\theta,k_0\cos\theta)\equiv A(\theta). \qquad (3)$$

Now the Fourier integral of Eq.2 becomes one dimensional:

$$u(x,z)=\frac{2\pi}{\lambda}\int_0^{2\pi} A(\theta)e^{j\frac{2\pi}{\lambda}(x\sin\theta+z\cos\theta)}d\theta. \qquad (4)$$

The change of the spatial coordinates x and z in polar form as $x=r\sin\phi$ and $z=r\cos\phi$ simplifies the integral in Eq. 4 to

$$\begin{aligned} u(r,\phi) &= \frac{2\pi}{\lambda}\int_0^{2\pi} A(\theta)e^{j\frac{2\pi}{\lambda}(r\sin\phi\sin\theta+r\cos\phi\cos\theta)}d\theta \\ &= \frac{2\pi}{\lambda}\int_0^{2\pi} A(\theta)e^{j\frac{2\pi}{\lambda}r\cos(\theta-\phi)}d\theta \qquad (5)\end{aligned}$$

We arrived at a rather simple form that does not impose any restrictions on the generated field. However, this expression is far from a computational formula since it involves integration and does not depend on finite number of elements. The Fourier transform of the field is nonzero only on a circle and therefore $A(\theta)$ can be considered as 2π-periodic with respect to θ. Hence $A(\theta)$ can be represented by the complex Fourier series:

$$A(\theta)=\sum_{m=-\infty}^{\infty} c_m e^{jm\theta} \qquad (6)$$

Inserting Eq.6 in Eq.5 leads to

$$\begin{aligned} u(r,\phi) &= \frac{2\pi}{\lambda}\int_0^{2\pi}\sum_{m=-\infty}^{\infty} c_m e^{jm\theta}e^{j\frac{2\pi}{\lambda}r\cos(\theta-\phi)}d\theta \\ &= \frac{2\pi}{\lambda}\sum_{m=-\infty}^{\infty} c_m\int_0^{2\pi} e^{jm\theta}e^{j\frac{2\pi}{\lambda}r\cos(\theta-\phi)}d\theta. \qquad (7)\end{aligned}$$

The integral inside the summation of the last line can be written as a Bessel function of the first kind [6] after making the substitution $\alpha=\theta-\phi$:

$$\begin{aligned} u(r,\phi) &= \frac{2\pi}{\lambda}\sum_{m=-\infty}^{\infty} c_m\int_{-\phi}^{2\pi-\phi} e^{jm(\alpha+\phi)}e^{j\frac{2\pi}{\lambda}r\cos\alpha}d\alpha \\ &= \frac{2\pi}{\lambda}\sum_{m=-\infty}^{\infty} c_m e^{jm\phi}\int_0^{2\pi} e^{jm\alpha}e^{j\frac{2\pi}{\lambda}r\cos\alpha}d\alpha \\ &= \frac{2\pi}{\lambda}\sum_{m=-\infty}^{\infty} c_m e^{jm\phi}2\pi j^m J_m\left(\tfrac{2\pi}{\lambda}r\right). \qquad (8)\end{aligned}$$

Here by $J_m(t)$ is denoted the m-th Bessel function of the first kind [6], whose integral form is

$$J_m(t)=\frac{1}{2\pi j^m}\int_0^{2\pi} e^{jm\alpha}e^{jt\cos\alpha}d\alpha. \qquad (9)$$

The last line of Eq. 8 can be simplified to obtain the final form of the discrete diffraction model:

$$u(r,\phi)=\frac{4\pi^2}{\lambda}\sum_{m=-\infty}^{\infty} c_m e^{jm(\phi+\frac{\pi}{2})}J_m\left(\tfrac{2\pi}{\lambda}r\right) \qquad (10)$$

There exist efficient and accurate numerical algorithms to calculate Bessel functions [7], which can be utilized in the model. Equation 10 can be used to calculate the field at any point (r,ϕ) if the coefficients c_m of the Fourier series of $A(\theta)$ are known. Note that there are no practical restrictions for the desired field. There were no assumptions during the derivation, except that the field is a wave, i.e. it satisfies the Helmholtz wave equation and, consequently, its Fourier transform is nonzero only on the circle with radius $\frac{2\pi}{\lambda}$.

Equation 10 is in fact a representation of the diffraction field $u(r,\phi)$ as a linear combination of the functions:

$$\psi_m(r,\phi)=e^{jm(\phi+\frac{\pi}{2})}J_m\left(\tfrac{2\pi}{\lambda}r\right). \qquad (11)$$

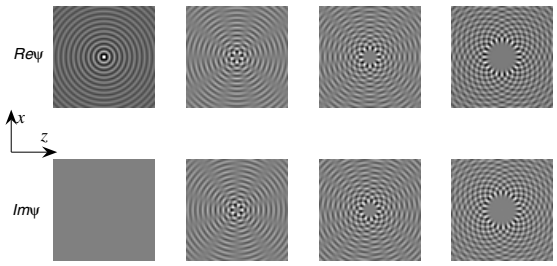


Fig. 1. Plots of some basis functions $\psi_m(r,\phi)$ in cartesian coordinate system (x,z) for $m = 0, 5, 10, 20$ (from left to right). The top and the bottom rows depict the real and imaginary parts, respectively.

It is interesting to note that these functions are *separable*. They are product of a Bessel function which depends only on r and an exponent which depends only on ϕ. The separability property can be used to show that the functions $\psi_m(r,\phi)$ are mutually *orthogonal*:

$$\begin{aligned}
&\langle \psi_m(r,\phi), \psi_n(r,\phi)\rangle = \int_0^{2\pi}\int_0^{\infty} \psi_m(r,\phi)\psi_n^*(r,\phi) d\phi dr \\
= &\int_0^{2\pi}\int_0^{\infty} e^{jm(\phi+\frac{\pi}{2})} J_m\left(\tfrac{2\pi}{\lambda}r\right) e^{-jn(\phi+\frac{\pi}{2})} J_n\left(\tfrac{2\pi}{\lambda}r\right) d\phi dr \\
= &\int_0^{\infty} J_m\left(\tfrac{2\pi}{\lambda}r\right) J_n\left(\tfrac{2\pi}{\lambda}r\right) dr \int_0^{2\pi} e^{j(m-n)(\phi+\frac{\pi}{2})} d\phi \\
= &\ 0 \text{ for } m \neq n. \qquad (12)
\end{aligned}$$

Therefore the set $\{\psi_m(r,\phi)\}_{m=-\infty}^{\infty}$ forms an orthogonal basis. As derived, the basis is suitable for decomposition of signals representing light fields. Figure 1 shows some basis functions for $m = 0, 5, 10$ and 20. The functions are shown with respect to the cartesian coordinate system (x,z). As one can see, they are smooth circular oscillatory functions, whose action moves away from the origin when the number m increases.

3. RECONSTRUCTION FROM NON-UNIFORMLY DISTRIBUTED SAMPLES

For numerical computation the infinite summation in the model from Eq. 10 is impractical. A finite dimensional version of the model can be obtained by taking only the partial sum of M basis elements:

$$u(r,\phi) = \frac{4\pi^2}{\lambda} \sum_{m=-\lfloor M/2 \rfloor}^{\lfloor (M-1)/2 \rfloor} c_m e^{jm(\phi+\frac{\pi}{2})} J_m\left(\tfrac{2\pi}{\lambda}r\right). \qquad (13)$$

We will use the polar model of Eq. 13 for the problem of non-uniform sampling of diffraction fields [2]. The problem can be stated for polar coordinates as reconstruction of $u(r,\phi)$ from a finite set of s sampling points $\{(r_i,\phi_i)\}_{i=1}^{s}$. The field can be calculated if the coefficients c_m are known. A system of s equations for c_m can be constructed by writing Eq. 13 for each point in the irregular sampling set:

$$u(r_i,\phi_i) = \frac{4\pi^2}{\lambda} \sum_{m=-\lfloor M/2 \rfloor}^{\lfloor M/2 \rfloor} c_m e^{jm(\phi_i+\frac{\pi}{2})} J_m\left(\tfrac{2\pi}{\lambda}r_i\right), i = 1...s. \qquad (14)$$

This system is linear and can be stated in matrix form:

$$\mathbf{u} = \mathbf{J}\mathbf{c}, \qquad (15)$$

where $\mathbf{c} = [c_{-\lfloor \frac{M}{2} \rfloor}, c_{-\lfloor \frac{M}{2} \rfloor+1}, ..., c_{\lfloor \frac{M-1}{2} \rfloor}]^T$ is the unknown vector of the coefficients and the vector of given samples is $\mathbf{u} = [u(r_1,\phi_1), u(r_2,\phi_2), ..., u(r_s,\phi_s)]^T$. $\mathbf{J}$ is the $s \times M$ reconstruction matrix

$$\begin{aligned}
\mathbf{J} = \{\mathcal{J}_{i,l}\} = \{e^{j(l-\lfloor \frac{M}{2} \rfloor -1)(\phi_i+\frac{\pi}{2})} J_{l-\lfloor \frac{M}{2} \rfloor -1}\left(\tfrac{2\pi}{\lambda}r_i\right)\}, \\
i = 1, ..., s, l = 1, ..., M. \qquad (16)
\end{aligned}$$

The straightforward approach to solve Eq. 15 is to take the pseudo-inverse. It has two different forms - for the over-determined case ($s > M$) and the under-determined case ($s < M$). In this work the latter case is ignored, because then the field can never be reconstructed, and therefore the results are not of practical interest. The major drawback of solving the problem by computing the pseudo-inverse is the high computational costs which grow when the number of given samples s increases. In linear programming [8] there exists a myriad of iterative algorithms for solving linear systems. Conjugate gradient method applied on normal equations (CGN) [8] is one of the most powerful algorithms for inversion of rectangular matrices. It proceeds as follows:

1. compute the matrix $\mathbf{J}$ according to Eq. 16 and $\mathbf{b} = \mathbf{J}^H\mathbf{u}$;
2. initialize $\hat{\mathbf{c}}^{[0]}$ arbitrary, $\mathbf{g}_0 = \mathbf{b} - \mathbf{J}^H\mathbf{J}\mathbf{c}$ and $\mathbf{d}_0 = -\mathbf{g}_0$
3. for $n = 1$ to n_{it}
   (a) $\alpha = \frac{\mathbf{g}_n^T\mathbf{g}_n}{\mathbf{d}_n^T\mathbf{J}^H\mathbf{J}\mathbf{d}_n}$
   (b) $\hat{\mathbf{c}}^{[n+1]} = \hat{\mathbf{c}}^{[n]} + \alpha\mathbf{d}_n$
   (c) $\mathbf{g}_{n+1} = \mathbf{g}_n + \alpha\mathbf{J}^H\mathbf{J}\mathbf{d}$
   (d) $\gamma = -\frac{\mathbf{g}_{n+1}^T\mathbf{g}_{n+1}}{\mathbf{g}_n^T\mathbf{g}_n}$
   (e) $\mathbf{d}_{n+1} = -\mathbf{g}_{n+1} + \gamma\mathbf{d}_n$
   end
4. reconstruct the diffraction field $u(r,\phi)$ from the estimated Fourier coefficient vector $\hat{\mathbf{c}}$ with Eq. 13.

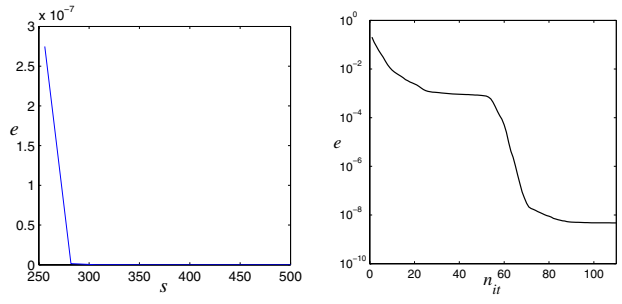
4. EXPERIMENTAL RESULTS

The non-uniform sampling problem outlined in section 3 was simulated in two experiments in order to evaluate the finite dimensional model of Eq. 13. In the experiments the given samples were generated by Eq. 14, where $M = 256$ coefficients c_m were chosen as a Gaussian pulse centered at the origin. The positions of the samples are randomly chosen inside a spatial rectangle, centered at the origin and lying in the half-plane $z > 0$. Assessment of the results is based on the normalized error between the original $\mathbf{c}$ and reconstructed $\hat{\mathbf{c}}$ coefficient vectors - $e = \|\mathbf{c} - \hat{\mathbf{c}}\|_2 / \|\mathbf{c}\|_2$.

Samples whose positions are mirror images with respect to $z = 0$ of the positions of the given samples were added to complete the initial information during the experiments. The values of the new samples are complex conjugate to the original values, as can be seen from Eq. 14. The addition of the new samples speeds up the convergence of CGN.

The goal of the first experiment is to track the behavior of CGN for different numbers of given samples. The number of iterations is kept fixed to the dimensionality of the problem $n_{it} = 256$, because this number is theoretically sufficient [8]. Figure 2(a) illustrates this experiment. As expected, the drop-off value of the curve is reached for values of s slightly higher than M. The second experiment illustrates the convergence of the CGN method (Figure 2(b)). In this experiment the number of iterations n_{it} is increased from zero to 200 while the number of given samples s is kept fixed to 384. This number is chosen to be $1.5M$, because in the first experiment this is the value where the error curve in Figure 2(a) saturates.

During the experiments for each value of s, 50 different vectors $\hat{\mathbf{c}}$ are reconstructed using s randomly chosen data points. Each reconstructed vector corresponds to a different random choice of the positions of the s known samples. The final error estimate for a value of s is an average of the errors of all 50 choices.



(a) (b)

Fig. 2. Experimental results (a) Normalized error e for different number of known samples s when the iterations are $n_{it} = 256$ (b) Convergence of the CGN algorithm when the number of given samples is $s = 384$.

5. CONCLUSION

The main goal of this paper is the derivation of discrete diffraction model suitable for non-uniform sampling and reconstruction of monochromatic light fields. It is practically feasible because its formulation is in terms of Bessel functions, for which there are accurate computational algorithms. The main advantage of the model is that it does not impose any restrictions on the field. Moreover, the model is a separable orthogonal basis where any diffraction field can be decomposed. The model was evaluated for reconstruction of diffraction fields from a set of non-uniformly distributed samples. The reconstruction converges fast for number of given samples slightly larger than the dimensionality of the model.

6. REFERENCES

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